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The Classical and Improved Euler-Jacobi Formula and Polynomial Vector Fields in $\mathbb{R}^n$

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Abstract
The classical and the new Euler-Jacobi formulae for simple and double points provide an algebraic relation between the singular points of a polynomial vector field and their topological indices. Using these formulae we obtain the geometrical configuration of the singular points together with their topological indices for several classes of polynomial differential systems in $\mathbb{R}^n$ when these differential systems, having the maximum number of singular points, either all their singular points are simple, or at most one singular point is double (i.e. it has multiplicity two).

Keywords Euler-Jacobi formula · Singular points · Topological index · Polynomial differential systems

Mathematics Subject Classification Primary 34A05 · Secondary 34C05 · 37C10

1 Introduction and Statement of the Main Results

Consider the polynomial differential system in $\mathbb{R}^n$

$$\dot{x}_i = P_i(x_1, \dots, x_n), \quad i = 1, \dots, n \quad n \geq 2 \tag{1}$$

where $P_i(x_1, \dots, x_n)$ are real polynomials such that $\deg(P_i) = 1$ for $i \geq 3$, and either $\deg(P_1) = 1$ and $\deg(P_2) = m$ with $m \in \mathbb{N}$, or $\deg(P_1) = 2$ and $\deg(P_2) = m$ with $m = 2, 3, 4, 5$. Moreover we assume that the $n-2$ hyperplanes $P_i = 0$ for $i \geq 3$ intersect in a two dimensional plane Σ contained in $\mathbb{R}^n$. We assume that system (1) has either m singular points, or $m-1$ singular points one of these singular points is double if $\deg(P_1) = 1$, and

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if $\deg(P_1) = 2$ then it has either $2m$ singular points, or $2m - 1$ singular points and one of these singular points is double. In the case in which there are $\deg(P_1)m$ finite singular points we use the classical Euler-Jacobi formula (a proof of the classical Euler-Jacobi formula can be found in [1]), and the case in which there are $\deg(P_1)m - 1$ finite singular points, the classical Euler-Jacobi formula is not valid anymore but Gasull and Torregrosa in [6] provided a generalization of the classical Euler-Jacobi formula in the case that the system has one double point and we will use such a formula. Using these formulae we obtain all the possible distributions of the singular points of system (1) when it has either $\deg(P_1)m$, or $\deg(P_1)m - 1$ singular points with $m \in \mathbb{N}$ when either $\deg(P_1) = 1$, or $m = 2, 3, 4, 5$ and $\deg(P_1) = 2$.

Since all the singular points of the differential system (1) are contained in the plane Σ we can restrict the study of the configurations of the singular points of the differential system (1) to study the configuration of the singular points of system (1) restricted to the plane Σ. Note that the plane Σ is not necessarily invariant by the flow of system (1). On the plane Σ we can reduce system (1) to the planar polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{2}$$

where $P(x, y)$ and $Q(x, y)$ are real polynomials such that either $\deg(P) = 1$ and $\deg(Q) = m$ with $m \in \mathbb{N}$, or $\deg(P) = 2$ and $\deg(Q) = m$ with $m = 2, 3, 4, 5$.

It follows from geometry that given two analytic curves $g = 0$ and $f = 0$ and a point p such that $f(p) = g(p) = 0$, p is *simple* if and only if the determinant of the Jacobian matrix of f and g at p, i.e.

$$J(f, g)(p) := J(p) = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\Big|_{(x,y)=p}$$

is different from zero, and that it is *double* if and only if $J(p) = 0$ and $I(f, g)(p) := I(p) \neq 0$ where

$$\begin{aligned} I(f, g)(p) := I(p) = &\left(\frac{\partial f}{\partial y}\right)^2 \left(\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial x \partial x^2} - \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial x^2}\right) \\ &- 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial x \partial y} - \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial x \partial y}\right) \\ &+ \left(\frac{\partial f}{\partial x}\right)^2 \left(\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y^2} - \frac{\partial g}{\partial x}\frac{\partial^2 f}{\partial y^2}\right)\Big|_{(x,y)=p}. \end{aligned}$$

For a proof see Lemma 2.2 of [6]. Moreover it is well-known that for planar polynomial differential systems (2), a simple singular point p has index 1 (if $J(p) > 0$), or -1 (if $J(p) < 0$) (see for instance [12]), and that a double singular point of our system has index zero.

It was proved in [5] and [7] that in the case of polynomial differential systems (2) the absolute value of the sum of the topological indices of all singular points is either 0 or 2 if m is even, and it is 0 if m is odd.

Consider a differential system formed by two real polynomials of degrees 2 and m respectively in the variables x and y. If the set of singular points of that system (that we denote by A) contains exactly $2m$ elements, then the Jacobian determinant evaluated at each zero does not vanish (see again [12]) and for any polynomial R of degree less than or equal to $m - 1$ we have

$$\sum_{a \in A} \frac{R(a)}{J(a)} = 0. \tag{3}$$

Using this classical Euler-Jacobi formula in [8] the authors characterized the number and distribution of the singular points of the polynomial differential systems (2) with $m = 2, 3, 4, 5$ when these systems have $2m$ finite singular points.

Now we consider the case in which one of the singular points is double. We will use the new Euler-Jacobi formula for double points proved in [6] which can be stated as follows. We need the following notation. We write the polynomial differential system (2) with $\deg P = 2$ and $\deg Q = m$ as

$$P(x, y) = P_{10}x + P_{01}y + P_{20}x^2 + P_{11}xy + P_{02}y^2,$$
$$Q(x, y) = Q_{10}x + Q_{01}y + Q_{20}x^2 + Q_{11}xy + Q_{02}y^2 + \dots,$$

and given a polynomial R we also write it as

$$R(x, y) = R_{00} + R_{10}x + R_{01}y + R_{20}x^2 + R_{11}xy + R_{02}y^2 + \dots$$

The next result is proved in Theorem 3.2 of [6] for two real polynomials of degrees n and m. We state it here for the case in which $n = 2$ when the system has $2m - 1$ finite singular points.

Theorem 1 *Consider a differential system of two real polynomials of degrees* 2 *and* $m \geq 2$ *respectively with* $m \geq 2$ *in the variables* x *and* y. *If the set of all singular points of the system (that we denote by* A*) contains exactly* $2m-1$ *elements (*$2m-2$ *being simple and one double that without loss of generality we can assume it is at the origin), then for any polynomial* R *of degree less than or equal to* $m - 1$ *we have*

$$\sum_{a \in A_S} \frac{R(a)}{J(a)} + S(0) = 0, \tag{4}$$

where A_S *denotes the set of simple singular points of the system and* $S(0)$ *is equal to*

$$S(0) = \frac{4P_{10}R_{00}N}{I(0)^2} + \frac{2P_{10}(P_{10}R_{01} - P_{01}R_{10})}{I(0)},$$

where

$$\begin{aligned} N = {} & P_{10}^3(Q_{10} - P_{10}Q_{03}) - P_{10}^2P_{01}(Q_{10} - P_{10}Q_{12}) \\ & + P_{10}P_{01}^2(Q_{10} - P_{10}Q_{21}) - P_{01}^3(Q_{10} - P_{10}Q_{30}) \\ & + P_{10}^3(Q_{11}P_{02} - P_{11}Q_{02}) - 2P_{10}^2P_{01}(Q_{20}P_{02} - P_{20}Q_{02}) \\ & + P_{10}P_{01}^2(Q_{20}P_{11} - P_{20}Q_{11}), \end{aligned}$$

where $Q_{3,0} = Q_{2,1} = Q_{1,2} = Q_{0,3} = 0$ *in case* $m = 2$.

Before we state the main results of this paper we need to introduce some notations.

Let $X = (P, Q)$ be the vector field associated to the differential system (2). We denote by $A_X = A$ the set of points $p \in \mathbb{R}^2$ such that $X(p) = 0$. Given a finite subset B of $\mathbb{R}^2$, we denote by $\hat{B}$, $\partial \hat{B}$ and $\#B$ its convex hull, the boundary of the convex hull, and its cardinal, respectively.

Set $A_0 = A$ and $A_{i+1} = A_i \cap \partial \widehat{A_i}$ for $i \geq 0$. Note that there exists a positive integer q such that $A_q \neq \emptyset$ and $A_{q+1} = \emptyset$.

We say that A has the *configuration* $(K_1; K_2; \dots; K_q)$ if $K_i = \#A_i$ for $i = 1, \dots, q$. We say that A has the configuration $(K_1; K_2; \dots; K_r; *)$ if we do not specify for the values of K_i for i between $r+1$ and q. We also say that the singular points of X belonging to $A_i \cap \partial \hat{A}_i$ are on the *i-th level*.

