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Reducibility of 1-D Quantum Harmonic Oscillator with New Unbounded Oscillatory Perturbations

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Abstract

Enlightened by Lemma 1.7 in Liang and Luo (J Differ Equ 270:343–389, 2021), we prove a similar lemma which is based upon oscillatory integrals and Langer's turning point theory. From it we show that the Schrödinger equation

$$\begin{aligned} &\mathrm{i}\partial_t u = -\partial_x^2 u + x^2 u + \epsilon \langle x\rangle^{\mu} \sum_{k\in\Lambda} \big(a_k(\omega t)\sin(k|x|^{\beta}) + b_k(\omega t)\cos(k|x|^{\beta})\big)\, u, \\ &u = u(t,x),\ x\in\mathbb{R},\ \beta>1, \end{aligned}$$

can be reduced in $\mathcal{H}^1(\mathbb{R})$ to an autonomous system for most values of the frequency vector ω, where $\Lambda \subset \mathbb{R}\backslash\{0\}$, $|\Lambda| < \infty$ and $\langle x\rangle := \sqrt{1+x^2}$. The functions $a_k(\theta)$ and $b_k(\theta)$ are analytic on $\mathbb{T}^n_\sigma$ and $\mu \geq 0$ will be chosen according to the value of β. Comparing with Liang and Luo (J Differ Equ 270:343–389, 2021), the novelty is that the phase functions of oscillatory integral are more degenerate when $\beta > 1$.

Keywords Reducibility · 1D quantum harmonic oscillator · Langer's turning point theory

1 Introduction of the Main Results

1.1 Main Theorem

Following [25] we continue to consider the reducibility for the time dependent Schrödinger equation

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$$\begin{aligned} &\mathrm{i}\partial_t u = H_\epsilon(\omega t)u, \quad x \in \mathbb{R}, \\ &H_\epsilon := -\partial_{xx} + x^2 + \epsilon X(x, \omega t), \end{aligned} \tag{1.1}$$

where

$$X(x,\theta) = \langle x\rangle^\mu \sum_{k\in\Lambda} \left(a_k(\theta)\sin(k|x|^\beta) + b_k(\theta)\cos(k|x|^\beta)\right)$$

with $\Lambda \subset \mathbb{R}\backslash\{0\}$, $|\Lambda| < \infty$ and $\langle x\rangle := \sqrt{1+x^2}$. The functions $a_k(\theta)$ and $b_k(\theta)$ are analytic on $\mathbb{T}^n_\sigma = \{a + b\mathrm{i} \in \mathbb{C}^n/2\pi\mathbb{Z}^n : |b| < \sigma\}$ with $\sigma > 0$ and $\beta > 1$ and $\mu \geq 0$ will be chosen in the following. We first introduce some functions and spaces.

Hermite Functions The harmonic oscillator $T = -\partial_{xx} + x^2$ has eigenfunctions $(h_m)_{m\geq 1}$, so called the Hermite functions, namely

$$Th_m = (2m-1)h_m, \quad \|h_m\|_{L^2(\mathbb{R})} = 1, \quad m \geq 1. \tag{1.2}$$

Linear Spaces For $s \geq 0$ denote by $\mathcal{H}^s$ the domain of $T^{\frac{s}{2}}$ endowed by the graph norm. For $s < 0$, the space $\mathcal{H}^s$ is the dual of $\mathcal{H}^{-s}$. Particularly, for $s \geq 0$ a integer we have

$$\mathcal{H}^s = \{f \in L^2(\mathbb{R}) : x^\alpha \partial^\beta f \in L^2(\mathbb{R}),\ \forall\, \alpha, \beta \in \mathbb{N}_0,\ \alpha + \beta \leq s\}.$$

We also define the complex weighted-ℓ^2-space $\ell^2_s := \{\xi = (\xi_m \in \mathbb{C}, m \geq 1) : \sum_{m\geq 1} m^s|\xi_m|^2 < \infty\}$. To a function $u \in \mathcal{H}^s$ we associate the sequence ξ of its Hermite coefficients by the formula $u = \sum_{m\geq 1}\xi_m h_m(x)$. In the following we will identify the space $\mathcal{H}^s$ with ℓ^2_s by endowing both space the norm

$$\|u\|_{\mathcal{H}^s} = \|\xi\|_s = \left(\sum_{m\geq 1} m^s|\xi_m|^2\right)^{\frac{1}{2}}.$$

Define

$$l_* = l(\beta,\mu) = \begin{cases} \frac{1}{4}\left(\frac{\beta}{6} - \mu\right), & 1 < \beta < 2, \\ \frac{1}{4}\left(\frac{2}{9} - \mu\right), & \beta = 2, \\ \frac{1}{4}\left(\frac{\beta-2}{4\beta-2} - \mu\right), & \beta > 2. \end{cases} \tag{1.3}$$

Then we can state our main theorem.

Theorem 1.1 *Assume $a_k(\theta)$ and $b_k(\theta)$ are analytic on $\mathbb{T}^n_\sigma$ with $\sigma > 0$ and $\beta > 1$ and μ satisfies*

$$0 \leq \mu < \begin{cases} \frac{\beta}{6}, & 1 < \beta < 2, \\ \frac{2}{9}, & \beta = 2, \\ \frac{\beta-2}{4\beta-2}, & \beta > 2. \end{cases} \tag{1.4}$$

There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there is a closed set $D_\epsilon \subset D_0 = [0, 2\pi]^n$ of asymptotically full measure such that for all $\omega \in D_\epsilon$, the linear Schrödinger equation (1.1) reduces to a linear autonomous equation in $\mathcal{H}^1$.
More precisely, for any $\omega \in D_\epsilon$ there exists a linear isomorphism $\Psi^\infty_{\omega,\epsilon}(\theta) \in \mathfrak{L}(\mathcal{H}^{s'})$ with $0 \leq s' \leq 1$, analytically dependent on $\theta \in \mathbb{T}^n_{\sigma/2}$ and unitary on $L^2(\mathbb{R})$, where $\Psi^\infty_{\omega,\epsilon} - \mathrm{Id} \in \mathfrak{L}(\mathcal{H}^0, \mathcal{H}^{2l_*}) \cap \mathfrak{L}(\mathcal{H}^{s'})$ and a bounded Hermitian operator $Q \in \mathfrak{L}(\mathcal{H}^1)$ such that

$t \mapsto u(t,\cdot) \in \mathcal{H}^1$ satisfies (1.1) if and only if $t \mapsto v(t,\cdot) = \Psi^{\infty}_{\omega,\epsilon} u(t,\cdot) \in \mathcal{H}^1$ satisfies the autonomous equation

$$\mathrm{i}\partial_t v = -v_{xx} + x^2 v + \epsilon Q(v),$$

furthermore, there are constants $C, K > 0$ such that

$$\begin{aligned}
&\mathrm{Meas}(D_0 \backslash D_\epsilon) \le C\epsilon^{\frac{3l_*}{2(2l_*+5)(2l_*+1)}},\\
&\|Q\|_{\mathfrak{L}(\mathcal{H}^p,\mathcal{H}^{p+4l_*})} + \|\partial_\omega Q\|_{\mathfrak{L}(\mathcal{H}^p,\mathcal{H}^{p+4l_*})} \le K, \quad \omega \in D_\epsilon,\ p \in \mathbb{N},\\
&\|\Psi^{\infty}_{\omega,\epsilon}(\theta) - \mathrm{Id}\|_{\mathfrak{L}(\mathcal{H}^0,\mathcal{H}^{2l_*})}, \|\Psi^{\infty}_{\omega,\epsilon}(\theta) - \mathrm{Id}\|_{\mathfrak{L}(\mathcal{H}^{s'})} \le C\epsilon^{\frac{1}{3}}, \quad (\omega,\theta) \in D_\epsilon \times \mathbb{T}^n_{\sigma/2}.
\end{aligned}$$

Consequently, Theorem 1.1 follows in the considered range of parameters the $\mathcal{H}^1$ norms of the solutions are all bounded forever and the spectrum of the corresponding operator is pure point.

1.2 Related Results and a Critical Lemma

In the following we recall some relevant reducibility results. For 1-D quantum harmonic oscillators('QHO' for short) with periodic or quasi-periodic in time bounded perturbations see [11, 15, 23, 28, 39, 40] .

In [5] Bambusi and Graffi proved the reducibility of 1-D Schrödinger equation with an unbounded time quasiperiodic perturbation in which the potential grows at infinity like $|x|^{2l}$ with a real $l > 1$ and the perturbation is bounded by $1 + |x|^\beta$ with $\beta < l - 1$. The reducibility in the limiting case $\beta = l - 1$ was proved by Liu and Yuan in [30]. Recently, the results in [5, 30] have been improved by Bambusi in [1, 2], in which he firstly obtained the reducibility results for 1-D QHO with unbounded perturbations.

It seems that the reducibility method in [1, 2] is hard to be applied for 1-D Schrödinger equations with the unbounded oscillatory perturbations(see remark 2.7 in [1]). The authors [25, 27] solved this problem by Langer's turning point and oscillatory integral estimates. We remark that the critical step in [25] is to build up a decay estimate of the integral $\int_{\mathbb{R}} \langle x\rangle^\mu e^{\mathrm{i}kx} h_m(x)\overline{h_n(x)}dx$, in which the phase functions of oscillatory integral are $\phi_{mn}(x) := \zeta_m(x) - \zeta_n(x) + kx$, where $\zeta_m(x) = \int_{X_m}^x \sqrt{\lambda_m - t^2}dt$ with $X_m = \sqrt{2m-1} = \sqrt{\lambda_m}$.

Comparing with [25], in this paper the phase functions $\Psi_{mn}(x) := \zeta_m(x) - \zeta_n(x) + kx^\beta$ with $\beta > 1$ are more degenerate. For $1 < \beta \le 2$, we use a similar method as [25]. The most difficult part as [25] is the integral $\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx$ where $\nu_2 = 1 - \frac{\beta}{3}$ for $1 < \beta < 2$ and $\nu_2 = \frac{5}{9}$ when $\beta = 2$. We have to discuss different cases in order to obtain a suitable lower bound of the derivatives of the phase function. For $\beta > 2$ we find a new simple proof which follows from Corollary 3.2 in [24], Lemma 6.1 and a straightforward computation. As Lemma 1.7 in [25] we have the following.

Lemma 1.2 *Assume $h_m(x)$ satisfies (1.2). For any $k \neq 0$,*

$$\left|\int_{\mathbb{R}} \langle x\rangle^\mu e^{\mathrm{i}k|x|^\beta} h_m(x)\overline{h_n(x)}dx\right| \le C \cdot C_{k,\beta}(mn)^{-l(\beta,\mu)}, \quad m, n \ge 1$$

for some absolute constant $C > 0$*, where* $\mu \geq 0, \beta > 1$*,* $l(\beta, \mu)$ *defined in* (1.3) *and*

$$C_{k,\beta} = \begin{cases} |\beta(\beta-1)(\beta-2)k|^{-\frac{1}{3}} \vee |k|^{-1} \vee |k|^{\frac{1}{4-2\beta}}, & 1 < \beta < 2, \\ |k|^{-1} \vee 1, & \beta = 2, \\ |\beta k|^{-1} \vee 1, & \beta > 2. \end{cases}$$

Remark 1.3 In fact $l(1, \mu) = \frac{1}{12} - \frac{\mu}{4}$ has been proved in [25].

