Journal of Dynamics and Differential Equations (2022) 34:1729–1768
https://doi.org/10.1007/s10884-021-10056-8

Lattice Structures for Attractors III

W. D. Kalies[1] · K. Mischaikow[2] · R. C. A. M. Vandervorst[3]

Received: 26 November 2019 / Revised: 12 June 2021 / Accepted: 19 July 2021 /
Published online: 12 October 2021

Abstract
The theory of bounded, distributive lattices provides the appropriate language for describing directionality and asymptotics in dynamical systems. For bounded, distributive lattices the general notion of 'set-difference' taking values in a semilattice is introduced, and is called the *Conley form.* The Conley form is used to build concrete, set-theoretic models of spectral spaces, or Priestley spaces, of bounded, distributive lattices and their finite coarsenings. Such representations formulate and compute order-theoretic models of dynamical systems such as Morse decompositions and Morse representations, which may be regarded as global characteristics of a dynamical system.

Keywords Booleanization · Conley Form · Morse Decomposition · Distributive Lattice · Birkhoff-Stone Representation Theorem

Mathematics Subject Classification 37B25 · 06D05 · 37B35

1 Prelude

Perhaps the simplest characterization of global dynamics is in terms of recurrent and non-recurrent dynamics. A systematic approach to this decomposition began with Smale [37] in the context of Axiom A diffeomorphisms. For general dynamical systems, Conley [10] established the concept of a Morse decomposition that uses the nonrecurrent dynamics to define a partial order on a finite collection of invariant sets that contain the recurrent dynam-

The first author was partially supported by NSF grant NFS-DMS-0914995 and ARO grant W911NF1810306. The second author was partially supported by NSF grants 1521771, 1622401, 1839294, 1841324, 1934924, by NIH-1R01GM126555-01 as part of the Joint DMS/NIGMS Initiative to Support Research at the Interface of the Biological and Mathematical Science and contracts, DARPA contract HR001117S0003-SD2-FP-011, and a grant from the Simons Foundation. The authors are also grateful for visiting positions at the MBI in Columbus and the CRM in Montreal.

✉ W. D. Kalies
wkalies@fau.edu

1 Florida Atlantic University, Boca Raton, FL 33431, USA

2 Rutgers University, Piscataway, NJ 08854-8019, USA

3 Vrije Universiteit Amsterdam, 1081 HV, Amsterdam, The Netherlands

ics. Aside from its generality, Morse decompositions have the advantage that they can be defined without a priori assumptions or understanding of detailed structures of the dynamics. In fact, as we demonstrate in this paper, it can be defined without fixing a dynamical system. Consequently, it is an extremely general tool, but this generality has led to subtle variances in its definition and how it is has been employed. A goal of this paper is to clarify (and rectify) this subtlety and give a proper order theoretic formulation of Morse decomposition.

Within the framework of Conley's approach to dynamical systems a global understanding of the dynamics is codified in the form of a chain complex with a boundary operator called the connection matrix [16,21,33,35]. Two fundamental facts associated with connection matrices are as follows. First, they respect the partial order associated with a Morse decomposition. Second, their computation depends on identifying an index filtration, which, as observed by Robbin and Salamon [35], is a finite lattice of attracting blocks. This duality between posets that capture the gradient-like nature of the dynamics and lattices that provide insight into the global organization of the dynamics is explored in a series of papers [23–26] that develop an algebraic representation of the nonrecurrent structure of nonlinear dynamics. This paper builds on these earlier works, however for the convenience of the reader we include an appendix in which essential concepts, results, and notation are recalled.

Nevertheless, we expect that the formal algebra of lattice theory employed in this paper is foreign to typical practitioners of dynamics and thus in an attempt to provide context for the results presented here we begin with four concrete examples: the first two, are indicative of how Conley theory is used in the classical analysis of dynamical systems, the third, is representative of these techniques in the computational analysis of dynamical systems; and the fourth is indicative of how these ideas may play a role in the analysis of data-driven dynamics.

Example 1 Consider the special form of the Cahn-Hilliard equation

$$u_t = (-\epsilon^2 u_{xx} + u^3 - u)_{xx}, \quad (x,t) \in [-1,1] \times \mathbb{R}$$

with boundary conditions $u_x(\pm 1, t) = u_{xxx}(\pm 1, t) = 0$. We recall a few relevant facts about the associated dynamics, cf. [31] for details and references therein. This equation has a global compact attractor X on which the dynamics is given by a flow $\varphi\colon \mathbb{R} \times X \to X$. There is an energy functional which is strictly decreasing along solutions that are not equilibria. As a function of the parameter ϵ, the complete set of equilibria are known. In particular, given a positive integer N there is a range of ϵ for which there are $2N+1$ equilibria, $\mathsf{M} := \{M_{n^\pm} \mid n = 0, \ldots, N-1\} \cup M_N$, where M_N is the solution $u \equiv 0$. Furthermore, for any $u \in X \setminus \mathsf{M}$ there exist $0 \leq n < m \leq N$ such that $\lim_{t\to\infty} \varphi(t,u)$ is either M_{n^+} or M_{n^-}, and $\lim_{t\to-\infty} \varphi(t,u)$ is either M_{m^+} or M_{m^-} (or 0 if $m = N$). Notice that the partial order

$$M_{n^\pm} < M_N, \quad \text{and} \quad M_{n^\pm} < M_{m^\pm}, \quad \text{for all } 0 \leq n < m \leq N$$

is compatible with the description of the possible heteroclinic orbits (Fig. 1).

This use of a partially ordered set (poset) to organize global dynamical information about the equilibria (recurrent dynamics) is extremely useful, if for no other reason that it allows one (as was done in [31]) to address the next natural question: what is the structure of the heteroclinic orbits between equilibria?

We capture the above mentioned global information via the following definition.

Definition 1 Let $\varphi\colon \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space X. A *Morse representation* of φ is a finite poset $(\mathsf{M}, \leqslant)$ where M consists of mutually disjoint,

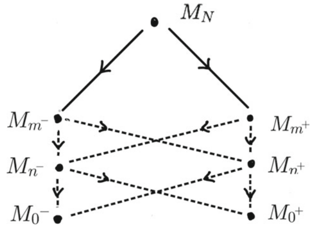


Fig. 1 The poset structure of M reveals that the equilibrium points are not linearly ordered

nonempty, compact invariant sets, called *Morse sets*, with the property that for each $x \in X$ there exists $M \in \mathsf{M}$ such that $\omega(x) \subset M$, and for each complete orbit γ_x with $x \notin \bigcup_{M \in \mathsf{M}} M$ there exist $M < M'$ such that $\omega(x) \subset M$ and $\alpha_o(\gamma_x^-) \subset M'$.[1]

For a fixed $\epsilon > 0$ the set of equilibria M in Example 1 provides a Morse representation for the Cahn-Hilliard flow on X. To indicate why we have introduced this new terminology of Morse representation we turn to the next example.

Example 2 In [27] J. Mallet-Paret considers a class of scalar delay-differential equations of the form

$$\dot{x}(t) = -f(x(t), x(t-1)) \tag{1}$$

where f satisfies appropriate conditions. Relevant facts for this system are that there exists a global compact attractor X, an induced flow $\varphi\colon \mathbb{R} \times X \to X$, and associated with this flow is a discrete valued Lyapunov function $V\colon X \setminus \{0\} \to \{1, 3, \ldots, 2n+1, \ldots\}$, i.e. $V(\varphi(t,x)) \quad V(x)$ for all $t > 0$. Define the invariant subsets

$$M_n := \left\{ x \in X \setminus \{0\} \;\middle|\; V(\varphi(t,x)) = n,\ \forall t \in \mathbb{R},\ \text{and } 0 \notin \alpha(x) \cup \omega(x) \right\}.$$

It follows from [27, Thm. B] that there is a positive integer N^* (determined by the linearization of the solution $x \equiv 0$) and another odd integer N ($N^* \quad N$) such that if $x \in X$, then either $x \in M_n$ for some $n \leqslant N$, or there exists $j < k \leqslant N$ such that $\omega(x) \subset M_j$ and $\alpha(x) \subset M_k$, cf. [27] for the definition of M_{N^*}.[2] Thus, the poset structure induced by the dynamics is the usual ordering on the integers.

Similar to Example 1 this result reduces the study of the global dynamics of (1) to identifying the dynamics of each invariant set M_n and the connecting orbits between these invariant sets. For example, in [27] it is shown that for $n < N^*$ each M_n contains a periodic orbit, and in [14,28] existence and structure of connecting orbits for the set of orbits for which $V \quad N^*$ is demonstrated. Observe that these results do not extend to orbits associated with $V > N^*$ as such results will depend on more detailed assumptions concerning f. In particular, if $n > N^*$, then it is possible that $M_n = \varnothing$ for multiple indices and therefore

$$M_1, M_3, \ldots, M_{N^*}, \ldots, M_N$$

does not have the structure of a partially ordered set in general.

In both examples an appropriate partial order structure provides a framework in which to try to understand the global dynamics, cf. [28,31]. This leads us to the following definition where it is not longer necessary to focus on nonempty invariant sets.

[1] The set $\alpha_o(\gamma_x^-)$ indicates the orbital alpha limit set for γ_x^-, cf. [25, Prop. 2.13]. An overview of the most important topological and algebraic notions used here is given in Appendix 1.

[2] The set $M_{N^*} = \{0\}$ whenever the origin is hyperbolic. In that case N^* is an even integer.

Definition 2 A *Morse decomposition* is an order-embedding $\pi : \mathsf{M} \hookrightarrow \mathsf{P}$, where $(\mathsf{M}, \leqslant)$ is a Morse representation of φ and $(\mathsf{P}, \leqslant)$ is a finite poset.

This definition implies that if $\pi : \mathsf{M} \hookrightarrow \mathsf{P}$ is a Morse decomposition then for each $x \in X$ there exists $p \in \mathsf{P}$ such that $\omega(x) \subset \pi^{-1}(p)$, and for each complete orbit γ_x with $x \notin \bigcup_{M \in \mathsf{M}} M$ there exist $p < p'$ such that $\omega(x) \subset \pi^{-1}(p)$ and $\alpha_o(\gamma_x^-) \subset \pi^{-1}(p')$.

The definition of Morse decomposition is the same in spirit as the definition in [10]. The main difference is that we do not 'label' the Morse sets by P but instead allow only the nonempty Morse sets for the poset M and keep track of a poset P to account for a global order structure of the flow, cf. Remark 1. A Morse decomposition is de facto the poset P. In Example 2 we define $\mathsf{P} = \{1, 3, \ldots, N^*, \ldots, N\}$ equipped with the linear order induced by the integers and define the poset (Morse representation) $\mathsf{M} := \{M_n \neq \varnothing \mid n \in \mathsf{P}\}$ and

$$M_n \leqslant M_m \iff n \leqslant m.$$

Note that M is a well-defined poset since all elements are necessarily distinct. We will return to Definition 2 and discuss it in a proper mathematical context in Sect. 9. In upcoming examples we show that this formulation of Morse decomposition is convenient in many ways and is the correct order-theoretic formulation of Conley's concept of Morse decomposition.

Example 3 The third example arises from a study of the dynamics of a nonlinear population model, $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-0.1(x_1+x_2)} \\ 0.7 x_1 \end{pmatrix} \tag{2}$$

where the parameter $\theta = (\theta_1, \theta_2) \in [8, 37] \times [3, 50]$. Because this is an ecological model it is unreasonable to assume precise knowledge of parameters, thus it is of interest to understand what happens over large ranges of parameters. However, this system is known to exhibit extremely complicated dynamics and bifurcations, cf. [38]. Therefore, from the perspective of applications a classical analytic analysis at a given parameter value is of limited value and any detailed analysis is dependent upon numerical computations.

The strategy adopted in [2] is to decompose parameter space into a uniform grid of 50×50 rectangles indexed by $\mathcal{Z}$. The portion of phase space on which computations are performed is a rectangle X that is divided uniformly into 2^{24} subrectangles that are indexed by $\mathcal{X}$. We use the notation $|\cdot|$ to pass from the indexing set to the topological region, i.e. given $\zeta \in \mathcal{Z}$, $|\zeta| \subset [8, 37] \times [3, 50]$ and given $\xi \in \mathcal{X}$, $|\xi| \subset X$. For a fixed $\zeta \in \mathcal{Z}$ rigorous bounds on f_θ are used to construct a combinatorial multivalued map $\mathcal{F}_\zeta : \mathcal{X} \rightrightarrows \mathcal{X}$ that satisfies $f_\theta(|\xi|) \subset \left|\mathcal{F}_\zeta(\xi)\right|$ for all $\theta \in |\zeta|$. The result of computations using $\mathcal{F}_\zeta$ produces a poset $\left(\mathsf{SC}(\mathcal{F}_\zeta), \leqslant_\zeta\right)$ which is defined as follows. The set $\mathsf{SC}(\mathcal{F}_\zeta)$ is the poset of strongly connected components of $\mathcal{F}_\zeta$ regarded as a digraph.

