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Admissibility via Evolution Semigroups

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Abstract
We establish the equivalence of various admissibility properties for an evolution family and its associated evolution semigroups on several Banach spaces. As an application, we describe how these results can give further relations between hyperbolicity for evolution families and evolution semigroups.

Keywords Admissibility · Evolution semigroups

Mathematics Subject Classification Primary: 37D99

1 Introduction

1.1 Admissibility Properties

Our main aim is to discuss the correspondence between the notions of admissibility for evolution families and their evolution semigroups. It turns out that using appropriate spaces for each of them these notions become equivalent. We consider spaces of bounded continuous functions, continuous functions with bounded exponential growth, and integrable functions. In particular, the equivalence between the notions of admissibility for evolution families and evolution semigroups leads to several new criteria of hyperbolicity for the original evolution family as a perturbation of an *autonomous* linear dynamics even if the original dynamics is nonautonomous.

We start by recalling the concept of admissibility, which essentially goes back to Perron in [17]. More precisely, he showed that if the equation

$$x' = A(t)x + f(t),$$

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with $A(t)$ varying continuously with $t \geq 0$, has a bounded solution on $\mathbb{R}^n$ for any bounded continuous function $f: \mathbb{R}_0^+ \to \mathbb{R}^n$, then any bounded solution of the linear equation $x' = A(t)x$ tends to zero when time approaches $+\infty$. Related results for discrete time were first obtained by Li in [12]. For some early contributions we refer to the books by Massera and Schäffer [15] (see also [14]) and by Dalec′kiĭ and Kreĭn [8]. Related results for discrete time were obtained by Coffman and Schäffer in [7]. See also [11] for some early results in infinite-dimension. For detailed lists of references, we refer to [1,6].

More generally, we can consider different spaces in which we look for the perturbations and for the solutions. Consider an evolution family $U(t, s)$, for $t \geq s$, composed of linear maps on a Banach space X. These can be obtained from a linear equation $x' = A(t)x$ for some linear operators $A(t)$ varying continuously with t and possibly unbounded. We say that a pair of Banach spaces (C, D) is *admissible* if for each $f \in C$ there exists a unique $x \in D$ satisfying

$$x(t) = U(t, s)x(s) + \int_s^t U(t, \tau) f(\tau)\, d\tau \quad \text{for } t \geq s. \tag{1}$$

It is this notion of admissibly that we consider in various situations, and in particular for the pairs of spaces

$$(C_0(X), C(X)), \quad (C_0(X), C_0(X)) \quad \text{and} \quad (L^p(X), L^p(X)), \tag{2}$$

where $C(X)$, $C_0(X)$ and $L^p(X)$ are sets of functions $u: \mathbb{R} \to X$: the set of bounded continuous functions, the set of continuous functions vanishing at infinity, and the set of measurable functions whose pth power is integrable (identified if they are equal almost everywhere).

We continue by recalling the notion of evolution semigroup. Again, let $U(t, s)$ be an evolution family on a Banach space X. We define an evolution semigroup S_t on each appropriate space of functions $u: \mathbb{R} \to X$ by

$$(S_t u)(s) = U(s, s - t)u(s - t)$$

for all $t \geq 0$ and $s \in \mathbb{R}$. We emphasize that the evolution semigroup gives rise to an autonomous dynamics even if the original dynamics is nonautonomous, although at the expense of considering an infinite-dimensional space even when X is finite-dimensional.

1.2 Motivation and Advantages of our Work

Sometimes, evolution semigroups allow giving simpler proofs of known results, by first passing some problem at the level of the evolution family $U(t, s)$ and its perturbations to the level of the evolution semigroup. Another less immediate application is that passing to the evolution semigroup we transform not only the original dynamics but also its properties into corresponding ones at the semigroup level. Sometimes this is quite helpful in finding right nonautonomous notions, in some appropriate sense. A major example of such a correspondence is the study of hyperbolicity and its various variations that goes back to Mather in [16]. Today the theory of semigroups is an important tool in the theory of differential equations (see for example [9,19]).

A main motivation for our work is that it is in general simpler to verify an admissibility property for an autonomous dynamics, sometimes even on an infinite-dimensional space. This causes that it is convenient to have characterizations of admissibility in terms of an autonomous dynamics. There is however another motivation that is equally important. In order to explain it, we recall that often admissibility and hyperbolicity are well related (see

[1,6] and the references therein). There are however quite general types of hyperbolicity that are much harder to relate to admissibility properties, such as tempered hyperbolicity for a cocycle with nonzero Lyapunov exponents that occurs naturally in the context of smooth ergodic theory (see [2,5]).

On the other hand, one should expect such admissibility properties to play an important role in discussing for example shadowing or Ulam stability properties in the context of smooth ergodic theory, particularly its nonautonomous version. Part of the problem seems to be what are the right or at least reasonable nonautonomous notions that should correspond to the usual autonomous properties. Thus, one can expect that the autonomous admissibility notions considered in our paper and some appropriate modifications play a relevant role in discussing nonautonomous versions of some results of smooth ergodic theory, with the advantage that some important parts of the theory have been extended successfully to infinite-dimensional spaces such as Oseledets' multiplicative ergodic theorem and certain applications to invariant manifold theory (see [4,13]).

Coming back to the motivation for our work, it follows from the former discussion that it is of interest to obtain autonomous characterizations of admissibility properties without involving hyperbolicity and indeed the main advantage of our work is that we give direct streamlined proofs of the equivalence of the notions of admissibility for evolution families and their evolution semigroups. It turns out that this requires some new nontrivial elements, of which the most delicate is the construction of appropriate functions at the level of evolution semigroups that somehow correspond to given functions at the level of evolutions families.

1.3 Brief Description of our Results

As already noted above, the main aim of our paper is to transfer in a faithful manner the admissibility properties mentioned above such as for the pairs in (2) to *equivalent* admissibility properties at the level of evolution semigroups. There are two main reasons why this is nontrivial:

1. One cannot know a priori what are the right Banach spaces on which one should consider the evolution semigroup. Indeed, if we consider for example a larger space for the perturbations, then we may need to reduce the space on which we look for the solutions so that these are unique and so that the admissibility property holds.
2. It is in general difficult to deduce equivalent properties, since while at the level of evolution semigroups there is plenty information, although often perhaps too much that somehow needs to be localized, at the level of the evolution family it needs instead to be globalized, often with a quite different approach as is the case in our work.

As an illustration of our results, we formulate a particular case of Theorem 4 for continuous functions. For simplicity we write $D_0(X) = C_0(C_0(X))$.

Theorem 1 *Let $U(t,s)$ be an evolution family composed of linear maps on X such that $\|U(t,s)\| \leq \kappa e^{\alpha(t-s)}$ for all $t \geq s$ and some $\kappa, \alpha > 0$. Then the following properties are equivalent:*

1. *For each $f \in C_0(X)$ there exists a unique $x \in C_0(X)$ satisfying (1);*
2. *For each $F \in D_0(X)$ there exists a unique $u \in D_0(X)$ such that*

$$u(t) = S_{t-s}u(s) + \int_s^t S_{t-\tau} F(\tau)\, d\tau \quad \text{for } t \geq s.$$

We emphasize that a major problem is to identify the space $D_0(X)$ in the second property that ensures the equivalence of the two properties. More generally, we shall consider families of norms $\|\cdot\|_t$, for $t \in \mathbb{R}$, and we shall consider spaces $C(X)$, $C_0(X)$ and $L^p(X)$ defined in terms of these norms. This essentially corresponds to consider the nonuniform exponential behavior that is ubiquitous in smooth ergodic theory.

It turns out that these admissibility properties are also related to the hyperbolicity of the original evolution family as well as of the associated evolutions semigroups. As an application, we shall also explore this relation, by giving further relations between hyperbolicity for evolution families and evolution semigroups (see Sect. 7).

We note that some results could be obtained using hyperbolicity. Namely, Theorem 4 is an immediate consequence of Theorem 7 in [3] together with the equivalence of the corresponding admissibility and hyperbolicity properties. On the other hand, for a family of norms $\|\cdot\|_t = \|\cdot\|$ independent of t our characterizations of admissibility can be obtained as consequences of former results in the area (see [6] for details building on work in [10]). Nevertheless, our proofs are new. Our main interest is precisely to characterize admissibility without using hyperbolicity. Incidentally, a family of norms $\|\cdot\|_t = \|\cdot\|$ essentially amounts to consider uniform hyperbolicity, while for nonuniform hyperbolicity it is convenient to consider other norms (see [2] for a detailed description building on the multiplicative ergodic theorem).

2 Basic Notions

Let $\mathcal{F}(X)$ be the set of all continuous maps $T\colon X \to X$ on a Banach space $X = (X, \|\cdot\|)$ and write

$$\Pi = \big\{(t, s) \in \mathbb{R}^2 : t \geq s\big\}.$$

An *evolution family* on X is a family $\mathcal{V} = (V(t, s))_{(t,s)\in\Pi}$ of maps in $\mathcal{F}(X)$ such that:

1. $V(t, t) = \mathrm{Id}$ and $V(t, s)V(s, r) = V(t, r)$ for $t, s, r \in \mathbb{R}$ with $t \geq s \geq r$;
2. The map $(t, s) \mapsto V(t, s)(x)$ is continuous on Π for each $x \in X$.