In the end we review some relative results. Eliasson–Kuksin [13] initiated to prove the reducibility for PDEs in high dimension. See [22, 26] for higher-dimensional QHO with bounded potential. The first reducibility result for n-D QHO was proved in [7] by Bambusi-Grébert-Maspero-Robert. Towards other PDEs with unbounded perturbations see the reducibility results by Montalto [35] for linear wave equations on $\mathbb{T}^d$ and Bambusi, Langella and Montalto [3] for transportation equations [18]. Feola and Grébert [19] set up a reducibility result for a linear Schrödinger equation on the sphere S^n with unbounded potential [20].

The reducibility results usually imply the boundedness of Sobolev norms. Delort [12] constructed a $t^{s/2}$- polynomial growth for 1-D QHO with certain time periodic perturbation [32]. Basing on a Mourre estimate, Maspero [33] proved similar results for 1-D QHO and half - wave equation on $\mathbb{T}$ and the instability is stable in some sense. For a polynomial periodic or quasi-periodic perturbations relative with 1-D QHO we refer to [7, 21, 29, 31]. For 2-D QHO with perturbation which is decaying in t, Faou-Raphaël [17] constructed a solution whose $\mathcal{H}^1-$norm presents logarithmic growth with t. For 2-D Schrödinger operator Thomann [38] constructed explicitly a traveling wave whose Sobolev norm presents polynomial growth with t, based on the study in [36] for linear Lowest Landau equations (LLL) with a time-dependent potential. There are also many literatures, e.g. [4, 6, 8–10, 16, 34, 41], which are closely relative to the upper growth bound of the solution in Sobolev space.

Our article is organized as follows: in Sect. 2 we state the reducibility theorem, i.e. Theorem 2.1. In Sect. 3, through checking all the assumptions in Theorem 2.1 we prove Theorem 1.1. In Sect. 4 we prove Lemma 1.2 for $1 < \beta \leq 2$ and the case for $\beta > 2$ is delayed in Sect. 5. Some auxiliary lemmas are presented in the "Appendix".

Notation We use the notations $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and $\mathbb{T}^n_\sigma = \{a + b\mathrm{i} \in \mathbb{C}^n/2\pi\mathbb{Z}^n : |b| < \sigma\}$. For Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ we denote by $\mathfrak{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and write $\mathfrak{L}(\mathcal{H}_1, \mathcal{H}_1)$ as $\mathfrak{L}(\mathcal{H}_1)$ for simplicity.

2 A KAM Theorem

Following [14, 23] we introduce the KAM Theorem from [26] especially for 1-D case.

2.1 Setting

Linear spaces. For $p \geq 0$ we define $X_p := \ell^2_p \times \ell^2_p = \{\zeta = (\zeta_a = (\xi_a, \eta_a) \in \mathbb{C}^2)_{a\in\mathbb{N}}, \|\zeta\|_p < \infty\}$ with $\|\zeta\|_p^2 = \sum_{a\in\mathbb{N}} a^p(|\xi_a|^2 + |\eta_a|^2)$. We equip the space with the symplectic structure i $\sum_{a\in\mathbb{N}} d\xi_a \wedge \eta_a$.

Infinite matrices. Denote by $\mathcal{M}_\alpha$ the set of infinite matrices $A : \mathbb{N}\times\mathbb{N} \mapsto \mathbb{C}$ with the norm $|A|_\alpha := \sup_{a,b\in\mathbb{N}}(ab)^\alpha|A_a^b| < \infty$. We also denote $\mathcal{M}_\alpha^+$ be the subspace of $\mathcal{M}_\alpha$ satisfying that an infinite matrix $A \in \mathcal{M}_\alpha^+$ if $|A|_{\alpha+} := \sup_{a,b\in\mathbb{N}}(ab)^\alpha(1 + |a - b|)|A_a^b| < \infty$.

In fact one can prove that for all $\alpha > 0$, a matrix in $\mathcal{M}_\alpha^+$ defines a bounded operator on ℓ_0^2. However, when $\alpha \in (0, \frac{1}{2})$, we can't insure that $\mathcal{M}_\alpha \subset \mathfrak{L}(\ell_0^2, \ell_s^2)$ for any $s \in \mathbb{R}$. This means that Px makes no sense when the perturbation operator $P \in \mathcal{M}_\alpha$ and $x \in \ell_0^2$. Fortunately, from Lemma 2.1 in [22] or Lemma 2.2 in [26] one can show $\mathcal{M}_\alpha \subset \mathfrak{L}(\ell_1^2, \ell_{-1}^2)$ and thus the reducibility in $\mathcal{H}^1$ can be built up in Theorem 1.1 instead of L^2.

Parameters. In this paper ω will play the role of a parameter belonging to $D_0 = [0, 2\pi]^n$. All the constructed maps will depend on ω with $\mathcal{C}^1$ regularity. When a map is only defined on a Cantor subset of D_0 the regularity is understood in Whitney sense.

A class of quadratic Hamiltonians. Let $D \subset D_0, \alpha > 0$ and $\sigma > 0$. We denote by $\mathcal{M}_\alpha(D, \sigma)$ the set of mappings as $\mathbb{T}_\sigma^n \times D \ni (\theta, \omega) \mapsto Q(\theta, \omega) \in \mathcal{M}_\alpha$ which is real analytic on $\theta \in \mathbb{T}_\sigma^n$ and $\mathcal{C}^1$ continuous on $\omega \in D$. And we endow this space with the norm $[Q]_\alpha^{D,\sigma} := \sup_{\omega\in D, |\Im\theta|<\sigma |k|=0,1} |\partial_\omega^k Q(\theta, \omega)|_\alpha$.

The subspace of $\mathcal{M}_\alpha(D, \sigma)$ formed by $F(\theta, \omega)$ such that $\partial_\omega^k F(\theta, \omega) \in \mathcal{M}_\alpha^+$, $|k| = 0, 1$, is denoted by $\mathcal{M}_\alpha^+(D, \sigma)$ and endowed with the norm $[F]_{\alpha+}^{D,\sigma} := \sup_{\omega\in D, |\Im\theta|<\sigma |k|=0,1} |\partial_\omega^k F(\theta, \omega)|_{\alpha+}$. Besides, the subspace of $M_\alpha(D, \sigma)$ that are independent of θ will be denoted by $\mathcal{M}_\alpha(D)$ and for $N \in \mathcal{M}_\alpha(D)$,

$$[N]_\alpha^D := \sup_{\omega\in D, |k|=0,1} |\partial_\omega^k N(\omega)|_\alpha.$$

2.2 The Reducibility Theorem

In this section we present an abstract reducibility theorem for a quadratic Hamiltonian quasiperiodic in time of the form

$$H(t, \xi, \eta) = \langle \xi, N\eta\rangle + \epsilon\langle \xi, P(\omega t)\eta\rangle, \quad (\xi, \eta) \in X_1 \subset X_0, \tag{2.1}$$

and the corresponding Hamiltonian system

$$\begin{cases} \dot{\xi} = -\mathrm{i}N\xi - \mathrm{i}\epsilon P^T(\omega t)\xi, \\ \dot{\eta} = \mathrm{i}N\eta + \mathrm{i}\epsilon P(\omega t)\eta, \end{cases} \tag{2.2}$$

where $N = \operatorname{diag}\{\lambda_a, a \in \mathbb{N}\}$ satisfies the following spectrum assumptions:

Hypothesis A1-Asymptotics There exist positive constants c_0, c_1, c_2 such that

$$c_1 a \geq \lambda_a \geq c_2 a \text{ and } |\lambda_a - \lambda_b| \geq c_0|a - b|, \ \forall\, a, b \in \mathbb{N}.$$

Hypothesis A2-Second Melnikov Condition in Measure Estimates There exist positive constants α_1, α_2 and c_3 such that the following holds: for each $0 < \kappa < \frac{1}{4}$ and $K > 0$ there exists a closed subset $D' = D'(\kappa, K) \subset D_0$ with $\mathrm{Meas}(D_0 \backslash D') \leq c_3 K^{\alpha_1}\kappa^{\alpha_2}$ such that for all $\omega \in D'$, $k \in \mathbb{Z}^n$ with $0 < |k| \leq K$ and $a, b \in \mathbb{N}$ we have $|k\cdot\omega + \lambda_a - \lambda_b| \geq \kappa(1 + |a - b|)$.

Then we can state our reducibility results.

Theorem 2.1 *Given a nonautonomous Hamiltonian* (2.1)*, we assume that* $(\lambda_a)_{a\in\mathbb{N}}$ *satisfies Hypothesis* A1–A2 *and* $P(\theta) \in \mathcal{M}_\alpha(D_0, \sigma)$ *with* $\alpha, \sigma > 0$*. Let* $\gamma_1 = \max\{\alpha_1, n + 3\}$ *and* $\gamma_2 = \frac{\alpha\alpha_2}{2\alpha\alpha_2+5}$*, then there exists* $\epsilon_* > 0$ *such that for all* $0 \leq \epsilon < \epsilon_*$ *there are*

(i) *a Cantor set* $D_\epsilon \subset D_0$ *with* $Meas(D_0 \backslash D_\epsilon) \leq C\epsilon^{\frac{3\delta\alpha}{2\alpha+1}}$ *for a* $\delta \in (0, \frac{\gamma_2}{24})$*;*

(ii) *a* $\mathcal{C}^1$ *family in* $\omega \in D_\epsilon$*(in Whitney sense), linear, unitary, analytically dependent on* $\theta \in \mathbb{T}_{\sigma/2}^n$ *and symplectic coordinate transformation* $\Phi_\omega^\infty(\theta) : X_0 \mapsto X_0$, $(\omega, \theta) \in D_\epsilon \times \mathbb{T}_{\sigma/2}^n$, *of the form*

$$(\xi_+, \eta_+) \mapsto (\xi, \eta) = \Phi_\omega^\infty(\theta)(\xi_+, \eta_+) = (\overline{M}_\omega(\theta)\xi_+, M_\omega(\theta)\eta_+),$$

where $\Phi_\omega^\infty(\theta) - \mathrm{Id}$ *satisfies for* $0 \le s' \le 1$

$$\|\Phi_\omega^\infty(\theta) - \mathrm{Id}\|_{\mathfrak{L}(X_0, X_{2\alpha})}, \|\Phi_\omega^\infty(\theta) - \mathrm{Id}\|_{\mathfrak{L}(X_{s'})} \le C\epsilon^{\frac{1}{3}};$$

(iii) *a* $\mathcal{C}^1$ *family of autonomous quadratic Hamiltonian in normal forms*

$$H_\infty(\xi_+, \eta_+) = \langle \xi_+, N_\infty(\omega)\eta_+\rangle = \sum_{m\ge1} \lambda_m^\infty \xi_{+,m}\eta_{+,m}, \quad \omega \in D_\epsilon,$$

where $N_\infty(\omega) = diag\{\lambda_m^\infty, m \in \mathbb{N}\}$ *is diagonal and is close to* N *in the sense of*

$$[N_\infty(\omega) - N]_\alpha^{D_\epsilon} \le C\epsilon,$$

such that

$$H(t, \Phi_\omega^\infty(\omega t)(\xi_+, \eta_+)) = H_\infty(\xi_+, \eta_+),\ t \in \mathbb{R},\ (\xi_+, \eta_+) \in X_1,\ \omega \in D_\epsilon.$$

