Journal of Dynamics and Differential Equations (2023) 35:1099–1131
https://doi.org/10.1007/s10884-022-10194-7

Propagation Phenomena for Nonlocal Dispersal Equations in Exterior Domains

Shao-Xia Qiao[1,2] · Wan-Tong Li[1] · Jian-Wen Sun[1]

Received: 7 August 2019 / Revised: 12 May 2022 / Accepted: 13 July 2022 /
Published online: 29 July 2022

Abstract
This paper is concerned with the spatial propagation of bistable nonlocal dispersal equations in exterior domains. We first obtain the existence and uniqueness of an entire solution which behaves like a planar traveling wave front for large negative time. Then, when the entire solution comes to the interior domain, the profile of the front will be disturbed. However, the disturbance is local in space for finite time, which means the disturbance disappears as its location is far away from the interior domain. Furthermore, we prove that the solution can gradually recover its planar wave profile uniformly in space and continue to propagate in the same direction for large positive time provided that the interior domain is compact and convex. Our work generalizes the local (Laplace) diffusion results obtained by Berestycki et al. (2009) to the nonlocal dispersal setting by using new known Liouville results and Lipschitz continuity of entire solutions due to Li et al. (2010).

Keywords Entire solution · Nonlocal dispersal · Exterior domain · Maximum principle

Mathematics Subject Classification 35K57 · 35R20 · 92D25

1 Introduction

This paper is concerned with the following bistable nonlocal dispersal equation

$$u_t(x,t) = \int_{\Omega} J(x-y)[u(y,t)-u(x,t)]dy + f(u(x,t)),\ x \in \Omega, \tag{1.1}$$

where $\Omega = \mathbb{R}^N \setminus K$ and K is a compact subset of $\mathbb{R}^N$ and the dispersal kernel function J is nonnegative. Throughout the paper, we make the following assumptions.

Submitted to JDDE on August 7, 2019.

✉ Wan-Tong Li
wtli@lzu.edu.cn

[1] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, People's Republic of China

[2] School of Mathematics and Statistics, Henan University, Kaifeng 475001, People's Republic of China

(J) The kernel function $J \in C^1(\mathbb{R}^N)$ is radially symmetric and compactly supported such that

$$J(x) \geq 0 \text{ for } x \in \mathbb{R}^N, \;\; J(0) > 0 \text{ and } \int_{\mathbb{R}^N} J(y)dy = 1.$$

(F) $f \in C^{1,1}([0, 1])$ and there exists $\theta \in (0, 1)$ such that

$$f(0) = f(1) = f(\theta) = 0, \;\; f(s) < 0 \text{ in } (0, \theta), \;\; f(s) > 0 \text{ in } (\theta, 1),$$
$$\int_0^1 f(s)ds > 0, \;\; f'(0) < 0, \;\; f'(1) < 0, \;\; f'(\theta) > 0,$$

and

$$f'(s) < \inf_{x\in\Omega} \int_{\Omega} J(x-y)dy < 1 \tag{1.2}$$

for $s \in [0, 1]$.

It follows from the assumption (F) that there exists $L_f > 0$ such that

$$|f(u+v) - f(u) - f(v)| \leq L_f uv \text{ for } 0 \leq u, v \leq 1.$$

On the other hand, we know that $\inf_{x\in\Omega} \int_{\Omega} J(x-y)dy \geq 1/2$ when K is convex. In this case, we can see that (1.2) is automatically satisfied if $f'(s) \leq 1/2$ for all $s \in [0, 1]$. Moreover, under the assumption (1.2), some nonlocal Liouville type results were established by Brasseur et al. [7]. Besides, note that K is compact, without loss of generality, we may assume that

$$K \subset \{x \in \mathbb{R}^N : x_1 \leq 0\} \text{ or } K \subset \{\mathbb{R}^N \setminus \text{supp}(J)\} \cap \{x \in \mathbb{R}^N : x_1 \leq 0\}.$$

It is well-known that the classical diffusion problem in exterior domain is established by the seminal works of Berestycki et al. [4] and Bouhours [6]. In order to study how a planar wave front propagates around an obstacle, they considered the following semi-linear parabolic problem

$$\begin{cases} u_t = \Delta u + f(u), \;\; x \in \Omega, \\ \nu \cdot \nabla u = 0, \;\; x \in \partial\Omega, \end{cases} \tag{1.3}$$

where ν denotes the outward unit normal to the smooth exterior domain Ω. More precisely, they proved how a planar traveling front can eventually recover its profile after disturbed by an obstacle K, leaving the obstacle behind. Bouhours [6] further obtained the robustness for the Liouville type results in [4]. Later on, Guo et al. [21] showed that the global mean speed of the entire solution constructed in [4] is the speed of traveling waves in homogeneous environment. More recently, Guo and Monobe [22] extended the results in [4] to V-shaped front. Hoffman et al. [27] considered a similar problem for two dimensional lattice differential equations with directionally convex obstacles.

The nonlocal dispersal equation has got numerous scholars interested, in view of its extensive use to describe the long range effects of spatial structure in biology, physics and chemistry [1, 2, 11, 16–19]. Moreover, the problems of nonlocal dispersal equations in exterior domains have attracted much attention recently. In particular, Cortázar et al. [12–14] considered the asymptotic behaviors of the solutions to linear equations. Brasseur et al. [7, 8] have established some Liouville results for such nonlocal obstacle problems and found that the stationary solutions of (1.1) converging to 1 as $|x| \to +\infty$ is indeed 1 for compact convex obstacle K.

When the obstacle K is empty, there have been many works devoted to the traveling wave solutions and entire solutions for (1.1) and its local dispersal counterpart in recent decades.

In particular, the authors of [2, 17, 34, 40] have obtained the monotone traveling wave and its asymptotic behaviors for (1.1) with bistable type nonlinearities f, and Chen [17] showed the uniqueness and stability of the traveling wave. For other types of nonlinear terms in (1.1), one can refer to [11, 18, 19, 29, 36, 37, 39] and references therein. For the results on local dispersal equations, readers can consult [15, 20, 23, 24, 28, 35, 38, 42, 43] and references therein.

If $\Omega = \mathbb{R}^N$ in (1.1), let $u(x,t) = \phi(x_1 + ct)$ and $z = x_1$, we have

$$\begin{cases} \int_{\mathbb{R}} J_1(z-y)[\phi(y) - \phi(z)]dy - c\phi'(z) + f(\phi(z)) = 0, \ z \in \mathbb{R}, \\ \phi(-\infty) = 0, \ \phi(+\infty) = 1, \\ 0 < \phi(z) < 1, \ z \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $c > 0$ and $J_1(x_1) = \int_{\mathbb{R}^{N-1}} J(x_1, y_2, y_3, \cdots, y_N)dy$. Then it follows from [2, 34] that there exists a unique real number $c > 0$ such that (1.4) admits a solution ϕ. In fact, the solution ϕ is the unique monotone planar traveling wave solution of (1.1). Besides, $\phi(z)$ satisfies

$$\begin{cases} \alpha_0 e^{\lambda z} \le \phi(z) \le \beta_0 e^{\lambda z}, \ z \le 0, \\ \alpha_1 e^{-\mu z} \le 1 - \phi(z) \le \beta_1 e^{-\mu z}, \ z > 0, \end{cases} \tag{1.5}$$

where $\alpha_0, \ \alpha_1, \ \beta_0$ and β_1 are some positive constants, λ and μ are the positive roots of

$$c\lambda = \int_{\mathbb{R}} J_1(y)e^{-\lambda y}dy - 1 + f'(0), \ c\mu = \int_{\mathbb{R}} J_1(y)e^{-\mu y}dy - 1 + f'(1),$$

and

$$\begin{cases} \gamma_0 e^{\lambda z} \le \phi'(z) \le \delta_0 e^{\lambda z}, \ z \le 0, \\ \gamma_1 e^{-\mu z} \le \phi'(z) \le \delta_1 e^{-\mu z}, \ z > 0 \end{cases} \tag{1.6}$$

for some constants $\gamma_0, \ \gamma_1, \ \delta_0$ and $\delta_1 > 0$. However, if the domain is not the whole space (such as (1.1), (1.3)), there is no classical traveling wave front. Therefore, it is naturally to consider the generalization of traveling fronts. In fact, such extensions have been introduced in [3, 5, 30]. In particular, the transition wave front, as a fully general notion of traveling front, has been widely established in many works [10, 25, 26, 31–33, 44, 45]. It is interesting to point out that the entire solution constructed in [4, 27] is indeed a generalized transition front.

In the present paper, we are interested to consider the nonlocal dispersal problem (1.1) in exterior domains. The main ingredient of this paper is to obtain a unique entire solution of (1.1) which behaves as planar wave fronts for large negative and positive time. More precisely, we first prove the existence and uniqueness of the entire solution like a planar wave front for large negative time by sub- and super-solutions method. Moreover, we find that the entire solution also approaches planar wave fronts as x is far away from K. Finally, we shall investigate the procedure how the front goes through K and eventually recovers its shape. Due to the lack of compactness of nonlocal operators, which is necessary to establish the uniqueness and asymptotic behaviors as $|x| \to +\infty$ of the entire solution, the methods and techniques adopted here are different from that in [4, 27], and additional difficulties appear when the entire solution is constructed. So motivated by the recent work of Li et al. [29], we establish the Lipschitz continuity in space variable x of entire solutions to the nonlocal problem (1.1) in exterior domains. Then we can discuss the uniqueness and asymptotic behaviors of entire solutions of (1.1). In addition, the appearance of convolution term and interior domain leads to the fact that planar wave fronts are not the solutions of (1.1), which causes much trouble in verifying the sub- and super-solutions when we construct the entire solution. Particularly, nonlocal dispersal equations admit no explicit fundamental solutions as Laplacian dispersal equations. Therefore, the sub- and super-solutions in [4, 27] are not suitable here to study the asymptotic behaviors of such an entire solution for large positive time. Consequently, we have to construct new sub- and super-solutions inspired by [4, 27] to investigate the asymptotic behaviors of the entire solution to (1.1). At last, our results show that the geometric shape of the interior domain affects the propagation of planar wave fronts.

Now we are ready to state the main result of this paper.

Theorem 1.1 *Assume that (F) and (J) hold. Let (ϕ, c) be the unique solution of* (1.4)*, and K be a compact subset of $\mathbb{R}^N$. Then there exists an entire solution $u(x,t)$ to* (1.1) *with $0 < u(x,t) < 1$ and $u_t(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, and satisfying that*

$$u(x,t) - \phi(x_1 + ct) \to 0 \text{ as } t \to -\infty \text{ uniformly in } x \in \overline{\Omega}, \tag{1.7}$$

and as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$. Moreover, if K is convex, then we have

$$u(x,t) - \phi(x_1 + ct) \to 0 \text{ as } t \to +\infty \text{ uniformly in } x \in \overline{\Omega}.$$

In particular, the condition (1.7) *determines a unique entire solution of* (1.1).

Remark 1.2 It follows from Brasseur and Coville [9, Thorem 10] that the entire solution constructed in Theorem 1.1 is a generalized transition almost-planar front with global mean speed c.

Remark 1.3 The techniques and ideas developed in this paper can be modified to treat a much more general case for deformations of K (see [7, Definition 1.2]). Let $K \subset \mathbb{R}^N$ be a compact convex set with non-empty interior and let $\{K_\epsilon\}_{0<\epsilon\le 1}$ be a family of $C^{0,\alpha}$ ($\alpha \in (0,1]$) deformations of K. Assume that

$$\max_{s\in[0,1]} f'(s) < \inf_{0<\epsilon\le 1} \inf_{x\in\mathbb{R}^N\setminus K_\epsilon} \|J(x-\cdot)\|_{L^1(\mathbb{R}^N\setminus K_\epsilon)}.$$

Then there exists $\epsilon_0 \in (0,1]$ such that the conclusions in Theorem 1.1 also hold true with K replaced by K_ϵ for $\epsilon \in (0, \epsilon_0]$.

In this paper, under assumptions that K is a compact convex set and the nonlocal dispersal kernel is compactly supported and radially symmetric, we establish the existence of an entire solution for the nonlocal dispersal equation (1.1) in the exterior domain Ω, which behaves like a planar traveling front for large negative and positive time. It is naturally to ask if the kernel function is not compactly supported, whether the entire solution we construct in this paper exists. In addition, we conjecture that when the obstacle K just being a compact subset of $\mathbb{R}^N$ is not convex, the entire solution of (1.1) can not recover its shape uniformly in space, but it converges to the nonconstant stationary solution in any bounded subset of $\mathbb{R}^N$ containing K. We shall study these in a future work.

This paper is organized as follows. In Sect. 2, we consider the Cauchy problem and establish the comparison principle for (1.1). Then the entire solution is constructed in Sect. 3. In Sect. 4, we study the behaviors of the entire solution far away from K in space for finite time. Section 5 is devoted to discussing the asymptotic behavior of the entire solution as time goes to positive infinity.

2 Preliminaries

2.1 The Nonlocal Cauchy Problem

We first consider the nonlocal Cauchy problem

$$\begin{cases} u_t(x,t) = \int_\Omega J(x-y)[u(y,t)-u(x,t)]dy + f(u(x,t)), \ x \in \Omega, \ t \geq 0, \\ u(x,0) = u_0(x), \ x \in \Omega, \end{cases} \tag{2.1}$$

where Ω is a subset of $\mathbb{R}^N$. We call $u(x,t)$ a solution of (2.1), if it satisfies

$$u(x,t) = u_0(x) + \int_0^t \int_\Omega J(x-y)[u(y,s)-u(x,s)]dyds + \int_0^t f(u(x,s))ds \tag{2.2}$$

for $x \in \Omega$ and $t \geq 0$. Then we have the following theorem.

Theorem 2.1 *Suppose that (J) holds and $f \in C^{1,1}(\mathbb{R})$. Then, for any $u_0 \in L^1(\Omega)$, there exists a unique solution $u \in C([0,t_0], L^1(\Omega))$ to (2.2) for some $t_0 > 0$.*

Proof For every $\omega \in C([0,t_0], L^1(\Omega))$, we define the norm

$$|||\omega||| = \max_{0 \leq t \leq t_0} \|\omega(\cdot,t)\|_{L^1(\Omega)},$$

and the operator

$$\mathcal{T}w(x,t) = u_0(x) + \int_0^t \int_\Omega J(x-y)[w(y,s)-w(x,s)]dyds + \int_0^t f(w(x,s))ds.$$

It is easily seen that

$$|||\mathcal{T}w||| \leq \|u_0\|_{L^1(\Omega)} + [2+L]t_0|||w|||,$$

here

$$L = \sup_{\tau \in [-|||\omega|||, |||\omega|||]} |f'(\tau)|,$$

which means $\mathcal{T}$ maps $C([0,t_0], L^1(\Omega))$ into $C([0,t_0], L^1(\Omega))$. On the other hand, note that

$$\begin{aligned} \mathcal{T}u(x,t) - \mathcal{T}v(x,t) = & \int_0^t \int_\Omega J(x-y)[u(y,s)-v(y,s)+v(x,s)-u(x,s)]dyds \\ & + \int_0^t [f(u(x,s)) - f(v(x,s))]ds. \end{aligned}$$

It follows that

$$|||\mathcal{T}u - \mathcal{T}v||| \leq 2t_0|||u-v||| + t_0L|||u-v||| \leq (2+L)t_0|||u-v|||.$$

In fact, let t_0 be sufficiently small such that $(2+L)t_0 < 1$. Then one can obtain that $\mathcal{T}$ is a strict contraction mapping in $C([0,t_0], L^1(\Omega))$.