We want to be more precise and study also the indices of the singular points of $X = (P, Q)$ with P and Q as in (2). Then we substitute each K_i by the sign of the indices of the points of A_i, i.e. instead of K_i in the configuration we write the string $(s_1^i, s_2^i, \ldots, s_{K_i}^i)$ where $s_i^j \in \{+, -, 0\}$. When $\hat{A}_i$ is a polygon, the starting s_1^i will be the point with multiplicity two (denoted by p_0) if such point is in the i-th level and the signs $s_2^i, \ldots, s_{K_i}^i$ are the signs of the list of positive or negative indices that follow the point with multiplicity two in counterclockwise or clockwise sense according with the largest list of points with the same index between the two lists closest to the point with multiplicity two. In case that both closest lists have the same length, we choose the one with the second largest closest list, and so one. In fact when there are ℓ equal consecutive signs, for instance if they are $+$, then instead of $+ \cdots +$ ℓ-times we shall write $\ell+$. In the case that the i-th level does not contain the point with multiplicity two, then s_1^i is the length of the largest list of positive or negative indices of the singular point in the i-th level. The numbers $s_2^i, \ldots, s_{K_1}^i$ are chosen following the previous criteria changing the point with multiplicity two by s_1^i.

When $\hat{A}_i$ is a segment, which does not contain the point with multiplicity two, we identify all the list of signs of this segment cyclically, i.e. after one endpoint it follows the other endpoint. The signs of the strings are ordered starting at one of the endpoints. Then we start for the endpoint having the larger list of signs independently if this list is formed by plus or minus signs. In case that the length of the list of signs of both endpoints are equal, then we choose to start with the endpoint whose second list is larger, and so on.

If $\hat{A}_i$ is a segment containing the point with multiplicity two again we identify all the list of signs of this segment cyclically, i.e. after one endpoint it follows the other endpoint. Then the starting sign in the list is the sign 0 of the point of multiplicity two, and after it we choose the largest list closest to p_0. In case that the two lists of signs separated by p_0 have the same length, then we choose to start with the list near p_0 whose second list is larger, and so on.

With these notations we can state the main results of the paper. The first main result is when $\deg(P) = 1$.

Theorem 2 *For the polynomial differential* (2) *with* $\deg(P) = 1$ *and* $m \in \mathbb{N}$ *having* $2m$ *singular points, the following statements hold.*

(a) *If it has m singular points then only the following configurations are possible:*

 (a.1) $(m) = (2+, -, +, -, \ldots, +, -)$ *or* $(m) = (2-, +, -, +, \ldots, +)$ *if m is odd;*
 (a.2) $(m) = (+, -, +, -, \ldots, +, -)$ *or* $(m) = (-, +, -, +, \ldots, -, +)$ *if m is even.*

(b) *If it has* $m - 1$ *singular points with one of them double then only the following configurations are possible:*

 (b.1) $(m - 1) = (0, +, -, +, -, \ldots, +, -, +)$ *or* $(m) = (0, -, +, -, +, \ldots, +, -)$ *if m is odd;*
 (b.2) $(m - 1) = (0, +, -, +, -, \ldots, +, -)$ *or* $(m) = (0, -, +, -, +, \ldots, -, +)$ *if m is even.*

Moreover there are examples of all these configurations.

The proof of Theorem 2 is given in Sect. 3.

From now on we consider the cases in which $\deg(P) = 2$. The first main result under these conditions, which is the second main result of this paper, is when system (2) has $2m$ finite singular points.

Theorem 3 *For the polynomial differential system* (2) *with* $\deg(P) = 2$*,* $m = 2, 3, 4, 5$ *and with* $2m$ *singular points, the following statements hold.*

(a) *If* $m = 2$ *then only the following two configurations are possible*

(i) $(4) = (+, -, +, -)$,
(ii) $(3; 1) = (3+; -)$, $(3-; +)$,

and there exist examples of such configurations.

(b) *If* $m = 3$ *then only the following two configurations are possible*

(i) $(6) = (+, -, +, -, +, -)$,
(ii) $(4; 2) = (2+, 2-; +, -)$,
(iii) $(3; 3) = (2+, -; 2-, +)$, $(2-, +; 2+, -)$,

and there exist examples of such configurations.

(c) *If* $m = 4$ *then only the following configurations are possible*

(i) $(8) = (+, -, +, -, +, -, +, -)$,
(ii) $(5; 3) = (4+, -; -, +, -)$, $(4-, +; +, -, +)$, $(2+, -, +, -; -, +, -)$, $(2-, +, -, +; +, -, +)$,
(iii) $(4; 4) = (+, -, +, -; +, -, +, -)$,
(iv) $(4; 3; 1) = (4+; 3-; +)$, $(4-; 3+; -)$,
(v) $(3; 5) = (3+; 2-, +, -, +)$, $(3-; 2+, -, +, -)$,

and there exist examples of such configurations.

(d) *If* $m = 5$ *then only the following configurations are possible*

(i) $(10) = (+, -, +, -, +, -, +, -, +, -)$,
(ii) $(6; 4) = (2+, 2-, +, -; +, -, +, -)$, $(2+, 2+, -, +; +, -, +, -)$,
(iii) $(4; 6) = (2+, 2-; +, -, +, -, +, -)$,
(iv) $(4; 4; 2) = (2+, 2-; 2+, 2-; +, -)$,
(v) $(4; 3; 3) = (2+, 2-; 2-, +; 2+, -)$, $(2+, 2-; 2+, -; 2-, +)$,
(vi) $(3; 7) = (2-, +; 2+, -, +, -, +, -)$, $(2+, -; 2-, +, -, +, -, +)$,

and there exist examples of such configurations.

The case $m = 2$ of Theorem 3 is the well-known Berlinskii's Theorem proved in [2] and reproved in [4] using the Euler-Jacobi formula. The case $m = 3$ was proved in [4]. The cases $m = 4, 5$ were proved in [8]. So we do not need to prove Theorem 3. However the configurations on that papers were counted in a slightly different way.

The second main result in the paper takes into account the case in which system (2) has $\deg(P) = 2$ and $2m - 1$ finite singular points and $m = 5$.

Theorem 4 *For the polynomial differential* (2) *with* $\deg(P) = 2$*,* $m = 2, 3, 4, 5$ *and with* $2m - 1$ *singular points with one double singular point, the following statements hold.*

(a) *If* $m = 2$ *then only the configurations* $(3) = (0, +, -)$*, and* $(0, 2+)$ *are possible, and there exist examples of such configurations.*

(b) *If* $m = 3$ *then only the following two configurations are possible*

(i) $(5) = (0, +, -, +, -)$,
(ii) $(4; 1) = (0, 2+, -; -)$, $(0, 2-, +; +)$, $(2+, 2-; 0)$,
(iii) $(3; 2) = (0, +, -; +, -)$, $(2+, -; 0, -)$, $(2-, +; 0, +)$,

and there exist examples of such configurations.

(c) *If* $m = 4$ *then only the following configurations are possible*

(i) $(7) = (0, +, -, +, -, +, -)$,
(ii) $(5; 2) = (0, 2+, -, +; +, -)$, $(0, 2-, +, -; +, -)$, $(0, 3+, -; +, -)$, $(0, 3-, +; +, -)$, $(4+, -; 0, -)$, $(4-, +; 0, +)$, $(0, -, +, -, +; +, -)$, $(0, +, -, +, -; +, -)$, $(0, +, 2-, +; +, -)$, $(0, -, 2+, -; +, -)$, $(2-, +, -, +; 0, +)$, $(2+, -, +, -; 0, -)$,
(iii) $(4; 3) = (0, 3+; 2-, +)$, $(0, 3-; 2+, -)$, $(4+; 0, 2-)$, $(4-; 0, 2+)$, $(0, +, -, +; 2-, +)$, $(0, -, +, -; 2+, -)$, $(+, -, +, -; 0, +, -)$,
(iv) $(3; 4) = (0, 2+; +, -, +, -)$, $(0, 2-; +, -, +, -)$, $(3+; 0, -, +, -)$, $(3-; 0, +, -, +)$, $(3+; 0, 2-, +)$, $(3-; 0, 2+, -)$, $(0, +, -; +, -, +, -)$,

and there exist examples of such configurations.