3 Application to the Quantum Harmonic Oscillator

In this section we will prove Theorem 1.1 by applying Theorem 2.1 to the original Eq. (1.1). Following the strategies in [13], we expand u on the Hermite basis $(h_m)_{m\ge1}$ as well as $\bar{u}$ by the following formula

$$u = \sum_{m\ge1} \xi_m h_m, \quad \bar{u} = \sum_{m\ge1} \eta_m \bar{h}_m.$$

Therefore the Eq. (1.1) is equivalent to a nonautonomous Hamiltonian system

$$\begin{cases} \dot{\xi}_m = -\mathrm{i}\frac{\partial H}{\partial \eta_m} = -\mathrm{i}(2m-1)\xi_m - \mathrm{i}\epsilon\left(P^T(\omega t)\xi\right)_m, \\ \dot{\eta}_m = \mathrm{i}\frac{\partial H}{\partial \xi_m} = \mathrm{i}(2m-1)\eta_m + \mathrm{i}\epsilon\left(P(\omega t)\eta\right)_m, \end{cases} \quad m \ge 1, \tag{3.1}$$

where

$$H(t, \xi, \eta) = \langle \xi, N\eta\rangle + \epsilon\langle \xi, P(\omega t)\eta\rangle, \quad (\xi, \eta) \in X_1 \subset X_0,$$

and $N = \mathrm{diag}\{2m-1, m \ge 1\}$ and

$$\begin{aligned} P_m^n(\omega t) = &\sum_{k\in\Lambda} a_k(\omega t) \int_{\mathbb{R}} \langle x\rangle^\mu \sin k|x|^\beta h_m(x)\overline{h_n(x)}dx \\ &+ \sum_{k\in\Lambda} b_k(\omega t) \int_{\mathbb{R}} \langle x\rangle^\mu \cos k|x|^\beta h_m(x)\overline{h_n(x)}dx, \end{aligned} \tag{3.2}$$

where the frequencies $\omega \in D_0 = [0, 2\pi]^n$ are the external parameters.

The spectrum assumptions can be easily checked by the following two lemmas.

Lemma 3.1 *When* $\lambda_a = 2a - 1, a \in \mathbb{N}$, *Hypothesis* A1 *holds true with* $c_0 = c_2 = 1$ *and* $c_1 = 2$.

Lemma 3.2 *When* $\lambda_a = 2a - 1, a \in \mathbb{N}$, *Hypothesis* A2 *holds true with* $\alpha_1 = n + 1, \alpha_2 = 1, c_3 = c(n)$ *and* $D_0 = [0, 2\pi]^n$,

$$D' = \{\omega \in [0, 2\pi]^n : |k \cdot \omega + j| \ge \kappa(1 + |j|),\ \forall\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n \backslash \{0\}\}.$$

The following lemma is a direct corollary of Lemma 1.2.

Lemma 3.3 *Assume that $a_k(\theta)$ and $b_k(\theta)$ are analytic on $\mathbb{T}^n_\sigma$ for any nonzero $k \in \Lambda$ with $\sigma > 0$ and $\beta > 1$ and μ satisfies (1.4), then there exists $\alpha = l(\beta, \mu) > 0$ such that the matrix function $P(\theta)$ defined by (3.2) is analytic from $\mathbb{T}^n_\sigma$ into $\mathcal{M}_\alpha$.*

Proof of Theorem 1.1. Expanding the Hermite basis $(h_m)_{m\geq 1}$, the Schrödinger equation (1.1) becomes Hamiltonian system (3.1), which is the form of Eq. (2.2) with $\lambda_a = 2a - 1$. By lemmas above, we can apply Theorem 2.1 to (3.1) with $\gamma_1 = n + 3$, $\gamma_2 = \frac{\alpha}{2\alpha+5}$ and $\delta = \frac{\gamma_2}{48}$. This follows Theorem 1.1.

More precisely, in new variables given in Theorem 2.1, $(\xi, \eta) = (\overline{M}_\omega \xi_+, M_\omega \eta_+)$, system (3.1) is conjugated into an autonomous system of the form:

$$\begin{cases} \dot{\xi}_{+,a} = -\mathrm{i}\lambda_a^\infty(\omega)\xi_{+,a}, \\ \dot{\eta}_{+,a} = \mathrm{i}\lambda_a^\infty(\omega)\eta_{+,a}, \end{cases} \qquad a \in \mathbb{N}.$$

Therefore the solution subject to the initial datum $(\xi_+(0), \eta_+(0))$ reads

$$(\xi_+(t), \eta_+(t)) = (e^{-\mathrm{i}tN_\infty}\xi_+(0), e^{\mathrm{i}tN_\infty}\eta_+(0)), \quad t \in \mathbb{R},$$

where $N_\infty = \mathrm{diag}\{\lambda_a^\infty, \ a \geq 1\}$. Then the solution of (1.1) with the initial datum $u_0(x) = \sum_{a\geq 1} \xi_a(0) h_a(x) \in \mathcal{H}^1$ is formulated by $u(t, x) = \sum_{a\geq 1} \xi_a(t) h_a(x)$ with $\xi(t) = \overline{M}_\omega(\omega t) e^{-\mathrm{i}tN_\infty} M_\omega^T(0)\xi(0)$, where we use the fact $\left(\overline{M}_\omega\right)^{-1} = M_\omega^T$ since M is unitary.

Now we define the coordinate transformation $\Psi_\omega^\infty(\theta)$ by

$$\Psi_\omega^\infty(\theta)\left(\sum_{a\geq 1} \xi_a h_a(x)\right) := \sum_{a\geq 1}\left(M_\omega^T(\theta)\xi\right)_a h_a(x) = \sum_{a\geq 1} \xi_{+,a} h_a(x).$$

Then we have $u(t, x)$ satisfies (1.1) if and only if $v(t, x) = \Psi_\omega^\infty(\omega t)u(t, x)$ satisfies the autonomous equation $\mathrm{i}\partial_t v = -v_{xx} + x^2 v + \epsilon Q(v)$, where

$$\epsilon Q\left(\sum_{a\geq 1} \xi_a h_a(x)\right) = \sum_{a\geq 1}((N_\infty - N_0)\xi)_a h_a(x) = \sum_{a\geq 1}(\lambda_a^\infty - \lambda_a)\xi_a h_a(x).$$

The rest estimates are standard (see Lemma 3.4 in [25] for the details). □

4 Proof of Lemma 1.2 When $1 < \beta \leq 2$

For reader's convenience, we will use the notations in [25]. In the whole section we will always suppose $\mu \geq 0$ and don't point it out in the following lemmas.

The eigenfunction of the quantum oscillator operator T is $h_n(x) = (n!2^n\pi^{\frac{1}{2}})^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} H_n(x)$, where $H_n(x)$ is the nth Hermite polynomial. Since $h_n(x)$ is an even (or odd) function when n is odd (or even), we only need to estimate

$$\int_0^{+\infty} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx, \quad 1 \leq m \leq n. \tag{4.1}$$

By Lemma 4.4 and Remarks 4.5, 4.6 in [25], when $m > m_0$,

$$h_m(x) = (\lambda_m - x^2)^{-\frac{1}{4}} \left(\frac{\pi\zeta_m}{2}\right)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(\zeta_m)$$

$$+(\lambda_m - x^2)^{-\frac{1}{4}} \left(\frac{\pi \zeta_m}{2}\right)^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(\zeta_m) O\left(\frac{1}{\lambda_m}\right)$$
$$:= \psi_1^{(m)}(x) + \psi_2^{(m)}(x),$$

where $\zeta_m(x) = \int_{X_m}^{x} \sqrt{\lambda_m - t^2}dt$ with $X_m^2 = \lambda_m (X_m > 0)$. Otherwise, when $m \le m_0$, then $h_m(x) = \psi_1^{(m)}(x) + \psi_2^{(m)}(x)$ for $x > 2X_{m_0}$, where $\psi_1^{(m)}(x) = (\lambda_m - x^2)^{-\frac{1}{4}}(\frac{\pi\zeta_m}{2})^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}(\zeta_m)$ and $|\psi_2^{(m)}(x)| \le \frac{C}{x^2}|\psi_1^{(m)}(x)|$. Following the same strategies in [25] we distinguish 3 cases to estimate (4.1):

I. $m, n < C_0 := 2^8 m_0^3$;
II. $m \le m_0$ and $n \ge C_0$;
III. $m, n > m_0$.

4.1 The Estimates for Case I and Case II

Lemma 4.1 *When $n, m < C_0$,*

$$\left|\int_0^{+\infty} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{(mn)^{\frac{1}{4}(\frac{\beta}{6}-\mu)}}.$$

Proof When $x \le X_0$, from Hölder inequality and $n, m < C_0$, we have

$$\left|\int_0^{X_0} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \le X_0^\mu \le \frac{C}{(mn)^{\frac{1}{4}(\frac{\beta}{6}-\mu)}}.$$

where X_0 is a positive constant depending on C_0 only. When $x > X_0$, $|X_m^2 - x^2|^{-\frac{1}{4}} < 1$, we have $\left|\sqrt{\frac{\pi\zeta_m}{2}} H^{(1)}_{\frac{1}{3}}(\zeta_m)\right| \le e^{-|\zeta_m|}$ by Lemma 5.4 in [25]. By Lemma 5.5 in [25] we have $|\zeta_m| \ge \frac{2\sqrt{2}}{3} X_m^{\frac{1}{2}}(x - X_m)^{\frac{3}{2}} \ge x - X_0$ for $x > X_0$. Thus

$$\left|\int_{X_0}^{+\infty} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \le \int_{X_0}^{+\infty} \langle x\rangle^\mu e^{-2(x-X_0)}dx \le Ce^{2X_0} \le \frac{C}{(mn)^{\frac{1}{4}(\frac{\beta}{6}-\mu)}}.$$

□

Lemma 4.2 *For $m \le m_0$ and $n \ge C_0$ and $\mu \ge 0$,*

$$\left|\int_0^{+\infty} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{(mn)^{\frac{1}{4}(\frac{\beta}{6}-\mu)}}.$$

Proof We divide the integral into two parts.

$$\int_0^{+\infty} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx = \int_0^{X_n^{\frac{1}{3}}} + \int_{X_n^{\frac{1}{3}}}^{+\infty}.$$

Since $x > 2X_{m_0}$, we have

$$|h_m(x)| \le 2(x^2 - X_m^2)^{-\frac{1}{4}} \left|\sqrt{\frac{\pi\zeta_m}{2}} H^{(1)}_{\frac{1}{3}}(\zeta_m)\right| \le 2e^{-|\zeta_m|}.$$

On the other hand, for $x \in [0, X_n^{\frac{1}{3}}]$, one has $|h_n(x)| \le C(X_n^2 - x^2)^{-\frac{1}{4}}$. Note $1 < \beta < 2$, it follows

$$\left|\int_0^{X_n^{\frac{1}{3}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le C\int_0^{X_n^{\frac{1}{3}}} \langle x\rangle^{\mu}(X_n^2 - x^2)^{-\frac{1}{4}}dx \le CX_n^{-\frac{1}{6}+\frac{\mu}{3}} \le \frac{C}{(mn)^{\frac{1}{4}(\frac{\beta}{6}-\mu)}}.$$

When $x \ge X_n^{\frac{1}{3}} \ge 2X_{m_0}$, from Lemma 5.5 in [25], $e^{-|\zeta_m|} \le e^{-C(x-X_m)}$. Note $\|h_n(x)\|_{L^2} = 1$, from Hölder inequality,

$$\left|\int_{X_n^{\frac{1}{3}}}^{+\infty} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le C\left(\int_{X_n^{\frac{1}{3}}}^{+\infty} \langle x\rangle^{2\mu} e^{-Cx}dx\right)^{\frac{1}{2}} \le e^{-CX_n^{\frac{1}{3}}}.$$

□

4.2 The Estimate for Case III

In the following we will turn to the complicated case when $m, n > m_0$. We divide the integral into two parts $\int_0^{+\infty} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx = \int_0^{X_n} + \int_{X_n}^{+\infty}$. We first go to the latter case $\int_{X_n}^{+\infty}$.