To extend the solution to $[0,+\infty)$, we can take $u(x,t_0) \in L^1(\Omega)$ as the initial datum and further obtain a solution in $[t_0, 2t_0]$. Then by iterating this procedure, we get a solution in $[0,+\infty)$. □

2.2 Comparison Principle

Theorem 2.2 *Suppose that the assumptions of Theorem 1.1 hold and* $u(x,0)$, $v(x,0)$, $u_0(x) \in L^\infty(\Omega)$. *Furthermore, if* $u(x,t)$, $v(x,t) \in C^1([0,+\infty), L^\infty(\Omega))$ *are uniformly bounded and satisfy*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \left(\displaystyle\int_\Omega J(x-y)[u(y,t)-u(x,t)]dy\right) + f(u(x,t)) \geq 0, \ (x,t) \in \Omega \times (0,+\infty), \\ u(x,0) \geq u_0(x), \ x \in \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} - \left(\displaystyle\int_\Omega J(x-y)[v(y,t)-v(x,t)]dy\right) + f(v(x,t)) \leq 0, \ (x,t) \in \Omega \times (0,+\infty), \\ v(x,0) \leq u_0(x), \ x \in \Omega, \end{cases}$$

respectively, then

$$u(x,t) \geq v(x,t) \ \text{in} \ \Omega \times [0,+\infty).$$

Proof Define $W(x,t) = u(x,t) - v(x,t)$, it follows that $W(x,0) \geq 0$ and

$$\begin{aligned} W_t(x,t) &\geq \int_\Omega J(x-y)[W(y,t)-W(x,t)]dy + f(u(x,t)) - f(v(x,t)) \\ &= \int_\Omega J(x-y)[W(y,t)-W(x,t)]dy + F(x,t)W(x,t), \end{aligned} \tag{2.3}$$

where

$$F(x,t) = \int_0^1 f'(v(x,t) + \theta W(x,t))d\theta.$$

Suppose that there exist $t_* > 0$ and $x_* \in \Omega$ such that $W(x_*,t_*) < 0$. Denote $\theta_* = -W(x_*,t_*)$, we can take $\epsilon > 0$ and $K' > 0$ such that $\theta_* = \epsilon e^{2K't_*}$. Let

$$T_* := \sup\left\{\tau \geq 0 \mid W(x,t) > -\epsilon e^{2K't} \text{ for all } x \in \Omega, \ 0 \leq t \leq \tau\right\},$$

then we have $0 < T_* \leq t_*$ since the facts $W(x,\cdot) \in C^1(0,\infty)$ and $W(x,0) \geq 0$. Moreover, it follows that

$$\inf_\Omega W(x,T_*) = -\epsilon e^{2K'T_*}.$$

Without loss of generality, we may assume that $0 \in \Omega$ and $W(0,T_*) < -\frac{7}{8}\epsilon e^{2K'T_*}$.

Consider now the function

$$W^-(x,t,\beta) = -\epsilon\left(\frac{3}{4} + \beta Z(x)\right)e^{2K't},$$

in which $\beta > 0$ is a parameter and $Z \in L^\infty(\mathbb{R}^N)$ with $Z(0) = 1$, $\lim\limits_{|x|\to+\infty} Z(x) = 3$, $1 \leq Z(x) \leq 3$. Take $\beta_* \in \left(\frac{1}{8}, \frac{1}{4}\right]$ as the minimal value of β for which $W(x,t) \geq W^-(x,t)$ holds for all $(x,t) \in \Omega \times [0,T_*]$. Since

$$\lim_{|x|\to+\infty} W^-(x,t,\beta_*) = -\epsilon\left(\frac{3}{4} + 3\beta_*\right)e^{2K't} < -\frac{9}{8}\epsilon e^{2K't},$$

there exist $x^* \in \Omega$ and $0 < t_0 < T_*$ such that $W(x^*, t_0) = W^-(x^*, t_0, \beta_*)$. The definition of β_* now implies that

$$W_t(x^*, t_0) \leq W_t^-(x^*, t_0, \beta_*).$$

In addition, by the fact that $W(x, t) \geq W^-(x, t, \beta_*)$ for all $(x, t) \in \Omega \times [0, T_*]$, we have

$$\int_\Omega J(x^* - y)[W(y, t_0) - W(x^*, t_0)]dy \geq \int_\Omega J(x^* - y)[W^-(y, t_0, \beta_*) - W^-(x^*, t_0, \beta_*)]dy.$$

It follows from (2.3) that

$$\begin{aligned} -\frac{7}{4}\epsilon K' e^{2K't_0} \geq & W_t^-(x^*, t_0, \beta_*) \geq W_t(x^*, t_0) \\ & \geq \int_\Omega J(x^* - y)[W^-(y, t_0, \beta_*) - W^-(x^*, t_0, \beta_*)]dy \\ & \quad + F(x^*, t_0)W^-(x^*, t_0, \beta_*). \end{aligned}$$

In particular, it follows from the assumptions of Theorem 2.2 that $u(x, t)$ and $v(x, t)$ are uniformly bounded and $f'(v(x, t) + \theta W(x, t))$ is also bounded, which means there is some $\mathfrak{M} > 0$ such that $|f'(v(x, t) + \theta W(x, t))| < \mathfrak{M}$ and $|F(x, t)| < \mathfrak{M}$ for $x \in \Omega$ and $t > 0$. Then we obtain

$$\begin{aligned} -\frac{7}{4}\epsilon K' e^{2K't_0} & \geq \beta_*\epsilon \int_\Omega J(x^* - y)[Z(x^*) - Z(y)]dy e^{2K't_0} + F(x^*, t_0)W^-(x^*, t_0, \beta_*) \\ & \geq -\epsilon \left[2\beta_* + \left(3\beta_* + \frac{3}{4}\right)\mathfrak{M}\right] e^{2K't_0} \\ & \geq -\left(\frac{1}{2} + \frac{3\mathfrak{M}}{2}\right)\epsilon e^{2K't_0}. \end{aligned}$$

This leads to a contradiction upon choosing K' to be sufficiently large. □

Corollary 2.3 *Under the assumptions of Theorem 2.2, let $u(x, t)$ and $v(x, t)$ be solutions of (1.1) with initial values $u(x, 0)$ and $v(x, 0)$, respectively. If $u(x, 0) \geq v(x, 0)$ and $u(x, 0) \not\equiv v(x, 0)$, then $u(x, t) > v(x, t)$ for all $x \in \Omega$ and $t > 0$.*

Proof It is sufficient to show that $u(x, t) > v(x, t)$ for all $x \in \Omega$ and $t \in (0, t_0]$ for some $t_0 > 0$. In fact, if $u(x, t) > v(x, t)$ for all $x \in \Omega$ and $t \in (0, t_0]$, we can similarly have $u(x, t) > v(x, t)$ for all $x \in \Omega$ and $t \in [t_0, 2t_0]$. Then by repeating this process, we can obtain the results of this lemma. Now, denote

$$w(x, t) = u(x, t) - v(x, t), \quad \tilde{w}(x, t) = e^{pt}w(x, t) - \epsilon t,$$

where $\epsilon,\ p > 0$ are real numbers and p is sufficiently large such that

$$p + F(x, t) - 2 \geq 0 \text{ for all } (x, t) \in \Omega \times [0, +\infty),$$

with $F(x, t)$ being defined as that in the proof of Theorem 2.2. Suppose that, by contradiction, there is $(x_*, t_*) \in \Omega \times (0, t_0]$ such that $w(x_*, t_*) = 0$. It then follows that $\inf_{x\in\overline{\Omega}, t\in(0,t_0]} \tilde{w}(x, t) < 0$. Furthermore, one can find a sequence (x_n, t_n) such that $t_n \to \tilde{t}_*$ and

$$\lim_{n\to\infty} \tilde{w}(x_n, t_n) = \inf_{x\in\overline{\Omega}, t\in(0,t_0]} \tilde{w}(x, t) < 0.$$

Observe that

$$\begin{aligned}\tilde{w}_t(x,t) =& pw(x,t)e^{pt} + e^{pt}w_t(x,t) - \epsilon \\ \geq& pw(x,t)e^{pt} + e^{pt}\left(\int_\Omega J(x-y)[w(y,t)-w(x,t)]dy + F(x,t)w(x,t)\right) - \epsilon \\ =& \int_\Omega J(x-y)[\tilde{w}(y,t)-\tilde{w}(x,t)]dy + (p+F(x,t))[\tilde{w}(x,t)-\epsilon t] - \epsilon.\end{aligned}$$

Then we have that

$$\begin{aligned}&\tilde{w}(x_n,t_n) - \tilde{w}(x_n,0) \\ &\geq \int_0^{t_n}\left[\int_\Omega J(x_n-y)\tilde{w}(y,s)dy - \tilde{w}(x_n,s) + (p+F(x,s))[\tilde{w}(x_n,s)-\epsilon s]\right]ds - \epsilon t_n \\ &\geq \int_0^{t_n}\int_\Omega J(x_n-y)\tilde{w}(y,s)dyds + t_0\Bigg[p - 1 + \sup_{x\in\Omega,t\in(0,t_0]} F(x,s) \\ &\quad - \frac{\epsilon\left[\frac{t_0}{2}\left(\sup_{x\in\Omega,t\in(0,t_0]}F(x,s)+p\right)+1\right]}{\inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t)}\Bigg]\inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t).\end{aligned}$$

Letting $n\to\infty$, it follows that

$$\begin{aligned}&\inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t) \\ &\quad\geq t_0\left[p + \sup_{x\in\Omega,t\in(0,t_0]}F(x,s) - \frac{\epsilon\left[\frac{t_0}{2}\left(\sup_{x\in\Omega,t\in(0,t_0]}F(x,s)+p\right)+1\right]}{\inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t)}\right] \\ &\qquad\inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t).\end{aligned}$$

Choose $t_0 > 0$ being sufficiently small such that

$$t_0\left[p + \sup_{x\in\Omega,t\in(0,t_0]}F(x,s) - \frac{\epsilon\left[\frac{t_0}{2}\left(\sup_{\Omega\times(0,t_0]}F(x,s)+p\right)+1\right]}{\inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t)}\right] < 1,$$

which implies that $\inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t) > \inf_{x\in\overline{\Omega},t\in(0,t_0]}\tilde{w}(x,t)$, since $\inf_{x\in\overline{\Omega},t\in(0,t_0]} < 0$. Thus we have finished the proof. □

3 Existence and Uniqueness of the Entire Solution

This section is devoted to establishing the existence and uniqueness of an entire solution to (1.1) which behaves as a planar traveling front until it approaches the interior domain K. Since the profile ϕ in (1.4) is monotone increasing and unique in the translation sense, without loss of generality, we further assume that $\phi(0) \leq \theta$ and $\phi''(\xi) \geq 0$ for $\xi \leq 0$. The main result of this section is stated as follows.

Theorem 3.1 *Assume that (F) and (J) hold and let (ϕ, c) be the unique solution of (1.4). If $K \subset \{x \in \mathbb{R}^N : x_1 \leq 0\} \cap \mathbb{R}^N \backslash supp(J)$, then there exists an entire solution $U(x,t)$ of (1.1) satisfying*

$$0 < U(x,t) < 1, \quad U_t(x,t) > 0 \textit{ for all } (x,t) \in \overline{\Omega}\times\mathbb{R}$$

and

$$U(x,t) \to \phi(x_1+ct) \text{ as } t \to -\infty \text{ uniformly in } x \in \overline{\Omega}. \tag{3.1}$$

Moreover, condition (3.1) *determines a unique entire solution of* (1.1).

In this section, the radial symmetry of $J(\cdot)$ can be released to $J(x) = J(-x)$. Moreover, the convexity and compactness of the obstacle are not required, while the boundedness of K is necessary. We prove Theorem 3.1 by constructing sub- and super-solutions.

3.1 Construction of the Entire Solution

To establish the entire solution, we shall construct some suitable sub- and super-solutions. Inspired by [4], we define the sub-solution

$$W^-(x,t) = \begin{cases} \phi(x_1+ct-\xi(t)) - \phi(-x_1+ct-\xi(t)), \ x_1 \geq 0, \\ 0, \ x_1 < 0, \end{cases}$$

and the super-solution

$$W^+(x,t) = \begin{cases} \phi(x_1+ct+\xi(t)) + \phi(-x_1+ct+\xi(t)), \ x_1 \geq 0, \\ 2\phi(ct+\xi(t)), \ x_1 < 0, \end{cases}$$

here $\xi(t)$ is the solution of the following equation

$$\dot{\xi}(t) = Me^{\lambda_0(ct+\xi)}, \ t < -T, \ \xi(-\infty) = 0,$$

where $M, \ \lambda_0$ and T are positive constants to be specified later. A direct calculation yields that

$$\xi(t) = \frac{1}{\lambda_0} \ln \frac{1}{1 - c^{-1}Me^{\lambda_0 ct}}.$$

For the function $\xi(t)$ to be defined, one must have $1 - c^{-1}Me^{\lambda_0 ct} > 0$. In addition, we suppose that

$$ct + \xi(t) \leq 0 \text{ for } -\infty < t \leq T.$$

Thus set $T := \frac{1}{\lambda_0 c} \ln \frac{c}{c+M}$. Moreover, it follows from (1.5) that there exist two positive numbers K_ϕ and k_ϕ such that

$$\left|\phi(x_1) - C_\phi e^{\lambda x_1}\right| \leq K_\phi e^{(k_\phi+\lambda)x_1} \text{ for all } x_1 \leq 0.$$

Then the following proposition holds.

Proposition 3.2 *Assume that* $\lambda_0 < \min\{\lambda, k_\phi\}$ *and* $K \subset \{x \in \mathbb{R}^N : x_1 \leq 0\} \cap \mathbb{R}^N \backslash supp(J)$. *Then there exists a sufficiently large number* $M > 0$ *such that* $W^-(x,t)$ *and* $W^+(x,t)$ *are sub- and super-solutions of* (1.1) *in the time range* $-\infty < t \leq T_1$ *for some* $T_1 \in (-\infty, T]$.

The proof of this lemma will be given in Appendix for the coherence of this paper.

Now we are in a position to construct the entire solution. Let $u_n(x,t)$ be the unique solution of (1.1) for $t \geq -n$ with initial data

$$u_n(x,-n) = W^-(x,-n).$$

Since $W^-(x,t)$ is a sub-solution, it is not difficult to show that the sequence $\{u_n(x,t)\}_{n=1}^{\infty}$ is nondecreasing in n. Choose some constant $T^* > 0$ such that $c > \dot{\xi}(t)$ for $t \leq -\max\{T^*, T_1\}$.

