



Instability of H^1 -stable Periodic Peakons for the μ -Camassa-Holm Equation

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Abstract

By applying the method of characteristics, we prove that the periodic peakons of the μ -Camassa-Holm (μ CH) equation are unstable under $W^{1,\infty}$ -perturbations. Also, we show that small initial $W^{1,\infty}$ -perturbations of the above periodic peakons can lead to the finite time blow-up in the nonlinear evolution of the μ CH equation.

Keywords μ -Camassa-Holm equation · Periodic peakon · Instability

1 Introduction

In this paper, we are concerned with the dynamical stability of peaked periodic waves to the following integrable μ CH equation

$$\mu(u_t) - u_{xxt} = -2\mu(u)u_x + 2u_x u_{xx} + uu_{xxx}, \quad t > 0, x \in S^1 = \mathbb{R}/\mathbb{Z}, \quad (1)$$

where $u(t, x)$ is a real-valued spatially periodic function and $\mu(u) = \int_{S^1} u(t, x) dx$ denotes its mean. This equation was introduced in [1] as an integrable equation arising in the study of the diffeomorphism group of the circle, and it describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic field and self-interaction.

The μ CH equation (1) is a midway equation between the following well-known Camassa-Holm (CH) equation [2, 3]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (2)$$

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and the Hunter-Saxton (HS) equation [4]

$$-u_{txx} = 2u_x u_{xx} + uu_{xxx}. \tag{3}$$

All the three equations (1)-(3) share many common properties. For instance, they are all completely integrable in the sense that they all have a Lax pair representation, a bi-Hamiltonian structure, and an infinite sequence of conservation laws (see [1, 6, 7]). Similar to Eqs.(2) and (3), Eq.(1) can be also regarded as a geodesic equation with respect to a right invariant Riemannian metric induced by the μ inner product

$$\langle f, g \rangle_\mu = \mu(f)\mu(g) + \int_{S^1} f'(x)g'(x)dx. \tag{4}$$

Moreover, they all can be written as Euler-Arnold equations on the regular dual of the Lie algebra $T_e Diff(S^1) = vect(S^1)$ of smooth vector fields on the circle (see [9–11]). Besides, they all admit wave breaking phenomenon, i.e. the solution remains bounded but its slope becomes unbounded in finite time, even though it admits initially smooth solutions (see [4–6, 8, 12–14, 24]).

Set $A(\varphi) = \mu(\varphi) - \varphi_{xx}$, then the μ CH equation (1) can be rewritten in the following conservation form

$$u_t + uu_x = -A^{-1}\partial_x \left(2\mu(u)u + \frac{1}{2}u_x^2 \right), \tag{5}$$

where $A^{-1} = (\mu - \partial_x^2)^{-1}$ is the operator defined by $A^{-1}u(x) = g * u = \int_{S^1} g(x-y)u(y)dy$, and the Green’s function $g(x)$ of the operator A^{-1} is given by the explicit form

$$g(x) = \frac{1}{2}(x^2 - x) + \frac{13}{12}, \quad x \in S^1, \tag{6}$$

and is extended periodically to the real line. In other words,

$$g(x - x') = \frac{1}{2}(x - x')^2 - \frac{1}{2}|x - x'| + \frac{13}{12}, \quad x, x' \in S^1,$$

which shows that $g \in H_{per}^1(S^1) \cap W^{1,\infty}(S^1)$ is a piecewise C^1 function with the maximum at $g(0) = g(1) = \frac{13}{12}$ and the minima at $g(\frac{1}{2}) = \frac{23}{24}$.

More recently, Khesin et al. [1] studied the well-posedness and wave breaking of the μ CH equation (1). They proved that the periodic Cauchy problem is locally well posed in H^s for $s > \frac{3}{2}$. Moreover, they also showed that the solutions breakdown in finite time provided that the initial data u_0 satisfies $4|\mu(u_0)| \leq |u'_0|_{L^2}$. Following closely the ideas used in [19, 20], Tiğlay [25] studied the periodic Cauchy problem for μ CH equation (1) and proved the existence and uniqueness of global conservative weak solutions. This global conservative solution is consistent with the three conserved quantities of the μ CH equation:

$$H_0[u] = \int_{S^1} u dx, \quad H_1[u] = \int_{S^1} (\mu(u)u + u_x^2) dx, \quad H_2[u] = \int_{S^1} (\mu(u)u^2 + \frac{1}{2}uu_x^2) dx. \tag{7}$$

An important feature of the CH equation is that it admits peakon solutions. Furthermore, its peakon solutions are all orbitally stable in the non-periodic case [15] as well as in the periodic case [17, 18]. Analogous to the CH equation, it was shown in [8] that the μ CH equation also admits peakons of the form:

$$u(x, t) = c\phi(x - ct), \quad \phi(x) = \frac{1}{26}(12x^2 + 23), \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{8}$$

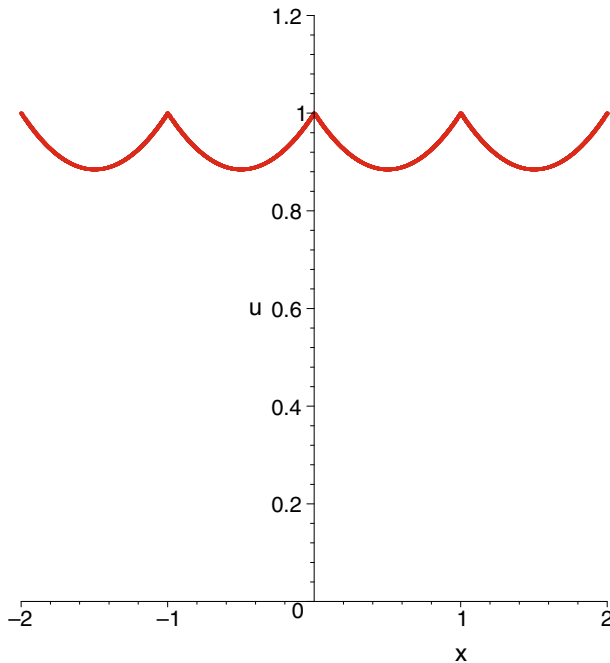


Fig. 1 The periodic peakon $\varphi(x)$ of Eq. (1) for $c = 1$

where $c \in \mathbb{R}$ and ϕ is extended periodically to the real line. It should be noted here that the above peakon solution (8) can be reformulated as

$$u(x, t) = c\varphi(x - ct), \quad \varphi(x) = \frac{6}{13}x(x - 1) + 1, \quad x \in S^1, \tag{9}$$

where we identify S^1 with the interval $[0, 1]$ and view functions on S^1 as periodic functions on the real line of period one (see Fig. 1). We can easily find that it admits the following relationship between the Green’s function $g(x)$ and $\varphi(x)$:

$$\varphi(x) = \frac{12}{13}g(x). \tag{10}$$

In other words, the corresponding traveling peaked periodic wave equation of the μ CH equation (1) can be rewritten as

$$-\frac{13}{12}g' + gg' + g' * (2\mu(g)g + \frac{1}{2}g'^2) = 0, \quad x \in S^1, \tag{11}$$

where the nonlocal equation is piecewise C^1 on both sides from the peak at $x = 0$ or $x = 1$. Notice that this traveling peaked periodic wave equation (11) has nothing to do with the wave speed c , then we can take $c = 1$ for simplicity in the rest of the paper.

By virtue of the inequalities related to the three conservation laws (7), it was proved in [21] that the periodic peakon $u(x, t) = \phi(x - t)$ is dynamically stable under small perturbations in the energy space $H^1(S^1)$. Liu, Qu and Zhang [22] proved that the periodic peakons of the modified μ CH equation are orbitally stable. Qu, Zhang, Liu and Liu [23] further proved that the periodic peakons of the generalized μ CH equation are orbitally stable. We reproduce here the corresponding result from [21].