4.2.1 The Integral on $[X_n, +\infty)$

Lemma 4.3 *For $m_0 < m \le n$,*

$$\left|\int_{X_n}^{+\infty} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{m^{\frac{1}{12}-\frac{\mu}{4}} n^{\frac{1}{12}-\frac{\mu}{4}}} \le \frac{C}{(mn)^{\frac{1}{4}(\frac{\beta}{6}-\mu)}}.$$

We first estimate the integral on $[2X_n, +\infty]$. The following result is clear from [25].

Lemma 4.4 *For $m_0 < m \le n$,*

$$\left|\int_{2X_n}^{+\infty} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le e^{-Cn}.$$

For the integral on $[X_n, 2X_n]$, we prove that

Lemma 4.5 *For $m_0 < m \le n$,*

$$\left|\int_{X_n}^{2X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{m^{\frac{1}{12}-\frac{\mu}{4}} n^{\frac{1}{12}-\frac{\mu}{4}}}.$$

Proof As [25], we only need to estimate the following integral $I := \int_{X_n}^{2X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx$ since the rest three ones are higher order. I can be divided into two parts as

$$|I| = \left|\left(\int_{X_n+X_n^{\frac{1}{3}}}^{2X_n} + \int_{X_n}^{X_n+X_n^{\frac{1}{3}}}\right)\langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right|$$

$$\leq C X_n^{\mu}\Big(\int_{X_n+X_n^{\frac{1}{3}}}^{2X_n}+\int_{X_n}^{X_n+X_n^{\frac{1}{3}}}\Big)\left|\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}\right|dx.$$

From Lemma 5.5 in [25], when $x \geq X_n + X_n^{\frac{1}{3}}$, $|\zeta_n| \geq \frac{2\sqrt{2}}{3}X_n^{\frac{1}{2}}(x-X_n)^{\frac{3}{2}} \geq \frac{2\sqrt{2}}{3}X_n$. Thus

$$\begin{aligned}\int_{X_n+X_n^{\frac{1}{3}}}^{2X_n}\left|\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}\right|dx &\leq C\int_{X_n+X_n^{\frac{1}{3}}}^{2X_n}(x^2-\lambda_m)^{-\frac{1}{4}}(x^2-\lambda_n)^{-\frac{1}{4}}e^{-|\zeta_n|}dx\\ &\leq Ce^{-\frac{2\sqrt{2}}{3}X_n}\int_{X_n+X_n^{\frac{1}{3}}}^{2X_n}(x^2-\lambda_n)^{-\frac{1}{2}}dx \leq Ce^{-\frac{2\sqrt{2}}{3}X_n}.\end{aligned}$$

For the second part,

$$\begin{aligned}&\int_{X_n}^{X_n+X_n^{\frac{1}{3}}}\left|\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}\right|dx\\ &\leq C\int_{X_n}^{X_n+X_n^{\frac{1}{3}}}(x^2-\lambda_m)^{-\frac{1}{4}}(x^2-\lambda_n)^{-\frac{1}{4}}dx \leq C\int_{X_n}^{X_n+X_n^{\frac{1}{3}}}(x^2-\lambda_n)^{-\frac{1}{2}}dx\\ &\quad \leq CX_n^{-\frac{1}{2}}\int_{X_n}^{X_n+X_n^{\frac{1}{3}}}(x-X_n)^{-\frac{1}{2}}dx \leq CX_n^{-\frac{1}{3}}.\end{aligned}$$

It follows $|I| \leq CX_n^{\mu-\frac{1}{3}} \leq \frac{C}{m^{\frac{1}{12}-\frac{\mu}{4}}n^{\frac{1}{12}-\frac{\mu}{4}}}$. □

Combining with the above two lemmas we finish Lemma 4.3.

In the following we will estimate the integral on $[0, X_n]$, which is the most complicated case. Note $m_0 < m \leq n$, the following two cases have to be considered respectively: I. $X_n > 2X_m$; II. $X_m \leq X_n \leq 2X_m$.

4.2.2 The Integral Estimate on $[0, X_n]$ When $X_n > 2X_m$

Our aim in this part is to build the following

Lemma 4.6 *For $k \neq 0$, if $X_n > 2X_m$ and $1 < \beta \leq 2$, then*

$$\left|\int_0^{X_n}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}h_m(x)\overline{h_n(x)}dx\right| \leq \frac{C(|k|^{\iota}\vee 1)}{m^{\frac{1}{8}-\frac{\mu}{4}}n^{\frac{1}{12}-\frac{\mu}{4}}},$$

where $m_0 < m \leq n$ and $\iota = \begin{cases}\frac{1}{4-2\beta}, & 1<\beta<2,\\ 0, & \beta = 2.\end{cases}$

As [25] we will use the following notation in the remained parts. We denote $f_m(x) = \int_0^{\infty}e^{-t}t^{-\frac{1}{6}}\left(1+\frac{\mathrm{i}t}{2\zeta_m}\right)^{-\frac{1}{6}}dt$ and $f_n(x) = \int_0^{\infty}e^{-t}t^{-\frac{1}{6}}\left(1+\frac{\mathrm{i}t}{2\zeta_n}\right)^{-\frac{1}{6}}dt$. When $x \in [0, X_m]$, from a straightforward computation we have

$$\begin{aligned}\psi_1^{(m)}(x) &= (X_m^2-x^2)^{-\frac{1}{4}}\sqrt{\frac{\pi\zeta_m}{2}}H_{\frac{1}{3}}^{(1)}(\zeta_m)\\ &= (X_m^2-x^2)^{-\frac{1}{4}}\frac{e^{\mathrm{i}(\zeta_m-\frac{\pi}{6}-\frac{\pi}{4})}}{\Gamma\left(\frac{5}{6}\right)}\int_0^{\infty}e^{-t}t^{-\frac{1}{6}}\left(1+\frac{\mathrm{i}t}{2\zeta_m}\right)^{-\frac{1}{6}}dt\end{aligned}$$

$$= C(X_m^2 - x^2)^{-\frac{1}{4}} e^{\mathrm{i}\zeta_m(x)} f_m(x).$$

Similarly, $\overline{\psi_1^{(n)}(x)} = C(X_n^2 - x^2)^{-\frac{1}{4}} e^{-\mathrm{i}\zeta_n(x)} \overline{f_n(x)}$. For $x \in [0, X_m]$, denote $\Psi(x) = (X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} \cdot f_m(x)\overline{f_n(x)}$ and $g(x) = (\zeta_n(x) - \zeta_m(x) - kx^\beta)' = \sqrt{X_n^2 - x^2} - \sqrt{X_m^2 - x^2} - k\beta x^{\beta-1}$,then

$$\begin{aligned}\Psi'(x) = &\frac{1}{2} x(X_m^2 - x^2)^{-\frac{5}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} \cdot f_m(x)\overline{f_n(x)}\\ &+ \frac{1}{2} x(X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{5}{4}} \cdot f_m(x)\overline{f_n(x)}\\ &+ (X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} \cdot \left(f_m'(x)\overline{f_n(x)} + f_m(x)\overline{f_n'(x)}\right).\end{aligned}$$

When $x \in [0, X_m]$, $|f_m(x)| \le \Gamma(\frac{5}{6})$ and $|f_n(x)| \le \Gamma(\frac{5}{6})$. Thus,

Corollary 4.7 *For $x \in [0, X_m)$ and $m \le n$,*

$$\begin{aligned}\left|\Psi'(x)\right| \le &C\Big(x(X_m^2 - x^2)^{-\frac{5}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} + x(X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{5}{4}}\\ &+ \frac{(X_m^2 - x^2)^{\frac{1}{4}} (X_n^2 - x^2)^{-\frac{1}{4}}}{X_m(X_m - x)^3} + \frac{(X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{\frac{1}{4}}}{X_n(X_n - x)^3}\Big)\\ = &C\big(J_1 + J_2 + J_3 + J_4\big) \le C(J_1 + J_3).\end{aligned}$$

We first estimate the integral on $[0, X_m - X_m^{-\frac{1}{3}}]$.

Lemma 4.8 *For $k \ne 0$, $1 < \beta < 2$, if $X_n > 2X_m$, then*

$$\left| \int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)} dx \right| \le \frac{C(|k|^{\frac{1}{4-2\beta}} \vee 1)}{m^{\frac{1}{8}-\frac{\mu}{4}} n^{\frac{1}{8}-\frac{\mu}{4}}},$$

where $m_0 < m \le n$.