For simplicity of the notation, we shall write $V(t, s)(x) = V(t, s)x$. When all maps $V(t, s)$ are linear, we shall also say that $\mathcal{V}$ is a *linear evolution family*.

A *semigroup* on a Banach space Y is a family $\mathcal{T} = (T_t)_{t\geq 0}$ of maps in $\mathcal{F}(Y)$ (the set of all continuous maps $T\colon Y \to Y$) such that

$$T_0 = \mathrm{Id} \quad \text{and} \quad T_t \circ T_s = T_{t+s} \quad \text{for } t, s \geq 0. \tag{3}$$

A semigroup $\mathcal{T}$ on Y is called a C_0 *semigroup* or a *strongly continuous semigroup* on Y if

$$\lim_{t\searrow 0} T_t u = u \quad \text{for } u \in Y.$$

To each evolution family $\mathcal{V}$ on X we associate semigroups $\mathcal{T} = \mathcal{T}|_{Y_X}$ on certain Banach spaces Y_X composed of functions $u\colon \mathbb{R} \to X$. Namely, we define

$$(T_t u)(s) = V(s, s-t)u(s-t) \quad \text{for } t \geq 0,\ s \in \mathbb{R},\ u \in Y_X,$$

provided that

$$T_t(Y_X) \subset Y_X \quad \text{for all } t \geq 0.$$

One can easily verify that property (3) holds. The semigroup $\mathcal{T}$ is called the *evolution semigroup* of $\mathcal{V}$ on Y_X.

3 Evolution Families

In this section we introduce a family of evolution families that later on are used to study an admissibility property.

Let $\mathcal{U} = (U(t,s))_{(t,s)\in\Pi}$ be a linear evolution family on a Banach space X. Given a locally integrable function $f\colon \mathbb{R} \to X$, for each pair $(s, x_s) \in \mathbb{R} \times X$ we define a function $x\colon [s, +\infty) \to X$ by

$$x(t) = U(t,s)x_s + \int_s^t U(t,\tau)f(\tau)\,d\tau \tag{4}$$

for $t \geq s$ (note that x is well defined because the integrand is continuous). We shall write it in the form $x(t) = V(t,s)x_s$, for $t \geq s$, and we consider the family of maps $\mathcal{V} = (V(t,s))_{(t,s)\in\Pi}$.

Theorem 2 *$\mathcal{V}$ is an evolution family on X.*

Proof For $t = s$ we have $x(t) = x_t$ and so $V(t,t) = \mathrm{Id}$. Moreover, for any $t, s, r \in \mathbb{R}$ with $t \geq s \geq r$ we have

$$\begin{aligned}
V(t,s)V(s,r)x_r &= U(t,s)V(s,r)x_r + \int_s^t U(t,\tau)f(\tau)\,d\tau \\
&= U(t,s)\left(U(s,r)x_r + \int_r^s U(s,\tau)f(\tau)\,d\tau\right) \\
&\quad + \int_s^t U(t,\tau)f(\tau)\,d\tau \\
&= U(t,r)x_r + \int_r^s U(t,\tau)f(\tau)\,d\tau + \int_s^t U(t,\tau)f(\tau)\,d\tau \\
&= U(t,r)x_r + \int_r^t U(t,\tau)f(\tau)\,d\tau = V(t,r)x_r,
\end{aligned}$$

which establishes the first property in the notion of an evolution family. For the second property, we note that

$$\begin{aligned}
V(t,s)x - V(\bar{t},\bar{s})x &= U(t,s)x - U(\bar{t},\bar{s})x \\
&\quad + \int_s^t U(t,\tau)f(\tau)\,d\tau - \int_{\bar{s}}^{\bar{t}} U(\bar{t},\tau)f(\tau)\,d\tau
\end{aligned} \tag{5}$$

for all $(t,s), (\bar{t},\bar{s}) \in \Pi$ and $x \in X$. We must show that

$$V(t,s)x \to V(\bar{t},\bar{s})x \quad \text{when } (t,s) \to (\bar{t},\bar{s}).$$

Since $\mathcal{U}$ is an evolution family, we already know that

$$U(t,s)x \to U(\bar{t},\bar{s})x \quad \text{when } (t,s) \to (\bar{t},\bar{s}).$$

So it suffices to show that the difference of integrals in (5) converges to zero when $(t,s) \to (\bar{t},\bar{s})$. Assume that

$$\bar{s} - \delta \leq s \leq t \leq \bar{t} + \delta \quad \text{for some } \delta > 0.$$

In view of the second property in the notion of an evolution family and the uniform boundedness principle, we have

$$c := \sup\left\{\|U(t,\tau)\| : \bar{s} - \delta \leq \tau \leq t \leq \bar{t} + \delta\right\} < \infty.$$

Now we consider the functions

$$g_{s,t}(\tau) = \chi_{[s,t]}(\tau)U(t,\tau)f(\tau),$$

which satisfy

$$\|g_{s,t}(\tau)\| \le c\chi_{[s,t]}(\tau)\|f(\tau)\|.$$

Since f is locally integrable, it follows from Lebesgue's dominated convergence theorem that

$$\int_s^t U(t,\tau)f(\tau)\,d\tau = \int_{\mathbb{R}} g_{s,t}(\tau)\,d\tau \to \int_{\mathbb{R}} g_{\bar{s},\bar{t}}(\tau)\,d\tau = \int_{\bar{s}}^{\bar{t}} U(\bar{t},\tau)f(\tau)\,d\tau$$

when $(t,s) \to (\bar{t},\bar{s})$. Therefore, the map $(t,s) \mapsto V(t,s)x$ is continuous on Π for each $x \in X$ and so $\mathcal{V}$ is an evolution family on X. □

We say that Eq. (4) *generates* the evolution family $\mathcal{V}$.

Now let $\|\cdot\|_t$, for $t \in \mathbb{R}$, be a family of norms on the Banach space X with the map $t \mapsto \|x\|_t$ continuous for each $x \in X$ such that

$$\|x\| \le \|x\|_t \le R(t)\|x\| \quad \text{for } t \in \mathbb{R},\ x \in X \tag{6}$$

for some continuous function $R\colon \mathbb{R} \to \mathbb{R}^+$. We say that the linear evolution family $\mathcal{U}$ is *exponentially bounded with respect to the norms* $\|\cdot\|_t$ if there exist $\alpha, \kappa > 0$ such that

$$\|U(t,s)x\|_t \le \kappa e^{\alpha(t-s)}\|x\|_s \quad \text{for } t \ge s,\ x \in X.$$

The following result is a straightforward consequence of identity (4).

Proposition 1 *If the evolution family $\mathcal{U}$ is exponentially bounded with respect to the norms $\|\cdot\|_t$, then the evolution family $\mathcal{V}$ satisfies*

$$\|V(t,s)x - V(t,s)y\|_t \le \kappa e^{\alpha(t-s)}\|x-y\|_s \tag{7}$$

for all $t \ge s$ and $x, y \in X$.

4 Admissibility I: Continuous Functions

In this section and in the following two we show how to transfer a certain admissibility property from evolution families to evolution semigroups and vice-versa. Here we consider evolution semigroups defined on a space of continuous functions.

4.1 Evolution Semigroups

We consider evolution semigroups on some appropriate Banach spaces. Let $C(X)$ be the set of all continuous functions $x\colon \mathbb{R} \to X$ such that

$$\|x\|_C := \sup_{t\in\mathbb{R}}\|x(t)\|_t < \infty.$$

We note that $C(X)$ is a Banach space when endowed with the norm $\|\cdot\|_C$. Let $C_0(X) \subset C(X)$ be the closed subspace of all functions $x \in C(X)$ with

$$\lim_{|t|\to\infty}\|x(t)\|_t = 0.$$

Using the evolution family $\mathcal{V}$ generated by Eq. (4) for a given function $f \in C(X)$, we define an operator $T_t : C(X) \to X^{\mathbb{R}}$ for each $t \geq 0$ by

$$(T_t u)(s) = V(s, s-t)u(s-t) \quad \text{for } s \in \mathbb{R},\ u \in C(X).$$

Note that the maps T_t need not be linear.