(d) *If* $m = 5$ *then only the following configurations are possible*

(i) $(9) = (0, +, -, +, -, +, -, +, -)$,
(ii) $(6; 3) = (0, 2-, +, -; 2+, -)$, $(0, 2+, -, +; 2-, +)$, $(0, +, 2-, +, -; 2+, -)$, $(0, -, 2+, -, +; 2-, +)$, $(2+, 2-, +, -; 0, +, -)$,
(iii) $(5; 4) = (0, 2+, 2-; +, -, +, -)$, $(0, +, 2-, +; +, -, +, -)$, $(0, -, 2+, -; +, -, +, -)$, $(0, +, -, +, -; +, -, +, -)$,
(iv) $(4; 5) = (0, 2+, -; -, +, -, +, -)$, $(0, 2-, +; +, -, +, -, +)$ $(2+, 2-; 0, +, -, +, -)$,
(v) $(4; 4; 1) = (2+, 2-; 2+, 2-; 0)$, $(2+, 2-; 0, 2-, +; +)$, $(2+, 2-; 0, 2+, -; -)$, $(0, 2+, -; 3-, +; +)$, $(0, 2-, +; 3+, -; -)$.
(vi) $(4; 3; 2) = (2+, 2-; 2-, +; 0, +)$, $(2+, 2-; 2+, -; 0, -)$, $(2+, 2-; 0, +, -; +, -)$, $(0, 2-, +; 2+, -; +, -)$, $(0, 2+, -; 2-, +; +, -)$,
(vii) $(3; 6) = (0, +, -; +, -, +, -, +, -)$, $(2-, +; 0, +, -, +, -, +)$, $(2+, -; 0, -, +, -, +, -)$,

and there exist examples of such configurations.

The cases $m = 2$ and $m = 3$ were proved in [6]. The case $m = 4$ was proved in [11]. In these two papers with the configurations counted in a slightly different way. In the present paper we will prove the case $m = 5$, see also [9, 10].

Note that the configuration of the singular points of the differential system (2) studied in Theorems 2–4 are the configurations of the singular points of the differential system (1), but the information on the indices of these singular points are only for the restriction of system (1) to the plane Σ, i.e. for system (2).

2 Preliminarities

In the proof of Theorem 4 we will use the following auxiliary result proved in [3].

Lemma 5 *Let* (P, Q) *be a polynomial vector field with* $\max(\deg P, \deg Q) = n$. *If* (P, Q) *has* $n + 1$ *singular points on the straight line* $L(x, y) = 0$, *then this line is full of singular points.*

First observe that if a configuration exists for a polynomial vector field $X = (P, Q)$, then it is possible to construct the same configuration but interchanging the index $+1$ by the index -1. For doing that it is enough to take the vector field $Y = (-P, Q)$ instead of the vector field $X = (P, Q)$. So we can restrict ourselves to the cases in which $\sum_{a \in A} i_X(a) \geq 0$.

Assume that the vector field (P, Q) has degrees 2 and 5 (respectively), 8 simple singular points and 1 double singular point p_0. We can consider p_0 at the origin and denote by $p_1, \ldots, p_8$ the simple singular points. Clearly p_0 has index 0 and the other singular points have index ± 1. During the proof of statement (d) of Theorem 4 we will denote by p_j the singular point for which there is no information about its index, by p_j^+ the singular points having positive index, and by p_j^- the singular points having negative index. Also we will denote by $L_{i,j}$ the straight line $L_{p_i,p_j}(x, y) = 0$ through the points p_i and p_j, and we will denote by L_i a straight line through a singular point p_i such that for any singular point q with $q \neq p_i$ we have $L_i(q) \neq 0$ and $L_i(p_i) = 0$.

It was proved in [7], see also [5], that in the case of polynomial vector fields of degree $(1, m)$ with m odd it holds that $|\sum_{a \in A} i_X(a)| = 1$, and $\sum_{a \in A} i_X(a) = 0$ if m is even. Moreover in the case of polynomial vector fields of degree $(2, m)$ with m odd it holds that $\sum_{a \in A} i_X(a) = 0$ and if m is even then $|\sum_{a \in A} i_X(a)| \in \{0, 2\}$.

3 Proof of Theorem 2

By statement (a) of Theorem 4 we have that if m is even then there are $m/2$ singular points with index $+1$ and $m/2$ singular points with index -1. If m is odd, by the explanation in the previous section, we can assume that there are $(m+1)/2$ points with index $+1$ and $(m-1)/2$ singular points with index -1.

Since P has degree 1, $P(x, y) = 0$ is a straight line and the m finite singular points of system (2) are on this straight line. Therefore when there are m finite singular points, the unique possible configuration is (m) because any convex hull of m points on a straight line has all points in the boundary of the convex hull. For the same reason when there are $m - 1$ finite singular points the unique possible configuration is $m - 1$.

We first study the configuration (m). Assume that the subscripts of the points in A are in such a way that $p_1, \ldots, p_m$ are ordered in $\partial \hat{A}$ consecutively. Take

$$C_1 = L_1 \cdots L_{m-2}, \quad C_{m-1} = L_3 L_4 \cdots L_m$$

and for $i = 2, \ldots, m-2$,

$$C_i = L_1 \cdots L_{m-i-1} L_{m-i+2} \cdots L_m,$$

where all the straight lines which appear in the C_i's for $i = 1, \ldots, m-1$ are parallel.

Then the Euler Jacobi formula (4) applied with $R = C_i$ yields

$$\frac{C_i(p_{m-i})}{J(p_{m-i})} + \frac{C_i(p_{m-i+1})}{J(p_{m-i+1})} = 0.$$

Since all the points $p_1, \ldots, p_m$ are in a straight line, the polynomial $C_i(x, y)$ has the same sign on the two points p_{m-i} and p_{m-i+1}. So $J(p_{m-i})J(p_{m-i+1}) < 0$ for all $i = 1, \ldots, m-1$. Hence the indices of p_{m-i} and p_{m-i+1} are different for $i = 1, \ldots, m-1$ providing the configurations (a.1) and (a.2) of statement (a) of Theorem 2.

The configurations of statement (a) can be realized intersecting a straight line $P(x, y) = 0$ with m parallel straight lines $Q(x, y) = 0$. This completes the proof of statement (a) of Theorem 2.

Now we prove statement (b). Assume that the subscripts of the points in A are in such a way that $p_0, p_1, \ldots, p_m$ are ordered in $\partial \hat{A}$ consecutively. Take

$$C_1 = L_0^2 L_1 \cdots L_{m-3}, \quad C_{m-2} = L_0^2 L_3 L_4 \cdots L_{m-1}$$

and for $i = 2, \ldots, m-3$,

$$C_i = L_0^2 L_1 \cdots L_{m-i-1} L_{m-i+2} \cdots L_{m-1},$$

where all the straight lines which appear in the C_i's for $i = 1, \ldots, m-2$ are parallel.

Then the Euler Jacobi formula (4) applied with $R = C_i$ yields

$$\frac{C_i(p_{m-i})}{J(p_{m-i})} + \frac{C_i(p_{m-i+1})}{J(p_{m-i+1})} = 0.$$

Since all the points $p_0, p_1, \ldots, p_m$ are in a straight line, the polynomial $C_i(x, y)$ has the same sign on the two points p_{m-i} and p_{m-i+1}. So $J(p_{m-i})J(p_{m-i+1}) < 0$ for all $i = 1, \ldots, m-2$. Hence the indices of p_{m-i} and p_{m-i+1} are different for $i = 1, \ldots, m-2$ providing the configurations (b.1) and (b.2) of statement (b) of Theorem 2.

The configurations of statement (b) can be realized intersecting a straight line $P(x, y) = 0$ with m straight lines, $Q(x, y) = 0$, being $m-1$ parallel straight lines and the other straight line intersecting P and one of the other $m-1$ straight lines in the same point. This completes the proof of statement (b) of Theorem 2.

4 Proof of Statement (*d*) of Theorem 4

In principle we could have the configurations (9), (8; 1), (7; 2), (6; 3), (5; 4), (5; 3; 1), (4; 5), (4; 4; 1), (4; 3; 2), (3; 6), (3; 5; 1), (3; 4; 2), (3; 3; 3) and (2; $*$). Note that since the polynomial P has degree two, $P(x, y) = 0$ is a conic and the nine singular points of system (2) are on this conic. Clearly configurations of the form (2+; $*$) cannot occur because the seven singular points would be on a straight line, and by Lemma 5 this straight line will be full of singular points, a contradiction. Moreover the configurations (8; 1) and (7; 2) are only possible if the conic $P = 0$ is formed by two straight lines intersecting at a real point but then either seven or six singular points would be on the same straight line, and by Lemma 5 this straight line will be full of singular points, a contradiction. The configuration (6; 3) is only possible with two straight lines intersecting at a real point. The configuration (5; 4) is only possible with two straight lines intersecting at a real point. The configuration (5; $*$; $*$) is not possible because any convex hull of 5 points on a conic with five points at least in the first level can only be supported by a conic formed by two straight lines intersecting to a real point and this configuration do not support having points in the 2nd-level. The configurations (4; 5), (4; 4; 1) and (4; 3; 2) are only possible with either a hyperbola or two straight lines intersecting at a real point, and the configuration (3; 6) is only possible with a hyperbola. The configurations (3; 5; 1), (3; 4; 2), (3; 3; 3) are not possible since no real conic (ellipse, parabola, hyperbola, two parallel straight lines, two straight lines intersecting in a real point, one double straight line, two parallel straight lines, or one point) do not support such configurations and the configurations.

