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Existence, Stability and Regularity of Periodic Solutions for Nonlinear Fokker–Planck Equations

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Abstract
We consider a class of nonlinear Fokker–Planck equations describing the dynamics of an infinite population of units with mean-field interaction. Relying on a slow–fast viewpoint and on the theory of approximately invariant manifolds we obtain the existence of a stable periodic solution for the PDE, consisting of probability measures. Moreover we establish the existence of a smooth isochron map in the neighborhood of this periodic solution.

Keywords Mean-field systems · Nonlinear Fokker–Planck equation · McKean–Vlasov process · periodic behavior · Normally hyperbolic manifolds · Isochron map

Mathematics Subject Classification 35K55 · 35Q84 · 37N25 · 60K35 · 82C31 · 92B20

1 Introduction

1.1 The Model

We are interested in this paper in the existence, stability and regularity of periodic solutions to the following nonlinear PDE on $\mathbb{R}^d$ ($d \geq 1$):

$$\partial_t u_t = \nabla \cdot \left(\sigma^2 u_t\right) + \nabla \cdot \left(K \left(x - \int_{\mathbb{R}^d} y u_t(\mathrm{d}y)\right) u_t\right) - \delta \nabla \cdot (F(x) u_t) . \tag{1.1}$$

Here, $t \geq 0 \mapsto u_t$ is a probability measure-valued process on $\mathbb{R}^d$, $K = \mathrm{diag}(k_1, \dots, k_d)$ and $\sigma = (\sigma_1, \dots, \sigma_d)$ and are diagonal matrices with positive coefficients and $F : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth bounded function with bounded derivatives. Equation (1.1) has a natural probabilistic

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interpretation: if u_0 is a probability distribution on $\mathbb{R}^d$, it is well known [30, 37] that u_t is the law of the McKean–Vlasov process X_t where $X_0 \sim u_0$ and

$$dX_t = \delta F(X_t)\,\mathrm{d}t - K\,(X_t - \mathbb{E}[X_t])\,\mathrm{d}t + \sqrt{2}\sigma\,\mathrm{d}B_t. \tag{1.2}$$

The dynamics of the process $(X_t)_{t\geq 0}$ is the superposition of a local part $\delta F(X_t)\mathrm{d}t$, where $\delta > 0$ is a scaling parameter, a linear interaction term $K\,(X_t - \mathbb{E}[X_t])\,\mathrm{d}t$, modulated by the intensity matrix K, and an additive noise given by a standard Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}^d$. The difficulty in the analysis of (1.2) lies in its nonlinear character: X_t interacts with its own law, more precisely its own expectation $\mathbb{E}\,[X_t]$. The long-time dynamics of (1.2) is a longstanding issue in the literature. In particular, the existence of stable equilibria for (1.1) (that is invariant measures for (1.2)) has been studied for various choices of dynamics, interaction and regimes of parameters δ, K, σ, mostly in a context where the corresponding particle dynamics defined in (1.3) below is reversible (see e.g. [7, 11, 39] for further details and references).

The question we address in the present paper concerns the existence of periodic solutions to nonlinear equations such as (1.1). In this case, a major difficulty lies in the fact that the underlying microscopic dynamics is not reversible. From an applicative perspective, the emergence of periodicity in such models relates in particular to chemical reactions (Brusselator model [35]), neurosciences [2, 9, 14, 17, 20, 21, 27, 28, 33], and statistical physics (e.g. spin-flip models [13, 16], see also [12], where the model considered is in fact not mean-field, but the Ising model with dissipation). An example of particular interest concerns the FitzHugh–Nagumo model [2, 34] (take $d = 2$ and $F(x, y) = \left(x - \frac{x^3}{3} - y, \frac{1}{c}\,(x + a - by)\right)$ with chosen constants $a \in \mathbb{R}$ and $b, c > 0$), commonly used as a prototype for excitability in neuronal models [26] or in physics [3]. Roughly speaking, excitability refers to the ability for a neuron to emit spikes (oscillations) in the presence of perturbations (such as noise and/or external input) whereas this neuron would be at rest (steady state) without perturbation. The long-time dynamics of (1.1) in the FizHugh-Nagumo case has been the subject of several previous works (existence of equilibria [31, 33] or periodic solutions [27, 28]) under various asymptotics of the parameters (δ, K, σ). A crucial feature in this context is the influence of noise and interaction in the emergence and stability of periodic solutions: generically, some balance has to be found in the intensity of noise and interaction that one needs to put in the system in order to observe oscillations (see [26–28] for further details).

1.1.1 Stability Properties and Regular Isochron Map

The purpose of the present paper is to complement the previous results concerning the existence of periodic orbits for (1.1) with accurate stability properties for this periodic solution and with the existence of a sufficiently regular isochron map, properties that are absent in the previous works cited above. We obtain these additional properties by applying a result concerning normally hyperbolic invariant manifolds in Banach spaces proved by Bates, Lu and Zeng [5]. The technical counterpart is that we require assumptions on F and σ that are somehow stricter than the ones used in [27, 28, 33, 35], in the sense that we are considering a field F that is bounded together with all its derivatives (the analog term in the Brusselator and FitzHugh–Nagumo models grows polynomially) as well as nondegenerate noise on all components (while in [28, 33] the noise is only present in one of the two variables).

1.1.2 Large Time Asymptotics for the Mean-Field Particle System

Standard propagation of chaos results [37] show that (1.2) is the natural limit of the following mean-field particle system

$$dX_{i,t} = \delta F(X_{i,t})\,\mathrm{d}t - K\left(X_{i,t} - \frac{1}{N}\sum_{j=1}^{N} X_{j,t}\right)\mathrm{d}t + \sqrt{2}\sigma\,\mathrm{d}B_{i,t}, \tag{1.3}$$

in the sense that one can easily couple (1.3) and (1.2) by choosing the same realization of the noise, so that the resulting error is of order $\frac{1}{\sqrt{N}}$ as $N \to \infty$, at least on any $[0, T]$ with T that can be arbitrarily large but fixed independently from N. At the level of the whole particle system, this boils down to the convergence as $N \to \infty$ of the empirical measure $u_{N,t} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_{i,t}}$ to u_t, solution to (1.1). Hence, supposing that (1.1) has a periodic solution $\left(\Gamma_t^\delta\right)_{t\geq 0}$, if the empirical measure $u_{N,0}$ is initially close to $\Gamma^\delta_{\theta_0}$ for some initial phase θ_0, $u_{N,t}$ has, for N large, a behavior close to being periodic, since it stays close to $\Gamma^\delta_{\theta_0+t}$.

The companion paper [29] of the present work is concerned with the behavior of the empirical measure $u_{N,t}$ on a time scale T that is no longer bounded, but of order N. We show in [29] that $u_{N,Nt}$ is close to $\Gamma^\delta_{\theta_0+Nt+\beta_t^N}$, where β_t^N is a random process in $\mathbb{R}$ whose weak limit as $N \to \infty$ has constant drift and diffusion coefficient. This kind of result was already obtained in [8, 15] in the case of the plane rotators model (mean-field noisy interacting oscillators defined on the circle), for which at the scale Nt the empirical measure has a diffusive behavior along the curve of stationary points. Our aim in [29] is to get similar results for models like (1.1) that are defined in $\mathbb{R}^d$, and are not reversible (while the plane rotators model is). As we will explain in more detail later, the additional stability and regularity results concerning periodic solution to (1.1) obtained in the present paper are crucial for the study of long time behavior of the mean-field particle systems (1.3) made in [29].

1.2 Slow–Fast Viewpoint and Application to the FitzHugh–Nagumo Model

We give in this paragraph informal intuition on the possibility of emergence of periodic solutions to (1.1). The point of view we adopt here is a slow–fast approach, based on the assumption that the parameter δ in (1.1) is small, as it was already the case in [27, 28]. More precisely, the linear character of the interaction term in (1.1) allows us to decompose the dynamics of (1.1) into its expectation $m_t = \int_{\mathbb{R}^d} x u_t(x)$ and its centered version $p_t(x) = u_t(x - m_t)$: (1.1) is equivalent to the system

$$\begin{cases} \partial_t p_t = \mathcal{L} p_t - \nabla\cdot(p_t(\delta F_{m_t} - \dot{m}_t)) \\ \dot{m}_t = \delta \int_{\mathbb{R}^d} F_{m_t}\,\mathrm{d}p_t \end{cases}, \tag{1.4}$$

where

$$\mathcal{L}u = \nabla\cdot(\sigma^2 \nabla f) + \nabla\cdot(Kxf)\,, \tag{1.5}$$

and

$$F_m(x) := F(x+m). \tag{1.6}$$

Remark that (p_t, m_t) is the weak limit as $N \to \infty$ of the process $\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{Y_{i,t}}, m_{N,t}\right)$, where

$$m_{N,t} = \frac{1}{N}\sum_{i=1}^{N} X_{i,t}, \quad \text{and} \quad Y_{i,t} = X_{i,t} - m_{N,t}. \tag{1.7}$$

In this set-up, p_t is the fast variable, while m_t is the slow one. For $\delta = 0$, this system reduces to

$$\begin{cases} \partial_t p_t^0 = \mathcal{L} p_t^0 \\ m_t^0 = m_0 \end{cases}, \tag{1.8}$$

so $p_t^0 = e^{t\mathcal{L}} p_0$ is the distribution of an Ornstein-Uhlenbeck process, and thus converges exponentially fast to ρ, the density of the Gaussian distribution on $\mathbb{R}^d$ with mean 0 and variance $\sigma^2 K^{-1}$ (see Proposition 1.1 for more details on the contraction properties of $\mathcal{L}$):

$$\rho(x) := \frac{1}{\left((2\pi)^d \det(\sigma^2 K^{-1})\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2} x \cdot \left(\sigma^2 K^{-1}\right)^{-1} x\right), \; x \in \mathbb{R}^d. \tag{1.9}$$

So heuristically, taking δ small, in a first approximation p_t stays close to ρ while m_t satisfies

$$\dot{m}_t \approx \delta \int_{\mathbb{R}^d} F_{m_t}(x)\rho(x)\,\mathrm{d}x = \delta \int_{\mathbb{R}^d} F(x)\rho(m_t - x)\,\mathrm{d}x = \delta (F * \rho)(m_t). \tag{1.10}$$

For the non-centered PDE (1.1) this approximation means that u_t is close to a Gaussian distribution with variance $\sigma^2 K^{-1}$ and mean m_t, where the dynamics of m_t is governed at first order by (1.10). Following this heuristics, we expect a periodic behavior for the system (1.4) if the approximate dynamics of m_t is itself periodic. In this spirit, the main hypothesis we will adopt below is that the following equation

$$\dot{z}_t = \delta \int_{\mathbb{R}^d} F_{z_t}(x)\rho(x)\,\mathrm{d}x = \delta \left\langle F_{z_t}, \rho \right\rangle \tag{1.11}$$

admits a periodic solution $(\alpha_t^\delta)_{t \in [0, \frac{T_\alpha}{\delta}]}$, for some $T_\alpha > 0$, that we suppose to be stable (more details on the notion of stability we consider will be given in Sect. 1.4). In Proposition 1.7 we will show that under these hypotheses, the manifold $\widetilde{\mathcal{M}}^\delta = (\rho, \alpha_t^\delta)_{t\in[0,T_\alpha/\delta]}$ is approximately invariant for (1.4).

Let us now describe a situation where the above heuristics is true: in [27, 28] we considered the classical FitzHugh–Nagumo model defined by $d = 2$ and

$$F(x, y) = \left(x - \frac{x^3}{3} - y, \frac{1}{c}(x + a - by)\right). \tag{1.12}$$

A direct calculation shows that in that case, with $K = \mathrm{diag}(k_1, k_2)$ and $\sigma = \mathrm{diag}(\sigma_1, \sigma_2)$,

$$\int_{\mathbb{R}^d} F_{z_1,z_2}(x, y)\rho(x, y)\,\mathrm{d}x\ \mathrm{d}y = \left(\left(1 - \frac{\sigma_1^2}{k_1}\right) z_1 - \frac{z_1^3}{3} - z_2, \frac{1}{c}(z_1 + a - bz_2)\right), \tag{1.13}$$

which defines again a FitzHugh–Nagumo model. The additional factor $\frac{\sigma_1^2}{k_1}$ in (1.13) reflects the influence of noise and interaction in the mean-field system (1.2). For an accurate choice of parameters (take e.g. $a = \frac{1}{3}$, $b = 1$ and $c = 10$), it can be shown that the dynamics of the mean value (1.11) has a unique steady state when $\frac{\sigma_1^2}{k_1} = 0$ whereas it admits a stable periodic solution for $\frac{\sigma_1^2}{k_1}$ not too small and not too large, for example $\frac{\sigma_1^2}{k_1} = 0.2$. We refer to [27], § 3.4 for more details on the corresponding bifurcations). The purpose of [27, 28] was to show that the heuristics developed above is true, i.e. the periodicity of (1.11) propagates to (1.4). This emergence of periodic behavior induced by noise and interaction is a signature of excitability: the system (1.1) exhibits a periodic behavior induced by the combined effect of noise and interaction, which is not present in the isolated system $\dot{z}_t = F(z_t)$. We refer to [27] for a discussion and references on this phenomenon.

As already said, the point of this present work is to go beyond the existence of oscillations for (1.1), that is to prove regularity for the dynamics around such a limit cycle. Unfortunately the FitzHugh Nagumo model does not satisfy the hypotheses of this present work, since it has polynomial growth at infinity. However it is easy to see that if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth non-increasing function that satisfies $\psi(t) = 1$ for $t \leq 1$ and $\psi(t) = 0$ for $t \geq 2$, then for any $\varepsilon > 0$ the function $x \mapsto F(x)\psi(\varepsilon|x|)$ satisfies our hypotheses, and that $z \mapsto \int_{\mathbb{R}^d} F_z(x)\psi(\varepsilon|x+z|)\rho(x)dx$ converges to $z \mapsto \int_{\mathbb{R}^d} F_z(x)\rho(x)dx$ in $C^1(\mathcal{B}(0, R), \mathbb{R}^d)$ for any ball $\mathcal{B}(0, R)$ centered at 0 with radius R. So, relying on classical results on normally hyperbolic manifolds [18, 19, 40] (a definition of this notion will be provided in Sect. 1.4), if (1.11) admits a stable limit cycle, then it will also be the case replacing F with $x \mapsto F(x)\psi(\varepsilon|x|)$ for ε small enough.

1.3 Weighted Sobolev Norms

We present in this section the Sobolev spaces that we will use in the paper. Let us denote by $|x|_A = (x \cdot Ax)^{1/2}$ the Euclidean norm twisted by some positive matrix A, and, for any $\theta \in \mathbb{R}$, let us define the weight w_θ by

$$w_\theta(x) = \exp\left(-\frac{\theta}{2}\,|x|^2_{K\sigma^{-2}}\right). \tag{1.14}$$

Recall here that $K = \mathrm{diag}(k_1, \dots, k_d)$ and $\sigma = (\sigma_1, \dots, \sigma_d)$, with $k_i, \sigma_i > 0$ for all $i = 1, \dots, d$. Define in particular

$$k_{\min} := \min(k_1, \dots, k_d) \text{ and } k_{\max} := \max(k_1, \dots, k_d), \tag{1.15}$$

$$\sigma_{\min} := \min(\sigma_1, \dots, \sigma_d) \text{ and } \sigma_{\max} := \max(\sigma_1, \dots, \sigma_d). \tag{1.16}$$

We denote as L^2_θ the L^2-space with weight w_θ, that is with norm

$$\|h\|_{L^2_\theta} = \left(\int_{\mathbb{R}^d} |h(x)|^2\, w_\theta(x)\mathrm{d}x\right)^{\frac{1}{2}}. \tag{1.17}$$

For any $\theta > 0$ we consider the Ornstein-Uhlenbeck operator

$$\mathcal{L}^*_\theta f = \nabla \cdot (\sigma^2 \nabla f) - \theta K x \cdot \nabla f. \tag{1.18}$$

It is well know (see for example [1]) that $\mathcal{L}^*_\theta$ admits the following decomposition: for all $l \in \mathbb{N}^d$,

$$\mathcal{L}^*_\theta \psi_l = -\lambda_l \psi_l, \quad \text{with} \quad \lambda_l = \theta \sum_{i=1}^d k_i l_i \quad \text{and} \quad \psi_l(x) := \psi_{l,\theta}(x)$$
$$= \prod_{i=1}^d h_{l_i}\left(\sqrt{\frac{\theta k_i}{\sigma_i^2}} x_i\right), \tag{1.19}$$

where h_n is the n^{th} renormalized Hermite polynomial:

$$h_n(x) = \frac{(-1)^n}{\sqrt{n!}(2\pi)^{\frac{1}{4}}} e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left\{ e^{-\frac{x^2}{2}} \right\}. \tag{1.20}$$

The family $(\psi_{l,\theta})_{l\in\mathbb{N}^d}$ is an orthonormal basis of L^2_θ. For f, g with decompositions $f = \sum_{l\in\mathbb{N}^d} f_l \psi_l$ and $g = \sum_{l\in\mathbb{N}^d} g_l \psi_l$, we consider the scalar products

$$\langle f, g\rangle_{H^r_\theta} = \left\langle (a_\theta - \mathcal{L}^*_\theta)^r f, \bar{g} \right\rangle_{L^2_\theta} = \sum_{l\in\mathbb{N}^d} (a_\theta + \lambda_l)^r f_l \bar{g}_l, \tag{1.21}$$

where $a_\theta = \theta\,\mathrm{Tr}K$ and denote by H^r_θ the completion of the space of smooth function u satisfying $\|u\|_{H^r_\theta} < \infty$. The choice of the constant a_θ is made to simplify some technical proofs given in the "Appendix 1" (see the proof of Proposition A.2). Another choice of positive constant would produce an equivalent norm. From Lemma A.1 it is clear that $\left\|\partial_{x_i} f\right\|_{H^r_\theta} \leq \|f\|_{H^{r+1}_\theta}$, and that, if $n \in \mathbb{N}$, the norm $\|f\|_{H^n_\theta}$ is in fact equivalent to

$$\sqrt{\sum_{l\in\mathbb{N}^d,\, \sum_{i=1}^d l_i \leq n} \left\| \partial_{x_1}^{l_1} \dots \partial_{x_d}^{l_d} f \right\|^2_{L^2_\theta}}. \tag{1.22}$$

We denote by H^{-r}_θ the dual of H^r_θ. Relying on a "pivot" space structure (for more details, see "Appendix 1"), the product $\langle u, f\rangle_{H^{-r}_\theta, H^r_\theta}$ can be identified with the flat L^2 product $\langle u, f\rangle$: $L^2_{-\theta}$ can be seen as a subset of H^{-r}_θ, and for all $f \in H^r_\theta$ and $u \in L^2_{-\theta}$ we have

$$\langle u, f\rangle_{H^{-r}_\theta, H^r_\theta} = \langle u, f\rangle. \tag{1.23}$$

This identification allows us to view the operator $\mathcal{L}_\theta$ defined by

$$\mathcal{L}_\theta u = \nabla\cdot(\sigma^2\nabla f) + \nabla\cdot(\theta K x f), \tag{1.24}$$

seen as an operator in H^{-r}_θ, as the adjoint of $\mathcal{L}^*_\theta$, seen as an operator in H^r_θ. This is in particular the case for $\mathcal{L} = \mathcal{L}_1$, whose contraction properties will be crucial in the results given in this paper.

Our aim in this paper is to give the existence of a periodic solution for (1.4) viewing p_t as an element of H^{-r}_θ. The necessity of considering H^{-r}_θ instead of simply taking H^{-r}_1 goes back to the companion paper [29], in which we study the long time behavior of the empirical measure $u_{N,t}$ in the same functional space. Since this empirical measure involves a sum of Dirac distributions, it can be seen as an element of H^{-r}_θ for $r > d/2$, and we have $\|\delta_x\|_{H^{-r}_\theta} \leq C w_{\frac{\theta}{4-\eta}}(x)$ for $\eta > 0$ (see Lemma 2.1 in [29]). Some moment estimates, obtained in [29], lead us to bound terms of the form $\mathbb{E}\left[w_{\frac{m\theta}{4-\eta}}(Y_{i,t})\right]$ with m large and $Y_{i,t}$ defined in (1.7). Since we consider cases where $Y_{i,t}$ has a distribution close to ρ given by (1.9), for this expectation to be bounded we need to consider small values of θ. We need therefore to work in H^{-r}_θ for general θ and not only for $\theta = 1$.

Due to the spectral decomposition (1.19), it is well known (see for example [23]) that the semi-group $e^{t\mathcal{L}}$ satisfies, for $\lambda < k_{\min}$ (recall (1.15)) and $u \in H^{-r}_1$ with $\int u = 0$, the contraction property

$$\left\|e^{t\mathcal{L}} u\right\|_{H^{-r}_1} \leq C t^{-\frac{\alpha}{2}} e^{-t\lambda} \|u\|_{H_1^{-(r+\alpha)}}. \tag{1.25}$$

By obtaining similar estimates (see the following Proposition, which is a particular case of the slightly more general Proposition A.3), we will be able to work in the space H^{-r}_θ with any value of θ smaller than 1, but with the constraint of considering values of r larger than a $r_0 > 0$ (independent of θ).