Proposition 2 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then the following properties hold:*

1. *For each $f \in C(X)$ we have*
$$T_t(C(X)) \subset C(X) \quad \textit{for each } t \geq 0;$$
2. *For each $f \in C_0(X)$ we have*
$$T_t(C_0(X)) \subset C_0(X) \quad \textit{for each } t \geq 0.$$

Proof Take $f, u \in C(X)$. Given $t \geq 0$ and $s, \sigma \in \mathbb{R}$, we have

$$\begin{aligned} &\|V(s, s-t)u(s-t) - V(\sigma, \sigma-t)u(\sigma-t)\| \\ &\quad \leq \|V(s, s-t)u(s-t) - V(s, s-t)u(\sigma-t)\| \\ &\qquad + \|V(s, s-t)u(\sigma-t) - V(\sigma, \sigma-t)u(\sigma-t)\|. \end{aligned} \tag{8}$$

In view of (6) and (7) we obtain

$$\begin{aligned} &\|V(s, s-t)u(s-t) - V(s, s-t)u(\sigma-t)\| \\ &\quad \leq \kappa e^{\alpha t}\|u(s-t) - u(\sigma-t)\|_{s-t} \\ &\quad \leq \kappa e^{\alpha t} R(s-t)\|u(s-t) - u(\sigma-t)\| \to 0 \end{aligned} \tag{9}$$

when $s \to \sigma$ (recall that the functions R and u are continuous). Finally, using the second property in the notion of an evolution family, it follows from (8) and (9) that

$$\|V(s, s-t)u(s-t) - V(\sigma, \sigma-t)u(\sigma-t)\| \to 0$$

when $s \to \sigma$. Moreover, we have

$$\begin{aligned} \|T_t u\|_C &= \sup_{s \in \mathbb{R}} \|V(s, s-t)u(s-t)\|_s \\ &\leq \sup_{s \in \mathbb{R}} \left\| U(s, s-t)u(s-t) + \int_{s-t}^{s} U(s, \tau) f(\tau)\, d\tau \right\|_s \\ &\leq \kappa e^{\alpha t} \sup_{s \in \mathbb{R}} \|u(s-t)\|_{s-t} + \sup_{s \in \mathbb{R}} \int_{s-t}^{s} \kappa e^{\alpha(s-\tau)} \|f(\tau)\|_\tau\, d\tau \\ &\leq \kappa e^{\alpha t}\|u\|_C + \frac{\kappa}{\alpha}(e^{\alpha t} - 1)\|f\|_C < \infty \end{aligned}$$

and so $T_t u \in C(X)$.

Now take $f, u \in C_0(X)$. By property 1, we already know that $T_t u \in C(X)$. Moreover,

$$\begin{aligned} \|(T_t u)(s)\|_s &= \left\| U(s, s-t)u(s-t) + \int_{s-t}^{s} U(s, \tau) f(\tau)\, d\tau \right\|_s \\ &\leq \kappa e^{\alpha t}\|u(s-t)\|_{s-t} + \int_{s-t}^{s} \kappa e^{\alpha(s-\tau)} \|f(\tau)\|_\tau\, d\tau \\ &\leq \kappa e^{\alpha t}\|u(s-t)\|_{s-t} + \frac{\kappa}{\alpha}(e^{\alpha t} - 1) \sup_{\tau \in [s-t, s]} \|f(\tau)\|_\tau \to 0 \end{aligned}$$

when $|s| \to \infty$ and so $T_t u \in C_0(X)$. This completes the proof of the proposition. □

The semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ is thus the evolution semigroup of $\mathcal{V}$ on $C(X)$ and also on the closed subspace $C_0(X)$.

4.2 Admissibility

Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then $\mathcal{U}$ generates an evolution semigroup $\mathcal{S} = (S_t)_{t\geq 0}$ on $C(X)$ defined by

$$(S_t u)(s) = U(s, s-t)u(s-t) \quad \text{for } t \geq 0,\ s \in \mathbb{R},\ u \in C(X).$$

By Proposition 2 with $f = 0$ in (4), indeed $S_t(C(X)) \subset C(X)$ for all $t \geq 0$.

Moreover, let $D(X)$ be the set of all continuous functions $v\colon \mathbb{R} \to C(X)$ such that

$$\|v\|_D := \sup_{t\in\mathbb{R}} \|v(t)\|_C < \infty.$$

We note that $D(X)$ is a Banach space when endowed with the norm $\|\cdot\|_D$. We also consider the closed subspace $D_0(X) \subset D(X)$ of all continuous functions $v\colon \mathbb{R} \to C_0(X)$ such that

$$\lim_{|t|\to\infty} \|v(t)\|_C = 0.$$

Note that

$$D(X) = C(C(X)) \quad \text{and} \quad D_0(X) = C_0(C_0(X)).$$

The following theorem is our main result.

Theorem 3 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then the following properties are equivalent:*

1. *For each $f \in C_0(X)$ there exists a unique $x \in C(X)$ such that*

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau \quad \text{for all } (t,s) \in \Pi; \tag{10}$$

2. *For each $F \in D_0(X)$ there exists a unique $u \in D(X)$ such that*

$$u(t) = S_{t-s}u(s) + \int_s^t S_{t-\tau}F(\tau)\,d\tau \quad \text{for all } (t,s) \in \Pi. \tag{11}$$

Proof We first prove an auxiliary result.

Lemma 1 *Given $F, u \in D(X)$, Eq. (11) holds if and only if*

$$u_{t-k}(t) = U(t,s)u_{s-k}(s) + \int_s^t U(t,w)f_k(w)\,dw \quad \text{for all } t,s,k \in \mathbb{R} \tag{12}$$

with $t \geq s$, where $f_k(t) = F(t-k)(t)$.

Proof of the lemma For simplicity of the notation we shall write $u(t) = u_t$. First assume that Eq. (11) holds, that is,

$$u_\tau = S_{\tau-\sigma}u_\sigma + \int_\sigma^\tau S_{\tau-w}F(w)\,dw \quad \text{for all } (\tau,\sigma) \in \Pi.$$

By the definition of S_t we have

$$\begin{aligned}u_\tau(t) &= (S_{\tau-\sigma}u_\sigma)(t) + \int_\sigma^\tau (S_{\tau-w}F(w))(t)\,dw\\ &= U(t, t-\tau+\sigma)u_\sigma(t-\tau+\sigma)\\ &\quad + \int_\sigma^\tau U(t, t-\tau+w)F(w)(t-\tau+w)\,dw.\end{aligned}$$

Taking $\tau = t-k$ and $\sigma = s-k$, we obtain

$$\begin{aligned}u_{t-k}(t) &= U(t,s)u_{s-k}(s)\\ &\quad + \int_{s-k}^{t-k} U(t, k+w)F(w)(k+w)\,dw\\ &= U(t,s)u_{s-k}(s) + \int_s^t U(t,w)F(w-k)(w)\,dw\\ &= U(t,s)u_{s-k}(s) + \int_s^t U(t,w)f_k(w)\,dw.\end{aligned} \tag{13}$$

Now assume that property (12) holds. Proceeding as in (13) and again by the definition of S_t, we have

$$\begin{aligned}u_{t-k}(t) &= U(t,s)u_{s-k}(s) + \int_s^t U(t,w)f_k(w)\,dw\\ &= U(t,s)u_{s-k}(s) + \int_{s-k}^{t-k} U(t,k+w)F(w)(k+w)\,dw\\ &= (S_{t-s}u_{s-k})(t) + \int_{s-k}^{t-k} (S_{t-k-w}F(w))(t)\,dw.\end{aligned}$$

Taking $k = t-\tau$, this equality can be rewritten as

$$u_\tau(t) = (S_{t-s}u_{s-t+\tau})(t) + \int_{s-t+\tau}^{\tau} (S_{\tau-w}F(w))(t)\,dw.$$

Finally, taking $s = \sigma + t - \tau$, we conclude that

$$u_\tau(t) = S_{\tau-\sigma}u_\sigma(t) + \int_\sigma^\tau (S_{\tau-w}F(w))(t)\,dw,$$

that is,

$$u_\tau = S_{\tau-\sigma}u_\sigma + \int_\sigma^\tau S_{\tau-w}F(w)\,dw,$$

as we wanted to show. □

We proceed with the proof of the theorem. We prove both implications separately.
($1 \Rightarrow 2$). Take $F \in D_0(X)$. For each $k \in \mathbb{R}$ we define a map $f_k\colon \mathbb{R} \to X$ by

$$f_k(t) = F(t-k)(t) \quad \text{for } t \in \mathbb{R}. \tag{14}$$

We show that $f_k \in C_0(X)$. Note that

$$\begin{aligned} \|f_k(t) - f_k(s)\| &= \|F(t-k)(t) - F(s-k)(s)\| \\ &\leq \|F(t-k)(t) - F(s-k)(t)\| \\ &\quad + \|F(s-k)(t) - F(s-k)(s)\| \\ &\leq \|F(t-k) - F(s-k)\|_C \\ &\quad + \|F(s-k)(t) - F(s-k)(s)\|. \end{aligned} \tag{15}$$

Since the maps $t \mapsto F(t-k)$ and $F(s-k)$ are continuous, letting $t \to s$ in (15), we find that $f_k(t) \to f_k(s)$ and so f_k is continuous. Moreover,

$$\lim_{|t|\to\infty} \|f_k(t)\|_t \leq \lim_{|t|\to\infty} \|F(t-k)\|_C = 0$$

since $F \in D_0(X)$, and so $f_k \in C_0(X)$. By property 1, there exists a unique solution $x_k \in C(X)$ of Eq. (10) with $f = f_k$, for each $k \in \mathbb{R}$. We shall use these functions to construct a solution of Eq. (11).