In the following discussing, without loss of generality, we assume that $n^* \geq T^*$. Then, we have

$$\frac{\partial u_n(x,t)}{\partial t} = \int_\Omega J(x-y)[u_n(y,t) - u_n(x,t)]dy + f(u_n(x,t)) \tag{3.2}$$

for $n \geq n^*$, $t \geq -n$ and $x \in \Omega$. Since $\frac{\partial W^-(x,t)}{\partial t} = 0$ for $x_1 \leq 0$ and

$$W_t^-(x,t) = (c - \dot{\xi}(t))(\phi'(x_1 + ct - \xi(t)) - \phi'(-x_1 + ct - \xi(t))) \geq 0$$

for $0 < x_1 \leq |ct - \xi(t)|$ and $t < -T^*$, it follows that

$$\begin{aligned}\frac{\partial u_n(x,-n)}{\partial t} &= \int_\Omega J(x-y)[u_n(y,-n) - u_n(x,-n)]dy + f(u_n(x,-n)) \\ &\geq \frac{\partial W^-(x,-n)}{\partial t} \\ &\geq 0\end{aligned}$$

for all $x_1 \leq |cn + \xi(-n)|$. Furthermore, by Corollary 2.3, $u_n(x,t)$ satisfies

$$\frac{\partial u_n(x,t)}{\partial t} > 0 \text{ for all } x_1 \leq |cn + \xi(-n)|,\ 0 < u_n(x,t) < 1 \text{ for all } t > -n,\ x \in \Omega,$$

and

$$W^-(x,t) < u_n(x,t) < W^+(x,t) \text{ for all } -n < t \leq T^* \text{ and } x \in \Omega.$$

In particular, since the sequence $\{u_n(x,t)\}_{n=n^*}^\infty$ being uniformly bounded in n satisfies (3.2) with $f \in C^{1,1}([0,1])$, we have that the sequence $\left\{\frac{\partial u_n(x,t)}{\partial t}\right\}_{n=n^*}^\infty$ is uniformly bounded. Then it follows that $\{u_n(x,t)\}_{n=n^*}^\infty$ is well-defined for each n and equicontinuous in t. Similarly, $\left\{\frac{\partial u_n(x,t)}{\partial t}\right\}_{n=n^*}^\infty$ is equicontinuous in t. Therefore, by Arzela-Ascolit theorem, for each fixed $x \in \Omega$, there exists a subsequence, still denoted by $\left\{u_n(x,t), \frac{\partial u_n(x,t)}{\partial t}\right\}_{n=n^*}^\infty$ such that

$$\left(u_n(x,t), \frac{\partial u_n(x,t)}{\partial t}\right) \to (u(x,t), u_t(x,t)) \text{ as } n \to +\infty, \tag{3.3}$$

where the convergence is locally uniform in $t \in \mathbb{R}$. Moreover, via diagonalization, take a subsequence of $\left\{u_n(x,t), \frac{\partial u_n(x,t)}{\partial t}\right\}_{n=n^*}^\infty$, which converges to some function $U(x,t)$. Since that $u_n(x,t)$ is the solution to (3.3) with initial value $u_n(x,-n) = W^-(x,-n)$ and that $U(x,t)$ is the limit of $u_n(x,t)$ as $n \to \infty$, we have that $U(x,t)$ is well defined for $t \in \mathbb{R}$. Then it follows from Lebesgue's dominated convergence theorem that

$$U_t(x,t) = \int_\Omega J(x-y)(U(y,t) - U(x,t))dy + f(U(x,t)),$$

and

$$U_t(x,t) \geq 0,\ 0 \leq U(x,t) \leq 1.$$

Besides, it follows from the definition of $W^-(x,t)$ and $W^+(x,t)$ that

$$\sup_{x\in\Omega} |U(x,t) - \phi(x_1 + ct)| \to 0 \text{ as } t \to -\infty.$$

Note that $U(x,t)$ is not a constant, by applying Corollary 2.3 to $U_t(x,t)$ and $U(x,t)$, one have that

$$U_t(x,t) > 0 \text{ and } 0 < U(x,t) < 1.$$

In addition, inspired by [29], we can show that $U(x,t)$ satisfies the following proposition.

Proposition 3.3 *Let $U(x,t)$ be the entire solution in Theorem 3.1. Then $U(x,t)$ satisfies*

$$|U(x+\eta,t) - U(x,t)| \le M'\eta \text{ for all } (x,t) \in \bar{\Omega} \times \mathbb{R}, \tag{3.4}$$

and

$$\left|\frac{\partial U(x+\eta,t)}{\partial t} - \frac{\partial U(x,t)}{\partial t}\right| \le M''\eta \text{ for all } (x,t) \in \bar{\Omega} \times \mathbb{R} \tag{3.5}$$

with $M' > 0$ and $M'' > 0$ being two real numbers.

Proof Since $\int_{\mathbb{R}^N} J(x)dx = 1,\ J(x) \ge 0$ and $J(x)$ is compactly supported, we have $J' \in L^1(\mathbb{R}^N)$. Furthermore, we get

$$\int_\Omega |J(x+\eta) - J(x)|dx = \int_\Omega \int_0^1 |\nabla J(x+\theta\eta)\eta|d\theta dx \le L_1|\eta| \text{ for some constant } L_1 > 0.$$

Let

$$m = \inf_{u\in[0,1]} \left(\inf_{x\in\Omega} \int_\Omega J(x-y)dy - f'(u)\right) > 0$$

and $v(t)$ be a solution of the equation

$$\begin{cases} v'(t) = L_1|\eta| - mv(t) \text{ for any } t > -n, \\ v(-n) = M|\eta| \end{cases}$$

for some $M \ge 2\sup_{\xi\in\mathbb{R}} |\phi'(\xi)|$. In addition, denote $V(x,t) = u_n(x+\eta,t) - u_n(x,t)$, where $u_n(x,t)$ is the solution of (1.1) with initial value $u_n(x,-n) = W^-(x,-n)$. Then

$$\begin{aligned} V_t(x,t) \le & \int_\Omega [J(x+\eta-y) - J(x-y)][u_n(y,t) - u_n(x+\eta,t)]dy \\ & - \inf_{x\in\Omega} \int_\Omega J(x-y)dyV(x,t) + f'(\overline{V})V(x,t), \end{aligned}$$

where $\overline{V}$ is between $u_n(x,t)$ and $u_n(x+\eta,t)$. Consequently, $V(x,t)$ satisfies

$$V_t(x,t) \le L_1|\eta| - mV(t) \text{ for } t > -n \text{ and } V(x,-n) \le M|\eta|.$$

Moreover, $|V(x,t)| \le v(t) \le M^*|\eta|$ for any $x \in \Omega,\ t \ge -n$ and $M^* = M + \frac{L_1}{m}$. Indeed,

$$0 < v(t) = e^{-m(t+n)}M|\eta| + \frac{L_1\eta}{m}\left(1 - e^{-m(t+n)}\right) < \left(M + \frac{L_1}{m}\right)|\eta| < M^*|\eta|$$

for any $x \in \mathbb{R}^N$ and $t \geq -n$. In particular, in view of $f'(s) < 1$ for $s \in [0, 1]$, there holds

$$\begin{aligned}
&\left|\frac{\partial u_n(x+\eta,t)}{\partial t} - \frac{\partial u_n(x,t)}{\partial t}\right| \\
&\leq \left|\int_\Omega [J(x+\eta-y) - J(x-y)]u_n(y,t)dy\right| + \left|[u_n(x+\eta,t) - u_n(x,t)]\right| \\
&\quad + \left|f'(\overline{V})[u_n(x+\eta,t) - u_n(x,t)]\right| \\
&\leq \int_\Omega \left|J(x+\eta-y) - J(x-y)\right|dy + (1 + \max_{s\in[0,1]} f'(s))\left|u_n(x+\eta,t) - u_n(x,t)\right| \\
&\leq [L_1 + 2M^*]|\eta|.
\end{aligned}$$

At last, since $u_n(x,t) \to U(x,t)$ locally uniformly in $t \in \mathbb{R}$ as $n \to +\infty$, we have

$$\begin{aligned}
|U(x+\eta,t) - U(x,t)| &\leq |U(x+\eta) - u_n(x+\eta,t)| + |u_n(x+\eta) - u_n(x,t)| \\
&\quad + |u_n(x,t) - U(x,t)| \\
&\leq (M^* + 2)|\eta|.
\end{aligned}$$

Now take $M' = M^*+2$, we can show that (3.4) and (3.5) hold by taking $M'' = L_1+2+2M^*$. □

3.2 Uniqueness of the Entire Solution

Now, we show the uniqueness of the entire solution constructed in Theorem 3.1. First, the following lemma is valid.

Lemma 3.4 *Assume that the settings of Theorem 1.1 hold. Then for any* $\varphi \in (0, \frac{1}{2}]$, *there exist constants* $T_\varphi = T_\varphi(\varphi) > 1$ *and* $K_\varphi = K_\varphi(\varphi) > 0$ *such that*

$$U_t(x,t) \geq K_\varphi \text{ for any } t \leq -T_\varphi \text{ and } x \in \Omega_\varphi(t),$$

where

$$\Omega_\varphi(t) = \{x \in \Omega : \ \varphi \leq U(x,t) \leq 1 - \varphi\}.$$

Proof It is easy to choose T_φ and M_φ such that $\Omega_\varphi(t) \subset \left\{x \in \Omega : |x_1 + ct| \leq M_\varphi\right\} \subset \{x \in \mathbb{R}^N : x_1 \geq 1\}$. Now suppose there exist sequences $t_k \in (-\infty, -T_\varphi]$ and $x^k := (x_1^k, x_2^k, ...x_N^k) \in \Omega_\varphi(t)$ such that

$$U_t(t_k, x^k) \to 0 \text{ as } k \to +\infty.$$

There are only two cases that can happen: $t_k \to -\infty$ or $t_k \to t_*$ for some $t_* \in (-\infty, -T_\varphi]$ as $k \to +\infty$. For the former case, denote

$$U_k(x,t) = U(x + x^k, t + t_k).$$

By Lemma 3.3, $\{U_k(x,t)\}_{k=1}^\infty$ is equicontinuous in $x \in \Omega$ and $t \in \mathbb{R}$. It follows that there exists a subsequence still denoted by $\{U_k(x,t)\}_{k=1}^\infty$ such that

$$U_k \to U_* \text{ as } k \to +\infty \text{ locally uniformly in } (x,t) \in \Omega \times \mathbb{R}$$

by Arzela-Ascolit theorem. Furthermore, U_* satisfies $\frac{\partial U_*(0,0)}{\partial t} = 0$. Applying strong maximum principle theorem to $\frac{\partial U_*(x,t)}{\partial t}$, We further have

$$\frac{\partial U_*(x,t)}{\partial t} \equiv 0 \text{ for all } t \leq 0 \text{ and } x \in \Omega.$$

However, this is impossible because

$$U_*(x,t) = \phi(x_1 + ct + a) \text{ for some } a \in [-M_\eta, M_\eta].$$

For the second case, x_1^k remains bounded by the definition of $\Omega_\varphi(t)$. Therefore, we assume that $x_1^k \to x_1^*$ as $k \to +\infty$ and let

$$U_k(x,t) := U(x + x^k, t).$$

Then, each $U_k(x,t)$ is defined for all $(x,t) \in (-\infty, -T_\varphi] \times \{x \in \mathbb{R}^N \mid x_1 \geq -1\}$ by the definition of $\Omega_\varphi(t)$. Similarly, there exists a subsequence, again denoted by $\{U_k\}_{k=1}^\infty$, such that

$$U_k \to U^* \text{ as } k \to +\infty \text{ locally uniformly in } (x,t) \in \Omega \times \mathbb{R} \text{ with } x_1 \geq -1,$$

for some function U^* satisfying (1.1) on $\{x \in \mathbb{R}^N \mid x_1 \geq -1\} \times (-\infty, -T_\varphi]$. Note that $U_t(x,t) > 0$, we have

$$\frac{\partial U^*}{\partial t}(0, t_*) = 0, \quad \frac{\partial U^*}{\partial t}(x,t) \geq 0 \text{ for } (x,t) \in \{x \in \mathbb{R}^N \mid x_1 \geq -1\} \times (-\infty, -T_\varphi].$$

Then we obtain $\frac{\partial U^*}{\partial t}(x,t) \equiv 0$ for $t \leq t_*$ by strong maximum principle, but this is impossible since

$$U^*(x,t) - \phi(x_1 + x_1^* + ct) \to 0 \text{ as } t \to -\infty \text{ uniformly in } \{x \in \mathbb{R}^N \mid x_1 \geq -1\}.$$

This ends the proof. □

Now we are ready to show the uniqueness of the entire solution. Suppose that there exists another entire solution $V(x,t)$ of (1.1) satisfying (3.1). Extend the function f as

$$f(s) = f'(0)s \text{ for } s \leq 0, \quad f(s) = f'(1)(s-1) \text{ for } s \geq 1$$

and choose $\eta > 0$ being sufficiently small such that

$$f'(s) \leq -\omega \text{ for } s \in [-2\eta, 2\eta] \cup [1-2\eta, 1+2\eta] \text{ and } \omega > 0.$$

Then for any $\epsilon \in (0, \eta)$, we can find $t_0 \in \mathbb{R}$ such that

$$\|V(\cdot,t) - U(\cdot,t)\|_{L^\infty(\Omega)} < \epsilon \text{ for } -\infty < t \leq t_0. \tag{3.6}$$

For each $t_0 \in (-\infty, T_\varphi - \sigma\epsilon]$, define

$$\tilde{U}^+(x,t) := U\left(x, t_0 + t + \sigma\epsilon\left(1 - e^{-\omega t}\right)\right) + \epsilon e^{-\omega t}$$

and

$$\tilde{U}^-(x,t) := U\left(x, t_0 + t - \sigma\epsilon\left(1 - e^{-\omega t}\right)\right) - \epsilon e^{-\omega t},$$

where the constant $\sigma > 0$ is specified later. Then by (3.6),

$$\tilde{U}^-(x,0) \leq V(x,t_0) \leq \tilde{U}^+(x,0) \text{ for all } x \in \Omega. \tag{3.7}$$

Next we show that $\tilde{U}^-(x,t)$ and $\tilde{U}^+(x,t)$ are sub- and super-solutions in $t \in [0, T_\varphi - t_0 - \sigma\epsilon]$, respectively. Let

$$\mathcal{L}w(x,t) := w_t(x,t) - \int_\Omega J(x-y)[w(y,t) - w(x,t)]dy + f(w(x,t)) \text{ for all } (x,t) \in \bar{\Omega} \times \mathbb{R}.$$

A straightforward calculation implies that

$$\begin{aligned}\mathcal{L}\tilde{U}^+(x,t) =& \sigma\epsilon\omega e^{-\omega t}U_t - \epsilon\omega e^{-\omega t} + f(U) - f\left(U + \epsilon e^{-\omega t}\right)\\ =& \epsilon e^{-\omega t}\left(\sigma\omega U_t - \omega - f'\left(U + \theta\epsilon e^{-\omega t}\right)\right),\end{aligned}$$

where $0 < \theta < 1$. For any $x \notin \Omega_\eta\left(x, t_0 + t + \sigma\epsilon\left(1 - e^{-\omega t}\right)\right)$, we can see

$$U + \theta\epsilon e^{-\omega t} \in [0, 2\eta] \cup [1 - \eta, 1 + \eta].$$

Consequently, $f'\left(U + \theta\epsilon e^{-\omega t}\right) \le -\omega$, which implies that

$$\mathcal{L}\tilde{U}^+(x,t) \ge \epsilon e^{-\omega t}(-\omega + \omega) = 0.$$

For $x \in \Omega_\eta\left(x, t_0 + t + \sigma\epsilon\left(1 - e^{-\omega t}\right)\right)$, by Lemma 3.4, there holds

$$\mathcal{L}\tilde{U}^+(x,t) \ge \epsilon e^{-\omega t}\left(\sigma\omega K_\eta - \omega - \max_{0\le s\le 1} f'(s)\right).$$

As a consequence, $\mathcal{L}\tilde{U}^+(x,t) \ge 0$ provided that σ is a sufficiently large number.

