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# Dynamics of Suspension Bridge Equation with Delay

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**Abstract**
Long-time dynamics of the solutions for the suspension bridge equation with constant and time-dependent delays have been investigated, but there are no works on suspension bridge equation with state-dependent delay. Thus, we first consider the existence of pullback attractor for the non-autonomous suspension bridge equation with state-dependent delay by using the contractive function methods.

**Keywords** Suspension bridge equation · State-dependent delay · Pullback $\mathcal{D}-$attractor · Contractive function

**Mathematics Subject Classification** 35B40, 37B55, 35K30

## 1 Introduction

Delay differential equations always play an important role in modeling a great variety of phenomena, such as viscoelasticity, neural networks, interaction of species, biomedicine, economy and many other fields. From the view of mathematics, we mainly concentrate on the well-posedness and asymptotic behavior of solutions for the delay differential equations. At the very beginning, the general theory of delay equations in infinite dimensional spaces started with [1,2] at the abstract level. In the last decades, the authors mainly investigated the parabolic-type models with constant and time-dependent delays [3–5].

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However, it appears that in many problems the constancy of the delay is just an extra case which makes the problem easier, it is not really well motivated by real world models. To describe a process more naturally, a new class of state-dependent delay models were introduced and studied recently. When the delay term depends on unknown variables in a equation, we call it a state-dependent delay differential equation. Partial differential equations with state-dependent delay have been essentially less investigated, see the discussions in references [6–9] where they considered the parabolic case. Chueshov and Rezounenko [9] considered dynamics of second order in time evolution equations with state-dependent delay, and obtained the existence of global and exponential attractors.

In this paper, our main goal is to consider the following non-autonomous suspension bridge with state-dependent delay in $\Omega = [0, L]$

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \mu\partial_t u + ku^+ + f(u) + u(t - \pi[u^t]) = g(x,t), & x \in \Omega,\ t > \tau, \\ u(0,t) = u(L,t) = \partial_{xx}u(0,t) = \partial_{xx}u(L,t) = 0, & x \in \Omega,\ t \in [\tau - h, +\infty), \\ u(x,t) = \varphi(x, t-\tau), & x \in \Omega,\ t \in [\tau - h, \tau], \\ \partial_t u(x,t) = \partial_t \varphi(x, t-\tau), & x \in \Omega,\ t \in [\tau - h, \tau]. \end{cases} \tag{1.1}$$

The model (1.1) describes the vibration of the road bed in the vertical direction, where $u(x,t)$ denotes the deflection in the downward direction; $\pi$ is a mapping defined on solutions with values in some interval $[0, h]$, $h > 0$ presents the (maximal) retardation time; $\mu\partial_t u$ represents the viscous damping, and $\mu$ is a positive constant; $ku^+$ represents the restoring force, $k > 0$ denotes the spring constant, the function $u^+(x,t) = \max\{u(x,t), 0\}$. $\tau$ is the initial time. $\varphi$ is the initial datum. The term $u(t - \pi[u^t])$ models effect of the Winkler type foundation with delay response and $u^t \equiv u(t+\theta),\ \ \theta \in [-h, 0]$.

In the last several decades, the spectacular collapse of the Tacoma narrow bridge has successfully attracted many of engineers, physicists, and mathematicians. They tried their best to explain such an amazing event. A one-dimensional simply supported beam suspended by hangers was modelled as a suspension bridge in [10] by Lazer and McKenna, which it described the vibration of the roadbed in the vertical plane. We note that the dynamics of suspension bridge without delay effects were studied by many authors. For instance, in [10–12], the authors proved existence of periodic solutions, property of travelling wave solutions and the numerical mountain pass solutions for the suspension bridge equation. Zhong et al. have investigated systematically the long-time behavior of solution for both the single and the coupled suspension bridge equations in [13–15]. Bochicchio et al. studied the existence of the global attractor for the Kirchhoff suspension bridge equation and obtained a regularity result of attractor, see [16] for details. The existence of global attractors for the suspension bridge with linear memory was achieved by Kang in [17].

With respect to the suspension bridge equations with constant or time-dependent delay, see [18–21] and references therein. The author [18] obtained the existence of the finite dimensional global attractors under the condition of $0 < |a_1| < a_0$. In [19], the authors studied the non-autonomous suspension bridge with time delay

$$\partial_{tt}u + \partial_{xxxx}u + \mu\partial_t u + ku^+ + g(u) = F(t, u_t) + f(x,t),$$

and proved the existence of pullback attractors when the delay term $F(t, u_t)$ is driven by a function with very weak assumptions. In addition, the authors proved the existence of uniform attractors for the nonlinear plate modelling suspension bridges with delay, and investigated the existence of strong pullback attractors for suspension bridge with variable delay (see [20,21] for details).

In this paper, we pay attention to the existence of pullback $\mathcal{D}-$attractor for the suspension bridge with state-dependent delay. Comparing with constant or time-dependent delays, the state-dependent delay will bring new difficulties in analysis, including the well-posedness and corresponding a priori estimates, thus the results about the systems with state-dependent delay are not so rich as that for other kinds of delay differential equations. Regarding problem (1.1), in order to guarantee the uniqueness of solutions, we have to choose a certain appropriate $C^1$-type space, and there need an additional term in energy functional as a compensator for the delay term in the proof of a priori estimates. Finally, we obtain the compactness of the process by using contractive function methods, which is different from reference [22], where they proved the existence of global and exponential attractors of system by using quasi-stable method.

The layout of this paper as follows. In Sect. 2, we define some functions setting and iterate some useful lemmas and abstract results about pullback $\mathcal{D}-$attractor. In Sect. 3, we make a priori estimates and establish well-posedness of problem (1.1). In Sect. 4, we prove the existence of pullback $\mathcal{D}-$attractor for (1.1).

## 2 Preliminaries

Firstly, Let $V, H$ be real Hilbert space. Define $D(A) = \{u \in V, Au \in H : u(0,t) = u(L,t) = \partial_{xx}u(0,t) = \partial_{xx}u(L,t) = 0\}$, where $A = \triangle^2 = \partial_{xxxx}$, then $A : \ D(A) \to H$ is a strictly positive self-adjoint operator. For any $s \in \mathbb{R}$, the scale of Hilbert spaces generated the powers of $A$ is introduced as follows:

$$V_s = D(A^{\frac{s}{4}}), \ \ (u, v)_{V_s} = (A^{\frac{s}{4}}u, A^{\frac{s}{4}}v), \ \ \|u\|^2_{V_s} = \|A^{\frac{s}{4}}u\|^2.$$

When $s = 0$, denote $V_0 = \mathrm{H} = L^2(\Omega)$, when $s = 2$, $V_2 = H^1_0(\Omega) \cap H^2(\Omega)$, the scalar product and the norm acting on $V_0$ and $V_2$ are denoted as follows,

$$(u, v)_{V_0} = (u, v), \quad \|u\|^2_{V_0} = \|u\|^2,$$

$$(u, v)_{V_2} = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v), \ \ \|u\|^2_{V_2} = \|A^{\frac{1}{2}}u\|^2 = \|\triangle u\|^2.$$

In particular, we denote as $|Au|$ the norm of $D(A)$. It is obvious that $V_4 \subset V_2 \subset V_0 = V_0^* \subset V_2^*$, here $V_0^*, \ V_2^*$ are the dual space of $V_0, \ V_2$ respectively, and each space is dense in the following one and the injections are continuous.

By the Poincaré inequality, we have

$$\lambda_1 \|u\|^2_{V_s} \leq \|u\|^2_{V_{s+1}}, \ \ \forall u \in V_{s+1},$$

where $\lambda_1^2$ is the first eigenvalue of $A$.

We will denote by $C_X$ the Banach space $C([-h, 0]; X)$, equipped with the sup-norm, for an element $v \in C_X$, its norm will be written as $\|v\|_{C_X} = \sup_{\theta\in[-h,0]} \|v(\theta)\|_X$.

Firstly, we introduce phase space

$$W \equiv C([-h, 0]; V_2) \cap C^1([-h, 0]; V_0),$$

its norm will be written as

$$\|\varphi\|_W = \|\varphi\|_{C_{V_2}} + \|\partial_t\varphi\|_{C_{V_0}}, \ \ \forall \varphi \in W.$$

***Remark 2.1*** [9] The main example of state-depenent delay term is

$$M(\phi) = \phi(-\pi(\phi)), \ \ \phi \in C([-h, 0]; H),$$

where $\pi$ maps $C([-h, 0]; H)$ into some interval $[0, h]$. We note that this delay term $M$ is not locally Lipschitz in the classical space of continuous functions $C([-h, 0]; H)$. This may lead to the non-uniqueness of solutions and make the study of differential equations with state-dependent delays essentially differ from the ones with constant or time-dependent delays. In order to prove the well-posedness of the system it requires additional assumptions. The main approach to $C^1-$solutions of general delay equations is the so-called "solution manifold method" which assumes some type of compatibility condition (see [23,28]). There is also an alternative approach avoiding compatibility condition (see [7]). Hence, it is important to deal with spaces in which we can ensure Lipschitz property for the above mapping. So in the paper, we choose a Banach space $W$ defined above as a phase space allows us to guarantee local Lipschitz property for the delay term. This phase space takes into account the natural "displacement-velocity" relation from the very beginning.

Secondly, we assume that nonlinear term $f$ satisfies the following conditions:

$$\liminf_{|s|\to\infty} \frac{F(s)}{s^2} \geq 0, \tag{2.1}$$

where $F(s) = \int_0^s f(r)dr$.

$$\liminf_{|s|\to\infty} \frac{sf(s) - C_0F(s)}{s^2} \geq 0, \quad C_0 > 0, \tag{2.2}$$

and

$$\limsup_{|s|\to\infty} \frac{|f'(s)|}{|s|^p} = 0, \tag{2.3}$$

where $0 \leq p < \infty$. For every $\eta > 0$, there exists $C_\eta > 0$, such that

$$\|u\|^2 \leq C_\eta + \eta(\|A^{\frac{1}{2}}u\|^2 + \int_\Omega F(u(x))dx). \tag{2.4}$$

Furthermore, we suppose that external force $g \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$ satisfies

$$\int_{-\infty}^t e^{\gamma s}\|g(s)\|^2 ds < \infty, \ \ \forall t \in \mathbb{R}, \tag{2.5}$$

where $\gamma > 0$ will be determined later.

Finally, we assume that the mapping $\pi : W \to [0, h]$ is locally Lipschitz, i.e., for any $R > 0$, there exists $C_R > 0$, such that

$$|\pi(\varphi_1) - \pi(\varphi_2)| \leq C_R\|\varphi_1 - \varphi_2\|_W, \tag{2.6}$$

for every $\varphi_1, \varphi_2 \in W, \ \ \|\varphi_j\|_W \leq R, \ \ j = 1, 2.$

In the following, we iterate some useful lemmas and abstract results (see [24–26]).

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\widetilde{D} = \{D(t); t \in \mathbb{R}\} \subset \mathcal{P}(X)$, and $\mathcal{P}(X)$ be a class of nonempty closed subsets of $X$.

Let $\{S(t, \tau)\}_{t\geq\tau}$ be a process (or a two-parameter operator group) on a metric space $X$, i.e., a family of continuous mappings $\{S(t, \tau)\}_{t\geq\tau}$, $-\infty < \tau \leq t < +\infty$: $X \to X$, satisfy

$$S(\tau, \tau)x = x, \ \ \forall x \in X;$$

$$S(t,\tau)=S(t,r)S(r,\tau)\ \ \forall\,\tau\le r\le t.$$

**Definition 2.2** The process $\{S(t,\tau)\}_{t\ge\tau}$ is said to be pullback $\mathcal{D}-$asymptotically compact, if for any $t\in\mathbb{R}$, $\widetilde{D}\in\mathcal{D}$, any sequence $\tau_n\to-\infty$, $x_n\in D(\tau_n)$, the sequence $\{S(t,\tau_n)x_n\}_{n=1}^{\infty}$ is precompact in $X$.

