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Asymptotic Profiles for Positive Solutions in Periodic-Parabolic Problem

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Abstract
In this paper, we are interested in the positive periodic solutions of the periodic-parabolic problem

$$\begin{cases} u_t = \Delta u + \lambda u - a(x,t)u^p & \text{in } \Omega \times (0,T], \\ Bu = 0 & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases}$$

where Ω is a $C^{2+\mu}$ bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $\lambda > 0$ is a real parameter, $p > 1$ is constant, $a \in C^{\mu,\mu/2}(\bar{\Omega} \times [0,T])$ is positive and T-periodic in t. We establish that the positive solution has a *"blow-up" phenomenon* due to large λ or small $a(x,t)$. By analyzing the sharp profiles, we find that the linear part λu and nonlinear part $a(x,t)u^p$ make quite different effects on the limiting behavior of positive periodic solutions. The second aim is then to investigate the sharp connections between linear and nonlinear parts on the asymptotic behavior of positive periodic solutions. Our study exhibits that the linear part plays a determined role. We also study the asymptotic profiles of periodic-parabolic problem with nonlocal dispersal. We find that the asymptotic profiles are different between two kinds of diffusion problems.

Keywords Reaction-diffusion · Nonlocal dispersal · Positive solution · Periodic profile

Mathematics Subject Classification 35B40 · 35K57 · 35B30

1 Introduction and Main Results

We consider the periodic-parabolic problem

$$\begin{cases} u_t = \Delta u + \lambda u - a(x,t)u^p & \text{in } \Omega \times (0,T], \\ Bu = 0 & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u(x,T) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where Ω is a $C^{2+\mu}$ bounded domain in $\mathbb{R}^N$ ($N \geq 1$), $\lambda > 0$ is a real parameter, $p > 1$ is constant and $a \in C^{\mu,\mu/2}(\bar{\Omega} \times [0, T])$ is positive and T-periodic in t. In (1.1), the boundary operator B is given by

$$Bu = \alpha_0 u_\nu + \beta_0 u,$$

here ν is the unit outward normal to $\partial\Omega$ and either $\alpha_0 = 0$, $\beta_0 = 1$ (the Dirichlet boundary condition) or $\alpha_0 = 1$, $\beta_0 \geq 0$ (the Neumann or Robin boundary conditions). Problem (1.1) is a basic model used in the study of diversity phenomena in the applied sciences (see, e.g. [2, 6, 7, 23, 24, 27, 28, 31]). It is also the paradigmatic model in population dynamics, the periodic logistic model [13, 23, 29]. In this context, Ω is the region inhabited by the population with species u and we are interested to the positive periodic solutions. Under the above assumptions, the periodic problem (1.1) as well as the corresponding elliptic problems were well studied, see [15–18, 24, 29, 30] and references therein. We know from the seminal works of Hess [23, 24] that there exists a unique positive periodic solution $\theta_\lambda \in C^{2+\mu,1+\mu/2}(\Omega \times [0, T])$ to (1.1) bifurcating from $(\lambda, u) = (\lambda_1^B(\Omega), 0)$ which is unbounded. Hereafter, $\lambda_1^B(\Omega)$ will stand for the principal eigenvalue of

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Our aim is to analyze the global structure of positive periodic solution $\theta_\lambda(x, t)$ of (1.1) with respect parameter λ. The asymptotic profile in the exterior of Ω is given by the following result.

Theorem 1.1 *Assume that $\theta_\lambda \in C^{2+\mu,1+\mu/2}(\Omega \times [0, T])$ is the unique positive solution of (1.1) for $\lambda > \lambda_1^B(\Omega)$. Let $K \subset \Omega$ be a compact subset of Ω, then*

$$\lim_{\lambda\to\infty} \lambda^{\frac{1}{1-p}} \theta_\lambda(x, t) = \left[\frac{1}{a(x, t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } K \times [0, T]. \tag{1.3}$$

As a direct conclusion of Theorem 1.1, we have

$$\lim_{\lambda\to\infty} \theta_\lambda(x, t) = \infty \text{ uniformly in } K \times [0, T].$$

Thus we know that the positive solution of (1.1) has a *"blow-up" phenomenon* due to the "large" linear part λu of reaction function. Note that the result in elliptic problem was proved by Fraile et al. [20], see also [9, 12]. In fact, we know that large linear part shall make a basic change on the asymptotic behavior.

On the other hand, if $a(x, t) \equiv 0$, then the nonlinear problem (1.1) reduces to the linear eigenvalue Eq. (1.2). In this case, the only positive solution of (1.2) is the principal eigenfunction and it is interesting to study the sharp changes of positive periodic solutions when $a(x, t)$ vanishes. To this end, we consider the following perturbation periodic-parabolic problem

$$\begin{cases} u_t = \Delta u + \lambda u - a^\varepsilon(x, t)u^p & \text{in } \Omega \times (0, \infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, t) = u(x, t + T) & \text{in } \Omega \times [0, \infty), \end{cases} \tag{1.4}$$

where $a^\varepsilon \in C^{\mu,\mu/2}(\bar{\Omega} \times [0, T])$ is T-periodic for $\varepsilon > 0$, $a(x, t)$ is as in (1.1) and

$$\lim_{\varepsilon\to 0+} \frac{a^\varepsilon(x, t)}{\varepsilon^\alpha} = a(x, t) \text{ uniformly in } \bar{\Omega} \times [0, T], \tag{1.5}$$

here $\alpha > 0$ denotes the quenching speed of nonlinear term in reaction function. It follows from the pioneering work of Hess [24] that there exists a unique positive periodic solution $\theta^\varepsilon(x,t)$ to (1.4) if and only if $\lambda > \lambda_1^B(\Omega)$, provided $\varepsilon > 0$ is small. Moreover, the positive solution $\theta^\varepsilon(x,t)$ is continuous with respect to ε. We shall analyze the asymptotic behavior of $\theta^\varepsilon(x,t)$ as follows.

Theorem 1.2 *Let $\theta^\varepsilon \in C^{2+\mu,1+\mu/2}(\Omega \times [0,T])$ be the unique positive periodic solution of (1.4) for $\lambda > \lambda_1^B(\Omega)$ and $K \subset \Omega$ be a compact subset of Ω. Then*

$$\lim_{\varepsilon\to 0+} \theta^\varepsilon(x,t) = \infty \text{ uniformly in } K \times [0,T],$$

and

$$\lim_{\varepsilon\to 0+} \varepsilon^{\frac{\alpha}{p-1}}\theta^\varepsilon(x,t) = v(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T], \tag{1.6}$$

where $v(x,t)$ stands for the unique positive periodic solution of (1.1).