Similarly, we can show $\mathcal{L}\tilde{U}^-(x,t) \le 0$ in $\Omega \times [0, T_\varphi - t_0 - \sigma\epsilon]$. In view of this and (3.7), we see that

$$\tilde{U}^-(x,t) \le V(x, t + t_0) \le \tilde{U}^+(x,t) \text{ for all } (x,t) \in \Omega \times [0, T_\varphi - t_0 - \sigma\epsilon].$$

Letting $t + t_0$ be replaced by t, the inequality above can be rewritten as

$$\begin{aligned}&U\left(x, t - \sigma\epsilon\left(1 - e^{-\omega(t-t_0)}\right)\right) - \epsilon e^{-\omega(t-t_0)}\\ &\le V(x,t) \le U\left(x, t + \sigma\epsilon\left(1 - e^{-\omega(t-t_0)}\right)\right) + \epsilon e^{-\omega(t-t_0)}\end{aligned}$$

for all $(x,t) \in \Omega \times [t_0, T_\varphi - \sigma\epsilon]$ and $t_0 \in (-\infty, T_\varphi - \sigma\epsilon]$. As $t_0 \to -\infty$, we obtain that

$$U(x, t - \sigma\epsilon) \le V(x,t) \le U(x, t + \sigma\epsilon) \text{ for all } (x,t) \in \Omega \times (-\infty, T_\varphi - \sigma\epsilon].$$

By the comparison principle, the inequality holds for $t \in \mathbb{R}$ and $x \in \Omega$. Letting $\epsilon \to 0$, since σ is independent of the choice of ϵ, we have $V(x,t) \equiv U(x,t)$.

4 Behaviors Far Away from the Interior Domain

In this section, we are going to figure out what the entire solution, constructed in Theorem 3.1, is like far away from the interior domain.

Theorem 4.1 *Assume that (F) and (J) hold. Let (ϕ, c) be the unique solution of (1.4) and $u(x,t)$ be a solution of*

$$\begin{cases} u_t(x,t) = \int_\Omega J(x-y)[u(y,t) - u(x,t)]dy + f(u(x,t)), & (x,t) \in \overline{\Omega} \times \mathbb{R},\\ 0 \le u(x,t) \le 1, & (x,t) \in \overline{\Omega} \times \mathbb{R}\end{cases} \tag{4.1}$$

such that

$$\sup_{x\in\bar{\Omega}} |u(x,t) - \phi(x_1 + ct)| \to 0 \textit{ as } t \to -\infty.$$

Then, for any sequence $(x_n')_{n\in\mathbb{N}} \in \mathbb{R}^{N-1}$ *such that* $|x_n'| \to +\infty$ *as* $n \to +\infty$*, there holds*

$$u(x_1, x' + x_n', t) \to \phi(x_1 + ct) \text{ as } n \to +\infty,$$

locally uniformly with respect to $(x, t) = (x_1, x', t) \in \mathbb{R}^N \times \mathbb{R}$*.*

Proof In order to prove this theorem we need to extend the function f as that in Sect. 3, and let $A > 0$ be sufficiently large such that

$$\phi(\xi) \le \frac{\eta}{2} \text{ for } \xi \le -A, \quad \phi(\xi) \ge 1 - \frac{\eta}{2} \text{ for } \xi \ge A.$$

Denote $\delta = \min_{[-A,A]} \phi'(\xi) > 0$ and take $T_\phi > 0$ such that for any $\epsilon \in (0, \frac{\eta}{2})$, we have that

$$|u(x, t) - \phi(x_1 + ct)| \le \epsilon \text{ for all } t \le -T_\phi \text{ and } x \in \Omega.$$

Now we are in a position to show this theorem. Under the assumptions of Theorem 4.1, let $(x_n')_{n\in\mathbb{N}} \in \mathbb{R}^{N-1}$ be a sequence such that $|x_n'| \to +\infty$ as $n \to +\infty$. And denote $u_n(x, t) = u(x_1, x' + x_n', t)$ for each $t \in \mathbb{R}$ and $x = (x_1, x') \in \Omega - (0, x_n')$. Since $0 \le u \le 1$, K is compact and $\left\{u_n(x, t), \frac{\partial u_n(x,t)}{\partial t}\right\}_{n=1}^{\infty}$ is equicontinuous in $x \in \Omega$ and $t \in \mathbb{R}$ by Proposition 3.3, then there exists a subsequence, still denoted by $\left\{u_n(x, t), \frac{\partial u_n(x,t)}{\partial t}\right\}_{n=1}^{\infty}$, such that

$$u_n(x, t) \to \mathfrak{u}(x, t), \ \frac{\partial u_n(x, t)}{\partial t} \to \mathfrak{u}_t(x, t) \text{ as } n \to +\infty,$$

locally uniformly in $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. In addition, for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, we further have that $0 \le \mathfrak{u}(x, t) \le 1$ and

$$\mathfrak{u}_t(x, t) = \int_{\mathbb{R}^N} J(x - y)[\mathfrak{u}(y, t) - \mathfrak{u}(x, t)]dy + f(\mathfrak{u}(x, t)), \tag{4.2}$$

since J is compactly supported, K is compact and $|x_n'| \to +\infty$ as $n \to \infty$. In addition, recall that

$$u_n(x, t) - \phi(x_1 + ct) \to 0 \text{ as } t \to -\infty,$$

uniformly in $x \in \overline{\Omega}$, the function $\mathfrak{u}(x, t)$ satisfies

$$\mathfrak{u}(x, t) - \phi(x_1 + ct) \to 0 \text{ as } t \to -\infty \text{ locally uniformly in } \mathbb{R}^N.$$

Now define two functions $\underline{u}(x, t)$ and $\overline{u}(x, t)$ as follows

$$\underline{u}(x, t) = \phi(\xi_-(x, t)) - \epsilon e^{-\omega(t-t_0)}, \ \overline{u}(x, t) = \phi(\xi_+(x, t)) + \epsilon e^{-\omega(t-t_0)}, \ t \ge t_0, \ x \in \mathbb{R}^N,$$

where

$$t_0 \le -T, \ \xi_\pm(x, t) = x_1 + ct \pm 2\epsilon\|f'\|\delta^{-1}\omega^{-1}\left[1 - e^{-\omega(t-t_0)}\right].$$

Then the following lemma holds, whose proof is left to Appendix as a regular argument. □

Lemma 4.2 *The functions* $\underline{u}(x, t)$ *and* $\overline{u}(x, t)$ *are sub- and super-solutions to* (4.2) *for* $t \ge t_0$*, respectively.*

By the comparison theorem and let $\epsilon \to 0$, we have $\mathfrak{u}(x, t) \equiv \phi(x_1 + ct)$. Since the limit is uniquely determined, the sequence $\{u_n(x, t)\}_{n=1}^{\infty}$ converges to $\phi(x_1 + ct)$ locally uniformly in $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ as $n \to +\infty$. Then the proof of Theorem 4.1 is completed. □

Theorem 4.3 *Suppose that all the assumptions in Theorem 4.1 hold. Then the solution $u(x,t)$ of (4.1) in Theorem 4.1 satisfies*

$$|u(x,t)-\phi(x_1+ct)| \to 0 \text{ as } |x| \to +\infty \text{ locally uniformly in } t \in \mathbb{R}.$$

Proof Extend f as that in Sect. 3. Then define $f_\delta(u) = f(u-\delta)$, (c_δ, ϕ_δ) satisfies

$$c_\delta(\phi_\delta)'(x) = \int_{\mathbb{R}^N} J(x-y)[\phi_\delta(y)-\phi_\delta(x)]dy + f_\delta(\phi_\delta(x)),$$

and

$$\phi_\delta(-\infty) = \delta,\ \phi_\delta(+\infty) = 1+\delta,$$

where $c_\delta > 0$ and $\delta > 0$ is sufficiently small. Now we are going to show the theorem in three steps.

Step 1. For $x_1 \gg 1$, since

$$\sup_{x\in\bar{\Omega}} |u(x,t)-\phi(x_1+ct)| \to 0 \text{ as } t \to -\infty,$$

there exists a sufficiently large number $T_1^* \geq 0$ such that

$$|u(x,t)-\phi(x_1+ct)| \leq \frac{\epsilon}{2} \text{ for all } x \in \Omega \text{ and } t \leq -T_1^*.$$

In particular, for any $x \in \Omega$, let $N_1 \gg 1$ such that

$$\phi(x_1 - cT_1^*) \geq 1-\frac{\epsilon}{2},\ u(x,-T_1^*) \geq 1-\epsilon \text{ for all } x_1 \geq N_1.$$

Since $\phi' > 0,\ u_t(x,t) > 0$, we have

$$|u(x,t)-\phi(x_1+ct)| < \epsilon \text{ for all } x_1 \geq N_1 \text{ and } t \geq -T_1^*.$$

Step 2. For $x_1 \ll -1$, let $\delta = \frac{1}{2}\epsilon$. Similarly as the first step, choose $T_2 > 0$ and $N_2 \gg 1$ such that

$$u(x,t) \leq \delta \text{ for all } x_1 \leq -N_2 \text{ and } t \leq -T_2,\ \text{particularly},\ u(x,-T_2) \leq \delta.$$

Obviously, there exists $x_0 \in \mathbb{R}$ such that $\phi_\delta(x_0) = 1$. Then

$$u(x,-T_2) \leq \phi_\delta(x_1+x_0+N_2) \text{ for all } x \in \Omega.$$

Since that $\phi_\delta(x,t)$ is increasing, we have

$$\phi_\delta(x_1+c_\delta(t+T_2)+x_0+N_2) \geq \phi_\delta(-N_2+x_0+N_2) = 1 \text{ for all } x_1 \geq -N_2 \text{ and } t \geq -T_2.$$

Then, by applying the comparison principle (see [41]) for $t \geq -T_2$ and $x_1 \leq -N_2$, one have that

$$u(x,t) \leq \phi_\delta(x_1+x_0+N_2+c_\delta(t+T_2)) \text{ for all } x_1 \leq -N_2 \text{ and } t \geq -T_2.$$

In particular, since $\phi_\delta(-\infty) = \delta = \frac{1}{2}\epsilon$, for any $\tau \geq 0$, there exists a $N_3 \gg 1$ such that

$$0 \leq u(x,t) \leq 2\delta < \epsilon,\ 0 < \phi(x_1+ct) \leq \epsilon \text{ for all } x_1 \leq -N_3 \text{ and } \tau \geq t \geq -T_2,$$

which again shows

$$|u(x,t)-\phi(x_1+ct)| \leq \epsilon \text{ for all } x_1 \leq -N_3 \text{ and } t \leq \tau.$$

Step 3. For $|x'| \gg 1$, it follows from the Theorem 4.1 that one can choose $N_4 > 0$ being sufficiently large such that

$$|u(x,t) - \phi(x_1 + ct)| \le \epsilon \text{ holds true for } t \text{ in any bounded interval,}$$

whenever $|x'| > N_4$ and $x_1 \in [-N_3, N_1]$. This finishes the proof. □

5 The Behavior for the Large Time

In this section we intend to investigate the behavior of the solution constructed in Theorem 3.1 for large positive time. For this goal, we establish the following result.

Theorem 5.1 *Suppose that (F) and (J) hold. Let (ϕ, c) be the unique solution of (1.4), $t_0 \in \mathbb{R}$ and $u(x,t)$ be a solution of*

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)[u(y,t) - u(x,t)]dy + f(u(x,t)), & (x,t) \in \overline{\Omega} \times [t_0, +\infty), \\ 0 \le u(x,t) \le 1, & (x,t) \in \overline{\Omega} \times [t_0, +\infty). \end{cases} \tag{5.1}$$

And assume that, for any $\epsilon > 0$, there is a number $t_\epsilon \ge t_0$ and a compact set $K_\epsilon \subset \overline{\Omega}$ such that

$$|u(x,t_\epsilon) - \phi(x_1 + ct_\epsilon)| \le \epsilon \textit{ for all } x \in \overline{\Omega \backslash K_\epsilon},$$

and

$$u(x,t) \ge 1 - \epsilon \textit{ for all } t \ge t_\epsilon \textit{ and } x \in \partial\Omega = \partial K_\epsilon.$$

Then

$$\sup_{x \in \overline{\Omega}} |u(x,t) - \phi(x_1 + ct)| \to 0 \textit{ as } t \to +\infty.$$

The most important ingredient of the proof is to construct suitable sub- and super-solutions. This process is such cumbersome that will be divided several parts. Enlightened by Hoffman [27], we first construct a function $z(t)$ in the following lemma, which plays an important role in the construction of sub- and super-solutions.

Lemma 5.2 *For any $0 < \eta_z < \ln 2$, there are two constants $\mathcal{I} = \mathcal{I}(\eta_z) > 0$ and $K_0 = K_0(\eta_z) > 0$ such that for any $t_1 \ge 0$, there exists a C^1-smooth function $\tilde{z}(t) : [0, +\infty) \to \mathbb{R}$ that satisfies the following properties.*

(i) *For all $t \ge 0$, the inequalities $\tilde{z}'(t) \ge -\eta_z \tilde{z}(t)$ and $0 < \tilde{z}(t) \le \tilde{z}(0) = 1$ hold.*
(ii) *In addition, $\tilde{z}(t) \ge K_0(1 + t - t_1)^{-\frac{3}{2}}$ for all $t \ge t_1$ and $\int_0^{+\infty} \tilde{z}(t)dt < \mathcal{I}$.*

Proof First we define

$$P_-(x) = -\frac{1}{3}\eta_z^2\left(x + \eta_z^{-1}\right)^2 + 1, \ \ P_+(x) = \frac{\nu\eta_z}{3}\left(x - \eta_z^{-1}\right)^2 + \frac{2}{3} - \frac{\nu}{3\eta_z}, \ \ 0 < \nu < \eta_z.$$

Let $l_P(\eta_z) = 1/\eta_z$. Then, by a direct calculation, it is easy to show that for any fixed $0 < \eta_z < \ln 2$ and $0 < \nu \le \eta_z$, $P_-(x)$ satisfies that

$$P_-(-l_P) = 1, \quad P_-'(-l_P) = 0, \quad P_-(0) = \frac{2}{3}, \quad P_-'(0) = -\frac{2}{3}\eta_z,$$

with

$$-\eta_z P_-(x) \le P_-'(x) \le 0, \quad -l_P \le x \le 0,$$

and $P_+(x)$ satisfies that

$$P_+(0) = \frac{2}{3}, \quad P_+'(0) = -\frac{2}{3}\nu, \quad P_+'(l_P) = 0,$$

$P_+(l_P) \ge \frac{1}{3}$ with

$$-\eta_z P_+(x) \le P_+'(x) \le 0, \ 0 \le x \le l_P.$$

In addition, denote

$$z_1(t) = \begin{cases} e^{-\eta_z t}, & 0 \le t \le \frac{3}{2}\eta_z^{-1} - 1, \\ \eta_z^{-\frac{3}{2}} \left(\frac{3}{2}\right)^{\frac{3}{2}} e^{\eta_z - \frac{3}{2}} (1+t)^{-\frac{3}{2}}, & t \ge \frac{3}{2}\eta_z^{-1} - 1. \end{cases}$$

Now if $0 < t_1 < 3\eta_z^{-1}$, then let $\tilde{z}(t) = z_1(t)$, otherwise we define the function $\tilde{z}(t)$ on five different intervals. In particular, define $\nu = \frac{-z_1'(t_1 - 3\eta_z^{-1})}{z_1(t_1 - 3\eta_z^{-1})}$, which implies $0 < \nu \le \eta_z$. Then, let

$$\tilde{z}(t) = \begin{cases} z_1(t), & 0 \le t \le t_1 - 3\eta_z^{-1}, \\ z_1\left(t_1 - 3\eta_z^{-1}\right) P_+\left(t - \left(t_1 - 3\eta_z^{-1}\right)\right), & t_1 - 3\eta_z^{-1} \le t \le t_1 - 2\eta_z^{-1}, \\ P_-(t - t_1), & t_1 - \eta_z^{-1} \le t \le t_1, \\ \frac{2}{3} z_1(t - t_1), & t \ge t_1. \end{cases}$$

It remains to specify $\tilde{z}(t)$ in $\left[t_1 - 2\eta_z^{-1}, t_1 - \eta_z^{-1}\right]$. This can be done by choosing an arbitrary C^1-smooth function, under the constraints

$$\tilde{z}\left(t_1 - 2\eta_z^{-1}\right) = z_1\left(t_1 - 3\eta_z^{-1}\right) P_+\left(\eta_z^{-1}\right), \ z\left(t_1 - \eta_z^{-1}\right) = 1, \ \tilde{z}'\left(t_1 - 2\eta_z^{-1}\right) = z'\left(t_1 - \eta_z^{-1}\right) = 0$$

and

$$\tilde{z}'(t) \ge 0, \quad t_1 - 2\eta_z^{-1} \le t \le t_1 - \eta_z^{-1}.$$

Then we finish the proof since the properties (i) and (ii) are valid by a direct calculation. □

Remark 5.3 It is not difficult to see that $z(1) \ge \frac{1}{2}$ from $0 < \eta_z < \ln 2$ and the first statement in Lemma 5.2.