In short the unique possible configurations are: (9) (realized with either an ellipse, or two straight lines intersecting at a point), (6; 3) and (5; 4) (both realized with two straight lines intersecting at a real point), (4; 5) (realized with either a hyperbola or two straight lines intersecting at a real point) and (4; 4; 1), (4; 3; 2), (3; 6) (all realized with a hyperbola). We study them separately.

Configuration (9). We first show that two consecutive points p_{k_1}, p_{k_2} none of them being p_0 must have opposite index, otherwise applying formula (4) with $R(x, y) = L_{p_0,p_{k_3}} L_{p_0,p_{k_4}} L_{p_{k_5},p_{k_6}} L_{p_{k_7} p_{k_8}}$ with p_{k_i} for $i \in \{3, \ldots, 8\}$ being all different and different from p_{k_1}, p_{k_2} we get

$$\frac{R(p_{k_1})}{J(p_{k_1})} + \frac{R(p_{k_2})}{J(p_{k_2})} = 0,$$

which is a contradiction because $R(p_{k_1})R(p_{k_2}) > 0$ and $J(p_{k_1}) = J(p_{k_2})$.

In short, the only possible configurations are $(0, +, -, +, -, +, -, +, -)$ (and of course $(0, -, +, -, +, -, +, -, +)$). System (2) with

$$\begin{aligned} P(x, y) &= xy, \\ Q(x, y) &= -96 + 224x + 219.2y - 190x^2 - 180y^2 + 75x^3 + 224xy^2 + 68y^3 \\ &\quad -14x^4 - 224x^3y + 224x^2y^2 + 224xy^3 - 12y^4 + x^5 + 224x^4y \\ &\quad +224x^3y^2 + 224x^2y^3 + 0.8y^5, \end{aligned}$$

has the singular points

$$(4, 0),\ (0, 5),\ (0, 4),\ (0, 3),\ (0, 2),\ (0, 1),\ (1, 0),\ (2, 0),\ (3, 0),$$

in the configuration $(0, +, -, +, -, +, -, +, -)$ (we recall that the other configuration can be obtained reversing the sing in P).

Configuration (6; 3). We note that the configuration (6; 3) only can be realized with a conic formed by two straight lines (R_1 and R_2) intersecting at a point q. Without loss of generality we can assume that five singular points are in R_1 and five or four singular points are in R_2 depending if the intersection point q is a singular point or not. Note that all points of R_1 are in the 1-st level.

If q is not a singular point then p_0 is in R_2, otherwise applying formula (4) with $R(x, y) = R_1 L_{p_0,q_1} L_{p_0,q_2} L_{p_0,q_3}$ where q_1, q_2 and q_3 are three singular points on R_2. Then we reach a contradiction. Moreover if q is a singular point then $q = p_0$. Indeed, if p_0 is in R_1 and $p_0 \neq q$ the previous argument also provides in this case a contradiction. Also if $p_0 \neq q$ is in R_2 we can repeat the same argument. So $q = p_0$.

We separate the proof in two cases.

Csse 1: q is not a singular point. Therefore $p_0 \in R_2$.

Let p_{k_1} and p_{k_2} be two consecutive singular points in R_1 not separated by q. Applying formula (4) to $R(x, y) = R_2 L_{p_0,p_{k_3}} L_{p_0,p_{k_4}} L_{p_0,p_{k_5}}$ where p_{k_j} for $j = 3, 4, 5$ are singular points in R_1 and different from p_{k_1}, p_{k_2}, we get that p_{k_1} and p_{k_2} have different index.

Let p_{k_1} and p_{k_2} be two singular points in R_1 such that the segment having them as endpoints contains the point q and not other singular points. Applying formula (4) to $R(x, y) = R_2 L_{p_0,p_{k_3}} L_{p_0,p_{k_4}} L_{p_0,p_{k_5}}$ where p_{k_j} for $j = 3, 4, 5$ are singular points in R_1 and different from p_{k_1}, p_{k_2}, we get that p_{k_1} and p_{k_2} have the same index.

Let p_{ℓ_1} and p_{ℓ_2} be two consecutive singular points in R_2 different from p_0. Applying formula (4) to $R(x, y) = R_1 L_{p_0,p_{k_1}} L_{p_0,p_{k_2}} L_{p_{\ell_3},p_{k_5}}$ where p_{k_j} for $j = 1, 2, 3$ are singular points in R_1 and p_{ℓ_3} is a singular point in R_2 and different from p_{ℓ_1} and p_{ℓ_2}, we get that p_{ℓ_1} and p_{ℓ_2} have different index.

Let p_{ℓ_1} and p_{ℓ_2} be two singular points in R_2 such that the segment having them as endpoints contains only a singular point, which is p_0. Applying formula (4) to $R(x, y) = R_1 L_{p_0,p_{k_1}} L_{p_0,p_{k_2}} L_{p_{\ell_3},p_{k_5}}$ where p_{k_j} for $j = 1, 2, 3$ are singular points in R_1 and p_{ℓ_3} is a singular point in R_2 and different from p_{ℓ_1} and p_{ℓ_2}, we get that p_{ℓ_1} and p_{ℓ_2} have different index.

Csse 2: q is a singular point. Then $q = p_0$.

Let p_{k_1} and p_{k_2} be two consecutive singular points in R_1 different from p_0. Applying formula (4) to $R(x, y) = R_2^2 L_{p_{\ell_1},p_{k_3}} L_{p_{\ell_2},p_{k_4}}$ where p_{k_j} for $j = 3, 4$ are singular points in

R_1 and different from p_{k_1}, p_{k_2} and p_{ℓ_1}, p_{ℓ_2} singular points in R_2, we get that p_{k_1} and p_{k_2} have different index.

Let p_{k_1} and p_{k_2} be two singular points in R_1 such that the segment having them as endpoints contains only a singular point, which is p_0. Applying formula (4) to $R(x, y) = R_2^2 L_{p_{\ell_1}, p_{k_3}} L_{p_{\ell_2}, p_{k_4}}$ where p_{k_j} for $j = 3, 4$ are singular points in R_1 and different from p_{k_1}, p_{k_2} and p_{ℓ_1}, p_{ℓ_2} singular points in R_2, we get that p_{k_1} and p_{k_2} have different index.

Let p_{ℓ_1} and p_{ℓ_2} be two consecutive singular points in R_2 different from p_0. Applying formula (4) to $R(x, y) = R_1^2 L_{p_{\ell_3}, p_{k_1}} L_{p_{\ell_4}, p_{k_2}}$ where p_{k_j} for $j = 1, 2$ are singular points in R_1 and p_{ℓ_3}, p_{ℓ_4} singular points in R_2 and different from p_{ℓ_1} and p_{ℓ_2}, we get that p_{ℓ_1} and p_{ℓ_2} have different index.

The possible configurations are $(0, 2-, +, -, +; 2+, -)$, $(0, +, 2-, +, -; 2+, -)$, $(2+, 2-, +, -; 0, +, -)$ (and of course, $(0, 2+, -, +, -; 2-, +)$, $(0, -, 2+, -, +; 2-, +)$).

System (2) with

$$P(x, y) = x(y + 1),$$
$$Q(x, y) = x + x^2 + xy + \frac{1}{36}y^2 - \frac{3}{8}x^3 + \frac{7}{3}x^2y - \frac{5}{8}xy^2 + \frac{17}{72}y^3 + x^4 + x^3y + x^2y^2 + xy^3 + \frac{5}{8}y^4 + x^5 - \frac{5}{12}x^4y + x^3y^2 + x^2y^3 + xy^4 + \frac{1}{2}y^5,$$

has the singular points

$(0, 0)$, $(-1, -1)$, $(-2/3, -1)$, $(-1/2, -1)$, $(-1/4, -1)$, $(1, -1)$ in the 1st level, and $(0, -2/3)$, $(0, -1/3)$, $(0, -1/4)$ in the 2nd level,

in the configuration $(0, 2-, +, -, +; 2+, -)$ (configuration $(0, 2+, -, +.-; 2-, +)$ can be obtained reversing the sing in P).