We continue to write $u(t) = u_t$ for each function $u \colon \mathbb{R} \to C(X)$. By Lemma 1, if $u \in D(X)$ is a solution of Eq. (11), then $t \mapsto u_{t-k}(t)$ is a solution of Eq. (10) with $f = f_k$, for each $k \in \mathbb{R}$. So, necessarily, $u_{t-k}(t) = x_k(t)$ for all $t, k \in \mathbb{R}$, that is, $u_t(s) = x_{s-t}(s)$ for all $t, s \in \mathbb{R}$. In particular, this implies that (11) has at most one solution $u \in D(X)$. This leads us to define

$$u_t(s) = x_{s-t}(s) \quad \text{for } t, s, \in \mathbb{R}. \tag{16}$$

We show below that $u \in D(X)$. Then it follows from Lemma 1 that u is a solution of Eq. (11) and as noted above it is automatically unique.

Now let R be the linear operator defined by $Rx = f$ on the domain composed of the functions $x \in C(X)$ for which there exists $f \in C_0(X)$ satisfying (10). We show that R is a well-defined closed operator. To show that R is well-defined, take $g \in C_0(X)$ such that

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)g(\tau)\,d\tau$$

for all $(t,s) \in \Pi$. Then

$$\frac{1}{t-s} \int_s^t U(t,\tau)(f(\tau) - g(\tau))\,d\tau = 0$$

for all $(t,s) \in \Pi$ with $t \neq s$. Since $f, g \in C_0(X)$, one can show that the function

$$\tau \mapsto U(t,\tau)(f(\tau) - g(\tau))$$

is continuous. For completeness we give the argument. Take $\bar{\tau} \in \mathbb{R}$ and write $h = f - g$. We have

$$\begin{aligned} \|U(t,\tau)h(\tau) - U(t,\bar{\tau})h(\bar{\tau})\| &\leq \|U(t,\tau)h(\tau) - U(t,\tau)h(\bar{\tau})\| \\ &\quad + \|U(t,\tau)h(\bar{\tau}) - U(t,\bar{\tau})h(\bar{\tau})\| \end{aligned}$$

whenever $(t,\tau), (t,\bar{\tau}) \in \Pi$. Now assume that $\tau, \bar{\tau} \in [s-\delta, t]$ for some $\delta > 0$. In view of the second property in the notion of an evolution family and the uniform boundedness principle, we have

$$c := \sup\big\{\|U(t,\tau)\| : \tau \in [s-\delta, t]\big\} < \infty.$$

Therefore,

$$\|U(t,\tau)h(\tau) - U(t,\bar{\tau})h(\bar{\tau})\| \le c\|h(\tau) - h(\bar{\tau})\| + \|[U(t,\tau) - U(t,\bar{\tau})]h(\bar{\tau})\|$$

and so, letting $\tau \to \bar{\tau}$ we conclude that the map $(t,\tau) \mapsto U(t,\tau)h(\tau)$ is continuous. Hence, letting $s \nearrow t$, we find that $f(t) = g(t)$ for all $t \in \mathbb{R}$. This shows that the operator R is well defined.

To show that R is closed, let $(x^\ell)_{\ell\in\mathbb{N}}$ be a sequence in the domain of R converging to $x \in C(X)$ such that $f^\ell = Rx^\ell$ converges to $f \in C_0(X)$. Then

$$\begin{aligned} x(t) - U(t,s)x(s) &= \lim_{\ell\to\infty}\big(x^\ell(t) - U(t,s)x^\ell(s)\big) \\ &= \lim_{\ell\to\infty}\int_s^t U(t,\tau)f^\ell(\tau)\,d\tau \end{aligned} \tag{17}$$

for all $(t,s) \in \Pi$. We have

$$\left\|\int_s^t U(t,\tau)f^\ell(\tau)\,d\tau - \int_s^t U(t,\tau)f(\tau)\,d\tau\right\| \le d\|f^\ell - f\|_C,$$

where

$$d := \sup\big\{\|U(t,\tau) : \tau \in [s,t]\big\} < \infty \tag{18}$$

in view of the uniform boundedness principle. Thus, letting $\ell \to \infty$ in (17) we find that

$$x(t) - U(t,s)x(s) = \int_s^t U(t,\tau)f(\tau)\,d\tau$$

for all $(t,s) \in \Pi$. This shows that $Rx = f$ and so x is in the domain of R. Hence, the operator R is closed. By the closed graph theorem, R is bounded. Moreover, by property 1 the operator R is onto and invertible. It follows from the open mapping theorem that it has a bounded inverse.

Now we show that $u \in D(X)$. First we prove that $u_t \in C(X)$ for each $t \in \mathbb{R}$. Note that

$$\begin{aligned} \|u_t(s) - u_t(\bar{s})\| &= \|x_{s-t}(s) - x_{\bar{s}-t}(\bar{s})\| \\ &\le \|x_{s-t}(s) - x_{\bar{s}-t}(s)\| + \|x_{\bar{s}-t}(s) - x_{\bar{s}-t}(\bar{s})\| \\ &\le \|x_{s-t} - x_{\bar{s}-t}\|_C + \|x_{\bar{s}-t}(s) - x_{\bar{s}-t}(\bar{s})\|. \end{aligned} \tag{19}$$

Letting $s \to \bar{s}$, the second term on the right-hand side tends to zero. For the first term we note that

$$\begin{aligned} \|x_{s-t} - x_{\bar{s}-t}\|_C &= \|R^{-1}(f_{s-t} - f_{\bar{s}-t})\|_C \\ &\le \|R^{-1}\| \cdot \|f_{s-t} - f_{\bar{s}-t}\|_C. \end{aligned} \tag{20}$$

We have

$$f_{s-t}(\tau) - f_{\bar{s}-t}(\tau) = F(\tau - s + t)(\tau) - F(\tau - \bar{s} - t)(\tau)$$

and thus,

$$\|f_{s-t} - f_{\bar{s}-t}\|_C \le \sup_{\tau\in\mathbb{R}}\|F(\tau - s + t) - F(\tau - \bar{s} - t)\|_C. \tag{21}$$

Since $F \in D_0(X)$, the map $\tau \mapsto F(\tau)$ is uniformly continuous and so, letting $s \to \bar{s}$ the right-hand side of (21) tends to zero. Hence, it follows from (20) that $x_{s-t} \to x_{\bar{s}-t}$ in $C(X)$

when $s \to \bar{s}$. By (19) we conclude that u_t is continuous. Moreover,

$$\|u_t\|_C = \sup_{s\in\mathbb{R}} \|x_{s-t}(s)\|_s \le \sup_{k\in\mathbb{R}} \|x_k\|_C \le \|R^{-1}\| \sup_{k\in\mathbb{R}} \|f_k\|_C$$

and

$$\begin{aligned}\|f_k\|_C &= \sup_{t\in\mathbb{R}} \|F(t-k)(t)\|_t \le \sup_{t,s\in\mathbb{R}} \|F(t)(s)\|_s \\ &= \sup_{t\in\mathbb{R}} \|F(t)\|_C = \|F\|_D < \infty,\end{aligned}$$

which shows that

$$\|u_t\|_C \le \|R^{-1}\| \cdot \|F\|_D < \infty$$

and so $u_t \in C(X)$. Finally, since

$$u_t(s) - u_\tau(s) = x_{s-t}(s) - x_{s-\tau}(s),$$

we obtain

$$\begin{aligned}\|u_t - u_\tau\|_C &= \sup_{s\in\mathbb{R}} \|x_{s-t}(s) - x_{s-\tau}(s)\|_s \\ &\le \sup_{s\in\mathbb{R}} \|x_{s-t} - x_{s-\tau}\|_C \\ &\le \|R^{-1}\| \sup_{s\in\mathbb{R}} \|f_{s-t} - f_{s-\tau}\|_C.\end{aligned}$$

Hence, it follows from (21) and the uniform continuity of the map $\tau \mapsto F(\tau)$ that $u_t \to u_\tau$ in $C(X)$ when $t \to \tau$. Therefore, the map $t \mapsto u_t$ is continuous and so $u \in D(X)$.

(2 ⇒ 1). Take $f \in C_0(X)$ and define a function $F\colon \mathbb{R} \to C_0(X)$ by

$$F(t)(s) = \frac{1}{1+(t-s)^2} f(s) \quad \text{for all } t, s \in \mathbb{R}. \tag{22}$$

We show that $F \in D_0(X)$. Note first that

$$F(t)(s) - F(\tau)(s) = \left(\frac{1}{1+(t-s)^2} - \frac{1}{1+(\tau-s)^2}\right) f(s)$$

and so

$$\begin{aligned}\|F(t) - F(\tau)\|_C &= \sup_{s\in\mathbb{R}} \frac{|(\tau-s)^2 - (t-s)^2|}{[1+(t-s)^2]\cdot[1+(\tau-s)^2]} \|f\|_C \\ &\le \sup_{s\in\mathbb{R}} \frac{|(\tau-t)(\tau+t-2s)|}{[1+(t-s)^2]\cdot[1+(\tau-s)^2]} \|f\|_C.\end{aligned}$$

For $t \in [\tau-\delta, \tau+\delta]$ we have

$$\|F(t) - F(\tau)\|_C \le \sup_{s\in\mathbb{R}} \frac{|\tau-t|(2|\tau|+\delta+2|s|)}{1+(\tau-s)^2} \|f\|_C \tag{23}$$

and so, letting $t \to \tau$ we find that $F(t) \to F(\tau)$ in $C(X)$. Moreover, we have

$$\|F(t)\|_C = \sup_{s\in\mathbb{R}} \frac{1}{1+(t-s)^2} \|f(s)\|_s.$$

Given $\epsilon > 0$, take $\rho > 0$ such that $\|f(s)\|_s < \epsilon$ whenever $|s| \geq \rho$. Then

$$\|F(t)\|_C \leq \sup_{s\in[-\rho,\rho]} \frac{1}{1+(t-s)^2}\|f\|_C + \epsilon \to \epsilon$$

when $|t| \to \infty$. It follows from the arbitrariness of ϵ that $F \in D_0(X)$.