Theorem 1 [21] *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $u \in C([0, T], H^1(S^1))$ is a solution to the μ CH equation (1) with*

$$\|u(\cdot, 0) - \phi\|_{H^1(S^1)} < \delta,$$

then

$$\|u(\cdot, t) - \phi(\cdot - \xi(t) - \frac{1}{2})\|_{H^1(S^1)} < \varepsilon, \quad t \in [0, T],$$

where $\xi(t)$ is any point where the function $u(\cdot, t)$ attains its maximum.

The purpose of this work is to study the problem of stability of the periodic peakon in the time evolution of the μ CH equation. More precisely, we will prove that the peaked periodic waves are all unstable in despite of the linear evolution or nonlinear evolution for its perturbations. It should be noted here that Theorem 1 only implies that H^1 -norm of the solution is sufficiently close to H^1 -norm of a translated peaked periodic wave, which certainly does not exclude the growth of $W^{1,\infty}$ -norm of the perturbation. As a matter of fact, we shall prove that the $W^{1,\infty}$ -norm of the perturbation can blow up in a finite time.

Most recently, the question of stability of peakons beyond the H^1 orbital stability [15–18, 21] had been studied for some different important models. For instance, Natali and Pelinovsky [26] proved that peaked solitary waves of the CH equation are strong unstable with respect to piecewise C^1 perturbations. Madiyeva and Pelinovsky [27] further proved that the peaked periodic waves for the CH equation are strongly unstable with respect to $W^{1,\infty}$ perturbations. Similarly to [26] and [27], Chen and Pelinovsky [28] proved that the peakon solutions for the Novikov equation were also unstable under $W^{1,\infty}$ perturbations. Moreover, they also established the local well-posedness result of the initial-value problem in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. It should be noted here that the study performed in [26–28] is inspired by the work on smooth and peaked periodic waves in the reduced Ostrovsky equation [30, 31], in which smooth periodic waves were linearly and nonlinearly stable, whereas peaked periodic waves were proved to be linearly unstable. Due to the lack of global well-posedness in the space H^1 , the question of nonlinear instability of peaked periodic waves for the reduced Ostrovsky equation remains open.

Our main result of this paper is reformulated as the following theorem.

Theorem 2 *For every $\delta > 0$ there exists a $t_1 > 0$ and $u_0 \in H^1_{per}(S^1) \cap W^{1,\infty}(S^1)$ satisfying*

$$\|u_0 - \phi\|_{H^1(S^1)} + \|u'_0 - \phi'\|_{L^\infty(S^1)} < \delta,$$

such that the local solution $u \in C([0, T], H^1_{per}(S^1) \cap W^{1,\infty}(S^1))$ to the μ CH equation (1) with the initial date u_0 and $T > t_1$ satisfies

$$\|u_x(\cdot, t_1) - \phi'(\cdot - \xi(t_1))\|_{L^\infty(S^1)} \geq 1,$$

where $\xi(t)$ is a point of peak of the function $u(\cdot, t)$ on S^1 . Moreover, there exist u_0 such that the maximal existence time T is finite.

The proof of Theorem 2 mainly relies on the method of characteristics, and is inspired by the pioneer work of [26, 27] where the instability of the peakons and periodic peakons of the CH equation is proved. The present work here is similar, but there are also differences. The main difference lies in that in [27] the Green function $g(x)$ associated with the convolution operator $(f * g)(x) = \int_{S^1} f(x - y)g(y)dy$ can be represented explicitly as $g(x) = \frac{\cosh(\frac{1}{2} - |x|)}{2 \sinh(\frac{1}{2})}$, $x \in [-\frac{1}{2}, \frac{1}{2}]$, whereas here the Green function is $g(x) =$

$\frac{1}{2}(x^2 - |x|) + \frac{13}{12}, x \in [-\frac{1}{2}, \frac{1}{2}]$. However, these two Green functions also have common characters, that is, both of them are piecewise smooth. Thus, it turns out to be enough to ascertain that in some sense that the μ CH equation still keeps similar dynamical evolution behaviour as the CH equation. In addition, it should be remarked here that the above Green function for the μ CH equation is piecewise smooth over the interval $[-\frac{1}{2}, \frac{1}{2}]$ and $x = 0$ is the peak, and thus we have to split this interval into two subintervals $[-\frac{1}{2}, 0)$ and $(0, \frac{1}{2}]$ and analyze the corresponding problem separately. This will make the process more tedious. For the sake of convenience in researching the problems, in view of translation invariance one can shift the interval from $[-\frac{1}{2}, \frac{1}{2}]$ to $[0, 1]$, and then the corresponding Green function becomes $g(x) = \frac{1}{2}(x^2 - x) + \frac{13}{12}$. It is easily seen that the Green function now is sufficiently smooth over $(0, 1)$ and its two peaks 0 and 1 are exactly the two endpoints, which is more convenient for us to analyze the problem. Therefore, we choose the Green function for the μ CH equation to be $g(x) = \frac{1}{2}(x^2 - x) + \frac{13}{12}, x \in [0, 1]$ throughout this paper.

The remainder of this paper is organized as follows. In Sect. 2, we define weak solutions to the μ CH equation and derive the evolution equations for perturbations near the peaked periodic wave. In Sect. 3, we study the linearized evolution equations for perturbations to the peaked periodic wave. By taking an approach similar to [27], we simplify the linearized evolution equation and solve it explicitly by using the method of characteristics. Moreover, we also show that the linearized instability in H^1 is related to the two conservative quantities $H_1(u)$ and $H_2(u)$ in (7). In Sect. 4, we study the nonlinear evolution equations for peaked perturbations to the peaked periodic wave. By applying the method of characteristics, we prove that the $W^{1,\infty}$ norm of the perturbation can grow and even blows up in finite time. Similar to [26, 27], we confirm once again that the passage from the linear to the nonlinear theory is false in H^1 as was stated in [16].

2 Peaked Periodic Waves as Weak Solutions

We first rewrite the Cauchy problem of the μ CH equation (1) in the form

$$\begin{cases} u_t + uu_x + Q[u] = 0, & t > 0, \\ u|_{t=0} = u_0, \end{cases} \tag{12}$$

where

$$Q[u](x) := \int_{S^1} g'(x - y) \left(2\mu(u)u(y) + \frac{1}{2}[u'(y)]^2 \right) dy, \quad x \in S^1. \tag{13}$$

The properties of $Q[u](x)$ are described in the following lemma.

Lemma 1 *If $u \in H^1_{per}(S^1)$, then $Q[u] \in C^0_{per}(S^1)$. Furthermore, if $u \in W^{1,\infty}(S^1)$, then $Q[u]$ is Lipschitz on S^1 .*

Proof $Q[u](x)$ can be rewritten as the explicit form

$$\begin{aligned} Q[u](x) &= \int_0^x \left(x - y - \frac{1}{2}\right) \left(2\mu(u)u(y) + \frac{1}{2}[u'(y)]^2\right) dy \\ &\quad + \int_x^1 \left(x - y + \frac{1}{2}\right) \left(2\mu(u)u(y) + \frac{1}{2}[u'(y)]^2\right) dy. \end{aligned}$$

Each integral is continuous since it is given by an integral of the absolutely integrable function if $u \in H^1_{per}(S^1)$. If $u \in H^1_{per}(S^1) \cap W^{1,\infty}(S^1)$, then $2\mu(u)u + \frac{1}{2}u^2$ is also bounded, so that each integral is Lipschitz on S^1 . \square

We say that $u \in C([0, T], H^1_{per}(S^1) \cap W^{1,\infty}(S^1))$ is a weak solution to the Cauchy problem (12) for some maximal existence time $T > 0$ if

$$\int_0^T \int_{S^1} \left(u\psi_t + \frac{u^2}{2}\psi_x - Q[u]\psi \right) dxdt + \int_{S^1} u_0(x)\psi(0, x)dx = 0 \tag{14}$$

is satisfied for every test function $\psi \in C^1([0, T] \times S^1)$ such that $\psi(T, \cdot) = 0$.