System (2) with

$$P(x, y) = -x(y + 1),$$
$$Q(x, y) = x + x^2 + xy - \frac{1}{36}y^2 - \frac{11}{8}x^3 + \frac{7}{6}x^2y + \frac{3}{8}xy^2 - \frac{17}{72}y^3 + x^4 + x^3y + x^2y^2 + xy^3 - \frac{5}{8}y^4 + x^5 + \frac{11}{12}x^4y + x^3y^2 + x^2y^3 + xy^4 - \frac{1}{2}y^5,$$

has the singular points

$(0, 0)$, $(1, -1)$, $(-1, -1)$, $(-1/2, -1)$, $(-1/4, -1)$, $(2/3, -1)$ in the 1st level, and $(0, -2/3)$, $(0, -1/3)$, $(0, -1/4)$ in the 2nd level,

in the configuration $(0, +, 2-, +, -; 2+, -)$ (configuration $(0, -, 2+, -, +; 2-, +)$ can be obtained reversing the sing in P).

System (2) with

$$P(x, y) = x(1 + y),$$
$$Q(x, y) = x - y/9 + x^2 - \frac{19}{18}y^2 - \frac{16}{9}x^3 - \frac{65}{18}x^2y - \frac{14}{9}xy^2 - \frac{11}{3}y^3 + x^3y + xy^3 - \frac{11}{2}y^4 + x^5 + \frac{1}{2}x^4y + x^2y^3 - 3y^5,$$

has the singular points

$(1, -1)$, $(0, 0)$, $(-2, -1)$, $(1/3, -1)$, $(1/2, -1)$, $(2/3, -1)$ in the 1st level, and

$$(0,-2/3),\ (0,-1/2),\ (0,-1/3) \text{ in the 2nd level,}$$

in the configuration $(2+, 2-, +, -; 0, +, -)$.

Configuration $(5; 4)$. We note that the configuration $(5; 4)$ only can be realized with a conic formed by two straight lines (R_1 and R_2) intersecting at a point q. Without loss of generality we can assume that four singular points are in R_1 (all in the 1st-level) and five singular points are in R_2.

Note that the intersection point q is not a singular point otherwise one of the straight lines would have six singular points, and by Lemma 5 this straight line will be full of singular points, a contradiction.

We note that p_0 is in R_1. Otherwise if p_0 is in R_2 then applying (4) with $R(x,y) = R_2 L_{p_0,q_1} L_{p_0,q_2} L_{p_0,q_3}$ where q_1, q_2 and q_3 are three singular points on R_1 we reach to a contradiction. So p_0 must be in R_1, and so in the 1st-level.

Let p_{k_1} and p_{k_2} be two consecutive singular points in R_1 different from p_0 and not separated by q. Applying formula (4) to $R(x,y) = R_2 L_{p_0,p_{\ell_1}} L_{p_0,p_{\ell_2}} L_{p_{\ell_3},p_{k_3}}$ where p_{k_3} is a singular point in R_1 different from p_{k_1} and p_{k_2}, and p_{ℓ_j} for $j = 1, 2, 3$ are singular points in R_2, we get that p_{k_1} and p_{k_2} have different index.

Let p_{k_1} and p_{k_2} be two consecutive singular points in R_1 such that the segment having them as endpoints does not contain q and contains only a singular point, which is p_0. Applying formula (4) to $R(x,y) = R_2 L_{p_0,p_{\ell_1}} L_{p_0,p_{\ell_2}} L_{p_{\ell_3},p_{k_3}}$ where p_{k_3} is a singular point in R_1 different from p_{k_1} and p_{k_2}, and p_{ℓ_j} for $j = 1, 2, 3$ are singular points in R_2, we get that p_{k_1} and p_{k_2} have different index.

Let p_{k_1} and p_{k_2} be two consecutive singular points in R_1 different from p_0 and separated by q. Applying formula (4) to $R(x,y) = R_2 L_{p_0,p_{\ell_1}} L_{p_0,p_{\ell_2}} L_{p_{\ell_3},p_{k_3}}$ where p_{k_3} is a singular point in R_1 different from p_{k_1} and p_{k_2}, and p_{ℓ_j} for $j = 1, 2, 3$ are singular points in R_2, we get that p_{k_1} and p_{k_2} have the same index.

Let p_{k_1} and p_{k_2} be two consecutive singular points in R_1 such that the segment having them as endpoints contains only q and one singular point which is p_0. Applying formula (4) to $R(x,y) = R_2 L_{p_0,p_{\ell_1}} L_{p_0,p_{\ell_2}} L_{p_{\ell_3},p_{k_3}}$ where p_{k_3} is a singular point in R_1 different from p_{k_1} and p_{k_2}, and p_{ℓ_j} for $j = 1, 2, 3$ are singular points in R_2, we get that p_{k_1} and p_{k_2} have the same index.

Let p_{ℓ_1} and p_{ℓ_2} be two consecutive singular points in R_2. Applying formula (4) to $R(x,y) = R_1 L_{p_0,p_{\ell_3}} L_{p_0,p_{\ell_4}} L_{p_0,p_{\ell_5}}$ where p_{ℓ_j} for $j = 3, 4, 5$ are singular points in R_2 and different from p_{ℓ_1}, p_{ℓ_2}, we get that p_{ℓ_1} and p_{ℓ_2} have different index.

The possible configurations are $(0, 2+, 2-; +, -, +, -)$, $(0, +, 2-, +; +, -, +, -)$, $(0, +, -, +, -; +, -, +, -)$ (and of course $(0, -, 2+, -; +, -, +, -)$),

The differential system (2) with

$$P(x,y) = x(1+y),$$
$$Q(x,y) = x - \frac{1}{12}y + x^2 + xy - \frac{7}{8}y^2 - \frac{5}{4}x^3 + \frac{9}{4}x^2y + \frac{1}{4}xy^2 - \frac{79}{24}y^3 + x^4 + x^3y + x^2y^2 + xy^3 - \frac{21}{4}y^4 + x^5 + x^3y^2 + x^2y^3 + xy^4 - 3y^5,$$

has the singular points

$$(1,-1),\ (0,0),\ (-1,-1),\ (-1/2,-1),\ (1/2,-1) \text{ in the1st level, and}$$
$$(0,-2/3),\ (0,-1/2),\ (0,-1/3),\ (0,-1/4) \text{ in the 2nd level,}$$

in the configuration $(0, 2+, 2-; +, -, +, -)$.

The differential system (2) with

$$\begin{aligned} P(x,y) &= x(1+y),\\ Q(x,y) &= x - \frac{1}{24}y + x^2 - \frac{7}{16}y^2 - \frac{5}{4}x^3 + \frac{21}{8}x^2y - \frac{3}{4}xy^2 - \frac{79}{48}y^3 + x^3y + x^2y^2\\ &\quad + xy^3 - \frac{21}{8}y^4 + x^5 - \frac{1}{2}x^4y + x^3y^2 + xy^4 - \frac{3}{2}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1,-1),\ (0,0),\ (-1,-1),\ (-1/2,-1),\ (1/2,-1) \text{ in the 1st level, and}\\ &(0,-2/3),\ (0,-1/2),\ (0,-1/3),\ (0,-1/4) \text{ in the 2nd level,} \end{aligned}$$

in the configuration $(0,+,2-,+;+,-,+,-)$ (note that $(0,-,2+,-;+,-,+,-)$ can be obtained reversing the sing in P).

The differential system (2) with

$$\begin{aligned} P(x,y) &= x(y+1),\\ Q(x,y) &= x + \frac{1}{18}y + x^2 + xy + \frac{7}{12}y^2 - \frac{5}{3}x^3 + x^2y + \frac{2}{3}xy^2 + \frac{79}{36}y^3 + x^4 + x^3y\\ &\quad + x^2y^2 + xy^3 + \frac{7}{2}y^4 + x^5 + \frac{5}{6}x^4y + x^3y^2 + x^2y^3 + xy^4 + 2y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1,-1),\ (0,0),\ (-1,-1),\ (1/3,-1),\ (1/2,-1) \text{ in the 1st level, and}\\ &(0,-2/3),\ (0,-1/2),\ (0,-1/3),\ (0,-1/4) \text{ in the 2nd level,} \end{aligned}$$

in the configuration $(0,+,-,+,-;+,-,+,-)$.