The above mentioned rigorous bounds imply the following result. For any fixed $\theta \in |\zeta|$ define $\mathsf{M}_\theta := \left\{M_\theta = \mathrm{Inv}(|\mathcal{S}|, f_\theta) \neq \varnothing \mid \mathcal{S} \in \mathsf{SC}(\mathcal{F}_\zeta)\right\}$, where $\mathrm{Inv}(|\mathcal{S}|, f_\theta)$ is defined to be the maximal invariant set in $|\mathcal{S}|$ under the dynamics f_θ, and

$$M_\theta \leqslant M'_\theta \iff \mathcal{S} \leqslant \mathcal{S}'.$$

Consequently, the application $M_\theta \mapsto \mathcal{S}$ defines the Morse decomposition $\pi : \mathsf{M}_\theta \hookrightarrow \mathsf{SC}(\mathcal{F}_\zeta)$ of the maximal invariant set in X under f_θ. As in Example 2 there are no a priori guarantees that M_θ is isomorphic to $\mathsf{SC}(\mathcal{F}_\zeta)$. The set $\mathsf{SC}(\mathcal{F}_\zeta)$ provides the organization of global dynamic information. Again, this suggests that the object of fundamental interest is the poset $\left(\mathsf{SC}(\mathcal{F}_\zeta), \leqslant_\zeta\right)$ that codifies the dynamics between regions $\left\{|\mathcal{S}| \mid \mathcal{S} \in \mathsf{SC}(\mathcal{F}_\zeta\right\}$. The latter is called a *Morse tessellation*, cf. Sect. 9.1 and $\pi : \mathsf{M}_\theta \hookrightarrow \mathsf{SC}(\mathcal{F}_\zeta)$ is a *tessellated Morse decomposition*, cf. Defn. 8.

The down-sets in the poset $\mathsf{SC}(\mathcal{F}_\zeta)$ yield the finite distributive lattice $\mathsf{O}(\mathsf{SC})$ which is isomorphic to the lattice of forward invariant sets $\mathsf{Invset}^+(\mathcal{F}_\zeta)$ for $\mathcal{F}_\zeta$, i.e. subsets $\mathcal{U} \subset \mathcal{X}$ such that $\mathcal{F}_\zeta(\mathcal{U}) \subset \mathcal{U}$, cf. App. 2 and [23]. The realization of a down-set gives an isolating block for f_θ, i.e. $f_\theta(|\mathcal{U}|) \subset \operatorname{int}|\mathcal{U}|$ for all $\theta \in |\zeta|$. These blocks form a finite lattice N_ζ of attracting blocks which is also referred to as an index lattice, cf. [15]. From this lattice it is possible to extract index pairs, and thus, for $\theta \in |\zeta|$ the Conley index of $|\mathcal{S}|$ can be determined for every $\mathcal{S} \in \mathsf{SC}(\mathcal{F}_\zeta)$. To be more specific, every $\mathcal{S} \in \mathsf{SC}(\mathcal{F}_\zeta)$ is uniquely determined by $\mathcal{S} = \mathcal{U} \smallsetminus \overleftarrow{\mathcal{U}}$, where $\mathcal{U}$ is a join-irreducible element in $\mathsf{Invset}^+(\mathcal{F}_\zeta)$ and $\overleftarrow{\mathcal{U}}$ its unique predecessor, cf. Sect. 8.1. Since $|\mathcal{S}|$ is an isolating neighborhood for all $\theta \in |\zeta|$, standard continuation arguments guarantee that the associated Conley index is independent of $\theta \in |\zeta|$, cf. [10]. The importance of this is that the Conley index provides information about the structure of the dynamics of $\operatorname{Inv}(|\mathcal{S}|, f_\theta)$, e.g. existence of fixed points, periodic orbits, chaotic dynamics, etc. cf. [32]. In particular, if the Conley index of $|\mathcal{S}|$ is nontrivial, then $\operatorname{Inv}(|\mathcal{S}|, f_\theta) \neq \varnothing$ for all $\theta \in |\zeta|$. However, a trivial Conley index allows for the possibility that for some values of $\theta \in \zeta$, $\operatorname{Inv}(|\mathcal{S}|, f_\theta) = \varnothing$, and for other values of $\theta \in \zeta$, $\operatorname{Inv}(|\mathcal{S}|, f_\theta) \neq \varnothing$. This indicates why using this framework rigorous global computations can be performed over large regions of parameter space without directly addressing the issue of bifurcations. This also suggests that in the setting of computational analysis of dynamics just considering Morse representations is too restrictive, but Morse decompositions have the required flexibility.

As in the case of Examples 1 and 2 , the poset structure of Definition 2 provides a language in which to understand the global dynamics of Example 3. In this example, the more detailed understanding of the dynamics makes use of an associated lattice of attracting blocks. But Definition 2 is necessarily more general and does not require any insight into this lattice structure. In fact, it is precisely because the explicit structure of the lattice of attracting blocks is unknown, that it is difficult to determine the global structure of connecting orbits for differential equations such as those of Examples 1 and 2 . We resolve this with the introduction of the concept of a tesselated Morse decomposition in Sect. 9 (see discussion below). We point out that in this setting the Morse representation always allows an associated lattice of attracting blocks and therefore a tessellated Morse decomposition,cf. [15,25,26].

Example 4 For the final example consider the interval $[-2, 2] \subset \mathbb{R}$ which is meant to represent a portion of phase space for some unknown process on which we can make experimental observations. We make two assumptions.

(i) The dynamics can be modeled by a continuous evolution equation.
(ii) Repeated experiments indicate that at $x = -2$ and $x = 1$ the vector field is positive, and at $x = -1$ and $x = 2$ the vector field is negative.

Observe that the finite distributive lattice N shown in Fig. 2(left) can be interpreted as a lattice of attracting blocks. This lattice contains considerable information about the dynamics, but as follows naturally from Birkhoff's theorem [12] is best organized by focusing on the poset of join irreducible elements $\{[-2, -1], [1, 2], [-2, 2]\}$ where the partial order is given by inclusion. Observe that this partial order is isomorphic to that shown in Fig. 2(right) wherein it is easily seen that the dynamics exhibits bistability. Furthermore, the regions $T(p_0) := [-2, -1] = \operatorname{cl}([-2, -1] \smallsetminus \varnothing)$, $T(p_+) := [1, 2] = \operatorname{cl}([1, 2] \smallsetminus \varnothing)$, and $T(p_-) := [-1, 1] = \operatorname{cl}([-2, 2] \smallsetminus ([-2, -1] \cup [1, 1]))$ defined in terms of the difference

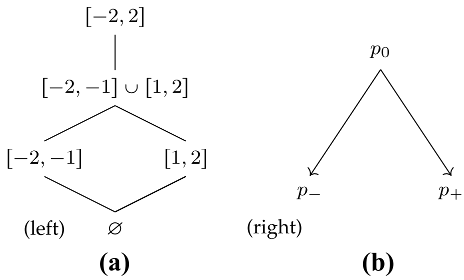


Fig. 2 **a** Lattice of attracting blocks for Example 4. **b** Partial order structure for join irreducible elements of lattice of attracting blocks

between join irreducible elements and their immediate predecessors forms a tessellation of phase space. This tessellation can in turn be used to compute Conley indices.

It is worth noting that this example is extremely general as the family of evolution equations to which these arguments may be applied include autonomous and nonautonomous differential equation, continuous random dynamical systems, and multivalued dynamical systems. But, essential to these arguments is the ability to pass between lattice and poset information.

In contrast to the first three examples, in Example 4 the starting point is a lattice of attracting blocks from which the poset structure that characterizes nonrecurrent dynamics is recovered. However, what all four examples have in common is the use of poset information as a tool for organizing global dynamics and the need (explicit or implicit) for lattice information to derive finer structures of dynamics.

Definition 2 captures the essential poset information for all these examples. Our goal is to demonstrate that this definition arises naturally from our perspective that lattices provide an appropriate framework for organizing and characterizing global dynamics. However, we do not claim that this is obvious. To motivate why this is a challenge consider Example 4. Here there is no choice, but to start with the lattice of attracting blocks. However, the definition of the set $\{T(p)\}$ that provides a tessellation of phase space made use of set difference which is not an operation with the algebraic setting of lattice theory. Recall that in a Boolean algebra $(\mathsf{B}, \vee, \wedge, ^{\neg}, 0, 1)$ the derived operation *set-difference* on B is given by $a \smallsetminus b := a \wedge b^{\neg}$. A major portion of this paper is dedicated to the construction and characterization of the natural analogue for bounded, distributive lattices.

In particular, in Sect. 3, given a bounded, distributive lattice L we define a notion of set-difference, called the *canonical Conley form on* L via $\mathsf{B}(\mathsf{L})$, where $\mathsf{B}(\mathsf{L})$ denotes the Booleanization of L, cf. Sect. 2. There is a natural embedding $j \colon \mathsf{L} \to \mathsf{B}(\mathsf{L})$, and since $\mathsf{B}(\mathsf{L})$ is a Boolean algebra, we define the canonical Conley form, $\mathsf{C}^{\mathsf{L}} \colon \mathsf{L} \times \mathsf{L} \to \mathsf{B}(\mathsf{L})$, by

$$\mathsf{C}^{\mathsf{L}}(a, b) := j(a) \smallsetminus j(b).$$

An important observation is that the range $\mathsf{B}^{\updownarrow}(\mathsf{L}) := \mathsf{C}^{\mathsf{L}}(\mathsf{L} \times \mathsf{L})$ is a meet semilattice.

However, abstract knowledge of the existence of C^{L} and $\mathsf{B}^{\updownarrow}(\mathsf{L})$ is of limited value. In Sect. 4 we introduce *Conley forms on* L *in* I. These are semilattice morphisms $\mathsf{C} \colon \mathsf{L} \times \mathsf{L} \to \mathsf{I}$ where I is an explicit meet semilattice consisting of structures of interest, and $\mathsf{C} = \gamma \circ \mathsf{C}^{\mathsf{L}}$ for some injective semilattice homomorphism γ. Any Conley form on L is isomorphic to C^{L} and is remarkably characterized by the following three properties:

(Absorption) $\mathsf{C}(a \vee b, a) = \mathsf{C}(b, a)$ and $\mathsf{C}(a, a \wedge b) = \mathsf{C}(a, b)$ for all $a, b \in \mathsf{L}$.
(Distributivity) $\mathsf{C}(a \wedge c, b \vee d) = \mathsf{C}(a, b) \wedge \mathsf{C}(c, d)$ for all $a, b, c, d \in \mathsf{L}$;
(Monotonicity) $\mathsf{C}(a, b) = \mathsf{C}(0, 1)$ implies $a \leqslant b$ for $a, b \in \mathsf{L}$.

This leads to the following result, cf. Theorem 2.

Theorem 1 *Let* $\mathrm{C}\colon \mathrm{L}\times\mathrm{L}\to\mathrm{I}$ *and* $\mathrm{C}'\colon \mathrm{L}\times\mathrm{L}\to\mathrm{I}'$ *be Conley forms. Then, there exists a meet semilattice isomorphism* $g\colon \mathrm{C}(\mathrm{L}\times\mathrm{L})\to\mathrm{C}'(\mathrm{L}\times\mathrm{L})$ *such that* $\mathrm{C}'=g\circ\mathrm{C}$.

Lattices of primary interest, such as the lattice of closed attracting blocks $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$ and the lattice of attractors $\mathsf{Att}(\varphi)$, are may be infinite. Thus, the proof of Theorem 1 relies on the compactness of the spectrum $\Sigma(\mathsf{L})$ in the Priestley topology, cf. Sect. 2.

As is demonstrated in Sect. 5, with the Conley form we are able to identify Morse sets (see Definition 1) from the lattice structures of attractors and repellers. In particular, in Example 7 we use the Conley form on $\mathsf{Att}(\varphi)$ in $\mathsf{Invset}(\varphi)$, the lattice of invariant sets, to define $\mathsf{Morse}(\varphi)$, the semilattice of Morse sets of φ. More precisely, the range of the Conley form on $\mathsf{Att}(\varphi)$ defines the semilattice of Morse sets

$$\mathsf{Morse}(\varphi):=\mathsf{C}_{\mathsf{Att}}(\mathsf{Att}(\varphi)\times\mathsf{Att}(\varphi)) \tag{3}$$

where $\mathsf{C}_{\mathsf{Att}}(A,A'):=A\cap A'^{*}$ and A'^{*} is the dual repeller to the attractor A', cf. [10, II.5.3.E]. Example 8 uses the same formalism to define Morse sets for the dynamics generated by a combinatorial multivalued map or relation $\mathcal{F}$.

As is shown in Sect. 6, homomorphisms between bounded distributive lattices lead to homomorphisms between Conley forms. This provides us with a tool to analyze the global structure of invariant sets. For example, if $\mathsf{INbhd}(\varphi)$ and $\mathsf{Isol}(\varphi)$ are the meet semilattices of isolating neighborhoods and isolated invariant sets respectively, cf. Sect. 7.2, then the following diagram shows how the Conley forms on $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$ and $\mathsf{Att}(\varphi)$ define isolated invariant sets

$$\begin{array}{ccc} \mathsf{ABlock}_{\mathscr{C}}(\varphi)\times\mathsf{ABlock}_{\mathscr{C}}(\varphi) & \xrightarrow{\ \mathsf{C}\ } & \mathsf{INbhd}(\varphi) \\ {\scriptstyle \omega\times\omega}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Inv}} \\ \mathsf{Att}(\varphi)\times\mathsf{Att}(\varphi) & \xrightarrow{\ \mathsf{C}_{\mathsf{Att}}\ } & \mathsf{Isol}(\varphi) \end{array} \tag{4}$$

where

$$\mathsf{C}(U,U')=U\cap U'^{c}=U\setminus U'\text{for }.U,U'\in\mathsf{ABlock}_{\mathscr{C}}(\varphi).$$

The remainder of the paper uses the tools developed in Sects. 2–4 and 6 to provide algebraic representations of structures of global dynamics via appropriately chosen Conley forms. In Sect. 7 we provide partitions of phase space, called *Morse tiles*, in the context of continuous and combinatorial dynamics. Furthermore, we discuss Morse tiles in the context of regular closed sets as these provide a useful computational structure.