Proposition 1.1 *For all $0 < \theta \leq 1$ the operator $\mathcal{L}$ is sectorial and generates an analytical semi-group in H_θ^{-r}. Moreover we have the following estimates: for any $\alpha \geq 0$, $r \geq 0$ and $\lambda < k_{\min}$ there exists a constant $C_{\mathcal{L}} > 0$ such that for all $u \in H_\theta^{-(r+\alpha)}$,*

$$\left\| e^{t\mathcal{L}} u \right\|_{H_\theta^{-r}} \leq C_{\mathcal{L}} \left(1 + t^{-\alpha/2} e^{-\lambda t}\right) \|u\|_{H_\theta^{-(r+\alpha)}}, \tag{1.26}$$

and for $r \geq 1$,

$$\left\| e^{t\mathcal{L}} \nabla u \right\|_{H_\theta^{-r}} \leq C_{\mathcal{L}} t^{-\frac{1}{2}} e^{-\lambda t} \|u\|_{H_\theta^{-r}}. \tag{1.27}$$

Moreover for all $r \geq 0$, $0 < \varepsilon \leq 1$ and $s \geq 0$,

$$\left\| \left(e^{(t+s)\mathcal{L}} - e^{t\mathcal{L}} \right) u \right\|_{H_\theta^{-r}} \leq C_{\mathcal{L}} s^{\varepsilon} t^{-\frac{1}{2}-\varepsilon} e^{-\lambda t} \|u\|_{H_\theta^{-(r+1)}}. \tag{1.28}$$

Finally, there exists $r_0 > 0$ such that for any $0 < \theta \leq 1$, for all $r > r_0$, $t > 0$ and all $u \in H_\theta^{-r}$ satisfying $\int u = 0$,

$$\left\| e^{t\mathcal{L}} u \right\|_{H_\theta^{-r}} \leq C_{\mathcal{L}} e^{-\lambda t} \|u\|_{H_\theta^{-r}}. \tag{1.29}$$

1.4 Main Results

With the notation $\mu_t := (p_t, m_t)$ the system (1.4) becomes

$$\begin{cases} \partial_t p_t = \mathcal{L} p_t + \delta G_1(\mu_t) \\ \dot{m}_t = \delta G_2(\mu_t) \end{cases}, \tag{1.30}$$

where

$$G(\mu) = G(p, m) = \begin{pmatrix} G_1(p, m) \\ G_2(p, m) \end{pmatrix} = \begin{pmatrix} -\nabla \cdot \left(p \left(F_m - \int F_m p\right)\right) \\ \int F_m p \end{pmatrix}. \tag{1.31}$$

We place ourselves on the space $\mathbf{H}_\theta^r := H_\theta^r \times \mathbb{R}^d$ endowed with the scalar product

$$\left\langle (f, m), (g, m') \right\rangle_{\mathbf{H}_\theta^r} := \langle f, g \rangle_{H_\theta^r} + m \cdot m'. \tag{1.32}$$

We will denote $\mathbf{H}_\theta^{-r}$ the dual of $\mathbf{H}_\theta^r$. Clearly $\mathbf{H}_\theta^{-r} = H_\theta^{-r} \times \mathbb{R}$ and, relying as above on a "pivot" space structure, the product $\langle (\nu, h), (\phi, \psi) \rangle_{\mathbf{H}_\theta^{-r}, \mathbf{H}_\theta^r}$ can be identified with the flat scalar product

$$\left\langle\!\left\langle (\nu, h); (\varphi, \psi) \right\rangle\!\right\rangle = \langle \nu, \varphi \rangle + h \cdot \psi. \tag{1.33}$$

The following theorem states the existence and uniqueness of mild solutions of (1.30). Its proof, given in Sect. 2, relies on classical arguments, due to the fact that $G : \mathbf{H}_\theta^{-r+1} \to \mathbf{H}_\theta^{-r}$ is locally Lispchitz and $\mathcal{L}$ is sectorial (see [36]).

Theorem 1.2 *For any initial condition $\mu = (p, m) \in \mathbf{H}_\theta^{-r}$ with $\int_{\mathbb{R}^d} p = 1$ there exists a unique maximal mild solution $\mu_t := (p_t, m_t) = T^t(\mu)$ to (1.30) on $[0, t_c]$ for some $t_c > 0$, which satisfies $t \mapsto T^t(\mu) \in \mathcal{C}\left([0, t_c) ; \mathbf{H}_\theta^{-r}\right)$.*

Moreover, $\mu \mapsto T^t(\mu)$ is C^2, and for any $R > 0$, there exists a $\delta(R) > 0$ such that for all $0 \le \delta \le \delta(R)$ and $\mu_0 = (p_0, m_0)$ satisfying $\|p_0 - \rho\|_{H_\theta^{-r}} \le R$ the solution $T^t(\mu_0)$ is well defined for all $t \ge 0$ and there exists a $C(R) > 0$ such that

$$\sup_{t\ge0} \|p_t\|_{H_\theta^{-r}} \le C(R). \tag{1.34}$$

Remark 1.3 Since we are interested in the existence of a periodic solution made of probability distributions, we will only consider initial conditions (p_0, m_0) satisfying $\int_{\mathbb{R}^d} p_0 = 1$, and the conservation of mass will induce that $\int_{\mathbb{R}^d} p_t = 1$ for all t. In the same spirit, we will only apply the differential of the semi-group $DT^t(\mu)$ to elements $\nu = (\eta, n) \in \mathbf{H}_\theta^{-r}$ that satisfy $\int_{\mathbb{R}^d} \eta = 0$.

As it was previously mentioned, we suppose in the following that the ordinary differential equation (1.11) admits a stable periodic solution $(\alpha_t^\delta)_{t\in[0,\frac{T_\alpha}{\delta}]}$. To state more precisely this hypothesis we rely on Floquet formalism (see for example [38]): let us denote by $\pi_{u+t,u}^\delta$ the principal matrix solution associated to the periodic solution α^δ, that is the solution to

$$\partial_t \pi_{u+t,u}^\delta = \delta\langle DF_{\alpha_{u+t}^\delta}, \rho\rangle \pi_{u+t,u}^\delta, \ \pi_{u,u}^\delta = I. \tag{1.35}$$

The process $\pi_{u+t,u}^\delta$ characterizes the linearized dynamics around $(\alpha_t^\delta)_{t\in[0,\frac{T_\alpha}{\delta}]}$: more precisely it corresponds to the differentiation of the flow of (1.11) with respect to the initial condition, at time t and initial point α_u^δ. We will suppose that this linearized dynamics is a contraction on a supplementary space of the tangent space to $(\alpha_t^\delta)_{t\in[0,\frac{T}{\delta}]}$. More precisely, the stability of the periodic solution $(\alpha_t^\delta)_{t\in[0,\frac{T_\alpha}{\delta}]}$ is expressed by the following hypothesis: we suppose that there exist projections $P_u^{\delta,c}$ and $P_u^{\delta,s}$ for all $u \in \mathbb{R}$ with $u \mapsto P_u^{\delta,c}$ and $u \mapsto P_u^{\delta,s}$ smooth and $\frac{T_\alpha}{\delta}$-periodic, that satisfy $P_u^{\delta,s} + P_u^{\delta,c} = I$ ($P_u^{\delta,c}$ being a projection on $\mathrm{vect}(\dot\alpha_u^\delta)$), that commute with π^δ, i.e.

$$P_u^{\delta,s} + P_u^{\delta,c} = I, \quad P_{u+t}^{\delta,s}\pi_{u+t,u}^\delta = \pi_{u+t,u}^\delta P_u^{\delta,s}, \tag{1.36}$$

and such that there exist positive constants c_α, C_α and λ_α such that for any $n \in \mathbb{R}^d$

$$\left|\pi_{u+t,u}^\delta P_u^{\delta,s} n\right| \le C_\alpha e^{-\delta\lambda_\alpha t}|n| \quad \text{and} \quad c_\alpha|n| \le \left|\pi_{u+t,u}^\delta P_u^{\delta,c} n\right| \le C_\alpha|n|. \tag{1.37}$$

For more details on the construction of these projections, see [38, Section 3.6] or [28, Section 3]. Remark that the factor δ in (1.11) is responsible for a change of time-scale for the dynamics, and induces the factor δ in the rate of contraction in (1.37) (the smaller δ, the slower the dynamics, the period being then T_α/δ since $\alpha_t^\delta = \alpha_{\delta t}^1$). The effect of this factor on the projections is only a change of parametrization: $P_u^{\delta,s}$ and $P_u^{\delta,c}$ are defined on $[0, T_\alpha/\delta)$, and $P_{u/\delta}^{\delta,s} = P_u^{1,s}$, $P_{u/\delta}^{\delta,c} = P_u^{1,c}$ for $u \in [0, T_\alpha)$.

With these hypotheses $(\alpha_t^\delta)_{t\in[0,\frac{T}{\delta}]}$ is in fact a simple example of Normally Hyperbolic Invariant Manifold (NHIM). We follow here the definition given in [4] for this concept: on a Banach space $\mathbf{X}$, a smooth compact connected manifold $\mathbf{M}$ is said to be a normally hyperbolic invariant manifold for a continuous semi flow $\mathbf{T}$ (such that $u \mapsto \mathbf{T}^t(\mu)$ is C^1 for all $t \ge 0$) if

(1) $\mathbf{T}(\mathbf{M}) \subset \mathbf{M}$ for all $t \ge 0$,
(2) For each $m \in \mathbf{M}$ there exists a decomposition $\mathbf{X} = \mathbf{X}_m^c + \mathbf{X}_m^u + \mathbf{X}_m^s$ of closed subspaces with $\mathbf{X}_m^c$ the tangent space to $\mathbf{M}$ at m,
(3) For each $m \in \mathbf{M}$ and $t \ge 0$, denoting $m_1 = \mathbf{T}^t(m)$, we have $D\mathbf{T}^t(m)_{|\mathbf{X}_m^\iota} : \mathbf{X}_m^\iota \to \mathbf{X}_{m_1}^\iota$ for $\iota = c, u, s$, and $D\mathbf{T}^t(m)_{|\mathbf{X}_m^u}$ is an isomorphism from $\mathbf{X}_m^u$ to $\mathbf{X}_{m_1}^u$.

(4) There exists a $t_0 \geq 0$ and a $\lambda > 0$ such that, for all $t \geq t_0$,

$$\lambda \inf\{|D\mathbf{T}^t(m)[x^u]| : x^u \in \mathbf{X}^u,\ |x^u| = 1\} > \max\left\{1, \left\| D\mathbf{T}^t(m)_{|\mathbf{X}^c_m} \right\|\right\}, \quad (1.38)$$

$$\lambda \min\{1, \inf |D\mathbf{T}^t(m)[x^c]| : x^c \in \mathbf{X}^c_m,\ |x^c| = 1\}\} > \left\| D\mathbf{T}^t(m)_{|\mathbf{X}^s_m} \right\|. \quad (1.39)$$

The inequality (1.38) implies that the semi flow $\mathbf{T}^t$ is expansive at m in the direction $\mathbf{X}^u_m$ at a rate strictly larger than on $\mathbf{M}$, while (1.39) shows implies that it is contractive at m in the direction $\mathbf{X}^s_m$ at a rate greater than on $\mathbf{M}$.

This kind of structure is known to be robust under perturbation of the semi-flow: it has been shown in [18, 19] for flows in $\mathbb{R}^d$, and then generalized in [24] in the case of Riemannian manifolds and in [4, 36] in the infinite dimensional setting. An improvement of these classical results has been obtained in [5] by Bates, Lu and Zeng, who showed that if a system admits a manifold that is approximately invariant and approximately normally hyperbolic (a precise definition of these notions will be given in Sect. 1.5), then the system possesses an actual normally hyperbolic invariant manifold in a neighborhood of the approximately invariant one.

We will rely on this deep result in our work. Here, the slow–fast viewpoint described in Sect. 1.2 suggests that for δ small the manifold (recall the definition of ρ in (1.9) and that (α_t) is a T_α-periodic solution to (1.11))

$$\widetilde{\mathcal{M}}^\delta := \{(\rho, \alpha_t) : t \in [0, T_\alpha)\} \quad (1.40)$$

is an approximately invariant manifold which is approximately normally hyperbolic (without unstable direction). This statement will be written rigorously in Sect. 1.5, and proved in Sect. 3. This idea will allow us to prove for δ small enough the existence of a stable periodic solution to (1.4), as an actual normally hyperbolic invariant manifold in a neighborhood of $\widetilde{\mathcal{M}}^\delta$. For a stable periodic solution, conditions (1.38) and (1.39) reduce to the fact that $DT^t(m)$ is bounded from above and below in the direction of the tangent space to the invariant manifold defined by the periodic solution, and is contractive on a stable direction.

Theorem 1.4 *There exists $\delta_0 > 0$ such that for r_0 given in Proposition 1.1 and for all $r \geq r_0$, $\delta \in (0, \delta_0)$ and $\theta \in (0, 1]$ the system (1.4) admits a periodic solution*

$$\left(\Gamma^\delta_t\right)_{t\in[0,T_\delta]} := (q^\delta_t, \gamma^\delta_t)_{t\in[0,T_\delta]} \quad (1.41)$$

in $\mathbf{H}^{-r}_\theta$ with period $T_\delta > 0$. Moreover q^δ_t is a probability distribution for all $t \geq 0$, and $t \mapsto \partial_t \Gamma^\delta_t$ and $t \mapsto \partial^2_t \Gamma^\delta_t$ are in $C([0, T_\delta), \mathbf{H}^{-r}_\theta)$.

Denoting

$$\mathcal{M}^\delta := \{\Gamma^\delta_t : t \in [0, T_\delta)\} \quad (1.42)$$

and

$$\Phi_{u+s,u}(\nu) = DT^s(\Gamma^\delta_u)[\nu] \quad (1.43)$$

there exist families of projections $\Pi^{\delta,c}_u$ and $\Pi^{\delta,s}_t$ that commute with Φ, i.e. that satisfy

$$\Pi^{\delta,\iota}_{u+t}\Phi_{u+t,u} = \Phi_{u+t,u}\Pi^{\delta,\iota}_u, \quad \text{for } \iota = c, s. \quad (1.44)$$

Moreover $\Pi^{\delta,c}_t$ is a projection on the tangent space to $\mathcal{M}^\delta$ at Γ^δ_t, $\Pi^{\delta,c}_t + \Pi^{\delta,s}_t = I_d$, $t \mapsto \Pi^{\delta,c}_t \in C^1([0, T_\delta), \mathcal{B}(\mathbf{H}^{-r}_\theta))$, and there exist positive constants $c_{\Phi,\delta}$, $C_{\Phi,\delta}$ and λ_δ such that

$$c_{\Phi,\delta} \left\| \Pi^{\delta,c}_u(\nu) \right\|_{\mathbf{H}^{-r}_\theta} \leq \left\| \Phi_{u+t,u}\Pi^{\delta,c}_u(\nu) \right\|_{\mathbf{H}^{-r}_\theta} \leq C_{\Phi,\delta} \left\| \Pi^{\delta,c}_u(\nu) \right\|_{\mathbf{H}^{-r}_\theta}, \quad (1.45)$$

$$\left\| \Phi_{u+t,u}\Pi^{\delta,s}_u(\nu) \right\|_{\mathbf{H}^{-r}_\theta} \leq C_{\Phi,\delta}\, t^{-\frac{\alpha}{2}} e^{-\lambda_\delta t} \left\| \Pi^{\delta,s}_u(\nu) \right\|_{\mathbf{H}^{-(r+\alpha)}_\theta}, \quad (1.46)$$

and

$$\left\| \Phi_{u+t,u} \nu \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{\Phi,\delta} \left(1 + t^{-\frac{\alpha}{2}} e^{-\lambda_\delta t} \right) \|\nu\|_{\mathbf{H}_\theta^{-(r+\alpha)}} . \tag{1.47}$$

Remark 1.5 The invariant manifold $\mathcal{M}^\delta$ is located at a distance of order δ from the approximately invariant manifold $\widetilde{\mathcal{M}}^\delta$ given in (1.40) and the period T_δ is close to T_α/δ (the period of the slow system (1.11)). Moreover λ_δ is of order δ due to the fact that z_t contracts around α_t with rate $\delta\lambda_\alpha$ (recall (1.37)).

In [5] it is in addition proven that the stable manifold of the actual NHIM (in our case $\mathcal{M}^\delta$ is attractive, the stable manifold is in fact a neighborhood $\mathcal{W}^\delta$ of $\mathcal{M}^\delta$) is foliated by invariant foliations: $\mathcal{W}^\delta = \cup_{m \in \mathcal{M}^\delta} \mathcal{W}_m^\delta$, where $\nu \in \mathcal{W}_m^\delta$ if and only if $T^t(\nu) - T^t(m)$ converges to 0 exponentially fast. This implies the existence of an isochron map $\Theta^\delta : \mathcal{W}^\delta \to \mathbb{R}/T_\delta\mathbb{Z}$ that satisfies $\Theta^\delta(\nu) = t$ if $\nu \in \mathcal{W}^\delta_{\Gamma_t^\delta}$. The deep general result of [5] ensures that Θ^δ is Hölder continuous, which is not entirely satisfactory in view of the companion paper [29], in which we aim to apply Itô's Lemma to $\Theta^\delta(u_{N,t})$. However, the fact that in the present case we simply deal with a stable periodic solution allow us to prove that Θ^δ has in our particular case C^2 regularity, as stated in the following theorem.

Theorem 1.6 *Recall the definitions of the flow T^t associated to (1.30) in Theorem 1.2 and of the manifold $\mathcal{M}^\delta$ in Theorem 1.4. For r and δ as in Theorem 1.4, there exists a neighborhood $\mathcal{W}^\delta \in \mathbf{H}_\theta^{-r}$ of $\mathcal{M}^\delta$ and a C^2 mapping $\Theta^\delta : \mathcal{W}^\delta \to \mathbb{R}/T_\delta\mathbb{Z}$ that satisfies, for all $\mu \in \mathcal{W}^\delta$, denoting $\mu_t = T^t\mu$,*

$$\Theta^\delta(\mu_t) = \Theta^\delta(\mu) + t \quad mod\ T_\delta, \tag{1.48}$$

and there exists a positive constant $C_{\Theta,\delta}$ such that, for all $\mu \in \mathcal{W}^\delta$ with $\mu_t = T^t\mu$,

$$\left\| \mu_t - \Gamma^\delta_{\Theta^\delta(\mu)+t} \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{\Theta,\delta} e^{-\lambda_\delta t} \left\| \mu - \Gamma^\delta_{\Theta^\delta(\mu)} \right\|_{\mathbf{H}_\theta^{-r}} . \tag{1.49}$$

Moreover Θ^δ satisfies, for all $\mu \in \mathcal{W}^\delta$,

$$\left\| D^2\Theta^\delta(\mu) - D^2\Theta^\delta\left(\Gamma^\delta_{\Theta^\delta(\mu)}\right) \right\|_{\mathcal{BL}(\mathbf{H}_\theta^{-r})} \leq C_{\Theta,\delta} \left\| \mu - \Gamma^\delta_{\Theta^\delta(\mu)} \right\|_{\mathbf{H}_\theta^{-r}} , \tag{1.50}$$

where $\mathcal{BL}(\mathbf{H}_\theta^{-r})$ denotes the space of bounded operators $\mathcal{A} : \mathbf{H}_\theta^{-r} \to \mathbf{H}_\theta^{-r}$.

1.5 An Approximately Invariant Manifold that is Approximately Normally Hyperbolic

In view of the slow–fast formalism described in Sect. 1.2, our aim is to view $\widetilde{\mathcal{M}}^\delta$ given by (1.40) as an approximately invariant and approximately normally hyperbolic manifold, in the sense of [5].

In fact the result of [5] is stated for dynamical systems taking values in a Banach space, while we will consider here solutions (p_t, m_t) to (1.4) elements of $\mathbf{H}_\theta^{-r}$ that satisfy $\int_{\mathbb{R}^d} p_t = 1$ (since we are interested in probability distributions, recall Remark 1.3), so we will rather consider an affine space. It will not pose any problem, since $(p_t - \rho, m_t)$ is an element of $\left\{(v, m) \in \mathbf{H}_\theta^{-r} : \int_{\mathbb{R}^d} v = 0\right\}$ which is a Banach space.

Following the notations of [5] we set (recall (1.9) and (1.11))

$$\psi(t) := (\rho, \alpha_t^\delta),\ t \in \mathbb{R}/\frac{T}{\delta}\mathbb{Z}. \tag{1.51}$$

With this notation we have $\widetilde{\mathcal{M}}^\delta = \psi\left(\mathbb{R}/\frac{T}{\delta}\mathbb{Z}\right)$ (recall its definition in (1.40)). We will consider the projections $\widetilde{\Pi}_u^{\delta,s}$ and $\widetilde{\Pi}_u^{\delta,c}$ defined for $(p,m) \in \mathbf{H}_\theta^{-r}$ by

$$\widetilde{\Pi}_u^{\delta,s}(p,m) = (p, P_u^{\delta,s}m), \qquad \widetilde{\Pi}_u^{\delta,c}(p,m) = (0, P_u^{\delta,c}m), \tag{1.52}$$

where $P_t^{\delta,s}$ and $P_t^{\delta,c}$ are the projections defined in Sect. 1.4. The subspaces $\widetilde{\mathbf{X}}_u^{\delta,c} = \widetilde{\Pi}_u^{\delta,c}(\mathbf{H}_\theta^{-r})$ and $\widetilde{\mathbf{X}}_u^{\delta,s} = \widetilde{\Pi}_u^{\delta,s}(\mathbf{H}_\theta^{-r})$ will correspond to the approximately tangent space and stable space of $\widetilde{\mathcal{M}}^\delta$. It is clear that for each $t \in [0, \frac{T}{\delta})$ we have

$$\mathbf{H}_\theta^{-r} = \widetilde{\mathbf{X}}_t^{\delta,c} \oplus \widetilde{\mathbf{X}}_t^{\delta,s}. \tag{1.53}$$

Consider τ such that

$$e^{-\lambda_\alpha \tau} \leq \frac{c_\alpha}{8C_\alpha}, \tag{1.54}$$

where $c_\alpha, C_\alpha, \lambda_\alpha$ are given by (1.46). The following proposition states that $\widetilde{\mathcal{M}}^\delta$ satisfies the hypotheses given in [5], making it an approximately invariant and approximately normally hyperbolic manifold.