The above basic results in Theorems 1.1 and 1.2 provide us that the positive periodic solution of (1.1) admits a *"blow-up" phenomenon* when the linear term λu is large or the nonlinear term $a(x,t)u^p$ is small. According to (1.3) and (1.6), we know that the asymptotic profiles are different in two cases. We then need to analyze the sharp effects between linear and nonlinear parts on the positive periodic solutions of (1.1). More precisely, we want to know which one plays a more important role. To do this, we consider the periodic-parabolic problem

$$\begin{cases} u_t = \Delta u + \frac{\lambda}{\varepsilon^\beta}u - a^\varepsilon(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega \times [0,\infty), \end{cases} \tag{1.7}$$

where $\beta > 0$, $a^\varepsilon \in C^{\mu,\mu/2}(\bar{\Omega} \times [0,T])$ is T-periodic in t for $\varepsilon > 0$ and a^ε satisfies (1.5).

We are ready to state the main result on the sharp profiles of positive solutions to (1.7).

Theorem 1.3 *Let $\theta^\varepsilon \in C^{2+\mu,1+\mu/2}(\Omega \times [0,T])$ be the unique positive periodic solution of (1.7) for $\lambda > \lambda_1^B(\Omega)$ and $K \subset \Omega$ be a compact subset of Ω. Then*

$$\lim_{\varepsilon\to 0+} \theta^\varepsilon(x,t) = \infty \text{ uniformly in } K \times [0,T],$$

and

$$\lim_{\varepsilon\to 0+} \varepsilon^{\frac{\alpha+\beta}{p-1}}\theta^\varepsilon(x,t) = \left[\frac{\lambda}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } K \times [0,T]. \tag{1.8}$$

The above theorem gives the sharp effects of linear and nonlinear parts of reaction function on the positive periodic solutions of (1.1). It is clear that both the linear and nonlinear parts make a change on the sharp blow-up profiles. However, we can see that the nonlinear part only change the blow-up speed. Furthermore, it follows from (1.3), (1.6) and (1.8) that the linear part plays a determined role on the sharp limiting profiles. Note that the periodic-parabolic problem with small diffusion rate has been studied by Daners and López-Gómez [14]. It follows from [14, Theorem 1.3] that the unique positive periodic solution converges to the positive solution of the corresponding kinetic equation without diffusion. Note also that the asymptotic profiles of positive periodic solutions are quite different between small diffusion rate and large growth rate.

On the other hand, we know that the classical reaction-diffusion equation is usually used to model diffusion with local or short effects [19]. Since the diffusion may take place between non-adjoint places, the research in nonlocal dispersal equation has attracted much attention in recently years [4, 8, 10, 11, 35, 38, 40, 43]. Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative and symmetric function. It is known that the nonlocal dispersal equation

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)[u(y,t)-u(x,t)]\,dy \text{ in } \mathbb{R}^n \times (0,\infty), \tag{1.9}$$

and variations of it, arise in the study of different dispersal process in material science, ecology, neurology and genetics (see, for instance, [21, 26, 33]). As stated in [19, 25], if $u(y,t)$ is thought of as the density at location y at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from y to x, then $\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy$ denotes the rate at which individuals are arriving to location x from all other places and $\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to all other places. Thus the right hand side of (1.9) is the change of density $u(x,t)$. There has been attracted considerable interest in the study of nonlocal dispersal equations recently, for example, the papers [5, 22, 37, 39, 41] and references therein.

In the second part of this paper, we consider the asymptotic profiles for positive periodic solutions of nonlocal dispersal problems. To do this, we study the nonlocal dispersal periodic-parabolic equation

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \lambda u - a(x,t)u^p & \text{in } \bar\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar\Omega \times [0,\infty), \end{cases} \tag{1.10}$$

where Ω is a bounded domain of $\mathbb{R}^N$ ($N \ge 1$), $\lambda > 0$ is a real parameter, $p > 1$ is constant and $a \in C(\bar\Omega \times [0,T])$ is positive and T-periodic in t. In the rest of this paper, we make the following assumption.

(H) $J \in C(\mathbb{R}^N)$ is a nonnegative, symmetric function such that $\int_{\mathbb{R}^N} J(y)\,dy = 1$ and $J(0) > 0$.

We know that there exists a unique positive solution $\omega_\lambda \in C(\bar\Omega \times [0,T])$ to (1.10) bifurcating from $(\lambda, u) = (\lambda_p(\Omega), u)$ which is unbounded, see Rawal and Shen [34], Sun et al. [36, 42]. Hereafter, $\lambda_p(\Omega)$ will stand for the principal eigenvalue of nonlocal problem

$$\int_\Omega J(x-y)u(y)\,dy - u(x) = -\lambda u \text{ in } \bar\Omega,$$

whose existence and properties are obtained in [5, 22, 25]. Since the nonlocal dispersal equation shares many properties with the reaction-diffusion equation, we will investigate the sharp behavior of positive solutions of (1.10) when the linear part is large or nonlinear part is small. However, there is a deficiency of regularity theory and compact property for nonlocal dispersal operators, the study of sharp behavior of (1.10) is quite different to (1.1), [10, 36]. We shall obtain the asymptotic behavior for nonlocal dispersal problem (1.10) by the means of nonlocal estimates and comparison arguments.

The next theorem is the limiting behavior of positive solutions of (1.10) when $\lambda \to \infty$.

Theorem 1.4 *Assume that $\omega_\lambda \in C(\bar\Omega \times [0,T])$ is the unique positive solution of (1.10) for $\lambda > \lambda_p(\Omega)$. Then*

$$\lim_{\lambda\to\infty} \theta_\lambda(x,t) = \infty \text{ uniformly in } \bar\Omega \times [0,\infty),$$

and

$$\lim_{\lambda\to\infty} \lambda^{\frac{1}{1-p}} \theta_\lambda(x,t) = \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } \bar\Omega \times [0,\infty). \tag{1.11}$$

As a complementary conclusion of Theorem 1.1 and Theorem 1.4, we know that the large linear growth term λu causes quite different asymptotic behaviors in two kinds of periodic problems, where the solution blows up in the whole domain Ω for large λ in the case of nonlocal problem.

Now let us consider the positive periodic solution of nonlocal problem

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \lambda u - a^\varepsilon(x,t)u^p & \text{in } \bar{\Omega}\times(0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar{\Omega}\times[0,\infty), \end{cases} \tag{1.12}$$

where $a^\varepsilon \in C(\bar{\Omega}\times[0,T])$ is T-periodic in t for $\varepsilon > 0$ and a^ε satisfies (1.5). In this case, we know that there exists a unique positive periodic solution $\omega^\varepsilon \in C^{0,1}(\bar{\Omega}\times[0,T])$ to (1.12) if and only if $\lambda > \lambda_p(\Omega)$, provided $\varepsilon > 0$ is small. Moreover, the positive solution $\omega^\varepsilon(x,t)$ is continuous with respect to ε [38]. We shall analyze the behavior of $\omega^\varepsilon(x,t)$ as $\varepsilon \to 0$.