In Sects. 8 and 9 we turn to the goal mentioned earlier in this introduction: an explicit description of the relationship between the order relations on Morse decompositions and the lattice structures of attractors and repellers. As is discussed in Sect. 8.1, every Morse representation can be generated from a finite sublattice of attractors A with the associated Morse representation given by

$$\mathsf{M}(\mathsf{A}):=\left\{\mathsf{C}_{\mathsf{Att}}(A,\overleftarrow{A})\mid A\in\mathsf{J}(\mathsf{A})\right\}$$

where $\mathsf{J}(\mathsf{A})$ denotes the set of join-irreducible elements of A, and $\overleftarrow{A}$ is the unique immediate predecessor of A in A. Given a finite sublattice N of attracting blocks, and the surjective homomorphism $\omega\colon\mathsf{N}\twoheadrightarrow\mathsf{A}$, we obtain a dual order-embedding $\pi:\mathsf{M}(\mathsf{A})\hookrightarrow\mathsf{T}(\mathsf{N})$ where

$$\mathsf{T}(\mathsf{N}):=\{\mathsf{C}^{b}(N,\overleftarrow{N})=N\setminus\overleftarrow{N}\mid N\in\mathsf{J}(\mathsf{N})\}$$

via the Conley form on N. The map $\pi : \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$[3] is dual to $\omega \colon \mathsf{N} \twoheadrightarrow \mathsf{A}$ and is referred to as a *tesselated Morse decomposition*, cf. Theorem 6 and Definition 2. In Corollary 4 we give a definition of Morse tessellation and tessellated Morse decomposition independent of a lattice of attracting blocks/neighborhoods.

Remark 1 A Morse decomposition is the multiple set analogue of an isolated invariant set S and its (closed) isolating neighborhood $N \subset X$, i.e. a closed subset $N \subset X$ such that $S = \mathrm{Inv}(N) \subset \mathrm{int}\, N$. Given a Morse tessellation $(\mathsf{T}, \leqslant)$ one is tempted to consider $\{\mathrm{Inv}(T) \mid T \in \mathsf{T}\}$ as candidate Morse decomposition. However, the latter is not a poset in general since the empty set may occur multiple times. Instead by considering $\mathsf{M}{:=}\{\mathrm{Inv}(T) \neq \varnothing \mid T \in \mathsf{T}\}$ as the Morse representation realized by T we obtain the order embedding $\pi : \mathsf{M} \hookrightarrow \mathsf{T}$ by setting $\mathrm{Inv}(T) \leqslant \mathrm{Inv}(T')$ if and only if $T \leqslant T'$. This is the correct order theoretic notion of Morse decomposition. The mapping $T \mapsto \mathrm{Inv}(T)$ acts as left-inverse to π and as semilattice homomorphism $\mathrm{Inv} \colon \mathsf{MTile}(\varphi) \to \mathsf{Morse}(\varphi)$, cf. Sect. 7.2. In Example 2 we described the notion of Morse decomposition $\pi : \mathsf{M} \hookrightarrow \mathsf{P}$ with respect to an abstract poset P used to define invariant sets. If we employ the main result in [26] we find a Morse tessellations $\mathsf{T} \cong \mathsf{P}$ such that the Morse decompositions may be regarded as tessellated Morse decompositions.

As is mentioned earlier, the ideas from Conley theory are being used in the context of rigorous computations and data analysis, and thus a fundamental question is how does the dynamics captured by a relation $\mathcal{F}$ compare to the dynamics of a continuous system φ? We address this question in Sect. 9.2. Closed regular sets, e.g. triangulations or regular CW-complexes, provide a wide variety of discretizations of phase space for continuous dynamical systems, and as is shown in Sect. 7.3, is rich enough to capture the lattice of attractors of a continuous system φ. This leads us to consider the span

$$\mathscr{R}(X) \xleftarrow{\ \supset\ } \mathsf{ABlock}_{\mathscr{R}}(\varphi) \xrightarrow{\ \omega\ } \mathsf{Att}(\varphi), \tag{5}$$

where $\mathscr{R}(X)$ are the regular closed sets in X and $\mathsf{ABlock}_{\mathscr{R}}(\varphi) \subset \mathsf{ABlock}_{\mathscr{C}}(\varphi)$ are the regular closed attracting blocks for φ, cf. App. 3. Let $\mathscr{R}_0$ be a finite subalgebra of $\mathscr{R}(X)$. Let $\mathcal{X}$ be an indexing set for the atoms of $\mathscr{R}_0$. We use $|\cdot| \colon \mathcal{X} \to \mathscr{R}_0$ to denote the geometric realization, i.e. $|\xi|$ is the associated atom in $\mathscr{R}_0$. Observe that this extends to $|\cdot| \colon \mathsf{Set}(\mathcal{X}) \to \mathscr{R}_0 \subset \mathscr{R}(X)$. Finally, consider a binary relation $\mathcal{F} \subset \mathcal{X} \times \mathcal{X}$. Ideally, we have the existence of the following commutative diagram, that we refer to as a *commutative combinatorial model* for φ:

$$\begin{array}{ccccc} \mathscr{R}(X) & \xleftarrow{\ \subset\ } & \mathsf{ABlock}_{\mathscr{R}}(\varphi) & \xrightarrow{\ \omega\ } & \mathsf{Att}(\varphi) \\ \uparrow {\scriptstyle |\cdot|} & & \uparrow {\scriptstyle |\cdot|} & & \uparrow {\scriptstyle \omega(|\cdot|)} \\ \mathsf{Set}(\mathcal{X}) & \xleftarrow{\ \subset\ } & \mathsf{Invset}^+(\mathcal{F}) & \xrightarrow{\ \boldsymbol{\omega}\ } & \mathsf{Att}(\mathcal{F}), \end{array} \tag{6}$$

where $\mathsf{Set}(\mathcal{X})$ denotes the lattice of subsets of $\mathcal{X}$. Theorem 7 provides an exact characterization of the properties of $\mathcal{F}$ such that (6) commutes.

The above description takes the perspective that φ is the object of primary importance and $\mathcal{F}$ is derived in order to study the dynamics of φ computationally. However, if one begins with data, then there are a variety of methods by which one can derive a relation $\mathcal{F}$. In this setting Theorem 7 provides constraints on continuous models φ that are compatible with the data. An open problem, but of increasing relevance in an age of data driven science, is to derive techniques for choices of maps or differential equations that generate φ.

[3] cf. Eqn. (44) for a precise definition of the map π.

We conclude by noting that we have restricted our attention in this paper to single-valued, continuous dynamical systems and to combinatorial dynamical systems. There are, of course, other models for continuous dynamics, e.g. set-valued [1,3,4,29], and for combinatorial dynamics, e.g. combinatorial vector fields [22,34]. It is our belief that the algebraic structures developed in this paper can be applied equally well in these other settings.

2 Booleanization

In this section we describe two algebraic principles, Booleanization and duality. These tools are fundamental to the description of the algebraic structures of global dynamics. Denote the categories of bounded, distributive lattices and posets by **BDLat** and **Poset** respectively. A bounded, distributive lattice has unique neutral elements 0 and 1, and in **BDLat** all lattice homomorphisms preserve 0 and 1, and all sublattices contain 0 and 1.

There are two functors that relate **BDLat** and **Poset**. The *down-set functor* $\mathsf{O}\colon \mathbf{Poset} \Rightarrow \mathbf{BDLat}$ is a contravariant functor that assigns to a poset P the bounded, distributive lattice of down-sets denoted by $(\mathsf{O}(\mathsf{P}), \cup, \cap)$. Recall that a *down-set* I in a poset P is defined via the property that $p \in I$ and $q \leqslant p$ implies $q \in I$. The *spectral functor* $\Sigma\colon \mathbf{BDLat} \Rightarrow \mathbf{Poset}$ is a contravariant functor that assigns to a bounded, distributive lattice L the poset $\big(\Sigma(\mathsf{L}), \subset\big)$ of the prime ideals in L called the *spectrum* of L. Recall that an *ideal* in a bounded, distributive lattice is a down-set I that is closed under join, i.e. $a, b \in I$ implies $a \vee b \in I$. An ideal I is a *prime ideal* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$. The prime ideals are exactly the pre-images $I = f^{-1}(0)$ where $f\colon \mathsf{L} \to \mathbf{2}$ is a lattice homomorphism and $\mathbf{2}$ is the lattice of two elements $\{0, 1\}$. For a detailed treatment of basic lattice theory and the functors O and Σ, see [12,36]. A classical result due to Birkhoff states that a bounded, distributive lattice is isomorphic to a sublattice of $\mathsf{Set}(\Sigma(\mathsf{L}))$, where $\mathsf{Set}(\Sigma(\mathsf{L}))$ denotes the algebra of subsets of $\Sigma(\mathsf{L})$. The map

$$
\begin{aligned}
j\colon \mathsf{L} &\to \mathsf{O}(\Sigma(\mathsf{L}))\\
a &\mapsto j(a) = \{I \in \Sigma(\mathsf{L}) \mid a \notin I\}
\end{aligned}
$$

defines such an embedding. The map j is not surjective in general. However, when L is finite, it is surjective, and this fact is called the Birkhoff Representation Theorem for finite, distributive lattices [36, Theorem 6.6]. In the case that L is a Boolean algebra, Stone introduced a topology on the spectrum in order to characterize the image of j, and this characterization is known as the Stone Representation Theorem [36, Theorem 10.18]. The idea underlying the Stone representation is that since the clopen sets in a topological space form a Boolean algebra, one can topologize $\Sigma(\mathsf{L})$ so that the image of j is the algebra of clopen sets, cf. [12,36].

For bounded, distributive lattices, Priestley introduced a topology on the spectrum that determines the image of j. Priestley's topology is induced by the basis

$$
\{j(a) \smallsetminus j(b) \mid a, b \in \mathsf{L}\},
$$

where $j(a) \smallsetminus j(b) := j(a) \cap j(b)^c$ is set-difference. Since $j(a)$, $j(b)^c$ are basic open sets, by choosing b as the zero element of the lattice L (respectively a as the unit element of L) we see that for every $a, b \in \mathsf{L}$ both $j(a)$ and the complement of $j(b)$, as basic sets, are open. Hence, every set of the form $j(a)$ for $a \in \mathsf{L}$ is clopen. Note that each $j(a)$ is a down-set so that the image of j is a sublattice of the down-sets of $\Sigma(\mathsf{L})$. The Priestley Representation Theorem characterizes the image of j as the clopen down-sets of $\Sigma(\mathsf{L})$ denoted by

$$
\mathsf{O}^{\mathrm{clp}}(\Sigma(\mathsf{L})) = \mathsf{B}^{\downarrow}(\mathsf{L}) := \{j(a) \mid a \in \mathsf{L}\},
$$

and L is isomorphic to $\mathsf{B}^{\downarrow}(\mathsf{L})$ via the map $j\colon \mathsf{L} \to \mathsf{B}^{\downarrow}(\mathsf{L})$, cf. [12,36]. Here $\mathsf{O}^{\mathrm{clp}}$ may be regarded a contravariant functor $\mathsf{O}^{\mathrm{clp}}\colon \mathbf{Pries} \Rightarrow \mathbf{BDLat}$ from category of Priestley spaces, **Pries**, to the category of bounded, distributive lattices, **BDLat**, cf. [12,36]. If L is a finite lattice then so is $\Sigma(\mathsf{L})$. Therefore the Priestley topology on $\Sigma(\mathsf{L})$ is discrete in that case. As a consequence, every subset of prime ideals is clopen, and the image of j consists of all down sets of prime ideals.

The spectrum $(\Sigma(\mathsf{L}), \subseteq)$ is a poset and with the Priestley topology the spectrum is a compact and totally order-separated topological space, called a *Priestley space*. Priestley spaces are necessarily Hausdorff and 0-dimensional. The Priestley Representation Theorem states that **BDLat** is dually equivalent to **Pries**. Birkhoff's theorem motivates the question of obtaining a smallest Boolean algebra in which the lattice embeds. Such a Boolean algebra is called a *Booleanization*, and a general procedure to obtain a specific Booleanization is based on the Priestley Representation Theorem [36, Theorem 10.15], cf. [5,30], [39, Defn. 9.5.5].

Let $\mathsf{B}(\mathsf{L})$ be the Boolean algebra of all clopen subsets of $\Sigma(\mathsf{L})$. The above construction yields the following Booleanization theorem.