**Definition 2.3** It is said that $\widetilde{B}\in\mathcal{D}$ is pullback $\mathcal{D}-$absorbing set for the process $\{S(t,\tau)\}_{t\ge\tau}$, if for any $t\in\mathbb{R}$ and any $\widetilde{D}\in\mathcal{D}$, there exists a $\tau_0=\tau_0(t,\widetilde{D})\le t$ such that

$$S(t,\tau)D(\tau)\subset\widetilde{B}(t),\ \forall\tau\le\tau_0(t,\widetilde{D}).$$

**Lemma 2.4** *A family $\widetilde{\mathcal{A}}=\{\mathcal{A}(t);t\in\mathbb{R}\}\subset\mathcal{P}(X)$ is said to be a pullback $\mathcal{D}-$attractor for the process $\{S(t,\tau)\}_{t\ge\tau}$ in $X$, if it satisfies*

*(1) $\mathcal{A}(t)$ is compact in $X$ for all $t\in\mathbb{R}$;*
*(2) $\widetilde{\mathcal{A}}$ is pullback $\mathcal{D}-$attracting in $X$, i.e.,*

$$\lim_{\tau\to-\infty}dist_X(S(t,\tau)D(\tau),\mathcal{A}(t))=0$$

*for all $\widetilde{D}\in\mathcal{D}$ and all $t\in\mathbb{R}$;*
*(3) $\mathcal{A}$ is invariant, i.e., $S(t,\tau)\mathcal{A}(\tau)=\mathcal{A}(t)$, for $-\infty<\tau\le t<+\infty$.*

**Definition 2.5** Let $(X,\|\cdot\|)$ be a Banach space and $\widetilde{B}=\{B(t);t\in\mathbb{R}\}$ be a subset of $X$. We call a function $\Phi(\cdot,\cdot)$ defined on $X\times X$ to be a contractive function on $\widetilde{B}\times\widetilde{B}$, if for any sequence $\{x_n\}_{n=1}^{\infty}\subset\widetilde{B}$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}\subset\{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\Phi(x_{n_k},x_{n_l})=0.$$

We denote the set of all contractive functions on $\widetilde{B}\times\widetilde{B}$ by $Contr(\widetilde{B})$.

**Theorem 2.6** *Let $\{S(t,\tau)\}_{t\ge\tau}$ be a process on Banach space $X$ and have a pullback $\mathcal{D}-$absorbing set $\widetilde{B}=\{B(t);t\in\mathbb{R}\}$. Moreover, assume that for any $\varepsilon>0$, there exist $T=T(t,\widetilde{B},\varepsilon)=t-\tau$ and $\Phi_{t,T}(\cdot,\cdot)\in Contr(\widetilde{B})$ such that*

$$\|S(t,t-T)x-S(t,t-T)y\|\le\varepsilon+\Phi_{t,T}(x,y),\ \ \forall x,y\in B(\tau),$$

*where $\Phi_{t,T}$ depends on $t$ and $T$. Then $\{S(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}-$asymptotically compact in $X$.*

**Theorem 2.7** *Let $\{S(t,\tau)\}_{t\ge\tau}$ be a process on Banach space $X$. Then $\{S(t,\tau)\}_{t\ge\tau}$ has a pullback $\mathcal{D}-$attractor in $X$, provided that the following conditions hold:*

*(i) $\{S(t,\tau)\}_{t\ge\tau}$ has a pullback $\mathcal{D}-$absorbing set $\widetilde{B}$ in $X$;*
*(ii) $\{S(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}-$asymptotically compact in $\widetilde{B}$.*

**Lemma 2.8** *[22] Let $A$ is a linear positive self-adjoint operator with discrete spectrum on separate Hilbert space. Then*

$$\|A^{\alpha}e^{-tA}\|\le(\frac{\alpha}{et})^{\alpha},\ t>0,\ \alpha\ge0.$$

*In particular,* $0^0=1$.

**Lemma 2.9** *[22] Let $s > \sigma$, Then the space $H_s$ is compactly embedded into $H_\sigma$. This means that every sequence bounded in $H_s$ is relatively compact in $H_\sigma$.*

**Theorem 2.10** *[26] (Lumer-Phillips) Let $A$ is dissipative and there exists $\lambda > 0$, such that $R(\lambda I - A) = X$. Then $A$ is the infinitesimal generator of a strongly continuous semigroup in $\mathcal{L}(X)$.*

# 3 Well-Posedness

## 3.1 A Priori Estimate

In this section, in order to obtain well-podedness associated to our problem (1.1), firstly, we make a priori estimate. At the same time, in the sequel $C,\ C_i (i = 1, 2, \cdots)$ denotes arbitrary positive constants, which may be different from line to line and even in the same line.

**Definition 3.1** A vector function

$$u(t) \in C([\tau - h, T]; V_2) \cap C^1([\tau - h, T]; V_0)$$

is said to be a weak solution of the problem (1.1) on the interval $[\tau, T]$, if $u(x, t)$ satisfies:

(i) $u(t) = \varphi(t - \tau),\ \forall t \in [\tau - h, \tau]$;
(ii) $\forall v \in V_2$, we have that

$$(\partial_{tt} u, v) + (\Delta u, \Delta v) + (\mu \partial_t u, v) + (k u^+, v) \\ +(f(u), v) + (u(t - \pi[u^t]), v) = (g(t), v).$$

**Lemma 3.2** *Assume that $f$ satisfy (2.1)-(2.2) and (2.4), $g \in L^2_{loc}(\mathbb{R}; H)$ satisfy (2.5). For any $\mu_0$, there exists $h_0 = h(\mu_0) > 0$, such that $(\mu, h) \in [\mu_0, +\infty) \times (0, h_0]$. Then the solution $(u, \partial_t u)$ of equation (1.1) satisfies the following estimates*

$$\begin{aligned} &\|\partial_t u(t)\|^2 + \|A^{\frac{1}{2}} u(t)\|^2 \le 4e^{-\gamma(t-\tau)}(E(\tau) + \alpha h \|\varphi\|_W^2) \\ &+ \frac{8}{\mu} e^{-\gamma t} \int_{-\infty}^t e^{\gamma s} \|g(s)\|^2 ds + C, \end{aligned} \tag{3.1}$$

*where $C = \frac{4}{\gamma}(2\gamma K_1 + 2\varepsilon K_2 + \frac{4}{\mu} C_\eta)$.*

***Proof*** Taking the scalar product in $V_0$ of (1.1) with $z = \partial_t u + \varepsilon u$ $(\varepsilon > 0)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|z\|^2 + \|A^{\frac{1}{2}} u\|^2) + \varepsilon \|A^{\frac{1}{2}} u\|^2 + (\mu - \varepsilon)\|z\|^2 - \varepsilon(\mu - \varepsilon)(u, z) + k(u^+, z) \\ &= -(f(u), z) - (u(t - \pi[u^t]), z) + (g(t), z). \end{aligned} \tag{3.2}$$

By using Young inequality, Hölder and Poincaré inequalities, choosing $\varepsilon$ small enough, such that

$$\varepsilon \|A^{\frac{1}{2}} u\|^2 + (\mu - \varepsilon)|v|^2 - \varepsilon(\mu - \varepsilon)(u, v) \ge \frac{\varepsilon}{2}\|A^{\frac{1}{2}} u\|^2 + \frac{3\mu}{4}\|z\|^2, \tag{3.3}$$

$$k(u^+, z) = \frac{1}{2}\frac{d}{dt} k\|u^+\|^2 + \varepsilon k \|u^+\|^2. \tag{3.4}$$

According to (2.1),(2.2) and Poincaré inequality, there exist constants $K_1, K_2 > 0$ such that

$$\int_\Omega F(u)dx + \frac{1}{8}\|A^{\frac{1}{2}}u\|^2 \geq -K_1, \qquad \forall u \in V_2, \tag{3.5}$$

$$(f(u), u) - C_0\int_\Omega F(u)dx + \frac{1}{8}\|A^{\frac{1}{2}}u\|^2 \geq -K_2, \quad \forall u \in V_2. \tag{3.6}$$

Then

$$\begin{aligned} -(f(u), z) =& -(f(u), \partial_t u) - \varepsilon(f(u), u) \\ \leq& -\frac{d}{dt}\int_\Omega F(u)dx - \varepsilon C_0\int_\Omega F(u)dx + \frac{\varepsilon}{8}\|A^{\frac{1}{2}}u\|^2 + \varepsilon K_2. \end{aligned} \tag{3.7}$$

Furthermore, exploiting Hölder and Young inequalities, it's easy to get that

$$(u(t-\pi[u^t]), z) + (g(t), z) \leq \frac{1}{\mu}\|u(t-\pi[u^t])\|^2 + \frac{1}{\mu}\|g(t)\|^2 + \frac{\mu}{2}\|z\|^2. \tag{3.8}$$

Substituting (3.3)-(3.4) and (3.7)-(3.8) into (3.2), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|z\|^2 + \|A^{\frac{1}{2}}u\|^2 + k\|u^+\|^2 + 2\int_\Omega F(u)dx) \\ &+ \frac{3\varepsilon}{8}\|A^{\frac{1}{2}}u\|^2 + \frac{\mu}{4}\|z\|^2 + k\varepsilon\|u^+\|^2 + \varepsilon C_0\int_\Omega F(u)dx \\ &\leq \frac{1}{\mu}\|g(t)\|^2 + \frac{1}{\mu}\|u(t-\pi[u^t])\|^2 + \varepsilon K_2. \end{aligned} \tag{3.9}$$

By Hölder inequality and variable substitution, it yields

$$\begin{aligned} \|u(t-\pi[u^t])\|^2 =&\|u(t) - \int_{t-\pi[u^t]}^t \partial_t u(s)ds\|^2 \\ \leq& 2\|u(t)\|^2 + 2\int_{t-h}^t \|\partial_t u(s)\|^2 ds \\ \leq& 2\|u(t)\|^2 + 2h\int_0^h \|\partial_t u(t-r)\|^2 dr. \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9), we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|z\|^2 + \|A^{\frac{1}{2}}u\|^2 + k\|u^+\|^2 + 2\int_\Omega F(u)dx) \\ &+ \frac{3\varepsilon}{8}\|A^{\frac{1}{2}}u\|^2 + \frac{\mu}{4}\|z\|^2 + k\varepsilon\|u^+\|^2 + \varepsilon C_0\int_\Omega F(u)dx \\ &\leq \frac{1}{\mu}\|g(t)\|^2 + \frac{2}{\mu}\|u(t)\|^2 + \frac{2h}{\mu}\int_0^h \|\partial_t u(t-r)\|^2 dr + \varepsilon K_2. \end{aligned}$$

Applying condition (2.4), taking $\eta = \frac{\varepsilon\mu}{8}$, then

$$\frac{2}{\mu}\|u(t)\|^2 \leq \frac{2}{\mu}C_\eta + \frac{\varepsilon}{4}\|A^{\frac{1}{2}}u\|^2 + \frac{\varepsilon}{4}\int_\Omega F(u)dx.$$