Following [27], in order to consider peaked periodic wave solutions with a peak on S^1 placed at the point $x = \xi(t)$ for every $t \in [0, T]$, we introduce the following function class

$$C^1_{\xi} := \{u \in H^1_{per}(S^1) \cap W^{1,\infty}(S^1) : u_x \in C^1(S^1 \setminus \{\xi\})\}. \tag{15}$$

Similarly to [27], the location of the peak moves with its local characteristic speed.

Lemma 2 *Assume that $u \in C([0, T], H^1_{per}(S^1) \cap W^{1,\infty}(S^1))$ is a weak solution to the μ CH equation in the form (12) for some $T > 0$ with a jump of u_x across $x = \xi(t)$ such that $u(t, \cdot) \in C^1_{\xi(t)}$, $t \in [0, T]$. Then, we have $\xi \in C^1(0, T)$ and $\xi'(t) = u(t, \xi(t))$, for $t \in [0, T]$.*

Note that the identity (11) can be checked directly by explicit computation, this then verifies the validity of the peakons $u(x, t) = \varphi(x - t)$ as solutions to the μ CH equation.

In order to study the evolution dynamics near the peaked periodic wave, we need to search a weak solution $u \in C([0, T], H^1_{per}(S^1) \cap W^{1,\infty}(S^1))$ to the μ -CH equation in the form (14), for which there exists $\xi(t) = t + a(t) \in S^1$ for $t \in [0, T]$ such that $u(t, \cdot) \in C^1_{\xi(t)}$ for $t \in [0, T]$. We now decompose this weak solution as in the form:

$$u(t, x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in [0, T], \quad x \in S^1, \tag{16}$$

where $a(t)$ is the deviation of the peak position from its unperturbed position, and $v(t, x)$ is the perturbation to the peaked periodic wave φ . By Lemma 2, we can know that $a \in C^1(0, T)$ satisfies the equation

$$\frac{da}{dt} = v(t, 0), \quad t \in [0, T]. \tag{17}$$

Substituting (16) and (17) into the Cauchy problem (12) yields the following Cauchy problem for the peaked perturbation v to the peakon φ :

$$\begin{cases} v_t = (1 - \frac{12}{13}g)v_x + \frac{12}{13}(v|_{x=0} - v)g_x + (v|_{x=0} - v)v_x \\ - \frac{24}{13}g' * (\bar{v}g + v + \frac{1}{2}g_x v_x) - Q[v], \quad t \in [0, T], \\ v|_{t=0} = v_0, \end{cases} \tag{18}$$

where

$$\bar{v} = \int_{S^1} v(t, x)dx.$$

Here we have used the relation (10)($\varphi(x) = \frac{12}{13}g(x)$) and replaced $x - t - a(t)$ by x thanks to the translational invariance of the system (12) with (13).

3 Linearized Evolution Near the Periodic Peakon

In this section, we study the linearized evolution equation, which can be obtained by the truncation of the nonlinear equation in system (18) at the linear terms in v :

$$v_t = \left(1 - \frac{12}{13}g\right)v_x + \frac{12}{13}(v|_{x=0} - v)g_x - \frac{24}{13}g' * (\bar{v}g + v + \frac{1}{2}g_x v_x). \tag{19}$$

By using straightforward computation, we can easily obtain the following result, which allows us to simplify the nonlocal terms in (19).

Lemma 3 For $v \in H^1_{per}(S^1)$, we have

$$g' * (\bar{v}g + v + \frac{1}{2}g_x v_x) = \frac{x}{12}(x + 1)(5 - 2x)\bar{v} - \frac{1}{2} \int_0^x v(y)dy + \frac{1}{2}(x - \frac{1}{2})[v(0) - v(x)]. \tag{20}$$

From Lemma 3, we can rewrite the Cauchy problem for the linear equation (19) in the following equivalent form:

$$\begin{cases} v_t = \frac{6}{13}(x - x^2)v_x - \frac{2x(x+1)(5-2x)}{13}\bar{v} + \frac{12}{13}w, & t > 0, \\ v|_{t=0} = v_0, \end{cases} \tag{21}$$

where

$$w(t, x) := \int_0^x v(t, y)dy, \quad \bar{v}(t) := \int_0^1 v(t, y)dy = w(t, 1). \tag{22}$$

We now consider the linearized Cauchy problem (21) in the space $C^1_{0,1}$ defined by (15) with $\xi = 0$ and $\xi = 1$. The next result shows that both $v(t, 0)$ and $\bar{v}(t)$ don't depend on t .

Lemma 4 Assume that there exists a solution $v \in C(\mathbb{R}^+, C^1_{0,1})$ to the Cauchy problem (21). Then, $v(t, 0) = v_0(0)$ and $\bar{v}(t) = \bar{v}_0$ for every $t \in \mathbb{R}^+$.

Proof If $v \in C(\mathbb{R}^+, C^1_{0,1})$, then $w \in C(\mathbb{R}^+, C^1(S^1))$ so that $w(t, 0) = 0$ in view of (22). Thus, it follows from (21) that

$$\lim_{x \rightarrow 0^+} v_t(t, x) = \lim_{x \rightarrow 1^-} v_t(t, x) = 0, \quad t > 0,$$

due to $v(t, \cdot) \in C^1_{0,1}$ for every $t \in \mathbb{R}^+$. Hence, $v(t, 0) = v_0(0)$ for every $t \in \mathbb{R}^+$.

Integrating the evolution equation (21) with respect to x on S^1 and using integration by parts, we get for every $t \in \mathbb{R}^+$:

$$\begin{aligned} \frac{d}{dt} \bar{v}(t) &= \int_0^1 \left[\frac{6}{13}(x - x^2)v_x - \frac{2}{13}x(x + 1)(5 - 2x)\bar{v} + \frac{12}{13}w(t, x) \right] dx \\ &= -\frac{6}{13} \int_0^1 (1 - 2x)v(t, x)dx - \frac{2}{13}\bar{v} \int_0^1 x(x + 1)(5 - 2x)dx \\ &\quad + \frac{12}{13} \int_0^1 w(t, x)dx \\ &= -\frac{6}{13}\bar{v} + \frac{12}{13} \int_0^1 xv(t, x)dx - \frac{6}{13}\bar{v} + \frac{12}{13} \int_0^1 w(t, x)dx. \end{aligned}$$

$$\begin{aligned}
 &= -\frac{12}{13}\bar{v} + \frac{12}{13}w(t, 1) - \frac{12}{13} \int_0^1 w(t, x)dx + \frac{12}{13} \int_0^1 w(t, x)dx \\
 &= 0.
 \end{aligned}$$

Therefore, $\bar{v}(t) = \bar{v}(0)$ for every $t \in \mathbb{R}^+$. □

Following the idea of [26–28], we proceed to solve the linearized problem (21) by using the method of characteristics. For this, we first define the characteristic curves $X(t, s)$ as

$$\begin{cases} \frac{dX}{dt} = \frac{6}{13}(X^2 - X), \\ X(0, s) = s. \end{cases} \tag{23}$$

For any fixed $s \in S^1$, the initial-value problem (23) has a unique solution since the right hand side of (23) is Lipschitz. Moreover, it follows that

$$X_s(t, s) = \exp\left(\frac{6}{13} \int_0^t (2X(\tau, s) - 1)d\tau\right) \tag{24}$$

thus $X(t, \cdot)$ is a diffeomorphism on S^1 for any $t \in \mathbb{R}^+$.