Now, we are in the position to construct the sub- and super-solutions.

5.1 Sub-solution

In this part, we construct a sub-solution to (5.1). Now define

$$\tilde{u}(x,t) = u(x, t - 1 + t_\epsilon), \ u^-(x,t) = \phi(x_1 + c(t - 1 + t_\epsilon) - \theta(x', t) - Z(t)) - z(t),$$

where $\theta(x', t) = \beta t^{-\alpha} e^{\frac{-|x'|^2}{\gamma t}}$ with $\alpha,\ \gamma > 1$ being two real numbers, $z(t) = \epsilon_1 \tilde{z}(t)$ with $\epsilon_1 = 2\epsilon$ and $Z(t) = K_z \int_0^t z(\tau)d\tau$ with $K_z > 0$ being a large number. $z(t)$ is defined in Lemma 5.2. It follows from the definition of $u^-(x, t)$ that

$$u^-(x, 1) \le \phi\left(x_1 + ct_\epsilon - \beta e^{\frac{-|x'|^2}{\gamma}} - Z(1)\right) \le \tilde{u}(x, 1) = u(x, t_\epsilon) \text{ for all } x \in K_\epsilon.$$

Thanks to $\phi(-\infty) = 0,\ \phi' > 0, 0 < u(x, t) < 1$ and $\min_{x \in K_\epsilon} u(x, t) > 0$, the last inequality holds provided that β is sufficiently large. If $x \in \mathbb{R}^N \backslash K_\epsilon$, then by $z(1) \ge \frac{1}{2}\epsilon_1 = \epsilon$ (from Remark 5.3), we have that

$$u^-(x, 1) \le \phi(x_1 + ct_\epsilon) - z(1) \le \phi(x_1 + ct_\epsilon) - \epsilon \le u(x, t_\epsilon) = \tilde{u}(x, 1).$$

As a consequence, there holds $u^-(x, 1) \le \tilde{u}(x, 1)$ for any $x \in \overline{\Omega}$.

Lemma 5.4 *The inequality $\mathcal{L}u^-(x, t) \le 0$ holds for all $x \in \Omega$ and $t \ge 1$, where*

$$\mathcal{L}u^-(x, t) = u_t^-(x, t) - \int_\Omega J(x - y)[u^-(y, t) - u^-(x, t)]dy - f(u^-(x, t)).$$

Proof Since $u^-(x, t) = \phi(\xi(x, t)) - z(t)$, where $\xi(x, t) = x_1 + c(t - 1 + t_\epsilon) - \beta t^{-\alpha} e^{\frac{-|x'|^2}{t\gamma}} - Z(t)$, we have

$$u_t^-(x, t) = \phi'(\xi(x, t))(c - \theta_t(x', t) - Z') - z'(t),$$

and

$$\int_\Omega J(x - y)[u^-(y, t) - u^-(x, t)]dy = \int_\Omega J(x - y)[\phi(\xi(y, t)) - \phi(\xi(x, t))]dy.$$

Denote $\mathcal{D}\phi = \int_{\mathbb{R}^N} J(x-y)[\phi(\xi(y, t))-\phi(\xi(x, t))]dy$. Then, applying mean value theorem, we get that

$$\begin{aligned}\mathcal{D}\phi = &\int_{\mathbb{R}^N} J(y)[\phi(\xi(x, t) - y_1) - \phi(\xi(x, t))]dy + \int_{\mathbb{R}^N} J(y)\Big[\phi\Big(x_1 - y_1 + c(t - 1 + t_\epsilon)\\ &- \beta t^{-\alpha} e^{\frac{-|x'-y'|^2}{t\gamma}} - Z(t)\Big) - \phi(\xi(x, t) - y_1)\Big]dy\\ \ge & c\phi'(\xi(x, t)) - f(\phi(\xi(x, t))) - C^0\phi'(\xi(x, t))\beta t^{-\alpha}\\ &\int_{\mathbb{R}^N} J(y)|y'|\frac{2|x' - \tilde{\theta}y'|}{t\gamma} e^{-\frac{|x'-\tilde{\theta}y'|^2}{t\gamma}} dy,\end{aligned}$$

where $0 < \tilde{\theta} < 1$. In particular,

$$\mathcal{D}\phi \ge c\phi'(\xi(x, t)) - f(\phi(\xi(x, t))) - C' t^{-\alpha-\frac{1}{2}}\phi'(\xi(x, t)).$$

Therefore, we have

$$\begin{aligned}\mathcal{L}u^-(x, t) =& u_t^-(x, t) - \int_\Omega J(x - y)[u^-(y, t) - u^-(x, t)]dy - f(u^-(x, t))\\ \le & f(\phi(\xi(x, t))) - f(\phi(\xi(x, t)) - z(t)) + \Big(C' t^{-\alpha-\frac{1}{2}} - Z'(t) - \theta_t(x', t)\Big)\phi'(\xi(x, t))\\ &+ \int_K J(x - y)[\phi(\xi(y, t)) - \phi(\xi(x, t))]dy - z'(t).\end{aligned}$$

Similarly as previous, one can get

$$\int_K J(x-y)[\phi(\xi(y,t)-\phi(\xi(x,t))] \geq -C^K t^{-\alpha-\frac{1}{2}}\phi'(\xi(x,t)) \text{ for some } C^K > 0.$$

It follows that

$$\begin{aligned}\mathcal{L}u^-(x,t) \leq & f(\phi(\xi(x,t))) - f(\phi(\xi(x,t)) - z(t)) \\ & + \left[(C'+C^K)t^{-\alpha-\frac{1}{2}} - Z'(t) - \theta_t(x',t)\right]\phi'(\xi(x,t)) - z'(t).\end{aligned}$$

Now we go further to show $\mathcal{L}u^-(x,t) \leq 0$ in two cases.

Case 1. We assume that $|\xi(x,t)| \gg 1$ such that $\phi(\xi(x,t)) \in [0,\eta] \cup [1-\eta, 1]$, where η is sufficiently small to ensure that $f'(s) \leq -\sigma < 0$ for any $s \in [0,\eta] \cup [1-\eta, 1]$ and $\sigma > 2\eta_z$ with η_z defined as that in Lemma 5.2. Then, since the function $z(t)$ constructed in Lemma 5.2 satisfies

$$z'(t) \geq -\eta_z z(t), \quad z(t) \geq K_0(1+t-t_1)^{-\frac{3}{2}} \text{ for } t_1 \geq 0,$$

there holds

$$\begin{aligned}\mathcal{L}u^-(x,t) \leq & (-\sigma+\eta_z)z(t) - \phi'(\xi(x,t)) \\ & \left[K_z z(t) + \left(\frac{|x'|^2}{\gamma t} - \alpha\right)t^{-1}\theta(x',t) - (C'+C^K)t^{-\alpha-\frac{1}{2}}\right] \\ \leq & -\eta_z K_0 t^{-\frac{3}{2}} + \left(\alpha\beta + C^K + C'\right)t^{-\alpha-\frac{1}{2}}\phi'(\xi(x,t)) \\ \leq & -\frac{1}{2}\eta_z K_0 t^{-\frac{3}{2}}.\end{aligned}$$

Indeed, since $\alpha > 1$ and $|\xi(x,t)|$ is sufficiently large such that $(\alpha\beta+C^K+C')\phi'(\xi) \leq \frac{1}{2}\eta_z K_0$, the last inequality above holds obviously.

Case 2. Let $\phi(\xi(x,t)) \in [\eta, 1-\eta]$ in this case. Since $\phi'(\xi) > 0$ for all $\xi \in \mathbb{R}$, we may choose $\tau_0 > 0$ being sufficiently small such that $\phi'(\xi) \geq \tau_0 > 0$. Denote $\max\limits_{[\eta,1-\eta]} f'(s) = \delta^0 > 0$. Then

$$\begin{aligned}\mathcal{L}u^-(x,t) \leq & \delta^0 z(t) + \eta_z z(t) - \left[K_z z(t) + \left(\frac{|x'|^2}{t\gamma} - \alpha\right)t^{-1}\theta(x',t) - \left(C^K + C'\right)t^{-\alpha-\frac{1}{2}}\right] \\ & \phi'(\xi(x,t)) \\ \leq & \left(-K_z\tau_0 + \delta^0 + \eta_z\right)z(t) + \left(\alpha\beta + C^K + C'\right)t^{-\alpha-\frac{1}{2}}\phi'(\xi(x,t)) \\ \leq & \left[-K_z\tau_0 + \delta^0 + \eta_z + \left(\alpha\beta + C^K + C'\right)\frac{1}{K_0}\|\phi'\|_\infty\right]z(t).\end{aligned}$$

Let K_z be sufficiently large such that $-K_z\tau_0+\delta^0+\eta_z+\left(\alpha\beta + C^K + C'\right)\frac{1}{K_0}\|\phi'\|_\infty \leq -\frac{1}{2}\eta_z$. One hence have that $\mathcal{L}u^-(x,t) \leq -\frac{1}{2}\eta_z z(t)$. The proof is finished. □

5.2 Super-solution

This part is devoted to verifying the super-solution defined as

$$u^+(x,t) = \phi(\psi(x,t)) + z(t),$$

where

$$\psi(x,t)=x_1+c(t-1+t_\epsilon)+\theta^1(x',t)+Z(t),\ \theta^1(x',t)=\beta^+t^{-\alpha^+}e^{-\frac{|x'|^2}{t\gamma}}$$

with $\alpha^+>1$ and $\beta^+>0$ being a large number. It follows from the definition of $u^+(x,t)$ that

$$u^+(x,1)\geq\phi\left(x_1+ct_\epsilon+\beta^+e^{\frac{-|x'|^2}{\gamma}}+Z(1)\right)\geq\tilde{u}(x,1)=u(x,t_\epsilon)$$

for $x\in K_\epsilon$. Since $\phi(-\infty)=0,\ \phi'>0,\ 0<u(x,t)<1$ and $\max_{x\in K_\epsilon}u(x,t)<1$, the last inequality holds provided that β^+ is sufficiently large. If $x\in\mathbb{R}^N\backslash K_\epsilon$, then by $z(1)\geq\frac{1}{2}\epsilon_1=\epsilon$ (from Remark 5.3), we have

$$u^+(x,1)\geq\phi(x_1+ct_\epsilon)+z(1)\geq\phi(x_1+ct_\epsilon)+\epsilon\geq u(x,t_\epsilon)=\tilde{u}(x,1).$$

As a consequence, there holds $u^+(x,1)\geq\tilde{u}(x,1)$ for any $x\in\overline{\Omega}$.

Lemma 5.5 *The inequality $\mathcal{L}u^+(x,t)\geq 0$ holds for all $x\in\Omega$ and $t\geq 1$, where*

$$\mathcal{L}u^+(x,t)=u_t^+(x,t)-\int_\Omega J(x-y)[u^+(y,t)-u^+(x,t)]dy-f(u^+(x,t)).$$

Proof It follows from a direct calculation that

$$u_t^+(x,t)=\left(c+\theta_t^1+Z'(t)\right)\phi'(\psi)+z'(t),$$

and

$$\begin{aligned}&\int_\Omega J(x-y)[u^+(y,t)-u^+(x,t)]dy\\&=\int_{\mathbb{R}^N}J(x-y)[\phi(\psi(y,t))-\phi(\psi(x,t))]dy\\&-\int_K J(x-y)[\phi(\psi(y,t))-\phi(\psi(x,t))]dy.\end{aligned}$$

Note that

$$c\phi'(\psi(x,t))=\int_{\mathbb{R}^N}J(y)[\phi(\psi(x,t)-y_1)-\phi(\psi(x,t))]dy+f(\phi(\psi(x,t))),$$

we have

$$\begin{aligned}u_t^+(x,t)=&(\theta_t'+Z'(t))\phi'(\psi(x,t))\\&+\int_{\mathbb{R}^N}J(y)[\phi(\psi(x,t)-y_1)-\phi(\psi(x,t))]dy+f(\phi(\psi(x,t)))+z'(t).\end{aligned}$$

Then it follows that

$$\begin{aligned}\mathcal{L}u^+(x,t)=&(\theta_t'+Z'(t))\phi'(\psi(x,t))+\int_{\mathbb{R}^N}J(y)[\phi(\psi(x,t)-y_1)-\phi(\psi(x,t))]dy\\&+f(\phi(\psi(x,t)))+z'(t)-\int_{\mathbb{R}^N}J(x-y)[\phi(\psi(y,t))-\phi(\psi(x,t))]dy\\&+\int_K J(x-y)[\phi(\psi(y,t))-\phi(\psi(x,t))]dy-f(u^+(x,t)),\end{aligned}$$

and we obtain

$$\begin{aligned}\mathcal{L}u^+(x,t) =& (\theta_t' + Z'(t))\phi'(\psi(x,t)) + \int_{\mathbb{R}^N} J(y)[\phi(\psi(x,t)-y_1) - \phi(\psi(x-y,t))]dy \\ &+ \int_K J(x-y)[\phi(\psi(y,t)) - \phi(\psi(x,t))]dy \\ &+ f(\phi(\psi(x,t))) + z'(t) - f(u^+(x,t)).\end{aligned}$$

Now we focus on all the integral items above denoted by

$$\begin{aligned}I :=& \int_{\mathbb{R}^N} J(y)[\phi(\psi(x,t)-y_1) - \phi(\psi(x-y,t))]dy \\ &+ \int_K J(x-y)[\phi(\psi(y,t)) - \phi(\psi(x,t))]dy.\end{aligned}$$

By the same progress as the calculation of $u^-(x,t)$, we have

$$I \geq -M' t^{-\alpha^+ - \frac{1}{2}} \phi'(\psi(x,t)).$$

Therefore,

$$\begin{aligned}\mathcal{L}u^+(x,t) \geq& \left(\theta_t^1 + Z'(t) - M' t^{-\alpha^+ - \frac{1}{2}}\right) \phi'(\psi(x,t)) + z'(t) \\ &+ f(\phi(\psi(x,t))) - f(u^+(x,t)).\end{aligned}$$

Next we are going to show $\mathcal{L}u^+(x,t) \geq 0$ in two cases.