Hence, we get

$$\frac{1}{2}\frac{d}{dt}(\|z\|^2+\|A^{\frac{1}{2}}u\|^2+k\|u^+\|^2+2\int_\Omega F(u)dx)$$
$$+\frac{\varepsilon}{8}\|A^{\frac{1}{2}}u\|^2+\frac{\mu}{4}\|z\|^2+k\varepsilon\|u^+\|^2+\varepsilon(C_0-\frac{1}{4})\int_\Omega F(u)dx$$
$$\leq\frac{1}{\mu}\|g(t)\|^2+\frac{2h}{\mu}\int_0^h\|\partial_t u(t-r)\|^2dr+\varepsilon K_2+\frac{2}{\mu}C_\eta.$$

Set

$$E(t)=\|z\|^2+\|A^{\frac{1}{2}}u\|^2+k\|u^+\|^2+2\int_\Omega F(u)dx+2K_1\geq 0. \tag{3.11}$$

Thus,

$$\begin{aligned}\frac{d}{dt}E(t)\leq&-\frac{\varepsilon}{4}\|A^{\frac{1}{2}}u\|^2-\frac{\mu}{2}\|z\|^2-2k\varepsilon\|u^+\|^2-2\varepsilon(C_0-\frac{1}{4})\int_\Omega F(u)dx\\&+\frac{2}{\mu}\|g(t)\|^2+\frac{4h}{\mu}\int_0^h\|\partial_t u(t-r)\|^2dr+2\varepsilon K_2+\frac{4}{\mu}C_\eta.\end{aligned} \tag{3.12}$$

We define

$$V(t)=E(t)+\frac{\alpha}{h}\int_0^h\int_{t-s}^t\|\partial_t u(r)\|^2drds,$$

where $\alpha>0$, and

$$E(t)\leq V(t)\leq E(t)+\alpha\int_0^h\|\partial_t u(t-r)\|^2dr, \tag{3.13}$$

$$\frac{d}{dt}V(t)=\frac{d}{dt}E(t)+\alpha\|\partial_t u(t)\|^2-\frac{\alpha}{h}\int_0^h\|\partial_t u(t-s)\|^2ds.$$

Moreover,

$$\|\partial_t u\|^2=\|\partial_t u+\varepsilon u-\varepsilon u\|^2\leq 2\|\partial_t u+\varepsilon u\|^2+2\varepsilon^2\|u\|^2\leq 2\|z\|^2+\frac{2\varepsilon^2}{\lambda_1}\|A^{\frac{1}{2}}u\|^2.$$

Thus, we have

$$\frac{d}{dt}V(t)\leq\frac{d}{dt}E(t)+2\alpha\|z\|^2+\frac{2\alpha\varepsilon^2}{\lambda_1}\|A^{\frac{1}{2}}u\|^2-\frac{\alpha}{h}\int_0^h\|\partial_t u(t-s)\|^2ds. \tag{3.14}$$

Substituting (3.12) into (3.14), we can deduce that

$$\begin{aligned}&\frac{d}{dt}V(t)+\varepsilon(\frac{1}{4}-\frac{2\alpha\varepsilon}{\lambda_1})\|A^{\frac{1}{2}}u\|^2+(\frac{\mu}{2}-2\alpha)\|z\|^2+2k\varepsilon\|u^+\|^2+2\varepsilon(C_0-\frac{1}{4})\int_\Omega F(u)dx\\&\leq\frac{2}{\mu}\|g(t)\|^2+(\frac{4h}{\mu}-\frac{\alpha}{h})\int_0^h\|\partial_t u(t-r)\|^2dr+2\varepsilon K_2+\frac{4}{\mu}C_\eta.\end{aligned} \tag{3.15}$$

Choosing $\varepsilon>0$ small enough, $\alpha=\frac{\mu}{8}$ and $C_0>\frac{1}{4}$, such that $\frac{1}{4}-2\alpha\varepsilon\lambda_1^{-1}>0$, $\frac{\mu}{2}-2\alpha>0$, $C_0-\frac{1}{4}>0$.

Taking $\gamma = \min\{\varepsilon(\frac{1}{4} - 2\alpha\varepsilon\lambda_1^{-1}), \frac{\mu}{2} - 2\alpha, 2k\varepsilon, 2\varepsilon(C_0 - \frac{1}{4})\}$. According to (3.11), we obtain that

$$\begin{aligned}\frac{d}{dt}V(t) + \gamma E(t) \leq \frac{2}{\mu}\|g(t)\|^2 + (\frac{4h}{\mu} - \frac{\alpha}{h})\int_0^h \|\partial_t u(t-r)\|^2 dr \\ + 2\gamma K_1 + 2\varepsilon K_2 + \frac{4}{\mu}C_\eta.\end{aligned} \tag{3.16}$$

Furthermore, using (3.13), we can get

$$\frac{d}{dt}V(t) + \gamma V(t) \leq \frac{2}{\mu}\|g(t)\|^2 + (\gamma\alpha + \frac{4h}{\mu} - \frac{\alpha}{h})\int_0^h \|\partial_t u(t-r)\|^2 dr + 2\gamma K_1 + 2\varepsilon K_2 + \frac{4}{\mu}C_\eta.$$

When $h < \frac{\mu}{8}$, $\frac{4h}{\mu} - \frac{\alpha}{h} < 0$, choosing $\gamma$ small enough, such that $\gamma\alpha + \frac{4h}{\mu} - \frac{\alpha}{h} \leq 0$, then

$$\frac{d}{dt}V(t) + \gamma V(t) \leq \frac{2}{\mu}\|g(t)\|^2 + 2\gamma K_1 + 2\varepsilon K_2 + \frac{4}{\mu}C_\eta. \tag{3.17}$$

Multiplying (3.17) by $e^{\gamma t}$, then integrating over $[\tau, t]$, we can obtain that

$$\begin{aligned}V(t) \leq e^{-\gamma(t-\tau)}V(\tau) + \frac{2}{\mu}e^{-\gamma t}\int_\tau^t e^{\gamma s}\|g(s)\|^2 ds + \frac{1}{\gamma}(2\gamma K_1 \\ + 2\varepsilon K_2 + \frac{4}{\mu}C_\eta).\end{aligned} \tag{3.18}$$

According to (3.5),(3.11), it yields

$$\begin{aligned}E(t) = \|z\|^2 + \|A^{\frac{1}{2}}u\|^2 \\ +k\|u^+\|^2 + 2\int_\Omega F(u)dx + 2K_1 \geq \frac{1}{4}(\|z\|^2 + \|A^{\frac{1}{2}}u\|^2 + k\|u^+\|^2),\end{aligned}$$

and

$$V(t) \geq E(t) \geq \frac{1}{4}(\|z\|^2 + \|A^{\frac{1}{2}}u\|^2 + k\|u^+\|^2).$$

From (3.18), we have

$$\begin{aligned}\|\partial_t u(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 \leq & 4e^{-\gamma(t-\tau)}V(\tau) + \frac{8}{\mu}e^{-\gamma t}\int_\tau^t e^{\gamma s}\|g(s)\|^2 ds + C \\ \leq & 4e^{-\gamma(t-\tau)}(E(\tau) + \alpha h\|\varphi\|_W^2) + \frac{8}{\mu}e^{-\gamma t}\int_{-\infty}^t e^{\gamma s}\|g(s)\|^2 ds + C,\end{aligned}$$

where $C = \frac{4}{\gamma}(2\gamma K_1 + 2\varepsilon K_2 + \frac{4}{\mu}C_\eta)$. The proof is completed. □

***Remark 3.3*** The energy functional constructed in Lemma 3.2 contains $\frac{\alpha}{h}\int_0^h\int_{t-s}^t \|\partial_t u(r)\|^2 dr ds$, which as a compensator for the delay term in equations. We can also see that the restriction on the delay time $h$ has the form $h \leq c\mu_0$, $c > 0$, because large time lag may destabilize the system. Thus we can increase the damping coefficient to make the system stable.

### 3.2 Existence and Uniqueness

Set $U(t) = (u(t); v(t))$, we can rewrite (1.1) as the following first order differential equation in the space $\mathcal{H} = V_2 \times V_0$

$$\begin{cases} \frac{d}{dt}U(t) + \mathcal{L}U(t) = \mathcal{N}(U^t), & (x,t) \in \Omega \times (\tau, +\infty), \\ U = 0, & (x,t) \in \partial\Omega \times (\tau - h, +\infty), \\ U(x,t) = \Phi(x, t-\tau), & (x,t) \in \Omega \times [\tau - h, \tau], \end{cases} \tag{3.19}$$

where $\Phi = (\varphi; \partial_t \varphi),\ \varphi \in W$, here the operator $\mathcal{L}$ and the mapping $\mathcal{N}$ are defined by

$$\begin{aligned} &\mathcal{L}z = (-v(t); Au + \mu v(t)), \quad U = (u; v) \in D(\mathcal{L}) \equiv D(A) \times D(A^{\frac{1}{2}}), \\ &\mathcal{N}(\Phi) = (0; -f(\varphi(\tau)) - k(\varphi(\tau)) - \varphi(-\pi[\varphi]) + g(x,t)). \end{aligned} \tag{3.20}$$

One can show (see, e.g., [27]) that the operator $\mathcal{L}$ generates exponentially stable $C_0$-semigroup $\{e^{-t\mathcal{L}} : t \geq 0\}$ in $\mathcal{H}$.

**Definition 3.4** A mild solution of (1.1) on an interval $[\tau, T]$ is defined as a function

$$u \in C([\tau - h, T]; V_2) \cap C^1([\tau - h, T]; V_0),$$

such that $u(\theta) = \varphi(\theta),\ \theta \in [\tau - h, \tau]$ and $U(t) = (u(t); \partial_t u(t))$ satisfies

$$U(t) = e^{-(t-\tau)\mathcal{L}} U(\tau) + \int_\tau^t e^{-(t-s)\mathcal{L}} \mathcal{N}(U^s) ds, \quad t \in [\tau, T]. \tag{3.21}$$

**Theorem 3.5** *Let assumptions (2.3) and (2.6) hold true. Then for any $\varphi_i \in W$, $\|\varphi_i\|_W \leq C$, $i = 1, 2$, there exists $\tau < T_{\max} \leq \infty$, and a unique mild solution $U(t) \equiv (u(t); \partial_t u(t))$ of (1.1) on the interval $[\tau, T_{\max}]$, $T_{\max} = \infty$ or $\lim_{t \to T_{\max}-} \|u^t\|_W = \infty$.*

***Proof*** For fixed $\sigma > 0$, we consider $B_\sigma = \{U \in C([\tau, T]; \mathcal{H}) : \|U - \bar{V}\|_{C([\tau,T];\mathcal{H})} \leq \sigma\}$, where $\bar{V} = e^{-(t-\tau)\mathcal{L}} \Phi(\tau)$. We define the mapping $K : C([\tau, T]; \mathcal{H}) \to C([\tau, T]; \mathcal{H})$ as follows:

$$[KU](t) = \bar{V}(t) + \int_\tau^t e^{-(t-s)\mathcal{L}} \mathcal{N}(U^s) ds, \quad t \in [\tau, T].$$