Proof First we estimate the main term of the integral

$$\int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)} dx = C \int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x\rangle^\mu e^{\mathrm{i}(\zeta_m - \zeta_n + kx^\beta)} \Psi(x) dx,$$

by method of oscillating integral estimate, where $\Psi(x) = (X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} \cdot f_m(x)\overline{f_n(x)}$. We discuss two different cases.
Case 1: $k \le \frac{X_n^{2-\beta}}{8}$. In this case, we have

$$g(x) \ge \sqrt{X_n^2 - x^2} - \sqrt{\frac{X_n^2}{4} - x^2} - \beta k X_n^{\beta-1} \ge \frac{X_n}{2} - 2 \cdot \frac{X_n^{2-\beta}}{8} X_n^{\beta-1} \ge \frac{X_n}{4}.$$

Thus, by Lemma 6.1,

$$\left| \int_0^{X_m - X_m^{-\frac{1}{3}}} e^{\mathrm{i}\frac{\zeta_m - \zeta_n + kx^\beta}{X_n} X_n} \langle x\rangle^\mu \Psi(x) dx \right|$$

$$\le CX_n^{-1}\left(\left|\left(\langle x\rangle^{\mu}\Psi\right)(X_m - X_m^{-\frac{1}{3}})\right| + \int_0^{X_m - X_m^{-\frac{1}{3}}} \left|\left(\langle x\rangle^{\mu}\Psi\right)'(x)\right| dx\right)$$

$$\le CX_n^{-1}\left(X_m^{\mu}\left|\Psi(X_m - X_m^{-\frac{1}{3}})\right| + \int_0^{X_m - X_m^{-\frac{1}{3}}} \left(2\langle x\rangle^{\mu}(J_1 + J_3) + \mu\langle x\rangle^{\mu-1}\frac{x}{\langle x\rangle}|\Psi(x)|\right) dx\right)$$

$$\le CX_n^{-1}X_m^{\mu}\left(\left|\Psi(X_m - X_m^{-\frac{1}{3}})\right| + \int_0^{X_m - X_m^{-\frac{1}{3}}} (J_1 + J_3)\, dx\right)$$

$$+ C\mu X_n^{-1}\int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x\rangle^{\mu-1}|\Psi(x)|\, dx.$$

Clearly,

$$X_m^{\mu}\left|\Psi(X_m - X_m^{-\frac{1}{3}})\right| \le CX_m^{\mu}\left(X_m^2 - (X_m - X_m^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}\left(X_n^2 - (X_m - X_m^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}$$
$$\le CX_m^{-\frac{1}{3}+\mu},$$

and

$$\int_0^{X_m - X_m^{-\frac{1}{3}}} \mu\langle x\rangle^{\mu-1}|\Psi(x)|\, dx \le C\left(X_m^2 - (X_m - X_m^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}$$
$$\left(X_n^2 - (X_m - X_m^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}\int_0^{X_m} \mu x^{\mu-1}dx$$
$$\le C\left(X_m^2 - (X_m - X_m^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}\left(X_n^2 - (X_m - X_m^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}X_m^{\mu} \le CX_m^{-\frac{1}{3}+\mu},$$

together with

$$\int_0^{X_m - X_m^{-\frac{1}{3}}} J_1 dx \le C\int_0^{X_m - X_m^{-\frac{1}{3}}} x(X_m^2 - x^2)^{-\frac{5}{4}}(X_m^2 - x^2)^{-\frac{1}{4}}dx \le CX_m^{-\frac{1}{3}},$$

and

$$\int_0^{X_m - X_m^{-\frac{1}{3}}} J_3 dx \le CX_m^{-1}\int_0^{X_m - X_m^{-\frac{1}{3}}} (X_m - x)^{-3}dx \le CX_m^{-\frac{1}{3}}.$$

So we obtain $\left|\int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \le CX_m^{-\frac{1}{3}+\mu}X_n^{-1}$. Now we turn to remained three terms. Since $m_0 < m \le n$,

$$\left|\int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}\psi_2^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \le C\int_0^{X_m - X_m^{-\frac{1}{3}}} X_m^{-2+\mu}(X_m^2 - x^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}}dx$$
$$\le CX_m^{-\frac{3}{2}+\mu}X_n^{-\frac{1}{2}} \le Cn^{-\frac{1}{4}+\frac{\mu}{2}}.$$

Similarly, when $m_0 < m \leq n$, we have

$$\left| \int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x \rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x) \overline{\psi_2^{(n)}(x)} dx \right| \leq C n^{-1+\frac{\mu}{2}}$$

and

$$\left| \int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x \rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_2^{(m)}(x) \overline{\psi_2^{(n)}(x)} dx \right| \leq C n^{-1+\frac{\mu}{2}}.$$

Thus,

$$\left| \int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x \rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{n^{\frac{1}{4}-\frac{\mu}{2}}} \leq \frac{C}{m^{\frac{1}{8}-\frac{\mu}{4}} n^{\frac{1}{8}-\frac{\mu}{4}}}, \quad m_0 < m \leq n.$$

Case 2: $k > \frac{X_n^{2-\beta}}{8} > 0$.
Since $m \leq n$, we have $2n \leq (8k)^{\frac{2}{2-\beta}} + 1$. It follows that

$$\left| \int_0^{X_m - X_m^{-\frac{1}{3}}} \langle x \rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x) \overline{h_n(x)} dx \right| \leq C X_m^{\mu} \leq C X_m^{\mu} \frac{m^{\frac{1}{8}} n^{\frac{1}{8}}}{m^{\frac{1}{8}} n^{\frac{1}{8}}} \leq \frac{C k^{\frac{1}{4-2\beta}}}{m^{\frac{1}{8}-\frac{\mu}{4}} n^{\frac{1}{8}-\frac{\mu}{4}}}.$$

Combining with these two cases we finish the proof. □

Lemma 4.9 *For $k \neq 0$, if $X_n > 2X_m$, then*

$$\left| \int_0^{X_m - X_m^{\frac{2}{3}}} \langle x \rangle^{\mu} e^{\mathrm{i}kx^{2}} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{\frac{1}{8}-\frac{\mu}{4}} n^{\frac{1}{8}-\frac{\mu}{4}}},$$

where $m_0 < m \leq n$.

Proof We first estimate the main part of the integral. By the oscillating integral estimate,

$$\int_0^{X_m - X_m^{\frac{2}{3}}} \langle x \rangle^{\mu} e^{\mathrm{i}kx^{2}} \psi_1^{(m)}(x) \overline{\psi_1^{(n)}(x)} dx = C \int_0^{X_m - X_m^{\frac{2}{3}}} \langle x \rangle^{\mu} e^{\mathrm{i}(\zeta_m - \zeta_n + kx^2)} \Psi(x) dx,$$

where $\Psi(x) = (X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} \cdot f_m(x) \overline{f_n(x)}$. Since $g''(x) \geq g''(0) = \frac{1}{X_m} - \frac{1}{X_n} \geq \frac{1}{2} X_m^{-1}$, by Lemma 6.1,

$$\left| \int_0^{X_m - X_m^{\frac{2}{3}}} e^{\mathrm{i} \frac{\zeta_m - \zeta_n + kx^2}{X_n} X_n} \langle x \rangle^{\mu} \Psi(x) dx \right|$$
$$\leq C X_m^{\frac{1}{3}} \left(\left| \left(\langle x \rangle^{\mu} \Psi \right) (X_m - X_m^{\frac{2}{3}}) \right| + \int_0^{X_m - X_m^{\frac{2}{3}}} \left| \left(\langle x \rangle^{\mu} \Psi \right)' (x) \right| dx \right)$$

$$\leq C X_m^{\frac{1}{3}} \left(X_m^{\mu} \left| \Psi(X_m - X_m^{\frac{2}{3}}) \right| + \int_0^{X_m - X_m^{\frac{2}{3}}} \left(2\langle x\rangle^{\mu} (J_1 + J_3) + \mu \langle x\rangle^{\mu-1} \frac{x}{\langle x\rangle} |\Psi(x)| \right) dx \right)$$

$$\leq C X_m^{\frac{1}{3}} X_m^{\mu} \left(\left| \Psi(X_m - X_m^{\frac{2}{3}}) \right| + \int_0^{X_m - X_m^{\frac{2}{3}}} (J_1 + J_3)\, dx \right)$$

$$+ C\mu X_m^{\frac{1}{3}} \int_0^{X_m - X_m^{\frac{2}{3}}} \langle x\rangle^{\mu-1} |\Psi(x)|\, dx.$$

The estimate comes from three terms. Clearly, for $x \in [0, X_m - X_m^{\frac{2}{3}}]$ we have

$$|\Psi(x)| \leq C(X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{1}{4}}$$
$$\leq C(X_m X_n)^{-\frac{1}{4}} (X_m - x)^{-\frac{1}{4}} (X_n - x)^{-\frac{1}{4}} \leq C X_m^{-\frac{5}{12}} X_n^{-\frac{1}{2}}.$$

It follows that

$$X_m^{\mu} \left| \Psi(X_m - X_m^{\frac{2}{3}}) \right| \leq C X_m^{\mu - \frac{5}{12}} X_n^{-\frac{1}{2}},$$

and

$$\int_0^{X_m - X_m^{\frac{2}{3}}} \mu \langle x\rangle^{\mu-1} |\Psi(x)|\, dx \leq C X_m^{-\frac{5}{12}} X_n^{-\frac{1}{2}} \int_0^{X_m} \langle x\rangle^{\mu-1} dx \leq C X_m^{\mu - \frac{5}{12}} X_n^{-\frac{1}{2}},$$

together with

$$\int_0^{X_m - X_m^{\frac{2}{3}}} J_1 dx \leq C \int_0^{X_m - X_m^{\frac{2}{3}}} x (X_m^2 - x^2)^{-\frac{5}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} dx \leq C X_m^{-\frac{5}{12}} X_n^{-\frac{1}{2}},$$

and

$$\int_0^{X_m - X_m^{\frac{2}{3}}} J_3 dx \leq C X_m^{-\frac{3}{4}} X_n^{-\frac{1}{2}} \int_0^{X_m - X_m^{\frac{2}{3}}} (X_m - x)^{-\frac{11}{4}} dx \leq C X_m^{-\frac{5}{12}} X_n^{-\frac{1}{2}},$$

we obtain $\left| \int_0^{X_m - X_m^{\frac{2}{3}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2} \psi_1^{(m)}(x) \overline{\psi_1^{(n)}(x)} dx \right| \leq C X_m^{-\frac{1}{12} + \mu} X_n^{-\frac{1}{2}} \leq C(X_m X_n)^{\frac{\mu}{2} - \frac{1}{4}}$. The estimate of rest parts of the integral is similar with Lemma 4.8. Thus,

$$\left| \int_0^{X_m - X_m^{\frac{2}{3}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{\frac{1}{8} - \frac{\mu}{4}} n^{\frac{1}{8} - \frac{\mu}{4}}}, \quad m_0 < m \leq n.$$

□

Lemma 4.10 *If $X_n > 2X_m$ and $1 < \beta \leq 2$, then*

$$\left| \int_{X_m - X_m^{\nu_1}}^{X_m} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{\frac{3}{16} - \frac{3}{16}\nu_1 - \frac{\mu}{4}} n^{\frac{3}{16} - \frac{3}{16}\nu_1 - \frac{\mu}{4}}},$$

where $m_0 < m \le n$ *and*

$$\nu_1 = \begin{cases} -1/3, & 1 < \beta < 2, \\ 2/3, & \beta = 2. \end{cases}$$

Proof First,

$$\left|\int_{X_m - X_m^{\nu_1}}^{X_m} \langle x\rangle^{\mu} e^{ikx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \le C\int_{X_m - X_m^{\nu_1}}^{X_m} \langle x\rangle^{\mu}(X_m^2 - x^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}}dx$$
$$\le CX_m^{-\frac{1}{4}+\mu}(X_n^2 - X_m^2)^{-\frac{1}{4}}\int_{X_m - X_m^{\nu_1}}^{X_m}(X_m - x)^{-\frac{1}{4}}dx$$
$$\le CX_m^{-\frac{1}{4}+\mu}(X_n^2 - \frac{X_n^2}{4})^{-\frac{1}{4}}X_m^{\frac{3}{4}\nu_1} \le CX_m^{-\frac{3}{8}+\frac{3}{8}\nu_1+\frac{\mu}{2}}X_n^{-\frac{3}{8}+\frac{3}{8}\nu_1+\frac{\mu}{2}}.$$

Similarly,

$$\left|\int_{X_m - X_m^{\nu_1}}^{X_m} \langle x\rangle^{\mu} e^{ikx^{\beta}} \psi_{j_1}^{(m)}(x)\overline{\psi_{j_2}^{(n)}(x)}dx\right| \le CX_m^{-\frac{3}{8}+\frac{3}{8}\nu_1+\frac{\mu}{2}}X_n^{-\frac{3}{8}+\frac{3}{8}\nu_1+\frac{\mu}{2}}, \quad j_1, j_2 \in \{1, 2\}.$$

Thus we finish the proof. □

Lemma 4.11 *When* $X_n > 2X_m$ *and* $1 < \beta \le 2$*,*

$$\left|\int_{X_m}^{X_n} \langle x\rangle^{\mu} e^{ikx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{m^{\frac{1}{8}-\frac{\mu}{4}}n^{\frac{1}{12}-\frac{\mu}{4}}},$$

where $m_0 < m \le n$.