Case 1. Let $|\psi(x,t)| \gg 1$ such that $\phi(\psi(x,t)) \in [0,\eta] \cup [1-\eta, 1]$, where $\eta > 0$ is sufficiently small to ensure that $f'(s) \leq -\sigma < 0$ for any $s \in [0,\eta] \cup [1-\eta, 1]$. Since

$$\left(K_z + \frac{\beta^+}{\gamma}\right) z(t) \geq \theta_t^1 + Z'(t) - M' t^{-\alpha^+ - \frac{1}{2}} \geq K_z z(t) - (\alpha^+ \beta^+ + M') t^{-\alpha^+ - \frac{1}{2}}$$

and $|\psi(x,t)| \gg 1$, we obtain that

$$\left| \left(\theta_t^1 + Z'(t) - M' t^{-\alpha^+ - \frac{1}{2}}\right) \phi'(\psi(x,t)) \right| \leq \frac{1}{2} \eta_z z(t),$$

which implies

$$\mathcal{L}u^+(x,t) \geq -\frac{1}{2} \eta_z z(t) - \eta_z z(t) + \sigma z(t) \geq 0,$$

since $\eta_z < \frac{1}{2}\sigma$ and $\alpha^+ > 1$.

Case 2. Since $\phi(\psi(x,t)) \in [\eta, 1-\eta]$ we have $\phi'(\psi(x,t)) \geq \tau_0 > 0$. Denote $\min_{s \in [0,1]} f'(s) = -\delta' < 0$. Then, we obtain

$$\begin{aligned}\mathcal{L}u^+(x,t) \geq& \left[K_z z(t) - (\alpha^+ \beta^+ + M') t^{-\alpha^+ - \frac{1}{2}}\right] \tau_0 - \delta' z(t) - \eta_z z(t) \\ \geq& \left(K_z - \frac{\alpha^+ \beta^+ + M'}{K_0}\right) \tau_0 z(t) - \delta' z(t) - \eta_z z(t).\end{aligned}$$

Recall the fact $\alpha^+ > 1$, it follows that $K_0 t^{-\alpha^+ - \frac{1}{2}} < z(t)$. Moreover, one can take $K_z > 0$ being sufficiently large such that $\left(K_z - \frac{\alpha^+ \beta^+ + M'}{K_0}\right) \tau_0 - \delta' - \eta_z \geq 0$. Then we have that $\mathcal{L}u^+(x,t) \geq 0$. The proof is finished. □

The proof of Theorem 5.1 From the Lemmas 5.4 and 5.5, for $t \geq t_\epsilon$, we have that

$$\begin{aligned}
&\inf_{x\in\bar{\Omega}} [u(x,t) - \phi(x_1 + ct)] \\
&\quad = \inf_{x\in\bar{\Omega}} [\tilde{u}(x, t+1-t_\epsilon) - \phi(x_1 + ct)] \\
&\quad \geq \inf_{x\in\bar{\Omega}} [\phi(x_1 + ct - \theta(x', t+1-t_\epsilon) - Z(t+1-t_\epsilon)) - z(t+1-t_\epsilon) - \phi(x_1+ct)] \\
&\quad \geq -[\beta(t+1-t_\epsilon)^{-\alpha} + Z(t+1-t_\epsilon)]\|\phi'\|_{L^\infty(\mathbb{R})} - z(t+1-t_\epsilon) \\
&\quad \geq -\beta(t+1-t_\epsilon)^{-\alpha}\|\phi'\|_{L^\infty(\mathbb{R})} - z(t+1-t_\epsilon) - \epsilon_1\mathcal{I}\|\phi'\|_{L^\infty(\mathbb{R})},
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{x\in\bar{\Omega}}[u(x,t) - \phi(x_1 + ct)] \\
&\quad = \sup_{x\in\bar{\Omega}}[\tilde{u}(x, t+1-t_\epsilon) - \phi(x_1 + ct)] \\
&\quad \leq \sup_{x\in\bar{\Omega}}[\phi(x_1 + ct + \theta^1(x', t+1-t_\epsilon) + Z(t+1-t_\epsilon)) + z(t+1-t_\epsilon) - \phi(x_1+ct)] \\
&\quad \leq [\beta^+(t+1-t_\epsilon)^{-\alpha} + Z(t+1-t_\epsilon)]\|\phi'\|_{L^\infty(\mathbb{R})} + z(t+1-t_\epsilon) \\
&\quad \leq \beta^+(t+1-t_\epsilon)^{-\alpha}\|\phi'\|_{L^\infty(\mathbb{R})} + z(t+1-t_\epsilon) + \epsilon_1\mathcal{I}\|\phi'\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

For any sufficiently small $\eta_s > 0$, take $0 < \epsilon_1 < \frac{\eta_s}{\mathcal{I}\|\phi'\|_{L^\infty}}$. Then by the construction of $z(t)$, we see that

$$\liminf_{t\to\infty} \inf_{x\in\bar{\Omega}} [u(x,t) - \phi(x_1+ct)] \geq -\eta_s,$$

and

$$\limsup_{t\to\infty} \sup_{x\in\bar{\Omega}}[u(x,t) - \phi(x_1+ct)] \leq \eta_s.$$

Since η_s is arbitrary, we have $\lim_{t\to\infty} \sup_{x\in\bar{\Omega}} |u(x,t) - \phi(x_1+ct)| = 0$. Thus, we finish the proof of Theorem 5.1. □

5.3 Proofs of Theorem 1.1

It follows from [7, Theorems 2.4 and 2.6] that under the conditions (F) and (J), the unique solution of the stationary problem of (1.1)

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N\setminus K} J(x-y)[v(y) - v(x)]dy + f(v(x)) = 0, \; x \in \mathbb{R}^N \setminus K(\text{ or } K_\epsilon), \\ 0 \leq v(x) \leq 1, \;\; x \in \mathbb{R}^N \setminus K(\text{ or } K_\epsilon), \\ \displaystyle\sup_{\mathbb{R}^N\setminus K} v(x) = 1 \end{cases} \tag{5.2}$$

is $v(x) \equiv 1$ in $\overline{\mathbb{R}^N\setminus K}$ (or $\overline{\mathbb{R}^N\setminus K_\epsilon}$).

Now we are in position to show the main theorem. It is obvious that if the solution $U(x,t)$ of (1.1) constructed in Theorem 3.1 satisfies the conditions in Theorem 5.1 together with Theorem 4.3, then the conclusions in Theorem 1.1 hold. Indeed, since $U_t(x,t) > 0$ and $U(x,t)$ is Lipschitz continuous in $x \in \mathbb{R}^N\setminus K$, we know that as time tends to positive infinity, $U(x,t)$ locally uniformly converges to some continuous function $V(x)$, without loss of generality, which can be deemed as a uniformly continuous function by [7, Lemma 3.2] because of the assumption (F). Furthermore, we claim that $V(x)$ satisfies (5.2). Therefore, we have $V(x) \equiv 1$ for all $x \in \mathbb{R}^N \backslash K$. In fact, it is sufficient to show $\sup_{\mathbb{R}^N \backslash K} V(x) = 1$. In view of $u(x,t) - \phi(x_1 + ct) \to 0$ as $t \to -\infty$ in Theorem 3.1, for any small $\epsilon' > 0$, there exist $t_{\epsilon'} < 0$ and $X_1 > 0$ being sufficiently large such that

$$\phi(x + ct_{\epsilon'}) \geq 1 - \frac{\epsilon'}{2} \text{ for all } x_1 \geq X_1,$$

and

$$|u(x,t) - \phi(x_1 + ct)| \leq \frac{\epsilon'}{2} \text{ for all } x \in \mathbb{R}^N \backslash K \text{ and } t \leq t_{\epsilon'}.$$

Thus

$$u(x,t) \geq 1 - \epsilon' \text{ for all } x_1 \geq X_1 \text{ and } t \geq t'_{\epsilon},$$

which implies that $V(x) \geq 1 - \epsilon'$ due to $u_t(x,t) > 0$. Since that ϵ' is actually arbitrary, one has that

$$\sup_{\mathbb{R}^N \backslash K} V(x) = 1.$$

Therefore, $V(x) \equiv 1$ for all $x \in \mathbb{R}^N \backslash K$. Hence, $u(x,t) \to 1$ as $t \to +\infty$ for all $x \in \mathbb{R}^N \backslash K$. It follows that for any $\epsilon > 0$, there are some $t_\epsilon > 0$ being sufficiently large and $K_\epsilon \subset \overline{\Omega}$ with $K \subset K_\epsilon$ such that

$$u(x,t) \geq 1 - \epsilon, \text{ for all } t \geq t_\epsilon, \ x \in \partial K_\epsilon.$$

In addition, it follows from Theorem 4.3 that

$$|u(x,t_\epsilon) - \phi(x_1 + ct_\epsilon)| < \epsilon, \text{ for all } x \in \overline{\mathbb{R}^N \setminus K_\epsilon}.$$

Thus, from Theorem 5.1, we have

$$\lim_{t \to \infty} \sup_{x \in \overline{\Omega}} |u(x,t) - \phi(x_1 + ct)| = 0.$$

Moreover, since $u(x,t) - \phi(x_1 + ct) \to 0$ as $t \to \pm\infty$ uniformly in $x \in \Omega$ and Theorem 4.3, we have that $u(x,t) - \phi(x_1 + ct) \to 0$ as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$. The proof of Theorem 1.1 is finished. Similarly, we know the results of Remark 1.2 hold.

Acknowledgements We are grateful to the anonymous referee for his/her careful reading and valuable suggestions which led to an improvement of our original manuscript. When this paper is revised, we find that Brasseur and Coville [9] also considered the obstacle problem and they have cited our work. Research of W.-T. Li was partially supported by NSF of China (11731005) and NSF of Gansu (21JR7RA537). Research of J.-W. Sun was partially supported by FRFCU (lzujbky-2021-52) and NSF of Gansu (21JR7RA535).

6 Appendix

6.1 Proof of Proposition 3.2

In this subsection we intend to show the results of Proposition 3.2. For convenience we define the operator $\mathcal{L}$ as follows

$$\mathcal{L}\omega(x,t) = \omega_t(x,t) - \int_\Omega J(x-y)[\omega(y,t)-\omega(x,t)]dy - f(\omega(x,t)).$$

We further show that $W^-(x,t)$ is a sub-solution. A straightforward computation shows that

$$\begin{aligned} &\mathcal{L}W^-(x,t) \\ &= \begin{cases} -\displaystyle\int_\Omega J(x-y)W^-(y,t)dy, & x_1 < 0, \\ (c-\dot{\xi}(t))[\phi'(x_1+ct-\xi(t)) - \phi'(-x_1+ct-\xi(t))] - \displaystyle\int_\Omega J(x-y)[W^-(y,t) \\ \quad - W^-(x,t)]dy - f(\phi(x_1+ct-\xi(t)) - \phi(-x_1+ct-\xi(t))), & x_1 \geq 0. \end{cases} \end{aligned}$$

For $x_1 < 0$, since $J(x) \geq 0$ and $W^- \geq 0$, we have

$$\mathcal{L}W^- = -\int_\Omega J(x-y)W^-(y,t)dy \leq 0.$$

For $x_1 \geq 0$, in view of that

$$\begin{aligned} &\int_\Omega J(x-y)[W^-(y,t)-W^-(x,t)]dy \\ &= \int_{\mathbb{R}^N} J(x-y)[W^-(y,t)-W^-(x,t)]dy - \int_K J(x-y)[W^-(y,t)-W^-(x,t)]dy \\ &= \int_{\mathbb{R}^N\cap\{y_1>0\}} J(x-y)[W^-(y,t)-W^-(x,t)]dy \\ &\quad + \int_{\mathbb{R}^N\cap\{y_1<0\}} J(x-y)[W^-(y,t)-W^-(x,t)]dy \\ &\quad - \int_K J(x-y)[W^-(y,t)-W^-(x,t)]dy \\ &\geq \int_{\mathbb{R}^N} J(x-y)[(\phi(y_1+ct-\xi(t)) - \phi(-y_1+ct-\xi(t))) - (\phi(x_1+ct-\xi(t)) \\ &\quad - \phi(-x_1+ct-\xi(t)))]dy - \int_K J(x-y)[W^-(y,t)-W^-(x,t)]dy, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{L}W^- \leq &-\dot{\xi}(t)[\phi'(z_+(t)) - \phi'(z_-(t))] \\ &+ f(\phi(z_+(t))) - f(\phi(z_-(t))) - f(\phi(z_+(t)) - \phi(z_-(t))) \\ &+ \int_K J(x-y)[W^-(y,t)-W^-(x,t)]dy, \end{aligned}$$

where $z_+(t) = x_1+ct-\xi(t),\ z_-(t) = -x_1+ct-\xi(t)$. Recall that $K \subset \{x \in \mathbb{R}^N \mid x_1 \leq 0\}$, it follows that

$$W^-(y,t) = 0 \text{ for all } y \in K,$$

which implies that

$$\begin{aligned}\mathcal{L}W^-(x,t) \le & -\dot{\xi}(t)[\phi'(z_+(t)) - \phi'(z_-(t))] \\ & + f(\phi(z_+(t))) - f(\phi(z_-(t))) - f(\phi(z_+(t)) - \phi(z_-(t))).\end{aligned}$$

Now we go further to show $\mathcal{L}W^- \le 0$ in two subcases.

Case A: $0 < x_1 < -ct + \xi(t)$.

In this case the following lemma holds.

Lemma 6.1 *Suppose that (F) holds. Let (ϕ, c) be the unique solution of (1.4) and $\phi''(\xi) \ge 0$ for $\xi \le 0$. Then there exists $k_3 > 0$ such that*

$$\phi'(\xi_1) - \phi'(\xi_2) \ge k_3[\phi(\xi_1) - \phi(\xi_2)] \tag{6.1}$$

for $\xi_2 < \xi_1 < 0$.

Proof It follows from (1.6) that there exists some $\mathfrak{c} > 0$ such that

$$\frac{\phi'(\xi_2)}{\phi'(\xi_1)} \le \mathfrak{c}e^{\lambda(\xi_2 - \xi_1)}.$$

Then one can choose $M' > \frac{\ln 2\mathfrak{c}}{\lambda}$ and $(\xi_1, \xi_2) \in \mathbb{R}^2$ with $\xi_1 - M' < \xi_2 < \xi_1 < 0$ such that

$$\frac{\phi'(\xi_2)}{\phi'(\xi_1)} \le \mathfrak{c}e^{-\lambda M'} \le \frac{1}{2}.$$

If $\xi_1 - M' < \xi_2 < \xi_1 < 0$, then we have

$$\phi'(\xi_1) - \phi'(\xi_2) = \phi''(\theta^1)(\xi_1 - \xi_2), \ \phi(\xi_1) - \phi(\xi_2) = \phi'(\theta^2)(\xi_1 - \xi_2)$$

for some $(\theta^1, \theta^2) \in [\xi_2, \xi_1]^2$ with $|\theta^1 - \theta^2| < M'$. This and the facts $\phi''(\xi) \ge 0$ for $\xi \le 0$ and $\phi'(\psi) > 0$ for all $\psi \in \mathbb{R}$ imply that (6.1) holds true for $\xi_1 - M' < \xi_2 < \xi_1 < 0$.