Proposition 1.7 *Recall the definition of the flow T^t of (1.30) in Theorem 1.2. There exists $\delta_0 > 0$ such that for r_0 given in Proposition 1.1 and for all $r \geq r_0$, $\delta \in (0,\delta_0)$ and $\theta \in (0,1)$, the following assertions are true.*

(1) (Definition 2.1. in [5]) *There exists a positive constant κ_1 such that for all $u \in \mathbb{R}/\frac{T}{\delta}\mathbb{Z}$,*

$$\left\| T^{\frac{\tau}{\delta}}(\rho,\alpha_u) - (\rho, \alpha_{u+\frac{\tau}{\delta}}) \right\|_{\mathbf{H}_\theta^{-r}} \leq \kappa_1 \delta. \tag{1.55}$$

(2) (Hypothesis (H2) in [5]) *There exist positive constants $\kappa_2, \kappa_3, \kappa_4$ such that for all $s, t \in \mathbb{R}/\frac{T}{\delta}\mathbb{Z}$ such that $|s-u| \leq 1$, $|t-u| \leq 1$, and $\iota = s, c$,*

$$\left\| \widetilde{\Pi}_u^{\delta,\iota} \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})} \leq \kappa_2, \qquad \left\| \widetilde{\Pi}_u^{\delta,\iota} - \widetilde{\Pi}_s^{\delta,\iota} \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})} \leq \kappa_3 \left\| \psi(t) - \psi(s) \right\|_{\mathbf{H}_\theta^{-r}}, \tag{1.56}$$

and

$$\frac{\left\| \psi(t) - \psi(s) - \widetilde{\Pi}_s^{\delta,c}(\psi(t) - \psi(s)) \right\|_{\mathbf{H}_\theta^{-r}}}{\left\| \psi(t) - \psi(s) \right\|_{\mathbf{H}_\theta^{-r}}} \leq \kappa_4 \delta. \tag{1.57}$$

(4) (Hypothesis H3 in [5]) *There exists a positive constant κ_5 such that for all $u \in \mathbb{R}/\frac{T}{\delta}\mathbb{Z}$,*

$$\max \left\{ \left\| \widetilde{\Pi}_{u+\frac{\tau}{\delta}}^{\delta,c} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)_{|\widetilde{\mathbf{X}}_u^{\delta,s}} \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})}, \left\| \widetilde{\Pi}_{u+\frac{\tau}{\delta}}^{\delta,s} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)_{|\widetilde{\mathbf{X}}_u^{\delta,c}} \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})} \right\} \leq \kappa_5 \delta. \tag{1.58}$$

(5) (Hypothesis H3' and C3 in [5]) *There exist $a \in (0,1)$ and $\widetilde{\lambda} > 0$ such that for all $u \in \mathbb{R}/\frac{T}{\delta}\mathbb{Z}$,*

$$\left\| \left(\widetilde{\Pi}_{u+\frac{\tau}{\delta}}^{\delta,c} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)_{|\widetilde{\mathbf{X}}_u^{\delta,c}} \right)^{-1} \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})}^{-1} > a, \tag{1.59}$$

and

$$\left\| \widetilde{\Pi}_{u+\frac{\tau}{\delta}}^{\delta,s} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)_{|\widetilde{\mathbf{X}}_u^{\delta,s}} \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})} \leq \widetilde{\lambda} \min \left(1, \left\| \left(\widetilde{\Pi}_{u+\frac{\tau}{\delta}}^{\delta,c} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)_{|\widetilde{\mathbf{X}}_u^{\delta,c}} \right)^{-1} \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})}^{-1} \right), \tag{1.60}$$

(6) (Hypothesis H4 in [5]) *There exist positive constants* κ_6 *and* κ_7 *such that*

$$\left\| DT^{\frac{\tau}{\delta}}|_{\mathcal{V}(\widetilde{\mathcal{M}}^{\delta},1)} \right\|_{\mathcal{B}(\mathbf{H}_{\theta}^{-r})} \leq \kappa_6, \quad \left\| D^2T^{\frac{\tau}{\delta}}|_{\mathcal{V}(\widetilde{\mathcal{M}}^{\delta},1)} \right\|_{\mathcal{BL}\left((\mathbf{H}_{\theta}^{-r})^2,\mathbf{H}_{\theta}^{-r}\right)} \leq \kappa_7, \quad (1.61)$$

where $\mathcal{V}(\widetilde{\mathcal{M}}^{\delta}, R_0)$ *denote the* R_0*-neighborhood of* $\widetilde{\mathcal{M}}^{\delta}$.

(7) (Hypothesis H5 in [5]) *For any* $\varepsilon > 0$ *there exists* $\zeta > 0$ *such that for all* $\mu = (p, m) \in \mathcal{V}(\widetilde{\mathcal{M}}^{\delta}, 1)$ *and* $t \in [\frac{\tau}{\delta}, \frac{\tau}{\delta} + \zeta]$,

$$\left\| T^t(\mu) - T^{\frac{\tau}{\delta}}(\mu) \right\|_{\mathbf{H}_{\theta}^{-r}} \leq \varepsilon. \quad (1.62)$$

The first five items of Proposition 1.7 focus on properties of the semi-group $\left(T^{n\frac{\tau}{\delta}}\right)_{n\geq 0}$ discretized in time, showing that $\widetilde{\mathcal{M}}^{\delta}$ given by (1.40) is an approximately invariant manifold approximately normally hyperbolic for this semi-group, while the last item is an uniform in time bound that implies that this property is also true for the semi-group $\left(T^t\right)_{t\geq 0}$. More precisely (1) shows that $\widetilde{\mathcal{M}}^{\delta}$ is approximately invariant for the discrete semi-group, (2) shows that $\widetilde{\mathbf{X}}_u^{\delta,c}$ is an approximation of the tangent space to $\widetilde{\mathcal{M}}^{\delta}$ at (ρ, α_u) and that ψ does not twist too much, (3) implies that $\widetilde{\mathbf{X}}^{\delta,c}$ and $\widetilde{\mathbf{X}}^{\delta,s}$ are approximately invariant under $\left(DT^{n\frac{\tau}{\delta}}\right)_{n\geq 0}$, and (4) implies that $\left(DT^{n\frac{\tau}{\delta}}\right)_{n\geq 0}$ contracts more in the direction $\widetilde{\mathbf{X}}^{\delta,s}$ than in the direction $\widetilde{\mathbf{X}}^{\delta,c}$, while it does not contract too much in the direction $\widetilde{\mathbf{X}}^{\delta,c}$. (5) is a technical assumption useful in their proof.

Remark that we do not quote the hypothesis (H1) of [5] in this Proposition, since it is simply (1.53). Moreover in [5] the authors treat first the inflowing invariant case, and then the overflowing invariant case, while we are here interested in an actual invariant manifold (both inflowing and overflowing), which is why we mix hypotheses (Hi) and $(C3)$, as it is done in Theorem 6.5 of [5].

1.6 Structure of the Paper

The proof of Theorem 1.2 concerning the well-posedness of (1.4) is carried out in Sect. 2. Proposition 1.7 is proven in Sect. 3. The main result of existence of periodic solutions (Theorem 1.4) is proven in Sect. 4. The question of regularity of the isochron is addressed in Sect. 5. The "Appendix 1" gathers technical estimates on the Ornstein-Uhlenbeck operator and some Grönwall type lemmas are listed in "Appendix 1".

2 Proof of Theorem 1.2

We give in this section the existence, uniqueness and regularity result of Theorem 1.2. We rely here on classical arguments one can find for example in [36] or [23].

Proof of Theorem 1.2 Recall the definitions of G in (1.31), of the space $\mathbf{H}_{\theta}^{-r}$ in (1.32) and of F_m in (1.6). We first remark that $G : \mathbf{H}_{\theta}^{-r} \to \mathbf{H}_{\theta}^{-(r+1)}$ is locally Lispchitz. Indeed, for any $(p, m) \in \mathbf{H}_{\theta}^{-r}$ and any $(\varphi, \psi) \in \mathbf{H}_{\theta}^{r+1}$,

$$\langle\!\langle G(p, m), (\varphi, \psi) \rangle\!\rangle = -\left\langle p\left(F_m - \int F_m p\right), \nabla\varphi \right\rangle + \psi \cdot \int F_m p. \quad (2.1)$$

We have $\left|\int F_m p\right| \leq \|F_m\|_{H^r_\theta} \|p\|_{\mathbf{H}^{-r}_\theta}$, and due to the fact that all derivatives of F are bounded, $\|F_m\|_{H^r_\theta} \leq C_F$ independently on m. Moreover, due to the same reason, we have $\|F_m \cdot \nabla\varphi\|_{H^r_\theta} \leq C_F \|\nabla\varphi\|_{H^r_\theta}$ independently on m. This means that

$$\left|\left\langle\!\left\langle G(p,m), (\varphi, \psi)\right\rangle\!\right\rangle\right| \leq C\|p\|_{\mathbf{H}^{-r}_\theta} \left(1 + \|p\|_{\mathbf{H}^{-r}_\theta}\right) \|\varphi\|_{H^{r+1}_\theta} + C\|p\|_{\mathbf{H}^{-r}_\theta} |\psi|. \tag{2.2}$$

We deduce $\|G(\nu)\|_{\mathbf{H}^{-(\mathbf{r}+1)}} \leq C\|\nu\|_{\mathbf{H}^{-\mathbf{r}}} \left(1 + \|\nu\|_{\mathbf{H}^{-\mathbf{r}}}\right)$, and thus that G is locally Lipschitz.

Remark that when p is a probability distribution $\left|\int F_m p\right| \leq C_F \left|\int p\right| \leq C_F$, and in this case G is in fact globally Lipschitz.

Now, since the operator $\mathcal{L}$ (recall its definition in (1.5)) is sectorial in H^{-r}_θ, it also the case for the operator $\widetilde{\mathcal{L}}$ in $\mathbf{H}^{-r}_\theta$ defined by $\widetilde{\mathcal{L}}(p,m) = \mathcal{L}p$, and thus, applying [36, Theorem 47.8], for all initial conditions $\mu = (p,m) \in \mathbf{H}^{-r}_\theta$ there exists a unique maximal mild solution $\mu_t := (p_t, m_t) = T^t(\mu)$ to (1.30) defined on some time interval $[0, t_c)$ and which satisfies $t \mapsto T^t(\mu) \in \mathcal{C}\left([0, t_c)\, ; \mathbf{H}^{-r}_\theta\right)$.

Now, for $\mu = (p,m)$ and $\nu = (\eta, n)$, recalling the definition of G_1, G_2 given in (1.31), the Frechet differential of G at μ and applied to ν, denoted by $DG(\mu)[\nu]$, is given by

$$\begin{aligned} DG(\mu)[\nu] &= \begin{pmatrix} DG_1(\mu)[\nu] \\ DG_2(\mu)[\nu] \end{pmatrix} \\ &= \begin{pmatrix} -\nabla\cdot\left(\eta\left(F_m - \int F_m p\right)\right) - \nabla\cdot\left(p\left(DF_m[n] - \int F_m\eta - \int DF_m[n]p\right)\right) \\ \int F_m\eta + \int DF_m[n]p \end{pmatrix}. \end{aligned} \tag{2.3}$$

It satisfies, by similar arguments as above (in particular the fact that the derivatives of F_m can be bounded independently on m)

$$\|DG(\mu)[\nu]\|_{\mathbf{H}^{-(r+1)}_\theta} \leq C\left(1 + \|\mu\|_{\mathbf{H}^{-r}_\theta}\right) \|\nu\|_{\mathbf{H}^{-r}_\theta}, \tag{2.4}$$

and by [36, Theorem 49.2], $\mu \mapsto T^t(\mu)$ is Frechet differentiable, with derivative $DT^t(\mu)[\nu] = \nu_t := (\eta_t, n_t)$ the unique mild solution to

$$\begin{cases} \partial_t\eta_t = \mathcal{L}\eta_t + \delta DG_1(\mu_t)[\nu_t] \\ \dot{n}_t = \delta DG_2(\mu_t)[\nu_t] \end{cases}. \tag{2.5}$$

By [36, Theorem 47.5] the solution $\nu_t = (\eta_t, n_t)$ to (2.5) depends continuously on $\mu = (p,m)$, so that the flow $T^t(\mu)$ is C^1. One can proceed similarly for the second derivative. We have this time, for $\nu_i = (\eta_i, n_i)$, $i = 1, 2$,

$$\begin{aligned} D^2G_1(\mu)[\nu_1, \nu_2] = &-\nabla\cdot\left(\eta_1\left(DF_m[n_2] - \int F_m\eta_2 - \int DF_m[n_2]p\right)\right) \\ &- \nabla\cdot\left(\eta_2\left(DF_m[n_1] - \int F_m\eta_1 - \int DF_m[n_1]p\right)\right) \\ &- \nabla\cdot\left(p\Big(D^2F_m[n_1,n_2] - \int DF_m[n_1]\eta_2 - \int DF_m[n_2]\eta_1 \right. \\ &\left. - \int D^2F_m[n_1,n_2]p\Big)\right), \end{aligned} \tag{2.6}$$

and

$$D^2G_2(\mu)[\nu_1,\nu_2] = \int DF_m[n_1]\eta_2 + \int DF_m[n_2]\eta_1 + \int D^2F_m[n_1,n_2]p, \tag{2.7}$$

so that

$$\left\| D^2G(\mu)[\nu_1, \nu_2] \right\|_{\mathbf{H}_\theta^{-(r+1)}} \leq C \left(1 + \|\mu\|_{\mathbf{H}_\theta^{-r}} \right) \|\nu_1\|_{\mathbf{H}_\theta^{-r}} \|\nu_2\|_{\mathbf{H}_\theta^{-r}}, \tag{2.8}$$

and $T^t(\mu)$ is C^2 with $D^2T^t(\mu)[\nu_1, \nu_2] = \xi_t = (\xi_t^1, \xi_t^2)$ where $\xi_0 = 0$ and

$$\partial_t \xi_t = \left(\mathcal{L}\xi_t^1, 0 \right) + \delta DG(\mu_t)[\xi_t] + \delta D^2G(\mu_t)[\nu_{1,t}, \nu_{2,t}], \tag{2.9}$$

where $\nu_{i,t} = DT^t(\mu_0)[\nu_i]$ for $i = 1, 2$.

To prove that, for $R > 0$ and $\|p_0 - \rho\|_{H_\theta^{-r}} \leq R$, this solution is in fact globally defined when δ is taken small enough, remark that it satisfies

$$p_t = e^{t\mathcal{L}} p_0 + \int_0^t e^{(t-s)\mathcal{L}} \nabla \cdot (p_s(\delta F_{m_s} + \dot{m}_s))\, \mathrm{d}s, \tag{2.10}$$

and

$$\dot{m}_t = \delta \langle F_{m_t}, p_t \rangle. \tag{2.11}$$

The estimates obtained above imply directly $|\dot{m}_s| \leq \delta C_F \|p_s\|_{H_\theta^{-r}}$. Using Proposition 1.1 we get (for the constant $C_{\mathcal{L}}$ introduced in Proposition 1.1 and any $\lambda < k_{\min}$):

$$\|p_t\|_{H_\theta^{-r}} \leq C_{\mathcal{L}} \|p_0\|_{H_\theta^{-r}} + C_1 \int_0^t \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \left\| p_s(\delta F_{m_s} + \dot{m}_s) \right\|_{H_\theta^{-r}} \mathrm{d}s \tag{2.12}$$

$$\leq C_2 \left(\|p_0\|_{H_\theta^{-r}} + \delta \int_0^t \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \|p_s\|_{H_\theta^{-r}} \left(1 + \|p_s\|_{H_\theta^{-r}} \right) \mathrm{d}s \right). \tag{2.13}$$

Set $t_0 = \inf \left\{ t > 0 : \|p_t\|_{H_\theta^{-r}} \geq 2C_2 \left(R + \|\rho\|_{H_\theta^{-r}} \right) \right\}$. By continuity, $t_0 > 0$ and for all $t \in [0, t_0]$,

$$\|p_t\|_{H_\theta^{-r}} \leq C_2 \left(R + \|\rho\|_{H_\theta^{-r}} \right) + \delta \sqrt{\frac{\pi}{\lambda}} 2C_2 \left(R + \|\rho\|_{H_\theta^{-r}} \right) \left(1 + 2C_2 \left(R + \|\rho\|_{H_\theta^{-r}} \right) \right). \tag{2.14}$$

For the choice of $\delta > 0$ sufficiently small such that $\delta \sqrt{\frac{\pi}{\lambda}} 2 \left(1 + 2C_2 \left(R + \|\rho\|_{H_\theta^{-r}} \right) \right) < 1$, this yields that $t_0 = \infty$, so that (p_t, m_t) is a global solution. □

3 Proof of Proposition 1.7

In this section we give the proof of Proposition 1.7 which shows that $\widetilde{\mathcal{M}}^\delta$ given by (1.40) is an approximately invariant approximately normally hyperbolic manifold. We do not prove the assertions in the order they are given in Proposition 1.7.

Proof of Proposition 1.7 *Proof of (1).* Recall again the definitions of ρ in (1.9), of α_t periodic solution to (1.11) and of F_m in (1.6). Take $p_0 = \rho$ and $m_0 = \alpha_u$. We then have, from (1.4),

$$p_t - \rho = \int_0^t e^{(t-s)\mathcal{L}} \nabla \cdot (p_s(\delta F_{m_s} + \dot{m}_s))\, \mathrm{d}s, \tag{3.1}$$

and

$$\dot{m}_t - \dot{\alpha}_{u+t} = \delta \langle F_{m_t}, p_t \rangle - \delta \langle F_{\alpha_{u+t}}, \rho \rangle. \tag{3.2}$$

As it was already proved in the preceding section, we have $|\dot{m}_s| \leq C_F\delta\|p_s\|_{H_\theta^{-r}}$, and since Theorem 1.2 with $R = 1$ implies that, choosing δ small enough, $\|p_t\|_{H_\theta^{-r}} \leq C(1)$, we get from Proposition 1.1,

$$\begin{aligned}\|p_t - \rho\|_{H_\theta^{-r}} &\leq C_1 \int_0^t \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \left\|p_s(\delta F_{m_s} + \dot{m}_s)\right\|_{H_\theta^{-r}} \,\mathrm{d}s \\ &\leq C_1\delta \int_0^t \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \|p_s\|_{H_\theta^{-r}} \left(1 + \|p_s\|_{H_\theta^{-r}}\right) \mathrm{d}s \leq C_2\delta. \end{aligned} \tag{3.3}$$

Now since

$$\begin{aligned}\frac{1}{\delta}(\dot{m}_t - \dot{\alpha}_{u+t}) = \langle DF_{\alpha_{u+t}}, \rho\rangle(m_t - \alpha_{u+t}) + \langle F_{m_t} - F_{\alpha_{u+t}} - DF_{\alpha_{u+t}}(m_t - \alpha_{u+t}), \rho\rangle \\ + \langle F_{m_t}, p_t - \rho\rangle, \end{aligned} \tag{3.4}$$

we have the following mild representation (recall the definition of $\pi^\delta_{u+t,u}$ in (1.35) and that $m_0 = \alpha_u$):

$$\begin{aligned} &m_t - \alpha_{u+t} \\ &= \delta \int_0^t \pi^\delta_{u+t,u+s}\Big(\langle F_{m_s} - F_{\alpha_{u+s}} - DF_{\alpha_{u+s}}(m_s - \alpha_{u+s}), \rho\rangle + \langle F_{m_s}, p_s - \rho\rangle\Big)\,\mathrm{d}s, \end{aligned} \tag{3.5}$$

which leads to (recall that the derivatives of F are bounded and that (3.3) is valid for all $t \geq 0$):

$$|m_t - \alpha_{u+t}| \leq C_3\delta \int_0^t |m_s - \alpha_{u+s}|^2 \,\mathrm{d}s + C_3\delta^2 t. \tag{3.6}$$

Consider $t_1 = \inf\{t > 0 : |m_t - \alpha_{u+t}| \geq 2\tau C_4\delta\}$ (recall the definition of τ in (1.54)). By continuity, $t_1 > 0$ and for all $t \leq t_1$ we have

$$|m_t - \alpha_{u+t}| \leq (4\tau^2 C_3^3\delta^3 + C_3\delta^2)t, \tag{3.7}$$

which means that $t_1 \geq \frac{\tau}{\delta}$ for δ small enough, and implies (1).
Proof of (2). The first two points follow directly from the fact that the projections P^c_u defined in (1.36) are smooth. For the third point we have

$$\frac{\left\|\psi(t) - \psi(s) - \widetilde{\Pi}^{\delta,c}_s(\psi(t) - \psi(s))\right\|_{\mathbf{H}_\theta^{-r}}}{\|\psi(t) - \psi(s)\|_{\mathbf{H}_\theta^{-r}}} = \frac{\left|\alpha^\delta_t - \alpha^\delta_s - P^{\delta,c}_s(\alpha^\delta_t - \alpha^\delta_s)\right|}{\left|\alpha^\delta_t - \alpha^\delta_s\right|}, \tag{3.8}$$

and since

$$\alpha^\delta_t - \alpha^\delta_s = \alpha^1_{\delta t} - \alpha^1_{\delta s} = \delta(t-s)\frac{d}{du}\alpha^1_{u|u=\delta s} + O(\delta^2(t-s)), \tag{3.9}$$

and

$$P^{\delta,c}_s \frac{d}{du}\alpha^1_{u|u=\delta s} = P^{0,c}_{\delta s}\frac{d}{du}\alpha^1_{u|u=\delta s} = \frac{d}{du}\alpha^1_{u|u=\delta s}, \tag{3.10}$$

the term $\frac{\left\|\psi(t)-\psi(s)-\widetilde{\Pi}^{\delta,c}_s(\psi(t)-\psi(s))\right\|_{\mathbf{H}_\theta^{-r}}}{\|\psi(t)-\psi(s)\|_{\mathbf{H}_\theta^{-r}}}$ is indeed of order δ.