By property 2, there exists a unique $u \in D(X)$ satisfying (11). In view of Lemma 1, for each $k \in \mathbb{R}$ the function $x_k(t) = u_{t-k}(t)$ satisfies Eq. (10) with f replaced by

$$f_k(t) = F(t-k)(t) = \frac{1}{1+k^2} f(t) \quad \text{for all } \ t \in \mathbb{R}.$$

Note that $y_k = (1+k^2)x_k$ satisfies Eq. (10) for each $k \in \mathbb{R}$. Before proceeding, we show that x_k is continuous. We have

$$\begin{aligned}\|x_k(t) - x_k(s)\| &\leq \|u_{t-k}(t) - u_{s-k}(t)\| + \|u_{s-k}(t) - u_{s-k}(s)\| \\ &\leq \|u_{t-k} - u_{s-k}\|_C + \|u_{s-k}(t) - u_{s-k}(s)\|.\end{aligned}$$

Therefore, letting $t \to s$ we conclude that $x_k(t) \to x_k(s)$ since $u \in D(X)$. Moreover,

$$\begin{aligned}\|x_k\|_C &= \sup_{t\in\mathbb{R}} \|u_{t-k}(t)\|_t \leq \sup_{t,s\in\mathbb{R}} \|u(t)(s)\|_s \\ &= \sup_{t\in\mathbb{R}} \|u(t)\|_C = \|u\|_D < \infty.\end{aligned}$$

We also show that y_k is independent of k. Otherwise, take distinct $p, q \in \mathbb{R}$ with $y_p \neq y_q$. We define a function $v \colon \mathbb{R} \to C(X)$ by

$$v_t(s) = \frac{y_p(s)}{1+(s-t)^2} \quad \text{for all } \ t, s \in \mathbb{R}.$$

One can show as in (23) that $v \in D(X)$. Then $v_{t-k}(t) = y_p(t)/(1+k^2)$ satisfies Eq. (12) for all k and so by Lemma 1, v is a solution of Eq. (11). But in view of property 2, we must have $v = u$, which is impossible since then

$$x_q(t) = u_{t-q}(t) = v_{t-q}(t) = \frac{y_p(t)}{1+q^2}$$

and so $y_q = y_p$. This contradiction shows that $y := y_k$, which is a solution of Eq. (10), is independent of k. To show that Eq. (10) has a unique solution, assume that z was a different solution. As before, we define a function $w \in D(X)$ by

$$w_t(s) = \frac{z(s)}{1+(s-t)^2} \quad \text{for all } \ t, s \in \mathbb{R}.$$

Similarly, $w_{t-k}(t) = z(t)/(1+k^2)$ satisfies Eq. (12) for all k and so by Lemma 1, w is a solution of Eq. (11). But then both $u, w \in D(X)$ are solutions of Eq. (11), which by hypothesis has a single solution. This contradiction shows that indeed Eq. (10) has a unique solution.

This concludes the proof of the theorem. □

5 Admissibility II: Exponential Growth

In this section we transfer a certain admissibility property from evolution families to evolution semigroups and vice-versa on spaces of continuous functions with at most exponential growth with a given exponential rate.

5.1 Evolution Semigroups

Given $c \geq 0$, consider the set $E^c(X)$ of all continuous functions $x: \mathbb{R} \to X$ such that the function $x_c: \mathbb{R} \to X$ defined by $x_c(t) = e^{-c|t|}x(t)$ for $t \in \mathbb{R}$ is in $C_0(X)$. We note that $E^c(X)$ is a Banach space when endowed with the norm

$$\|x\|_{E^c} := \|x_c\|_C.$$

Using the evolution family $\mathcal{V}$ generated by Eq. (4) for a given function $f \in E^c(X)$, we define an operator $T_t: E^c(X) \to X^{\mathbb{R}}$ for each $t \geq 0$ by

$$(T_t u)(s) = V(s, s-t)u(s-t) \quad \text{for } s \in \mathbb{R},\ u \in E^c(X).$$

Proposition 3 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then for each $c \geq 0$ and $f \in E^c(X)$ we have*

$$T_t(E^c(X)) \subset E^c(X) \quad \textit{for each } t \geq 0.$$

Proof Take $f, u \in E^c(X)$. It follows as in the proof of Proposition 3 that the function $T_t u$ is continuous for each $t \geq 0$. Moreover, we have

$$\begin{aligned} e^{-c|s|}\|(T_t u)(s)\|_s &= e^{-c|s|}\left\|U(s, s-t)u(s-t) + \int_{s-t}^{s} U(s,\tau)f(\tau)\,d\tau\right\|_s \\ &\leq \kappa e^{(\alpha+c)t}e^{-c|s-t|}\|u(s-t)\|_{s-t} \\ &\quad + \int_{s-t}^{s} \kappa e^{(\alpha+|c|)(s-\tau)}e^{-c|\tau|}\|f(\tau)\|_\tau\,d\tau \\ &\leq \kappa e^{(\alpha+c)t}e^{-c|s-t|}\|u(s-t)\|_{s-t} \\ &\quad + \frac{\kappa}{\alpha+|c|}(e^{(\alpha+|c|)t}-1)\sup_{\tau\in[s-t,s]}(e^{-c|\tau|}\|f(\tau)\|_\tau) \to 0 \end{aligned}$$

when $|s| \to \infty$ and so $T_t u \in E^c(X)$. □

The semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ is thus the evolution semigroup of $\mathcal{V}$ on $E^c(X)$.

5.2 Admissibility

As a preparation for the result relating the admissibility properties we first establish a version of Theorem 3 in which we consider the same spaces for the perturbations and for the solutions. This corresponds to consider $c = 0$ and so the space $E^c(X) = C_0(X)$.

Theorem 4 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then the following properties are equivalent:*

1. *For each $f \in C_0(X)$ there exists a unique $x \in C_0(X)$ satisfying (10);*
2. *For each $F \in D_0(X)$ there exists a unique $u \in D_0(X)$ satisfying (11).*

Proof We prove both implications separately.

(1 ⇒ 2). Take $F \in D_0(X)$ and consider again the maps $f_k \in C_0(X)$ defined by (14) for each $k \in \mathbb{R}$. By property 1, there exists a unique solution $x_k \in C_0(X)$ of Eq. (10) with $f = f_k$. Again we consider the function $u_t(s) = x_{s-t}(s)$ as in (16). We shall show that $u \in D_0(X)$. As in the proof of Theorem 3, u is a solution of Eq. (11) and it is then the unique solution in $D_0(X)$.

We already know from the proof of Theorem 3 that $u \in D(X)$ and so it remains only to verify that $u_t \in C_0(X)$ for each $t \in \mathbb{R}$ and that

$$\lim_{|t|\to\infty} \|u_t\|_C = 0.$$

We first show that

$$\lim_{|k|\to\infty} \|f_k\|_C = 0. \tag{24}$$

Since

$$\lim_{|t|\to\infty} \|F(t)\|_C = 0,$$

for each $\epsilon > 0$ there exists $T > 0$ such that

$$\|F(t)(s)\|_s < \epsilon \quad \text{whenever} \quad |t| \ge T \quad \text{and} \quad s \in \mathbb{R}.$$

For each $t \in [-T, T]$ we take an open interval I_t centered at t and $s_t > 0$ such that

$$\|F(\tau)(s)\|_s < \epsilon \quad \text{whenever} \quad \tau \in I_t \text{ and } |s| \ge s_t.$$

Since $[-T, T]$ is compact, it has a finite cover by intervals I_{t_i} with $i = 1, \dots, n$ and we let $S = \max_{1\le i\le n} S_{t_i} < \infty$. Then

$$\|F(t)(s)\|_s < \epsilon \quad \text{whenever} \quad t \in [-T, T] \text{ and } |s| \ge S. \tag{25}$$

In particular, this implies that $\|f_k\|_C < \epsilon$ for any sufficiently large integer $|k|$ since the line $\{(t-k, t) : t \in \mathbb{R}\}$ is then outside $[-T, T] \times [-S, S]$. Property (24) follows now readily from the arbitrariness of ϵ.