Theorem 1.5 *Let $\omega^\varepsilon \in C(\bar{\Omega}\times[0,T])$ be the unique positive periodic solution of* (1.12) *for $\lambda > \lambda_p(\Omega)$. Then*

$$\lim_{\varepsilon\to 0+} \omega^\varepsilon(x,t) = \infty \text{ uniformly in } \bar{\Omega}\times[0,\infty),$$

and

$$\lim_{\varepsilon\to 0+} \varepsilon^{\frac{\alpha}{p-1}}\omega^\varepsilon(x,t) = v(x,t) \text{ uniformly in } \bar{\Omega}\times[0,\infty),$$

where $v(x,t)$ stands for the unique positive periodic solution of (1.10).

At last, we consider the nonlocal periodic problem

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \frac{\lambda}{\varepsilon^\beta}u - a^\varepsilon(x,t)u^p & \text{in } \bar{\Omega}\times(0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar{\Omega}\times[0,\infty), \end{cases} \tag{1.13}$$

where $\beta > 0$, $a^\varepsilon \in C(\bar{\Omega}\times[0,T])$ is T-periodic in t for $\varepsilon > 0$ and it satisfies (1.5). By studying the limiting behavior of the positive solutions to (1.13) as $\varepsilon \to 0$, we shall find the sharp connections between linear and nonlinear parts of reaction functions on the asymptotic behavior of positive periodic solutions of (1.10).

Theorem 1.6 *Let $\omega^\varepsilon \in C(\bar{\Omega}\times[0,T])$ be the unique positive periodic solution of* (1.13) *for $\lambda > \lambda_p(\Omega)$. Then*

$$\lim_{\varepsilon\to 0+} \omega^\varepsilon(x,t) = \infty \text{ uniformly in } \bar{\Omega}\times[0,\infty),$$

and

$$\lim_{\varepsilon\to 0+} \varepsilon^{\frac{\alpha+\beta}{p-1}}\omega^\varepsilon(x,t) = \left[\frac{\lambda}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}\times[0,\infty).$$

Remark 1.7 In Theorem 1.6, we obtain the sharp changes of positive periodic solutions to the nonlocal periodic-parabolic Eq. (1.13). From Theorems 1.3 and 1.6 we have that the reaction functions also play quite different roles between nonlocal and reaction-diffusion (local) problems. We can consider more general equations, the investigation is similar to the arguments of (1.7) and (1.13). Meanwhile, our results show that sharp changes between two kinds of equations are also different. Note that the periodic solution of nonlocal dispersal equations has a blow-up phenomenon in the whole $\bar{\Omega}$.

The rest of this paper is organized as follows. In Sect. 2, we investigate the sharp connections of linear and nonlinear parts of reaction functions to the reaction-diffusion equations. Sect. 3 is devoted to the sharp profiles of nonlocal periodic-parabolic problems.

2 The Periodic Reaction-Diffusion Problem

In this section, we shall consider the sharp profiles of positive periodic solutions to the classical reaction-diffusion equations (1.1), (1.4) and (1.7).

2.1 The Periodic Singular Perturbation Problem

In this subsection, we consider the positive periodic solution of

$$\begin{cases} u_t = \Delta u + \lambda u - a(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega \times [0,\infty), \end{cases} \tag{2.1}$$

when the parameter λ is large. Let $\theta_\lambda(x,t)$ be the unique positive periodic solution of (2.1) for $\lambda > \lambda_1^B(\Omega)$, we shall analyze the behavior of $\theta_\lambda(x,t)$ as $\lambda \to \infty$. Our main methods are based on the maximum principle of periodic-parabolic operators as well as the upper-lower solutions arguments, see e.g. [1, 3, 27] and references therein. The behavior of $\theta_\lambda(x,t)$ in the interior of Ω is given by the following result, here we adopt the method developed by Fraile et al. [20] in the study of elliptic problem.

Theorem 2.1 *Assume that $\theta_\lambda(x,t)$ is the unique positive solution of (2.1) for $\lambda > \lambda_1^B(\Omega)$. Let $K \subset \Omega$ be a compact subset of Ω, then*

$$\lim_{\lambda\to\infty} \theta_\lambda(x,t) = \infty \text{ uniformly in } K \times [0,\infty), \tag{2.2}$$

and

$$\lim_{\lambda\to\infty} \lambda^{\frac{1}{1-p}}\theta_\lambda(x,t) = \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } K \times [0,\infty). \tag{2.3}$$

Proof First note that for every $\lambda > \lambda_1^B(\Omega)$, there exists a unique positive solution $u_\lambda(x)$ to the semilinear elliptic problem

$$\begin{cases} \Delta u + \lambda u - a^*(x)u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

here

$$a^*(x) = \max_{t\in[0,T]} a(x,t).$$

It follows from the argument of upper-lower solutions that

$$\theta_\lambda(x,t) \geq u_\lambda(x)$$

for $(x,t) \in \Omega \times [0,\infty)$. On the other hand, we known that

$$\lim_{\lambda\to\infty} u_\lambda(x) = \infty \text{ uniformly in } K,$$

(see e.g. [12, 20]). This also implies (2.2).

Now we prove the second claim (2.3). The change of variable

$$v_\lambda(x,t) = \lambda^{\frac{1}{1-p}}\theta_\lambda(x,t)$$

transforms (2.1) into the singular perturbation problem

$$\begin{cases} (v_\lambda)_t = \Delta v_\lambda + \lambda[v_\lambda - a(x,t)v_\lambda^p] & \text{in } \Omega \times (0,\infty), \\ Bv_\lambda = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v_\lambda(x,t) = v_\lambda(x,t+T) & \text{in } \Omega \times [0,\infty). \end{cases} \tag{2.4}$$

Then we only need to show that

$$\lim_{\lambda\to\infty} v_\lambda(x,t) = \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } K \times [0,\infty). \tag{2.5}$$

We first take smooth function $a_\delta(x,t)$ for $\delta > 0$ such that

$$a^\delta(x,t) = a^\delta(x,t+T), \quad a^\delta(x,t) \geq a(x,t)\ gt;0$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$ and

$$\lim_{\delta\to 0+} a_\delta(x,t) = a(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T].$$