Configuration $(4;5)$. We note that the configuration $(4;5)$ can only be realized in a hyperbola or the conic formed by two straight lines intersecting to a real point. We do not consider the case that the conic is formed by two straight lines intersecting in a real point because the proof in this case is completely similar to the proof when the conic is a hyperbola. In this last case one branch of the hyperbola, B_1 has the two singular points p_8, p_9 ordered in counterclockwise sense which are both of them in the 1st level and the other branch of the hyperbola B_2 has the seven singular points $p_1, p_2, p_3, p_4, p_5, p_6, p_7$ ordered in counterclockwise sense. Note that p_1, p_7 are in the 1st level and p_2, p_3, p_4, p_5, p_6 are in the 2nd level. Note that one of the p_i's must be p_0 but we will make it explicit during the proof.

Assume that $p_0 \in B_2$. We show that two singular points p_{ℓ_1}, p_{ℓ_2} in B_2 none of them being p_0 either are consecutive, or are consecutive and such that the arc of the hyperbola having them as endpoints contains only a singular point, which is p_0, must have different index. Otherwise, applying formula (4) to $R(x,y) = L_{p_8,p_9} L_{p_0,p_{\ell_3}} L_{p_0,p_{\ell_4}} L_{p_{\ell_5},p_{\ell_6}}$ being p_{ℓ_i} for $i = 3,\ldots,6$ singular points in B_2 different from p_{ℓ_1}, p_{ℓ_2} we reach a contradiction.

Assume that $p_0 \in B_1$. Without loss of generality $p_0 = p_8$. Then two consecutive singular points p_{ℓ_1}, p_{ℓ_2} of the branch B_2 have different index. Indeed, applying formula (4) to $R(x,y) = L_{p_0,p_9} L_{p_0,p_{\ell_3}} L_{p_{\ell_4},p_{\ell_5}} L_{p_{\ell_6},p_{\ell_7}}$ being p_{ℓ_i} for $i = 3,\ldots,7$ singular points in B_2 different from p_{ℓ_1} and p_{ℓ_2} it follows that p_{ℓ_1} and p_{ℓ_2} have different index.

This characterized completely the indices of the singular points of the branch B_2.

Assume first that p_0 is in the 1st level. Then by the arguments above the unique possible configurations in the 2nd-level must be $(+,-,+,-,+)$ or $(-,+,-,+,-)$. If p_0 is in the branch B_1 then, again by the arguments above, the unique possible configuration is $(0, 2-, +; +, -, +, -, +)$ in the first case, and $(0, 2+, -; -, +, -, +, -)$ in the second case. If p_0 is in the branch B_2 then without loss of generality we can assume that it is p_7 and applying formula (4) to $R(x, y) = L_{p_0,p_8} L_{p_0,p_2} L_{p_3,p_4} L_{p_5,p_6}$ we get that p_1 and p_9 must have the same index and so the unique possible configurations are again $(0, 2-, +; +, -, +, -, +)$ and $(0, 2+, -; -, +, -, +, -)$.

If p_0 is in the 2nd-level it must be in B_2. Then the unique possible configurations in the 2nd level is $(0, +, -, +, -)$. Applying formula (4) to $R(x, y) = L_{p_7,p_8} L_{p_0,p_{\ell_1}} L_{p_0,p_{\ell_2}} L_{p_{\ell_3},p_{\ell_4}}$ with p_{ℓ_i} for $i = 1, \ldots, 4$ the singular points in B_2 different from p_0 and p_1, we get that p_1 and p_9 have the same index. In short the unique possible configuration is $(2+, 2-; 0, +, -, +, -)$.

The differential system (2) with

$$P(x, y) = x(1 + y),$$
$$Q(x, y) = -\frac{32}{117} y^2 + \frac{5}{18} x^3 + \frac{25}{18} x^2 y + \frac{13}{18} x y^2 - \frac{118}{117} y^3 - y^4 - x^5 - \frac{3}{2} x^4 y - \frac{2}{13} y^5,$$

has the singular points

$$(1, -1),\ (0, 0),\ (-1, -1),\ (0, -16/3) \text{ in the 1st level, and}$$
$$(0, -1/2),\ (0, -2/3),\ (1/3, -1),\ (1/2, -1),\ (2/3, -1) \text{ in the2nd level,}$$

in the configuration $(0, 2+, -; -, +, -, +, -)$ (the configuration $(0, 2-, +; +, -, +, -, +)$ can be obtained reversing the sing in P).

The differential system (2) with

$$P(x, y) = y(y - x), \quad Q(x, y) = x(x^2 - 1)(x^2 - 4),$$

has the singular points

$$(-2, -2),\ (-2, 0),\ (2, 2),\ (2, 0) \text{ in the 1st level, and}$$
$$(1, 0),\ (1, 1),\ (-1, -1),\ (-1, 0),\ (0, 0) \text{ in the 2nd level,}$$

in the configuration $(2+, 2-; 0, +, -, +, -)$.

Configuration $(4; 4; 1)$. We note that the configuration $(4; 4; 1)$ can only be realized in a hyperbola or the conic formed by two straight lines intersecting to a real point. We do not consider the case that the conic is formed by two straight lines intersecting in a real point because the proof in this case is completely similar to the proof when the conic is a hyperbola. In this last case one branch of the hyperbola, B_1 has the four singular points p_6, p_7, p_8, p_9 ordered in counterclockwise sense being p_6, p_9 in the 1st-level and p_7, p_8 in the 2nd level, and in the other branch B_2 of the hyperbola there are the five singular points p_1, p_2, p_3, p_4, p_5 ordered in counterclockwise sense. Note that p_1, p_5 are in the 1st level, p_2, p_4 are in the 2nd-level and p_3 is in the 3rd-level. Note that one of the p_i's must be p_0 but we will make it explicit during the proof.

Assume $p_0 \in B_2$. We show that two singular points p_{ℓ_1}, p_{ℓ_2} in B_2 none of them being p_0 that are either consecutive, or are consecutive and such that the arc of the hyperbola having them as endpoints contains only a singular point, which is p_0, must have different index. Otherwise, applying formula (4) to $R(x, y) = L_{p_6,p_9} L_{p_7,p_8} L_{p_0,p_{\ell_3}} L_{p_0,p_{\ell_4}}$ being p_{ℓ_3}, p_{ℓ_4} singular points in B_2 different from p_{ℓ_1}, p_{ℓ_2}, we reach a contradiction. Moreover, we show that two consecutive singular points p_{k_1}, p_{k_2} in B_1 have different index. Indeed, applying formula (4) to $R(x, y) = L_{p_0,p_{k_3}} L_{p_0,p_{k_4}} L_{p_{\ell_1},p_{\ell_2}} L_{p_{\ell_3},p_{\ell_4}}$ being p_{k_3}, p_{k_4} singular points in B_1 different from p_{k_1}, p_{k_2} and p_{ℓ_i} for $i = 1, \ldots, 4$ singular points in B_2 different from p_0, we get that p_{k_1} and p_{k_2} have different index.

Assume $p_0 \in B_1$. We show that two consecutive singular points $p_{\ell_1}, p_{\ell_2} \in B_2$ have different index. Indeed, applying the Euler-Jacobi formula (4) to $R(x, y) = L_{p_0,p_{\ell_3}} L_{p_0,p_{k_1}} L_{p_{k_2},p_{k_3}} L_{p_{\ell_4},p_{\ell_5}}$ being p_{k_i} for $i = 1, \ldots, 4$ the singular points in B_1 different from p_0, and p_{ℓ_i} for $i = 3, 4, 5$ singular points in B_2 different from p_{ℓ_1}, p_{ℓ_2} we reach a contradiction. Finally, we show that two singular points p_{k_1}, p_{k_2} in B_1 none of them being p_0 that are either consecutive, or are consecutive and such that the arc of the hyperbola having them as endpoints contains only a singular point, which is p_0, must have different index. Otherwise, applying formula (4) to $R(x, y) = L_{p_0,p_{k_3}} L_{p_0,p_{\ell_1}} L_{p_{\ell_2},p_{\ell_3}} L_{p_{\ell_4},p_{\ell_5}}$ being p_{k_3} a singular point in B_1 different from p_{k_1}, p_{k_2}, p_0 and p_{ℓ_i} for $i = 1, \ldots, 5$ the five singular points in B_2, we reach a contradiction.

If p_0 is in the 3rd-level then $p_0 = p_3$ and we show that p_5 and p_6 have the same index. Indeed, applying formula (4) to $R(x, y) = L_{p_7,p_8} L_{p_1,p_9} L_{p_0,p_4} L_{p_0,p_2}$ we get that p_5 and p_6 have the same index. So the unique possible configurations are $(2+, 2-; 2+, 2-; 0)$.