Proof of (5). We choose in the following $R_0 = 1$. For any $(p, m) \in \mathcal{V}(\widetilde{\mathcal{M}}^\delta, R_0)$, wich means in particular $\|p - \rho\|_{H_\theta^{-r}} \leq R_0$, we deduce from Theorem 1.2, if δ is small enough, that

$$\sup_{t \geq 0} \|p_t\|_{H_\theta^{-r}} \leq C(R_0). \tag{3.11}$$

This means in particular, since $m_t = m_0 + \delta \int_0^t \langle F_{m_s}, p_s \rangle ds$, that for $C_4, C_5 > 0$

$$\sup_{t \geq 0} |\dot{m}_t| \leq \delta C_4, \quad \text{and} \quad \sup_{t \in [0, \frac{\tau}{\delta}]} |m_t| \leq C_5, \tag{3.12}$$

where C_5 depends on τ. Now, using (2.5) we have, with $\mu_s = (p_s, m_s)$,

$$\eta_t = e^{t\mathcal{L}} \eta_0 + \delta \int_0^t e^{(t-s)\mathcal{L}} DG_1(\mu_s)[\eta_s, n_s] \, \mathrm{d}s, \tag{3.13}$$

and

$$n_t = n_0 + \delta \int_0^t DG_2(\mu_s)[\eta_s, n_s] \, \mathrm{d}s. \tag{3.14}$$

From (2.4) and Proposition 1.1 (recall that $\int_{\mathbb{R}^d} \eta_0 = 0$, see Remark 1.3), we obtain

$$\|\eta_t\|_{H_\theta^{-r}} \leq C_{\mathcal{L}} e^{-\lambda t} \|\eta_0\|_{H_\theta^{-r}} + C_6 \delta \int_0^t \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \left(\|\eta_s\|_{H_\theta^{-r}} + |n_s| \right) \mathrm{d}s, \tag{3.15}$$

and

$$|n_t| \leq |n_0| + C_6 \delta \int_0 \left(\|\eta_s\|_{H_\theta^{-r}} + |n_s| \right) \mathrm{d}s. \tag{3.16}$$

We deduce that, for $\nu_t = DT^t(p, m)[\nu_0] = (\eta_t, n_t)$,

$$\|\nu_t\|_{\mathbf{H}_\theta^{-r}} \leq C_7 \|\nu_0\|_{\mathbf{H}_\theta^{-r}} + C_8 \delta \int_0^t \left(1 + \frac{1}{\sqrt{t-s}} \right) \|\nu_s\|_{\mathbf{H}_\theta^{-r}} \, \mathrm{d}s. \tag{3.17}$$

Applying Lemma B.1, we get the desired bound for the $DT^{\frac{\tau}{\delta}}$ with $\kappa_6 = 2C_7 e^{3C_8 \tau}$, when δ is small enough.

For the second derivative, recall that $D^2 T^t(\mu)[\nu_1, \nu_2] = \xi_t = (\xi_t^1, \xi_t^2)$, where $\xi_0 = 0$ and (recall (2.9))

$$\xi_t^1 = \delta \int_0^t e^{(t-s)\mathcal{L}} \left(DG_1(\mu_s)[\xi_s] + D^2 G_1(\mu_s)[\nu_{1,s}, \nu_{2,s}] \right) \mathrm{d}s, \tag{3.18}$$

and

$$\xi_t^2 = \delta \int_0^t \left(DG_2(\mu_s)[\xi_s] + D^2 G_2(\mu_s)[\nu_{1,s}, \nu_{2,s}] \right) \mathrm{d}s \tag{3.19}$$

where $\mu_t = (p_t, m_t)$, and $\nu_{i,t} = DT^t(\mu_0)[\nu_i]$ for $i = 1, 2$. This induces for $t \in [0, \frac{\tau}{\delta}]$, recalling (2.4), (2.8) and since $\|\nu_{i,t}\|_{\mathbf{H}_\theta^{-r}} \leq \kappa_6 \|\nu_{i,0}\|_{\mathbf{H}_\theta^{-r}}$,

$$\left\|\xi_t^1\right\|_{H_\theta^{-r}} \leq \delta C_9 \int_0^t \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \left(\|\xi_s\|_{\mathbf{H}_\theta^{-r}} + \|\nu_{1,0}\|_{\mathbf{H}_\theta^{-r}} \|\nu_{2,0}\|_{\mathbf{H}_\theta^{-r}} \right) \mathrm{d}s, \tag{3.20}$$

and

$$\left|\xi_t^2\right| \leq \delta C_9 \int_0^t \left(\|\xi_s\|_{\mathbf{H}_\theta^{-r}} + \|\nu_{1,0}\|_{\mathbf{H}_\theta^{-r}} \|\nu_{2,0}\|_{\mathbf{H}_\theta^{-r}} \right) \mathrm{d}s. \tag{3.21}$$

So for $t \leq \frac{\tau}{\delta}$,

$$\|\xi_t\|_{\mathbf{H}_\theta^{-r}} \leq C_{10}\|\nu_{1,0}\|_{\mathbf{H}_\theta^{-r}}\|\nu_{2,0}\|_{\mathbf{H}_\theta^{-r}} + \delta C_{10}\int_0^t \left(1+\frac{e^{-\lambda(t-s)}}{\sqrt{t-s}}\right)\|\xi_s\|_{\mathbf{H}_\theta^{-r}}\,\mathrm{d}s, \quad (3.22)$$

and one deduces from Lemma B.1 that $\|\xi_t\|_{\mathbf{H}_\theta^{-r}} \leq \kappa_7\|\nu_{1,0}\|_{\mathbf{H}_\theta^{-r}}\|\nu_{2,0}\|_{\mathbf{H}_\theta^{-r}}$ with $\kappa_7 = 2C_{10}e^{3C_{10}\tau}$ for $t \leq \frac{\tau}{\delta}$ and δ small enough, which concludes the proof of (5).

Proof of (3). We are now interested in $DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)(\eta_0,n_0) = (\eta_{\frac{\tau}{\delta}}, n_{\frac{\tau}{\delta}}) = \nu_{\frac{\tau}{\delta}}$. From the proof of Point (3) we already know that $\sup_{t\in[0,\frac{\tau}{\delta}]}\|\nu_t\|_{\mathbf{H}_\theta^{-r}} \leq \kappa_6\|\nu_0\|_{\mathbf{H}_\theta^{-r}}$, which means, recalling (3.15), that

$$\begin{aligned}\|\eta_t\|_{H_\theta^{-r}} &\leq C_{\mathcal{L}}e^{-\lambda t}\|\eta_0\|_{H_\theta^{-r}} + C_{11}\delta\int_0^t \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}}\left(\|\eta_0\|_{H_\theta^{-r}}+|n_0|\right)\mathrm{d}s\\ &\leq C_{\mathcal{L}}e^{-\lambda t}\|\eta_0\|_{H_\theta^{-r}} + C_{12}\delta\left(\|\eta_0\|_{H_\theta^{-r}}+|n_0|\right). \end{aligned} \quad (3.23)$$

Moreover, since

$$\begin{aligned}\frac{1}{\delta}\dot{n}_t &= \langle DF_{\alpha_{u+t}}[n_t],\rho\rangle - \langle DF_{\alpha_{u+t}}[n_t] - DF_{m_t}[n_t],\rho\rangle + \langle DF_{m_t}[n_t], p_t-\rho\rangle\\ &\quad + \langle F_{m_t},\eta_t\rangle, \end{aligned} \quad (3.24)$$

we have the mild representation (recall again the definition of π in (1.35))

$$\begin{aligned} n_t &= \pi^\delta_{u+t,u}n_0 + \delta\int_0^t \pi^\delta_{u+t,u+s}\Big(-\langle DF_{\alpha_{u+s}}[n_s] - DF_{m_s}[n_s],\rho\rangle\\ &\quad + \langle DF_{m_s}[n_s],p_s-\rho\rangle + \langle F_{m_s},\eta_s\rangle\Big)\mathrm{d}s. \end{aligned} \quad (3.25)$$

From the proof of point (1), for $t \leq \frac{\tau}{\delta}$, $\|p_t-\rho\|_{H_\theta^{-r}}$ and $|m_t-\alpha_{u+t}|$ are of order δ, and thus we obtain (recall also that $\sup_{t\in[0,\frac{\tau}{\delta}]}|n_t| \leq \kappa_6\|\nu_0\|_{\mathbf{H}_\theta^{-r}}$):

$$\begin{aligned}\left|n_t - \pi^\delta_{u+t,u}n_0\right| &\leq C_{13}\delta\int_0^t\left(\|\eta_s\|_{H_\theta^{-r}} + \delta|n_0|\right)\mathrm{d}s\\ &\leq C_{13}\delta\int_0^t\left(C_{\mathcal{L}}e^{-\lambda s}\|\eta_0\|_{H_\theta^{-r}} + C_{12}\delta\left(\|\eta_0\|_{H_\theta^{-r}}+|n_0|\right)+\delta|n_0|\right)\mathrm{d}s\\ &\leq C_{14}\delta\left(\|\eta_0\|_{H_\theta^{-r}}+|n_0|\right). \end{aligned} \quad (3.26)$$

Suppose now that $(\eta_0,n_0)\in\widetilde{\mathbf{X}}^{\delta,s}_u$, that is $P^{\delta,c}_u n_0 = 0$ (recall the definitions of $\widetilde{\mathbf{X}}^{\delta,s}_u$ and $P^{\delta,c}_u$ in § 1.5). Then we have $P^{\delta,c}_{u+\frac{\tau}{\delta}}\pi^\delta_{u+\frac{\tau}{\delta},u}n_0 = P^{\delta,c}_u n_0 = 0$, and thus, recalling (3.26) and (3.23),

$$\left|P^{\delta,c}_{u+\frac{\tau}{\delta}}n_{\frac{\tau}{\delta}}\right| = \left|P^{\delta,c}_{u+\frac{\tau}{\delta}}\left(n_{\frac{\tau}{\delta}} - \pi^\delta_{u+\frac{\tau}{\delta},u}n_0\right)\right| \leq C_{15}\delta\left(\|\eta_0\|_{H_\theta^{-r}}+|n_0|\right). \quad (3.27)$$

This shows that

$$\left\|\widetilde{\Pi}^{\delta,c}_{u+\frac{\tau}{\delta}}DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)|_{\widetilde{\mathbf{X}}^{\delta,s}_u}\right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})} \leq C_{15}\delta. \quad (3.28)$$

On the other hand, suppose that $(\eta_0,n_0)\in\widetilde{\mathbf{X}}^{\delta,c}_u$, that is $\eta_0=0$ and $P^{\delta,s}_u n_0 = 0$. We then have directly $\left\|\eta_{\frac{\tau}{\delta}}\right\|_{H_\theta^{-r}} \leq C_{12}\delta|n_0|$, and since $P^{\delta,s}_{u+\frac{\tau}{\delta}}\pi^\delta_{u+\frac{\tau}{\delta},u}n_0 = P^{\delta,s}_u n_0 = 0$, from (3.26) we deduce

$$\left|P^s_{u+\frac{\tau}{\delta}} n_{\frac{\tau}{\delta}}\right| = \left|P^s_{u+\frac{\tau}{\delta}}\left(n_{\frac{\tau}{\delta}} - \pi^\delta_{u+\frac{\tau}{\delta},u} n_0\right)\right| \leq C_{16}\delta^2 \int_0^{\frac{\tau}{\delta}} |n_0|\,\mathrm{d}s \leq C_{16}\tau\delta|n_0|. \quad (3.29)$$

This means that

$$\left\|\widetilde{\Pi}^{\delta,s}_{u+\frac{\tau}{\delta}} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)|_{\widetilde{\mathbf{X}}^{\delta,c}_u}\right\|_{\mathcal{B}(\mathbf{H}^{-r}_\theta)} \leq (C_{12} + C_{16}\delta)\delta. \quad (3.30)$$

Proof of (4). On the one hand consider $(\eta_0, n_0) \in \widetilde{\mathbf{X}}^{\delta,s}_u$, that is $P^{\delta,c}_u n_0 = 0$. Then, considering δ small enough such that $C_{\mathcal{L}} e^{-\lambda\frac{\tau}{\delta}} \leq C_{12}\delta$, by (3.23) we obtain

$$\left\|\eta_{\frac{\tau}{\delta}}\right\|_{H^{-r}_\theta} \leq 2C_{12}\delta\left(\|\eta_0\|_{H^{-r}_\theta} + |n_0|\right). \quad (3.31)$$

Moreover, since $P^{\delta,s}_{u+\frac{\tau}{\delta}} \pi^\delta_{u+\frac{\tau}{\delta},u} n_0 = \pi^\delta_{u+\frac{\tau}{\delta},u} n_0$ and $P^{\delta,c}_u n_0 = 0$ we obtain, by (3.26) and (1.37),

$$\left|P^s_{u+\frac{\tau}{\delta}} n_{\frac{\tau}{\delta}}\right| \leq \left|P^s_{u+\frac{\tau}{\delta}} \pi^\delta_{u+\frac{\tau}{\delta},u} n_0\right| + \left|P^s_{u+\frac{\tau}{\delta}}\left(n_{\frac{\tau}{\delta}} - \pi^\delta_{u+\frac{\tau}{\delta},u} n_0\right)\right| \quad (3.32)$$

$$\leq C_\alpha e^{-\lambda_\alpha \tau}|n_0| + C_{17}\delta\left(\|\eta_0\|_{H^{-r}_\theta} + |n_0|\right). \quad (3.33)$$

We deduce that for δ small enough

$$\left\|\widetilde{\Pi}^{\delta,s}_{u+\frac{\tau}{\delta}} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)|_{\widetilde{\mathbf{X}}^{\delta,s}_u}\right\|_{\mathcal{B}(\mathbf{H}^{-r}_\theta)} \leq 2C_\alpha e^{-\lambda_\alpha \tau}. \quad (3.34)$$

On the other hand consider $(\eta_0, n_0) \in \widetilde{\mathbf{X}}^{\delta,c}_u$, which means $\eta_0 = 0$ and $P^{\delta,s}_u n_0 = 0$. Then similar arguments as above (recall that this time $\eta_0 = 0$) lead to

$$\left|P^{\delta,c}_{u+\frac{\tau}{\delta}}\left(n_{\frac{\tau}{\delta}} - \pi^\delta_{u+\frac{\tau}{\delta},u} n_0\right)\right| \leq C_{18}\delta|n_0|. \quad (3.35)$$

We then obtain, for δ small enough, recalling (1.37),

$$\left|P^c_{u+\frac{\tau}{\delta}} n_{\frac{\tau}{\delta}}\right| \geq (c_\alpha - C_{18}\delta)\,|n_0| \geq \frac{c_\alpha}{2}|n_0|. \quad (3.36)$$

This means in particular that $\widetilde{\Pi}^{\delta,c}_{u+\frac{\tau}{\delta}} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)|_{\widetilde{\mathbf{X}}^{\delta,c}_u}$, which is a linear mapping in finite dimensional spaces, is invertible and satisfies

$$\left\|\left(\widetilde{\Pi}^{\delta,c}_{u+\frac{\tau}{\delta}} DT^{\frac{\tau}{\delta}}(\rho,\alpha_u)|_{\widetilde{\mathbf{X}}^{\delta,c}_u}\right)^{-1}\right\|_{\mathcal{B}(\mathbf{H}^{-r}_\theta)} \leq \frac{2}{c_\alpha}. \quad (3.37)$$

We deduce (4) with $a = \frac{c_\alpha}{4}$ and $\widetilde{\lambda} = \frac{4C_\alpha e^{-\lambda_\alpha \tau}}{c_\alpha}$, recalling (1.54).

Proof of (6). For any initial condition $\mu = (p_0, m_0) \in \mathcal{V}(\widetilde{\mathcal{M}}^\delta, 1)$ recall that Theorem 1.2 implies $\sup_{t\geq 0}\|p_t\|_{H^{-r}_\theta} \leq C(1)$. Then for $\frac{\tau}{\delta} \leq t < t'$, $t' - t \leq \zeta$, for some $\zeta \leq 1$ to be chosen later, relying on (3.1), the following is true:

$$\begin{aligned}\|p_{t'} - p_t\|_{H^{-r}_\theta} \leq{}& \left\|\left(e^{t'\mathcal{L}} - e^{t\mathcal{L}}\right) p_0\right\|_{H^{-r}_\theta} \\ &+ \int_0^t \left\|\left(e^{(t'-s)\mathcal{L}} - e^{(t-s)\mathcal{L}}\right)\nabla\cdot(p_s(\delta F_{m_s} + \dot{m}_s))\right\|_{H^{-r}_\theta}\mathrm{d}s \\ &+ \int_t^{t'} \left\|e^{(t'-s)\mathcal{L}}\nabla\cdot(p_s(\delta F_{m_s} + \dot{m}_s))\right\|_{H^{-r}_\theta}\mathrm{d}s. \end{aligned} \quad (3.38)$$

Using Proposition 1.1, the first term above may be bounded as

$$\left\| \left(e^{t'\mathcal{L}} - e^{t\mathcal{L}}\right) p_0 \right\|_{H_\theta^{-r}} \leq C_{\mathcal{L}}(t'-t)^{\varepsilon'} \frac{e^{-\lambda t}}{t^{\frac{1}{2}+\varepsilon'}} \|p_0\|_{H_\theta^{-(r+1)}} \leq C_{19}\zeta^{\varepsilon'}\delta^{\frac{1}{2}+\varepsilon'} \frac{e^{-\lambda\frac{\tau}{\delta}}}{\tau^{\frac{1}{2}+\varepsilon'}}, \quad (3.39)$$

for some $\varepsilon' \in (0, 1)$. Concerning the second term,

$$\begin{aligned} &\int_0^t \left\| \left(e^{(t'-s)\mathcal{L}} - e^{(t-s)\mathcal{L}}\right) \nabla \cdot (p_s(\delta F_{m_s} + \dot{m}_s)) \right\|_{H_\theta^{-r}} \mathrm{d}s \\ &\leq C_{\mathcal{L}}(t'-t)^{\varepsilon'} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\frac{1}{2}+\varepsilon'}} \left\| p_s(\delta F_{m_s} + \dot{m}_s) \right\|_{H_\theta^{-r}} \mathrm{d}s \\ &\leq C_{20}\delta(t'-t)^{\varepsilon'} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\frac{1}{2}+\varepsilon'}} \|p_s\|_{H_\theta^{-r}} \left(1 + \|p_s\|_{H_\theta^{-r}}\right) \mathrm{d}s \\ &\leq C_{21}\delta\zeta^{\varepsilon'}. \end{aligned} \quad (3.40)$$

Now turning to the third term, relying again on Proposition 1.1,

$$\begin{aligned} &\int_t^{t'} \left\| e^{(t'-s)\mathcal{L}} \nabla \cdot (p_s(\delta F_{m_s} + \dot{m}_s)) \right\|_{H_\theta^{-r}} \mathrm{d}s \\ &\leq C_{22}\delta \int_t^{t'} \frac{e^{-\lambda(t'-s)}}{\sqrt{t'-s}} \|p_s\|_{H_\theta^{-r}} \left(1 + \|p_s\|_{H_\theta^{-r}}\right) \mathrm{d}s, \\ &\leq C_{23}\delta\zeta^{\frac{1}{2}}. \end{aligned} \quad (3.41)$$

Gathering (3.39), (3.40), (3.41) into (3.38) yields

$$\|p_{t'} - p_t\|_{H_\theta^{-r}} \leq \frac{\varepsilon}{2} \quad (3.42)$$

if $\zeta \leq 1$ is chosen sufficiently small.