Proof When $X_n > 2X_{m_0}$, $X_m + X_m^{-\frac{1}{3}} \le \frac{X_n}{2} + 1 \le \frac{3}{4}X_n$. It follows

$$\left|\int_{X_m}^{X_m + X_m^{-\frac{1}{3}}} \langle x\rangle^{\mu} e^{ikx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right|$$
$$\le CX_m^{\mu}\int_{X_m}^{X_m + X_m^{-\frac{1}{3}}}(x^2 - X_m^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}}dx$$
$$\le CX_m^{-\frac{1}{4}+\mu}\left(X_n^2 - (X_m + X_m^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}\int_{X_m}^{X_m + X_m^{-\frac{1}{3}}}(x - X_m)^{-\frac{1}{4}}dx$$
$$\le CX_m^{-\frac{1}{2}+\frac{\mu}{2}}X_n^{-\frac{1}{2}+\frac{\mu}{2}}.$$

From $X_n > 2X_m$, we have $X_n - X_n^{-\frac{1}{3}} \ge \frac{3}{2}X_m$, together with Lemma 5.5 in [25], thus

$$\left|\int_{X_m + X_m^{-\frac{1}{3}}}^{\frac{3}{2}X_m} \langle x\rangle^{\mu} e^{ikx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right|$$
$$\le CX_m^{\mu}\int_{X_m + X_m^{-\frac{1}{3}}}^{\frac{3}{2}X_m}(x^2 - X_m^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}}e^{i\zeta_m}dx$$
$$\le CX_m^{-\frac{1}{4}+\mu}\left(X_n^2 - (X_n - X_n^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}\int_{X_m + X_m^{-\frac{1}{3}}}^{\frac{3}{2}X_m}(x - X_m)^{-\frac{1}{4}}e^{-(x - X_m)}dx$$

$$\leq CX_m^{-\frac{1}{4}+\mu}X_n^{-\frac{1}{6}}\int_0^{\infty}t^{-\frac{1}{4}}e^{-t}dt \leq CX_m^{-\frac{1}{4}+\frac{\mu}{2}}X_n^{-\frac{1}{6}+\frac{\mu}{2}}.$$

When $x \geq \frac{3}{2}X_m$, $x - X_m \geq \frac{1}{3}x$, it follows

$$\begin{aligned}
&\left|\int_{\frac{3}{2}X_m}^{X_n-X_n^{-\frac{1}{3}}}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \\
&\leq C\int_{\frac{3}{2}X_m}^{X_n-X_n^{-\frac{1}{3}}}\langle x\rangle^{\mu}(x^2-X_m^2)^{-\frac{1}{4}}(X_n^2-x^2)^{-\frac{1}{4}}e^{\mathrm{i}\zeta_m}dx \\
&\leq CX_m^{-\frac{1}{4}}\left(X_n^2-(X_n-X_n^{-\frac{1}{3}})^2\right)^{-\frac{1}{4}}\int_{\frac{3}{2}X_m}^{X_n-X_n^{-\frac{1}{3}}}(x-X_m)^{-\frac{1}{4}+\mu}e^{-(x-X_m)}dx \\
&\leq CX_m^{-\frac{1}{4}}X_n^{-\frac{1}{6}}\int_0^{\infty}t^{-\frac{1}{4}+\mu}e^{-t}dt \leq CX_m^{-\frac{1}{4}}X_n^{-\frac{1}{6}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\int_{X_n-X_n^{-\frac{1}{3}}}^{X_n}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \\
&\leq C\int_{X_n-X_n^{-\frac{1}{3}}}^{X_n}x^{\mu}(x^2-X_m^2)^{-\frac{1}{4}}(X_n^2-x^2)^{-\frac{1}{4}}e^{\mathrm{i}\zeta_m}dx \\
&\leq C\left((X_n-X_n^{-\frac{1}{3}})^2-X_m^2\right)^{-\frac{1}{4}}X_n^{-\frac{1}{4}}\int_{X_n-X_n^{-\frac{1}{3}}}^{X_n}(X_n-x)^{-\frac{1}{4}}x^{\mu}e^{-\frac{1}{3}x}dx \\
&\leq CX_m^{-\frac{1}{2}}X_n^{-\frac{1}{4}}\int_{X_n-X_n^{-\frac{1}{3}}}^{X_n}(X_n-x)^{-\frac{1}{4}}dx \\
&\leq CX_m^{-\frac{1}{2}}X_n^{-\frac{1}{2}} \leq C(X_mX_n)^{-\frac{1}{2}+\frac{\mu}{2}}.
\end{aligned}$$

Combining with all the above, we have $\left|\int_{X_m}^{X_n}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \leq \frac{C}{m^{\frac{1}{8}-\frac{\mu}{4}}n^{\frac{1}{12}-\frac{\mu}{4}}}$. The rest estimates are similar as above. □

Combining with Lemmas 4.8, 4.9, 4.10, 4.11, we finish the proof of Lemma 4.6.

4.2.3 The Integral Estimate on $[0, X_n]$ When $X_m \leq X_n \leq 2X_m$

One can split the integral into

$$\int_0^{X_n}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}h_m(x)\overline{h_n(x)}dx = \left(\int_0^{X_m^{\frac{2}{3}}}+\int_{X_m^{\frac{2}{3}}}^{X_m-X_m^{\nu_2}}+\int_{X_m-X_m^{\nu_2}}^{X_n}\right)\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}h_m(x)\overline{h_n(x)}dx,$$

and estimate them respectively, where

$$\nu_2 = \begin{cases}1-\frac{\beta}{3}, & 1<\beta<2,\\ \frac{5}{9}, & \beta=2.\end{cases}$$

Our main aim in this part is to build the following two lemmas.

Lemma 4.12 *For* $X_m \le X_n \le 2X_m$ *and* $k \neq 0$,

$$\left|\int_0^{X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C(|k|^{-1} \vee 1)}{m^{\frac{1}{18}-\frac{\mu}{4}} n^{\frac{1}{18}-\frac{\mu}{4}}},$$

where $C > 0$, $m_0 < m \le n$.

Lemma 4.13 *For* $X_m \le X_n \le 2X_m$, $k \neq 0$ *and* $1 < \beta < 2$,

$$\left|\int_0^{X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C(|k|^{-1} \vee |\beta(\beta-1)(\beta-2)k|^{-\frac{1}{3}} \vee 1)}{m^{\frac{\beta}{24}-\frac{\mu}{4}} n^{\frac{\beta}{24}-\frac{\mu}{4}}},$$

where $C > 0$, $m_0 < m \le n$.

From a straightforward computation we have

Lemma 4.14 *For* $X_m \le X_n \le 2X_m$ *and* $1 < \beta \le 2$,

$$\left|\int_0^{X_m^{\frac{2}{3}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{m^{\frac{1}{12}-\frac{\mu}{4}} n^{\frac{1}{12}-\frac{\mu}{4}}},$$

where $C > 0$, $m_0 < m \le n$.

Next we estimate the integral on $[X_m^{\frac{2}{3}}, X_m - X_m^{\frac{5}{9}}]$, for which we discuss different cases as the following.

Lemma 4.15 *If* $X_m \le X_n \le 2X_m$, *when* $k > 0$ *and* $0 \le X_n^2 - X_m^2 \le kX_m^{\frac{4}{3}}$, *then*

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2} h_m(x)\overline{h_n(x)}dx\right| \le \frac{Ck^{-\frac{1}{2}}}{m^{\frac{7}{36}-\frac{\mu}{4}} n^{\frac{7}{36}-\frac{\mu}{4}}},$$

where $m_0 < m \le n$.

Proof We first estimate

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| = \left|C\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}(\zeta_m-\zeta_n+kx^2)}\Psi(x)dx\right|.$$

Notice that

$$\begin{aligned} g'(x) &\le \frac{kX_m^{\frac{4}{3}}X_m}{\sqrt{2X_m^{\frac{14}{9}} - X_m^{\frac{10}{9}}}\sqrt{2X_m^{\frac{14}{9}} - X_m^{\frac{10}{9}}}\left(\sqrt{2X_m^{\frac{14}{9}} - X_m^{\frac{10}{9}}} + \sqrt{2X_m^{\frac{14}{9}} - X_m^{\frac{10}{9}}}\right)} - 2k \\ &\le \frac{kX_m^{\frac{7}{3}}}{2X_m^{\frac{7}{3}}} - 2k = -\frac{3}{2}k, \end{aligned}$$

and by straightforward computation, $g''(x) \ge 0$. It follows $|g'(x)| \ge \frac{3}{2}k$ on $x \in [X_m^{\frac{2}{3}}, X_m - X_m^{\frac{5}{9}}]$. By Lemma 6.1,

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}\frac{2}{3k}(\zeta_m-\zeta_n+kx^2)\cdot\frac{3k}{2}} (X_m^2 - x^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}} \cdot f_m(x)\overline{f_n(x)}dx\right|$$

$$\leq Ck^{-\frac{1}{2}}\left[\left|(\langle x\rangle^{\mu}\Psi)(X_m - X_m^{\frac{5}{9}})\right| + \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \left|(\langle x\rangle^{\mu}\Psi)'(x)\right| dx\right]$$
$$\leq Ck^{-\frac{1}{2}}\left[X_m^{\mu}\left|\Psi(X_m - X_m^{\frac{5}{9}})\right| + \langle X_m\rangle^{\mu}\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \left|\Psi'(x)\right| dx\right.$$
$$\left. +\mu\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu-1}\left|\Psi(x)\right| dx\right].$$

We estimate the above terms one by one. Clearly,

$$\left|\Psi(X_m - X_m^{\frac{5}{9}})\right| \leq C\left(X_m^2 - (X_m - X_m^{\frac{5}{9}})^2\right)^{-\frac{1}{4}}\left(X_n^2 - (X_m - X_m^{\frac{5}{9}})^2\right)^{-\frac{1}{4}} \leq CX_m^{-\frac{7}{9}},$$

and $\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu-1}\left|\Psi(x)\right| dx \leq CX_m^{-\frac{7}{9}+\mu}$. Besides, we have $|\Psi'(x)| \leq C(J_1 + J_3)$ and

$$\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} J_1 dx \leq C\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} x(X_m^2 - x^2)^{-\frac{5}{4}}(X_n^2 - x^2)^{-\frac{1}{4}} dx \leq CX_m^{-\frac{7}{9}},$$

also, $\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} J_3 dx \leq CX_m^{-\frac{7}{9}}$. Combining with all the estimates, we obtain

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2}\psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \leq Ck^{-\frac{1}{2}}X_m^{-\frac{7}{9}+\mu}.$$

The estimates for the rest three terms are easier. In fact, by $m > m_0$,

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2}\psi_2^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \leq CX_m^{-2+\mu}\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} (X_m^2 - x^2)^{-\frac{1}{2}} dx$$
$$\leq CX_m^{-\frac{5}{2}+\mu}\int_0^{X_m - X_m^{\frac{5}{9}}} (X_m - x)^{-\frac{1}{2}} dx \leq CX_m^{-\frac{7}{9}+\mu}.$$

The estimates for the other two are similar. Thus,

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2} h_m(x)\overline{h_n(x)}dx\right| \leq \frac{Ck^{-\frac{1}{2}}}{m^{\frac{7}{36}-\frac{\mu}{4}} n^{\frac{7}{36}-\frac{\mu}{4}}}.$$

□

Lemma 4.16 *For $k > 0$, $X_m \leq X_n \leq 2X_m$, if $kX_m^{\frac{4}{3}} \leq X_n^2 - X_m^2$, then*

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^2} h_m(x)\overline{h_n(x)}dx\right| \leq \frac{Ck^{-\frac{1}{3}}}{m^{\frac{1}{18}-\frac{\mu}{4}} n^{\frac{1}{18}-\frac{\mu}{4}}},$$

where $m_0 < m \le n$.