If p_0 is in the 2nd-level then without loss of generality we can assume that it is p_4 (otherwise we can make the same argument if it is p_2). Applying formula (4) to $R(x, y) = L_{p_7,p_8} L_{p_1,p_9} L_{p_0,p_3} L_{p_0,p_2}$ we get that p_5 and p_6 have the same index. So the unique possible configurations are $(2+, 2-; 0, 2-, +; +)$ and of course $(2+, 2-; 0, 2+, -; -)$.

If p_0 is in the 1st-level then without loss of generality we can assume that it is p_5 (otherwise we can make the same argument if it is p_1). Applying formula (4) to $R(x, y) = L_{p_0,p_2} L_{p_0,p_6} L_{p_3,p_4} L_{p_7,p_8}$ we get that p_1 and p_9 have the same index. So the unique possible configurations are $(0, 2+, -; 3-, +; +)$ and of course $(0, 2-, +; 3+, -; -)$.

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(y + 1),\\ Q(x, y) &= x + \frac{1}{8}y + x^2 + xy + \frac{15}{16}y^2 - \frac{17}{16}x^3 + \frac{25}{8}x^2y + \frac{1}{16}xy^2 + \frac{39}{16}y^3 + x^4\\ &\quad + x^3y + x^2y^2 + xy^3 + \frac{5}{2}y^4 + x^5 - x^4y + x^3y^2 + x^2y^3 + xy^4 + \frac{3}{4}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(1, -1),\ (0, 0),\ (-2, -1),\ (0, -2) \text{ in the 1st level,}\\ &(1/4, -1),\ (0, -1/3),\ (-1, -1), (-1/4, -1) \text{ in the 2nd level, and}\\ &(0, -1/2) \text{ in the 3rd level,} \end{aligned}$$

in the configuration $(2+, 2-; 2+, 2-; 0)$.

The differential system (2) with

$$\begin{aligned} P(x, y) &= x(y + 1),\\ Q(x, y) &= x + \frac{1}{2}y + x^2 + xy + \frac{35}{12}y^2 - \frac{11}{8}x^3 + \frac{7}{6}x^2y + \frac{3}{8}xy^2 + \frac{38}{9}y^3 + x^4 + x^3y\\ &\quad + x^2y^2 + \frac{7}{3}y^4 + x^5 + \frac{11}{12}x^4y + x^3y^2 + xy^3 + x^2y^3 + xy^4 + \frac{4}{9}y^5, \end{aligned}$$

has the singular points

$$\begin{aligned} &(0, 0),\ (-1, -1),\ (0, -2),\ (1, -1) \text{ in the 1st level,}\\ &(0, -3/2),\ (-1/2, -1),\ (0, -1/4),\ (2/3, -1) \text{ in the 2nd level, and}\\ &(-1/4, -1) \text{ in the 3rd level,} \end{aligned}$$

in the configuration $(2+, 2-; 0, 2-, +; +)$ (the configuration $(2+, 2-; 0, 2+, -; -)$ can be obtained reversing the sing in P).

The differential system (2) with

$$P(x, y) = -x(y+1),$$
$$Q(x, y) = x - \frac{1}{4}y + x^2 + xy - \frac{3}{2}y^2 - \frac{17}{16}x^3 + \frac{25}{8}x^2y + \frac{1}{16}xy^2 - \frac{45}{16}y^3 + x^4 + x^3y + x^2y^2 + xy^3 - \frac{29}{16}y^4 + x^5 - x^4y + x^3y^2 + x^2y^3 + xy^4 - \frac{3}{8}y^5,$$

has the singular points

$(1, -1),\ (0, 0),\ (-2, -1),\ (0, -2)$ in the 1st level,
$(1/4, -1),\ (0, -1/3),\ (-1, -1),\ (-1/4, -1)$ in the 2nd level, and
$(0, -1/2)$ in the 3rd level,

in the configuration $(0, 2+, -; 3-, +; +)$ (the configuration $(0, 2-, +; 3+, -; -)$ can be obtained reversing the sing in P).

Configuration $(4; 3; 2)$. We note that the configuration $(4; 3; 2)$ can only be realized in a hyperbola or in a conic formed by two straight lines intersecting into a real point. We do not consider the case that the conic is formed by two straight lines intersecting in a real point because the proof in this case is completely similar to the proof when the conic is a hyperbola. In this last case one branch of the hyperbola, B_1 has the three singular points p_7, p_8, p_9 ordered in counterclockwise sense being p_7, p_9 in the 1st-level and p_8 in the 2nd level, and in the other branch B_2 of the hyperbola there are the six singular points $p_1, p_2, p_3, p_4, p_5, p_6$ ordered in counterclockwise sense. Note that p_1, p_6 are in the 1st level, p_2, p_5 are in the 2nd-level and p_3, p_4 are in the 3rd-level. Note that one of the p_i's must be p_0 but we will make it explicit during the proof.

Assume $p_0 \in B_2$. We show that two singular points p_{ℓ_1}, p_{ℓ_2} in B_2 none of them being p_0 that are either consecutive, or are consecutive and such that in the arc of the hyperbola having them as endpoints contains only a singular point, which is p_0, must have different index. Otherwise, applying formula (4) to $R(x, y) = L_{p_0,p_6} L_{p_7,p_9} L_{p_0,p_{\ell_3}} L_{p_{\ell_4},p_{\ell_5}}$ being $p_{\ell_3}, p_{\ell_4}, p_{\ell_5}$ singular points in B_2 different from p_{ℓ_1}, p_{ℓ_2}, we reach a contradiction. Moreover, we show that two consecutive singular points p_{k_1}, p_{k_2} in B_1 have different index. Indeed, applying formula (4) to $R(x, y) = L_{p_0,p_{k_3}} L_{p_0,p_{\ell_1}} L_{p_{\ell_2},p_{\ell_3}} L_{p_{\ell_4},p_{\ell_5}}$ being p_{k_3} a singular point in B_1 different from p_{k_1}, p_{k_2} and p_{ℓ_i} for $i = 1, \ldots, 5$ singular points in B_2 different from p_0, we get that p_{k_1} and p_{k_2} have different index.

Assume $p_0 \in B_1$. We show that two consecutive singular points $p_{\ell_1}, p_{\ell_2} \in B_2$ have different index. Indeed, applying the Euler-Jacobi formula (4) to $R(x, y) = L_{p_0,p_{k_1}} L_{p_0,p_{k_2}} L_{p_{\ell_3},p_{\ell_4}} L_{p_{\ell_5},p_{\ell_6}}$ being p_{k_i} for $i = 1, 2$ the singular points in B_1 different from p_0, and p_{ℓ_i} for $i = 3, \ldots, 6$ singular points in B_2 different from p_{ℓ_1}, p_{ℓ_2} we reach a contradiction. Finally, we show that two singular points p_{k_1}, p_{k_2} in B_1 none of them being p_0 that are either consecutive, or are consecutive and such that in the arc of the hyperbola having them as endpoints contains only a singular point, which is p_0, must have different index. Otherwise, applying formula (4) to $R(x, y) = L_{p_0,p_{\ell_1}} L_{p_0,p_{\ell_2}} L_{p_{\ell_3},p_{\ell_4}} L_{p_{\ell_5},p_{\ell_6}}$ being p_{ℓ_i} for $i = 1, \ldots, 6$ the six singular points in B_2, we reach a contradiction.

If p_0 is in the 3rd-level then the possible configurations are $(2+, 2-; 2-, +; 0, +)$ and $(2+, 2-; 2+, -; 0, -)$.

If p_0 is in the 2nd-level then it can be either in B_1 or B_2. If p_0 is in B_2 the unique possible configuration is $(2+, 2-; 0, +, -; +, -)$. If p_0 is in B_1 then it is p_8. Applying formula (4) to $R(x, y) = L_{p_3,p_0} L_{p_0,p_4} L_{p_5,p_2} L_{p_6,p_9}$ we get that p_1 and p_7 have the same index. Again the unique possible configuration is $(2+, 2-; 0, +, -; +, -)$.

If p_0 is in the 1st-level then it can be either in B_1 or B_2. If p_0 is in B_2 then the unique possible configurations are $(0, 2-, +; 2+, -; +, -)$ and $(0, 2+, -; 2-, +; +, -)$. If p_0 is in B_1 then without loss of generality we can assume that it is p_9 (if it is p_7 the arguments are analogous). Applying formula (4) to $R(x, y) = L_{p_0,p_7} L_{p_0,p_6} L_{p_5,p_1} L_{p_2,p_4}$ we get that p_3 and p_8 have different index. Again the unique possible configurations are $(0, 2-, +; 2+, -; +, -)$ and $(0, 2+, -; 2-, +; +, -)$.