We now turn turn to the control of the mean: since $\dot{m}_t = \delta \int F_{m_t} dp_t$ we have that for $t \leq t' \leq t + \zeta$,

$$m_{t'} - m_t = \delta \int_t^{t'} \langle F_{m_s}, p_s \rangle \,\mathrm{d}s.$$

Since we have the uniform bound $\sup_{s\geq 0} \|p_s\|_{H_\theta^{-r}} \leq C(1)$ and since F and its derivatives are bounded, the above quantity is easily bounded by some $C\delta(t'-t)$ which can be made smaller than $\varepsilon/2$, provided ζ is taken small enough. □

4 Proof of Theorem 1.4

Proof of Theorem 1.4 From Proposition 1.7 we know that the hypotheses needed in [5] are satisfied for δ small enough, which means that the system (1.4) admits a stable normally hyperbolic manifold $\mathcal{M}^\delta$ that is at distance δ from $\widetilde{\mathcal{M}}^\delta$. Indeed in [5] some constants η, χ, σ need to be small for their result to be true, but in our case these constants are of order δ, so we only need to suppose δ small enough. Moreover $\mathcal{M}^\delta$ is constructed at a distance δ_0 from $\widetilde{\mathcal{M}}^\delta$, with δ_0 chosen such that η/ε and ε/δ_0 are bounded for some $\varepsilon > 0$ (see [5], Theorem 4.2). Since in our case η is of order δ, we can take δ_0 of order δ, and $\mathcal{M}^\delta$ is indeed at distance δ from $\widetilde{\mathcal{M}}^\delta$.

The invariant manifold $\mathcal{M}^\delta$ is one dimensional, since $\widetilde{\mathcal{M}}^\delta$ is, so to prove that it corresponds to a periodic solution it is sufficient to prove that it does not possess any invariant point. But for any $(p_0, m_0) \in \mathcal{M}^\delta$ we have, since $\|p - \rho\|_{H_\theta^{-r}}$ and $|m_0 - \alpha_u^\delta|$ are of order δ for some $u \in [0, \frac{T_\alpha}{\delta}]$,

$$\dot{m}_0 = \delta \int F_{m_0} p_0 = \delta \int F_{\alpha_u} \rho + O(\delta^2) = \dot{\alpha}_u + O(\delta^2). \tag{4.1}$$

Since there exists $c > 0$ such that $|\dot{\alpha}_u^\delta/\delta| > c$ independently on u, we have $\dot{m}_0 \neq 0$ for the solutions starting from any point of $\mathcal{M}^\delta$, which means that $\mathcal{M}^\delta$ does not possess any fixed-point, and is thus defined by a periodic solution of positive period T_δ, that we denote $\Gamma_t^\delta = (q_t^\delta, \gamma_t^\delta)$ for $t \in [0, T_\delta]$.

Now, by the Herculean Theorem (see [36], Theorem 47.6), since $\mathcal{M}^\delta$ is invariant, Γ_t^δ is in fact an element of $\mathbf{H}_\theta^{-r+2}$ and, by [36] Theorem 48.5, $\partial_t \Gamma_{s+t}^\delta = (\partial_t q_{s+t}^\delta, \dot{\gamma}_{s+t}^\delta)$ is in $C([0, T_\delta), \mathbf{H}_\theta^{-r})$ and it is solution to

$$\begin{cases} \partial_t \eta_t = \mathcal{L}\eta_t + \delta DG_1(\Gamma_{s+t}^\delta)[\nu_t] \\ \dot{n}_t = \delta DG_2(\Gamma_{s+t}^\delta)[\nu_t] \end{cases}, \tag{4.2}$$

which means in particular that $\partial_t \Gamma_{s+t}^\delta = \Phi_{s+t,s} \partial_t \Gamma_s^\delta$. Now $\partial_t \Gamma_{s+t}^\delta$ is a periodic solution to (4.2), and the same arguments imply that $\partial_t^2 \Gamma_{s+t}^\delta$ is in $C([0, T_\delta), \mathbf{H}_\theta^{-r})$.

In addition, it is proved in [5] that $\mathcal{M}^\delta$ is foliated by C^1 invariant foliations: a neighborhood $\mathcal{W}^\delta$ of $\mathcal{M}^\delta$ satisfies the decomposition $\mathcal{W}^\delta = \cup_{s \in [0, T_\delta)} \mathcal{W}_s^\delta$, where $\mathcal{W}_s^\delta$ corresponds to the elements of $\mu \in \mathbf{H}_\theta^{-r}$ such that $T^{nT_\delta}(\mu)$ converges exponentially fast to Γ_s^δ as n goes to infinity. The projections $\Pi_s^{\delta,c}$ and $\Pi_s^{\delta,s}$ correspond then respectively to the projections on the tangent space to $\mathcal{M}^\delta$ and to $\mathcal{W}_s^\delta$ at Γ_s^δ. The linear operator $\Phi_{s+t,s}^\delta = DT^t(\Gamma_s^\delta)$ commutes then with these projections, and is bounded from above and below in the direction of the tangent space to $\mathcal{M}^\delta$, while it is contractive in the direction of the tangent space to stable foliations.

In addition to the contractive property, the regularization effect of Φ^δ given in (1.46) is a consequence of the fact that $\Phi_{t+s,s}\nu = \nu_t$ where $\nu_0 = \nu$ and $\nu_t = (\eta_t, n_t)$ is solution to

$$\begin{cases} \partial_t \eta_t = \mathcal{L}\eta_t + \delta DG_1(\Gamma_{s+t}^\delta)[\nu_t] \\ \dot{n}_t = \delta DG_2(\Gamma_{s+t}^\delta)[\nu_t] \end{cases}. \tag{4.3}$$

The operator $\widetilde{\mathcal{L}}(\eta, n) = (\mathcal{L}\eta, 0)$ is sectorial in $\mathbf{H}_\theta^{-r}$ and thus induces regularization properties for the solutions to (4.3), and thus for Φ^δ. More precisely we are in fact exactly in the situation of [23], Theorem 7.2.3 and the following remark. Indeed, for $s \in [0, T_\delta)$ we can define the operator $U_s^\delta = \Phi_{s+T_\delta,s}^\delta$, and we can deduce from above spectral properties for U_t^δ. Since Γ^δ is a periodic solution, U_s^δ admits 1 as eigenvalue, with eigenfunction $\partial_s \Gamma_s^\delta$ and corresponding projection $\Pi_s^{\delta,c}$, and due to the contractive property of Φ^δ the rest of the spectrum of U_s^δ is located in a disk centered at 0 with radius $e^{-\lambda_\delta T_\delta}$. We can then apply Theorem 7.2.3 and the following remark to obtain (1.46) (reducing slightly the value of λ_δ).

The C^1 regularity of $s \mapsto \Pi_s^{\delta,c}$ is not a direct consequence of the normally hyperbolic results of [5] (they prove that $\mathcal{W}_s^\delta$ has a Hölder regularity with respect to s), but since we are in the case of a periodic solution we have an explicit formula for $\Pi_s^{\delta,c}$: 1 is an isolated eigenvalue of U_t^δ, so for $\mathcal{C}_\varepsilon$ the circle centered at 1 with radius $\varepsilon > 0$, with ε small enough, we have

$$\Pi_s^{\delta,c} = \frac{1}{2i\pi} \int_{\mathcal{C}_\varepsilon} (\lambda - U_s^\delta)^{-1} \, \mathrm{d}\lambda. \tag{4.4}$$

But applying [23], Theorem 3.4.4., $t \mapsto U_s^\delta$ is C^1, with $\partial_s U_s^\delta \zeta = \zeta_{T_\delta} = (\zeta^1_{T_\delta}, \zeta^2_{T_\delta})$, where $\zeta_0 = \zeta$ and

$$\begin{cases} \partial_t \zeta_t^1 = \mathcal{L}\zeta_t^1 + \delta DG_1(\Gamma^\delta_{s+t})[\zeta_t] + \delta D^2 G_1(\Gamma^\delta_{s+t})[\partial_t \Gamma^\delta_{s+t}, \zeta_t] \\ \dot{\zeta}_t^2 = \delta DG_2(\Gamma^\delta_{s+t})[\zeta_t] + \delta D^2 G_2(\Gamma^\delta_{s+t})[\partial_t \Gamma^\delta_{s+t}, \zeta_t]. \end{cases}, \tag{4.5}$$

and thus $s \mapsto \Pi_s^{\delta,c}$ is also C^1.

It is not immediate that q_s^δ is a probability distribution, since we apply the results of [5] considering solutions $p_t \in H_\theta^{-r}$ satisfying $\int_{\mathbb{R}^d} p_t = 1$ but without any hypotheses on nonnegativity. However, $\widetilde{\mathcal{M}}^\delta$ is in the basin of attraction of $\mathcal{M}^\delta$, so any $(q_s^\delta, m_s^\delta) \in \mathcal{M}^\delta$ is the limit in $\mathbf{H}_\theta^{-r}$ of $(p_t, m_t) = T^t(\rho, \alpha_u^\delta)$ for some $u \in [0, \frac{T_\alpha}{\delta})$. So, since in this case p_t is a probability distribution (recall that it is the probability distribution of $X_t - \mathbb{E}[X_t]$, where X_t satisfies (1.2) with initial distribution ρ), we deduce that $\langle q_s^\delta, \varphi \rangle \geq 0$ for any smooth function φ with compact support, and thus q_s^δ is also a probability distribution. □

5 Proof of Theorem 1.6

Recall once again the definition of Γ^δ in (1.41) as well as the definition of the flow T^t in Theorem 1.2. As it was already explained in Sect. 1.4, the existence of the map Θ^δ is a consequence of the foliation property proved in [5]. Moreover Θ^δ satisfies the relation

$$\Gamma^\delta_{\Theta(\mu)} = \lim_{n\to\infty} T^{nT_\delta}\mu. \tag{5.1}$$

Our aim in the present section is to prove the C^2 regularity of Θ^δ. Following ideas from [22], we will prove uniform in time bounds for the first and second derivatives of the flow T^t, which will induce the regularity of

$$S(\mu) := \lim_{n\to\infty} T^{nT_\delta}\mu \tag{5.2}$$

and thus the regularity of Θ^δ.

Proof of Theorem 1.6 *Step 1* let us first show that for some constant $c_1 > 0$

$$\sup_{t\geq 0} \sup_{\mu\in\mathcal{V}(\mathcal{M}^\delta,\varepsilon)} \left\| DT^t(\mu) \right\|_{\mathcal{B}(\mathbf{H}_\theta^{-r})} \leq c_1, \tag{5.3}$$

where $\mathcal{V}(\mathcal{M}^\delta, \varepsilon) := \left\{ \mu \in \mathbf{H}_\theta^{-r},\ \mathrm{dist}_{\mathbf{H}_\theta^{-r}}\left(\mu, \mathcal{M}^\delta\right) < \varepsilon \right\}$ is a neighborhood of $\mathcal{M}^d$ (given by (1.42)) on which the trajectories are attracted to the cycle. For $\mu_0 = (p_0, m_0) \in \mathcal{V}(\mathcal{M}^\delta, \varepsilon)$ and $u = \Theta(\mu_0)$, denoting by $\nu_t = (\eta_t, n_t) = DT^t(\mu_0)[\nu_0]$ and recalling the definitions of G in (1.31) and of Φ in (1.43), we have,

$$\nu_t = \Phi^\delta_{u+t,u}\nu_0 + \delta \int_0^t \Phi^\delta_{u+t,u+s}\left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right)[\nu_s]\,\mathrm{d}s. \tag{5.4}$$

Let us now prove that there exists a constant C_G such that, for $\mu = (p, m)$ and $\Gamma = (q, \gamma)$,

$$\| DG(\mu) - DG(\Gamma)\|_{\mathcal{B}\left(\mathbf{H}_\theta^{-r}, \mathbf{H}_\theta^{-(r+1)}\right)} \leq C_G \|\mu - \Gamma\|_{\mathbf{H}_\theta^{-r}}. \tag{5.5}$$

We have, for $\nu = (\eta, n)$

$$(DG_1(\mu) - DG_1(\Gamma))[\nu] = -\nabla \cdot \big(\eta(F_m - F_\gamma)\big) + \nabla \cdot \left(\eta\left(\int F_m p - \int F_\gamma q\right)\right) \tag{5.6}$$

$$- \nabla \cdot \big(pDF_m[n] - qDF_\gamma[n]\big) \tag{5.7}$$

$$+ \nabla \cdot \left(p \int F_m \eta - q \int F_\gamma \eta\right) \tag{5.8}$$

$$+ \nabla \cdot \left(p \int DF_m[n]p - q \int DF_\gamma[n]q\right), \tag{5.9}$$

and

$$(DG_2(\mu) - DG_2(\Gamma))[\nu] = \int (F_m - F_\gamma)\eta + \int DF_m[n]p - \int DF_\gamma[n]q. \tag{5.10}$$

For the first term, we obtain

$$\left\|\nabla \cdot \big(\eta(F_m - F_\gamma)\big)\right\|_{H_\theta^{-(r+1)}} \leq C_1 \left\|\eta(F_m - F_\gamma)\right\|_{H_\theta^{-r}}, \tag{5.11}$$

and since, for $f \in H_\theta^r$,

$$\langle \eta(F_m - F_\gamma), f\rangle \leq \|\eta\|_{H_\theta^{-r}} \|(F_m - F_\gamma)f\|_{H_\theta^r} \leq C_2|m-\gamma|\|\eta\|_{H_\theta^{-r}}\|f\|_{H_\theta^r}, \tag{5.12}$$

where we have used the fact that all the derivatives of F are Lipschitz, we get, for some $C_3 > 0$,

$$\left\|\nabla \cdot \big(\eta(F_m - F_\gamma)\big)\right\|_{H_\theta^{-(r+1)}} \leq C_3|m-\gamma|\|\eta\|_{H_\theta^{-r}}. \tag{5.13}$$

For the second term, since

$$\left|\int F_m p - \int F_\gamma q\right| \leq \left|\int F_m(p-q)\right| + \left|\int (F_m - F_\gamma)q\right|$$
$$\leq C_4\left(\|p-q\|_{H_\theta^{-r}} + |m-\gamma|\right), \tag{5.14}$$

we have

$$\left\|\nabla \cdot \left(\eta\left(\int F_m p - \int F_\gamma q\right)\right)\right\|_{H_\theta^{-(r+1)}} \leq C_5\left(\|p-q\|_{H_\theta^{-r}} + |m-\gamma|\right). \tag{5.15}$$

The other terms can be tackled in a similar way. Now, since $\mu_0 \in \mathcal{W}_u^\delta$, we have for some $C_{\Gamma^\delta} > 0$,

$$\left\|\mu_s - \Gamma_{u+s}^\delta\right\|_{\mathbf{H}_\theta^{-r}} \leq C_{\Gamma^\delta} e^{-\lambda_\delta s}\left\|\mu_0 - \Gamma_u^\delta\right\|_{\mathbf{H}_\theta^{-r}}, \tag{5.16}$$

and from the estimates obtained above , we deduce

$$\|\nu_t\|_{\mathbf{H}_\theta^{-r}} \leq C_6\|\nu_0\|_{\mathbf{H}_\theta^{-r}} + C_6\delta\int_0^t \left(1 + (t-s)^{-\frac{1}{2}}e^{-\lambda_\delta(t-s)}\right)e^{-\lambda_\delta s}\|\nu_s\|_{\mathbf{H}_\theta^{-r}}\,\mathrm{d}s. \tag{5.17}$$

Applying Lemma B.2 for $\phi(u) = u^{-\frac{1}{2}}e^{-\lambda_\delta u}$, we obtain from (B.3) that

$$\sup_{t\geq 0}\|\nu_t\|_{\mathbf{H}_\theta^{-r}} \leq c_1\|\nu_0\|_{\mathbf{H}_\theta^{-r}}, \tag{5.18}$$

for some $c_1 > 0$.

Step 2 let us now show that $\left(DT^{nT_\delta}\right)_{n\geq 0}$ is a Cauchy sequence in the space $C\big(\mathcal{V}(\mathcal{M}^\delta, \varepsilon), \mathcal{B}\left(\mathbf{H}_\theta^{-r}\right)\big)$, which implies that $\mu \mapsto S(\mu)$ is C^1 (recall (5.2)).

For $n \geq m$ we have

$$\nu_{nT_\delta} - \nu_{mT_\delta} = \left(\Phi^\delta_{u+nT_\delta,u} - \Phi^\delta_{u+mT_\delta,u}\right)\nu_0 + \delta\int_{mT_\delta}^{nT_\delta}\Phi^\delta_{u+nT_\delta,u+s}\left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right)[\nu_s]\,\mathrm{d}s + \delta\int_0^{mT_\delta}\left(\Phi^\delta_{u+nT_\delta,u+s} - \Phi^\delta_{u+mT_\delta,u+s}\right)\left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right)[\nu_s]\,\mathrm{d}s. \tag{5.19}$$

For the first term, we get

$$\left\|\left(\Phi_{u+nT_\delta,u} - \Phi_{u+mT_\delta,u}\right)\nu_0\right\|_{\mathbf{H}_\theta^{-r}} = \left\|\left(\Phi_{u+nT_\delta,u} - \Phi_{u+mT_\delta,u}\right)\Pi_{\delta,u}\nu_0\right\|_{\mathbf{H}_\theta^{-r}} \leq C_7 e^{-\lambda_\delta m T_\delta}\left\|\nu_0\right\|_{\mathbf{H}_\theta^{-r}}. \tag{5.20}$$

For the second one, using (5.18),

$$\begin{aligned}
&\left\|\int_{mT_\delta}^{nT_\delta}\Phi^\delta_{u+nT_\delta,u+s}\big(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\big)[\nu_s]\,\mathrm{d}s\right\|_{\mathbf{H}_\theta^{-r}} \\
&\leq C_8\left\|\mu - \Gamma^\delta_u\right\|_{\mathbf{H}_\theta^{-r}}\left\|\nu_0\right\|_{\mathbf{H}_\theta^{-r}}\int_{mT_\delta}^{nT_\delta}\left(1 + (nT_\delta - s)^{-\frac{1}{2}}e^{-\lambda_\delta(nT_\delta - s)}\right)e^{-\lambda_\delta s}\,\mathrm{d}s \\
&= \frac{C_8\left\|\mu - \Gamma^\delta_u\right\|_{\mathbf{H}_\theta^{-r}}\left\|\nu_0\right\|_{\mathbf{H}_\theta^{-r}}}{\lambda_\delta}e^{-\lambda_\delta m T_\delta}\left(1 + e^{-\lambda_\delta(n-m)T_\delta}\left(2\lambda_\delta\sqrt{(n-m)T_\delta} - 1\right)\right) \\
&\leq C_9\left\|\mu - \Gamma^\delta_u\right\|_{\mathbf{H}_\theta^{-r}}\left\|\nu_0\right\|_{\mathbf{H}_\theta^{-r}}e^{-\lambda_\delta m T_\delta}.
\end{aligned} \tag{5.21}$$

For the last term, remark first that

$$\Phi^\delta_{u+nT_\delta,u+s} - \Phi^\delta_{u+mT_\delta,u+s} = \left(\Phi^\delta_{u+nT_\delta,u+mT} - I_d\right)\Pi^{\delta,s}_{u+mT_\delta}\Phi^\delta_{u+mT_\delta,u+s}, \tag{5.22}$$

so that, using again (5.18),

$$\begin{aligned}
&\left\|\int_0^{mT_\delta}\left(\Phi^\delta_{u+nT_\delta,u+s} - \Phi^\delta_{u+mT_\delta,u+s}\right)\left(DG(\mu_s) - DG(\Gamma_{u+s})\right)[\nu_s]ds\right\|_{\mathbf{H}_\theta^{-r}} \\
&\leq C_{10}\left\|\mu - \Gamma^\delta_u\right\|_{\mathbf{H}_\theta^{-r}}\left\|\nu_0\right\|_{\mathbf{H}_\theta^{-r}}\int_0^{mT_\delta}(mT_\delta - s)^{-\frac{1}{2}}e^{-\lambda_\delta(mT_\delta - s)}e^{-\lambda_\delta s}ds \\
&= 2C_{10}\left\|\mu - \Gamma^\delta_u\right\|_{\mathbf{H}_\theta^{-r}}\left\|\nu_0\right\|_{\mathbf{H}_\theta^{-r}}\sqrt{mT_\delta}e^{-\lambda_\delta m T_\delta}.
\end{aligned} \tag{5.23}$$

Since the constants above are uniform in $\mu \in \mathcal{V}$, we deduce that $\left(DT^{nT_\delta}\right)_{n\geq 0}$ is indeed a Cauchy sequence. Thus S is C^1 with $DS(\mu) = \lim_{n\to\infty} DT^{nT_\delta}(\mu)$.