Proof For $kX_m^{\frac{4}{3}} \le X_n^2 - X_m^2$, straightforward computation shows that $g'''(x) \ge 0$. So

$$g''(x) \ge g''(0) = \frac{1}{X_m} - \frac{1}{X_n} = \frac{X_n^2 - X_m^2}{X_m X_n (X_m + X_n)} \ge \frac{kX_m^{\frac{4}{3}}}{3X_m X_m X_n} \ge \frac{k}{6} X_m^{-\frac{5}{3}}.$$

Then by Lemma 6.1,

$$\begin{aligned}
&\left| \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^\mu e^{\mathrm{i}\frac{\zeta_m - \zeta_n + kx^2}{kX_m^{-\frac{5}{3}}} kX_m^{-\frac{5}{3}}} (X_n^2 - x^2)^{-\frac{1}{4}} (X_m^2 - x^2)^{-\frac{1}{4}} f_m(x)\overline{f_n(x)} dx \right| \\
&\le Ck^{-\frac{1}{3}} X_m^{\frac{5}{9}} \left[\left| (\langle x\rangle^\mu \Psi)(X_m - X_m^{\frac{5}{9}}) \right| + \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \left| (\langle x\rangle^\mu \Psi)'(x) \right| dx \right] \\
&\le Ck^{-\frac{1}{3}} X_m^{\frac{5}{9}} \Bigg[X_m^\mu \left| \Psi(X_m - X_m^{\frac{5}{9}}) \right| + X_m^\mu \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \left| \Psi'(x) \right| dx \\
&\quad + \mu \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\frac{5}{9}}} \langle x\rangle^{\mu-1} |\Psi(x)|\, dx \Bigg].
\end{aligned}$$

The rest part of the proof is similar with Lemma 4.15. □

Lemma 4.17 *For* $\forall k < 0$, $X_m \le X_n \le 2X_m$, *we have*

$$\left| \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)} dx \right| \le \frac{C(|k|^{-1} \vee 1)}{m^{\frac{1}{6}\beta - \frac{1}{24} + \frac{\nu_2}{8} - \frac{\mu}{4}} n^{\frac{1}{6}\beta - \frac{1}{24} + \frac{\nu_2}{8} - \frac{\mu}{4}}},$$

where $m_0 < m \le n$.

Proof We first estimate

$$\left| \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)} dx \right| = \left| C \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^\mu e^{\mathrm{i}(\zeta_m - \zeta_n + kx^\beta)} \Psi(x) dx \right|.$$

Notice that $g(x) = \sqrt{X_n^2 - x^2} - \sqrt{X_m^2 - x^2} - \beta kx^{\beta-1} \ge -kX_m^{\frac{2}{3}(\beta-1)}$ and $g'(x) > 0$, then by Lemma 6.1,

$$\begin{aligned}
&\left| \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^\mu e^{\mathrm{i}\frac{\zeta_m - \zeta_n + kx^\beta}{|k|X_m^{\frac{2}{3}(\beta-1)}} |k| X_m^{\frac{2}{3}(\beta-1)}} (X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{1}{4}} f_m(x)\overline{f_n(x)} dx \right| \\
&\le \frac{C}{|k|} X_m^{-\frac{2}{3}(\beta-1)} \left[\left| (\langle x\rangle^\mu \Psi)(X_m - X_m^{\nu_2}) \right| + \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} |(\langle x\rangle^\mu \Psi)'(x)| dx \right] \\
&\le C|k|^{-1} X_m^{-\frac{2}{3}(\beta-1)} \Bigg[X_m^\mu \left| \Psi(X_m - X_m^{\nu_2}) \right| + X_m^\mu \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} |\Psi'(x)| dx \\
&\quad + \mu \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^{\mu-1} |\Psi(x)| dx \Bigg].
\end{aligned}$$

The rest part of the proof is similar with Lemma 4.15. □

Finally, we have

Lemma 4.18 *For $k > 0$, $1 < \beta < 2$, $X_m \leq X_n \leq 2X_m$, we have*

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \leq \frac{C|\beta(\beta-1)(\beta-2)k|^{-\frac{1}{3}}}{m^{\frac{\beta}{24}-\frac{\mu}{4}} n^{\frac{\beta}{24}-\frac{\mu}{4}}},$$

where $m_0 < m \leq n$.

Proof First we estimate

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| = \left|C\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^{\mu} e^{\mathrm{i}(\zeta_m - \zeta_n + kx^{\beta})}\Psi(x)dx\right|.$$

Since

$$g''(x) \geq -\beta(\beta-1)(\beta-2)kx^{\beta-3} \geq -\beta(\beta-1)(\beta-2)kX_m^{\beta-3},$$

then by Lemma 6.1,

$$\left|\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^{\mu} e^{\mathrm{i}(\zeta_m - \zeta_n + kx^{\beta})} (X_m^2 - x^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}} \cdot f_m(x)\overline{f_n(x)}dx\right|$$
$$\leq C|\beta(\beta-1)(\beta-2)k|^{-\frac{1}{3}} X_m^{1-\frac{\beta}{3}} \left[\left|(\langle x\rangle^{\mu}\Psi)(X_m - X_m^{\nu_2})\right| + \int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu}} \left|(\langle x\rangle^{\mu}\Psi)'(x)\right| dx\right]$$
$$\leq C|\beta(\beta-1)(\beta-2)k|^{-\frac{1}{3}} X_m^{1-\frac{\beta}{3}} \left[X_m^{\mu}\left|\Psi(X_m - X_m^{\nu_2})\right| + X_m^{\mu}\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu}} \left|\Psi'(x)\right| dx\right.$$
$$\left. + \mu\int_{X_m^{\frac{2}{3}}}^{X_m - X_m^{\nu_2}} \langle x\rangle^{\mu-1} |\Psi(x)|\, dx\right].$$

The rest part of the proof is similar with Lemma 4.15. □

For the last part of the integral, we have

Lemma 4.19 *For $\forall k \neq 0$, $X_m \leq X_n \leq 2X_m$, $1 < \beta \leq 2$,*

$$\left|\int_{X_m - X_m^{\nu_2}}^{X_m} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \leq \frac{C}{m^{\frac{1}{8}-\frac{\nu_2}{8}-\frac{\mu}{4}} n^{\frac{1}{8}-\frac{\nu_2}{8}-\frac{\mu}{4}}}.$$

Here $m_0 < m \leq n$.

Proof First,

$$\left|\int_{X_m - X_m^{\nu_2}}^{X_m} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \leq C\int_{X_m - X_m^{\nu_2}}^{X_m} \langle x\rangle^{\mu}(X_m^2 - x^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}}dx$$
$$\leq CX_m^{\mu}\int_{X_m - X_m^{\nu_2}}^{X_m} (X_m^2 - x^2)^{-\frac{1}{2}}dx$$
$$\leq CX_m^{-\frac{1}{2}+\mu}\int_{X_m - X_m^{\nu_2}}^{X_m} (X_m - x)^{-\frac{1}{2}}dx$$
$$\leq CX_m^{-\frac{1}{2}+\mu} X_m^{\frac{\nu_2}{2}} \leq \frac{C}{m^{\frac{1}{8}-\frac{\mu}{4}-\frac{\nu_2}{8}} n^{\frac{1}{8}-\frac{\mu}{4}-\frac{\nu_2}{8}}}.$$

It follows $\left|\int_{X_m - X_m^{\nu_2}}^{X_m} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \leq \frac{C}{m^{\frac{1}{8}-\frac{\mu}{4}-\frac{\nu_2}{8}} n^{\frac{1}{8}-\frac{\mu}{4}-\frac{\nu_2}{8}}}$. □

Lemma 4.20 *For* $k \neq 0,\ X_m \le X_n \le 2X_m$ *and* $1 < \beta \le 2$, *we have*

$$\Big|\int_{X_m}^{X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\Big| \le \frac{C}{m^{-\frac{\nu_2}{16}+\frac{1}{8}-\frac{\mu}{4}} n^{-\frac{\nu_2}{16}+\frac{1}{8}-\frac{\mu}{4}}}.$$

Proof For the integral on $[X_m, X_n]$, we discuss two different cases:
Case 1.$X_n - X_n^{-\nu_2} \ge X_m + X_m^{-\nu_2}$. We split the integral into three parts. First,

$$\begin{aligned}
&\left|\int_{X_m}^{X_m+X_m^{-\nu_2}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \\
&\le \int_{X_m}^{X_m+X_m^{-\nu_2}} \langle x\rangle^{\mu}(x^2 - X_m^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}}dx \\
&\le C X_m^{\mu} X_m^{-\frac{1}{4}} X_n^{-\frac{1}{4}}(X_n - X_m - X_m^{-\nu_2})^{-\frac{1}{4}} \int_{X_m}^{X_m+X_m^{-\nu_2}} (x - X_m)^{-\frac{1}{4}}dx \\
&\le C X_m^{\mu} X_m^{-\frac{1}{4}} X_n^{-\frac{1}{4}} X_n^{\frac{\nu_2}{4}} X_m^{-\frac{3}{4}\nu_2} \le C n^{-\frac{1}{8}-\frac{\nu_2}{8}+\frac{\mu}{4}} m^{-\frac{1}{8}-\frac{\nu_2}{8}+\frac{\mu}{4}}.
\end{aligned}$$