Given $\varepsilon_0 > 0$ small such that

$$a_\delta(x,t) > \varepsilon_0$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$, we define

$$\hat{v}(x,t) = [a_\delta(x,t) - \varepsilon_0]^{\frac{1}{1-p}}.$$

Subsequently, we have

$$\begin{aligned} &\hat{v}_t - \Delta\hat{v} - \lambda[\hat{v} - a(x,t)\hat{v}^p] \\ &= \hat{v}_t - \Delta\hat{v} - \lambda[a_\delta(x,t) - \varepsilon_0]^{\frac{p}{1-p}}[a_\delta(x,t) - \varepsilon_0 - a(x,t)] \\ &\geq \hat{v}_t - \Delta\hat{v} + \varepsilon\lambda[a_\delta(x,t) - \varepsilon_0]^{\frac{p}{1-p}} \\ &\geq 0, \end{aligned}$$

provided λ is sufficiently large. In this case, we know that $\hat{v}(x,t)$ is an upper-solution to (2.4) and the comparison argument gives that

$$v(x,t) \leq \hat{v}(x,t) = \left[\frac{1}{a_\delta(x,t) - \varepsilon_0}\right]^{\frac{1}{p-1}}$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$. Hence

$$\limsup_{\lambda\to\infty} v(x,t) \leq \left[\frac{1}{a_\delta(x,t)}\right]^{\frac{1}{p-1}}.$$

Letting $\delta \to 0+$, one has

$$\limsup_{\lambda\to\infty} v(x,t) \leq \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}} \tag{2.6}$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$

On the other hand, given $x_* \in K$ and $R > 0$ such that

$$B_R(x_*) = \{x \in \Omega : |x - x_*| < R\} \subset \Omega.$$

Let λ_1^R be the unique positive eigenvalue of

$$\begin{cases} \Delta u = -\lambda u & \text{in } B_R(x_*), \\ u = 0 & \text{on } \partial B_R(x_*), \end{cases}$$

associated with a positive eigenfunction $\phi(x)$ such that $\|\phi\|_{L^\infty(B_R(x_*))} = 1$. Similarly, we take smooth function $a^\delta(x,t)$ for $\delta > 0$ such that

$$a^\delta(x,t) = a^\delta(x, t+T), \;\; a^\delta(x,t) \geq a(x,t) > 0$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$ and

$$\lim_{\delta \to 0+} a^\delta(x,t) = a(x,t) \text{ uniformly in } \bar{\Omega} \times [0, T].$$

Given $\varepsilon > 0$ small such that

$$a^\delta(x,t) > \varepsilon$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$. We define

$$\bar{v}(x,t) = \alpha[a^\delta(x,t) + \varepsilon]^{\frac{1}{1-p}} \phi(x),$$

where $\alpha > 1$ satisfying

$$\frac{a(x,t) + \varepsilon}{a(x,t)} > \alpha^{p-1}$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$. Accordingly, we have

$$\begin{aligned} &\bar{v}_t - \Delta \bar{v} - \lambda[\bar{v} - a(x,t)\bar{v}^p] \\ &\leq \bar{v}_t - \Delta \bar{v} - \lambda[a^\delta(x,t) + \varepsilon]^{\frac{p}{1-p}} \phi(x)[\alpha(a(x,t) + \varepsilon) - a(x,t)\alpha^p] \\ &\leq 0, \end{aligned}$$

provided λ is sufficiently large. Thus by the comparison principle, we get

$$v(x,t) \geq \bar{v}(x,t) = \alpha \left[\frac{1}{a^\delta(x,t) + \varepsilon}\right]^{\frac{1}{p-1}} \phi(x)$$

for $(x,t) \in \bar{B}_R(x_*) \times [0,\infty)$. Since $\|\phi\|_{L^\infty(B_R(x_*))} = 1$ and $\phi(x)$ is radially symmetric, we can find $R_1 \in (0, R)$ such that

$$\phi(x) \geq \frac{1}{\alpha}$$

for $x \in B_{R_1}(x_*)$. Hence

$$\liminf_{\lambda \to \infty} v(x,t) \geq \left[\frac{1}{a^\delta(x,t)}\right]^{\frac{1}{p-1}}$$

for $(x,t) \in \bar{B}_{R_1}(x_*) \times [0,\infty)$. Letting $\delta \to 0+$, one has

$$\liminf_{\lambda \to \infty} v(x,t) \geq \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}}$$

for $(x,t) \in \bar{B}_{R_1}(x_*) \times [0,\infty)$. Note that Ω is bounded, by a standard compactness argument we have

$$\liminf_{\lambda \to \infty} v(x,t) \geq \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}}$$

for $(x, t) \in K \times [0, \infty)$. Using (2.6) we know that (2.5) holds. The theorem is thus proved. □

2.2 The Effect of Nonlinear Functions

In this subsection, we study the perturbation periodic-parabolic problem

$$\begin{cases} u_t = \Delta u + \lambda u - a^\varepsilon(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega \times [0,\infty), \end{cases} \tag{2.7}$$

here $a^\varepsilon \in C^{\mu,\mu/2}(\bar{\Omega} \times [0,T])$ is T-periodic in t for $\varepsilon > 0$,

$$\lim_{\varepsilon\to 0+} \frac{a^\varepsilon(x,t)}{\varepsilon^\alpha} = a(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T],$$

and $a \in C^{\mu,\mu/2}(\bar{\Omega} \times [0,T])$ is positive and T-periodic in t. It follows from the classical results [17, 24] that there exists a unique positive periodic solution $\theta^\varepsilon(x,t)$ to (2.7) if and only if $\lambda > \lambda_1^B(\Omega)$, provided $\varepsilon > 0$ is small. Moreover, the positive solution θ^ε is continuous with respect to ε. We shall analyze the behavior of $\theta^\varepsilon(x,t)$ as $\varepsilon \to 0$. In this case, we find the effect of nonlinear part in reaction function on the positive periodic solutions.

Theorem 2.2 *Let $\theta^\varepsilon \in C^{2+\mu,1+\mu/2}(\Omega \times [0,T])$ be the unique positive periodic solution of (2.7) for $\lambda > \lambda_1^B(\Omega)$ and $K \subset \Omega$ be a compact subset of Ω. Then*

$$\lim_{\varepsilon\to 0+} \theta^\varepsilon(x,t) = \infty \textit{ uniformly in } K \times [0,\infty). \tag{2.8}$$

Proof Consider the semilinear equation

$$\begin{cases} \Delta u + \lambda u - b^\varepsilon(x)u^p = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

where

$$b^\varepsilon(x) = \max_{t\in[0,T]} a^\varepsilon(x,t).$$

It follows that there exists a unique positive solution $u^\varepsilon(x)$ to (2.9) for $\lambda > \lambda_1^B(\Omega)$ and $\varepsilon > 0$ is small. Since

$$\lim_{\varepsilon\to 0+} \frac{b^\varepsilon(x)}{\varepsilon^\alpha} = \max_{t\in[0,T]} a(x,t) \text{ uniformly in } \bar{\Omega},$$

we know from [37] that

$$\lim_{\varepsilon\to 0+} u^\varepsilon(x) = \infty \text{ uniformly in } K. \tag{2.10}$$

But $u^\varepsilon(x)$ is the unique positive solution of (2.9), it follows from the comparison argument that

$$\theta^\varepsilon(x,t) \geq u^\varepsilon(x)$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Hence (2.8) follows from (2.10). □

Set

$$\omega^\varepsilon(x,t) = \varepsilon^{\frac{\alpha}{p-1}} \theta^\varepsilon(x,t),$$

let us investigate the sharp blow-up profiles of positive solutions.