Let R_0 be the linear operator defined by $R_0 x = f$ on the domain composed of the functions $x \in C_0(X)$ for which there exists $f \in C_0(X)$ satisfying (10). One can show as in the proof of Theorem 3 that R_0 has a bounded inverse. Then

$$\begin{aligned}\lim_{|s|\to\infty} \|u_t(s)\|_s &= \lim_{|s|\to\infty} \|x_{s-t}(s)\|_s \le \lim_{|s|\to\infty} \|x_{s-t}\|_C \\ &\le \|R_0^{-1}\| \lim_{|s|\to\infty} \|f_{s-t}\|_C = 0\end{aligned}$$

in view of (24) and so $u_t \in C_0(X)$ for each $t \in \mathbb{R}$. Finally, we show that

$$\lim_{|t|\to\infty} \|u_t\|_C = \lim_{|t|\to\infty} \sup_{s\in\mathbb{R}} \|x_{s-t}(s)\|_s = 0. \tag{26}$$

Since $\|x_k\|_C \le \|R_0^{-1}\| \cdot \|f_k\|_C$, it follows from (24) that for each $\epsilon > 0$ there exists $K > 0$ such that

$$\|x_k(s)\|_s < \epsilon \quad \text{whenever} \quad |k| \ge K \quad \text{and} \quad s \in \mathbb{R}.$$

On the other hand, we already showed in the proof of Theorem 3 that the map $k \mapsto x_k$ is continuous, which follows from (21) together with the uniform continuity of the map $\tau \mapsto F(\tau)$. Since $x_k \in C_0(X)$, for each $k \in [-K, K]$ there exist an open interval J_k centered at k and $s_k > 0$ such that

$$\|x_l(s)\|_s < \epsilon \quad \text{whenever} \quad l \in J_k \quad \text{and} \quad |s| \ge s_k.$$

Since $[-K, K]$ is compact, one can proceed as above to find $S > 0$ such that

$$\|x_k(s)\|_s < \epsilon \quad \text{whenever} \quad k \in [-K, K] \quad \text{and} \quad |s| \ge S.$$

In particular, this implies that $\sup_{s\in\mathbb{R}} \|x_{s-t}(s)\|_s < \epsilon$ for any sufficiently large $|t|$ since the line $\{(s-t, s) : t \in \mathbb{R}\}$ is then outside $[-K, K] \times [-S, S]$. Property (26) follows now from the arbitrariness of ϵ and so $u \in D_0(X)$.

($2 \Rightarrow 1$). Take $f \in C_0(X)$ and consider the function $F \in D_0(X)$ defined by (22). By property 2, there exists a unique $u \in D_0(X)$ satisfying (11). We know from the proof of Theorem 3 that the function $x(t) = (1 + k^2)u_{t-k}(t)$ is independent of k and that it is the unique solution of Eq. (10) in $C(X)$. It remains to verify that $x \in C_0(X)$.

In the proof of the implication ($1 \Rightarrow 2$) we showed that given $F \in D_0(X)$ and $\epsilon > 0$, there exist $T, S > 0$ such that property (25) holds. Since $u \in D_0(X)$, given $\epsilon > 0$, there exist $T, S > 0$ such that

$$\|u_t(s)\|_s < \epsilon \quad \text{whenever} \ \ t \in [-T, T] \ \ \text{and} \ \ |s| \geq S.$$

Also as before, this implies that $\sup_{t\in\mathbb{R}} \|u_{t-k}(t)\|_s < \epsilon$ for any sufficiently large $|k|$. It thus follows from the arbitrariness of ϵ that $x(t) \to 0$ when $|t| \to \infty$ and so $x \in C_0(X)$. This completes the proof of the theorem. □

Using this result we are able to consider the space $E^c(X)$ for an arbitrary constant $c \geq 0$. Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Given $c \geq 0$, the evolution family $\mathcal{U}$ generates an evolution semigroup $\mathcal{S} = (S_t)_{t\geq 0}$ on $E^c(X)$ defined by

$$(S_t u)(s) = U(s, s-t)u(s-t) \quad \text{for} \ \ t \geq 0, \ \ s \in \mathbb{R}, \ \ u \in E^c(X).$$

By Proposition 3 with $f = 0$ in (4), indeed $S_t(E^c(X)) \subset E^c(X)$ for all $t \geq 0$. Moreover, let $F^c(X)$ be the set of all continuous functions $v \colon \mathbb{R} \to E^c(X)$ such that

$$\lim_{|t|\to\infty} \|v(t)\|_{E^c} = 0.$$

We note that $F^c(X)$ is a Banach space when endowed with the norm

$$\|v\|_{F^c} := \sup_{t\in\mathbb{R}} \|v(t)\|_{E^c} < \infty.$$

In other words, $F^c(X) = C_0(E^c(X))$.

Theorem 5 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then for each $c \geq 0$ the following properties are equivalent:*

1. *For each $f \in E^c(X)$ there exists a unique $x \in E^c(X)$ satisfying (10);*
2. *For each $F \in F^c(X)$ there exists a unique $u \in F^c(X)$ satisfying (11).*

Proof Take $f, x \in E^c(X)$. We consider the functions $f_c, x_c \in C_0(X)$ defined by

$$f_c(t) = e^{-c|t|} f(t) \quad \text{and} \quad x_c(t) = e^{-c|t|} x(t)$$

for $t \in \mathbb{R}$. Note that property (10) holds if and only if

$$x_c(t) = U_c(t, s)x_c(s) + \int_s^t U_c(t, \tau) f_c(\tau)\, d\tau \quad \text{for all } (t, s) \in \Pi, \tag{27}$$

where

$$U_c(t, s) = e^{-c|t|+c|s|} U(t, s).$$

Therefore, property 1 holds if and only if for each $f \in E^c(X)$ there exists a unique $x \in E^c(X)$ satisfying (27).

Notice that the evolution family formed by the linear operators $U_c(t, s)$ for $(t, s) \in \Pi$ and a given $c \geq 0$ is also exponentially bounded with respect to the norms $\|\cdot\|_t$. Since the maps $f \mapsto f_c$ and $x \mapsto x_c$ are bijections from $E^c(X)$ onto $C_0(X)$, it follows from Theorem 4 that property 1 holds if and only if for each $F \in D_0(X)$ there exists a unique $u \in D_0(X)$ satisfying

$$u(t) = S^c_{t-s}u(s) + \int_s^t S^c_{t-\tau} F(\tau)\, d\tau \quad \text{for all } (t, s) \in \Pi, \tag{28}$$

where

$$(S^c_t v)(r) = U_c(r, r-t)v(r-t) \quad \text{for } t \geq 0,\ r \in \mathbb{R},\ v \in C_0(X).$$

We have

$$e^{c|r|}(S^c_t v)(r) = U(r, r-t)e^{c|r-t|}v(r-t),$$

that is,

$$\gamma \circ S^c_t = S_t \circ \gamma, \quad \text{with } \gamma(v)(r) = e^{c|r|}v(r).$$

Letting $u_c(s) = \gamma(u(s))$ we have

$$\gamma(S^c_{t-s}u(s)) = S_{t-s}(\gamma(u(s))) = S_{t-s}u_c(s)$$

and so property (28) is equivalent to

$$u_c(t) = S_{t-s}u_c(s) + \int_s^t S_{t-\tau} F_c(\tau)\, d\tau \quad \text{for all } (t, s) \in \Pi, \tag{29}$$

where

$$F_c(t)(s) = \gamma(F(t))(s) = e^{c|s|}F(t)(s).$$

Since the maps $F \mapsto F_c$ and $u \mapsto u_c$ are bijections from $D_0(X)$ onto $F^c(X)$, it follows from (29) that property 1 holds if and only if property 2 holds. □

6 Admissibility III: Integrable Functions

In this section we show once more how to transfer a certain admissibility property from evolution families to evolution semigroups and vice-versa. Here we consider evolution semigroups defined on a L^p space.

6.1 Evolution Semigroups

For each $p \in [1, +\infty)$, let $L^p(X)$ be the set of all (Bochner) measurable functions $x : \mathbb{R} \to X$ such that

$$\|x\|_{L^p} = \left(\int_{\mathbb{R}} \|x(s)\|_s^p\, ds \right)^{1/p} < \infty$$

identified almost everywhere with respect to the Lebesgue measure. We note that $L^p(X)$ is a Banach space when endowed with the norm $\|\cdot\|_{L^p}$. Using the evolution family $\mathcal{V}$ generated by Eq. (4) for a given function $f \in L^p(X)$, we define an operator $T_t\colon L^p(X) \to X^{\mathbb{R}}$ for each $t \geq 0$ by

$$(T_t u)(s) = V(s, s-t)u(s-t) \quad \text{for } s \in \mathbb{R},\ u \in L^p(X),$$

again identifying the functions almost everywhere.