(I). For any $t \in [\tau, T]$, $U_1, U_2 \in B_\sigma$, we have

$$
\begin{aligned}
&\|[KU_1](t) - [KU_2](t)\|_{C([\tau,T];\mathcal{H})} \\
&= \| \int_\tau^t e^{-(t-s)\mathcal{L}} \mathcal{N}(U_1^s) ds - \int_\tau^t e^{-(t-s)\mathcal{L}} \mathcal{N}(U_2^s) ds \|_{C([\tau,T];\mathcal{H})} \\
&\le \int_\tau^t \|e^{-(t-s)\mathcal{L}}(f(u_2(s)) - f(u_1(s))\|_{C([\tau,T];H)} ds \\
&\quad + \int_\tau^t \|e^{-(t-s)\mathcal{L}}(ku_2^+(s) - ku_1^+(s))\|_{C([\tau,T];H)} ds \\
&\quad + \int_\tau^t \|e^{-(t-s)\mathcal{L}}(u_2(s - \pi[u_2^s]) - u_1(s - \pi[u_1^s]))\|_{C([\tau,T];H)} ds \\
&\le \int_\tau^t \|(f(u_2(s)) - f(u_1(s))\|_{C([\tau,T];H)} ds \\
&\quad + \int_\tau^t \|(ku_2^+(s) - ku_1^+(s))\|_{C([\tau,T];H)} ds \\
&\quad + \int_\tau^t \|(u_2(s - \pi[u_2^s]) - u_1(s - \pi[u_1^s]))\|_{C([\tau,T];H)} ds.
\end{aligned}
\tag{3.22}
$$

By (2.3), (3.1) and Sobolev embedding theorem, we know that $f(u)$, $f'(u)$ are uniformly bounded in $L^\infty$. That is, there exists a constant $K_3 > 0$, such that

$$
|f(u)|_{L^\infty} \le K_3, \ |f'(u)|_{L^\infty} \le K_3. \tag{3.23}
$$

Thus, using differential mean value theorem, we get

$$
\|f(u_2) - f(u_1)\| \le K_3 \|u_2 - u_1\| \le \tilde{M} \|A^{\frac{1}{2}}(u_2 - u_1)\|,
$$

and

$$
k\|u_2^+ - u_1^+\| \le kl\|u_2 - u_1\| \le \hat{M}\|A^{\frac{1}{2}}(u_2 - u_1)\|,
$$

where $k|u_1^+ - u_2^+| \le kl|u_1 - u_2|$, $l$ is a proper positive constant.

We know that $U_i \in B_\sigma$ and $\|U_i - \bar{V}\|_{C([\tau,T];\mathcal{H})} \le \sigma$, exploiting Lemma 2.8,

$$
\begin{aligned}
\|U_i\|_{C([\tau,T];\mathcal{H})} &= \max_{t\in[\tau,T]} (\|A^{\frac{1}{2}} u_i(t)\| + \|\partial_t u_i(t)\|) \\
&\le \sigma + \|\bar{V}\|_{C([\tau,T];\mathcal{H})} \\
&\le \sigma + \max_{t\in[\tau,T]} (\|A^{\frac{1}{2}} e^{-(t-\tau)\mathcal{L}} \varphi(\tau)\| + \|e^{-(t-\tau)\mathcal{L}} \partial_t \varphi(\tau)\|) \\
&\le \sigma + (\frac{1}{2e\tau})^{\frac{1}{2}} \|\varphi(\tau)\| + \|\partial_t \varphi(\tau)\| \triangleq \tilde{R},
\end{aligned}
\tag{3.24}
$$

then we have $\|A^{\frac{1}{2}} u_i(t)\| \le \tilde{R}, \ t \in [\tau, T], \ i = 1, 2.$

Hence

$$
\begin{aligned}
\|f(u_2) - f(u_1)\|_{C([\tau,T];H)} &\le \tilde{M}_{\tilde{R}} \max_{t\in[\tau,T]} \|A^{\frac{1}{2}}(u_2 - u_1)\| \\
&\le \tilde{M}_{\tilde{R}} \|U_2 - U_1\|_{C([\tau,T];\mathcal{H})},
\end{aligned}
\tag{3.25}
$$

and

$$
\|k(u_2^+ - u_1^+)\|_{C([\tau,T];H)} \le \hat{M}_{\tilde{R}} \|U_2 - U_1\|_{C([\tau,T];\mathcal{H})}. \tag{3.26}
$$

From (3.24), for $\tau \leq s \leq T$,

$$\begin{aligned}
\|u_i^s\|_W &= \max_{\theta\in[-h,0]} \|A^{\frac{1}{2}}u_i^s(\theta)\| + \max_{\theta\in[-h,0]} \|\partial_t u_i^s(\theta)\| \\
&= \max_{r\in[s-h,s]} \|A^{\frac{1}{2}}u_i(r)\| + \max_{r\in[s-h,s]} \|\partial_t u_i(r)\| \\
&\leq 2(\max_{r\in[\tau-h,\tau]} \|A^{\frac{1}{2}}u_i(r)\| + \max_{r\in[\tau-h,\tau]} \|\partial_t u_i(r)\|) \\
&\quad + 2(\max_{r\in[\tau,T]} \|A^{\frac{1}{2}}u_i(r)\| + \max_{r\in[\tau,T]} \|\partial_t u_i(r)\|) \\
&\leq 2\|\varphi\|_W + 2\max_{r\in[\tau,T]}(\|A^{\frac{1}{2}}u_i(r)\| + \|\partial_t u_i(r)\|) \\
&\leq 2\|\varphi\|_W + 2\|U_i\|_{C([\tau,T];\mathcal{H})} \leq 2\|\varphi\|_W + 2\tilde{R} \triangleq \hat{R}.
\end{aligned} \tag{3.27}$$

Since $u(t-\pi[u^t]) = u(t) - \int_{t-\pi[u^t]}^t \partial_t u(s)ds$, by (2.6) and (3.27)

$$\begin{aligned}
&\|u_2(s-\pi[u_2^s]) - u_1(s-\pi[u_1^s])\| \\
&\leq \|u_2(s-\pi[u_2^s]) - u_2(s-\pi[u_1^s])\| + \|u_2(s-\pi[u_1^s]) - u_1(s-\pi[u_1^s])\| \\
&= \|u_2(s) - \int_{s-\pi[u_2^s]}^s \partial_t u_2(r)dr - u_2(s) + \int_{s-\pi[u_1^s]}^s \partial_t u_2(r)dr\| \\
&\quad + \max_{\theta\in[-h,0]} \|u_2(s+\theta) - u_1(s+\theta)\| \\
&\leq |\int_{s-\pi[u_1^s]}^{s-\pi[u_2^s]} \|\partial_t u_2(r)\|dr| + \|u_2^s - u_1^s\|_W \\
&\leq \hat{R}|\pi[u_1^s] - \pi[u_2^s]| + \|u_2^s - u_1^s\|_W \leq (\hat{R}\cdot C_R + 1)\|u_2^s - u_1^s\|_W,
\end{aligned} \tag{3.28}$$

moreover,

$$\begin{aligned}
\|u_2^s - u_1^s\|_W &\leq \max_{r\in[s-h,\tau]}(\|A^{\frac{1}{2}}(u_2(r)-u_1(r))\| + \|\partial_t u_2(r) - \partial_t u_1(r)\|) \\
&\quad + \max_{r\in[\tau,s]}(\|A^{\frac{1}{2}}(u_2(r)-u_1(r))\| + \|\partial_t u_2(r) - \partial_t u_1(r)\|) \\
&\leq \max_{r\in[\tau-h,T]}(\|A^{\frac{1}{2}}(u_2(r)-u_1(r))\| + \|\partial_t u_2(r) - \partial_t u_1(r)\|) \\
&\leq 2\max_{r\in[\tau-h,\tau]}(\|A^{\frac{1}{2}}(u_2(r)-u_1(r))\| + \|\partial_t u_2(r) - \partial_t u_1(r)\|) \\
&\quad + 2\max_{r\in[\tau,T]}(\|A^{\frac{1}{2}}(u_2(r)-u_1(r))\| + \|\partial_t u_2(r) - \partial_t u_1(r)\|) \leq 2\|U_2 - U_1\|_W,
\end{aligned} \tag{3.29}$$

substituting (3.29) into (3.28), it yields

$$\|u_2(s-\pi[u_2^s]) - u_1(s-\pi[u_1^s])\|_{C([\tau,T];H)} \leq 2(\hat{R}\cdot C_R + 1)\|U_2 - U_1\|_{C([\tau,T];\mathcal{H})}. \tag{3.30}$$

Substituting (3.25),(3.26) and (3.30) into (3.22), we obtain that

$$\begin{aligned}&\|[KU_1](t)-[KU_2](t)\|_{C([\tau,T];\mathcal{H})}\\&\le\int_\tau^t(\tilde{M}_{\tilde{R}}+\hat{M}_{\tilde{R}}+2(\hat{R}\cdot C_R+1))\|U_2-U_1\|_{C([\tau,T];\mathcal{H})}ds\\&\le(T-\tau)\cdot(\tilde{M}_{\tilde{R}}+\hat{M}_{\tilde{R}}+2(\hat{R}\cdot C_R+1))\|U_2-U_1\|_{C([\tau,T];\mathcal{H})},\end{aligned}$$

choosing T, such that $(T-\tau)\cdot(\tilde{M}_{\tilde{R}}+\hat{M}_{\tilde{R}}+2(\hat{R}\cdot C_R+1))<1$.

(II). For all $t\in[\tau,T]$, $U\in B_\sigma$, and by (3.24)-(3.26),(3.30), we have

$$\begin{aligned}&\|[KU](t)-\bar{V}(t)\|_{C([\tau,T];\mathcal{H})}=\|\int_\tau^t e^{-(t-s)\mathcal{L}}\mathcal{N}(U^s)ds\|_{C([\tau,T];\mathcal{H})}\\&\le\int_\tau^t\|e^{-(t-s)\mathcal{L}}(-f(u(s)-ku^+-u(s-\pi[u^s])+g(x,s))\|_{C([\tau,T];H)}ds\\&\le\int_\tau^t(\|f(u(s))\|_{C([\tau,T];H)}\\&+k\|u^+\|_{C([\tau,T];H)}+\|u(s-\pi[u^s])\|_{C([\tau,T];H)}+\|g(x,s))\|_{C([\tau,T];H)})ds\\&\le(T-\tau)\cdot[(\tilde{M}_{\tilde{R}}+\hat{M}_{\tilde{R}}+2(\hat{R}\cdot C_R+1))\cdot\tilde{R}+\|g(s)\|_{C([\tau,T];H)}],\end{aligned}$$

choosing T, such that $(T-\tau)\cdot[(\tilde{M}_{\tilde{R}}+C_2+\hat{M}_{\tilde{R}}+(\hat{R}\cdot C_R+1))\cdot\tilde{R}+\|g(s)\|_{C([\tau,T];H)}]<\sigma$.

So, $K:B_\sigma\to B_\sigma$ is a contraction mapping based on (I)(II), then there exists a fixed point $U\in C([\tau,T];\mathcal{H})$.

Let

$$\bar{u}=\begin{cases}u(t), & t\in[\tau,T],\\ \varphi(t), & t\in[\tau-h,\tau],\end{cases}$$

and $\bar{u}\in C([\tau-h,T];V_2)\cap C^1([\tau-h,T];V_0)$, namely, $\bar{u}$ is a mild solution of (1.1) on interval $[\tau-h,T]$.