The differential system (2) with

$$P(x, y) = y^2 - (x-1)^2 + 1, \quad Q(x, y) = y(2x + y)(x - 3)(x + 4)(x + 5),$$

has the singular points

$$\begin{aligned}&(3, -\sqrt{3}),\ (3, \sqrt{3}),\ (-5, \sqrt{35}),\ (-5, -\sqrt{35}) \text{ in the 1st level,}\\ &(2, 0),\ (-4, 2\sqrt{6}),\ (-4, -2\sqrt{6}) \text{ in the 2nd level, and}\\ &(-2/3, 4/3),\ (0, 0) \text{ in the 3rd level,}\end{aligned}$$

in the configuration $(2+, 2-; 2-, +; 0, +)$ (the configuration $(2+, 2-; 2+, -; 0, -)$ can be obtained reversing the sing of P).

The differential system (2) with

$$\begin{aligned}P(x, y) &= x(y + 1),\\ Q(x, y) &= x + \frac{3}{8}y + \frac{43}{16}y^2 - \frac{1}{4}x^3 + \frac{5}{8}x^2y - \frac{7}{4}xy^2 + \frac{107}{16}y^3 + x^3y + x^2y^2 + \frac{27}{4}y^4\\ &\quad + x^5 - \frac{1}{2}x^4y + x^2y^3 + xy^4 + \frac{9}{4}y^5,\end{aligned}$$

has the singular points

$$\begin{aligned}&(1, -1),\ (0, 0),\ (-1, -1),\ (0, -3/2) \text{ in the 1st level,}\\ &(-1/2, -1),\ (1/2, -1),\ (0, -1/3) \text{ in the 2nd level, and}\\ &(0, -2/3),\ (0, -1/2) \text{ in the 3rd level,}\end{aligned}$$

in the configuration $(2+, 2-; 0, +, -; +, -)$.

The differential system (2) with

$$\begin{aligned}P(x, y) &= x(y + 1),\\ Q(x, y) &= x + \frac{1}{24}y + x^2 + xy + \frac{19}{48}y^2 - \frac{11}{8}x^3 + \frac{15}{8}x^2y + \frac{3}{8}xy^2 + \frac{61}{48}y^3 + x^4\\ &\quad + x^3y + x^2y^2 + xy^3 + \frac{37}{24}y^4 + x^5 + \frac{1}{4}x^4y + x^3y^2 + xy^4 + \frac{1}{2}y^5,\end{aligned}$$

has the singular points

$$\begin{aligned}&(-1, -1),\ (0, -2)\ (1, -1),\ (0, 0), \text{ in the 1st level,}\\ &(-1/4, -1),\ (1/2, -1),\ (0, -1/4) \text{ in the 2nd level, and}\\ &(0, -1/3),\ (0, -1/2) \text{ in the 3rd level,}\end{aligned}$$

in the configuration $(0, 2-, +; 2+, -; +, -)$ (the configuration $(0, 2+, -; 2-, +; +, -)$ can be obtained reversing the sing in P).

Configuration $(3; 6)$. We note that the configuration $(3; 6)$ only can be realized with a conic formed by a hyperbola. Moreover one branch B_1 of the hyperbola has one point p_9 (which is in the 1st-level) and the other branch B_2 of the hyperbola has eight points. We denote them by $p_1, \ldots, p_8$ ordered in counterclockwise sense. The points p_1, p_8 are in the 1sst level and the remaining points are in the 2nd-level.

If $p_0 \in B_1$, i.e., $p_0 = p_9$, then we show that two consecutive singular points in $p_{\ell_1}, p_{\ell_2} \in B_2$ have different index. Indeed, applying formula (4) to $R(x, y) = L_{p_0,p_{\ell_3}} L_{p_0,p_{\ell_4}} L_{p_{\ell_5},p_{\ell_6}} L_{p_{\ell_7},p_{\ell_8}}$ being p_{ℓ_i} for $i = 1, \ldots, 8$ singular points in B_2 different from p_{ℓ_1} and p_{ℓ_2}, we get that p_{ℓ_1} and p_{ℓ_2} have different index.

If $p_0 \in B_2$ we show that two singular points $p_{\ell_1}, p_{\ell_2} \in B_2$ none of them being p_0 that are either consecutive or are consecutive and such that in the arc of hyperbola having them as endpoints contains only a singular point, which is p_0 have different index. Otherwise applying formula (4) to $R(x, y) = L_{p_0,p_9} L_{p_0,p_{\ell_3}} L_{p_{\ell_4},p_{\ell_5}} L_{p_{\ell_6},p_{\ell_7}}$ being p_{ℓ_i} for $i = 1, \ldots, 7$ singular points in B_2 different from p_{ℓ_1}, p_{ℓ_2} and p_0, we get a contradiction.

If p_0 is in the 1st-level then the 2nd-level must be $(+, -, +, -, +, -)$ and in this case the 1st-level can only by $(0, +, -)$. Hence we have the configurations $(0, +, -; +, -, +, -, +, -)$.

If p_0 is in the 2nd-level the 2nd-level must be $(0, +, -, +, -, +)$ (or $(0, +, -, +, -, +)$) and in this case the 1st-level can only by $(2-, +)$ (or $(2+, -)$. Hence we have the configurations $(2-, +; 0, +, -, +, -, +)$ and $(2+, -; 0, -, +, -, +, -)$.

The differential system (2) with

$$P(x, y) = y^2 - (x - 1)^2 + 1, \quad Q(x, y) = x(x - 5)(x - 3)(x - 4)(x - 6),$$

has the singular points

$$\begin{aligned} &(0, 0),\ (6, -2\sqrt{6}),\ (6, 2\sqrt{6}) \text{ in the 1st level, and} \\ &(5, \sqrt{15}),\ (4, 2\sqrt{2}),\ (3, \sqrt{3}),\ (3, -\sqrt{3}),\ (4, -2\sqrt{2}),\ (5, -\sqrt{15}) \text{ in the 2nd level,} \end{aligned}$$

in the configuration $(0, +, -; +, -, +, -, +, -)$.

The differential system (2) with

$$P(x, y) = y^2 - (x - 1)^2 + 1, \quad Q(x, y) = -y(2x + y)(x + 3)(x + 4)(x + 5),$$

has the singular points

$$\begin{aligned} &(-5, -\sqrt{35}),\ (-5, \sqrt{35}),\ (2, 0) \text{ in the 1st level, and} \\ &(-4, -2\sqrt{6}), (-4, 2\sqrt{6}), (-3, -\sqrt{15}), (-3, \sqrt{15}), (-2/3, 4/3), (0, 0) \text{ in the 2nd level,} \end{aligned}$$

in the configuration $(2-, +; 0, +, -, +, -, +)$ (the configuration $(2+, -; 0, -, +, -, +, -)$ can be obtained reversing the sing in P).

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References

1. Arnold, V., Varchenko, A., Goussein-Zadé, S.: Singularités des applications différentialbes. Mir, Moscow (1982)
2. Berlinskii, A.N.: On the behavior of the integral curves of a differential equation. Izv. Vyssh. Uchebn. Zaved. Mat. **2**, 3–18 (1960)
3. Chicone, C., Jinghuang, T.: On general properties of quadratic systems. Amer. Math. Monthly **81**, 167–178 (1982)

4. Cima, A., Gasull, A., Manõsas, F.: Some applications of the Euler-Jacobi formula to differential equations. Proc. Amer. Math. **118**, 151–163 (1993)
5. Cima, A., Llibre, J.: Configurations of fans and nests of limit cycles for polynomial vector fields in the plane. J. Differential Equations **82**, 71–97 (1989)
6. Gasull, A., Torregrosa, J.: Euler-Jacobi formula for double points and applications to quadratic and cubic systems. Bull. Belg. Math. Soc. **6**, 337–346 (1999)
7. Khovanskii, A.G.: Index of a polynomial vector field. Funktsional Anal. i Prilozhen **13**, 49–58 (1979)
8. Llibre, J., Valls, C.: The Euler-Jacobi formula and the planar quadratic-quadrtic polynomial differential systems. Proc. Amer. Math. Soc. **149**, 135–141 (2020)
9. Llibre, J., Valls, C.: On the configurations of the singular points and their topological indices for the spatial quadratic polynomial differential systems. J. Differential Equations **269**, 10571–10586 (2020)
10. Llibre, J., Valls, C.: Configurations of the topological indices of planar polynomial differential systems of degree $(2, m)$. Results Math. **76**, 21–38 (2021)
11. Llibre, J., Valls, C.: The improved Euler-Jacobi formula and the planar polynomial vector fields of degree 2–4, to appear in Bull. Belg. Math. Soc. (2022)
12. Lloyd, N.G.: Degree Theory, Cambridge Tracks in Mathematics, 73. Cambridge University Pres, New York (1978)