Theorem 2.3 *Let $\theta^\varepsilon \in C^{2+\mu,1+\mu/2}(\Omega \times [0,T])$ be the unique positive periodic solution of (2.7) for $\lambda > \lambda_1^B(\Omega)$. Then*

$$\lim_{\varepsilon\to 0+} \omega^\varepsilon(x,t) = v(x,t) \textit{ uniformly in } \bar{\Omega} \times [0,\infty),$$

where $v(x,t)$ is the unique positive periodic solution of

$$\begin{cases} u_t = \Delta u + \lambda u - a(x,t)u^p & \textit{in } \Omega \times (0,\infty), \\ Bu = 0 & \textit{on } \partial\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \textit{in } \Omega \times [0,\infty). \end{cases} \tag{2.11}$$

Proof We first take $\delta > 0$ such that

$$a(x,t) > \delta > 0$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Then we choose $\varepsilon > 0$ small, denoted by $\varepsilon < \varepsilon_0$ such that

$$a(x,t) + 1 \geq \frac{a^\varepsilon(x,t)}{\varepsilon^\alpha} \geq a(x,t) - \delta > 0 \tag{2.12}$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Let $\hat{u}(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \Delta u + \lambda u - [a(x,t)+1]u^p & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega \times [0,\infty) \end{cases}$$

and $\bar{u}(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \Delta u + \lambda u - [a(x,t)+1]u^p & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega \times [0,\infty), \end{cases}$$

for $\lambda > \lambda_1^B(\Omega)$, respectively. Since $\omega^\varepsilon(x,t)$ satisfies

$$\begin{cases} \omega_t^\varepsilon = \Delta\omega^\varepsilon + \lambda\omega^\varepsilon - \frac{a^\varepsilon(x,t)}{\varepsilon^\alpha}[\omega^\varepsilon]^p & \text{in } \Omega \times (0,\infty), \\ B\omega^\varepsilon = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \omega^\varepsilon(x,t) = \omega^\varepsilon(x,t+T) & \text{in } \Omega \times [0,\infty), \end{cases} \tag{2.13}$$

by the comparison principle we have

$$0 < \bar{u}(x,t) \leq \omega^\varepsilon(x,t) \leq \hat{u}(x,t) \tag{2.14}$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Thus we know from (2.12) and (2.14) that the right-hand side of (2.13) is bounded in $L^\infty(\Omega)$, which is independent to ε. By the parabolic L^p estimates, for any $\tau \in (0,T)$, there exists $C = C_\tau > 0$ such that

$$\|\omega^\varepsilon\|_{C^{1+\mu,\frac{1+\mu}{2}}(\bar{\Omega}\times[\tau,T])} \leq C.$$

Thus we can use a diagonal argument, subject to a subsequence, to show that there has $v(x,t)$ such that

$$\lim_{\varepsilon\to 0+} \omega^\varepsilon(x,t) = v(x,t) \quad \text{in } C^{1,\frac{1}{2}}(\bar{\Omega} \times [\tau,T]),$$

and $v(x,t)$ is a weak solution to (2.11). By standard parabolic regularity we know that $v(x,t)$ satisfies (2.11) in the classical sense. Since there admits a unique solution to (2.11), we get

$$\lim_{\varepsilon\to 0+} \omega^{\varepsilon}(x,t) = v(x,t) \text{ uniformly in } \bar{\Omega}\times[0,\infty)$$

holds for the entire sequences. This also completes the proof. □

2.3 Linear vs Nonlinear

In this subsection, we study the positive periodic solution of

$$\begin{cases} u_t = \Delta u + \frac{\lambda}{\varepsilon^{\beta}} u - a^{\varepsilon}(x,t)u^p & \text{in } \Omega\times(0,\infty), \\ Bu = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega\times[0,\infty), \end{cases} \tag{2.15}$$

where $\beta > 0$, $a^{\varepsilon} \in C^{\mu,\mu/2}(\bar{\Omega}\times[0,T])$ is T-periodic in t for $\varepsilon > 0$, $a \in C^{\mu,\mu/2}(\bar{\Omega}\times[0,T])$ is positive and T-periodic in t and there exists $\alpha > 0$ such that

$$\lim_{\varepsilon\to 0+} \frac{a^{\varepsilon}(x,t)}{\varepsilon^{\alpha}} = a(x,t) \text{ uniformly in } \bar{\Omega}\times[0,T].$$

The main aim of this subsection is to investigate the sharp effect of linear and nonlinear parts in reaction functions on the asymptotic behavior of positive periodic solutions.

Letting

$$\omega^{\varepsilon}(x,t) = \varepsilon^{\frac{\alpha+\beta}{p-1}} \theta^{\varepsilon}(x,t),$$

we can establish the profile of $\omega^{\varepsilon}(x,t)$ as follows.

Theorem 2.4 *Let $\theta^{\varepsilon} \in C^{2+\mu,1+\mu/2}(\Omega\times[0,T])$ be the unique positive periodic solution of (2.15) for $\lambda > \lambda_1^B(\Omega)$ and $K \subset \Omega$ be a compact subset of Ω. Then*

$$\lim_{\varepsilon\to 0+} \theta^{\varepsilon}(x,t) = \infty \text{ uniformly in } K\times[0,\infty), \tag{2.16}$$

and

$$\lim_{\varepsilon\to 0+} \omega^{\varepsilon}(x,t) = \left[\frac{\lambda}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } K\times[0,\infty). \tag{2.17}$$

Proof Since (2.16) is followed by (2.17), we only need to show (2.17). A direct computation gives that $\omega^{\varepsilon}(x,t)$ satisfies

$$\begin{cases} \omega_t^{\varepsilon} = \Delta\omega^{\varepsilon} + \frac{\lambda}{\varepsilon^{\beta}}\omega^{\varepsilon} - \frac{a^{\varepsilon}(x,t)}{\varepsilon^{\alpha+\beta}}[\omega^{\varepsilon}]^p & \text{in } \Omega\times(0,\infty), \\ B\omega^{\varepsilon} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ \omega^{\varepsilon}(x,t) = \omega^{\varepsilon}(x,t+T) & \text{in } \Omega\times[0,\infty). \end{cases} \tag{2.18}$$

Take $l > 0$ small, we can find ε_0 such that

$$a(x,t) + l \geq \frac{a^{\varepsilon}(x,t)}{\varepsilon^{\alpha}} \geq a(x,t) - l > 0 \tag{2.19}$$

for $(x,t) \in \bar{\Omega}\times[0,T]$, provided $0 < \varepsilon \leq \varepsilon_0$. Let $\hat{u}_l^{\varepsilon}(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \Delta u + \frac{1}{\varepsilon^{\beta}}\left[\lambda u - (a(x,t) - l)u^p\right] & \text{in } \Omega\times(0,\infty), \\ Bu = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega\times[0,\infty), \end{cases}$$

and $\bar{u}_l^\varepsilon(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \Delta u + \frac{1}{\varepsilon^\beta}\left[\lambda u - (a(x,t)+l)u^p\right] & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega \times [0,\infty) \end{cases}$$

for $\lambda > \lambda_1^B(\Omega)$, respectively. Thanks to (2.19), by the comparison argument we have

$$0 < \bar{u}_l^\varepsilon(x,t) \le \omega^\varepsilon(x,t) \le \hat{u}_l^\varepsilon(x,t) \tag{2.20}$$

for $(x,t) \in \bar{\Omega} \times [0,T]$.