Before moving to the second derivative, let us have a closer look at DS. We have

$$\left\|\Pi^{\delta,s}_{u+nT_\delta}\nu_{nT_\delta}\right\|_{\mathbf{H}_\theta^{-r}} \leq \left\|\Pi^{\delta,s}_{u+nT_\delta}\Phi^\delta_{u+nT_\delta,u}\nu_0\right\|_{\mathbf{H}_\theta^{-r}} + \left\|\int_0^{nT_\delta}\Pi^{\delta,s}_{u+nT_\delta}\Phi^\delta_{u+nT_\delta,u+s}\left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right)[\nu_s]ds\right\|_{\mathbf{H}_\theta^{-r}}, \tag{5.24}$$

and we can bound the right hand side in three steps. Firstly,

$$\left\| \Pi^{\delta,s}_{u+nT_\delta} \Phi^{\delta}_{u+nT_\delta,u} \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{\Phi,\delta} e^{-\lambda_\delta n T_\delta} \left\| \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} . \tag{5.25}$$

Secondly, since $\sup_{\mu\in\mathcal{V}} \|DG(\mu)\|_{\mathcal{B}\left(\mathbf{H}_\theta^{-r},\mathbf{H}_\theta^{-(r+1)}\right)} \leq C_G$,

$$\begin{aligned}
&\left\| \int_0^{\frac{nT_\delta}{2}} \Pi^{\delta}_{u+nT_\delta} \Phi^{\delta}_{u+nT_\delta,u+s} \left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right) [\nu_s] \, \mathrm{d}s \right\|_{\mathbf{H}_\theta^{-r}} \\
&\leq C_{11} \left\| \mu - \Gamma^\delta_u \right\|_{\mathbf{H}_\theta^{-r}} \left\| \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} \int_0^{\frac{nT_\delta}{2}} (nT_\delta - s)^{-\frac{1}{2}} e^{-\lambda_\delta (nT_\delta - s)} \, \mathrm{d}s \\
&\leq C_{12} \left\| \mu - \Gamma^\delta_u \right\|_{\mathbf{H}_\theta^{-r}} n^{\frac{1}{2}} e^{-\frac{\lambda_\delta n T_\delta}{2}} \left\| \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} .
\end{aligned} \tag{5.26}$$

Thirdly, by similar arguments as above (replacing mT_δ with $n\frac{T_\delta}{2}$),

$$\begin{aligned}
&\left\| \int_{\frac{nT_\delta}{2}}^{nT_\delta} \Phi^{\delta}_{u+nT_\delta,u+s} \left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right) [\nu_s] \, \mathrm{d}s \right\|_{\mathbf{H}_\theta^{-r}} \\
&\leq C_{13} \left\| \mu - \Gamma^\delta_u \right\|_{\mathbf{H}_\theta^{-r}} e^{-\lambda_\delta n \frac{T_\delta}{2}} \left\| \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} .
\end{aligned} \tag{5.27}$$

We deduce that $\Pi_{\Theta(\mu)} DS(\mu) = 0$, so that DS has rank 1 and thus there exists a family of linear forms $l_\mu \in \mathcal{B}\left(\mathbf{H}_\theta^{-r}, \mathbb{R}\right)$ (that depend continuously on μ) such that, for $u = \Theta(\mu)$,

$$DS(\mu)[\nu] = l_\mu[\nu] \partial_u \Gamma_u , \tag{5.28}$$

and we have proved, for $\nu_t = DT^t(\mu)[\nu_0]$,

$$\left\| \nu_{nT_\delta} - l_\mu[\nu_0] \partial_u \Gamma_u \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{13} n^{\frac{1}{2}} e^{-\lambda_\delta n \frac{T_\delta}{2}} \left\| \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} . \tag{5.29}$$

With similar computations one can in fact show that

$$\left\| \nu_t - l_\mu[\nu_0] \partial_u \Gamma_{u+t} \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{14} t^{\frac{1}{2}} e^{-\lambda_\delta \frac{t}{2}} \left\| \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} . \tag{5.30}$$

In the case when $\mu = \Gamma^\delta_u$, we deduce in particular that

$$DS(\Gamma^\delta_u) = \Pi^{\delta,c}_u . \tag{5.31}$$

In fact, we have proved a more precise estimate: if $\nu^2_t = DT^t(\mu)[\nu_0]$, $\nu^1_t = DT^t(\Gamma^\delta_u)[\nu_0]$ with $u = \Theta(\mu)$, the estimates above lead to

$$\left\| \nu^2_t - \nu^1_t - \left(l_\mu[\nu_0] - l_{\Gamma^\delta_u}[\nu_0] \right) \partial_u \Gamma_{u+t} \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{15} \left\| \mu - \Gamma^\delta_u \right\|_{\mathbf{H}_\theta^{-r}} t^{\frac{1}{2}} e^{-\lambda_\delta \frac{t}{2}} \left\| \nu_0 \right\|_{\mathbf{H}_\theta^{-r}} . \tag{5.32}$$

Step 3 let us now show that for a constant $c_2 > 0$,

$$\sup_{t\geq 0} \sup_{\mu\in\mathcal{V}(\Gamma^\delta,\varepsilon)} \left\| D^2 T^t(\mu) \right\|_{\mathcal{BL}(\mathbf{H}_\theta^{-r})} \leq c_2 . \tag{5.33}$$

From (2.9), we deduce, for $\xi_t = D^2 T^t(\mu)[\nu, w]$, the following mild formulation (recall that $\xi_0 = 0$):

$$\xi_t = \delta \int_0^t \Phi^\delta_{u+t,u+s} \left(D^2 G(\mu_s)[\nu_s, w_s] + \left(DG(\mu_s) - DG(\Gamma^\delta_{u+s}) \right) \xi_s \right) \mathrm{d}s , \tag{5.34}$$

where $\nu_t = DT^t(\mu_0)[\nu]$, $w_t = DT^t(\mu_0)[w]$. With similar arguments as above, we obtain

$$\begin{aligned}&\left\| \int_0^t \Phi^\delta_{u+t,u+s} \left(D^2G(\mu_s)[\nu_s, w_s] - D^2G(\Gamma^\delta_{u+s})[\nu_s, w_s]\right) \mathrm{d}s \right\|_{\mathbf{H}_\theta^{-r}} \\ &\leq C_{16} \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} e^{-\lambda_\delta(t-s)}\right) \left\|\mu_s - \Gamma^\delta_{u+s}\right\|_{\mathbf{H}_\theta^{-r}} \|\nu_s\|_{\mathbf{H}_\theta^{-r}} \|w_s\|_{\mathbf{H}_\theta^{-r}} \mathrm{d}s \\ &\leq C_{16} \|\nu_0\|_{\mathbf{H}_\theta^{-r}} \|w_0\|_{\mathbf{H}_\theta^{-r}} . \end{aligned} \tag{5.35}$$

Remark now that

$$\partial_t^2 \Gamma^\delta_{u+t} = (\mathcal{L}\partial_t q^\delta_{u+t}, 0) + \delta DG(\Gamma^\delta_{u+t})[\partial_t \Gamma_{u+t}], \tag{5.36}$$

and

$$\partial_t^3 \Gamma^\delta_{u+t} = \left(\mathcal{L}\partial_t^2 q^\delta_{u+t}, 0\right) + \delta DG(\Gamma^\delta_{u+t})[\partial_t^2 \Gamma^\delta_{u+t}] + \delta D^2G(\Gamma^\delta_{u+t})[\partial_t \Gamma^\delta_{u+t}, \partial_t \Gamma^\delta_{u+t}], \tag{5.37}$$

and thus

$$\partial_t^2 \Gamma^\delta_{u+t} = \Phi^\delta_{u+t,u} \partial_t^2 \Gamma^\delta_u + \delta \int_0^t \Phi^\delta_{u+t,u+s} D^2G(\Gamma^\delta_{u+s})[\partial_s \Gamma^\delta_{u+s}, \partial_s \Gamma^\delta_{u+s}] ds. \tag{5.38}$$

So, in particular, since

$$\Pi^{\delta,c}_u \Phi^\delta_{u+T_\delta,u} \partial_u^2 \Gamma_u = \Pi^{\delta,c}_u (\partial_u^2 \Gamma_u), \tag{5.39}$$

we deduce from (5.38) that

$$\Pi^{\delta,c}_u \left(\int_0^{T_\delta} \Phi^\delta_{u+T_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\partial_s \Gamma^\delta_{u+s}, \partial_s \Gamma^\delta_{u+s}] ds \right) = 0. \tag{5.40}$$

Now, recalling (5.30),

$$\left\| \nu_t - l_\mu[\nu_0] \partial_u \Gamma_{u+t} \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{14} t^{\frac{1}{2}} e^{-\lambda_\delta \frac{t}{2}} \|\nu_0\|_{\mathbf{H}_\theta^{-r}} , \tag{5.41}$$

$$\left\| w_t - l_\mu[w_0] \partial_u \Gamma_{u+t} \right\|_{\mathbf{H}_\theta^{-r}} \leq C_{14} t^{\frac{1}{2}} e^{-\lambda_\delta \frac{t}{2}} \|w_0\|_{\mathbf{H}_\theta^{-r}} , \tag{5.42}$$

and we deduce

$$\begin{aligned}&\left\| \int_0^t \Phi_{u+t,u+s} D^2G(\mu_s)[\nu_s, w_s] ds \right. \\ &\qquad \left. - l_\mu[\nu_0] l_\mu[w_0] \int_0^t \Phi_{u+t,u+s} D^2G(\mu_s)[\partial_u \Gamma_{u+s}, \partial_u \Gamma_{u+s}] \, \mathrm{d}s \right\|_{\mathbf{H}_\theta^{-r}} \\ &\leq C_{17} \|\nu_0\|_{\mathbf{H}_\theta^{-r}} \|w_0\|_{\mathbf{H}_\theta^{-r}} . \end{aligned} \tag{5.43}$$

So, recalling (5.40), and since

$$\begin{aligned}&\left\| \Pi^{\delta,s}_{u+t} \int_0^t \Phi^\delta_{u+t,u+s} D^2G(\mu_s)[\partial_u \Gamma^\delta_{u+s}, \partial_u \Gamma^\delta_{u+s}] \, \mathrm{d}s \right\|_{\mathbf{H}_\theta^{-r}} \\ &\leq C_{18} \int_0^t (t-s)^{-\frac{1}{2}} e^{-\lambda_\delta(t-s)} ds \leq C_{19}, \end{aligned} \tag{5.44}$$

we deduce, coming back to (5.34), that

$$\|\xi_t\|_{\mathbf{H}_\theta^{-r}} \leq C_{19}\,\|\nu_0\|_{\mathbf{H}_\theta^{-r}}\,\|w_0\|_{\mathbf{H}_\theta^{-r}} + \delta\left\|\int_0^t \Phi^\delta_{u+t,u+s}\left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right)\xi_s\,\mathrm{d}s\right\|_{\mathbf{H}_\theta^{-r}}. \tag{5.45}$$

Relying again on (5.16), we deduce that, for some $c_2 > 0$,

$$\|\xi_t\|_{\mathbf{H}_\theta^{-r}} \leq c_2\,\|\nu_0\|_{\mathbf{H}_\theta^{-r}}\,\|w_0\|_{\mathbf{H}_\theta^{-r}}, \tag{5.46}$$

which implies (5.33).

Step 4 let us now prove that $\left(D^2T^{nT_\delta}\right)_{n\geq 0}$ is a Cauchy sequence in the space $C\left(\mathcal{V}(\mathcal{M}^\delta,\varepsilon), \mathcal{BL}\left(\mathbf{H}_\theta^{-r}\right)\right)$, which implies that $\mu \mapsto S(\mu)$ is C^2.

We have, for $n \geq m$,

$$\begin{aligned}
\xi_{nT_\delta} - \xi_{mT_\delta} = &\int_0^{nT_\delta} \Phi^\delta_{u+nT_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_s, w_s]\,\mathrm{d}s \\
&- \int_0^{mT_\delta} \Phi^\delta_{u+mT_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_s, w_s]\,\mathrm{d}s \\
&+ \int_{mT_\delta}^{nT_\delta} \Phi^\delta_{u+nT_\delta,u+s}\left(D^2G(\mu_s)[\nu_s,w_s] - D^2G(\Gamma^\delta_{u+s})[\nu_s,w_s]\right)\mathrm{d}s \\
&+ \int_0^{mT_\delta}\left(\Phi^\delta_{u+nT_\delta,u+s} - \Phi^\delta_{u+mT_\delta,u+s}\right) \\
&\times \left(D^2G(\mu_s)[\nu_s,w_s] - D^2G(\Gamma^\delta_{u+s})[\nu_s,w_s]\right)\mathrm{d}s \\
&+ \int_{mT_\delta}^{nT_\delta} \Phi^\delta_{u+nT_\delta,u+s}\left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right)\xi_s\,\mathrm{d}s \\
&+ \int_0^{nT_\delta}\left(\Phi^\delta_{u+nT_\delta,u+s} - \Phi^\delta_{u+mT_\delta,u+s}\right)\left(DG(\mu_s) - DG(\Gamma^\delta_{u+s})\right)\xi_s\,\mathrm{d}s.
\end{aligned} \tag{5.47}$$

Let us define

$$\begin{aligned}
R_n^{norm} &:= \int_0^{nT_\delta} \Pi^{\delta,s}_{u+nT_\delta}\Phi^\delta_{u+nT_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\partial_s\Gamma^\delta_{u+s}, \partial_s\Gamma^\delta_{u+s}]\,\mathrm{d}s \\
&= \sum_{j=0}^{n-1}\left(\Phi^\delta_{u+T_\delta,u}\Pi^{\delta,s}_u\right)^j \int_0^{T_\delta} \Phi^\delta_{u+T_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\partial_s\Gamma^\delta_{u+s}, \partial_s\Gamma^\delta_{u+s}]\,\mathrm{d}s,
\end{aligned} \tag{5.48}$$

and

$$R_n^{tang}[\nu_0, w_0] := \int_0^{nT_\delta} \Pi^{\delta,c}_{u+nT_\delta}\Phi^\delta_{u+nT_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_s, w_s]\,\mathrm{d}s \tag{5.49}$$

$$= \sum_{j=0}^{n-1}\int_0^{T_\delta} \Pi^{\delta,c}_{u+T_\delta}\Phi^\delta_{u+T_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_{jT_\delta+s}, w_{jT_\delta+s}]\,\mathrm{d}s. \tag{5.50}$$

It is clear that

$$\left\|R_n^{norm} - R_m^{norm}\right\|_{\mathbf{H}_\theta^{-r}} \leq Ce^{-\lambda_\delta mT_\delta}. \tag{5.51}$$

Now, for $j \geq 1$, recalling (5.40), (5.41) and (5.42) we have

$$\begin{aligned}
&\left\| \int_0^{T_\delta} \Pi^{\delta,c}_{u+T_\delta} \Phi^\delta_{u+T_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_{jT_\delta+s}, w_{jT_\delta+s}]\,\mathrm{d}s \right\|_{\mathbf{H}^{-r}_\theta} \\
&\leq C_{20} \|\nu_0\|_{\mathbf{H}^{-r}_\theta} \|w_0\|_{\mathbf{H}^{-r}_\theta} \int_0^{T_\delta} \left(1 + (T_\delta - s)^{-\frac{1}{2}} e^{-\lambda_\delta (T_\delta - s)}\right) (jT_\delta + s)^{\frac{1}{2}} e^{-\lambda_\delta \frac{jT_\delta + s}{2}}\,\mathrm{d}s \\
&\leq C_{21} \|\nu_0\|_{\mathbf{H}^{-r}_\theta} \|w_0\|_{\mathbf{H}^{-r}_\theta} (j+1)^{\frac{1}{2}} e^{-\lambda_\delta j \frac{T_\delta}{2}}, \qquad (5.52)
\end{aligned}$$

so that

$$\left\| R^{tan}_n[\nu_0, w_0] - R^{tang}_m[\nu_0, w_0] \right\|_{\mathbf{H}^{-r}_\theta} \leq C_{22} e^{-\lambda_\delta m \frac{T_\delta}{4}} \|\nu_0\|_{\mathbf{H}^{-r}_\theta} \|w_0\|_{\mathbf{H}^{-r}_\theta}. \qquad (5.53)$$

Using similar arguments as above, relying on (5.41) and (5.42),

$$\begin{aligned}
&\left\| \int_0^{nT_\delta} \Phi^\delta_{u+nT_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_s, w_s] ds - R^{tan}_n[\nu_0, w_0] - l_\mu[\nu_0] l_\mu[w_0] R^{norm}_n\,\mathrm{d}s \right\|_{\mathbf{H}^{-r}_\theta} \\
&\leq C_{23} \|\nu_0\|_{\mathbf{H}^{-r}_\theta} \|w_0\|_{\mathbf{H}^{-r}_\theta} \int_0^{nT_\delta} (nT_\delta - s)^{-\frac{1}{2}} e^{-\lambda_\delta (nT_\delta - s)} s^{\frac{1}{2}} e^{-\lambda_\delta \frac{s}{2}}\,\mathrm{d}s \\
&\leq C_{24} n^{\frac{1}{2}} e^{-\lambda_\delta n \frac{T_\delta}{2}} \|\nu_0\|_{\mathbf{H}^{-r}_\theta} \|w_0\|_{\mathbf{H}^{-r}_\theta}. \qquad (5.54)
\end{aligned}$$

With all these estimates we are able to tackle the first two lines of (5.47):

$$\begin{aligned}
&\left\| \int_0^{nT_\delta} \Phi^\delta_{u+nT_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_s, w_s]\,\mathrm{d}s - \int_0^{mT_\delta} \Phi^\delta_{u+mT_\delta,u+s} D^2G(\Gamma^\delta_{u+s})[\nu_s, w_s]\,\mathrm{d}s \right\|_{\mathbf{H}^{-r}_\theta} \\
&\leq C_{25} n^{\frac{1}{2}} e^{-\lambda_\delta n \frac{T_\delta}{4}} \|\nu_0\|_{\mathbf{H}^{-r}_\theta} \|w_0\|_{\mathbf{H}^{-r}_\theta}. \qquad (5.55)
\end{aligned}$$

The other terms can be treated in a straightforward way, with similar estimates as the ones used in Step 2 and Step 3. At the end, one obtains

$$\left\| \xi_{nT_\delta} - \xi_{mT_\delta} \right\|_{\mathbf{H}^{-r}_\theta} \leq C_{26} m^{\frac{1}{2}} e^{-\lambda_\delta m \frac{T_\delta}{4}}, \qquad (5.56)$$

with a constant C_{25} uniform in $\mathcal{V}$. Hence, $\mu \mapsto S(\mu)$ is thus C^2. Remark that we have in particular

$$D^2S(\Gamma_u)[\nu, w] = \lim_{n\to\infty} \int_0^{nT_\delta} \Phi_{u+nT_\delta,u+s} D^2G(\Gamma_{u+s})[\Phi_{u+s,u}\nu, \Phi_{u+s,u}w]\,\mathrm{d}s. \quad (5.57)$$

Step 5 from the previous steps, and the fact that $t \mapsto \Gamma^\delta_t$ is a C^2 bijection from $\mathbb{R}/T_\delta\mathbb{Z}$ to $\mathcal{M}^\delta$ implies that Θ^δ is itself C^2.

For the last estimate of the Theorem, let us denote $\xi^2_t = D^2T^t(\mu)[\nu, w]$, $\xi^1_t = D^2T^t(\Gamma^\delta_u)[\nu, w]$, $\nu^2_t = DT^t(\mu)[\nu]$, $\nu^1_t = DT^t(\Gamma^\delta_u)[\nu]$, $w^2_t = DT^t(\mu)[w]$ and $w^1_t =$

$DT^t(\Gamma_u^\delta)[w]$. We then have the decomposition

$$\begin{aligned}\xi_t^2-\xi_t^1 &= \delta\int_0^t \Phi^\delta_{u+t,u+s}\left(DG(\mu_s)-DG\left(\Gamma^\delta_{u+s}\right)\right)\xi_s^2\,\mathrm{d}s \\ &+\delta\int_0^t \Phi^\delta_{u+t,u+s}\left(D^2G(\mu_s)-D^2G\left(\Gamma^\delta_{u+s}\right)\right)[\nu_s^2,w_s^2]\,\mathrm{d}s \\ &+\delta\int_0^t \Phi^\delta_{u+t,u+s}D^2G\left(\Gamma^\delta_{u+s}\right)[\nu_s^2-\nu_s^1,w_s^2]\,\mathrm{d}s \\ &+\delta\int_0^t \Phi^\delta_{u+t,u+s}D^2G\left(\Gamma^\delta_{u+s}\right)[\nu_s^1,w_s^2-w_s^1]\,\mathrm{d}s.\end{aligned} \tag{5.58}$$

Following similar estimates as in the previous steps, relying in particular on (5.32), we obtain

$$\left\|\xi_t^2-\xi_t^1\right\|_{\mathbf{H}_\theta^{-r}} \le C_{27}\left\|\mu-\Gamma_u^\delta\right\|_{\mathbf{H}_\theta^{-r}}\|\nu\|_{\mathbf{H}_\theta^{-r}}\|w\|_{\mathbf{H}_\theta^{-r}}, \tag{5.59}$$

which implies indeed that $\left\|D^2\Theta^\delta(\mu)-D^2\Theta^\delta\left(\Gamma_u^\delta\right)\right\|_{\mathcal{BL}(\mathbf{H}_\theta^{-r})} \le C_{28}\left\|\mu-\Gamma_u^\delta\right\|_{\mathbf{H}_\theta^{-r}}$. □

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Appendix A. Ornstein–Uhlenbeck Operators

The aim of this Section is to give bounds for the operators and norms that where defined in Sect. 1.3. In the sequel, the following notation will be used: for any multi-index $l=(l_1,\dots,l_d)\in\mathbb{N}^d$ and $i\in\{1,\dots,d\}$, denote by

$$l_{\downarrow_i}=(l_1,\dots,l_{i-1},l_i-1,l_{i+1},\dots,l_d), \tag{A.1}$$

$$l_{\uparrow_i}=(l_1,\dots,l_{i-1},l_i+1,l_{i+1},\dots,l_d) \tag{A.2}$$

as the shifts w.r.t. the ith coordinate (multiple arrows notation such as $l_{\uparrow\uparrow_i}$ corresponding to iterated shifts).