When $\xi_2 + M' < \xi_1 < 0$, it follows that

$$\phi'(\xi_2) \le \frac{1}{2}\phi'(\xi_1).$$

This and the inequalities in (1.5) and (1.6) yield

$$\phi'(\xi_1) - \phi'(\xi_2) \ge \frac{1}{2}\phi'(\xi_1) \ge k_3\phi(\xi_1) \ge k_3[\phi(\xi_1) - \phi(\xi_2)].$$

Thus we finish the proof. □

Now we are ready to show $\mathcal{L}W^-(x,t) \le 0$. By Lemma 6.1, we have

$$\begin{aligned}\mathcal{L}W^-(x,t) \le & -\dot{\xi}(t)(\phi'(z_+(t)) - \phi'(z_-(t))) + L_f\phi(z_-(t))(\phi(z_+(t)) - \phi(z_-(t))) \\ \le & \left[-Mk_3e^{\lambda_0(ct+\xi(t))} + L_f\phi(z_-(t))\right][\phi(z_+(t)) - \phi(z_-(t))] \\ \le & \left[L_f\beta_0e^{\lambda(-x_1+ct-\xi(t))} - Mk_3e^{\lambda_0(ct+\xi(t))}\right][\phi(z_+(t)) - \phi(z_-(t))] \\ \le & e^{\lambda_0(ct+\xi(t))}\left[L_f\beta_0e^{(\lambda-\lambda_0)(ct+\xi(t))-2\lambda\xi(t)} - Mk_3\right][\phi(z_+(t)) - \phi(z_-(t))] \\ \le & (L_f\beta_0 - Mk_3)[\phi(z_+(t)) - \phi(z_-(t))] \\ \le & 0.\end{aligned}$$

The last inequality holds provided that $M \ge \frac{L_f\beta_0}{k_3}$.

Case B: $x_1 \geq -ct + \xi(t)$.
A direct calculation gives that

$$\begin{aligned}\mathcal{L}W^-(x,t) \leq & -Me^{\lambda_0(ct+\xi(t))}[\phi'(z_+(t)) - \phi'(z_-(t))] + L_f\phi(z_-(t))[\phi(z_+(t)) - \phi(z_-(t))] \\ \leq & L_f\phi(z_-(t)) - Me^{\lambda_0(ct-\xi(t))+2\lambda_0\xi(t)}[\phi'(z_+(t)) - \phi'(z_-(t))] \\ \leq & e^{-\lambda x_1+\lambda_0(ct-\xi(t))+2\lambda_0\xi(t)}\bigg[L_f\beta_0 e^{(\lambda-\lambda_0)(ct-\xi(t))-2\lambda_0\xi(t)} \\ & - M\left(\gamma_1 e^{(\lambda-\mu)x_1-\mu(ct-\xi(t))} - \delta_0 e^{\lambda(ct-\xi(t))}\right)\bigg] \\ \leq & e^{-\lambda x_1+\lambda_0(ct-\xi(t))+2\lambda_0\xi(t)}\left[L_f\beta_0 - M\left(\gamma_1 e^{(\lambda-\mu)x_1-\mu(ct-\xi(t))} - \delta_0 e^{\lambda(ct-\xi(t))}\right)\right].\end{aligned}$$

If $\lambda \geq \mu$, then

$$\begin{aligned}\mathcal{L}W^-(x,t) \leq & e^{-\lambda x_1+\lambda_0(ct-\xi(t))+2\lambda_0\xi(t)}\left[L_f\beta_0 - M\left(\gamma_1 e^{-\mu(ct-\xi(t))} - \delta_0 e^{\lambda(ct-\xi(t))}\right)\right] \\ \leq & 0\end{aligned}$$

for $ct - \xi(t) \ll -1$ and $M > 1$ is sufficiently large.
When $\lambda < \mu$, which means $|f'(1) - f'(0)| > 0$, there holds

$$\begin{aligned}& f(\phi(z_+(t)))) - f(\phi(z_-(t))) - f(\phi(z_+(t))) - \phi(z_-(t)))) \\ & = f'(\phi(z_+(t)))\phi(z_-(t))) - o(\phi^2(z_-(t)))) - f'(\phi(z_-(t))))\phi(z_-(t))) + o(\phi^2(z_-(t)))) \\ & \leq -k_4\phi(z_-(t))\end{aligned}$$

for $x_1 + ct - \xi(t) > L_2 > 0$ with L_2 being large enough, where $0 < k_4 < \frac{1}{2}|f'(1) - f'(0)|$. The inequality above follows from that $f'(\phi(z_+(t))) \to f'(1)$ and $f'(\phi(z_-(t))) \to f'(0)$ as $L_2 \to +\infty$. Then

$$\begin{aligned}\mathcal{L}W^-(x,t) \leq & Me^{\lambda_0(ct+\xi(t))}\phi'(z_-(t)) - k_4\phi(z_-(t)) \\ \leq & Me^{\lambda_0(ct+\xi(t))}\delta_0 e^{\lambda(-x_1+ct-\xi(t))} - k_4\alpha_0 e^{\lambda(-x_1+ct-\xi(t))} \\ \leq & e^{\lambda(-x_1+ct-\xi(t))}\left(Me^{\lambda_0(ct+\xi(t))} - k_4\alpha_0\right) \\ \leq & 0,\end{aligned}$$

provided that $ct + \xi(t) \ll -1$.
In addition, for $0 < x_1 + ct - \xi(t) < L_2$, there holds

$$\begin{aligned}\mathcal{L}W^-(x,t) \leq & e^{-\lambda x_1+\lambda_0(ct-\xi(t))+2\lambda_0\xi(t)}\bigg[L_f\beta_0 e^{(\lambda-\lambda_0)(ct-\xi(t))-2\lambda_0\xi(t)} \\ & - M\left(\gamma_1 e^{(\lambda-\mu)x_1-\mu(ct-\xi(t))} - \delta_0 e^{\lambda(ct-\xi(t))}\right)\bigg] \\ \leq & e^{-\lambda x_1+\lambda_0(ct-\xi(t))+2\lambda_0\xi(t)}\left[L_f\beta_0 - M\left(\gamma_1 e^{(\lambda-\mu)L_2}e^{-\lambda(ct-\xi(t))} - \delta_0 e^{\lambda(ct-\xi(t))}\right)\right].\end{aligned}$$

Since $ct - \xi(t) \ll -1$ and $M \gg 1$, we have $\mathcal{L}W^-(x,t) \leq 0$.

Next, we show the $W^+(x,t)$ is a super-solution. A straightforward computation shows that

$$\mathcal{L}W^+(x,t)=\begin{cases}2(c+\dot{\xi}(t))\phi'(x_1+ct)-f(2\phi(ct+\xi(t)))\\ -\int_\Omega J(x-y)[W^+(y,t)-W^+(x,t)]dy,\ x_1<0,\\ (c+\dot{\xi}(t))[\phi'(x_1+ct+\xi(t))+\phi'(-x_1+ct+\xi(t))]\\ \quad -\int_\Omega J(x-y)[W^+(y,t)-W^+(x,t)]dy-f(\phi(x_1+ct+\xi(t))\\ \quad +\phi(-x_1+ct+\xi(t))),\ x_1>0.\end{cases}$$

When $x_1\geq 0$, denote

$$\Gamma^+=\{x\in\mathbb{R}^N\mid y_1>0\},\ \Gamma^-=\{x\in\mathbb{R}^N\mid y_1\leq 0\}.$$

In view of that $K\subset\mathbb{R}^N\backslash\text{supp}(J)\cap\{x\in\mathbb{R}^N:\ x_1\leq 0\}$, one gets

$$\begin{aligned}&\int_\Omega J(x-y)[W^+(y,t)-W^+(x,t)]dy\\ &=\int_{\Omega\cap\Gamma^+}J(x-y)[(\phi(y_1+ct+\xi(t))+\phi(-y_1+ct+\xi(t)))\\ &\quad-(\phi(x_1+ct+\xi(t)+\phi(-x_1+ct+\xi(t))]dy\\ &\quad+\int_{\Omega\cap\Gamma^-}J(x-y)[2\phi(ct+\xi(t))-(\phi(x_1+ct+\xi(t))+\phi(-x_1+ct+\xi(t)))]dy\\ &=\int_{\mathbb{R}^N}J(x-y)[(\phi(y_1+ct+\xi(t))+\phi(-y_1+ct+\xi(t)))-(\phi(x_1+ct+\xi(t))\\ &\quad+\phi(-x_1+ct+\xi(t)))]dy+\int_{\Omega\cap\Gamma^-}J(x-y)[2\phi(ct+\xi(t))-(\phi(y_1+ct+\xi(t))\\ &\quad+\phi(-y_1+ct+\xi(t)))]dy\\ &=c(\phi'(x_1+ct+\xi(t))+\phi'(-x_1+ct+\xi(t)))-f(\phi(x_1+ct+\xi(t)))\\ &\quad-f(\phi(-x_1+ct+\xi(t)))\\ &\quad+\int_{\Omega\cap\Gamma^-}J(x-y)[2\phi(ct+\xi(t))-(\phi(y_1+ct+\xi(t))+\phi(-y_1+ct+\xi(t)))].\end{aligned}$$

Observe that, if $x_1>|ct+\xi(t)|>L$, where L is the diameter of the compact support of J, then the integral item of the last equality is equal to 0. Therefore, we obtain

$$\begin{aligned}\mathcal{L}W^+(x,t)=&\dot{\xi}[\phi'(x_1+ct+\xi(t))+\phi'(-x_1+xt+\xi(t))]+f(\phi(x_1+ct+\xi(t)))\\ &+f(\phi(-x_1+ct+\xi(t)))-f(\phi(x_1+ct+\xi(t))+\phi(-x_1+ct+\xi(t)))\\ \geq&\dot{\xi}\phi'(x_1+ct+\xi(t))-L_f\phi(x_1+ct+\xi(t))\phi(-x_1+ct+\xi(t))\\ \geq&e^{\lambda_0(ct+\xi(t))}\left(M\gamma_1e^{-\mu(x_1+ct+\xi(t))}-L_f\alpha_0e^{-\lambda x_1}e^{(\lambda-\lambda_0)(ct+\xi(t))}\right).\end{aligned}$$

If $\mu\leq\lambda$, by choosing $M\gamma_1\geq L_f\alpha_0$, it is obvious that $\mathcal{L}W^+(x,t)\geq 0$ with $x_1>|ct+\xi(t)|$ being sufficiently large.

For $\mu>\lambda$, we have $f'(1)<f'(0)$. Consider the case $x_1+ct+\xi(t)\geq L_0\gg 1$. Then $\phi(x_1+ct+\xi(t))\approx 1$ while $\phi(-x_1+ct+\xi(t))\approx 0$. Furthermore,

$$f(\phi(x_1+ct+\xi(t)))+f(\phi(-x_1+ct+\xi(t)))$$

$$
\begin{aligned}
&- f(\phi(x_1+ct+\xi(t))+\phi(-x_1+ct+\xi(t)))\\
&\geq \frac{1}{2}(f'(0)-f'(1))\phi(-x_1+ct+\xi(t))\\
&\geq 0,
\end{aligned}
$$

which implies that $\mathcal{L}W^+(x,t)\geq 0$. For the other case $x_1+ct+\xi(t)\leq L_0$, we know

$$
\mathcal{L}W^+(x,t)\geq e^{\lambda_0(ct+\xi(t))}\left(M\gamma_1 e^{-\mu L_0}-L_f\alpha_0 e^{-\lambda x_1}e^{(\lambda-\lambda_0)(ct+\xi(t))}\right).
$$

Since $\lambda_0<\lambda$, we obtain $\mathcal{L}W^+(x,t)\geq 0$ holds provided that $M\geq \frac{L_f\alpha_0}{\gamma_1}e^{\mu L_0}$.

For the case $0<x_1<|ct+\xi(t)|$, one can see that

$$
\begin{aligned}
&\int_{\Omega\cap\Gamma^-} J(x-y)[2\phi(ct+\xi(t))-(\phi(y_1+ct+\xi(t))+\phi(-y_1+ct+\xi(t)))]dy\\
&=\int_{\Omega\cap\{y_1<ct+\xi(t)\}} J(x-y)[2\phi(ct+\xi(t))-(\phi(y_1+ct+\xi(t))+\phi(-y_1+ct+\xi(t)))]dy\\
&\quad+\int_{\Omega\cap\{ct+\xi(t)<y_1<0\}} J(x-y)[2\phi(ct+\xi(t))-(\phi(y_1+ct+\xi(t))+\phi(-y_1+ct+\xi(t)))]dy\\
&:=I_1+I_2.
\end{aligned}
$$

Since $\phi(ct+\xi(t))\leq \frac{\theta}{2}$ for $ct+\xi(t)\ll -1$, $\phi(0)\leq\theta$, and $\phi'>0$, we get that $\phi(-y_1+ct+\xi(t))>\theta\geq 2\phi(ct+\xi(t))$ for $y_1<ct+\xi(t)$. It follows that $I_1\leq 0$. We know that

$$
\begin{aligned}
I_2\leq &\int_{\Omega\cap\{ct+\xi(t)<y_1<0\}} J(x-y)C_\phi e^{\lambda(ct+\xi(t))}\left(2-\left(e^{\lambda y_1}+e^{-\lambda y_1}\right)\right)dy\\
&+K_\phi e^{(k_\phi+\lambda)(ct+\xi(t))}\int_{\Omega\cap\{ct+\xi(t)<y_1<0\}} J(x-y)\left(2+e^{\lambda y_1}+e^{-\lambda y_1}\right)dy\\
\leq &C_0 e^{(k_\phi+\lambda)(ct+\xi(t))}.
\end{aligned}
$$

The first inequality is follows from that there exist two numbers $K_\phi>0$ and $k_\phi>0$ such that $\left|\phi(x_1)-C_\phi e^{\lambda x_1}\right|\leq K_\phi e^{(k_\phi+\lambda)x_1}$ for $x_1\leq 0$ which is easy to obtain by (1.5). Then we have

$$
\begin{aligned}
\mathcal{L}W^+\geq &Me^{\lambda_0(ct+\xi(t))}(\phi'(x_1+ct+\xi(t))+\phi'(-x_1+ct+\xi(t)))+f(\phi(x_1+ct+\xi(t)))\\
&+f(\phi(-x_1+ct+\xi(t)))-f(\phi(x_1+ct+\xi(t))+\phi(-x_1+ct+\xi(t)))\\
&-C_0e^{(k_\phi+\lambda)(ct+\xi(t))}\\
\geq &Me^{\lambda_0(ct+\xi(t))}(\phi'(x_1+ct+\xi(t))+\phi'(-x_1+ct+\xi(t)))\\
&-L_f\phi(x_1+ct+\xi(t))\phi(-x_1+ct+\xi(t))-C_0e^{(k_\phi+\lambda)(ct+\xi(t))}\\
\geq &e^{(\lambda_0+\lambda)(ct+\xi(t))}\left[2M\gamma_0-L_f\beta_0e^{(\lambda-\lambda_0)(ct+\xi(t))}-C_0e^{(k_\phi-\lambda_0)(ct+\xi(t))}\right].
\end{aligned}
$$

This gives that $\mathcal{L}W^+\geq 0$, provided $2M\alpha_0>L_f\beta_0+C_0$ and $\lambda_0<\min\{k_\phi,\lambda\}$.