On the other hand, by a similar argument as in the proof of Theorem 2.1, we know that

$$\lim_{\varepsilon\to 0+} \bar{u}_l^\varepsilon(x,t) = \left[\frac{\lambda}{a(x,t)+l}\right]^{\frac{1}{p-1}} \text{ uniformly in } K \times [0,\infty),$$

and

$$\lim_{\varepsilon\to 0+} \hat{u}_l^\varepsilon(x,t) = \left[\frac{\lambda}{a(x,t)-l}\right]^{\frac{1}{p-1}} \text{ uniformly in } K \times [0,\infty).$$

Thus (2.17) is obtained by (2.20), since $l > 0$ is arbitrary. □

The main results Theorems 1.1-1.3 follow from Theorems 2.1, 2.2, 2.3 and 2.4.

3 The Periodic Nonlocal Problem

In this section, we study the effects of linear and nonlinear parts of reaction functions on the positive periodic solutions of nonlocal dispersal equations. Further, we will find the sharp connections between linear and nonlinear on the asymptotic behavior of positive periodic solutions.

3.1 The Nonlocal Singular Perturbation Problem

In this subsection, we study the positive periodic solution of

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \lambda u - a(x,t)u^p & \text{in } \bar{\Omega} \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar{\Omega} \times [0,\infty). \end{cases} \tag{3.1}$$

Let $\theta_\lambda(x,t)$ be the unique positive periodic solution of (3.1) for $\lambda > \lambda_p(\Omega)$, we first analyze the behavior of $\theta_\lambda(x,t)$ as $\lambda \to \infty$.

Theorem 3.1 *Assume that $\theta_\lambda(x,t)$ is the unique positive solution of (3.1) for $\lambda > \lambda_p(\Omega)$. Then*

$$\lim_{\lambda\to\infty} \theta_\lambda(x,t) = \infty \text{ uniformly in } \bar{\Omega} \times [0,\infty), \tag{3.2}$$

and

$$\lim_{\lambda\to\infty} \lambda^{\frac{1}{1-p}}\theta_\lambda(x,t) = \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega} \times [0,\infty). \tag{3.3}$$

Proof

(i) The proof of (3.2) is similar as in Theorem 2.4 of [42], we omit it here.

(ii) We can see that by changing the variable

$$v_\lambda(x,t) = \lambda^{\frac{1}{1-p}}\theta_\lambda(x,t),$$

Eq. (3.1) transforms into the singular perturbation problem

$$\begin{cases}(v_\lambda)_t = \int_\Omega J(x-y)v_\lambda(y,t)\,dy - v_\lambda(x,t) + \lambda[v_\lambda - a(x,t)v_\lambda^p] & \text{in } \bar{\Omega}\times(0,\infty),\\ v_\lambda(x,t) = v_\lambda(x,t+T) & \text{in } \bar{\Omega}\times[0,\infty).\end{cases} \tag{3.4}$$

Thus we only need to show that

$$\lim_{\lambda\to\infty} v_\lambda(x,t) = \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega}\times[0,\infty). \tag{3.5}$$

Take smooth function $a_\delta(x,t)$ for $\delta > 0$ satisfying

$$a_\delta(x,t) = a_\delta(x,t+T),\ \ a(x,t) \geq a_\delta(x,t)$$

for $(x,t) \in \bar{\Omega}\times[0,\infty)$, and

$$\lim_{\delta\to 0+} a_\delta(x,t) = a(x,t) \text{ uniformly in } \bar{\Omega}\times[0,T].$$

Given $\varepsilon_0 > 0$ small such that

$$a_\delta(x,t) > \varepsilon_0$$

for $(x,t) \in \bar{\Omega}\times[0,\infty)$, we define

$$\hat{v}(x,t) = [a_\delta(x,t) - \varepsilon_0]^{\frac{1}{1-p}}.$$

Subsequently, we have

$$\begin{aligned}&\hat{v}_t - \int_\Omega J(x-y)\hat{v}(y,t)\,dy + \hat{v}(x,t) - \lambda[\hat{v} - a(x,t)\hat{v}^p]\\ &= \hat{v}_t - \int_\Omega J(x-y)\hat{v}(y,t)\,dy + \hat{v}(x,t) - \lambda[a_\delta(x,t)-\varepsilon_0]^{\frac{p}{1-p}}[a_\delta(x,t) - \varepsilon - a(x,t)]\\ &\geq \hat{v}_t - \int_\Omega J(x-y)\hat{v}(y,t)\,dy + \hat{v}(x,t) + \varepsilon_0\lambda[a_\delta(x,t)-\varepsilon_0]^{\frac{p}{1-p}}\\ &\geq 0,\end{aligned}$$

provided λ is sufficiently large. In this case, we know that $\hat{v}(x,t)$ is an upper-solution to (2.4) and the comparison argument gives that

$$v(x,t) \leq \hat{v}(x,t) = \left[\frac{1}{a_\delta(x,t)-\varepsilon}\right]^{\frac{1}{p-1}}$$

for $(x,t) \in \bar{\Omega}\times[0,\infty)$. Hence

$$\limsup_{\lambda\to\infty} v(x,t) \leq \left[\frac{1}{a_\delta(x,t)}\right]^{\frac{1}{p-1}}.$$

Letting $\delta \to 0+$, one has

$$\limsup_{\lambda\to\infty} v(x,t) \leq \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}} \tag{3.6}$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$.

On the other hand, given $x_* \in \Omega$ and $R > 0$ such that

$$B_R(x_*) = \{x \in \Omega : |x - x_*| \le R\} \subset \bar{\Omega}.$$

Let λ_1^R be the unique positive eigenvalue of

$$\int_\Omega J(x-y)u(y)\,dy - u(x) = -\lambda u \text{ in } B_R(x_*),$$

associated with a positive eigenfunction $\phi(x)$ such that $\|\phi\|_{L^\infty(B_R(x_*))} = 1$.