By using the standard method in [22]. The solution $u$ on the interval $[\tau,T]$ can be extended to the interval $[\tau,T+\epsilon]$, where $\epsilon$ depends on an upper bound for $\|u^T\|_W$, where, as above, $u^T=\{u(T+\theta):\theta\in[-h,0]\}$. This means that there is a maximal existence interval $[\tau,T_{max}]$. If $T_{max}<+\infty$ and $\lim_{t\to T_{max}-}\|u^t\|_W=+\infty$ is not true, then there exists a sequence $T_n\to T_{max-}$, such that $\|u^{T_n}\|_W\le C$, for all $n=1,2,\cdots$. Thus using $u^{T_n}$ as an initial data we can extend the solution to an interval $[\tau,T_n+\epsilon]$ for some $\epsilon>0$, which dose not depend on $n$. Since $T_n\to T_{max-}$, this means that we are able to extend the solution beyond $T_{max}$. □

**Theorem 3.6** *(Well-posedness) Let assumptions (2.1)-(2.6) hold true. Then for any* $\varphi_1,\varphi_2\in W$, $\|\varphi_j\|_W\le\varpi$, $j=1,2$, *there exists a unique global mild solution* $U(t)\equiv(u(t);\partial_t u(t))$ *of (1.1) on the interval* $[\tau,+\infty]$. *Moreover, for any* $\varpi>0$ *and* $T>\tau$ *there exists a positive constant* $C_{\varpi,T}$, *such that*

$$\|A^{\frac{1}{2}}(u_1(t)-u_2(t))\|^2+\|\partial_t u_1(t)-\partial_t u_2(t)\|^2\le C_{\varpi,T}\|\varphi_1-\varphi_2\|_W^2,\quad t\in[\tau,T].$$

***Proof*** (Existence) In Theorem 3.5, the local existence of and uniqueness of mild solution for equation (1.1) is obtained. Let $U=(u;\partial_t u)$ be a mild solution of (1.1) on the interval $[\tau-h,T_{max}]$ and

$$G(u)\equiv f(u(t))+u(t-\pi[u^t])\in C([\tau,T_{max}];H).$$

It is clear that we can consider $(u; \partial_t u)$ as a mild solution of the following linear non-delayed problem

$$\partial_{tt}u(t) + Au(t) + \mu\partial_t u(t) + ku^+(t) + G(u(t)) = g(t),$$
$$t \in [\tau, T_{max}], \quad (u(\tau), \partial_t u(\tau)) = (\varphi(\tau), \partial_t\varphi(\tau)).$$

Hence, one can see that $u(t)$ satisfies the following energy relation.

$$\begin{aligned} &E_1(t) + \mu\int_\tau^t \|\partial_t u(s)\|^2 ds \\ &\leq E_1(\tau) + \frac{2}{\mu}\int_\tau^t \|g(s)\|^2 ds + \frac{2}{\mu}\int_\tau^t \|u^s\|_W^2 ds,\ \tau \leq t \leq T_{max}, \end{aligned} \tag{3.31}$$

where

$$E_1(t) = \|\partial_t u\|^2 + \|A^{\frac{1}{2}}u\|^2 + k\|u^+\|^2 + 2\int_\Omega F(u)dx + 2K_1 \geq 0.$$

In fact, taking the scalar product in $V_0$ of (1.1) with $\partial_t u$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\partial_t u\|^2 + \|A^{\frac{1}{2}}u\|^2 + k\|u^+\|^2 + 2\int_\Omega F(u)dx) + \mu\|\partial_t u\|^2 \\ &= -(u(t - \pi[u^t]), \partial_t u) + (g(t), \partial_t u), \end{aligned} \tag{3.32}$$

integrating (3.32) over $[\tau, t]$, we can get energy relation (3.31).

On the other hand, for any $s \in [\tau, T_{max})$,

$$\begin{aligned} \|u^s\|_W &= \max_{\theta\in[-h,0]}\|A^{\frac{1}{2}}u^s(\theta)\| + \max_{\theta\in[-h,0]}\|\partial_t u^s(\theta)\| \\ &\leq \max_{r\in[s-h,\tau]}\|A^{\frac{1}{2}}u(r)\| + \max_{r\in[\tau,s]}\|A^{\frac{1}{2}}u(r)\| + \max_{r\in[s-h,\tau]}\|\partial_t u(r)\| + \max_{r\in[\tau,s]}\|\partial_t u(r)\| \\ &\leq \|\varphi\|_W + \sqrt{2}\sqrt{\max_{r\in[\tau,s]}\|A^{\frac{1}{2}}u(r)\|^2 + \max_{r\in[\tau,s]}\|\partial_t u(r)\|^2} \\ &\leq \|\varphi\|_W + 2\sqrt{\max_{r\in[\tau,s]}(\|A^{\frac{1}{2}}u(r)\|^2 + \|\partial_t u(r)\|^2)}. \end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.32), we obtain that

$$\begin{aligned} &\max_{r\in[\tau,s]}(\|A^{\frac{1}{2}}u(r)\|^2 + \|\partial_t u(r)\|^2) \\ &\leq C_1(E_1(\tau) + (t-\tau)\|\varphi\|_W^2 + \int_\tau^t \max_{r\in[\tau,s]}(\|A^{\frac{1}{2}}u(r)\|^2 + \|\partial_t u(r)\|^2)ds + \int_\tau^t \|g(s)\|^2 ds). \end{aligned}$$

By the integral Gronwall Lemma, $t < T_{max}$

$$\begin{aligned} &\max_{r\in[\tau,s]}(\|A^{\frac{1}{2}}u(r)\|^2 + \|\partial_t u(r)\|^2) \\ &\leq C_1(E_1(\tau) + \|\varphi\|_W^2 + \int_\tau^t \|g(s)\|^2 ds)^{e^{C_1(t-\tau)}}. \end{aligned} \tag{3.34}$$

For any $T > \tau$, $[\tau, T_{\max}) \subset [\tau, T]$, (3.34) holds true, then the solution of (1.1) can be extended to $[\tau, +\infty)$.

(Uniqueness) Set $w(t)=u_1(t)-u_2(t)$, then

$$\begin{aligned}\partial_{tt}w+Aw+\mu\partial_t w+ku_1^+-ku_2^++f(u_1)-f(u_2)\\+u_1(t-\pi[u_1^t])-u_2(t-\pi[u_2^t])=0.\end{aligned}\tag{3.35}$$

Taking the scalar product in $V_0$ of (3.35) with $\partial_t w$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}w\|^2+\|\partial_t w\|^2)+\mu\|\partial_t w\|^2\\&=-k(u_1^+-u_2^+,\partial_t w)-(f(u_1)-f(u_2),\partial_t w)-(u_1(t-\pi[u_1^t])-u_2(t-\pi[u_2^t]),\partial_t w).\end{aligned}\tag{3.36}$$

Furthermore, exploiting Hölder and Young inequalities, it yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|A^{\frac{1}{2}}w\|^2+\|\partial_t w\|^2)+\frac{\mu}{4}\|\partial_t w\|^2\\&\le\frac{k}{\mu}\|u_2^+-u_1^+\|^2+\frac{1}{\mu}\|f(u_2)-f(u_1)\|^2+\frac{1}{\mu}\|u_1(t-\pi[u_1^t])-u_2(t-\pi[u_2^t])\|^2,\end{aligned}\tag{3.37}$$

by (3.25),(3.26) and (3.30), we can get that

$$\begin{aligned}&k\|u_2^+-u_1^+\|^2+\|f(u_2)-f(u_1)\|^2+\|u_1(t-\pi[u_1^t])-u_2(t-\pi[u_2^t])\|^2\\&\le[\tilde{M}_{\tilde{\varpi}}^2+\hat{M}_{\tilde{\varpi}}^2+(\tilde{\varpi}\cdot C_{\tilde{\varpi}}+1)^2]\|w^t\|_W^2\\&\le[\tilde{M}_{\tilde{\varpi}}^2+\hat{M}_{\tilde{\varpi}}^2+(\tilde{\varpi}\cdot C_{\tilde{\varpi}}+1)^2](2\|\varphi_1-\varphi_2\|_W^2+4\max_{r\in[\tau,t]}(\|A^{\frac{1}{2}}w(r)\|^2+\|\partial_t w(r)\|^2)),\end{aligned}\tag{3.38}$$

where $\tilde{\varpi}=\max\{\|u_1^t\|_W,\|u_2^t\|_W\},\ \forall t\in[\tau,T]$.
Substituting (3.38) into (3.37), one can see that

$$\begin{aligned}&\frac{d}{dt}(\|A^{\frac{1}{2}}w\|^2+\|\partial_t w\|^2)+\frac{\mu}{2}\|\partial_t w\|^2\\&\le 4\mu[\tilde{M}_{\tilde{\varpi}}^2+\hat{M}_{\tilde{\varpi}}^2+(\tilde{\varpi}\cdot C_{\tilde{\varpi}}+1)^2](\|\varphi_1-\varphi_2\|_W^2+2\max_{r\in[\tau,t]}(\|A^{\frac{1}{2}}w(r)\|^2+\|\partial_t w(r)\|^2)).\end{aligned}$$

By using the Gronwall Lemma, we obtain that

$$\max_{r\in[\tau,t]}(\|A^{\frac{1}{2}}w(r)\|^2+\|\partial_t w(r)\|^2)\le C_{\tilde{\varpi},T}(\|\varphi_1-\varphi_2\|_W^2),\forall t\in[\tau,T].$$

The proof is completed. □

Now let's consider the existence of smooth solutions for system (1.1). In the following we show that under additional hypotheses mild solutions become strong. According to Corollary 2.6 in references [9], similarly, we have the following conclusion.

**Corollary 3.7** *Assume that the hypotheses of Theorem 3.6 be in force, for every $\varphi_1,\varphi_2\in W,\ \|\varphi_j\|_W\le\varpi,\ j=1,2,$*

$$|\pi(\varphi_1)-\pi(\varphi_2)|\le C_\varrho\|\varphi_1-\varphi_2\|_{C_{V_0}}\tag{3.39}$$

*hold. If the initial data function $\varphi(\theta)$ possesses the property*

$$\varphi(\tau)\in D(A),\quad \partial_t\varphi(\tau)\in V_2,\tag{3.40}$$

*for every $T>0$, then the solution satisfies the relations*

$$u(t)\in L^\infty(\tau,T;D(A)),\ \partial_t u(t)\in L^\infty(\tau,T;V_2),\ \partial_{tt}u(t)\in L^\infty(\tau,T;V_0).\tag{3.41}$$

*Furthermore, if in addition $G(u)$ is Fréchet differentiable and $\|G'(u)v\| \le C_r \|A^{\frac{1}{2}}v\|$, for every $u \in D(A)$ with $|Au| \le r$, then we have*

$$u(t) \in C(\mathbb{R}_+; D(A)),\ \partial_t u(t) \in C(\mathbb{R}_+; V_2),\ \partial_{tt} u(t) \in C(\mathbb{R}_+; V_0). \tag{3.42}$$

*For the proof of Corollary 3.7, we can refer to references [9,30].*

***Remark 3.8*** [9] The property in (3.39) means that $\pi$ is Lipschitz on subsets in $C([-h,0]; V_0)$ which are bounded in $W$. In order to obtain strong solutions, we need to assume an additional smoothness of initial data in the right end point of the interval $[\tau - h, \tau]$ only.

More precisely, if we let

$$\varphi \in W_{sm} = C^2(-h,0); V_0) \cap C^1(-h,0); V_2) \cap C(-h,0); D(A)). \tag{3.43}$$

Furthermore, if we have the following compatibility hypotheses

$$\partial_{tt}\varphi(\tau) + A\varphi(\tau) + \mu\partial_t\varphi(\tau) + k\varphi^+(\tau) + f(\varphi(\tau)) + \varphi(-\pi[\varphi]) = 0. \tag{3.44}$$

One can see that the set $\mathcal{M} = \{\varphi \in W_{sm} : \varphi \ \text{ satisfies } (3.44)\} \subset W$. Thus we can define the dynamics in a smoother space. The set $\mathcal{M}$ is an analog to the solution manifold used in [28].