Proposition 1 *(Theorem 10.19 in [36]) For every bounded distributive lattice* L*, the map*

$$\begin{aligned} j\colon \mathsf{L} &\to \mathsf{B}(\mathsf{L}) \\ a &\mapsto j(a) = \{\mathrm{I} \in \Sigma(\mathsf{L}) \mid a \notin \mathrm{I}\} \end{aligned} \tag{7}$$

is the unique lattice monomorphism with the property that for every homomorphism $h\colon \mathsf{L} \to \mathsf{E}$ *to a Boolean algebra* E *there exists a unique lattice homomorphism* $\mathsf{B}(h)\colon \mathsf{B}(\mathsf{L}) \to \mathsf{E}$ *such that* $\mathsf{B}(h) \circ j = h$*. The Boolean algebra* $\mathsf{B}(\mathsf{L})$ *is called the Booleanization of* L *and the mapping* $\mathsf{B}(h)\colon \mathsf{B}(\mathsf{L}) \to \mathsf{E}$ *is Boolean.*

Remark 2 In the case that L is a finite, distributive lattice then the Booleanization $\mathsf{B}(\mathsf{L})$ is the Boolean algebra of all subsets of $\Sigma(\mathsf{L})$.

Remark 3 Throughout the rest of this paper

$$j\colon \mathsf{L} \to \mathsf{B}(\mathsf{L})$$

denotes the specific lattice monomorphism of Proposition 1. Furthermore, when we are explicitly working with this monomorphism an element $a \in \mathsf{L}$ is denoted in *lower case* and its image in $\mathsf{B}(\mathsf{L})$ by A in *upper case* so that $A = j(a)$.

The Booleanization theorem above also applies to homomorphisms $h\colon \mathsf{K} \to \mathsf{L}$. Proposition 1 yields the following commutative diagram

$$\begin{array}{ccc} \mathsf{K} & \xrightarrow{\;h\;} & \mathsf{L} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ \mathsf{B}(\mathsf{K}) & \xrightarrow{\;\mathsf{B}(h)\;} & \mathsf{B}(\mathsf{L}) \end{array} \tag{8}$$

In particular $j\big(h(a)\big) = \mathsf{B}(h)(A)$ where $A = j(a)$.

Booleanization is a (covariant) functor $\mathsf{B}\colon \mathbf{BDLat} \Rightarrow \mathbf{Bool}$, left adjoint to the forgetful functor and which is obtained via the composition $\mathsf{Set}^{\mathrm{clp}} \circ \Sigma$, where $\mathsf{Set}^{\mathrm{clp}}\colon \mathbf{Pries} \Rightarrow \mathbf{Bool}$ is the clopen subset functor which gives the clopen subsets in a Priestley space and Σ is the spectral functor, cf. [36, Thm. 10.19], [39, Sect. 9.5]. Moreover, due to the compactness of the Priestley topology, each element of $\mathsf{B}(\mathsf{L})$ can be written as a finite union of the clopen convex sets

$$\mathsf{B}^{\updownarrow}(\mathsf{L}) := \{A \smallsetminus B \mid A, B \in \mathsf{B}^{\downarrow}(\mathsf{L})\},$$

cf. [36, Theorem 10.10] and [12, Lemma 11.22].

For finite, distributive lattices the spectrum $\Sigma(\mathsf{L})$ is order-isomorphic to the poset of join-irreducible elements $\mathsf{J}(\mathsf{L})$. A nonzero element $a \in \mathsf{L}$ is *join-irreducible* if a has a unique predecessor in L which is denoted by $\overleftarrow{a}$. The order-isomorphism $\mathsf{J}(\mathsf{L}) \to \Sigma(\mathsf{L})$ is given by the map $a \mapsto (\uparrow a)^c$. Every element in L can be written as a join of join-irreducible elements, for example

$$a = \bigvee_{\substack{a' \leqslant a \\ a' \in \mathsf{J}(\mathsf{L})}} a'.$$

Such join-representations are not unique, but each element has a unique irredundant join-representation, cf. [36, Thm. 4.30].

3 Lattice Forms

In general, lattices do not allow complements. However, in this section we define an operation on bounded, distributive lattices with the properties of the set difference operation.

Given a lattice L the set $\mathsf{L} \times \mathsf{L}$ has a natural meet semilattice structure defined by $(a, b) \wedge (c, d) := (a \wedge c, b \vee d)$, with neutral elements $0 = (0, 1)$ and $1 = (1, 0)$. It follows that

$$(a, b) \leqslant (c, d) \text{ if and only if } a \leqslant c \text{ and } b \leqslant d. \tag{9}$$

Definition 3 Let L be a lattice, and let I be a meet semilattice. A *lattice form on* L *represented in* I is a function $\rho \colon \mathsf{L} \times \mathsf{L} \to \mathsf{I}$ satisfying the property

(Absorption) $\rho(a \vee b, a) = \rho(b, a)$ and $\rho(a, a \wedge b) = \rho(a, b)$ for all $a, b \in \mathsf{L}$.

Example 5 If L is a Boolean algebra, then $(a, b) \mapsto \rho(a, b) = a \smallsetminus b := a \cap b^c$ defines a lattice form represented in L. This lattice form also satisfies the following properties

(Distributivity) $\rho(a \wedge c, b \vee d) = \rho(a, b) \wedge \rho(c, d)$ for all $a, b, c, d \in \mathsf{L}$;
(Monotonicity) $\rho(a, b) = \rho(0, 1)$ implies $a \leqslant b$ for $a, b \in \mathsf{L}$,

which are called *distributive* and *monotone* lattice forms respectively. A concrete example is the set difference operation in the Boolean algebra consisting of subsets of a set X denoted by $(\mathsf{Set}(X), \cup, \cap, {}^c, \varnothing, X)$.

Lemma 1 *From the distributivity property the following exchange property follows*

(Exchange) *:* $\rho(a, b) \wedge \rho(c, d) = \rho(a, d) \wedge \rho(c, b)$ *for all* $a, b, c, d \in \mathsf{L}$.

Proof From distributivity we have that

$$\rho(a, b) \wedge \rho(c, d) = \rho(a \wedge c, b \vee d) = \rho(a, d) \wedge \rho(c, b),$$

which proves the lemma. □

Proposition 2 *If* $\rho \colon \mathsf{L} \times \mathsf{L} \to \mathsf{I}$ *is a distributive lattice form, then* ρ *is a meet semilattice homomorphism.*

Proof By distributivity

$$\rho((a,b)\wedge(c,d)) = \rho(a\wedge c, b\vee d) = \rho(a,b)\wedge\rho(c,d),$$

which proves that ρ preserves meet operations. □

The following proposition lists a number of properties of distributive lattice forms.

Proposition 3 *Let $\rho\colon \mathsf{L}\times\mathsf{L}\to\mathsf{I}$ be a distributive lattice form. Then,*

(i) $\rho(a,b)\leqslant\rho(c,d)$ *for all* $a\leqslant c$ *and* $b\leqslant d$*;*
(ii) $\rho(0,1)\leqslant\rho(a,b)\leqslant\rho(1,0)$ *for all* $a,b\in\mathsf{L}$*;*
(iii) $\rho(0,a)=\rho(0,1)$ *and* $\rho(a,1)=\rho(0,1)$ *for all* $a\in\mathsf{L}$.

If in addition ρ is monotone, then

(iv) $\rho(a,b)=\rho(0,1)$ *if and only if* $a\leqslant b$*.*

Proof (i) From distributivity it follows that ρ is a meet semilattice homomorphism and thus order-preserving. The order on $\mathsf{L}\times\mathsf{L}$ is given by (9).
(ii) Apply (i) to the inequalities $0\leqslant a\leqslant 1$ and $1\leqslant b\leqslant 0$.
(iii) From absorption we have that $\rho(1,1)=\rho(1\vee 0,1)=\rho(0,1)$. Then by (i) and (ii) and absorption we have that

$$\rho(0,1)\leqslant\rho(0,a)\leqslant\rho(a,a)=\rho(a,a\wedge 1)=\rho(a,1)\leqslant\rho(1,1)=\rho(0,1).$$

(iv) By (i) and (ii), $\rho(0,1)\leqslant\rho(a,b)\leqslant\rho(a,a)=\rho(0,1)$, which shows that $\rho(a,b)=\rho(0,1)$. The other direction is monotonicity completing the proof. □

By Property (ii) in Proposition 3 the elements $0{:=}\rho(0,1)$ and $1{:=}\rho(1,0)$ are the neutral elements in the meet semilattice $\rho(\mathsf{L}\times\mathsf{L})\subset\mathsf{I}$. However, in general the semilattice I need not have neutral elements, and if there are neutral elements they need not coincide with $(0,1)$ and $(1,0)$.

Proposition 4 *Let $\rho\colon\mathsf{L}\times\mathsf{L}\to\mathsf{I}$ be a lattice form, and let $\mathsf{K}\subset\mathsf{L}$ be a sublattice. Then, the form $\rho|_{\mathsf{K}}\colon\mathsf{K}\times\mathsf{K}\to\mathsf{I}$ defined by restriction is a lattice form on K. The properties of distributivity and monotonicity are also preserved under restriction.*

Proof Absorption, distributivity, and monotonicity follow from the fact that K is a sublattice L. □

Definition 4 Let L be bounded, distributive lattice. The *canonical Conley form* on L is defined by

$$\begin{aligned} \mathsf{C}^\sigma\colon \mathsf{L}\times\mathsf{L} &\twoheadrightarrow \mathsf{B}^{\updownarrow}(\mathsf{L})\\ (a,b) &\mapsto \mathsf{C}^\sigma(a,b){:=}A\smallsetminus B, \end{aligned} \tag{10}$$

where $A=j(a)$ and $B=j(b)$ and $j\colon\mathsf{L}\to\mathsf{B}(\mathsf{L})$ is as defined in Proposition 1.

Proposition 5 *For a bounded, distributive lattice L the canonical Conley form is a monotone, distributive lattice form.*

Proof We first prove the absorption property. Observe that

$$\mathsf{C}^\sigma(a\vee b,a)=(A\cup B)\smallsetminus A=B\smallsetminus A=\mathsf{C}^\sigma(b,a)$$

and

$$\mathsf{C}^{\sigma}(a, a \wedge b) = A \smallsetminus (A \cap B) = A \smallsetminus B = \mathsf{C}^{\sigma}(a, b)$$

As for distributivity and monotonicity we have:

$$\begin{aligned}\mathsf{C}^{\sigma}(a \wedge c, b \vee d) &= (A \cap C) \smallsetminus (B \cup D) = (A \smallsetminus B) \cap (C \smallsetminus D)\\ &= \mathsf{C}^{\sigma}(a, b) \cap \mathsf{C}^{\sigma}(c, d)\end{aligned}$$

so that distributivity is satisfied. Observe that $\mathsf{C}^{\sigma}(a, b) = A \smallsetminus B = \varnothing$ implies that $A \subseteq B$. Since j is a lattice monomorphism, we conclude that a b. Hence, monotonicity is satisfied. □

4 The Conley form on Bounded, Distributive Lattices

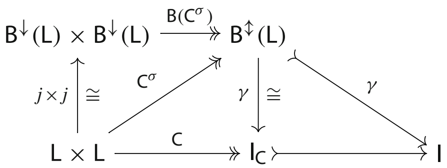
The canonical Conley form on a bounded distributive lattice L takes values in $\mathsf{B}^{\updownarrow}(\mathsf{L})$, an abstractly defined semilattice. For applications it is desirable to represent this form in particular meet semilattices. With this in mind let I be a meet semilattice, and $\gamma \colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \rightarrowtail \mathsf{I}$ be a meet injective semilattice homomorphism. Define

$$\begin{aligned}\mathsf{C} \colon \mathsf{L} \times \mathsf{L} &\to \mathsf{I}\\ (a, b) &\mapsto \mathsf{C}(a, b) := \gamma(A \smallsetminus B)\end{aligned} \tag{11}$$

and set $\mathsf{I}_{\mathsf{C}} := \gamma(\mathsf{B}^{\updownarrow}(\mathsf{L}))$. Observe that since γ is injective, $\gamma \colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}_{\mathsf{C}}$ is an isomorphism.

Lemma 2 C *is a monotone, distributive lattice form.*

Proof Since the canonical Conley form is distributive and monotone, the properties are transferred to C under the injection γ as in the following diagram



(12)

where $\mathsf{B}(\mathsf{C}^{\sigma})$ restricted to $\mathsf{B}^{\downarrow}(\mathsf{L}) \times \mathsf{B}^{\downarrow}(\mathsf{L})$ is given by $(j(a), j(b)) \mapsto j(a) \smallsetminus j(b)$ and is the Booleanization of C^{σ} via the composition $\mathsf{L} \times \mathsf{L} \twoheadrightarrow \mathsf{B}^{\updownarrow}(\mathsf{L}) \rightarrowtail \mathsf{B}(\mathsf{L})$. □

We now turn to the main result of this section that characterizes monotone, distributive lattice forms as representations of the canonical Conley form in a given meet semilattice.

Theorem 2 *Let* L *be a bounded, distributive lattice and let* I *be a meet semilattice. If* $\gamma \colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}$ *is a meet injective semilattice homomorphism, then* $\mathsf{C} = \gamma \circ \mathsf{C}^{\sigma}$ *is a monotone, distributive lattice form. Conversely, if* $\mathsf{C} \colon \mathsf{L} \times \mathsf{L} \to \mathsf{I}$ *is a monotone, distributive lattice form, then there exists an injective meet semilattice homomorphism* $\gamma \colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}$ *defined by*

$$\gamma(A \smallsetminus B) := \mathsf{C}(a, b), \tag{13}$$

such that $\mathsf{C} = \gamma \circ \mathsf{C}^{\sigma}$.