We then take smooth function $a^\delta(x,t)$ for $\delta > 0$ such that

$$a^\delta(x,t) = a^\delta(x,t+T), \ \ a^\delta(x,t) \ge a(x,t) > 0,$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$, and

$$\lim_{\delta \to 0+} a^\delta(x,t) = a(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T].$$

Given $\varepsilon > 0$ small such that

$$a^\delta(x,t) > \varepsilon$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$, we define

$$\bar{v}(x,t) = \alpha[a^\delta(x,t) + \varepsilon]^{\frac{1}{1-p}}\phi(x),$$

where $\alpha > 1$ satisfying

$$\frac{a(x,t)+\varepsilon}{a(x,t)} > \alpha^{p-1}$$

for $(x,t) \in \bar{\Omega} \times [0,\infty)$. Accordingly, we have

$$\begin{aligned}
&\bar{v}_t - \int_\Omega J(x-y)\bar{v}(y,t)\,dy + \bar{v}(x,t) - \lambda[\bar{v} - a(x,t)\bar{v}^p] \\
&\quad \le \bar{v}_t - \int_\Omega J(x-y)\bar{v}(y,t)\,dy + \bar{v}(x,t) - \lambda[a^\delta(x,t)+\varepsilon]^{\frac{p}{1-p}}\phi(x)[\alpha(a(x,t)+\varepsilon) - a(x,t)\alpha^p] \\
&\quad \le 0,
\end{aligned}$$

provided λ is sufficiently large. Thus by the comparison principle, we get

$$v(x,t) \ge \bar{v}(x,t) = \alpha\left[\frac{1}{a^\delta(x,t)+\varepsilon}\right]^{\frac{1}{p-1}}\phi(x)$$

for $(x,t) \in B_R(x_*) \times [0,\infty)$. Since $\phi \in C(B_R(x_*))$ and $\phi(x) > 0$ for $x \in B_R(x_*)$, we can find $R_1 \in (0,R)$ such that

$$\phi(x) \ge \frac{1}{\alpha}$$

for $x \in B_{R_1}(x_*)$. Hence

$$\liminf_{\lambda \to \infty} v(x,t) \ge \left[\frac{1}{a^\delta(x,t)}\right]^{\frac{1}{p-1}}$$

for $(x,t) \in B_{R_1}(x_*) \times [0,\infty)$. Letting $\delta \to 0+$, one has

$$\liminf_{\lambda \to \infty} v(x,t) \ge \left[\frac{1}{a(x,t)}\right]^{\frac{1}{p-1}}$$

for $(x, t) \in \bar{B}_{R_1}(x_*) \times [0, \infty)$. Since Ω is bounded, by standard step arguments we have

$$\liminf_{\lambda \to \infty} v(x, t) \geq \left[\frac{1}{a(x, t)} \right]^{\frac{1}{p-1}}$$

for $(x, t) \in \bar{\Omega} \times [0, \infty)$. It follows from (3.6) that (3.3) holds. □

3.2 The Effect of Nonlinear Functions

In this subsection, we study the perturbation periodic problem

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \lambda u - a^\varepsilon(x,t)u^p & \text{in } \Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \Omega \times [0,\infty), \end{cases} \tag{3.7}$$

where $a^\varepsilon \in C(\bar{\Omega} \times [0, T])$ is T-periodic in t for $\varepsilon > 0$ and

$$\lim_{\varepsilon \to 0+} \frac{a^\varepsilon(x, t)}{\varepsilon^\alpha} = a(x, t) \text{ uniformly in } \bar{\Omega} \times [0, T],$$

with $a \in C(\bar{\Omega} \times [0, T])$ is positive and T-periodic in t. Then we know from [34, 42] that there exists a unique positive periodic solution $\theta^\varepsilon(x, t)$ to (3.7) if and only if $\lambda > \lambda_p(\Omega)$, provided $\varepsilon > 0$ is small. Moreover, the positive solution θ^ε is continuous with respect to ε.

Theorem 3.2 *Let $\theta^\varepsilon \in C(\bar{\Omega} \times [0, T])$ be the unique positive periodic solution of (3.7) for $\lambda > \lambda_p(\Omega)$, then*

$$\lim_{\varepsilon \to 0+} \theta^\varepsilon(x, t) = \infty \text{ uniformly in } \bar{\Omega} \times [0, \infty). \tag{3.8}$$

Proof Consider the semilinear equation

$$\int_\Omega J(x-y)u(y)\,dy - u(x) + \lambda u - b^\varepsilon(x)u^p = 0 \text{ in } \Omega, \tag{3.9}$$

where

$$b^\varepsilon(x) = \max_{t \in [0,T]} a^\varepsilon(x, t).$$

We know that there exists a unique positive solution $u^\varepsilon(x)$ to (3.9) for $\lambda > \lambda_p(\Omega)$. Since

$$\lim_{\varepsilon \to 0+} \frac{b^\varepsilon(x)}{\varepsilon^\alpha} = \max_{t \in [0,T]} a(x, t) \text{ uniformly in } \bar{\Omega},$$

we know from [37] that

$$\lim_{\varepsilon \to 0+} u^\varepsilon(x) = \infty \text{ uniformly in } \bar{\Omega}.$$

But $u^\varepsilon(x)$ is the unique positive solution of (3.9), it follows from comparison argument that

$$\theta^\varepsilon(x, t) \geq u^\varepsilon(x)$$

for $(x, t) \in \bar{\Omega} \times [0, T]$ and we know that (3.9) holds. □

Letting

$$\omega^\varepsilon(x) = \varepsilon^{\frac{\alpha}{p-1}} \theta^\varepsilon(x, t),$$

we then can obtain the sharp blow-up profiles of positive solutions.

Theorem 3.3 *Let $\theta^\varepsilon \in C(\Omega \times [0,T])$ be the unique positive periodic solution of* (3.7) *for $\lambda > \lambda_p(\Omega)$, then*

$$\lim_{\varepsilon \to 0+} \omega^\varepsilon(x,t) = v(x,t) \text{ uniformly in } \bar{\Omega} \times [0,\infty),$$

where $v(x,t)$ is the unique positive periodic solution of the periodic problem

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \lambda u - a(x,t)u^p & \text{in } \bar{\Omega} \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar{\Omega} \times [0,\infty). \end{cases} \tag{3.10}$$