## 4 Pullback $\mathcal{D}-$attractors

### 4.1 Pullback $\mathcal{D}-$absorbing Set

Owing to Theorem 3.6, we can define an evolution process $S(t,\tau) : W \to W, t \ge \tau$. $S(t,\tau)\varphi = \{u^t(\cdot;\tau,\varphi) | u(\cdot) \text{ is a solution of (1.1) with } \varphi \in W\}$, and satisfy $u^\tau = \varphi$. Moreover, by Theorem 3.6, for any $t \ge \tau$, $(t,\tau,x) \to S(t,\tau)x,\ x \in W$ is a continuous process and generates a dynamical system $(S(t,\tau), W)$ (see [27,31]).

***Remark 4.1*** We can equivalently define the dynamical system on the linear space of vector-functions $\overline{W} \equiv \{\Phi = (\varphi, \partial_t\varphi) | \varphi \in W\} \subset C([-h,0]; V_2 \times V_0)$. In this notations evolution process reads $\bar{S}(t,\tau)\Phi \equiv U^t$ and we have $G : W \ni \varphi \mapsto (\varphi, \partial_t\varphi) \in \overline{W}$ satisfying $GS(t,\tau) = \bar{S}(t,\tau)G$.

**Lemma 4.2** *Suppose that assumptions (2.1)-(2.5) hold true. Then the solution $(u, \partial_t u)$ of equation (1.1) satisfies the following estimates*

$$\begin{aligned}
&\|u^t\|_W^2 \\
&\quad = \max_{\theta\in[-h,0]} \|z(t+\theta)\|^2 + \max_{\theta\in[-h,0]} \|A^{\frac{1}{2}}u(t+\theta)\|^2 \\
&\le 2\max_{\theta\in[-h,0]} (\|z(t+\theta)\|^2 + \|A^{\frac{1}{2}}u(t+\theta)\|^2) \\
&\le 8e^{-\gamma(t-\tau-h)}(E(\tau) + \alpha h\|\varphi\|_W^2) + \frac{16}{\mu}e^{-\gamma(t-h)}\int_{-\infty}^t e^{\gamma s}\|g(s)\|^2 ds + C,
\end{aligned} \tag{4.1}$$

*where $C = \frac{4}{\gamma}(\gamma K_1 + \varepsilon K_2 + \frac{2}{\mu}C_\eta)$.*

***Proof*** Taking the scalar product in $V_0$ of (1.1) with $z = \partial_t u + \varepsilon u$ ($\varepsilon > 0$), we have

$$\frac{1}{2}\frac{d}{dt}(\|z\|^2 + \|A^{\frac{1}{2}}u\|^2) + \varepsilon\|A^{\frac{1}{2}}u\|^2 + (\mu-\varepsilon)\|z\|^2 - \varepsilon(\mu-\varepsilon)(u,z) + k(u^+,z)$$
$$= -(f(u),z) - (u(t-\pi[u^t]),z) + (g(t),z).$$

Similar to the priori estimates in Sect. 3.1, we can get

$$\begin{aligned}\|z\|^2 + \|A^{\frac{1}{2}}u\|^2 + k\|u^+\|^2 &\le 4e^{-\gamma(t-\tau)}V(\tau) + \frac{8}{\mu}e^{-\gamma t}\int_\tau^t e^{\gamma s}\|g(s)\|^2 ds + C \\ \le & 4e^{-\gamma(t-\tau)}(E(\tau) + \alpha h\|\varphi\|_W^2) + \frac{8}{\mu}e^{-\gamma t}\int_\tau^t e^{\gamma s}\|g(s)\|^2 ds + C,\end{aligned} \tag{4.2}$$

where $C = \frac{4}{\gamma}(2\gamma K_1 + \varepsilon K_2 + \frac{2}{\mu}C_\eta)$.

Now setting $t+\theta$ instead of $t$ (where $\theta \in [-h, 0]$) in (4.2), there holds

$$\begin{aligned}&\|A^{\frac{1}{2}}u(t+\theta)\|^2 + \|z(t+\theta)\|^2 + k\|u^+(t+\theta)\|^2 \\ &\le 4e^{-\gamma(t+\theta-\tau)}(E(\tau) + \alpha h\|\varphi\|_W^2) + \frac{8}{\mu}e^{-\gamma(t+\theta)}\int_\tau^{t+\theta} e^{\gamma s}\|g(s)\|^2 ds + C \\ &\le 4e^{-\gamma(t-\tau-h)}(E(\tau) + \alpha h\|\varphi\|_W^2) + \frac{8}{\mu}e^{-\gamma(t-h)}\int_\tau^t e^{\gamma s}\|g(s)\|^2 ds + C.\end{aligned} \tag{4.3}$$

Furthermore, from (4.3) we get

$$\begin{aligned}\|u^t\|_W^2 &= \max_{\theta\in[-h,0]}\|z(t+\theta)\|^2 + \max_{\theta\in[-h,0]}\|A^{\frac{1}{2}}u(t+\theta)\|^2 \\ &\le 2\max_{\theta\in[-h,0]}(\|z(t+\theta)\|^2 + \|A^{\frac{1}{2}}u(t+\theta)\|^2) \\ &\le 8e^{-\gamma(t-\tau-h)}(E(\tau) + \alpha h\|\varphi\|_W^2) + \frac{16}{\mu}e^{-\gamma(t-h)}\int_{-\infty}^t e^{\gamma s}\|g(s)\|^2 ds + C.\end{aligned}$$

For any $\gamma > 0$, we denote by $\mathcal{D}_\gamma$ the class of all families of nonempty subsets $\mathcal{D} = \{D(t); t \in \mathbb{R}\} \subset \mathcal{P}(W)$ such that

$$\lim_{\tau\to-\infty}(e^{\gamma\tau}\sup_{u\in D(\tau)}\|u\|_W^2) = 0.$$

□

**Theorem 4.3** *(pullback $\mathcal{D}_\gamma$−absorbing set) Let assumptions of Lemma 4.2 be in force. Then the family of bounded sets $\mathcal{D}_1 = \{D_1(t); t \in \mathbb{R}\}$ with $D_1(t) = B(0, \sqrt{C}r(t))$, where*

$$r^2(t) = Ce^{-\gamma(t-h)}\int_{-\infty}^t e^{\gamma s}\|g(s)\|^2 ds + C < \infty, \tag{4.4}$$

*is pullback $\mathcal{D}_\gamma$−absorbing set for the process $\{S(t,\tau)\}$ and $\mathcal{D}_1 \in \mathcal{D}_\gamma$.*

***Proof*** That $\mathcal{D}_1$ is pullback $\mathcal{D}_\gamma$−absorbing set for the problem (1.1) is an immediate consequence of (4.1) in Lemma 4.2.

Thanks to (4.4), we have $e^{\gamma t}r^2(t) \to 0$, as $t \to -\infty$. Then $\mathcal{D}_1$ belongs to $D_\gamma$. □

### 4.2 Pullback $\mathcal{D}-$asymptotically Compact

In order to prove pullback $\mathcal{D}-$asymptotically compact, we furthermore assume that there exists $\delta > 0$, the delay term satisfies for any $R > 0$, $\phi_i,\ i = 1, 2$, there exists $L_R > 0$, such that $\|\phi_i\|_W \le R$, one has

$$|\pi(\phi_1) - \pi(\phi_2)| \le L_R \max_{\theta\in[-h,0]} \|A^{\frac{1}{2}-\delta}(\phi_1(\theta) - \phi_2(\theta))\|. \tag{4.5}$$

***Remark 4.4*** In this paper, we consider the one-dimensional suspension bridge equation with state-dependent delay, but we still need to assume that the delay term satisfies the above condition when we use the contractive function methods to verify asymptotical compactness of the process $\{S(t, \tau)\}_{t\ge\tau}$.

**Lemma 4.5** *Suppose that (2.1)-(2.6) hold true. Then the process $\{S(t, \tau)\}_{t\ge\tau}$ corresponding to (1.1) is pullback $\mathcal{D}$-asymptotically compact.*

***Proof*** For any fixed $T \in \mathbb{R}$. Let $(u_i(t), \partial_t u_i(t))$ be the solution of (1.1) corresponding to the initial data $(\varphi_i(x, \theta), \partial_t\varphi_i(x, \theta)) \in D_1(\tau) \times D_1(\tau)(i = 1, 2,\ \theta \in [-h, 0])$.

Set $w = u_1(t) - u_2(t)$, then

$$(w(\theta); \partial_t w(\theta)) = (\varphi_1(\theta); \partial_t\varphi_1(\theta)) - (\varphi_2(\theta); \partial_t\varphi_2(\theta)),\ \theta \in [-h, 0],\ x \in [0, L],$$

satisfies

$$\begin{aligned} \partial_{tt}w + \partial_{xxxx}w + \mu\partial_t w = k(u_2^+ - u_1^+) + f(u_2(t)) - f(u_1(t)) \\ + u_2(t - \pi[u_2^t]) - u_1(t - \pi[u_1^t]). \end{aligned} \tag{4.6}$$

We define

$$E_w(t) = \frac{1}{2}(\|A^{\frac{1}{2}}w\|^2 + \|\partial_t w\|^2).$$

Multiplying (4.6) by $e^{\gamma t}\partial_t w$ and integrating it over $\Omega$, thanks to Young and Hölder inequalities, we get

$$\begin{aligned} &\frac{d}{dt}e^{\gamma t}E_w(t) + \frac{\mu}{2}e^{\gamma t}\|\partial_t w\|^2 \le e^{\gamma t}(f(u_2) - f(u_1), \partial_t w) + \frac{k^2l^2}{\mu}e^{\gamma t}\|w\|^2 \\ &+ \gamma e^{\gamma t}E_w(t) + \frac{1}{\mu}e^{\gamma t}\|u_2(t - \pi[u_2^t]) - u_1(t - \pi[u_1^t]]\|^2. \end{aligned} \tag{4.7}$$

Furthermore, integrating (4.7) over $[s, T]$, it yields

$$\begin{aligned} &e^{\gamma T}E_w(T) - e^{\gamma s}E_w(s) + \frac{\mu}{2}\int_s^T e^{\gamma\xi}\|\partial_t w(\xi)\|^2 d\xi \\ &\le \gamma\int_s^T e^{\gamma\xi}E_w(\xi)d\xi + \frac{k^2l^2}{\mu}\int_s^T e^{\gamma\xi}\|w(\xi)\|^2 d\xi \\ &\quad + \frac{1}{\mu}\int_s^T e^{\gamma\xi}\|u_2(\xi - \pi[u_2^\xi]) - u_1(\xi - \pi[u_1^\xi]]\|^2 d\xi \\ &\quad + \int_s^T e^{\gamma\xi}(f(u^2) - f(u^1), \partial_t w(\xi))d\xi, \end{aligned} \tag{4.8}$$

and then integrating (4.8) over $[T-\tau, t]$ with respect to $s$, we can see that

$$\begin{aligned}
&\tau e^{\gamma T}E_w(T)-\int_{T-\tau}^{T}e^{\gamma s}E_w(s)ds+\frac{\mu}{2}\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|\partial_t w(\xi)\|^2d\xi ds\\
&\le\gamma\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}E_w(\xi)d\xi ds+\frac{k^2l^2}{\mu}\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|w(\xi)\|^2d\xi ds\\
&\quad+\frac{1}{\mu}\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|u_2(\xi-\pi[u_2^{\xi}])-u_1(\xi-\pi[u_1^{\xi}]]\|^2d\xi ds\\
&\quad+\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),\partial_t w(\xi))d\xi ds.
\end{aligned}\tag{4.9}$$