Proof Combining Lemma 2 with Lemmas 3 and 4 below proves the theorem. □

Lemma 3 *Let* $\rho\colon \mathsf{L} \times \mathsf{L} \to \mathsf{I}$ *be a lattice form. Then,* $\gamma\colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}$ *given by*

$$\gamma(A \setminus B) := \rho(a, b)$$

is a well-defined function.

Proof We need to prove that if $A \setminus B = A' \setminus B'$, then $\rho(a, b) = \rho(a', b')$. Observe that, since ρ is a lattice form, we have $\rho(a, b) = \rho(a, a \wedge b)$ and thus we may assume without loss of generality, by possibly replacing b by $a \wedge b$, that $b \leqslant a$. The same holds for $b' \leqslant a'$. Since $e := A \setminus B = A' \setminus B'$ we have

$$\begin{aligned} (A \cup A') \setminus (B \cup B') &= (A \cup A') \cap (B^c \cap B'^c) \\ &= \big((A \setminus B) \cap B'^c\big) \cup \big((A' \setminus B') \cap B^c\big) \\ &= (e \setminus B') \cup (e \setminus B) = e \cup e = e. \end{aligned}$$

Therefore assume without loss of generality that $B \subset A \subset A'$ and $B \subset B' \subset A'$. Since $A' = e \cup B'$ and $A = e \cup A$, we have that $A \cup B' = e \cup A \cup B' = A' \cup A = A'$. Similarly, $A = e \cup B$ and thus $A \cap B' = (e \cup B) \cap B' = B$. This implies

$$a' = a \vee b' \quad \text{and} \quad b = a \wedge b'. \tag{14}$$

Using the characterization in (14) and absorption, we have

$$\rho(a', b') = \rho(a \vee b', b') = \rho(a, b') = \rho(a, a \wedge b') = \rho(a, b),$$

which completes the proof. □

Lemma 4 *Let* $\mathsf{C}\colon \mathsf{L} \times \mathsf{L} \to \mathsf{I}$ *be a monotone, distributive lattice form. Then, the map* $\gamma\colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}$ *defined in (13) is an injective meet semilattice homomorphism.*

Proof We start with showing that γ preserves the meet operation. By Proposition 2, both C^σ and C induce meet semilattice homomorphisms $\mathsf{L} \times \mathsf{L} \to \mathsf{I}$. Then,

$$\begin{aligned} \gamma\big((A \setminus B) \cap (C \setminus D)\big) &= \gamma\big((A \cap C) \setminus (B \cup D)\big) = \mathsf{C}(a \wedge c, b \vee d) \\ &= \mathsf{C}(a, b) \wedge \mathsf{C}(c, d) = \gamma(A \setminus B) \wedge \gamma(C \setminus D). \end{aligned}$$

By Proposition 3(ii), the function γ satisfies $\gamma(\varnothing) = \mathsf{C}(0, 1) = 0$ and $\gamma(\Sigma(\mathsf{L})) = \mathsf{C}(1, 0) = 1$, the neutral elements in the range $\gamma(\mathsf{B}^{\updownarrow}(\mathsf{L}))$. Moreover, Proposition 3(iv) implies $\gamma(A \setminus B) = \mathsf{C}(a, b) = 0$ if and only if $a \quad b$ if and only if $A \setminus B = \mathsf{C}^\sigma(a, b) = \varnothing$. Thus, $\gamma^{-1}(0) = \varnothing$.

It remains to show that γ is injective. Suppose $a, b, a', b' \in \mathsf{L}$ such that

$$\mathsf{C}(a, b) = \gamma(A \setminus B) = \gamma(A' \setminus B') = \mathsf{C}(a', b') \quad \text{for} \quad A \setminus B \neq A' \setminus B'.$$

Since $\gamma^{-1}(0) = \varnothing$, it follows that $A \setminus B \neq \varnothing \neq A' \setminus B'$ and $\mathsf{C}(a, b) = \mathsf{C}(a', b') \neq 0$. Let $C = A \setminus B$, $D = A' \setminus B' \in \mathsf{B}^{\updownarrow}(\mathsf{L})$, then $\gamma(C) = \gamma(D) \neq 0$. Recall that $\mathsf{B}^{\updownarrow}(\mathsf{L}) \subset \mathsf{B}(\mathsf{L})$ and thus

$$(C \cup D) \setminus (C \cap D) = (C \setminus D) \cup (D \setminus C) \neq \varnothing,$$

since $C \neq D$. Therefore, either $C \setminus D \neq \varnothing$ or $D \setminus C \neq \varnothing$, and we assume without loss of generality the former holds. From the description of the Priestley topology in Sect. 2 every clopen subset of $\Sigma(\mathsf{L})$ is a finite union of elements of $\mathsf{B}^{\updownarrow}(\mathsf{L})$. Therefore there exist sets $\{E_i \in \mathsf{B}^{\updownarrow}(\mathsf{L}) \mid i = 1, \ldots, n\}$ such that $C \setminus D = \bigcup_i E_i$. This implies that for $j \in \{1, \ldots, n\}$

$$\varnothing \neq E_j \subset C \quad \text{and} \quad E_j \cap D = \varnothing.$$

Observe that, since γ is a semilattice homomorphism,

$$\gamma(E_j) = \gamma(E_j \cap C) = \gamma(E_j) \cap \gamma(C) = \gamma(E_j) \cap \gamma(D) = \gamma(E_j \cap D) = \gamma(\varnothing) = 0,$$

which is a contradiction since $\gamma^{-1}(0) = \varnothing$. □

Corollary 1 *Suppose* I, I' *are meet semilattices,* $\mathsf{C}, \mathsf{C}' \colon \mathsf{L} \times \mathsf{L} \to \mathsf{I}, \mathsf{I}'$ *are monotone, distributive lattice forms. Let* $\gamma, \gamma' \colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \rightarrowtail \mathsf{I}, \mathsf{I}'$ *be the meet injective semilattice homomorphisms given by Theorem 2. Then,* $\mathsf{C}' = g \circ \mathsf{C}$ *where*

$$g = \gamma' \circ \gamma^{-1} \colon \mathsf{I}_{\mathsf{C}} \to \mathsf{I}'_{\mathsf{C}'}, \tag{15}$$

is an isomorophism.

There exits only one monotone, distributive lattice form up to isomorphisms which yields the equivalence class of monotone, distributive lattice forms and leads to the following definition.

Definition 5 Let L be a bounded, distributive lattice, and let $\gamma \colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}$ be a injective meet semilattice homomorphism. The *Conley form on* L *via* γ is

$$\mathsf{C} := \gamma \circ \mathsf{C}^{\sigma} \colon \mathsf{L} \times \mathsf{L} \to \gamma(\mathsf{C}^{\sigma}(\mathsf{L} \times \mathsf{L})) = \mathsf{I}_{\mathsf{C}}.$$

We refer to I_{C} as the convexity *semilattice*. Often it is the meet semilattice I that is important, and the specific map γ is implicitly defined from Theorem 2, in which case we refer to *a representation of the Conley form in* I. If there is no ambiguity about the semilattice I or homomorphism γ we simply write $a - b := \mathsf{C}(a, b)$ to denote the Conley form for ease of notation.

Remark 4 Observe that if L is embedded in a Boolean algebra E, then there is a natural representation of the Conley form in E itself with $\mathsf{C}^b \colon \mathsf{L} \times \mathsf{L} \to \mathsf{E}$ given by

$$\mathsf{C}^b(a, b) = a \smallsetminus b$$

as in Example 5. The Conley form C^b also implies a natural decomposition of elements in L which have a finite join-representation of the form $a = \bigvee_{\substack{a' \leqslant a \\ a' \in \mathsf{J}(\mathsf{L})}} a'$. For such elements $a = \bigvee_{\substack{a' \leqslant a \\ a' \in \mathsf{J}(\mathsf{L})}} (a' \smallsetminus \overleftarrow{a}')$

The decomposition given in Remark 4 can be extended to lattices embedded into another lattice where the Boolean structure is replaced by a lattice form. Let L and K be bounded distributive lattices with $\mathsf{L} \subset \mathsf{K}$ and let $\rho \colon \mathsf{L} \times \mathsf{L} \to \mathsf{K}$ be a lattice form with the following additivity property

(Additivity) $\rho(a, b) \vee b = a$, for all $b \leqslant a$.

This yields the following extension of the decomposition statement in Remark 4.

Proposition 6 *If* $a = \bigvee_{\substack{a' \leqslant a \\ a' \in \mathsf{J}(\mathsf{L})}} a'$ *is finite join-representation, then*

$$a = \bigvee_{\substack{a' \leqslant a \\ a' \in \mathsf{J}(\mathsf{L})}} \rho(a', \overleftarrow{a}') = \bigvee_{\substack{a' \leqslant a \\ a' \in \mathsf{J}(\mathsf{L})}} \gamma(a' - \overleftarrow{a}'), \tag{16}$$

where $a - b$ *is the Conley form on* L *in a semilattice* I *and* $\gamma \colon \mathsf{I} \to \mathsf{K}$ *is given in Lemma 3 and Corollary 1.*

Proof Let a'' be a maximal element in $\{a' \in \mathsf{J}(\mathsf{L}) \mid a' \leqslant a\}$, then $\overleftarrow{a}'' \leqslant \bigvee\{a' \in \mathsf{J}(\mathsf{L}) \mid a' \leqslant a, a' \neq a''\}$. The additivity property of ρ and induction on a' give

$$\begin{aligned} a &= a'' \vee \bigvee\{a' \in \mathsf{J}(\mathsf{L}) \mid a' \leqslant a, a' \neq a''\} \\ &= \rho(a'', \overleftarrow{a}'') \vee \bigvee\{a' \in \mathsf{J}(\mathsf{L}) \mid a' \leqslant a, a' \neq a''\} = \bigvee_{\substack{a' \leqslant a \\ a' \in \mathsf{J}(\mathsf{L})}} \rho(a', \overleftarrow{a}'). \end{aligned}$$

The latter statement in (16) follows from Lemma 3 and Corollary 1. □

Remark 5 If C is a Conley form on L represented in a semilattice I, then L is naturally embedded in the convexity semilattice I_C, it has a natural dual lattice in I_C and the notion of 'complement' or 'dual' is well-defined. The embedding of L into I is given by $a \mapsto \mathsf{C}(a, 0)$ and the dual of a is defined as $a^* := \mathsf{C}(1, a)$. The dual lattice is given as $\mathsf{L}^* = \{a^* \mid a \in \mathsf{L}\}$. As a consequence, L and L^* may be regarded as lattices in the same 'universe' I. Note that from distributivity we have that

$$\mathsf{C}(a, b) = \mathsf{C}(a \wedge 1, 0 \vee b) = \mathsf{C}(a, 0) \wedge \mathsf{C}(1, b) = a \wedge b^*,$$

which proves that every Conley form can be characterized this way. For a homomorphism $h \colon \mathsf{K} \to \mathsf{L}$ there exists an induced dual anti-homomorphism $h^* \colon \mathsf{K}^* \to \mathsf{L}^*$ given by $h^*(a^*) = h(a)^*$ and

$$\begin{array}{ccc} \mathsf{K} & \xrightarrow{\ *\ } & \mathsf{K}^* \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathsf{L} & \xrightarrow{\ *\ } & \mathsf{L}^* \end{array} \tag{17}$$

5 Examples of Conley Forms

In this paper we are interested in representations of the Conley form in the context of lattices of attractors and repellers, and we now provide some examples.

Example 6 In the context of invertible dynamical systems, attractors, repellers, and invariant sets all have lattice structures induced by intersection and union in the Boolean algebra $\mathsf{Set}(X)$. Indeed, $\mathsf{Invset}(\varphi)$ is a complete (atomic) Boolean subalgebra of $\mathsf{Set}(X)$, and it contains all attractors and repellers, cf. App. 1. Therefore, the Booleanizations of these lattices are isomorphic to a subalgebra of $\mathsf{Invset}(\varphi)$ by Proposition 1. In particular, in light of Remark 4

$$\begin{aligned} \mathsf{C}^b \colon \mathsf{Invset}(\varphi) \times \mathsf{Invset}(\varphi) &\to \mathsf{Invset}(\varphi) \\ (S, S') &\mapsto \mathsf{C}^b(S, S') = S \smallsetminus S'. \end{aligned} \tag{18}$$

By Proposition 4 the restrictions of C^b to $\mathsf{Att}(\varphi)$ and $\mathsf{Rep}(\varphi)$ are representations of the Conley forms of these lattices in $\mathsf{Invset}(\varphi)$.