Solving (23) explicitly, we obtain

$$X(t, s) = \frac{s}{s + (1 - s)e^{\frac{6}{13}t}}, \quad t \in \mathbb{R}^+, \quad s \in [0, 1] \tag{25}$$

such that

$$\lim_{s \rightarrow 0^+} X(t, s) = 0 \quad \text{and} \quad \lim_{s \rightarrow 1^-} X(t, s) = 1.$$

Substituting $v = w_x$ into (21) yields

$$w_{tx} = \frac{6}{13}(x - x^2)w_{xx} - \frac{2x(x + 1)(5 - 2x)}{13}\bar{v} + \frac{12}{13}w, \quad x \in [0, 1]$$

which can be integrated with respect to x as follows:

$$w_t = \frac{6}{13}(x - x^2)w_x + \frac{6}{13}(2x - 1)w + \frac{x^2(x^2 - 2x - 5)}{13}\bar{v}, \quad x \in [0, 1]$$

where the integration constant is zero thanks to the boundary condition $w(t, 0) = 0$ for every $t \in \mathbb{R}^+$.

Along each characteristic curve $x = X(t, s)$ satisfying (23), function $W(t, s) := w(t, X(t, s))$ satisfies the initial-value problem:

$$\begin{cases} \frac{dW}{dt} = \frac{6}{13}(2X - 1)W + \frac{X^2(X^2 - 2X - 5)}{13}\bar{v}, \\ W(0, s) = w_0(s), \end{cases} \tag{26}$$

where $w_0(x) = \int_0^x v_0(y)dy$. After simple computation, we can find that

$$\frac{\partial X}{\partial s}(t, s) = \frac{e^{\frac{6}{13}t}}{[s + (1 - s)e^{\frac{6}{13}t}]^2}, \quad t \in \mathbb{R}^+, \quad s \in [0, 1], \tag{27}$$

such that

$$\lim_{s \rightarrow 0^+} \frac{\partial X}{\partial s}(t, s) = e^{-\frac{6}{13}t} \quad \text{and} \quad \lim_{s \rightarrow 1^-} \frac{\partial X}{\partial s}(t, s) = e^{\frac{6}{13}t}.$$

Integrating (26) with an integrating factor yields

$$W(t, s) = \frac{\partial X}{\partial s} \left[w_0(s) - \frac{\bar{v}}{6}s^2(s + 5) - \frac{\bar{v}}{6} \cdot \frac{s^2(1 - s)}{s + (1 - s)e^{\frac{6}{13}t}} + \bar{v}s^2e^{-\frac{6}{13}t} \right], \tag{28}$$

$t \in \mathbb{R}^+, \quad s \in [0, 1],$

such that

$$\lim_{s \rightarrow 0^+} W(t, s) = 0 \quad \text{and} \quad \lim_{s \rightarrow 1^-} W(t, s) = \bar{v}.$$

Along each characteristic curve $x = X(t, s)$ satisfying (23), $V(t, s) := v(t, X(t, s))$ satisfies the initial-value problem:

$$\begin{cases} \frac{dV}{dt} = \frac{12}{13}W(t, s) + \frac{2X(2X^2 - 3X - 5)}{13}\bar{v}, \\ V(0, s) = v_0(s). \end{cases} \tag{29}$$

By using the chain rule

$$\frac{\partial W}{\partial s}(t, s) = V(t, s) \frac{\partial X}{\partial s}(t, s),$$

we can obtain

$$\begin{aligned} V(t, s) = v_0(s) - \frac{3s^2 + 10s}{6}\bar{v} + 2s\bar{v}e^{-\frac{6}{13}t} - \frac{[2s(s - 1)^2e^{\frac{6}{13}t} + s^2(1 - 2s)]\bar{v}}{6[s + (1 - s)e^{\frac{6}{13}t}]^2} \\ + \frac{2(e^{\frac{6}{13}t} - 1)}{(1 - e^{\frac{6}{13}t})s + e^{\frac{6}{13}t}} [w_0(s) - \frac{s^2(s + 5)}{6}\bar{v} - \frac{s^2(1 - s)\bar{v}}{6(s + (1 - s)e^{\frac{6}{13}t})} \\ + \bar{v}s^2e^{-\frac{6}{13}t}] \end{aligned} \tag{30}$$

such that

$$\lim_{s \rightarrow 0^+} V(t, s) = v_0(0) \quad \text{and} \quad \lim_{s \rightarrow 1^-} V(t, s) = v_0(1) = v_0(0).$$

The next lemma follows from the solutions to the initial-value problems (23),(26) and (29).

Lemma 5 *For every $v_0 \in C^1_{0,1}$, there exists the unique solution $v \in C^1(\mathbb{R}^+, C^1_{0,1})$ to the Cauchy problem (21).*

Proof Firstly, since $v_0 \in C^1_{0,1}$, the map $(0, 1) \ni s \mapsto V(t, s) \in \mathbb{R}$ is C^1 for every $t \in \mathbb{R}^+$. Secondly, since the right-hand side of (30) implies that $V(t, 0) = v_0(0) = v_0(1) = V(t, 1)$, then $V(t, \cdot) \in C^1_{0,1}$ for every $t \in \mathbb{R}^+$. Finally, it follows from (27) that the change of coordinates $\mathbb{R}^+ \times (0, 1) \ni (t, s) \rightarrow (t, X) \in \mathbb{R}^+ \times (0, 1)$ is a diffeomorphism. Combining this with the compactness of S^1 leads to that the solution $v(t, \cdot) = V(t, s = X^{-1}(t, \cdot)) \in C^1_{0,1}$ for every $t \in \mathbb{R}^+$. □

By analyzing the exact solution (30), it is shown in the following lemma that $v(t, \cdot)$ remains bounded in the L^∞ norm.

Lemma 6 *Assume that $v_0 \in C^1_{0,1}$ in the Cauchy problem (21). Then, there exists $M > 0$ such that*

$$\|v(t, \cdot)\|_{L^\infty(S^1)} \leq M, \quad t \in \mathbb{R}^+. \tag{31}$$

Proof By Lemma 5, the unique solution to the Cauchy problem (21) with $v_0 \in C^1_{0,1}$ satisfies $v(t, \cdot) \in C^1_{0,1}$ for every $t \in \mathbb{R}^+$. Using Sobolev’s embedding theorem leads to $v(t, \cdot) \in L^\infty(S^1)$ for every $t \in \mathbb{R}^+$. So we only need to show that (31) holds true uniformly for $t \in \mathbb{R}^+$.

Since $X(t, \cdot)$ is a diffeomorphism on S^1 for any $t \in \mathbb{R}^+$, we have

$$\|v(t, \cdot)\|_{L^\infty(S^1)} = \max_{s \in [0,1]} |V(t, s)|,$$

Note that $V(t, 0) = V(t, 1) = v_0(0) \in L^\infty$, then it suffices to consider the exact solution (30) for $s \in (0, 1)$ and $t \in \mathbb{R}^+$.