On the other hand, multiplying (4.6) by $e^{\gamma t}w$ and integrating it over $\Omega$, applying Young and Hölder inequalities, we arrive at

$$\begin{aligned}
&\frac{d}{dt}(e^{\gamma t}(\partial_t w,w))+\frac{1}{2}e^{\gamma t}\|A^{\frac{1}{2}}w\|^2\\
&\le(\gamma-\mu)e^{\gamma t}(\partial_t w,w)+kle^{\gamma t}\|w\|^2+e^{\gamma t}\|\partial_t w\|^2\\
&\quad+e^{\gamma t}(f(u_2)-f(u_1),w)+\frac{1}{2\lambda_1}e^{\gamma t}\|u_2(t-\pi[u_2^t])-u_1(t-\pi[u_1^t])\|^2.
\end{aligned}\tag{4.10}$$

Integrating (4.10) over $[s, T]$, one can see that

$$\begin{aligned}
&e^{\gamma T}(\partial_t w(T),w(T))+\frac{1}{2}\int_{s}^{T}e^{\gamma\xi}\|A^{\frac{1}{2}}w(\xi)\|^2d\xi\\
&\le e^{\gamma s}(\partial_t w(s),w(s))+\int_{s}^{T}e^{\gamma\xi}\|\partial_t w(\xi)\|^2d\xi\\
&\quad+kl\int_{s}^{T}e^{\gamma\xi}\|w(\xi)\|^2d\xi+(\gamma-\mu)\int_{s}^{T}e^{\gamma\xi}(\partial_t w(\xi),w(\xi))d\xi\\
&\quad+\frac{1}{2\lambda_1}\int_{s}^{T}e^{\gamma\xi}\|u_2(\xi-\pi[u_2^{\xi}])-u_1(\xi-\pi[u_1^{\xi}])\|^2d\xi\\
&\quad+\int_{s}^{T}e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),w(\xi))d\xi.
\end{aligned}\tag{4.11}$$

Integrating (4.11) over $[T-\tau, T]$ again, we have that

$$\begin{aligned}
&\tau e^{\gamma T}(\partial_t w(T),w(T))+\frac{1}{2}\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|A^{\frac{1}{2}}w(\xi)\|^2d\xi ds\\
&\le\int_{T-\tau}^{T}e^{\gamma s}(\partial_t w(s),w(s))ds+(\gamma-\mu)\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(\partial_t w(\xi),w(\xi))d\xi ds\\
&\quad+kl\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|w(\xi)\|^2d\xi ds+\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|\partial_t w(\xi)\|^2d\xi ds\\
&\quad+\frac{1}{2\lambda_1}\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|u_2(\xi-\pi[u_2^{\xi}])-u_1(\xi-\pi[u_1^{\xi}])\|^2d\xi ds\\
&\quad+\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),w(\xi))d\xi ds.
\end{aligned}\tag{4.12}$$

Multiplying (4.12) by $\gamma$ and substituting result into (4.9), it follows that

$$
\begin{aligned}
&\tau e^{\gamma T}E_w(T)-\int_{T-\tau}^{T}e^{\gamma s}E_w(s)ds+\frac{1}{2}(\mu-3\gamma)\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|\partial_t w(\xi)\|^2d\xi ds\\
&\le\gamma\int_{T-\tau}^{T}e^{\gamma s}(\partial_t w(s),w(s))ds+\gamma(\gamma-\mu)\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(\partial_t w(\xi),w(\xi))d\xi ds\\
&\quad+(kl\gamma+\frac{k^2l^2}{\mu})\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|w(\xi)\|^2d\xi ds-\gamma\tau e^{\gamma T}(\partial_t w(T),w(T))\\
&\quad+(\frac{\gamma}{2\lambda_1}+\frac{1}{\mu})\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|u_2(\xi-\pi[u_2^\xi])-u_1(\xi-\pi[u_1^\xi])\|^2d\xi ds\\
&\quad+\gamma\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),w(\xi))d\xi ds\\
&\quad+\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),\partial_t w(\xi))d\xi ds.
\end{aligned}
\tag{4.13}
$$

Integrating (4.10) over $[T-\tau,T]$ with respect to $t$, it yields

$$
\begin{aligned}
\frac{1}{2}\int_{T-\tau}^{T}e^{\gamma\xi}\|A^{\frac{1}{2}}w(\xi)\|^2d\xi\le{}&(\gamma-\mu)\int_{T-\tau}^{T}e^{\gamma\xi}(\partial_t w(\xi),w(\xi))d\xi\\
&+e^{\gamma(T-\tau)}(\partial_t w(T-\tau),w(T-\tau))-e^{\gamma T}(\partial_t w(T),w(T))\\
&+\frac{1}{2\lambda_1}\int_{T-\tau}^{T}e^{\gamma\xi}\|u_2(\xi-\pi[u_2^\xi])-u_1(\xi-\pi[u_1^\xi])\|^2d\xi\\
&+\int_{T-\tau}^{T}e^{\gamma\xi}\|\partial_t w(\xi)\|^2d\xi+kl\int_{T-\tau}^{T}e^{\gamma\xi}\|w(\xi)\|^2d\xi\\
&+\int_{T-\tau}^{T}e^{\gamma\xi}(f(u_2)-f(u_1),w(\xi))d\xi.
\end{aligned}
\tag{4.14}
$$

Noticing that $6\gamma<\mu$. Substituting (4.14) into (4.13), we get that

$$
\begin{aligned}
&\tau e^{\gamma T}E_w(T)+\int_{T-\tau}^{T}e^{\gamma\xi}E_w(\xi)d\xi\\
&\le(3\gamma-2\mu)\int_{T-\tau}^{T}e^{\gamma s}(\partial_t w(s),w(s))ds-\gamma\tau e^{\gamma T}(\partial_t w(T),w(T))\\
&\quad+\gamma(\gamma-\mu)\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(\partial_t w(\xi),w(\xi))d\xi ds+2kl\int_{T-\tau}^{T}e^{\gamma\xi}\|w(\xi)\|^2d\xi\\
&\quad+(kl\gamma+\frac{k^2l^2}{\mu})\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|w(\xi)\|^2d\xi ds-2e^{\gamma T}(\partial_t w(T),w(T))\\
&\quad+(\frac{\gamma}{2\lambda_1}+\frac{1}{\mu})\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}\|u_2(\xi-\pi[u_2^\xi])-u_1(\xi-\pi[u_1^\xi])\|^2d\xi ds\\
&\quad+\gamma\int_{T-\tau}^{T}\int_{s}^{T}e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),w(\xi))d\xi ds+3\int_{T-\tau}^{T}e^{\gamma\xi}\|\partial_t w(\xi)\|^2d\xi\\
&\quad+2\int_{T-\tau}^{T}e^{\gamma\xi}(f(u_2)-f(u_1),w(\xi))d\xi+2e^{\gamma(T-\tau)}(\partial_t w(T-\tau),w(T-\tau))
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),\partial_t w(\xi))d\xi ds\\
&+\frac{1}{\lambda_1}\int_{T-\tau}^{T} e^{\gamma\xi}\|u_2(\xi-\pi[u_2^{\xi}])-u_1(\xi-\pi[u_1^{\xi}])\|^2 d\xi. \qquad (4.15)
\end{aligned}
$$

Multiplying (4.8) by $\frac{6}{\mu}$ and integrating result over $[T-\tau, T]$ with respect to $t$, we obtain that

$$
\begin{aligned}
&\frac{6}{\mu}e^{\gamma T}E_w(T)+3\int_{T-\tau}^{T} e^{\gamma\xi}\|\partial_t w(\xi)\|^2 d\xi\\
&\le \frac{6}{\mu}e^{\gamma(T-\tau)}E_w(T-\tau)+\frac{6\gamma}{\mu}\int_{T-\tau}^{T} e^{\gamma\xi}E_w(\xi)d\xi\\
&\quad+\frac{6}{\mu^2}\int_{T-\tau}^{T} e^{\gamma\xi}\|u_2(\xi-\pi[u_2^{\xi}])-u_1(\xi-\pi[u_1^{\xi}]\|^2 d\xi\\
&\quad+\frac{6}{\mu}\int_{T-\tau}^{T} e^{\gamma\xi}(f(u_2)-f(u_1),\partial_t w(\xi))d\xi\\
&\quad+\frac{6k^2l^2}{\mu^2}\int_{T-\tau}^{T} e^{\gamma\xi}\|w(\xi)\|^2 d\xi. \qquad (4.16)
\end{aligned}
$$

Substituting (4.16) into (4.15), we have that

$$
\begin{aligned}
E_w(T)\le{}& \frac{3\gamma-2\mu}{\tau}e^{-\gamma\tau}\int_{T-\tau}^{T} e^{\gamma s}(\partial_t w(s),w(s))ds+\frac{6}{\tau\mu}e^{-\gamma\tau}E_w(T-\tau)\\
&+\frac{2}{\tau}e^{-\gamma\tau}(\partial_t w(T-\tau),w(T-\tau))-(\frac{2}{\tau}+\gamma)(\partial_t w(T),w(T))\\
&+\frac{\gamma(\gamma-\mu)}{\tau}e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}(\partial_t w(\xi),w(\xi))d\xi ds\\
&+\frac{1}{\tau}(kl\gamma+\frac{k^2l^2}{\mu})e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}\|w(\xi)\|^2 d\xi ds\\
&+\frac{\gamma}{\tau}e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),w(\xi))d\xi ds\\
&+\frac{1}{\tau}e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}(f(u_2(\xi))-f(u_1(\xi)),\partial_t w(\xi))d\xi ds\\
&+\frac{6}{\tau\mu}e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma\xi}(f(u_2)-f(u_1),\partial_t w(\xi))d\xi\\
&+\frac{2}{\tau}e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma\xi}(f(u_2)-f(u_1),w(\xi))d\xi\\
&+\frac{2}{\tau}(\frac{3k^2l^2}{\mu^2}+kl)e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma\xi}\|w(\xi)\|^2 d\xi\\
&+\frac{1}{\tau}(\frac{1}{\lambda_1}+\frac{6}{\mu^2})e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma\xi}\|u_2(\xi-\pi[u_2^{\xi}])-u_1(\xi-\pi[u_1^{\xi}])\|^2 d\xi\\
&+\frac{1}{\tau}(\frac{\gamma}{2\lambda_1}+\frac{1}{\mu})e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}\|u_2(\xi-\pi[u_2^{\xi}])-u_1(\xi-\pi[u_1^{\xi}])\|^2 d\xi ds.
\end{aligned}
$$

(4.17)

By Hölder inequality and (4.1) in Lemma 4.2, we can get

$$\begin{aligned} &e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma s}(\partial_t w(s), w(s))ds \\ &\leq (e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma s}\|\partial_t w(s)\|^2 ds)^{\frac{1}{2}} \cdot (e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma s}\|w(s)\|^2 ds)^{\frac{1}{2}} \\ &\leq C(e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma s}\|w(s)\|^2 ds)^{\frac{1}{2}} \\ &\leq C(\int_{T-\tau}^{T} \|w(s)\|^2 ds)^{\frac{1}{2}}, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma \xi}(\partial_t w(\xi), w(\xi))d\xi ds \\ &\leq \tau(e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma s}\|\partial_t w(s)\|^2 ds)^{\frac{1}{2}} \cdot (e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma s}\|w(s)\|^2 ds)^{\frac{1}{2}} \\ &\leq C\tau(\int_{T-\tau}^{T} \|w(s)\|^2 ds)^{\frac{1}{2}}. \end{aligned} \tag{4.19}$$