We first prove the following lemma, which shows the link between the norm $\|f\|_{H_\theta^r}$ and the space derivatives.

Lemma A.1 *For all $\theta>0$, there exists explicit positive constants C_1, C_2 such that for all $r\ge 0$:*

$$C_1\left(\|u\|^2_{H_\theta^r}+\sum_{i=1}^d\left\|\partial_{x_i}u\right\|^2_{H_\theta^r}\right)\le\|u\|^2_{H_\theta^{r+1}}\le C_2\left(\|u\|^2_{H_\theta^r}+\sum_{i=1}^d\left\|\partial_{x_i}u\right\|^2_{H_\theta^r}\right). \tag{A.3}$$

Proof Recall the definitions of ψ_l in (1.19) and of h_n in (1.20). For u with decomposition $u=\sum_{l\in\mathbb{N}^d}u_l\,\psi_l$, we have $\partial_{x_i}u=\sum_{l\in\mathbb{N}^d}u_l\,\partial_{x_i}\psi_l$, and straightforward calculations using the fact that $h_n'(x)=\sqrt{n}h_{n-1}(x)$ show that

$$\partial_{x_i}\psi_l=\sqrt{l_i}\sqrt{\frac{\theta k_i}{\sigma_i^2}}\psi_{l_{\downarrow_i}}\mathbf{1}_{l_i\ge 1}, \tag{A.4}$$

where we used the notation (A.1). Then we have the decomposition

$$\partial_{x_i} u = \sqrt{\frac{\theta k_i}{\sigma_i^2}} \sum_{l \in \mathbb{N}^d} \sqrt{l_i} \mathbf{1}_{l_i \geq 1} u_l \, \psi_{l_{\downarrow_i}}, \tag{A.5}$$

so that, by definition of the H_θ^r-norm in (1.21) (recall in particular that $a_\theta = \theta \mathrm{Tr} K$)

$$\|u\|_{H_\theta^r}^2 + \sum_{i=1}^d \|\partial_{x_i} u\|_{H_\theta^r}^2 = \sum_{l \in \mathbb{N}^d} u_l^2 \left((a_\theta + \lambda_l)^r + \sum_{i=1}^d l_i \mathbf{1}_{l_i \geq 1} \frac{\theta k_i}{\sigma_i^2} (a_\theta + \lambda_l - \theta k_i)^r \right). \tag{A.6}$$

Let us prove the upper bound in (A.3): note that for $l_i \geq 1$, we have $\lambda_l \geq \theta k_i$. Hence, since for all $\mu \geq 0$, $r \geq 0$, $\lambda \geq \mu$, $(a_\theta + \lambda - \mu)^r \geq a_\theta^r \frac{(a_\theta+\lambda)^r}{(a_\theta+\mu)^r}$, we deduce that

$$\begin{aligned} \|u\|_{H_\theta^r}^2 + \sum_{i=1}^d \|\partial_{x_i} u\|_{H_\theta^r}^2 &\geq \sum_{l \in \mathbb{N}^d} u_l^2 (a_\theta + \lambda_l)^r \left(1 + \sum_{i=1}^d l_i \mathbf{1}_{l_i \geq 1} \frac{\theta k_i a_\theta^r}{\sigma_i^2 (a_\theta + \theta k_i)^r} \right), \\ &\geq \sum_{l \in \mathbb{N}^d} u_l^2 (a_\theta + \lambda_l)^r \left(1 + \frac{a_\theta^r}{\sigma_{\max}^2 (a_\theta + \theta k_{\max})^r} \lambda_l \right), \end{aligned}$$

so that the upper bound in (A.3) is true for $C_2 := \max\left(\frac{\sigma_{\max}^2 (a_\theta + \theta k_{\max})^r}{a_\theta^r}, a_\theta \right)$. Concerning the lower bound in (A.3), we have from (A.6),

$$\|u\|_{H_\theta^r}^2 + \sum_{i=1}^d \|\partial_{x_i} u\|_{H_\theta^r}^2 \leq \sum_{l \in \mathbb{N}^d} u_l^2 (a_\theta + \lambda_l)^r \left(1 + \frac{\lambda_l}{\sigma_{\min}^2} \right),$$

where $\sigma_{\min}$ is given in (1.16), so that the upper bound holds for $C_1 := \frac{1}{\min(\sigma_{\min}^2, a_\theta)}$. □

For all $\theta > 0$, the operator $-\mathcal{L}_\theta^*$ (recall its definition (1.18) and its decomposition (1.19)) is sectorial in L_θ^2 and generates a semigroup $e^{t\mathcal{L}_\theta^*}$ satisfying (see e.g. [23]) for all $\alpha \geq 0$, $r \geq 0$, and $\lambda < \theta \min(k_1, \ldots, k_d)$, there exists some $C > 0$ such that for all $f \in H_\theta^r$,

$$\left\| e^{t\mathcal{L}_\theta^*} f \right\|_{H_\theta^{r+\alpha}} \leq C \left(1 + t^{-\alpha/2} e^{-\lambda t} \right) \|f\|_{H_\theta^r}, \tag{A.7}$$

and for all $f \in H_\theta^r$ such that $\int f w_\theta = 0$,

$$\left\| e^{t\mathcal{L}_\theta^*} f \right\|_{H_\theta^{r+\alpha}} \leq C t^{-\frac{1}{2}} e^{-\lambda t} \|f\|_{H_\theta^r}. \tag{A.8}$$

Let $\theta' > 0$. The point of the following result is to state similar contraction results for $\mathcal{L}_{\theta'}^*$ in H_θ^r, in the case $\theta' \neq \theta$:

Proposition A.2 *For all $0 < \theta \leq \theta'$ the following is true: the operator $\mathcal{L}_{\theta'}^*$ generates an analytic semigroup in H_θ^r and for all $r \geq 0$, $\alpha \geq 0$ and $\lambda < \theta k_{\min}$, there exists a constant $C > 0$ such that for all $f \in H_\theta^r$ and $t > 0$*

$$\left\| e^{t\mathcal{L}_{\theta'}^*} f \right\|_{H_\theta^{r+\alpha}} \leq C \left(1 + t^{-\alpha/2} e^{-\lambda t} \right) \|f\|_{H_\theta^r}, \tag{A.9}$$

and for all $r \geq 1$,

$$\left\| \nabla e^{t\mathcal{L}^*_{\theta'}} f \right\|_{H^r_\theta} \leq C t^{-\frac{1}{2}} e^{-\lambda t} \|f\|_{H^r_\theta}. \tag{A.10}$$

Moreover for all $r \geq 0$, $0 < \varepsilon \leq 1$ *and* $s \geq 0$,

$$\left\| \left(e^{(t+s)\mathcal{L}^*_{\theta'}} - e^{t\mathcal{L}^*_{\theta'}} \right) f \right\|_{H^{r+1}_\theta} \leq C s^\varepsilon t^{-\frac{1}{2}-\varepsilon} e^{-\lambda t} \|f\|_{H^r_\theta}. \tag{A.11}$$

Finally, there exists $r_0 > 0$ *such that for all* $r > r_0$, $t > 0$ *and all* $f \in H^r_\theta$,

$$\left\| e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta} \right\|_{H^r_\theta} \leq e^{-\lambda t} \left\| f - \int_{\mathbb{R}^d} f w_\theta \right\|_{H^r_\theta}. \tag{A.12}$$

Proof of Proposition A.2 First remark that for all smooth test function u

$$\left(\mathcal{L}^*_\theta - \mathcal{L}^*_{\theta'} \right) u = (\theta' - \theta) K x \cdot \nabla u. \tag{A.13}$$

Recalling the decomposition (1.19),since $h'_n(x) = \sqrt{n} h_{n-1}(x)$ and $x h_{n-1}(x) = \sqrt{n} h_n(x) + \sqrt{n-1} h_{n-2}(x)$ (see e.g. [6], p.102), we get, for all $l \in \mathbb{N}$,

$$\begin{aligned} \left(\mathcal{L}^*_\theta - \mathcal{L}^*_{\theta'} \right) \psi_{\theta,l} &= (\theta' - \theta) \sum_{i=1}^d k_i \sqrt{\frac{\theta k_i}{\sigma_i^2}} \sqrt{l_i} x_i \psi_{\theta, l_{\downarrow_i}} && \text{(A.14)} \\ &= (\theta' - \theta) \sum_{i=1}^d k_i \left(l_i \psi_{\theta,l} + \sqrt{l_i(l_i - 1)} \psi_{\theta, l_{\downarrow\downarrow_i}} \right), && \text{(A.15)} \end{aligned}$$

where we used the notation (A.1) and (A.2) and the convention $\psi_l = 0$ if $l_i < 0$ for some $i \in \{1, \ldots, d\}$. In particular we have, recalling that $\lambda_{\theta,l} = \theta \sum_{i=1}^d k_i l_i$,

$$-\mathcal{L}^*_{\theta'} \psi_{\theta,l} = \frac{\theta' \lambda_{\theta,l}}{\theta} \psi_{\theta,l} + (\theta' - \theta) \sum_{i=1}^d k_i \sqrt{l_i(l_i - 1)} \psi_{\theta, l_{\downarrow\downarrow_i}}, \tag{A.16}$$

So, we deduce that for $f = \sum_l f_l \psi_{\theta,l}$, with $f_l \in \mathbb{C}$ for all l,

$$\left\| \left(\mathcal{L}^*_\theta - \frac{\theta}{\theta'} \mathcal{L}^* \right) f \right\|^2_{H^r_\theta} = \left(1 - \frac{\theta}{\theta'} \right)^2 \sum_l (a_\theta + \lambda_{\theta,l})^r \left| \sum_{i=1}^d \theta k_i \sqrt{(l_i + 1)(l_i + 2)} f_{l_{\uparrow\uparrow_i}} \right|^2.$$

Setting

$$I(r, f) := \sum_l (a_\theta + \lambda_{\theta,l})^r \left(\sum_{i=1}^d \theta k_i \sqrt{(l_i + 1)(l_i + 2)} f_{l_{\uparrow\uparrow_i}} \right)^2. \tag{A.17}$$

Using Jensen's inequality, we obtain (recalling that $a_\theta = \theta \mathrm{Tr}(K)$),

$$
\begin{aligned}
I(r, f) &\leq \sum_{l_1,\dots,l_d=0}^{+\infty} \left(a_\theta + \lambda_{\theta,l}\right)^r \left(\sum_{i=1}^{d} \theta k_i (l_i+1)\right) \left(\sum_{i=1}^{d} \theta k_i (l_i+2) \left|f_{l_{\uparrow\uparrow_i}}\right|^2\right) \\
&= \sum_{l_1,\dots,l_d=0}^{+\infty} \left(a_\theta + \lambda_{\theta,l}\right)^{r+1} \left(\sum_{i=1}^{d} \theta k_i (l_i+2) \left|f_{l_{\uparrow\uparrow_i}}\right|^2\right) \\
&= \sum_{l_1,\dots,l_d=2}^{+\infty} \left(a_\theta + \lambda_{\theta,l-2}\right)^{r+1} \left(\sum_{i=1}^{d} \theta k_i l_i \left|f_{l_1-2,\dots,l_i,l_{i+1}-2,l_d-2}\right|^2\right) \\
&= \sum_{i=1}^{d} \sum_{l_1,\dots,l_d=2}^{+\infty} \left(a_\theta + \lambda_{\theta,l-2}\right)^{r+1} \theta k_i l_i \left|f_{l_1-2,\dots,l_i,l_{i+1}-2,l_d-2}\right|^2 \\
&= \sum_{i=1}^{d} \sum_{l_i=2}^{+\infty} \sum_{\substack{l_p=0 \\ p\neq i}}^{+\infty} \left(a_\theta + \lambda_{\theta,l_{\downarrow\downarrow_i}}\right)^{r+1} \theta k_i l_i \left|f_l\right|^2 .
\end{aligned}
$$

Now we use that $\lambda_{\theta,l_{\downarrow\downarrow_i}} \leq \lambda_{\theta,l}$ for any l and i, so that

$$
I(r, f) \leq \sum_{i=1}^{d} \sum_{l_i=2}^{+\infty} \sum_{\substack{l_p=0 \\ p\neq i}}^{+\infty} \left(a_\theta + \lambda_{\theta,l}\right)^{r+1} \theta k_i l_i \left|f_l\right|^2 , \tag{A.18}
$$

This sum is anyway smaller than

$$
\begin{aligned}
I(r, f) &\leq \sum_{i=1}^{d} \sum_{l_i=0}^{+\infty} \sum_{\substack{l_p=0 \\ p\neq i}}^{+\infty} \left(a_\theta + \lambda_{\theta,l}\right)^{r+1} \theta k_i l_i \left|f_l\right|^2 = \sum_{l_1=0}^{+\infty} \dots \sum_{l_d=0}^{+\infty} \left(a_\theta + \lambda_{\theta,l}\right)^{r+1} \lambda_{\theta,l} \left|f_l\right|^2 , \\
&\leq \sum_{l_1=0}^{+\infty} \dots \sum_{l_d=0}^{+\infty} \left(a_\theta + \lambda_{\theta,l}\right)^{r+2} \left|f_l\right|^2 = \left\|(a_\theta - \mathcal{L}_\theta^*) f\right\|_{H_\theta^r}^2 .
\end{aligned} \tag{A.19}
$$

Coming back to (A.17), we obtain

$$
\left\|\left(\mathcal{L}_\theta^* - \frac{\theta}{\theta'}\mathcal{L}_{\theta'}^*\right) f\right\|_{H_\theta^r} \leq \left(1 - \frac{\theta}{\theta'}\right) \left\|(a_\theta - \mathcal{L}_\theta^*) f\right\|_{H_\theta^r} , \tag{A.20}
$$

which implies in particular that

$$
\left\|\left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}_{\theta'}^*\right) f\right\|_{H_\theta^r} \leq \left(2 - \frac{\theta}{\theta'}\right) \left\|(a_\theta - \mathcal{L}_\theta^*) f\right\|_{H_\theta^r} = \left(2 - \frac{\theta}{\theta'}\right) \left\|f\right\|_{H_\theta^{r+2}} . \tag{A.21}
$$

Let us now look for a lower bound. Using $(a+b)^2 \geq \frac{\varepsilon}{1+\varepsilon}a^2 - \varepsilon b^2$ ($\varepsilon > 0$), we get

$$\begin{aligned}\left\| \left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right) f \right\|^2_{H^r_\theta} &= \sum_l (a_\theta + \lambda_{\theta,l})^r \Bigg((a_\theta + \lambda_{\theta,l}) f_l \\ &\quad + \left(1 - \frac{\theta}{\theta'}\right) \sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, f_{l_{\uparrow\uparrow_i}} \Bigg)^2 \\ &\geq \frac{\varepsilon}{1+\varepsilon} \sum_l (a_\theta + \lambda_{\theta,l})^{r+2} f_l^2 - \varepsilon \left(1 - \frac{\theta}{\theta'}\right)^2 I(r,f). \end{aligned} \tag{A.22}$$

Hence, recalling (A.19), we obtain

$$\left\| \left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right) f \right\|^2_{H^r_\theta} \geq \varepsilon \left(\frac{1}{1+\varepsilon} - \left(1 - \frac{\theta}{\theta'}\right)^2 \right) \left\| (a_\theta - \mathcal{L}^*_\theta) f \right\|^2_{H^r_\theta}. \tag{A.23}$$

So for $\varepsilon > 0$ small enough (depending only on θ, θ'), there exists a constant $c_{\theta,\theta'} > 0$ such that we have $\left\| \left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right) f \right\|_{H^r_\theta} \geq c_{\theta,\theta'} \left\| (a_\theta - \mathcal{L}^*_\theta) f \right\|_{H^r_\theta}$. This means, together with (A.21), that

$$c_{\theta,\theta'} \|f\|_{H^{r+2}_\theta} \leq \left\| \left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right) f \right\|_{H^r_\theta} \leq \left(2 - \frac{\theta}{\theta'}\right) \|f\|_{H^{r+2}_\theta}. \tag{A.24}$$

In particular 0 is in the resolvent set of $a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}$, and $a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}$ has a compact resolvent, since it is the case for $a_\theta - \mathcal{L}^*_\theta$. So $\mathcal{L}^*_{\theta'}$ has a discrete spectrum, composed of a sequence of eigenvalues with modulus going to infinity. But any eigenfunction ψ of $\mathcal{L}^*_{\theta'}$ in H^r_θ is also an eigenfunction of $\mathcal{L}^*_{\theta'}$ in $H^r_{\theta'}$, and thus the eigenvalues of $\mathcal{L}^*_{\theta'}$ in H^r_θ are the $\lambda_{l,\theta'}$'s, with associated eigenfunctions the $\psi_{l,\theta'}$'s. So in particular $\mathcal{L}^*_{\theta'}$ is sectorial, and thus generates an analytic semigroup $e^{t\mathcal{L}^*_{\theta'}}$ in H^r_θ.

Let us now prove that the interpolation spaces induced by $\mathcal{L}^*_{\theta'}$ and $\mathcal{L}^*_\theta$ in H^r_θ are equivalent. Since the operator $\left(\frac{\theta}{\theta'}\mathcal{L}^*_{\theta'} - \mathcal{L}^*_\theta\right)\left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right)^{-1}$ (and thus $(1 + (\theta\mathcal{L}^* - \mathcal{L}^*_\theta)(a_\theta - \theta\mathcal{L}^*)^{-1})^\alpha$ for $\alpha \geq 0$) is bounded in H^r_θ, we obtain

$$\begin{aligned}\|f\|_{H^{r+\alpha}_\theta} &= \left\| \left(a_\theta - \mathcal{L}^*_\theta\right)^{\alpha/2} f \right\|_{H^r_\theta} \\ &= \left\| \left(1 + \left(\frac{\theta}{\theta'}\mathcal{L}^*_{\theta'} - \mathcal{L}^*_\theta\right)\left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right)^{-1}\right)^{\alpha/2} \left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right)^{\alpha/2} f \right\|_{H^r_\theta} \\ &\leq C \left\| \left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right)^{\alpha/2} f \right\|_{H^r_\theta}. \end{aligned} \tag{A.25}$$

The inverse bound $\left\| \left(a_\theta - \frac{\theta}{\theta'}\mathcal{L}^*_{\theta'}\right)^{\alpha/2} f \right\|_{H^r_\theta} \leq C\|f\|_{H^{r+\alpha}_\theta}$ follows from similar arguments.

We are now in condition to prove (A.7), applying [23], Th. 1.4.3. Indeed, $\mathcal{L}^*_{\theta'}$ has a real spectrum located on the left of $-\theta' k_{\min}$ on the subspace of H^r_θ generated by the eigenfunctions $\psi_{l,\theta'}$ with $l \neq 0$, so applying this Theorem we get, denoting $\mathcal{P}_{\theta'} f = f - \frac{\int_{\mathbb{R}^d} f w_{\theta'}}{\left(\int_{\mathbb{R}^d} w_{\theta'}\right)^2}$ the projection on this subspace (which is an element of $\mathcal{B}(H^r_\theta)$ for $0 < \theta \le \theta'$),

$$\left\| e^{t\mathcal{L}^*_{\theta'}} \mathcal{P}_{\theta'} f \right\|_{H^{r+\alpha}_\theta} \le C_1 \left\| \left(a_\theta - \frac{\theta}{\theta'} \mathcal{L}^*_{\theta'} \right)^{\alpha/2} e^{t\mathcal{L}^*_{\theta'}} \mathcal{P}_{\theta'} f \right\|_{H^r_\theta} \le C_2 t^{-\alpha/2} e^{-\lambda t} \| f \|_{H^r_\theta} . \tag{A.26}$$

This implies (A.7), since $e^{t\mathcal{L}^*_{\theta'}} (1 - \mathcal{P}_{\theta'}) f = (1 - \mathcal{P}_{\theta'}) f$.