Proposition 4 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then for each $f \in L^r(X)$ with $1 \leq r \leq p < \infty$, we have*

$$T_t(L^p(X)) \subset L^p(X) \quad \textit{for each } t \geq 0.$$

Proof Take $u \in L^p(X)$. By Minkowski's inequality we have

$$\begin{aligned}
\|T_t u\|_{L^p} &= \left(\int_{\mathbb{R}} \|V(s, s-t)u(s-t)\|_s^p \, ds\right)^{1/p} \\
&\leq \left(\int_{\mathbb{R}} \|U(s, s-t)u(s-t)\|_s^p \, ds\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}} \left\| \int_{s-t}^{s} U(s, \tau) f(\tau)\, d\tau \right\|_s^p \, ds\right)^{1/p} \\
&\leq \kappa e^{\alpha t} \left(\int_{\mathbb{R}} \|u(s-t)\|_{s-t}^p \, ds\right)^{1/p} \\
&\quad + \kappa \left(\int_{\mathbb{R}} \left(\int_{s-t}^{s} e^{\alpha(s-\tau)} \|f(\tau)\|_\tau \, d\tau\right)^p \, ds\right)^{1/p} \\
&= \kappa e^{\alpha t} \|u\|_{L^p} + \kappa \left(\int_{\mathbb{R}} \left(\int_{s-t}^{s} e^{\alpha(s-\tau)} \|f(\tau)\|_\tau \, d\tau\right)^p \, ds\right)^{1/p}.
\end{aligned}$$

and so

$$\|T_t u\|_{L^p} \leq \kappa e^{\alpha t} \|u\|_{L^p} + \kappa e^{\alpha t} \left(\int_{\mathbb{R}} \left(\int_{s-t}^{s} \|f(\tau)\|_\tau \, d\tau\right)^p \, ds\right)^{1/p}. \tag{30}$$

First assume that $r = p$. For $p = 1$ we obtain

$$\begin{aligned}
\|T_t u\|_{L^1} &\leq \kappa e^{\alpha t} \|u\|_{L^1} + \kappa e^{\alpha t} \int_{\mathbb{R}} \int_{s-t}^{s} \|f(\tau)\|_\tau \, d\tau \, ds \\
&= \kappa e^{\alpha t} \|u\|_{L^1} + \kappa e^{\alpha t} \int_{\mathbb{R}} \int_{\tau}^{\tau+t} \|f(\tau)\|_\tau \, ds \, d\tau \\
&= \kappa e^{\alpha t} \|u\|_{L^1} + \kappa t e^{\alpha t} \|f\|_{L^1} < \infty.
\end{aligned}$$

On the other hand, for $p > 1$ and $q \in (1, +\infty)$ such that $1/p + 1/q = 1$, it follows from Hölder's inequality that

$$\int_{s-t}^{s} \|f(\tau)\|_\tau \, d\tau \leq t^{1/q} \left(\int_{s-t}^{s} \|f(\tau)\|_\tau^p \, d\tau\right)^{1/p}$$

and again by (30) we obtain

$$
\begin{aligned}
\|T_t u\|_{L^p} &\le \kappa e^{\alpha t}\|u\|_{L^p} + \kappa t^{1/q} e^{\alpha t}\left(\int_{\mathbb{R}}\int_{s-t}^{s}\|f(\tau)\|_\tau^p\, d\tau\, ds\right)^{1/p} \\
&\le \kappa e^{\alpha t}\|u\|_{L^p} + \kappa t^{1/q} e^{\alpha t}\left(\int_{\mathbb{R}}\int_{\tau}^{\tau+t}\|f(\tau)\|_\tau^p\, ds\, d\tau\right)^{1/p} \\
&= \kappa e^{\alpha t}\|u\|_{L^p} + \kappa t e^{\alpha t}\|f\|_{L^p} < \infty.
\end{aligned}
$$

This shows that $T_t u \in L^p(X)$ when $r = p$.

Now assume that $r < p$ and take $\sigma \in (1, +\infty)$ such that $1/r + 1/\sigma = 1$. It follows from Hölder's inequality that

$$
\int_{s-t}^{s}\|f(\tau)\|_\tau\, d\tau \le t^{1/\sigma}\left(\int_{s-t}^{s}\|f(\tau)\|_\tau^r\, d\tau\right)^{1/r}
$$

and so

$$
\begin{aligned}
\left(\int_{s-t}^{s}\|f(\tau)\|_\tau\, d\tau\right)^p &\le t^{p/\sigma}\left(\int_{s-t}^{s}\|f(\tau)\|_\tau^r\, d\tau\right)^{p/r} \\
&\le t^{p/\sigma}\|f\|_{L^r}^{p/r-1}\int_{s-t}^{s}\|f(\tau)\|_\tau^r\, d\tau.
\end{aligned}
$$

Hence, it follows from (30) that

$$
\begin{aligned}
\|T_t u\|_{L^p} &\le \kappa e^{\alpha t}\|u\|_{L^p} + \kappa t^{1/\sigma} e^{\alpha t}\|f\|_{L^r}^{1/r-1/p}\left(\int_{\mathbb{R}}\int_{s-t}^{s}\|f(\tau)\|_\tau^r\, d\tau\, ds\right)^{1/p} \\
&= \kappa e^{\alpha t}\|u\|_{L^p} + \kappa t^{1/\sigma} e^{\alpha t}\|f\|_{L^r}^{1/r-1/p}\left(\int_{\mathbb{R}}\int_{\tau}^{\tau+t}\|f(\tau)\|_\tau^r\, ds\, d\tau\right)^{1/p} \\
&= \kappa e^{\alpha t}\|u\|_{L^p} + \kappa t^{1/\sigma+1/p} e^{\alpha t}\|f\|_{L^r}^{1/r-1/p+r/p} < \infty,
\end{aligned}
$$

which shows that $T_t u \in L^p(X)$ when $r < p$. □

The semigroup $\mathcal{T} = (T_t)_{t\ge 0}$ is thus the evolution semigroup of $\mathcal{V}$ on $L^p(X)$ for each $p \in [1, +\infty)$.

6.2 Admissibility

Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then $\mathcal{U}$ generates an evolution semigroup $\mathcal{S} = (S_t)_{t\ge 0}$ on $L^p(X)$ defined by

$$
(S_t u)(s) = U(s, s-t)u(s-t) \quad \text{for } t \ge 0,\ s \in \mathbb{R},\ u \in L^p(X).
$$

By Proposition 4 with $f = 0$ in (4), indeed $S_t(L^p(X)) \subset L^p(X)$ for all $t \ge 0$.

Moreover, for each $p \in [1, +\infty)$ let $M^p(X) = L^p(L^p(X))$ be the set of all (Bochner) measurable functions $v\colon \mathbb{R} \to L^p(X)$ such that

$$
\|v\|_{M^p} := \left(\int_{\mathbb{R}}\|v(t)\|_{L^p}^p\, dt\right)^{1/p} = \left(\int_{\mathbb{R}}\int_{\mathbb{R}}\|v_t(s)\|_s^p\, ds\right)^{1/p} < \infty,
$$

identified almost everywhere with respect to the Lebesgue measure. We note that $M^p(X)$ is a Banach space when endowed with the norm $\|\cdot\|_{M^p}$.

Theorem 6 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then the following properties are equivalent:*

1. *For each $f \in L^p(X)$ there exists a unique $x \in L^p(X)$ satisfying (10);*
2. *For each $F \in M^p(X)$ there exists a unique $u \in M^p(X)$ satisfying (11).*

Proof We note that the statement in Lemma 1 also holds for $F \in M^p(X)$ and $u \in M^p(X)$. We proceed with the proof of the theorem.

(1 ⇒ 2). Take $F \in M^p(X)$. For each $k \in \mathbb{R}$ we define a map $f_k : \mathbb{R} \to X$ by (14). We have

$$\begin{aligned}\int_{\mathbb{R}} \|f_k\|_{L^p}^p\,dk &= \int_{\mathbb{R}}\int_{\mathbb{R}} \|F(s-k)(s)\|_s^p\,ds\,dk\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \|F(t)(\tau)\|_\tau^p\,d\tau\,dt = \|F\|_{M^p}^p < \infty\end{aligned} \tag{31}$$

and so $f_k \in L^p(X)$ for almost all $k \in \mathbb{R}$. By property 1, there exists a unique solution $x_k \in L^p(X)$ of Eq. (10) with $f = f_k$, for almost all $k \in \mathbb{R}$. Writing $u(t) = u_t$ for each function $u : \mathbb{R} \to L^p(X)$, we define

$$u_t(s) = x_{s-t}(s) \quad \text{for } t, s, \in \mathbb{R}.$$

We show below that $u \in M^p(X)$. Then it follows from Lemma 1 that u is a solution of Eq. (11) and as in the proof of Theorem 3 it is automatically unique.

Let R be the linear operator defined by $Rx = f$ on the domain composed of the functions $x \in L^p(X)$ for which there exists $f \in L^p(X)$ satisfying (10). As in the proof of Theorem 3, we show that R is a well-defined closed operator. To show that R is well-defined, take $g \in L^p(X)$ such that

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)g(\tau)\,d\tau$$

for all $(t,s) \in \Pi$. Then

$$\frac{1}{t-s}\int_s^t U(t,\tau)(f(\tau) - g(\tau))\,d\tau = 0$$

for all $(t,s) \in \Pi$ with $t \neq s$. Since the integrand is locally integrable (which follows from Hölder's inequality), letting $s \nearrow t$ it follows from the Lebesgue differentiation theorem that $f(t) = g(t)$ for almost every $t \in \mathbb{R}$. This shows that the operator R is well defined.