For $x_1<0$, we just deal with the case $-L<x_1<0$ because that for $x_1\leq -L$,

$$
\begin{aligned}
&\int_\Omega J(x-y)[W^+(y,t)-W^+(x,t)]dy\\
&\quad=\int_{\Gamma^+} J(x-y)[\phi(y_1+ct+\xi(t))+\phi(y_1+ct+\xi(t))-2\phi(ct+\xi(t))]dy=0.
\end{aligned}
$$

Since $\phi''(x) \geq 0$ for $x \leq 0$, we have that

$$
\begin{aligned}
&\int_{\Omega} J(x-y)[W^+(y,t) - W^+(x,t)]dy \\
&\leq c\phi'(ct+\xi(t)) - f(\phi(ct+\xi(t))) + \int_{-\infty}^{x_1} J_1(y_1)[\phi(y_1 - x_1 + ct + \xi(t)) \\
&\quad + \phi(x_1 - y_1 + ct + \xi(t)) - 2\phi(ct+\xi(t))]dy_1 \\
&\quad - \int_{\mathbb{R}} J_1(y_1)[\phi(ct+\xi(t) - y_1) - \phi(ct+\xi(t))]dy_1 \\
&= c\phi'(ct+\xi(t)) - f(\phi(ct+\xi(t))) + \int_{-\infty}^{0} J_1(y_1)[\phi(y_1 - x_1 + ct + \xi(t)) \\
&\quad + \phi(x_1 - y_1 + ct + \xi(t)) - 2\phi(ct+\xi(t))]dy_1 \\
&\quad + \int_{x_1}^{0} J_1(y_1)[2\phi(ct+\xi(t)) - \phi(y_1 - x_1 + ct + \xi(t)) - \phi(x_1 - y_1 + ct + \xi(t))] \\
&\quad - \int_{-\infty}^{0} J_1(y_1)[\phi(ct+\xi(t) - y_1) + \phi(ct+\xi(t) + y_1) - 2\phi(ct+\xi(t))]dy_1 \\
&\leq c\phi'(ct+\xi(t)) - f(\phi(ct+\xi(t))) + C_0 e^{(k_\phi+\lambda)(ct+\xi(t))} \\
&\quad + \int_{-\infty}^{0} J_1(y_1)[\phi(y_1 - x_1 + ct + \xi(t)) \\
&\quad + \phi(x_1 - y_1 + ct + \xi(t)) - \phi(ct+\xi(t) - y_1) - \phi(ct+\xi(t) + y_1)]dy_1.
\end{aligned}
$$

Observe that, if $x_1 < 0$ then $\left|\phi(x_1) - C_\phi e^{\lambda x_1}\right| \leq K_\phi e^{(k_\phi+\lambda)x_1}$. Thus there is $C_0 > 0$ such that

$$
\begin{aligned}
&\int_{-\infty}^{0} J_1(y_1)[\phi(y_1 - x_1 + ct + \xi(t)) + \phi(x_1 - y_1 + ct + \xi(t)) \\
&\quad - \phi(ct+\xi(t) - y_1) - \phi(ct+\xi(t) + y_1)]dy_1 \\
&\leq C_\phi e^{\lambda(ct+\xi(t))} \int_{-\infty}^{0} J_1(y_1)\left[\left(e^{\lambda(x_1-y_1)} + e^{\lambda(y_1-x_1)}\right) - \left(e^{\lambda y_1} + e^{-\lambda y_1}\right)\right]dy_1 \\
&\quad + 2K_\phi e^{(k_\phi+\lambda)(ct+\xi(t))} \int_{-\infty}^{0} J_1(y_1)\Big[\left(e^{(k_\phi+\lambda)(x_1-y_1)} + e^{(k_\phi+\lambda)(y_1-x_1)}\right) \\
&\quad - \left(e^{(k_\phi+\lambda)y_1} + e^{-(k_\phi+\lambda)y_1}\right)\Big]dy_1 \\
&\leq C_0 e^{(k_\phi+\lambda)(ct+\xi(t))}.
\end{aligned}
$$

The last inequality above holds true, since $x_1 < 0$ and $f(\upsilon) = \upsilon + \frac{1}{\upsilon}$ is monotonically increasing in $\upsilon \in (1, \infty)$. Then it follows that for $M \geq \frac{C_0}{\gamma_0}$,

$$
\begin{aligned}
\mathcal{L}W^+ &\geq 2\dot{\xi}(t)\phi'(ct+\xi(t)) + f(\phi(ct+\xi(t))) - f(2\phi(ct+\xi(t))) - C_0 e^{(k_\phi+\lambda)(ct+\xi(t))} \\
&\geq 2M\gamma_0 e^{(\lambda_0+\lambda)(ct+\xi(t))} - 2C_0 e^{(k_\phi+\lambda)(ct+\xi(t))} \\
&= e^{(k_\phi+\lambda)(ct+\xi(t))}(2M\gamma_0 - 2C_0) \\
&\geq 0.
\end{aligned}
$$

The second inequality follows from that $f'(s) < 0$ in $[\phi(ct+\xi(t)), 2\phi(ct+\xi(t))]$ for $ct+\xi(t) \ll -1$. The proof of Proposition 3.2 has been finished.

6.2 Proof of Lemma 4.2

We know that

$$\begin{aligned}\mathcal{M}\underline{u} :=&\underline{u}_t(x,t) - \int_{\mathbb{R}^N} J(x-y)[\underline{u}(y,t)-\underline{u}(x,t)]dy - f(\underline{u}(x,t))\\ =&(c-2\epsilon\|f'\|\delta^{-1}e^{-\omega(t-t_0)})\phi' + \epsilon\omega e^{-\omega(t-t_0)}\\ &-\int_{\mathbb{R}^N} J(x-y)[\phi(\xi_-(y,t))-\phi(\xi_-(x,t))]dy\\ &-f(\phi(\xi_-(x,t))-\epsilon e^{-\omega(t-t_0)})\\ =&-2\epsilon\|f'\|\delta^{-1}e^{-\omega(t-t_0)}\phi' + \epsilon\omega e^{-\omega(t-t_0)} + f(\phi(\xi_-(x,t)))\\ &-f\left(\phi(\xi_-(x,t))-\epsilon e^{-\omega(t-t_0)}\right).\end{aligned}$$

When $\xi_-(x,t)\in[-A,A]$, there holds $\phi'(\xi_-(x,t))\geq\delta$. Therefore,

$$\mathcal{M}\underline{u} \leq \epsilon e^{-\omega(t-t_0)}(-2\|f'\|+\omega+\|f'\|)\leq 0.$$

For $|\xi_-(x,t)|\geq A$, we have

$$\phi(\xi_-(x,t)), \underline{u}(x,t)\in[-\infty,\eta]\cup[1-\eta,+\infty].$$

Then $f'(s)\leq-\omega$ for $s\in[\phi(\xi_-(xt))-\epsilon e^{-\omega(t-t_0)}, \phi(\xi_-(xt))]$. Hence,

$$\mathcal{M}\underline{u}\leq\epsilon\omega e^{-\omega(t-t_0)}-\omega\epsilon e^{-\omega(t-t_0)}=0.$$

For $t_0\leq -T$, one get

$$\underline{u}(x,t_0)=\phi(x_1+ct)-\epsilon\leq u(x,t_0).$$

Until now, we have show the function $\underline{u}$ is a sub-solution to (4.2). Similarly one can show $\overline{u}$ is a super-solution to (4.2).

References

1. Andreu-Vaillo, F., Mazón, J.M., Rossi, J.D., Toledo-Melero, J.: Nonlocal Diffusion Problems. Mathematical Surveys and Monographs, AMS, Providence, Rhode Island (2010)
2. Bates, P.W., Fife, P.C., Ren, X.F., Wang, X.F.: Traveling waves in a convolution model for phase transitions. Arch. Rational Mech. Anal. **138**, 105–136 (1997)
3. Berestycki, H., Hamel, F.: Generalized travelling waves for reaction-diffusion equations, Perspectives in nonlinear partial differential equations. Amer. Math. Soc, Contemp. Math. **446**, 101–123 (2007)
4. Berestycki, H., Hamel, F., Matano, H.: Bistable traveling waves around an obstacle. Comm. Pure Appl. Math. **62**, 729–788 (2009)
5. Berestycki, H., Hamel, F.: Generalized transition waves and their properties. Comm. Pure Appl. Math. **65**, 592–648 (2012)
6. Bouhours, J.: Robustness for a Liouville type theorem in exterior domains. J. Dynam. Differential Equations **27**, 297–306 (2015)
7. Brasseur, J., Coville, J., Hamel, F., Valdinoci, E.: Liouville type results for a nonlocal obstacle problem. Proc. London Math. Soc. **119**, 291–328 (2019)
8. Brasseur, J., Coville, J.: A counterexample to the Liouville property of some nonlocal problems. Ann. Inst. H. Poincaré Anal. Non Linéaire **37**, 549–579 (2020)

9. Brasseur, J., Coville, J.: Propagation phenomena with nonlocal diffusion in presence of an obstacle. J. Dynam. Differential Equations (2021). https://doi.org/10.1007/s10884-021-09988-y
10. Bu, Z.-H., Guo, H., Wang, Z.-C.: Transition fronts of combustion reaction diffusion equations in $\mathbb{R}^N$. J. Dynam. Differential Equations **31**, 1987–2015 (2019)
11. Carr, J., Chmaj, A.: Uniqueness of travelling waves for nonlocal monostable equations. Proc. Amer. Math. Soc. **132**, 2433–2439 (2004)
12. Cortázar, C., Elgueta, M., Quirós, F., Wolanski, N.: Asymptotic behavior for a nonlocal diffusion equation in domains with holes. Arch. Rational Mech. Anal. **205**, 673–697 (2012)
13. Cortázar, C., Elgueta, M., Quirós, F., Wolanski, N.: Asymptotic behavior for a one-dimensional nonlocal diffusion equation in exterior domains. SIAM. J. Math. Anal. **48**, 1549–1574 (2016)
14. Cortázar, C., Elgueta, M., Quirós, F., Wolanski, N.: Asymptotic behavior for a nonlocal diffusion equation in exterior domains?: The critical two-dimensional case. J. Math. Anal. Appl. **436**, 586–610 (2016)
15. Chapuisat, G., Grenier, E.: Existence and nonexistence of traveling wave solutions for a bistable reaction-diffusion equation in an infinite cylinder whose diameter is suddenly increased. Comm. Partial Differential Equations **30**, 1805–1816 (2005)
16. Chasseigne, E., Chavesb, M., Rossi, J.D.: Asymptotic behavior for nonlocal diffusion equations. J. Math Pures Appl. **86**, 271–291 (2006)
17. Chen, X.: Existence, uniqueness and asymptotic stablility of traveling waves in nonlocal evolution equations. Adv. Differential Equations **2**, 125–160 (1997)
18. Coville, J., Dupaigne, L.: On a nonlocal reaction diffusion equation arising in population dynamics. Proc. Roy. Soc. Edinburgh Sect. **137**, 727–755 (2007)
19. Coville, J., Dávila, J., Martínez, S.: Nonlocal anisotropic dispersal with monostable nonlinearity. J. Differential Equations **244**, 3080–3118 (2008)
20. Guo, J.-S., Morita, Y.: Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations. Discrete Contin. Dyn. Syst. **12**, 193–212 (2005)
21. Guo, H., Hamel, F., Sheng, W.-J.: On the mean speed of bistable transition fronts in unbounded domains. J. Math. Pures Appl. **136**, 92–157 (2020)
22. Guo, H., Monobe, H.: V-shaped fronts around an obstacle. Math. Ann. **379**, 661–689 (2021)
23. Hamel, F., Nadirashvili, N.: Entire solution of the KPP eqution. Comm. Pure Appl. Math. **52**, 1255–1276 (1999)
24. Hamel, F., Nadirashvili, N.: Travelling fronts and entire solutions of the Fisher-KPP equation in R^N. Arch. Rational Mech. Anal. **157**, 91–163 (2001)
25. Hamel, F., Rossi, L.: Transition fronts for the Fisher-KPP equation. Trans. Amer. Math. Soc. **368**, 8675–8713 (2016)
26. Hamel, F.: Bistable transition fronts in $\mathbb{R}^N$. Adv. Math. **289**, 279–344 (2016)
27. Hoffman, A., Hupkes, H.J., Van Vleck, E.S.: Entire solutions for bistable lattice differential equations with obstacles. Mem. Amer. Math. Soc. **250**, (2017)
28. Kolmogorov, A.N., Petrovsky, I.G., Piskunov, N.S.: Étude de l'équation de la diffusion avec croissance de la quantité de la matière et son application à un problème biologique. Bull. Univ. Etat Mosc. Sr. Int. A **1**, 1-26 (1937)
29. Li, W.-T., Sun, Y.-J., Wang, Z.-C.: Entire solutions in the Fisher-KPP equation with nonlocal dispersal. Nonlinear Anal. Real World Appl. **11**, 2302–2313 (2010)
30. Shen, W.: Traveling waves in diffusive random media. J. Dynam. Differential Equations **16**, 1011–1060 (2004)
31. Shen, W., Shen, Z.: Transition fronts in time heterogeneous and random media of ignition type. J. Differential Equations **262**, 454–485 (2017)
32. Shen, W.: Stability of transition waves and positive entire solutions of Fisher-KPP equations with time and space dependence. Nonlinearity **30**, 3466–3491 (2017)
33. Sheng, W.-J., Guo, H.: Transition fronts of time periodic bistable reaction-diffusion equations in $\mathbb{R}^N$. J. Differential Equations **265**, 2191–2242 (2018)
34. Sun, Y.-J., Li, W.-T., Wang, Z.-C.: Entire solutions for nonlocal dispersal equations with bistable nonlineartiy. J. Differential Equations **251**, 551–581 (2011)
35. Sun, Y.-J., Zhang, L., Li, W.-T., Wang, Z.-C.: Entire solutions in nonlocal monostable equations: asymmetric case. Commun. Pure Appl. Anal. **18**, 1049–1072 (2019)
36. Wang, J.-B., Li, W.-T., Dong, F.-D., Qiao, S.-X.: Recent developments on spatial propagation for diffusion equations in shifting environments. Discrete Contin. Dyn. Syst. Ser. B **27**, 5101–5127 (2022)
37. Wang, Z.-C., Li, W.-T., Ruan, S.: Travelling wave fronts in reaction-diffusion systems with spatio-temporal delays. J. Differential Equations **222**, 185–232 (2006)
38. Weinberger, H.F.: Long-time behavior of a class of biological models. SIAM J. Math. Anal. **13**, 353–396 (1982)

39. Yagisita, H.: Existence and nonexistence of traveling waves for a nonlocal monostable equation. Publ. Res. Inst. Math. Sci. **45**, 925–953 (2009)
40. Yagisita, H.: Existence of traveling wave solutions for a nonlocal bistable equation:an abstract approach. Publ. Res. Inst. Math. Sci. **45**, 955–979 (2009)
41. Zhang, G.-B., Zhao, X.-Q.: Propagation phenomena for a two-species Lotka-Volterra strong competition system with nonlocal dispersal. Calc. Var. Partial Differential Equations **59**, (2020) Paper No.10, 34pp
42. Zhang, L., Li, W.-T., Wang, Z.-C.: Entire solution in an ignition nonlocal dispersal equation: asymmetric kernel. Sci. China Math. **60**, 1791–1804 (2017)
43. Zhang, L., Li, W.-T., Wang, Z.-C., Sun, Y.-J.: Entire solutions in nonlocal bistable equations: asymmetric case. Acta Math. Sin. **35**, 1771–1794 (2019)
44. Zlatoš, A.: Generalized traveling waves in disordered media: existence, uniqueness, and stability. Arch. Ration. Mech. Anal. **208**, 447–480 (2013)
45. Zlatoš, A.: Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations. J. Math. Pures Appl. **98**, 89–102 (2012)