For $s \in (0, 1)$ and $t \in \mathbb{R}^+$, we have

$$\left| \frac{2s(s-1)^2 e^{\frac{6}{13}t}}{6[s+(1-s)e^{\frac{6}{13}t}]^2} \right| = \frac{2s(s-1)^2 e^{-\frac{6}{13}t}}{6[s e^{-\frac{6}{13}t} + (1-s)]^2} \leq \frac{2s(s-1)^2}{6(1-s)^2} = \frac{s}{3} \leq \frac{1}{3},$$

and

$$\left| \frac{s^2(1-2s)}{6[s+(1-s)e^{\frac{6}{13}t}]^2} \right| \leq \frac{s^2|1-2s|}{6s^2} \leq \frac{1}{6},$$

which leads to

$$\left| \frac{[2s(s-1)^2 e^{\frac{6}{13}t} + s^2(1-2s)]\bar{v}}{6[s+(1-s)e^{\frac{6}{13}t}]^2} \right| \leq \left(\frac{1}{3} + \frac{1}{6} \right) |\bar{v}| = \frac{1}{2} |\bar{v}|.$$

Since $v_0, w_0 \in L^\infty(S^1)$, we only need to estimate the function $Y(t, s)$ denoted by

$$Y(t, s) := \frac{2(e^{\frac{6}{13}t} - 1)}{(1-s)e^{\frac{6}{13}t} + s}.$$

Notice that

$$\frac{\partial Y}{\partial s}(t, s) := \frac{2(e^{\frac{6}{13}t} - 1)^2}{[(1-s)e^{\frac{6}{13}t} + s]^2} > 0,$$

this means that $Y(t, s)$ is monotonically increasing with respect to s for any fixed $t \in \mathbb{R}^+$. Thus, for $s \in (0, 1)$ it follows that

$$0 \leq Y(t, s) \leq Y(t, 1^-) = 2(e^{\frac{6}{13}t} - 1) \leq 2e^{\frac{6}{13}t}.$$

On the other hand, for $s \in (0, 1)$ and $t \in \mathbb{R}^+$, we also have

$$0 \leq Y(t, s) = \frac{2(1 - e^{-\frac{6}{13}t})}{(1-s) + s e^{-\frac{6}{13}t}} \leq \frac{2}{1-s}.$$

Notice that

$$\begin{aligned} & w_0(s) - \frac{s^2(s+5)}{6} \bar{v} - \frac{s^2(1-s)\bar{v}}{6(s+(1-s)e^{\frac{6}{13}t})} + \bar{v}s^2 e^{-\frac{6}{13}t} \\ &= \int_1^s v_0(s') ds' + \left[\frac{6-s^2(s+5)}{6} - \frac{s^2(1-s)}{6(s+(1-s)e^{\frac{6}{13}t})} \right] \bar{v} + \bar{v}s^2 e^{-\frac{6}{13}t}. \end{aligned}$$

And then based on the estimates of the function $Y(t, s)$, we can derive that

$$\left| \bar{v}s^2 e^{-\frac{6}{13}t} Y(t, s) \right| \leq 2|\bar{v}|s^2 \leq 2|\bar{v}|$$

and

$$\left| \left(\int_1^s v_0(s') ds' \right) Y(t, s) \right| \leq \|v_0\|_{L^\infty(S^1)}(1-s) \cdot \frac{2}{1-s} = 2\|v_0\|_{L^\infty(S^1)}.$$

We further can obtain

$$\begin{aligned} & \left| \left[\frac{6-s^2(s+5)}{6} - \frac{s^2(1-s)}{6(s+(1-s)e^{\frac{6}{13}t})} \right] \bar{v}Y(t, s) \right| \\ & \leq |\bar{v}| \cdot \left[\frac{6-s^2(s+5)}{6} + \frac{s(1-s)}{6} \right] \cdot \frac{2}{1-s} \\ & = \frac{(1+s)(6+s)}{3} |\bar{v}| \leq \frac{14}{3} |\bar{v}|. \end{aligned}$$

A combination of all these estimates yields the bound (31). □

Lemma 6 implies that the peaked perturbations are linear stable in the L^∞ norm. In what follows, we show that the perturbations grow in the $W^{1,\infty}$ norm as is listed in the following Lemma.

Lemma 7 Assume that $v_0 \in C^1_{0,1}$ in the Cauchy problem (21). Then we have

$$\lim_{x \rightarrow 0^+} v_x(t, x) = v'_0(0^+)e^{\frac{6}{13}t} + (2v_0(0) - \frac{5}{3}\bar{v})(e^{\frac{6}{13}t} - 1) \tag{32}$$

and

$$\lim_{x \rightarrow 1^-} v_x(t, x) = v'_0(1^-)e^{-\frac{6}{13}t} + (2v_0(0) - \frac{5}{3}\bar{v})(1 - e^{-\frac{6}{13}t}). \tag{33}$$

Proof Differentiating (21) with respect to x yields

$$v_{tx} = \frac{6}{13}(x-x^2)v_{xx} + \frac{6}{13}(1-2x)v_x - \frac{2(-6x^2+6x+5)}{13}\bar{v} + \frac{12}{13}v.$$

Thus, along each characteristic curve $x = X(t, s)$ satisfies (23), function $\Phi(t, s) =: v_x(t, X(t, s))$ satisfies the initial-value problem:

$$\begin{cases} \frac{d\Phi}{dt} = \frac{6}{13}(1-2X)\Phi(t, s) + \frac{2(6X^2-6X-5)}{13}\bar{v} + \frac{12}{13}V(t, s), \\ \Phi(0, s) = v'_0(s). \end{cases} \tag{34}$$

Note that the coefficients in (34) are all $C^1(0, 1)$. Taking the limit $s \rightarrow 0^+$ in (34) yields

$$\frac{d}{dt}\Phi(t, 0^+) = \frac{6}{13}\Phi(t, 0^+) + \frac{12}{13}v_0(0) - \frac{10}{13}\bar{v}, \tag{35}$$

while the limit $s \rightarrow 1^-$ in (34) yields

$$\frac{d}{dt}\Phi(t, 1^-) = -\frac{6}{13}\Phi(t, 1^-) + \frac{12}{13}v_0(0) - \frac{10}{13}\bar{v}, \tag{36}$$

where we have used $V(t, 1) = V(t, 0) = v_0(0)$. Hence, the assertion of the Lemma 7 follows by solving the linear differential equations (35) and (36).

The exponential growth of $v_x(t, \cdot)$ as $t \rightarrow \infty$ at the right side of the peak $x = 0$ indicates the linear instability of the periodic peakon in the $W^{1,\infty}$ norm.

Following [21], we define the μ -inner product $\langle u, v \rangle_\mu$ and the associated μ -norm $\|u\|_\mu$ by

$$\langle u, v \rangle_\mu = \mu(u)\mu(v) + \int_{S^1} u_x v_x dx, \quad \|u\|_\mu^2 = \langle u, u \rangle_\mu = H_1[u], \quad u, v \in H^1(S^1).$$

The following Lemma shows that the peaked perturbations are linearly unstable in the μ -norm. This also means the instability of peaked perturbations in H^1 , since it is noted in [21, Lemma 3.5] that the μ -norm is equivalent to the $H^1(S^1)$ -norm.

Lemma 8 *Assume that $v_0 \in C_{0,1}^1$ in the Cauchy problem (21). Then we have*

$$\|v(t, \cdot)\|_\mu^2 = C_+ e^{\frac{6}{13}t} + C_0 + C_- e^{-\frac{6}{13}t}, \tag{37}$$

for some uniquely defined constants C_+, C_0, C_- .