Proof We first take $\delta > 0$ such that

$$a(x,t) > \delta > 0$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Then we choose $\varepsilon > 0$ small, denoted by $\varepsilon < \varepsilon_0$ such that

$$a(x,t) + \delta \geq \frac{a^\varepsilon(x,t)}{\varepsilon^\alpha} \geq a(x,t) - \delta > 0$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Let $\hat{u}_\delta(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \lambda u - [a(x,t) - \delta]u^p & \text{in } \bar{\Omega} \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar{\Omega} \times [0,\infty), \end{cases} \tag{3.11}$$

and $\bar{u}_\delta(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \lambda u - [a(x,t) + \delta]u^p & \text{in } \bar{\Omega} \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar{\Omega} \times [0,\infty) \end{cases} \tag{3.12}$$

for $\lambda > \lambda_p(\Omega)$, respectively. Since $\omega^\varepsilon(x,t)$ satisfies

$$\begin{cases} \omega^\varepsilon_t = \int_\Omega J(x-y)\omega^\varepsilon(y,t)\,dy - \omega^\varepsilon(x,t) + \lambda\omega^\varepsilon - \frac{a^\varepsilon(x,t)}{\varepsilon^\alpha}[\omega^\varepsilon]^p & \text{in } \Omega \times (0,\infty), \\ \omega^\varepsilon(x,t) = \omega^\varepsilon(x,t+T) & \text{in } \Omega \times [0,\infty), \end{cases}$$

from comparison argument we have

$$0 < \bar{u}_\delta(x,t) \leq \omega^\varepsilon(x,t) \leq \hat{u}_\delta(x,t) \tag{3.13}$$

for $(x,t) \in \bar{\Omega} \times [0,T]$, provided $\varepsilon < \varepsilon_0$.

A direct computation from (3.11) and (3.12) shows that $\bar{u}_\delta(x,t)$ and $\hat{u}_\delta(x,t)$ are monotone with respect to δ and

$$\lim_{\delta \to 0+} \bar{u}_\delta(x,t) = \lim_{\delta \to 0+} \hat{u}_\delta(x,t) = v(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T]. \tag{3.14}$$

Thus it follows from (3.13) that

$$\limsup_{\varepsilon \to 0+} \omega^\varepsilon(x,t) \leq \hat{u}_\delta(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T],$$

and

$$\liminf_{\varepsilon \to 0+} \omega^\varepsilon(x,t) \geq \bar{u}_\delta(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T].$$

Letting $\delta \to 0+$, we know from (3.14) that

$$\lim_{\varepsilon \to 0+} \omega^\varepsilon(x,t) = v(x,t) \text{ uniformly in } \bar{\Omega} \times [0,\infty).$$

□

3.3 Linear vs Nonlinear

In this subsection, we study the periodic-parabolic problem

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \frac{\lambda}{\varepsilon^\beta}u - a^\varepsilon(x,t)u^p & \text{in } \bar\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar\Omega \times [0,\infty), \end{cases} \tag{3.15}$$

where $\beta > 0$, $a^\varepsilon \in C(\bar\Omega \times [0,T])$ is positive and T-periodic in t for $\varepsilon > 0$ and there exists $\alpha > 0$ such that

$$\lim_{\varepsilon\to 0+} \frac{a^\varepsilon(x,t)}{\varepsilon^\alpha} = a(x,t) \text{ uniformly in } \bar\Omega \times [0,T],$$

with $a \in C(\bar\Omega \times [0,T])$ is positive and T-periodic in t. The main aim of this section is to investigate the sharp connections between linear and nonlinear parts of reaction function on the asymptotic behavior of positive periodic solutions of (3.15).

Theorem 3.4 *Let $\theta^\varepsilon \in C(\Omega \times [0,T])$ be the unique positive periodic solution of (3.15) for $\lambda > \lambda_p(\Omega)$. Set $\omega^\varepsilon(x,t) = \varepsilon^{\frac{\alpha+\beta}{p-1}}\theta^\varepsilon(x,t)$, then*

$$\lim_{\varepsilon\to 0+} \theta^\varepsilon(x,t) = \infty \textit{ uniformly in } \bar\Omega \times [0,\infty),$$

and

$$\lim_{\varepsilon\to 0+} \omega^\varepsilon(x,t) = \left[\frac{\lambda}{a(x,t)}\right]^{\frac{1}{p-1}} \textit{ uniformly in } \bar\Omega \times [0,\infty). \tag{3.16}$$

Proof We only need to show the second claim. We know from (3.15) that $\omega^\varepsilon(x,t)$ satisfies

$$\begin{cases} \omega^\varepsilon_t = \int_\Omega J(x-y)\omega^\varepsilon(y,t)\,dy - \omega^\varepsilon + \frac{\lambda}{\varepsilon^\beta}\omega^\varepsilon - \frac{a^\varepsilon(x,t)}{\varepsilon^{\alpha+\beta}}[\omega^\varepsilon]^p & \text{in } \bar\Omega \times (0,\infty), \\ \omega^\varepsilon(x,t) = \omega^\varepsilon(x,t+T) & \text{in } \bar\Omega \times [0,\infty). \end{cases}$$

Given $l > 0$ small, we can find ε_0 such that

$$a(x,t) + l \ge \frac{a^\varepsilon(x,t)}{\varepsilon^\alpha} \ge a(x,t) - l > 0 \tag{3.17}$$

for $(x,t) \in \bar\Omega \times [0,T]$, provided $0 < \varepsilon \le \varepsilon_0$. Let $\hat u^\varepsilon_l(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \frac{1}{\varepsilon^\beta}\left[\lambda u - (a(x,t)-l)u^p\right] & \text{in } \bar\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar\Omega \times [0,\infty), \end{cases}$$

and $\bar u^\varepsilon_l(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = \int_\Omega J(x-y)u(y,t)\,dy - u(x,t) + \frac{1}{\varepsilon^\beta}\left[\lambda u - (a(x,t)+l)u^p\right] & \text{in } \bar\Omega \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar\Omega \times [0,\infty) \end{cases}$$

for $\lambda > \lambda_p(\Omega)$, respectively. Using (3.17) and a comparison argument provide us

$$0 < \bar u^\varepsilon_l(x,t) \le \omega^\varepsilon(x,t) \le \hat u^\varepsilon_l(x,t) \tag{3.18}$$

for $(x,t) \in \bar\Omega \times [0,T]$.

On the other hand, by a similar proof as in the proof of Theorem 3.1, we know that

$$\lim_{\varepsilon\to 0+} \bar u^\varepsilon_l(x,t) = \left[\frac{\lambda}{a(x,t)+l}\right]^{\frac{1}{p-1}} \text{ uniformly in } \bar\Omega \times [0,\infty),$$

and

$$\lim_{\varepsilon \to 0+} \hat{u}_l^\varepsilon(x,t) = \left[\frac{\lambda}{a(x,t)-l}\right]^{\frac{1}{p-1}} \text{ uniformly in } \bar{\Omega} \times [0, \infty).$$

Thus (3.16) is obtained by (3.18), since $l > 0$ is arbitrary. □

The main results Theorems 1.4-1.6 follow from Theorems 3.1, 3.2, 3.3 and 3.4.

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Declarations

Conflict of interest There is no conflicts of interest to this work.

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