To show that R is closed, let $(x^\ell)_{\ell\in\mathbb{N}}$ be a sequence in the domain of R converging to $x \in L^p(X)$ such that $f^\ell = Rx^\ell$ converges to $f \in L^p(X)$. Then property (17) holds for all $(t,s) \in \Pi$. We have

$$\left\| \int_s^t U(t,\tau)f^\ell(\tau)\,d\tau - \int_s^t U(t,\tau)f(\tau)\,d\tau \right\| \le d(t-s)^{1/q}\|f^\ell - f\|_{L^p},$$

with d as in (18) and with $q \in (1,+\infty]$ such that $1/p + 1/q = 1$ (when $p = 1$ we make the convention that $(t-s)^{1/q} = 1$). Thus, letting $\ell \to \infty$ in (17) we find that

$$x(t) - U(t,s)x(s) = \int_s^t U(t,\tau)f(\tau)\,d\tau$$

for all $(t,s) \in \Pi$. This shows that $Rx = f$ and so x is in the domain of R. Hence, the operator R is closed and by property, it has a bounded inverse.

Now we show that $u \in M^p(X)$. We have

$$\begin{aligned}\int_{\mathbb{R}} \|u_t\|_{L^p}^p \, dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} \|x_{s-t}(s)\|_s^p \, ds \, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \|x_k(\tau)\|_\tau^p \, d\tau \, dk = \int_{\mathbb{R}} \|x_k\|_{L^p}^p \, dk.\end{aligned} \tag{32}$$

Since

$$\|x_k\|_{L^p} \le \|R^{-1}\| \cdot \|f_k\|_{L^p},$$

it follows from (31) that

$$\begin{aligned}\int_{\mathbb{R}} \|u_t\|_{L^p}^p \, dt &= \int_{\mathbb{R}} \|x_k\|_{L^p}^p \, dk \\ &\le \|R^{-1}\|^p \int_{\mathbb{R}} \|f_k\|_{L^p}^p \, dk \\ &= \|R^{-1}\|^p \|F\|_{M^p}^p < \infty\end{aligned}$$

and so $u \in M^p(X)$.

(2 $\Rightarrow$ 1). Take $f \in L^p(X)$ and define a function $F : \mathbb{R} \to L^p(X)$ by

$$F(t)(s) = \frac{1}{1+(t-s)^2} f(s) \quad \text{for all } t, s \in \mathbb{R}.$$

Note that $F \in M^p(X)$. Indeed,

$$\begin{aligned}\|F\|_{M^p}^p &= \int_{\mathbb{R}} \int_{\mathbb{R}} \|F(t)(s)\|_s^p \, ds \, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \|F(t)(s)\|_s^p \, dt \, ds \\ &= \int_{\mathbb{R}} \|f(s)\|_s^p \int_{\mathbb{R}} \frac{1}{(1+(t-s)^2)^p} \, dt \, ds \\ &= \int_{\mathbb{R}} \|f(s)\|_s^p \, ds \int_{\mathbb{R}} \frac{1}{(1+t^2)^p} \, dt \\ &= c_r \|f\|_{L^p}^p < \infty\end{aligned}$$

for some constant $c_p > 0$ that depends only on p. By property 2, there exists a unique $u \in M^p(X)$ satisfying (11). By Lemma 1, for each $k \in \mathbb{R}$ the function

$$x_k(t) = u_{t-k}(t) := u(t-k)(t)$$

satisfies Eq. (10) with f replaced by

$$f_k(t) = F(t-k)(t) = \frac{1}{1+k^2} f(t) \quad \text{for all } \ t \in \mathbb{R}.$$

Proceeding as in (32), we obtain

$$\int_{\mathbb{R}} \|x_k\|_{L^p}^p \, dk = \int_{\mathbb{R}} \|u_t\|_{L^p}^p \, dt = \|u\|_{M^p}^p < \infty$$

and so $x_k \in L^p(X)$ for almost all $k \in \mathbb{R}$. One can then show in a similar manner to that in the proof of Theorem 3 that $x(t) = (1+k^2)x_k(t)$ is the same for all k in a full measure set and that it is the unique solution of Eq. (10) in $L^p(X)$. This concludes the proof of the theorem. □

7 Hyperbolicity

In this section we consider the relation between the notions of admissibility and hyperbolicity, and how the former results can give further relations between hyperbolicity for evolution families and evolution semigroups.

We continue to consider a family of norms $\|\cdot\|_t$, for $t \in \mathbb{R}$, on a Banach space X. We say that an evolution family $\mathcal{U} = (U(t,s))_{(t,s)\in\Pi}$ of linear maps on X is *hyperbolic with respect to the norms* $\|\cdot\|_t$ if:

1. There exist projections $P_t \colon X \to X$, for $t \in \mathbb{R}$, such that
$$P_t U(t,s) = U(t,s) P_s$$
and, writing $Q_s = \mathrm{Id} - P_s$, the map
$$U(t,s)|_{\mathrm{Im}\, Q_s} \colon \mathrm{Im}\, Q_s \to \mathrm{Im}\, Q_t,$$
is onto and invertible for each $t \geq s$;
2. There exist constants $\lambda, N > 0$ such that
$$\|U(t,s)P_s x\|_t \leq N e^{-\lambda(t-s)} \|x\|_s$$
and
$$\|\bar{U}(s,t)Q_t x\|_s \leq N e^{-\lambda(t-s)} \|x\|_t$$
for $t \geq s$ and $x \in X$, where $\bar{U}(s,t) = (U(t,s)|_{\mathrm{Im}\, Q_s})^{-1}$.

The following statement relates admissibility and hyperbolicity in various situations (among many other equivalent properties considered in the area).

Proposition 5 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then the following properties are equivalent:*

1. *The evolution family $\mathcal{U}$ is hyperbolic with respect to the norms $\|\cdot\|_t$;*
2. *For each $f \in C_0(X)$ there exists a unique $x \in C_0(X)$ satisfying* (10)*;*
3. *For each $f \in L^p(X)$ there exists a unique $x \in L^p(X)$ satisfying* (10)*;*
4. *There exists $d > 0$ such that for each $c \in [0,d)$ and $f \in E^c(X)$ there exists a unique $x \in E^c(X)$ satisfying* (10).

These results can be obtained for example as in [10] replacing the family of norms $\|\cdot\|_t = \|\cdot\|$, for $t \in \mathbb{R}$, by an arbitrary family. This essentially corresponds to consider the nonuniform exponential behavior that is ubiquitous in smooth ergodic theory.

The following result is a simple consequence of the former Theorems 4, 5 and 6 together with Proposition 5.

Theorem 7 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then the following properties are equivalent:*

1. *The evolution family $\mathcal{U}$ is hyperbolic with respect to the norms $\|\cdot\|_t$;*
2. *For each $F \in D_0(X)$ there exists a unique $u \in D_0(X)$ satisfying* (11)*;*
3. *For each $F \in M^p(X)$ there exists a unique $u \in M^p(X)$ satisfying* (11)*;*
4. *There exists $d > 0$ such that for each $c \in [0,d)$ and $F \in F^c(X)$ there exists a unique $u \in F^c(X)$ satisfying* (11).

One can also establish relations with the hyperbolicity of the evolution semigroup on various Banach spaces. We recall that a semigroup $\mathcal{T} = (T_t)_{t \geq 0}$ of linear maps on a Banach space Y is said to be *hyperbolic* if:

1. There exists a projection $P \colon Y \to Y$ such that $PT_t = T_t P$ and, writing $Q = \mathrm{Id} - P$, the map
$$T_t|_{\mathrm{Im}\, Q} \colon \mathrm{Im}\, Q \to \mathrm{Im}\, Q,$$
is onto and invertible for each $t \geq 0$;
2. There exist constants $\lambda, N > 0$ such that
$$\|T_t P\| \leq N e^{-\lambda t} \quad \text{and} \quad \|\bar{T}_t Q\| \leq N e^{-\lambda t}$$
for $t \geq 0$, where $\bar{T}_t = (T_t|_{\mathrm{Im}\, Q})^{-1}$.

The equivalence of the notions of hyperbolicity for an evolution family $\mathcal{U}$ and its evolution semigroup $\mathcal{T}$ on $C_0(X)$ and on $L^p(X)$ lead to further equivalences to the former admissibility properties (see [6,18]). In particular, we have the following result.

Theorem 8 *Let $\mathcal{U}$ be a linear evolution family that is exponentially bounded with respect to the norms $\|\cdot\|_t$. Then $\mathcal{U}$ is hyperbolic with respect to the norms $\|\cdot\|_t$ if and only if there exists $d > 0$ such that the semigroup defined by*
$$(S_t^c v)(r) = e^{-c|r| + c|r-t|} U(r, r-t) v(r-t)$$
for $t \geq 0$, $r \in \mathbb{R}$ and $v \in C_0(X)$ is hyperbolic on $C_0(X)$ for each $c \in [0, d)$.

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