Finally, by (4.5), we get that

$$\begin{aligned} &\int_{T-\tau}^{T} e^{\gamma \xi}\|u_2(\xi-\pi[u_2^\xi]) - u_1(\xi-\pi[u_1^\xi])\|^2 d\xi \\ &\leq 2\int_{T-\tau}^{T} e^{\gamma \xi}\|u_2(\xi-\pi[u_2^\xi]) - u_2(\xi-\pi[u_1^\xi])\|^2 d\xi \\ &\quad + 2\int_{T-\tau}^{T} e^{\gamma \xi}\|u_2(\xi-\pi[u_1^\xi]) - u_1(\xi-\pi[u_1^\xi])\|^2 d\xi \\ &\leq 2\int_{T-\tau}^{T} e^{\gamma \xi}\|\int_{\xi-\pi[u_1^\xi]}^{\xi-\pi[u_2^\xi]} \partial_t u_2(r)dr\|^2 d\xi + 2\int_{T-\tau}^{T} e^{\gamma \xi}\max_{\theta\in[-h,0]}\|w(\xi+\theta)\|^2 d\xi \\ &\leq 2C_{T,\tau}^2\int_{T-\tau}^{T} e^{\gamma \xi}\|\pi[u_1^\xi]-\pi[u_2^\xi]\|^2 d\xi + 2\int_{T-\tau}^{T} e^{\gamma \xi}\max_{\theta\in[-h,0]}\|w(\xi+\theta)\|^2 d\xi \\ &\leq 2C_{T,\tau}^2 L_{C_{T,\tau}}\int_{T-\tau}^{T} e^{\gamma \xi}\max_{\theta\in[-h,0]}\|A^{\frac{1}{2}-\delta}w^\xi(\theta)\|^2 d\xi + 2\int_{T-\tau}^{T} e^{\gamma \xi}\max_{\theta\in[-h,0]}\|w(\xi+\theta)\|^2 d\xi. \end{aligned} \tag{4.20}$$

Similarly, we have

$$\begin{aligned} &\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma \xi}\|u_2(\xi-\pi[u_2^\xi]) - u_1(\xi-\pi[u_1^\xi])\|^2 d\xi ds \\ &\leq 2\tau C_{T,\tau}^2 L_{C_{T,\tau}}\int_{T-\tau}^{T} e^{\gamma \xi}\max_{\theta\in[-h,0]}\|A^{\frac{1}{2}-\delta}w^\xi(\theta)\|^2 d\xi + 2\tau\int_{T-\tau}^{T} e^{\gamma \xi}\max_{\theta\in[-h,0]}\|w(\xi+\theta)\|^2 d\xi. \end{aligned} \tag{4.21}$$

Let

$$
\begin{aligned}
&\Phi_{T,\tau}((\varphi_1, \partial_t\varphi_1), ((\varphi_2, \partial_t\varphi_2))\\
&= C(\frac{3\gamma-2\mu}{\tau}+\gamma(\gamma-\mu))(\int_{T-\tau}^{T}\|w(s)\|^2 ds)^{\frac{1}{2}} - (\frac{2}{\tau}+\gamma)(\partial_t w(T), w(T))\\
&\quad + \frac{2}{\tau}e^{-\gamma\tau}(\partial_t w(T-\tau), w(T-\tau)) + \frac{2}{\tau}(\frac{3k^2L^2}{\mu^2}+kL)e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma\xi}\|w(\xi)\|^2 d\xi\\
&\quad + [\frac{2}{\tau}(\frac{1}{\lambda_1}+\frac{6}{\mu^2}) + (\frac{\gamma}{\lambda_1}+\frac{2}{\mu})]e^{-\gamma T} C_{T,\tau}^2 L_{C_{T,\tau}} \int_{T-\tau}^{T} e^{\gamma\xi} \max_{\theta\in[-h,0]} \|A^{\frac{1}{2}-\delta} w(\xi+\theta)\|^2 d\xi\\
&\quad + [\frac{2}{\tau}(\frac{1}{\lambda_1}+\frac{6}{\mu^2}) + (\frac{\gamma}{\lambda_1}+\frac{2}{\mu})]e^{-\gamma T} \int_{T-\tau}^{T} e^{\gamma\xi} \max_{\theta\in[-h,0]} \|w(\xi+\theta)\|^2 d\xi\\
&\quad + \frac{\gamma}{\tau}e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}(f(u_2(\xi)) - f(u_1(\xi)), w(\xi)) d\xi ds\\
&\quad + \frac{1}{\tau}e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}(f(u_2(\xi)) - f(u_1(\xi)), \partial_t w(\xi)) d\xi ds\\
&\quad + \frac{1}{\tau}(kl\gamma + \frac{k^2l^2}{\mu})e^{-\gamma T}\int_{T-\tau}^{T}\int_{s}^{T} e^{\gamma\xi}\|w(\xi)\|^2 d\xi ds\\
&\quad + \frac{6}{\tau\mu}e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma\xi}(f(u_2) - f(u_1), \partial_t w(\xi)) d\xi\\
&\quad + \frac{2}{\tau}e^{-\gamma T}\int_{T-\tau}^{T} e^{\gamma\xi}(f(u_2) - f(u_1), w(\xi)) d\xi.
\end{aligned} \tag{4.22}
$$

Thus,

$$
E_w(T) \leq \frac{6}{\tau\mu}e^{-\gamma\tau}E_w(T-\tau) + \Phi_{T,\tau}((\varphi_1, \partial_t\varphi_1), ((\varphi_2, \partial_t\varphi_2)). \tag{4.23}
$$

Choosing $\tau_0 = \tau_0(t, \mathcal{D}_1, \varepsilon) < T$, such that for any $(\phi_i, \partial_t\phi_i) \in D_1 (T-\tau_0) \times D_1(T-\tau_0)(i = 1, 2)$, we have

$$
E_w(T) \leq \varepsilon + \Phi_{T,\tau}((\varphi_1, \partial_t\varphi_1), ((\varphi_2, \partial_t\varphi_2)).
$$

Subsequently, we will verify (4.22) is a contractive function.

Let $(u_n(t), \partial_t u_n(t))$ be the solution corresponding to initial data $(\phi_n, \partial_t\phi_n) \in D_1 (T-\tau_0) \times D_1(T-\tau_0)(n = 1, 2)$, $\forall T \in \mathbb{R}$, $D_1(T-\tau_0) \times D_1(T-\tau_0)$ is bounded in $V_2 \times V_0$. Then for any $s \in [T-\tau_0, T]$, $n \in \mathbb{N}$, we conclude that

$$
\|(u_n(s), \partial_t u_n(s))\|_{V_2\times V_0} \leq C_{t,\tau} < +\infty,
$$

here $C_{t,\tau}$ depends on $t$, $\tau$.

According to Alaoglu Theorem, without loss of generality, we can assume that

$$
u_n \to u \text{ weakly star in } L^\infty(T-\tau_0, T; V_2); \tag{4.24}
$$

$$
\partial_t u_n \to \partial_t u \text{ weakly star in } L^\infty(T-\tau_0, T; V_0). \tag{4.25}
$$

Hence

$$
u_n \to u \text{ in } L^2(T-\tau_0, T; V_0); \tag{4.26}
$$

$$
u_n(T-\tau_0) \to u(T-\tau_0),\ u_n(T) \to u(T) \text{ in } V_0. \tag{4.27}
$$

Now, we will deal with each term in (4.22) one by one. Firstly, from (4.25)-(4.27), we get

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T-\tau_0}^{T}\|u_n(r)-u_m(r)\|^2dr=0, \tag{4.28}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{\Omega}(\partial_t u_n(T-\tau)-\partial_t u_m(T-\tau))(u_n(\tau)-u_m(\tau))dx=0, \tag{4.29}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\|u_n(T)-u_m(T)\|^2=0. \tag{4.30}$$

Secondly, $f(u_m)\to f(u)$ weakly star in $L_2(T-\tau_0,T;V_0)$ and together with (4.25), (4.26),

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T-\tau_0}^{T}\int_{\Omega}(u_n(\xi)-u_m(\xi))(f(u_n(\xi))-f(u_m(\xi)))dxd\xi=0, \tag{4.31}$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T-\tau_0}^{T}\int_{\Omega}(\partial_t u_n(\xi)-\partial_t u_m(\xi))(f(u_n(\xi))-f(u_m(\xi)))dxd\xi=0. \tag{4.32}$$

At the same time, since $|\int_s^T\int_\Omega(\partial_t u_n(\xi)-\partial_t u_m(\xi))(f(u_n(\xi))-f(u_m(\xi)))dxd\xi|$ is bounded, for each $s\in[T-\tau_0,T]$, then by (4.32) and the Lebesgue dominated convergence Theorem, we have

$$\begin{aligned}&\lim_{n\to\infty}\lim_{m\to\infty}\int_{T-\tau_0}^{T}\int_s^T\int_{\Omega}(\partial_t u_n(\xi)-\partial_t u_m(\xi))(f(u_n(\xi))-f(u_m(\xi)))dxd\xi ds\\&=\int_{T-\tau_0}^{T}(\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_{\Omega}(\partial_t u_n(\xi)-\partial_t u_m(\xi))(f(u_n(\xi))-f(u_m(\xi)))dxd\xi)ds\\&=\int_{T-\tau_0}^{T}0ds=0.\end{aligned} \tag{4.33}$$

Similarly, we can obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T-\tau_0}^{T}\int_s^T\int_{\Omega}(u_n(\xi)-u_m(\xi))(f(u_n(\xi))-f(u_m(\xi)))dxd\xi ds=0. \tag{4.34}$$

Finally, since $\{u_n^\xi\}$ is bounded in $W$, it yields

$$\max_{\theta\in[-h,0]}\|A^{\frac{1}{2}}u_n^\xi(\theta)\|\le C_{T,\tau_0}<\infty,\ \xi\in[T-\tau_0,T].$$

Then by Lemma 2.9, sequence $\{A^{\frac{1}{2}-\delta}u_n^\xi(\theta)\}$ has convergent subsequence, hence

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T-\tau_0}^{T}e^{\gamma\xi}\max_{\theta\in[-h,0]}\|A^{\frac{1}{2}-\delta}(u_n(\xi+\theta)-u_m(\xi+\theta)\|^2d\xi=0, \tag{4.35}$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_{T-\tau_0}^{T}e^{\gamma\xi}\max_{\theta\in[-h,0]}\|(u_n(\xi+\theta)-u_m(\xi+\theta)\|^2d\xi=0. \tag{4.36}$$

Combining with (4.28)-(4.36), $\Phi_{T,\tau_0}(\cdot,\cdot)$ is a contractive function on $D_1(T-\tau_0)\times D_1(T-\tau_0)$. In view of Theorem 2.6 we conclude that the process $\{S(t,\tau)\}_{t\geq\tau}$ is pullback $\mathcal{D}-$asymptotically compact.

By Theorem 4.3 and Lemma 4.5, we know that the conditions of Theorem 2.7 are all satisfied. So we immediately obtain the following conclusion. □

**Theorem 4.6** *(Pullback $\mathcal{D}_\gamma-$attractors) Suppose that (2.1)-(2.6) and (4.5) hold true. Then the process $\{S(t,\tau)\}_{t\geq\tau}$ generated by problem (1.1) has a pullback $\mathcal{D}_\gamma-$attractor.*

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## Declarations

**Conflict of interest** The authors declare no conflict of interest.

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