Proof Firstly, the evolution equation for v and v_x can be written explicitly as

$$v_t = \frac{6}{13}(x - x^2)v_x + \frac{12}{13}w - \frac{2\bar{v}}{13}(-2x^3 + 3x^2 + 5x), \tag{38}$$

and

$$v_{tx} = \frac{6}{13}(x - x^2)v_{xx} - \frac{6}{13}(2x - 1)v_x + \frac{12}{13}v - \frac{2\bar{v}}{13}(-6x^2 + 6x + 5). \tag{39}$$

By multiplying (39) by v_x , integrating it on $[0, 1]$ and using integration by parts, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_x(t, \cdot)\|_{L^2(0,1)}^2 &= \int_0^1 \left[\frac{6}{13}(x - x^2)v_x v_{xx} + \frac{6}{13}(1 - 2x)v_x^2 + \frac{12}{13}v v_x \right] dx \\ &\quad - \int_0^1 \left[\frac{2\bar{v}}{13}(-6x^2 + 6x + 5)v_x \right] dx \\ &= \int_0^1 \left[\frac{3}{13}(2x - 1)v_x^2 + \frac{6}{13}(1 - 2x)v_x^2 + \frac{12\bar{v}}{13}(1 - 2x)v \right] dx \\ &= - \int_0^1 \varphi' \left[2\bar{v}v + \frac{v_x^2}{2} \right] dx, \end{aligned}$$

where we have used $v(t, 0) = v(t, 1)$ and φ is defined by (9). This is equivalent to

$$\frac{d}{dt} H_1[v] = -2 \int_0^1 \varphi' \left[2\bar{v}v + \frac{v_x^2}{2} \right] dx, \tag{40}$$

due to the independence of $\mu(v)$ with respect to t , where $H_1[v] := \mu(v)^2 + \int_{S^1} v_x^2 dx$ in line with (7).

Next, by multiplying (38) by $2\bar{v}\varphi + \frac{12}{13}v$ and (39) by φv_x , integrating it on $[0, 1]$ and using integration by parts, we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^1 \left(2\bar{v}v\varphi + \frac{6}{13}v^2 \right) dx &= \int_0^1 \left[2(2\varphi - 1)\varphi'\bar{v}v + \frac{24\bar{v}}{13}\varphi w \right] dx \\ &\quad - \int_0^1 \left[\frac{4\bar{v}^2}{13}(-2x^3 + 3x^2 + 5x)\varphi \right] dx + \frac{72}{169}\bar{v}^2 \\ &\quad + \frac{6}{13} \int_0^1 \left[\varphi'v^2 - \frac{4\bar{v}v}{13}(-2x^3 + 3x^2 + 5x) \right] dx \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \int_0^1 \frac{\varphi v_x^2}{2} dx &= - \int_0^1 \varphi' \left[\frac{v_x^2}{2} + \frac{6}{13}v^2 + \frac{2\bar{v}v}{13}(-6x^2 + 6x + 5) \right] dx \\ &\quad + \frac{12}{13} \int_0^1 (1 - 2x)\varphi\bar{v}v dx \end{aligned}$$

where we have used $w' = v$, $w(t, 1) = \bar{v}$. Adding these two identities yields

$$\frac{d}{dt} \int_0^1 \left[\varphi \left(\frac{v_x^2}{2} + 2\bar{v}v \right) + \frac{6}{13}v^2 \right] dx = - \int_0^1 \varphi' \left[2\bar{v}v + \frac{v_x^2}{2} \right] dx. \tag{41}$$

Now we define

$$P(t) := \int_0^1 \left[\varphi \left(\frac{v_x^2}{2} + 2\bar{v}v \right) + \frac{6}{13}v^2 \right] dx, \quad S(t) := \int_0^1 \varphi' \left[2\bar{v}v + \frac{v_x^2}{2} \right] dx. \tag{42}$$

Then from (40) and (41) we can derive that

$$2P(t) = H_1[v] + C_1, \tag{43}$$

where C_1 is an arbitrary constant.

Finally, by multiplying (38) by $2\bar{v}\varphi' + \frac{12}{13}v$ and (39) by $\varphi'v_x$, integrating it on $[0, 1]$ and using integration by parts, we obtain

$$\frac{d}{dt} \int_0^1 2\bar{v}v\varphi' dx = \int_0^1 \left[\frac{72}{169}\bar{v}v(6x^2 - 6x + 1) - \frac{24\bar{v}}{13}\varphi v \right] dx + \frac{24}{13}\bar{v}^2 - \frac{4\bar{v}^2}{13}c_0$$

and

$$\begin{aligned} \frac{d}{dt} \int_0^1 \frac{v_x^2}{2} \varphi' dx &= \frac{24}{169}v|_{x=0}(3v|_{x=0} - 5\bar{v}) \\ &\quad + \int_0^1 \left[-\frac{36}{169}(x^2 - x + \frac{1}{2})v_x^2 - \frac{72}{169}v^2 \right] dx \\ &\quad + \int_0^1 \left[\frac{24}{169}\bar{v}v(-18x^2 + 18x + 2) \right] dx \end{aligned}$$

where $c_0 = \int_0^1 \varphi'(-2x^3 + 3x^2 + 5x)dx$ is obviously a constant. Adding the two identities yields

$$\frac{d}{dt} S(t) = -\frac{12}{13}P(t) + \frac{60}{169}H_1[v] + C_2, \tag{44}$$

where C_2 is a constant defined by

$$C_2 := \frac{372 - 52c_0}{169}\bar{v}^2 + \frac{24}{169}v|_{x=0}(3v|_{x=0} - 5\bar{v}),$$

since both $v|_{x=0}$ and \bar{v} don't depend on t . By combining with equations (41),(43) and (44), we obtain the following system of differential equations

$$P'(t) = -S(t), \quad S'(t) = -\frac{36}{169}P(t) + C_3,$$

where $C_3 := C_2 - \frac{60}{169}C_1$. Thus, we have $S''(t) = (\frac{6}{13})^2S(t)$. This allows us to solve explicitly the above system of differential equations and its general solution is given by

$$S(t) = S_+e^{\frac{6}{13}t} + S_-e^{-\frac{6}{13}t}, \quad P(t) = -\frac{13}{6}S_+e^{\frac{6}{13}t} + \frac{13}{6}S_-e^{-\frac{6}{13}t} + \frac{169}{36}C_3,$$

where S_+ and S_- are arbitrary constants. Substituting it into (43) leads to (37) for some constants $C_+, C_0,$ and C_- . □

4 Nonlinear Evolution Near the Periodic Peakon

In this section, we study the nonlinear evolution equation (18) and prove Theorem 2.

Firstly, we derive the evolution equation for v_x and w . It follows that the Cauchy problem (18) can be rewritten as

$$\begin{cases} v_t = \frac{6}{13}(x - x^2)v_x - \frac{2x(x+1)(5-2x)}{13}\bar{v} + \frac{12}{13}w + (v|_{x=0} - v)v_x - Q[v] \\ v|_{t=0} = v_0, \end{cases} \tag{45}$$

where w and \bar{v} are still defined by (22) and

$$Q[v](x) := \int_{S^1} g'(x - y) \left[2\mu(v)v + \frac{1}{2}v_y^2 \right] (y)dy.$$

Differentiating (45) with respect to x yields

$$\begin{cases} v_{tx} = \frac{6}{13}(x - x^2)v_{xx} + \frac{6}{13}(1 - 2x)v_x - \frac{2(-6x^2+6x+5)}{13}\bar{v} + \frac{12}{13}w \\ \quad + (v|_{x=0} - v)v_{xx} - v_x^2 - P[v] + 2\bar{v}^2 + \frac{1}{2}\int_0^1 v_x^2 dx, \\ v_x|_{t=0} = v'_0, \end{cases} \tag{46}$$

where

$$P[v](x) := \int_{S^1} g(x - y) \left[2\mu(v)v + \frac{1}{2}v_y^2 \right] (y)dy,$$

and we have used the relation

$$\begin{aligned} \int_{S^1} g''(x - y) \left[2\mu(v)v + \frac{1}{2}v_y^2 \right] (y)dy &= \int_{S^1} g(x - y) \left[2\mu(v)v + \frac{1}{2}v_y^2 \right] (y)dy \\ &\quad - 2\bar{v}^2 - \frac{1}{2}\int_0^1 v_x^2 dx. \end{aligned}$$

Substituting $v = w_x$ into (45) leads to

$$w_{tx} = \frac{6}{13}(x - x^2)w_{xx} - \frac{2x(x + 1)(5 - 2x)}{13}\bar{v} + \frac{12}{13}w + (v|_{x=0} - v)w_{xx} - Q[v].$$

By integrating this evolution equation with respect to x and in view of that $w(t, 0) = 0$, we obtain

$$\begin{cases} w_t = \frac{6}{13}(x - x^2)w_x + \frac{6}{13}(2x - 1)w - \frac{x^2(-x^2+2x+5)}{13}\bar{v} \\ \quad - \frac{1}{2}(v|_{x=0} - v)^2 - P[v] + P[v]|_{x=0}, \\ w|_{t=0} = w_0, \end{cases} \tag{47}$$

where $w_0 := \int_0^x v_0(y)dy$.