Example 7 Let $\varphi \colon \mathbb{T} \times X \to X$ be an invertible dynamical system on a compact metric space. Let $S \subset X$ be a compact invariant set and define the unstable set: $W^u(S) := \{x \in X \mid \alpha(x) \subset S\}$, cf. App. 1. For compact invariant sets S, S' we have

$$W^u(S \cap S') = W^u(S) \cap W^u(S'),$$

cf. App. 1[Lemma 18] and App. 1[Remark 17]. By Example 6, $\mathsf{C}^b\colon \mathsf{Att}(\varphi)\times\mathsf{Att}(\varphi)\to\mathsf{Invset}(\varphi)$ given in (18) is a representation of the Conley form on $\mathsf{Att}(\varphi)$ in $\mathsf{Invset}(\varphi)$. To obtain an explict formula for C^b in terms of W^u, observe that if $A\in\mathsf{Att}(\varphi)$, then $W^u(A)=A$, and furthermore, by [25, Theorem 3.19] $A^c=W^u(A^*)$ where A^* is the dual repeller of A. Therefore,

$$\begin{aligned}\mathsf{C}^b(A,A') &= A\smallsetminus A' = A\cap A'^c\\ &= W^u(A)\cap W^u\left(A'^*\right)\\ &= W^u(A\cap A'^*).\end{aligned}$$

Clearly, $A\cap A'^*\in\mathsf{Morse}(\varphi)=\{A\cap R\mid A\in\mathsf{Att}(\varphi),\ R\in\mathsf{Rep}(\varphi)\}$, where $\mathsf{Morse}(\varphi)$ is the semilattice of Morse sets defined in (3). Since $\mathsf{Morse}(\varphi)$ is a subsemilattice of $\mathsf{Invset}(\varphi)$ and $W^u\colon\mathsf{Morse}(\varphi)\to\mathsf{Invset}(\varphi)$ is injective, cf. App. 1[Lemma 21],

$$\mathsf{C}_{\mathsf{Att}}(A,A') = A - A' := A\cap A'^* \tag{19}$$

is another (isomorphic) representation of the Conley form of $\mathsf{Att}(\varphi)$ in $\mathsf{Invset}(\varphi)$, cf. Theorem 2 and Corollary 1. Since the dual operator $^*\colon\mathsf{Att}(\varphi)\to\mathsf{Rep}(\varphi)$ is an anti-isomorphism, c.f. [25, Proposition 4.7],

$$\mathsf{Morse}(\varphi)=\mathsf{C}_{\mathsf{Att}}(\mathsf{Att}(\varphi)\times\mathsf{Att}(\varphi)).$$

Example 8 Consider a binary relation $\mathcal{F}\subset\mathcal{X}\times\mathcal{X}$ on a finite set $\mathcal{X}$, see App. 2. Theorem 3 establishes that the lattice form $\mathsf{C}_{\mathsf{Att}}(\mathcal{A},\mathcal{A}'):=\mathcal{A}\cap\mathcal{A}'^*$ is a representation of the Conley form on $\mathsf{Att}(\mathcal{F})$ in $\mathsf{Invset}(\mathcal{F})$. By [26, Diagram (5)] the dual operator $^*\colon\mathsf{Att}(\mathcal{F})\to\mathsf{Rep}(\mathcal{F})$ is an anti-isomorphism, therefore, as in Example 7

$$\mathsf{C}_{\mathsf{Att}}(\mathsf{Att}(\mathcal{F})\times\mathsf{Att}(\mathcal{F}))=\mathsf{Morse}(\mathcal{F}):=\{\mathcal{A}\cap\mathcal{R}\mid\mathcal{A}\in\mathsf{Att}(\mathcal{F}),\mathcal{R}\in\mathsf{Rep}(\mathcal{F})\},$$

where sets of the form $\mathcal{M}=\mathcal{A}\cap\mathcal{R}$ are called *Morse sets*.

Lemma 5 *Morse sets in* $\mathsf{Morse}(\mathcal{F})$ *are invariant.*

Proof Suppose $\xi\in\mathcal{A}\cap\mathcal{R}$. Since $\mathcal{A}\in\mathsf{Att}(\mathcal{F})$, we have $\mathcal{F}(\mathcal{A})=\mathcal{A}$. Therefore, there exists $\eta\in\mathcal{A}$ such that $\xi\in\mathcal{F}(\eta)$. Similarly, $\mathcal{R}\in\mathsf{Rep}(\mathcal{F})$ and hence $\mathcal{F}^{-1}(\mathcal{R})=\mathcal{R}$. Thus, $\eta\in\mathcal{F}^{-1}(\xi)\subset\mathcal{F}^{-1}(\mathcal{R})=\mathcal{R}$, and hence $\eta\in\mathcal{A}\cap\mathcal{R}$ and $\xi\in\mathcal{F}(\mathcal{A}\cap\mathcal{R})$. Therefore $\mathcal{A}\cap\mathcal{R}\subset\mathcal{F}(\mathcal{A}\cap\mathcal{R})$. The same argument applied to $\mathcal{F}^{-1}$ gives $\mathcal{A}\cap\mathcal{R}\subset\mathcal{F}^{-1}(\mathcal{A}\cap\mathcal{R})$, which implies that $\mathcal{A}\cap\mathcal{R}\in\mathsf{Invset}(\mathcal{F})$ by [24, Proposition 3.4]. □

Lemma 6 *Let* $\mathcal{U}\in\mathsf{Invset}^+(\mathcal{F})$ *and* $\mathcal{V}\in\mathsf{Invset}^-(\mathcal{F})$. *Then,* $\mathrm{Inv}(\mathcal{U}\cap\mathcal{V})=\boldsymbol{\omega}(\mathcal{U})\cap\boldsymbol{\alpha}(\mathcal{V})$.

Proof Since $\boldsymbol{\omega}(\mathcal{U})\cap\boldsymbol{\alpha}(\mathcal{V})\subset\mathcal{U}\cap\mathcal{V}$, Lemma 5 implies that $\boldsymbol{\omega}(\mathcal{U})\cap\boldsymbol{\alpha}(\mathcal{V})\subset\mathrm{Inv}(\mathcal{U}\cap\mathcal{V})$. Let $\mathcal{S}\subset\mathcal{U}\cap\mathcal{V}$ be an invariant set. Since $\mathrm{Inv}(\mathcal{U})\subset\boldsymbol{\omega}(\mathcal{U})\subset\mathcal{U}$ when $\mathcal{U}$ is forward invariant, $\mathcal{S}\subset\mathrm{Inv}(\mathcal{U})\subset\boldsymbol{\omega}(\mathcal{U})$. Similarly, $\mathcal{S}\subset\boldsymbol{\alpha}(\mathcal{V})$ and therefore $\mathcal{S}\subset\boldsymbol{\omega}(\mathcal{U})\cap\boldsymbol{\alpha}(\mathcal{V})$. □

Theorem 3 *The lattice form*

$$\mathsf{C}_{\mathsf{Att}}(\mathcal{A},\mathcal{A}'):=\mathcal{A}\cap\mathcal{A}'^* \tag{20}$$

is a representation of the Conley form in $\mathsf{Invset}(\mathcal{F})$.

Proof For attractors $\mathcal{A},\mathcal{A}'$ there exist $\mathcal{U},\mathcal{U}'\in\mathsf{Invset}^+(\mathcal{F})$ such that $\mathcal{A}=\boldsymbol{\omega}(\mathcal{U})\subset\mathcal{U}$ and $\mathcal{A}'=\boldsymbol{\omega}(\mathcal{U}')\subset\mathcal{U}'$. For the dual repellers $\mathcal{A}^*,\mathcal{A}'^*$ we have $\mathcal{A}^*=\boldsymbol{\alpha}(\mathcal{U}^c)\subset\mathcal{U}^c$ and $\mathcal{A}'*=$

$\boldsymbol{\alpha}(\mathcal{U}'^c) \subset \mathcal{U}'^c$. In particular we can choose $\mathcal{U} = \mathcal{A}$ and $\mathcal{U}' = \mathcal{A}'$. Observe that $\mathcal{A} \cap \mathcal{A}^* = \varnothing$. Indeed,

$$\mathcal{A} \cap \mathcal{A}^* = \boldsymbol{\omega}(\mathcal{U}) \cap \boldsymbol{\alpha}(\mathcal{U}^c) \subset \mathcal{U} \cap \mathcal{U}^c = \varnothing.$$

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathsf{Att}(\mathcal{F})$. Absorption is established by

$$\mathsf{C}_{\mathsf{Att}}(\mathcal{A} \vee \mathcal{B}, \mathcal{A}) = (\mathcal{A} \cup \mathcal{B}) \cap A^* = (\mathcal{A} \cap \mathcal{A}^*) \cup (\mathcal{B} \cap A^*) = \mathsf{C}_{\mathsf{Att}}(\mathcal{B}, \mathcal{A}),$$

and similarly

$$\begin{aligned} \mathsf{C}_{\mathsf{Att}}(\mathcal{A}, \mathcal{A} \wedge \mathcal{B}) &= \mathcal{A} \cap (\mathcal{A} \wedge \mathcal{B})^* = \mathcal{A} \cap (\mathcal{A}^* \cup \mathcal{B}^*) \\ &= (\mathcal{A} \cap \mathcal{A}^*) \cup (\mathcal{A} \cap \mathcal{B}^*) = \mathsf{C}_{\mathsf{Att}}(\mathcal{A}, \mathcal{B}). \end{aligned}$$

Since $\mathcal{A} \cap \mathcal{C} \in \mathsf{Invset}^+(\mathcal{F})$ and $\mathcal{B}^* \cap \mathcal{D}^* \in \mathsf{Invset}^-(\mathcal{F})$ we have, using Lemma 6,

$$\begin{aligned} \mathsf{C}_{\mathsf{Att}}(\mathcal{A}, \mathcal{B}) \wedge \mathsf{C}_{\mathsf{Att}}(\mathcal{C}, \mathcal{D}) &= \operatorname{Inv}\big((\mathcal{A} \cap \mathcal{B}^*) \cap (\mathcal{C} \cap \mathcal{D}^*)\big) \\ &= \operatorname{Inv}\big((\mathcal{A} \cap \mathcal{C}) \cap (\mathcal{B}^* \cap \mathcal{D}^*)\big) \\ &= \boldsymbol{\omega}(\mathcal{A} \cap \mathcal{C}) \cap \boldsymbol{\alpha}(\mathcal{B}^* \cap \mathcal{D}^*) = (\mathcal{A} \wedge \mathcal{C}) \cap (\mathcal{B}^* \wedge \mathcal{D}^*) \\ &= (\mathcal{A} \wedge \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D})^* = \mathsf{C}_{\mathsf{Att}}(\mathcal{A} \wedge \mathcal{C}, \mathcal{B} \cup \mathcal{D}), \end{aligned}$$

which proves distributivity. It remains to show that the lattice form is monotone. Assume $\mathcal{A}, \mathcal{A}' \in \mathsf{Att}(\mathcal{F})$ satisfy $\mathsf{C}_{\mathsf{Att}}(\mathcal{A}, \mathcal{A}') = \mathcal{A} \cap \mathcal{A}'^* = \varnothing$. Observe that $\mathcal{A}'^* \in \mathsf{Invset}^-(\mathcal{F})$ and $\mathcal{U}' := (\mathcal{A}'^*)^c \in \mathsf{Invset}^+(\mathcal{F})$. Since $\boldsymbol{\alpha}(\mathcal{U}'^c) = \mathcal{A}'^*$ we have that $\boldsymbol{\omega}(\mathcal{U}') = \mathcal{A}'$. Then,

$$\varnothing = \mathcal{A} \cap \mathcal{A}'^* = \mathcal{A} \setminus (\mathcal{A}'^*)^c,$$

which implies that $\mathcal{A} \subset (\mathcal{A}'^*)^c$ and therefore $\mathcal{A} = \boldsymbol{\omega}(\mathcal{A}) \subset \boldsymbol{\omega}((\mathcal{A}'^*)^c) = \mathcal{A}'$ which establishes monotonicity and completes the proof. □

6 Maps between Conley Forms

We now discuss the effect of a lattice homomorphism on lattice forms and the Conley form in particular. Theorems 2 and 4 (below) imply that the Conley form behaves as a Boolean homomorphism under a homomorphism between lattices. This confirms that the Conley form is a generalization of the set difference operator for bounded, distributive lattices.

Theorem 4 *Let* L *and* K *be bounded, distributive lattices and let* $h\colon \mathsf{K} \to \mathsf{L}$ *be a lattice homomorphism. For every representation of Conley forms on* L *and* K *if* $a - b = a' - b'$, *then*

$$h(a) - h(b) = h(a') - h(b').$$

Proof We use Diagram (8) for the Booleanization of h. By construction of the Conley form on K we have $a - b = a' - b'$ if and only if $A \setminus B = A' \setminus B'$. Similarly, for the Conley form on L we have $h(a) - h(b) = h(a') - h(b')$ if and only if $j(h(a)) \setminus j(h(b)) = j(h(a')) \setminus j(h(b'))$. By Diagram (8) the latter is equivalent to

$$\mathsf{B}(h)(A) \setminus \mathsf{B}(h)(B) = \mathsf{B}(h)(A') \setminus \mathsf{B}(h)(B').$$

Since $\mathsf{B}(h)$ is Boolean it holds that $\mathsf{B}(h)(A) \setminus \mathsf{B}(h)(B) = \mathsf{B}(h)(A \setminus B)$ which completes the proof. □

Remark 6 The canonical Conley forms on K and L are represented in Boolean algebras $\mathsf{B}(\mathsf{K})$ and $\mathsf{B}(\mathsf{L})$. The key idea in the above proof can be expressed as the fact that the Boolean map $\mathsf{B}(h)$ commutes with the canonical Conley forms, i.e.

$$\mathsf{C}^\sigma \circ (\mathsf{B}(h) \times \mathsf{B}(h)) = \mathsf{B}(h) \circ \mathsf{C}^\sigma.$$

Corollary 2 *Under the hypotheses of Theorem 4, h induces a map $\theta\colon \mathsf{I}_\mathsf{C} \to \mathsf{J}_\mathsf{C}$ given by*

$$\theta(a - b) := h(a) - h(b),$$

and θ is a meet semilattice homomorphism preserving both neutral elements as expressed in the commutative diagram

$$\begin{array}{ccccc} \mathsf{K}\times\mathsf{K} & \twoheadrightarrow & \mathsf{I}_\mathsf{C} & \rightarrowtail & \mathsf{I} \\ \Big\downarrow{\scriptstyle h\times h} & & \Big\downarrow{\scriptstyle \theta} & \searrow^{\theta} & \\ \mathsf{L}\times\mathsf{L} & \twoheadrightarrow & \mathsf{J}_\mathsf{C} & \rightarrowtail & \mathsf{J} \end{array} \tag{21}$$

Proof For the homomorphism property we argue as follows. Using the distributivity of the Conley forms on K and L we have

$$\begin{aligned} \theta\big((a-b)\wedge(c-d)\big) &= \theta\big((a\wedge c)-(b\vee d)\big) = h(a\wedge c) - h(b\vee d) \\ &= \big(h(a)\wedge h(c)\big) - \big(h(b)\vee h(d)\big) \\ &= \big(h(a)-h(b)\big)\wedge\big(h(c)-h(d)\big) \\ &= \theta(a-b)\wedge\theta(c-d). \end{aligned}$$

For the neutral elements we have

$$\theta(0-1) = h(0) - h(1) = 0 - 1,$$

and similarly $\theta(1-0) = 1-0$ which shows that θ preserves the neutral elements in I_C and J_C. □

Remark 7 In Theorem 4 we can relax the Conley form on L by a lattice form ρ and the map θ is still well-defined since only the absorption property is. As for Corollary 2 we still obtain a semilattice homomorphism θ if the Conley form on L is relaxed to a distributive lattice form.