The proof of (A.8) relies on the classical identity, valid for $f \in L^2_{\theta'}$,

$$e^{t\mathcal{L}^*_{\theta'}} f = \mathbb{E}\left[f\left(e^{-t\theta' K} x + \sqrt{1 - e^{-2t\theta' K}} G_{\theta'} \right) \right], \tag{A.27}$$

where $G_{\theta'}$ is a gaussian variable on $\mathbb{R}^d$ with mean 0 and variance $(\theta' K)^{-1}\sigma^2$. This implies directly, for $f \in H^r_\theta$ with $r \ge 1$,

$$\nabla e^{t\mathcal{L}^*_N} f = e^{-t\theta' K} e^{t\mathcal{L}^*_N} \nabla f, \tag{A.28}$$

and thus, recalling the definition of $k_{\min}$ in (1.15),

$$\begin{aligned} \left\| \nabla e^{t\mathcal{L}^*_N} f \right\|_{H^r_\theta} &\le e^{-\theta' k_{\min} t} \left\| e^{t\mathcal{L}^*_N} \nabla f \right\|_{H^r_\theta} \le C \left(1 + t^{-\frac{1}{2}} e^{-\lambda t} \right) e^{-\theta' k_{\min} t} \| \nabla f \|_{H^{r-1}_\theta} \\ &\le C' t^{-\frac{1}{2}} e^{-\lambda t} \| f \|_{H^r_\theta} . \end{aligned} \tag{A.29}$$

For the proof of the third assertion, since [23], Th. 1.4.3. implies that for $0 < \varepsilon \le 1$,

$$\left\| \left(e^{s\mathcal{L}^*_{\theta'}} - 1 \right) f \right\|_{H^r_\theta} \le C_\varepsilon s^\varepsilon \| f \|_{H^{r+2\varepsilon}_\theta} , \tag{A.30}$$

we obtain, since $\left(e^{s\mathcal{L}^*_{\theta'}} - 1 \right) \mathcal{P}_{\theta'} f = \left(e^{s\mathcal{L}^*_{\theta'}} - 1 \right) f$ and $\mathcal{P}_{\theta'}$ commutes with $e^{t\mathcal{L}^*_{\theta'}}$,

$$\begin{aligned} \left\| \left(e^{(t+s)\mathcal{L}^*_{\theta'}} - e^{t\mathcal{L}^*_{\theta'}} \right) f \right\|_{H^{r+1}_\theta} &= \left\| \left(e^{s\mathcal{L}^*_{\theta'}} - 1 \right) e^{t\mathcal{L}^*_{\theta'}} \mathcal{P}_{\theta'} f \right\|_{H^{r+1}_\theta} \le C_\varepsilon s^\varepsilon \left\| e^{t\mathcal{L}^*_{\theta'}} \mathcal{P}_{\theta'} f \right\|_{H^{r+1+2\varepsilon}_\theta} \\ &\le C s^\varepsilon t^{-\frac{1}{2}-\varepsilon} e^{-\lambda t} \| f \|_{H^r_\theta} . \end{aligned} \tag{A.31}$$

The last assertion is not a direct consequence of the estimates obtained above, since the hypothesis $\int_{\mathbb{R}^d} f w_\theta = 0$ is not well adapted to the eigenfunctions $\psi_{l,\theta'}$ of $\mathcal{L}^*_{\theta'}$. In particular, having $\int_{\mathbb{R}^d} f w_\theta = 0$ does not imply $\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta = 0$, while it is the case when $\theta = \theta'$. We will only be able to obtain this estimates for r large enough, via direct calculations. Remark first that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta} \right\|^2_{H^r_\theta} \\ &= \left\langle \mathcal{L}^*_{\theta'} e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} \mathcal{L}^*_{\theta'} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta}, e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta} \right\rangle_{H^r_\theta} . \end{aligned} \tag{A.32}$$

Recalling that $\mathcal{L}^*_{\theta'} a = 0$ and remarking that $\left\langle a, e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta} \right\rangle_{H^r_\theta} = 0$ for any constant a, we get

$$\frac{1}{2}\frac{d}{dt}\left\| e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta} \right\|^2_{H^r_\theta}$$
$$= \left\langle \mathcal{L}^*_{\theta'} \left(e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta} \right), e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta} \right\rangle_{H^r_\theta}, \tag{A.33}$$

so the proof of the last assertion reduces to the study of $\langle \mathcal{L}^*_{\theta'} f, f\rangle_{H^r_\theta}$ with $\int_{\mathbb{R}^d} f w_\theta = 0$. Now for f satisfying $\int_{\mathbb{R}^d} f w_\theta = 0$, with decomposition $f = \sum_{l\neq 0} f_l \psi_{l,\theta}$, we get

$$-\frac{\theta}{\theta'}\left\langle \mathcal{L}^*_{\theta'} f, f\right\rangle_{H^r_\theta} = \sum_{l\neq 0}(a_\theta + \lambda_{\theta,l})^r \lambda_{\theta,l}|f_l|^2$$
$$+ \left(1 - \frac{\theta}{\theta'}\right)\left(\sum_{l\neq 0}(a_\theta + \lambda_{\theta,l})^r \sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, f_l \bar{f}_{l_{\uparrow\uparrow_i}}\right). \tag{A.34}$$

Now remark that for the second term, using Cauchy–Schwarz inequality, we get

$$\left|\sum_{l\neq 0}(a_\theta + \lambda_{\theta,l})^r \sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, f_l \bar{f}_{l_{\uparrow\uparrow_i}}\right|$$
$$= \left|\sum_{l\neq 0}\left\{(a_\theta + \lambda_{\theta,l})^{r/2}\sqrt{\lambda_{\theta,l}}\, f_l\right\}\left\{\frac{(a_\theta + \lambda_{\theta,l})^{r/2}}{\sqrt{\lambda_{\theta,l}}}\sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, \bar{f}_{l_{\uparrow\uparrow_i}}\right\}\right|$$
$$\leq \left|\sum_{l\neq 0}(a_\theta + \lambda_{\theta,l})^r \lambda_{\theta,l}|f_l|^2\right|^{\frac{1}{2}}\left|\sum_{l\neq 0}\frac{(a_\theta + \lambda_{\theta,l})^r}{\lambda_{\theta,l}}\left(\sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, \bar{f}_{l_{\uparrow\uparrow_i}}\right)^2\right|^{\frac{1}{2}}.$$

Using Jensen's inequality

$$\left|\sum_{l\neq 0}\frac{(a_\theta + \lambda_{\theta,l})^r}{\lambda_{\theta,l}}\left(\sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, \bar{f}_{l_{\uparrow\uparrow_i}}\right)^2\right|$$
$$\leq \sum_{l\neq 0}\frac{(a_\theta + \lambda_{\theta,l})^r}{\lambda_{\theta,l}}\left(\sum_{i=1}^d \theta k_i (l_i+2)\right)\sum_{i=1}^d \theta k_i (l_i+2)\left|f_{l_{\uparrow\uparrow_i}}\right|^2,$$
$$= \sum_{l\neq 0}\frac{(a_\theta + \lambda_{\theta,l})^r}{\lambda_{\theta,l}}\left(2a_\theta + \lambda_{\theta,l}\right)\sum_{i=1}^d \theta k_i (l_i+2)\left|f_{l_{\uparrow\uparrow_i}}\right|^2,$$

Denoting by $\mathcal{N}_i := \upuparrows_i \left(\mathbb{N}^d \setminus \{0\}\right) = \left\{ l \in \mathbb{N}^d, l_i \geq 2, \sum_{j=1}^d l_j \geq 3 \right\}$, we obtain

$$
\begin{aligned}
&\left| \sum_{l \neq 0} \frac{\left(a_\theta + \lambda_{\theta,l}\right)^r}{\lambda_{\theta,l}} \left(\sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, \bar{f}_{l_{\upuparrows_i}} \right)^2 \right| \\
&\quad \leq 2 \sum_{i=1}^d \sum_{l \in \mathcal{N}_i} \frac{\left(a_\theta + \lambda_{\theta,l_{\downdownarrows_i}}\right)^r}{\lambda_{\theta,l_{\downdownarrows_i}}} \left(2a_\theta + \lambda_{\theta,l_{\downdownarrows_i}}\right) \theta k_i l_i \, |f_l|^2 \\
&\quad = \sum_{i=1}^d \sum_{l \in \mathcal{N}_i} \frac{\left(a_\theta + \lambda_{\theta,l} - 2\theta k_i\right)^r}{\lambda_{\theta,l} - 2\theta k_i} \left(2a_\theta + \lambda_{\theta,l} - 2\theta k_i\right) \theta k_i l_i \, |f_l|^2 \\
&\quad = \sum_{i=1}^d \sum_{l \in \mathcal{N}_i} b_{\theta,l,i} \left(a_\theta + \lambda_{\theta,l}\right)^r \theta k_i l_i \, |f_l|^2 ,
\end{aligned}
$$

where

$$
\begin{aligned}
b_{\theta,l,i} &:= \frac{\left(a_\theta + \lambda_{\theta,l} - 2\theta k_i\right)^r \left(2a_\theta + \lambda_{\theta,l} - 2\theta k_i\right)}{\left(a_\theta + \lambda_{\theta,l}\right)^r \left(\lambda_{\theta,l} - 2\theta k_i\right)} \\
&= \left(1 - \frac{2\theta k_i}{a_\theta + \lambda_{\theta,l}}\right)^r \left(1 + \frac{2a_\theta}{\lambda_{\theta,l} - 2\theta k_i}\right).
\end{aligned}
\tag{A.35}
$$

Now, for $l \in \mathcal{N}_i$, we have

$$
a_\theta + \lambda_{\theta,l} \leq d\theta k_{\max} + \lambda_{\theta,l} \leq (d+1) \frac{k_{\max}}{k_{\min}} \lambda_{\theta,l}, \tag{A.36}
$$

and

$$
\lambda_{\theta,l} - 2\theta k_i \geq \theta \left(\sum_{j=1}^d l_j - 2 \right) k_{\min} \geq \frac{k_{\min}}{3k_{\max}} \lambda_{\theta,l}, \tag{A.37}
$$

so that

$$
b_{\theta,l,i} \leq \left(1 - \frac{2k_{\min}^2}{(d+1)k_{\max}} \cdot \frac{1}{\sum_{j=1}^d k_j l_j}\right)^r \left(1 + \frac{6d\, k_{\max}^2}{k_{\min}} \frac{1}{\sum_{j=1}^d k_j l_j}\right). \tag{A.38}
$$

Now, observe that for $c_1, c_2 > 0$, $x \mapsto \left(1 - \frac{c_1}{x}\right)^r \left(1 + \frac{c_2}{x}\right)$ is strictly increasing with limit 1 as $x \to \infty$, provided that $r > \frac{c_2}{c_1}$. Hence, taking r large enough in (A.38) (r depending only on K and d, not on l, i and θ) we have $|b_{\theta,l,i}| \leq 1$, which means that the second term of the right-hand side of (A.34) is bounded as follows:

$$
\left| \sum_{l \neq 0} (a_\theta + \lambda_{\theta,l})^r \sum_{i=1}^d \theta k_i \sqrt{(l_i+1)(l_i+2)}\, f_l \bar{f}_{l_{\upuparrows_i}} \right| \leq \sum_{l \neq 0} (a_\theta + \lambda_{\theta,l})^r \lambda_{\theta,l} |f_l|^2. \tag{A.39}
$$

We deduce from (A.34) and this estimate that

$$
\left| -\frac{\theta}{\theta'} \left\langle \mathcal{L}_{\theta'}^* f, \bar{f} \right\rangle_{H_\theta^r} - \sum_{l \neq 0} (a_\theta + \lambda_{\theta,l})^r \lambda_{\theta,l} |f_l|^2 \right| \leq \left(1 - \frac{\theta}{\theta'}\right) \sum_{l \neq 0} (a_\theta + \lambda_{\theta,l})^r \lambda_{\theta,l} |f_l|^2, \tag{A.40}
$$

which means that

$$-\operatorname{Re}\left\langle \mathcal{L}^*_{\theta'} f,\, \bar{f}\right\rangle_{H^r_\theta} \geq \sum_{l\neq 0} (a_\theta + \lambda_{\theta,l})^r \lambda_{\theta,l} |f_l|^2 \geq \theta k_{\min} \|f\|^2_{H^r_\theta}. \tag{A.41}$$

This concludes the proof of Proposition A.2. □

As already stated in Sect. 1.3, we rely in this paper on a "pivot" space structure (see [10], pp. 81–82): observe first that for $u \in L^2_{-\theta}$, $v \in L^2_\theta \mapsto \int_{\mathbb{R}^d} uv\mathrm{d}x$ defines a continuous linear form on L^2_θ. Respectively, for $u \in \left(L^2_\theta\right)'$, the mapping $\psi \mapsto Tu(\psi) := \langle u\,,\, \psi w_{-\theta}\rangle$ defines a continuous linear form on L^2 (that is the usual L^2 space without weight, i.e. $w \equiv 1$ in (1.17)). By Riesz Theorem, there exists $v \in L^2$, such that $Tu(\psi) = \int v\psi = \int \widetilde{v}\widetilde{\psi}$, $\psi \in L^2$, for $\widetilde{v} := vw_{\theta/2} \in L^2_{-\theta}$, $\widetilde{\psi} = \psi w_{-\theta/2} \in L^2_\theta$. This observation permits the identification of $(L^2_\theta)'$ with $L^2_{-\theta}$ (and hence, $\langle\cdot,\cdot\rangle_{(L^2_\theta)'\times L^2_\theta}$ with $\langle\cdot,\cdot\rangle_{L^2} = \langle\cdot,\cdot\rangle$). Now, since $H^r_\theta \to L^2_\theta$ is dense, we have a dense injection $(L^2_\theta)' \to H^{-r}_\theta$. With the identification $(L^2_\theta)' \approx L^2_{-\theta}$, we obtain, for all $u \in L^2_{-\theta} \subset H^{-r}_\theta$ and all $f \in H^r_\theta$,

$$\langle u, f\rangle_{H^{-r}_\theta \times H^r_\theta} = \langle u, f\rangle.$$

Remark in particular that if $u \in L^2_{-\theta}$, then for all $f \in H^{r+1}_\theta$ we have

$$\left|\langle \partial_{x_i} u, f\rangle\right| = \left|-\langle u, \partial_{x_i} f\rangle\right| \leq C\|u\|_{H^{-r}_\theta} \|f\|_{H^{r+1}_\theta}, \tag{A.42}$$

so that if $u \in H^{-r}_\theta$ then $\nabla u \in H^{-(r+1)}_\theta$ with

$$\|\nabla u\|_{H^{-(r+1)}_\theta} \leq C\|u\|_{H^{-r}_\theta}. \tag{A.43}$$

With this structure since L^2_θ is reflexive, the closure of $\mathcal{L}_{\theta'}$ seen as an operator on $(L^2_\theta)'$ is the adjoint of $\mathcal{L}^*_{\theta'}$ ([25], Th. 5.29) and is thus sectorial and defines an analytical semi-group $e^{t\mathcal{L}_{\theta'}}$ in H^{-r}_θ. In the same way, since H^r_θ is reflexive, the adjoint of $e^{t\mathcal{L}^*_{\theta'}}$ seen as an operator on H^r_θ is $e^{t\mathcal{L}_{\theta'}}$ seen as an operator on H^{-r}_θ ([32], Cor. 10.6).

From Proposition A.2 and the structure described above we deduce directly the following estimates for the semi-group induced by $\mathcal{L}_{\theta'}$ (recall (1.24)) in H^{-r}_θ and $t > 0$.

Proposition A.3 *For all $0 < \theta \leq \theta'$ the operator $\mathcal{L}_{\theta'}$ is sectorial and generates an analytical semi-group in H^{-r}_θ. Moreover we have the following estimates: for any $r \geq 0$, $\alpha \geq 0$ and $\lambda < \theta k_{\min}$, there exists a constant $C > 0$ such that for all $u \in H^{-(r+\alpha)}_\theta$,*

$$\left\|e^{t\mathcal{L}_{\theta'}} u\right\|_{H^{-r}_\theta} \leq C\left(1 + t^{-\alpha/2} e^{-\lambda t}\right)\|u\|_{H^{-(r+\alpha)}_\theta}, \tag{A.44}$$

and for all $r \geq 1$,

$$\left\|e^{t\mathcal{L}_{\theta'}} \nabla u\right\|_{H^{-r}_\theta} \leq Ct^{-\frac{1}{2}} e^{-\lambda t}\|u\|_{H^{-r}_\theta}. \tag{A.45}$$

Moreover for all $r \geq 0$, $0 < \varepsilon \leq 1$ and $s \geq 0$,

$$\left\|\left(e^{(t+s)\mathcal{L}_{\theta'}} - e^{t\mathcal{L}_{\theta'}}\right) u\right\|_{H^{-r}_\theta} \leq Cs^\varepsilon t^{-\frac{1}{2}-\varepsilon} e^{-\lambda t}\|u\|_{H^{-(r+1)}_\theta}. \tag{A.46}$$

Finally, there exist $r_0 > 0$, $C > 0$ such that for any $0 < \theta \leq \theta'$, for all $r > r_0$, $t > 0$ and all $u \in H^{-r}_\theta$ satisfying $\int u = 0$,

$$\left\|e^{t\mathcal{L}_{\theta'}} u\right\|_{H^{-r}_\theta} \leq Ce^{-\lambda t}\|u\|_{H^{-r}_\theta}. \tag{A.47}$$

Proof of Proposition A.3 The spectral structure of $\mathcal{L}_{\theta'}$ follows directly from the one of $\mathcal{L}^*_{\theta'}$. To prove the first estimate of the proposition it is now sufficient to remark that for all $f \in H^r_\theta$, $u \in L^2_{-\theta}$,

$$\left|\langle e^{t\mathcal{L}_{\theta'}} u, f\rangle\right| = \left|\langle u, e^{t\mathcal{L}^*_{\theta'}} f\rangle\right| \leq C\left(1 + t^{-\alpha/2} e^{-\lambda t}\right) \|f\|_{H^r_\theta} \|u\|_{H_\theta^{-(r+\alpha)}}.$$

For the second point,

$$\left|\langle e^{t\mathcal{L}_{\theta'}} \nabla u, f\rangle\right| = \left|\langle u, \nabla e^{t\mathcal{L}^*_{\theta'}} f\rangle\right| \leq C t^{-\alpha/2} e^{-\lambda t} \|f\|_{H^r_\theta} \|u\|_{H_\theta^{-(r+\alpha)}}.$$

The third point follows from similar estimates. For the last point, remark that if $\langle u\ ,\ 1\rangle = 0$,

$$\begin{aligned} \left|\langle e^{t\mathcal{L}_{\theta'}} u, f\rangle\right| &= \left|\left\langle u, e^{t\mathcal{L}^*_{\theta'}} f\right\rangle\right| = \left|\left\langle u, e^{t\mathcal{L}^*_{\theta'}} f - \frac{\int_{\mathbb{R}^d} e^{t\mathcal{L}^*_{\theta'}} f w_\theta}{\int_{\mathbb{R}^d} w_\theta}\right\rangle\right| \\ &\leq C t^{-\alpha/2} e^{-\lambda t} \left\| f - \int f w_\theta \right\|_{H^r_\theta} \|u\|_{H_\theta^{-(r+\alpha)}}, \end{aligned} \tag{A.48}$$

and $\left\| f - \int f w_\theta \right\|_{H^r_\theta} \leq 2\|f\|_{H^r_\theta}$. □

Appendix B. Grönwall Lemma

Lemma B.1 *Let $t \mapsto y_t$ be a nonnegative and continuous function on $[0, T)$ satisfying, for all $t \in [0, T)$ and some positive constants c_0 and c_1,*

$$y_t \leq c_0 + c_1 \int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right) y_s \, \mathrm{d}s. \tag{B.1}$$

Then for all $t \in [0, T)$, $y_t \leq 2c_0 e^{\alpha t}$ with $\alpha = 2c_1 + 4c_1^2 \left(\Gamma\left(\frac{1}{2}\right)\right)^2$, where Γ is the usual special function $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$.

For the proof of this Lemma, see [20], Lemma 5.2.

Lemma B.2 *Let $a, b, \lambda > 0$ and ϕ a nonnegative measurable function on $[0, +\infty)$ such that ϕ is integrable on $[0, +\infty)$. Suppose that $t \geq 0 \mapsto u_t$ is a nonnegative function satisfying*

$$u_t \leq a + b \int_0^t (1 + \phi(t-s)) e^{-\lambda s} u_s \mathrm{d}s. \tag{B.2}$$

Then, there exists some constant $C(b, \phi) > 0$ such that

$$\sup_{t \geq 0} u_t \leq 2a \exp\left(\frac{C(b, \phi)}{\lambda}\right). \tag{B.3}$$

Proof of Lemma B.2 Define $A = A(b, \phi) \geq 0$ such that $\int_0^{+\infty} \phi(u)\mathbf{1}_{\{\phi(u)\geq A\}}\mathrm{d}u \leq \frac{1}{2b}$. Then, for all $v \leq t$

$$
\begin{aligned}
u_v &\leq a + b\int_0^v e^{-\lambda s}u_s\mathrm{d}s + b\int_0^v \phi(v-s)\mathbf{1}_{\{\phi(v-s)\geq A\}}e^{-\lambda s}u_s\mathrm{d}s \\
&\quad + b\int_0^v \phi(v-s)\mathbf{1}_{\{\phi(v-s)\leq A\}}e^{-\lambda s}u_s\mathrm{d}s, \\
&\leq a + b\int_0^v e^{-\lambda s}u_s\mathrm{d}s + b\sup_{s\leq v} u_s\int_0^v \phi(v-s)\mathbf{1}_{\{\phi(v-s)\geq A\}}\mathrm{d}s + bA\int_0^v e^{-\lambda s}u_s\mathrm{d}s, \\
&\leq a + b(1+A)\int_0^v e^{-\lambda s}u_s\mathrm{d}s + \frac{1}{2}u_v^* \leq a + b(1+A)\int_0^t e^{-\lambda s}u_s^*\mathrm{d}s + \frac{1}{2}u_t^*,
\end{aligned}
$$

where we have defined $u^*(s) := \sup_{r\leq s} u_r$. Since the last inequality is true for all $v \leq t$, we get

$$
u_t^* \leq 2a + 2b(1+A)\int_0^t e^{-\lambda s}u_s^*\mathrm{d}s.
$$

The usual Grönwall lemma applied to $t \mapsto u_t^*$ gives the conclusion, for $C(b, \phi) = 2b(1 + A(b, \phi))$. □

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