The following Lemma shows that \bar{v} still does not depend on t . However, $v|_{x=0}$ indeed depends on t .

Lemma 9 *Assume that there exists a solution $v \in C([0, T], C^1_{0,1})$ to the Cauchy problem (45). Thus, $\bar{v}(t) = \bar{v}_0$ for every $t \in [0, T]$.*

Proof By integrating the evolution equation (45) with respect to x on $[0, 1]$ and using the results presented in Lemma 4, we can know that $\frac{d}{dt}\bar{v}(t) = 0$ holds true if and only if

$$\int_0^1 Q[v](t, x)dx = \int_0^1 \int_0^1 g'(x - y)q[v](t, y)dydx = 0,$$

where both g' and $q[v](t, y) := 2\mu(v)v + \frac{1}{2}v_y^2$ are absolutely integrable.

By direct computation, we obtain

$$\int_0^1 g'(x - y)q[v](t, y)dy = (x + \frac{1}{2}) \int_0^1 q[v](t, y)dy - \int_0^1 yq[v](t, y)dy - \int_0^x q[v](t, y)dy,$$

so that

$$\begin{aligned} \int_0^1 Q[v](t, x)dx &= \int_0^1 q[v](t, y)dy - \int_0^1 yq[v](t, y)dy - \int_0^1 \int_0^x q[v](t, y)dydx \\ &= \int_0^1 (1 - y)q[v](t, y)dy - \int_0^1 dy \int_y^1 q[v](t, y)dx \\ &= \int_0^1 (1 - y)q[v](t, y)dy - \int_0^1 (1 - y)q[v](t, y)dy = 0. \end{aligned}$$

Therefore, $\bar{v}(t) = \bar{v}_0$ for every $t \in [0, T]$. □

Notice that the local well-posedness in $C^1_{0,1}$ for the Cauchy problem (45) has not been established in S^1 . Now, our attention is turned to get the local well-posedness result by using the method of characteristics as well as the theory of dynamical systems.

Firstly, the evolution problem (45) suggests us to introduce the characteristic curves $x = X(t, s)$ which satisfy the following evolution problem:

$$\begin{cases} \frac{dX}{dt} = \frac{6}{13}(X^2 - X) + v(t, X) - v(t, 0), \\ X(0, s) = s, \quad s \in S^1. \end{cases} \tag{48}$$

Assuming that $v(t, \cdot)$ for every $t \in [0, T]$, and then differentiating (48) with respect to $s \in (0, 1)$ yields

$$\begin{cases} \frac{d}{dt} \frac{\partial X}{\partial s} = [\frac{6}{13}(2X - 1) + v_x(t, X)] \frac{\partial X}{\partial s}, \\ X_s(0, s) = 1, \quad s \in (0, 1). \end{cases} \tag{49}$$

with the unique solution

$$X_s(t, s) = \exp \left(\int_0^t \left[\frac{6}{13} (2X(t', s) - 1) + v_x(t', X(t', s)) \right] dt' \right) > 0, \tag{50}$$

thus $X(t, \cdot)$ is invertible with respect to $s \in [0, 1]$ for $t \in [0, T)$. Moreover, the peak's locations at 0 and 1 are invariant in the time evolution if $v(t, \cdot) \in C^1_{0,1}$ for every $t \in [0, T)$, since $X(t, 0) = 0$ and $X(t, 1) = 1$ are equilibrium points of (48).

Next, we establish the following local well-posedness result for the Cauchy problem (45).

Lemma 10 *For every $v_0 \in C^1_{0,1}$, there exists the maximal existence time $T > 0$ (finite or infinite) and the unique solution $v \in C^1([0, T), H^1 \cap C^1_{0,1})$ to the Cauchy problem (45) that depends continuously on $v_0 \in C^1_{0,1}$.*

Proof Setting $V(t, s) := v(t, X(t, s))$, $W(t, s) := w(t, X(t, s))$ and $\Phi(t, s) := v_x(t, X(t, s))$, then along each characteristic curve $x = X(t, s)$ satisfying (48), the functions $V(t, s)$, $W(t, s)$ and $\Phi(t, s)$ satisfy the following Cauchy problem

$$\begin{cases} \frac{dV}{dt} = \frac{12}{13} W - \frac{2\bar{v}}{13} X(-2X^2 + 3X + 5) - Q[v](X), \\ V|_{t=0} = v_0(s), \end{cases} \tag{51}$$

$$\begin{cases} \frac{dW}{dt} = \frac{6}{13} (2X - 1)W - \frac{\bar{v}}{13} X^2(-X^2 + 2X + 5) + \frac{1}{2} [V^2 - (v|_{X=0})^2] - P[v](X) + P[v](0), \\ W|_{t=0} = w_0(s), \end{cases} \tag{52}$$

and

$$\begin{cases} \frac{d\Phi}{dt} = \frac{6}{13} (1 - 2X)\Phi - \frac{2\bar{v}}{13} (-6X^2 + 6X + 5) + \frac{12}{13} V - \Phi^2 \\ - P[v](X) + 2\bar{v}^2 + \frac{1}{2} \int_0^1 v_y^2 dy, \\ \Phi|_{t=0} = v'_0(s). \end{cases} \tag{53}$$

If $v \in C^1_{0,1}$, then by Lemma 1 we can know that $Q[v] \in C^0_{per}(S^1) \cap Lip(S^1)$. By Lemma 9 and in view of $\int_{S^1} Q[v](t, x) dx = \int_{S^1} P[v]_x(t, x) dx = 0$, we can derive that $P[v] \in C^1_{per}(S^1)$. Therefore, the nonlocal parts of the Cauchy problems (51), (52), and (53) are well-defined and we can regard $X \in [0, 1]$ as corresponding to $s \in [0, 1]$.

For $s \in [0, 1]$, we can rewrite the evolution equations (48),(51),(52) and (53) as the dynamical system

$$\frac{d}{dt} \begin{pmatrix} X \\ V \\ W \\ \Phi \end{pmatrix} = \begin{pmatrix} f^{(X)}(X, V) \\ f^{(V)}(X, V, W) \\ f^{(W)}(X, V, W) \\ f^{(\Phi)}(X, V, W, \Phi) \end{pmatrix} := F(X, V, W, \Phi),$$

with the initial datum

$$\begin{bmatrix} X \\ V \\ W \\ \Phi \end{bmatrix} \Big|_{t=0} = \begin{bmatrix} s \\ v_0(s) \\ w_0(s) \\ v'_0(s) \end{bmatrix},$$

and the boundary conditions

$$\begin{cases} X(t, 0) = 0, & X(t, 1) = 1, \\ V(t, 0) = V(t, 1) = V|_{s=0}, \\ W(t, 0) = 0, & W(t, 1) = \bar{v}, \end{cases}$$

where components of $F(X, V, W, \Phi)$ are given by

$$\begin{aligned} f^{(X)}(X, V) &= \frac{6}{13}(X^2 - X) + V - V|_{s=0}, \\ f^{(V)}(X, V, W) &= \frac{12}{13}W - \frac{2\bar{v}}{13}X(-2X^2 + 3X + 5) - Q[v](X), \\ f^{(W)}(X, V, W) &= \frac{6}{13}(2X - 1)W - \frac{\bar{v}}{13}X^2(-X^2 + 2X + 5) \\ &\quad + \frac{1}{2}[V^2 - (V|_{s=0})^2] - P[v](X) + P[v]|_{s=0}, \\ f^{(\Phi)}(X, V, W, \Phi) &= \frac{6}{13}(1 - 2X)\Phi - \frac{2\bar{v}}{13}(-6X^2 + 6X + 5) + \frac{12}{13}V - \Phi^2 \\ &\quad - P[v](X) + 2\bar{v}^2 + \frac{1}{2} \int_0^1 v_y^2 dy. \end{aligned}$$