By Corollary 2 and Remark 7 we can define the pullback of a lattice form. Let $h\colon \mathsf{K} \to \mathsf{L}$ be a lattice homomorphism and ρ be a lattice form on L. Then,

$$(h^\bullet\rho)(a,b) := \rho\big(h(a), h(b)\big),$$

defines a lattice form on K.

Corollary 3 *Let $h\colon \mathsf{K} \to \mathsf{L}$ be a lattice isomorphism, and let C be a representation of the Conley form on L in I, then $h^\bullet\mathsf{C}$ is a representation of the Conley form on K in I.*

Proof Distributivity follows from the proof of Corollary 2. By definition $(h^\bullet\mathsf{C})(a,b) = h(a) - h(b)$. By Theorem 2 to check that $h^\bullet\mathsf{C}$ is a representation of the Conley form on K we need to show monotonicity. Consider $(h^\bullet)(a,b) = (h^\bullet)(0,1)$ which is equivalent to $h(a) - h(b) = h(0) - h(1) = 0 - 1$ in I. This implies that $h(a) \leqslant h(b)$ and thus $a \leqslant b$ since h is an isomorphism. □

Remark 8 If $\mathsf{K} \subset \mathsf{L}$, then h may be regarded as a lattice embedding in which case $h^\bullet \mathsf{C}$ is the restriction of C to K, cf. Proposition 4.

When $h\colon \mathsf{K} \to \mathsf{L}$ is an anti-homomorphism, we define a pullback of a lattice form by

$$(h^\bullet \rho)(a, b) := \rho(h(b), h(a)), \tag{22}$$

which is justified by the following proposition.

Proposition 7 *Let $h\colon \mathsf{K} \to \mathsf{L}$ be a lattice anti-isomorphism and let C be a representation of the Conley form on L in I, then $h^\bullet \mathsf{C}$ is a representation of the Conley form on K in I.*

Proof To show that $h^\bullet \mathsf{C}$ is a Conley form we verify absorption, distributivity, and monotonicity. Consider

$$\begin{aligned}(h^\bullet \mathsf{C})(a \vee b, a) &= \mathsf{C}(h(a), h(a \vee b)) = \mathsf{C}(h(a), h(a) \wedge h(b)) \\ &= \mathsf{C}(h(a), h(b)) = (h^\bullet \mathsf{C})(b, a),\end{aligned}$$

and

$$\begin{aligned}(h^\bullet \mathsf{C})(a, a \wedge b) &= \mathsf{C}(h(a \wedge b), h(a)) = \mathsf{C}(h(a) \vee h(b), h(a)) \\ &= \mathsf{C}(h(b), h(a)) = (h^\bullet \mathsf{C})(a, b)\end{aligned}$$

which establishes absorption and

$$\begin{aligned}(h^\bullet \mathsf{C})(a \wedge c, b \vee d) &= \mathsf{C}(h(b \vee d), h(a \wedge c)) = \mathsf{C}(h(b) \wedge h(d), h(a) \vee h(c)) \\ &= \mathsf{C}(h(b), h(a)) \wedge \mathsf{C}(h(d), h(c)) = (h^\bullet \mathsf{C})(a, b) \wedge (h^\bullet \mathsf{C})(c, d)\end{aligned}$$

establishes distributivity. As for monotonicity we argue as follows. Suppose $(h^\bullet \mathsf{C})(a, b) = h^\bullet \mathsf{C}(0, 1)$, then $h(b) - h(a) = h(1) - h(0) = 0 - 1$. Therefore, $h(b) \leqslant h(a)$ which implies $a \leqslant b$ since lattice anti-isomorphisms are order-reversing. □

Example 9 Let $\varphi\colon \mathbb{T}^+ \times X \to X$ be a dynamical system that is not necessarily invertible. The arguments in Example 7 make use of the fact that $\mathsf{Invset}(\varphi)$ is a subalgebra of $\mathsf{Set}(X)$. For noninvertible dynamical systems, the meet lattice operation is not intersection, and hence $\mathsf{Invset}(\varphi)$ is not generally a sublattice of $\mathsf{Set}(X)$. Therefore, we need an alternative representation of a Conley form. By App. 1[Lemma 21]

$$\begin{aligned} W^s\colon \mathsf{Morse}(\varphi) &\to \mathsf{Invset}^\pm(\varphi) \\ S &\mapsto W^s(S) := \{x \in X \mid \omega(x) \subset S\}\end{aligned}$$

is an injective semilattice homomorphism. Since $\mathsf{Invset}^\pm(\varphi)$ is a Boolean algebra, following the same arguments as in Example 7, using W^s and App. 1[Lemma 18] instead, we obtain a representation of Conley form on $\mathsf{Rep}(\varphi)$ represented in $\mathsf{Invset}(\varphi)$ as

$$\mathsf{C}_{\mathsf{Rep}}(R, R') = R - R' := R \cap R'^*$$

with range $\mathsf{Morse}(\varphi) = \mathsf{C}_{\mathsf{Rep}}(\mathsf{Rep}(\varphi) \times \mathsf{Rep}(\varphi))$.

Since the dual operator $^*\colon \mathsf{Att}(\varphi) \to \mathsf{Rep}(\varphi)$ is an anti-isomorphism, c.f. [25, Proposition 4.7], Proposition 7 and Equation (22) imply that the pullback

$$(h^\bullet \mathsf{C}_{\mathsf{Rep}})(A, A') = \mathsf{C}_{\mathsf{Rep}}(A'^*, A^*) = A'^* \cap (A^*)^* = A \cap A'^*$$

gives the following representation of the Conley form on $\mathsf{Att}(\varphi)$ in $\mathsf{Invset}(\varphi)$

$$\begin{aligned}\mathsf{C}_{\mathsf{Att}}\colon \mathsf{Att}(\varphi) \times \mathsf{Att}(\varphi) &\to \mathsf{Morse}(\varphi) \\ (A, A') &\mapsto \mathsf{C}_{\mathsf{Att}}(A, A') = A - A' := A \cap A'^*,\end{aligned}$$

and $\mathsf{Morse}(\varphi) = \mathsf{C}_{\mathsf{Att}}(\mathsf{Att}(\varphi) \times \mathsf{Att}(\varphi))$.

Remark 9 In the remainder of the paper we will adopt the notation $\mathsf{C}_{\mathsf{Att}}(A, A') = A - A'$ and $\mathsf{C}_{\mathsf{Rep}}(R, R') = R - R'$ indicated by the distinguished Conley forms $\mathsf{C}_{\mathsf{Att}}$ and $\mathsf{C}_{\mathsf{Rep}}$.

7 Conley forms and Convexity Semilattices for Dynamical Systems

We refine Corollary 2 in the context of various forms of dynamics.

7.1 Combinatorial Systems

Define the meet semilattice of *Morse tiles* to be

$$\mathsf{MTile}(\mathcal{F}) := \mathsf{C}^b(\mathsf{Invset}^+(\mathcal{F}) \times \mathsf{Invset}^+(\mathcal{F}))$$

with $\mathsf{C}^b(\mathcal{U}, \mathcal{V}) = \mathcal{U} \smallsetminus \mathcal{V}$. Then Diagram (21) yields

$$\begin{array}{ccccc} \mathsf{Invset}^+(\mathcal{F}) \times \mathsf{Invset}^+(\mathcal{F}) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}(\mathcal{F}) & \xrightarrow{\subset} & \mathsf{Set}(\mathcal{X}) \\ \Big\downarrow{\scriptstyle \omega\times\omega} & & \Big\downarrow{\scriptstyle \theta} & \searrow^{\theta} & \\ \mathsf{Att}(\mathcal{F}) \times \mathsf{Att}(\mathcal{F}) & \xrightarrow{\mathsf{C}} & \mathsf{Morse}(\mathcal{F}) & \xrightarrow{\subset} & \mathsf{Invset}(\mathcal{F}) \end{array} \tag{23}$$

The semilattice homomorphism $\theta\colon \mathsf{MTile}(\mathcal{F}) \twoheadrightarrow \mathsf{Morse}(\mathcal{F})$ is defined by $\theta(\mathcal{U} \smallsetminus \mathcal{U}') = \boldsymbol{\omega}(\mathcal{U}) - \boldsymbol{\omega}(\mathcal{U}') = \mathcal{A} - \mathcal{A}'$ where $\mathcal{A} = \boldsymbol{\omega}(\mathcal{U})$ and $\mathcal{A}' = \boldsymbol{\omega}(\mathcal{U}')$. Since we have an explicit characterization of attractors via $\boldsymbol{\omega}$, we can further characterize θ.

Lemma 7 $\theta(\mathcal{U} \smallsetminus \mathcal{U}') = \mathrm{Inv}(\mathcal{U} \smallsetminus \mathcal{U}')$.

Proof By Lemma 6, $\mathcal{A} - \mathcal{A}' = \mathrm{Inv}(\mathcal{U} \smallsetminus \mathcal{U}')$. □

Lemma 7 in combination with Diagram (23) gives the following commutative diagram

$$\begin{array}{ccccc} \mathsf{Invset}^+(\mathcal{F}) \times \mathsf{Invset}^+(\mathcal{F}) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}(\mathcal{F}) & \xrightarrow{\subset} & \mathsf{Set}(\mathcal{X}) \\ \Big\downarrow{\scriptstyle \omega\times\omega} & & \Big\downarrow{\scriptstyle \mathrm{Inv}} & & \Big\downarrow{\scriptstyle \mathrm{Inv}} \\ \mathsf{Att}(\mathcal{F}) \times \mathsf{Att}(\mathcal{F}) & \xrightarrow{\mathsf{C}} & \mathsf{Morse}(\mathcal{F}) & \xrightarrow{\subset} & \mathsf{Invset}(\mathcal{F}) \end{array} \tag{24}$$

7.2 Dynamical Systems

Example 9 establishes a nontrivial representation of the Conley form on $\mathsf{Att}(\varphi)$ into $\mathsf{Morse}(\varphi)$. In this setting, Diagram (21) applied to the lattice of closed attracting blocks, $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$, yields

$$\begin{array}{ccccc} \mathsf{ABlock}_{\mathscr{C}}(\varphi) \times \mathsf{ABlock}_{\mathscr{C}}(\varphi) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}(\varphi) & \xrightarrow{\subset} & \mathsf{Set}(X) \\ \Big\downarrow{\scriptstyle \omega\times\omega} & & \Big\downarrow{\scriptstyle \theta} & \searrow^{\theta} & \\ \mathsf{Att}(\varphi) \times \mathsf{Att}(\varphi) & \xrightarrow{\mathsf{C}_{\mathsf{Att}}} & \mathsf{Morse}(\varphi) & \xrightarrow{\subset} & \mathsf{Invset}(\varphi) \end{array} \tag{25}$$

where $\mathsf{C}^b(U, U') = U \smallsetminus U'$ is a Conley form in $\mathsf{Set}(X)$ by Rmk. 4. The range is

$$\mathsf{MTile}(\varphi) := \mathsf{C}^b\left(\mathsf{ABlock}_{\mathscr{C}}(\varphi) \times \mathsf{ABlock}_{\mathscr{C}}(\varphi)\right)$$

which is called the meet semilattice of *Morse tiles*. Recall that a set $U \subset X$ is an *isolating neighborhood* if $\operatorname{Inv}(\operatorname{cl} U) \subset \operatorname{int} U$ and the associated *isolated invariant set* is $S = \operatorname{Inv}(\operatorname{cl} U)$. The set of isolating neighborhoods $\mathsf{INbhd}(\varphi)$ is a subsemilattice of $\mathsf{Set}(X)$ and the set of isolated invariant sets $\mathsf{Isol}(\varphi)$ is a subsemilattice of $\mathsf{Invset}(\varphi)$.

The meet semilattice homomorphism $\theta\colon \mathsf{MTile}(\varphi) \twoheadrightarrow \mathsf{Morse}(\varphi)$ can be explicitly characterized.