When $x \ge X_m + X_m^{-\nu_2}$, we have $\mathrm{i}\zeta_m \le -(x - X_m)$. It follows

$$\begin{aligned}
&\left|\int_{X_m+X_m^{-\nu_2}}^{X_n-X_n^{-\nu_2}} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \\
&\le C \int_{X_m+X_m^{-\nu_2}}^{X_n-X_n^{-\nu_2}} \langle x\rangle^{\mu}(x^2 - X_m^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}} e^{\mathrm{i}\zeta_m}dx \\
&\le C(2X_m)^{\mu} X_m^{-\frac{1}{4}} \left(X_n^2 - (X_n - X_n^{-\nu_2})^2\right)^{-\frac{1}{4}} \int_{X_m+X_m^{-\nu_2}}^{X_n-X_n^{-\nu_2}} (x - X_m)^{-\frac{1}{4}} e^{\mathrm{i}\zeta_m}dx \\
&\le C(2X_m)^{\mu} X_m^{-\frac{1}{4}} X_n^{\frac{\nu_2-1}{4}} \int_0^{\infty} t^{-\frac{1}{4}}e^{-t}dt \le C n^{-\frac{1}{8}+\frac{\nu_2}{16}+\frac{\mu}{4}} m^{-\frac{1}{8}+\frac{\nu_2}{16}+\frac{\mu}{4}}.
\end{aligned}$$

Finally, from

$$\begin{aligned}
&\left|\int_{X_n-X_n^{-\nu_2}}^{X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right| \\
&\le \int_{X_n-X_n^{-\nu_2}}^{X_n} \langle x\rangle^{\mu}(x^2 - X_m^2)^{-\frac{1}{4}}(X_n^2 - x^2)^{-\frac{1}{4}}dx \\
&\le C(2X_m)^{\mu} \left((X_n - X_n^{-\nu_2})^2 - X_m^2\right)^{-\frac{1}{4}} X_n^{-\frac{1}{4}} \int_{X_n-X_n^{-\nu_2}}^{X_n} (X_n - x)^{-\frac{1}{4}}dx \\
&\le C n^{-\frac{1}{8}-\frac{\nu_2}{8}+\frac{\mu}{4}} m^{-\frac{1}{8}-\frac{\nu_2}{8}+\frac{\mu}{4}},
\end{aligned}$$

it follows $\left|\int_{X_m}^{X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{(nm)^{\frac{1}{8}-\frac{\mu}{4}-\frac{\nu_2}{16}}}$.
Case 2. $X_n - X_n^{-\nu_2} < X_m + X_m^{-\nu_2}$. In fact, notice that the function $(x - X_m)^{-\frac{1}{4}}(X_n - x)^{-\frac{1}{4}}$ is symmetric on $[X_m, X_n]$, we obtain

$$\left|\int_{X_m}^{X_n} \langle x\rangle^{\mu} e^{\mathrm{i}kx^{\beta}} \psi_1^{(m)}(x)\overline{\psi_1^{(n)}(x)}dx\right|$$

$$
\begin{aligned}
&\le \int_{X_m}^{X_n} \langle x\rangle^{\mu}(x^2-X_m^2)^{-\frac14}(X_n^2-x^2)^{-\frac14}dx\\
&\le C(2X_m^{\mu})X_m^{-\frac14}X_n^{-\frac14}\int_{X_m}^{X_n}(x-X_m)^{-\frac14}(X_n-x)^{-\frac14}dx\\
&\le C(2X_m^{\mu})X_m^{-\frac14}X_n^{-\frac14}\int_{X_m}^{X_n}(x-X_m)^{-\frac12}dx\\
&\le CX_m^{\mu}X_m^{-\frac14}X_n^{-\frac14}(X_n-X_m)^{\frac12}\le (nm)^{\frac14\mu-\frac18-\frac14\nu_2}.
\end{aligned}
$$

Thus, $\left|\int_{X_m}^{X_n}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{m^{\frac18+\frac{\nu_2}{8}-\frac{\mu}{4}}n^{\frac18+\frac{\nu_2}{8}-\frac{\mu}{4}}}$. Combining with the above two cases, we finish the proof. □

Lemmas 4.12 and 4.13 follow directly by the lemmas in Sect. 4.2.3. Combining with all the lemmas in this section we finish the proof of Lemma 1.2 for $1<\beta\le 2$.

5 Proof of Lemma 1.2 When $\beta > 2$

In the following we will suppose that $m \le n$ without losing the generality. As the case $1<\beta\le 2$ we only need to estimate the integral on $[0,\infty]$. We first apply Theorem 3.1 in [24] to obtain the integral estimates on $[2X_m,\infty)$ as follows.

Lemma 5.1 *For $\mu \ge 0$, then we have*

$$
\left|\int_{2X_m}^{\infty}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{(mn)^{\frac14\left(\frac13-\mu\right)}}.
$$

Proof By Theorem 3.1 in [24] we have

$$
\begin{aligned}
&\left|\int_{2X_m}^{\infty}\langle x\rangle^{\mu}e^{\mathrm{ikx^{fi}}}h_m(x)\overline{h_n(x)}dx\right|\\
&\le C\int_{2X_m}^{\infty}\langle x\rangle^{-1}\langle x\rangle^{\mu+1}|h_m(x)||h_n(x)|dx\\
&\le C\|\langle x\rangle^{-1}\|_{L^2(\mathbb{R})}^{\frac12}\cdot\|\langle x\rangle^{\mu+1}h_m(x)\|_{L^2(\{x:|x|\ge 2X_m\})}^{\frac12}\cdot\|h_n\|_{L^\infty(\mathbb{R})}\\
&\le CX_m^{-\frac16}X_n^{-\frac16}\le C(mn)^{\frac14\left(\mu-\frac13\right)}.
\end{aligned}
$$

□

Next we consider the integral on $[0,2X_m]$. Define $\nu_3=\frac{5\beta-4}{2(\beta-1)(2\beta-1)}\in(0,1)$ when $\beta>2$. In the following we will discuss two different cases depending on whether $X_n^{\nu_3}\ge 2X_m$ or not.

The Case $X_n^{\nu_3}\ge 2X_m$ In this case we directly estimate the rest integral on $[0,2X_m]$.

Lemma 5.2 *For $X_n^{\nu_3}\ge 2X_m$ and $\beta>2$, then*

$$
\left|\int_0^{2X_m}\langle x\rangle^{\mu}e^{\mathrm{i}kx^{\beta}}h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{(mn)^{\frac14\left(\frac{\beta-2}{4\beta-2}-\mu\right)}}.
$$

Proof From Lemma 6.2,

$$\begin{aligned}
&\left|\int_0^{2X_m} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \\
&\le \int_0^{2X_m} \langle x\rangle^\mu |h_m(x)||h_n(x)|dx \le CX_m^\mu \int_0^{X_n^{\nu_3}} |h_m(x)||h_n(x)|dx \\
&\le CX_m^\mu \|h_m(x)\|_{L^2}\|h_n(x)\|_{L^p}\left(\int_0^{X_n^{\nu_3}} dx\right)^{\frac{1}{q}} \\
&\le CX_m^\mu X_n^{-\left(\frac{1}{2}-\frac{1}{p}\right)} X_n^{\frac{\nu_3}{q}} \le C(mn)^{\frac{1}{4}\left(\mu-\frac{\beta-2}{4\beta-2}\right)},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, $q=\frac{4\beta-3}{\beta-1}>4$. □

The Case $X_n^{\nu_3}<2X_m$ In this case we split the rest integral into two parts as follows.

Lemma 5.3 *For* $k\neq 0$, $X_n^{\nu_3}<2X_m$ *and* $\beta>2$, *then*

$$\left|\int_{X_n^{\nu_3}}^{2X_m} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C(|k\beta|^{-1}\vee 1)}{(mn)^{\frac{1}{4}\left(\frac{\beta-2}{4\beta-2}-\mu\right)}}.$$

Proof Denote $\psi(x)=\langle x\rangle^\mu h_m(x)h_n(x)$, $\phi(x)=x^\beta$. Notice that when $x\in[X_n^{\nu_3},2X_m]$, $|\phi'(x)|\ge \beta X_n^{\nu_3(\beta-1)}$. So by Lemma 6.1,

$$\left|\int_{X_n^{\nu_3}}^{2X_m} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \le C|k\beta|^{-1}X_n^{-\nu_3(\beta-1)}\left[|\psi(2X_m)|+\int_{X_n^{\nu_3}}^{2X_m}\left|\psi'(x)\right|dx\right],$$

where $|\psi(2X_m)|\le CX_m^\mu$, and $|\int_{X_n^{\nu_3}}^{2X_m}\langle x\rangle^{\mu-1}h_m(x)\overline{h_n(x)}dx|\le CX_m^{\mu-1}$. By Lemma 6.2, $\|h_m'(x)\|_{L^2}\le CX_m$. Thus,

$$\left|\int_{X_n^{\nu_3}}^{2X_m} \langle x\rangle^\mu h_m'(x)\overline{h_n(x)}dx\right| \le CX_m^\mu X_m = CX_m^{\mu+1},$$
$$\left|\int_{X_n^{\nu_3}}^{2X_m} \langle x\rangle^\mu h_m(x)\overline{h_n'(x)}dx\right| \le CX_m^\mu X_n \le CX_m^\mu X_n.$$

Combining with all the conclusions we have

$$\begin{aligned}
\left|\int_{X_n^{\nu_3}}^{2X_m} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| &\le C|k\beta|^{-1}X_n^{-\nu_3(\beta-1)+1}X_m^\mu \le C|k\beta|^{-1}(X_mX_n)^{\frac{\mu}{2}}X_n^{-\frac{\beta-2}{4\beta-2}} \\
&\le C|k\beta|^{-1}(mn)^{\frac{1}{4}\left(\mu-\frac{\beta-2}{4\beta-2}\right)}.
\end{aligned}$$

□

Lemma 5.4 *For* $X_n^{\nu_3}<2X_m$ *and* $\beta>2$, *then*

$$\left|\int_0^{X_n^{\nu_3}} \langle x\rangle^\mu e^{\mathrm{i}kx^\beta} h_m(x)\overline{h_n(x)}dx\right| \le \frac{C}{(mn)^{\frac{1}{4}\left(\frac{\beta-2}{4\beta-2}-\mu\right)}}.$$

Proof The proof is similar as Lemma 5.2, we omit it. □

Hence, combining the above four Lemmas we obtain Lemma 1.2 when $\beta>2$.

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6 Appendix

In the following we will introduce two technical lemmas without proof. The first lemma provides an estimate of oscillatory integral. For more details, see [37].

Lemma 6.1 [37] *Suppose ϕ is real-valued and smooth in (A, B), ψ is complex-valued, and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (A, B)$. Then*

$$\left|\int_A^B e^{\text{iˇŒ(x)}} \psi(x)dx\right| \leq c_k \lambda^{-1/k} \left[|\psi(B)| + \int_A^B |\psi'(x)|dx\right]$$

holds when:

(i) $k \geq 2$, *or*
(ii) $k = 1$ *and $\phi'(x)$ is monotonic. The bound c_k is independent of ϕ, ψ and λ.*

The second lemma shows that the L^p-norm of the eigenfunction of harmonic oscillating operator can be controlled by its L^2-norm.

Lemma 6.2 [24] *Sppose that $h(x)$ is the eigenfunction of Herimite operator with the corresponding eigenvalue μ^2. Then*

$$\|h\|_{L^p} \leq \mu^{\rho(p)} \|h\|_{L^2},$$

where

$$\rho(p) = \begin{cases} -(\frac{1}{2} - \frac{1}{p}), & 2 \leq p < 4, \\ -\frac{1}{3} + \frac{1}{3}(\frac{1}{2} - \frac{1}{p}), & 4 < p \leq \infty. \end{cases}$$

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