In particular, $V|_{s=0}$ satisfies

$$\frac{d}{dt} V|_{s=0} = -Q[v](0).$$

Denote $V(s) = v(X(s))$, $W(s) = w(X(s))$, $\Phi(s) = v_x(X(s))$. Thanks to invertibility of $X(t, \cdot)$ with respect to $s \in [0, 1]$ for $t \in [0, T)$, we have $V, W \in C^1_{0,1}$ and Φ is bounded and continuous. Thanks to the chain rule, it follows from $v \in H^1$ that $V, W \in L^2$ and $\Phi \in L^2$. Thus, the nonlocal term $\int_0^1 v_y^2 dy$ in $f^{(\Phi)}(X, V, W, \Phi)$ is locally Lipschitz in (X, V, W, Φ) for every $X \in [0, 1]$, $V, W \in L^2$, and $\Phi \in L^2$, thanks to integrability of v_x^2 , invertibility of $X(s)$ with respect to s and the following chain rule

$$\int_0^1 v_y^2 dy = \int_0^1 \Phi^2(s') X_s(s') ds'.$$

Note that $Q[v] \in C^0_{per}(S^1) \cap Lip(S^1)$ and $P[v] \in C^1_{per}(S^1)$, then it follows that the nonlocal terms in $F(q, V, W)$ are locally Lipschitz in (X, V, W, Φ) . Also, it is easily seen that all local terms in $F(X, V, W, \Phi)$ are locally Lipschitz in (X, V, W, Φ) . Therefore, the vector field $F(X, V, W, \Phi)$ of the dynamical system is locally Lipschitz in (X, V, W, Φ) on $[0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. By the existence and uniqueness theory for differential equations, there exists the unique local solution $X(\cdot, s), V(\cdot, s), W(\cdot, s), U(\cdot, s) \in C^1([0, T))$ to the Cauchy problem for some maximal existence time $T > 0$. The solution depends continuously on the initial data for every $s \in [0, 1]$. Moreover, since the initial data is $C^1(0, 1)$, then $X(t, \cdot), V(t, \cdot), W(t, \cdot), \Phi(t, \cdot) \in C^1(0, 1)$ for every $t \in [0, T)$. Thanks to preservation of the boundary conditions, the solution $V(t, \cdot) \in C^1(0, 1)$ can be extended to $V(t, \cdot) \in C^1_{0,1}$ on S^1 . Therefore, the invertibility of the transformation formula $V(t, s) = v(t, X(t, s))$ yields the unique solution $v \in C^1([0, T), H^1 \cap C^1_{0,1})$ which depends continuously on $v_0 \in C^1_{0,1}$. \square

Proof of Theorem 2. By Lemma 10, we can define the one-sided limits $\Phi^\pm \in C^1(0, T)$ by

$$\Phi^+ := \lim_{s \rightarrow 0^+} \Phi(t, s) = \lim_{s \rightarrow 0^+} v_x(t, X(t, s)),$$

$$\Phi^- := \lim_{s \rightarrow 1^-} \Phi(t, s) = \lim_{s \rightarrow 1^-} v_x(t, X(t, s)),$$

By taking the limits $s \rightarrow 0^+$ and $s \rightarrow 1^-$ in (53), and in view of $P[v](0) = P[v](1)$ we can know that for $t \in (0, T)$ the functions Φ^\pm satisfies:

$$\frac{d\Phi^\pm}{dt} = \pm \frac{6}{13} \Phi^\pm - \frac{10}{13} \bar{v} + \frac{12}{13} V|_{s=0} - (\Phi^\pm)^2 - P[v](0) + 2\bar{v}^2 + \frac{1}{2} \int_0^1 v_y^2 dy. \tag{54}$$

The instability argument relies on the behavior of $v_x(t, x)$ near the peak at $x = 0$ from the right side, where Lemma 7 has demonstrated the exponential growth of solutions v_x . Therefore, we now focus on the the study of evolution of Φ^+ .

It follows from the decomposition (16) that the initial bound presented in Theorem 2 can be rewritten as the form

$$\|v_0\|_{H^1(S^1)} + \|v'_0\|_{L^\infty(S^1)} < \delta. \tag{55}$$

Then picking Φ^+ in (54) and we obtain

$$\frac{d\Phi^+}{dt} = \frac{6}{13} \Phi^+ - \frac{10}{13} \bar{v} + \frac{12}{13} V|_{s=0} - (\Phi^+)^2 - P[v](0) + 2\bar{v}^2 + \frac{1}{2} \int_0^1 v_y^2 dy. \tag{56}$$

By Theorem 1, we know that for every small $\varepsilon > 0$, there exists $\nu(\varepsilon) > 0$ such that if $\|v_0\|_{H^1(S^1)} < \nu(\varepsilon)$, then $\|v(t, \cdot)\|_{H^1(S^1)} < \varepsilon$ for every $t \in [0, T)$. Therefore Sobolev embedding implies that

$$\|v(t, \cdot)\|_{L^\infty(S^1)} \leq r \|v(t, \cdot)\|_{H^1(S^1)} < r\varepsilon, \tag{57}$$

for a positive constant r .

Due to the bound (57), we have for sufficiently small ε

$$\begin{aligned} \left| -\frac{10}{13} \bar{v} + \frac{12}{13} V|_{s=0} - P[v](0) + 2\bar{v}^2 + \frac{1}{2} \int_0^1 v_y^2 dy \right| &\leq \frac{10}{13} r\varepsilon + \frac{12}{13} r\varepsilon \\ &\quad + \frac{25}{6} r^2 \varepsilon^2 + \frac{25}{24} \varepsilon^2 \\ &\leq r_1 \varepsilon \end{aligned}$$

for some ε -independent constant $r_1 > 0$. Thus, it follows from (56) that Φ^+ satisfies the following Riccati inequality

$$\frac{d\Phi^+}{dt} \leq -(\Phi^+ - \frac{3}{13})^2 + \frac{9}{169} + r_1 \varepsilon.$$

By performing simple analysis of this differential inequality (see [29]), we can know that if the initial datum satisfy

$$\Phi^+(0) < \frac{3 - \sqrt{9 + 169r_1\varepsilon}}{13}, \tag{58}$$

then $\Phi^+(t) \rightarrow -\infty$ in finite time. To this end, we only need to pick the initial datum $v_0 \in H^1 \cap C^1_{0,1}$ satisfying

$$\|v_0\|_{H^1} < \nu(\varepsilon), \quad \lim_{x \rightarrow 0^+} v'_0(x) = -5r_1\varepsilon, \quad \nu(\varepsilon) + 5r_1\varepsilon < \delta,$$

then (58) is satisfied, and therefore $v_x(t, 0) \rightarrow -\infty$ as $t \rightarrow T^*$ for some $T^* < \infty$. This implies that the maximal existence time T is finite. Meanwhile, this also implies that there exists $t_1 \in (0, T)$ such that $\|v_x(t_1, \cdot)\|_{L^\infty(0,1)} \geq 1$. So the proof of Theorem 2 is completed. \square

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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