Lemma 8 *For all* $U, U' \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$

$$A - A' = \theta(U \smallsetminus U') = \operatorname{Inv}(U \smallsetminus U') = \operatorname{Inv}\big(\operatorname{cl}(U \smallsetminus U')\big) \subset \operatorname{int}(U \smallsetminus U').$$

Furthermore, $\mathsf{MTile}(\varphi) \subset \mathsf{INbhd}(\varphi)$ *is a subsemilattice. In particular, Morse sets are isolated invariant sets.*

Proof Let $S \subset U \smallsetminus \operatorname{int} U' = U \cap \operatorname{cl}(U'^c)$ be an invariant set. Then $S \subset U$, and thus $A \cup S \subset U$. Since $A = \operatorname{Inv}(U)$ it follows that $S \subset A$. Similarly, $S \subset \operatorname{cl}(U'^c)$ and thus $A^* \cup S \subset \operatorname{cl}(U'^c)$. Since $A^* = \operatorname{Inv}^+\big(\operatorname{cl}(U'^c)\big)$ it follows that $S \subset A'^*$. Consequently, $A - A' = \operatorname{Inv}(U \smallsetminus \operatorname{int} U')$. Since $\operatorname{cl}(U \smallsetminus U') \subset U \cap \operatorname{cl}(U'^c) = U \smallsetminus \operatorname{int} U'$ it follows that

$$\operatorname{Inv}\big(\operatorname{cl}(U \smallsetminus U')\big) \subset \operatorname{Inv}(U \smallsetminus \operatorname{int} U') = A - A' \subset \operatorname{int}(U \cap U'^c) = \operatorname{int}(U \smallsetminus U'),$$

which proves that $U \smallsetminus U'$ is an isolating neighborhood. Because $A - A' \subset U \smallsetminus U' \subset \operatorname{cl}(U \smallsetminus U')$ it follows that $A - A' = \operatorname{Inv}(U \smallsetminus U') = \operatorname{Inv}\big(\operatorname{cl}(U \smallsetminus U')$. The fact that $\mathsf{MTile}(\varphi)$ is a subsemilattice of $\mathsf{Set}(X)$ implies it is a subsemilattice of $\mathsf{INbhd}(\varphi)$. □

Refining Diagram (25) based on Lemma 8 gives

$$\begin{array}{ccccc} \mathsf{ABlock}_{\mathscr{C}}(\varphi) \times \mathsf{ABlock}_{\mathscr{C}}(\varphi) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}(\varphi) & \xrightarrow{\subset} & \mathsf{INbhd}(\varphi) \\ \Big\downarrow{\scriptstyle \omega\times\omega} & & \Big\downarrow{\scriptstyle \mathrm{Inv}} & & \Big\downarrow{\scriptstyle \mathrm{Inv}} \\ \mathsf{Att}(\varphi) \times \mathsf{Att}(\varphi) & \xrightarrow{\mathsf{C}_{\mathsf{Att}}} & \mathsf{Morse}(\varphi) & \xrightarrow{\subset} & \mathsf{Isol}(\varphi). \end{array} \qquad (26)$$

The fact that $\operatorname{Inv}\colon \mathsf{INbhd}(\varphi) \twoheadrightarrow \mathsf{Isol}(\varphi)$ is a semilattice homomorphism follows from [25, Lemma 2.7].

Remark 10 In the above commutative diagram we could also have chosen to use the lattice of attracting neighborhoods in place of attracting blocks. In this case, the image of the Conley form is a larger subsemilattice of the isolating neighborhoods. In the next section we present Morse tiles in the setting of regular closed sets which arise naturally in computations, [23,26].

7.3 Regular Closed Sets

As indicated in [26], for computational purposes it is useful to define Conley forms in the setting of regular closed sets $\mathscr{R}(X)$, cf. App. 3. The set of closed regular sets that are attracting blocks is denoted by $\mathsf{ABlock}_{\mathscr{R}}(\varphi)$. The goal of this section is to prove that the following is a commutative diagram of lattice homomorphisms

$$
\begin{array}{ccccc}
\mathsf{ABlock}_{\mathscr{C}}(\varphi)\times\mathsf{ABlock}_{\mathscr{C}}(\varphi) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}(\varphi) & \xrightarrow{\subset} & \mathsf{IBlock}(\varphi)\\
\downarrow {\scriptstyle \#\#\times\#\#} & & \downarrow {\scriptstyle \theta_{\#\#}} & \searrow^{\theta_{\#\#}} & \\
\mathsf{ABlock}_{\mathscr{R}}(\varphi)\times\mathsf{ABlock}_{\mathscr{R}}(\varphi) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}_{\mathscr{R}}(\varphi) & \xrightarrow{\subset} & \mathsf{IBlock}_{\mathscr{R}}(\varphi)\\
\downarrow {\scriptstyle \omega\times\omega} & & \downarrow {\scriptstyle \mathrm{Inv}} & & \downarrow {\scriptstyle \mathrm{Inv}}\\
\mathsf{Att}(\varphi)\times\mathsf{Att}(\varphi) & \xrightarrow{\mathsf{C}_{\mathsf{Att}}} & \mathsf{Morse}(\varphi) & \xrightarrow{\subset} & \mathsf{Isol}(\varphi)
\end{array}
\tag{27}
$$

where $U^{\#\#}:=\mathrm{cl}\,\mathrm{int}\,U$.

Remark 11 Observe that since the top and bottom rows are as in (26) and the vertical maps are surjective, there is no information lost by working with regular closed sets.

Lemma 9 *If* $U\in\mathsf{ABlock}_{\mathscr{C}}(\varphi)$*, then* $U^{\#\#}\in\mathsf{ABlock}_{\mathscr{R}}(\varphi)$.

Proof By assumption $\varphi(t,U^{\#\#})\subset\varphi(t,U)\subset\mathrm{int}\,U=\mathrm{int}\,U^{\#\#}$ for all positive $t\in\mathbb{T}$, where the latter follows the fact that $\mathrm{int}\,U=U^{\perp\perp}=U^{\perp\perp\perp\perp}=\mathrm{int}\,\mathrm{cl}\,\mathrm{int}\,U=\mathrm{int}\,U^{\#\#}$ and $U^{\perp}=(\mathrm{cl}\,U)^c$. □

The map ${}^{\#\#}\colon\mathsf{ABlock}_{\mathscr{C}}(\varphi)\to\mathsf{ABlock}_{\mathscr{R}}(\varphi)$ is a lattice homomorphism by Lemma 22 and $\omega\colon\mathsf{ABlock}_{\mathscr{R}}(\varphi)\to\mathsf{Att}(\varphi)$ is a lattice homomorphism by [26, Theorem 3.15]. As a consequence, we obtain the following three commutative diagrams of lattice homomorphisms. First,

$$
\begin{array}{ccc}
\mathsf{ABlock}_{\mathscr{C}}(\varphi) & \xrightarrow{\#\#} & \mathsf{ABlock}_{\mathscr{R}}(\varphi)\\
{\scriptstyle \omega}\searrow & & \swarrow{\scriptstyle \omega}\\
 & \mathsf{Att}(\varphi) &
\end{array}
\tag{28}
$$

where the surjectivity of ${}^{\#\#}$ follows from $\mathsf{ABlock}_{\mathscr{R}}(\varphi)\subset\mathsf{ABlock}_{\mathscr{C}}(\varphi)$. Furthermore, by Diagram (21)

$$
\begin{array}{ccccc}
\mathsf{ABlock}_{\mathscr{C}}(\varphi)\times\mathsf{ABlock}_{\mathscr{C}}(\varphi) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}(\varphi) & \xrightarrow{\subset} & \mathsf{IBlock}(\varphi)\\
\downarrow {\scriptstyle \#\#\times\#\#} & & \downarrow {\scriptstyle \theta_{\#\#}} & \searrow^{\theta_{\#\#}} & \\
\mathsf{ABlock}_{\mathscr{R}}(\varphi)\times\mathsf{ABlock}_{\mathscr{R}}(\varphi) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}_{\mathscr{R}}(\varphi) & \xrightarrow{\subset} & \mathsf{IBlock}_{\mathscr{R}}(\varphi)
\end{array}
\tag{29}
$$

and

$$
\begin{array}{ccccc}
\mathsf{ABlock}_{\mathscr{R}}(\varphi)\times\mathsf{ABlock}_{\mathscr{R}}(\varphi) & \xrightarrow{\mathsf{C}^b} & \mathsf{MTile}_{\mathscr{R}}(\varphi) & \xrightarrow{\subset} & \mathsf{IBlock}_{\mathscr{R}}(\varphi)\\
\downarrow {\scriptstyle \omega\times\omega} & & \downarrow {\scriptstyle \theta_{\mathscr{R}}} & \searrow^{\theta_{\mathscr{R}}} & \\
\mathsf{Att}(\varphi)\times\mathsf{Att}(\varphi) & \xrightarrow{\mathsf{C}_{\mathsf{Att}}} & \mathsf{Morse}(\varphi) & \xrightarrow{\subset} & \mathsf{Isol}(\varphi).
\end{array}
\tag{30}
$$

The Conley form on $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$ is given by $\mathsf{C}^b(U,U')=U\smallsetminus U'$ for $U,U'\in\mathsf{ABlock}_{\mathscr{C}}(\varphi)$. The Conley form on $\mathsf{ABlock}_{\mathscr{R}}(\varphi)$ is given by

$$
\mathsf{C}^b(N,N')=N\wedge N'^{\#}=\mathrm{cl}\,(N\smallsetminus N')\quad\text{for } N,N'\in\mathsf{ABlock}_{\mathscr{R}}(\varphi),
\tag{31}
$$

where the latter follows from Lemma 23.

Consider the homomorphism ${}^{\#\#}\colon \mathsf{ABlock}_{\mathscr{C}}(\varphi) \to \mathsf{ABlock}_{\mathscr{R}}(\varphi)$. Via Corollary 2 and Eqn. (53) the induced meet semilattice homomorphism $\theta\colon \mathsf{MTile}(\varphi) \to \mathsf{MTile}_{\mathscr{R}}(\varphi)$ is given by

$$\theta_{\#\#}(U \smallsetminus U') = U^{\#\#} - U'^{\#\#} := U^{\#\#} \wedge U'^{\#\#\#} = \mathrm{cl}\,(N \smallsetminus N')$$

where $N := U^{\#\#}$. From Lemma 8, Corollary 2 and Lemma 23 we derive that

$$\theta_{\mathscr{R}}(N - N') = \omega(N) - \omega(N') = \mathrm{Inv}(N \smallsetminus N') = \mathrm{Inv}(\mathrm{cl}\,(N \smallsetminus N')) = \mathrm{Inv}(N - N'),$$

which proves the following lemma.

Lemma 10 $A - A' = \theta_{\mathscr{R}}(N - N') = \mathrm{Inv}(N - N')$.

Lemma 10 together with (30) yields the bottom half of (27). The duality between regular closed attracting and repelling blocks is given by the following lemma which explains regular closed Morse tiles as regular intersections of attracting blocks and repelling blocks and characterizes the duality in Diagram (17) in this case.

Lemma 11 $\mathsf{ABlock}_{\mathscr{R}}(\varphi) \overset{\#}{\longleftrightarrow} \mathsf{RBlock}_{\mathscr{R}}(\varphi)$.

Proof From [25, Lemma 3.17] we derive that if $U \in \mathsf{RBlock}_{\mathscr{R}}(X, \varphi)$, then $U^{\#}$ satisfies

$$\varphi(-t, U^{\#}) \subset \mathrm{int}\, U^c = U^c = \mathrm{int}\,\mathrm{cl}\, U^c = \mathrm{int}\, U^{\#}, \quad t < 0.$$

The latter follows from the fact that U^c is a regular open set. From [25, Lemma 3.17] we also derive that if $U \in \mathsf{ABlock}_{\mathscr{R}}(X, \varphi)$, then

$$\varphi(-t, U^{\#}) \subset U^c = \mathrm{int}\, U^{\#}, \quad t > 0,$$

which proves that $U^{\#} \in \mathsf{RBlock}_{\mathscr{R}}(X, \varphi)$. □

8 Representations of Lattices

We have shown that attractors in a dynamical system have the structure of a bounded, distributive lattice, which codifies algebraically the global structure of the dynamical system. From a dynamics point of view, this global structure has been alternatively described in terms of a poset of distinguished invariant sets, the order of which encodes the global structure. From an algebraic point of view, a bounded, distributive lattice is dually equivalent to a poset via Priestley duality as described in Section 2. Hence, the order on the Priestley space is dynamically defined, and the central issue is the representation of the Priestley space as a poset of invariant sets.

In the previous sections we have identified dynamically distinguished invariant sets, namely the Morse sets, which can be characterized as the image of the specific Conley form $\mathsf{C}_{\mathsf{Att}}$ on the lattice of attractors represented in the invariant sets. In particular, $\mathsf{Morse}(\varphi) \cong \mathsf{B}^{\updownarrow}(\mathsf{Att})$. The Conley form is designed to provide a representation of the semilattice structure of $\mathsf{B}^{\updownarrow}(\mathsf{L})$ in a more meaningful semilattice I. However, since the Booleanization functor $\mathsf{B} = \mathsf{O}^{\mathrm{clp}} \circ \mathsf{F} \circ \Sigma$ forgets the order on $\Sigma(\mathsf{L})$, a representation of $\Sigma(\mathsf{L})$ as a poset in I does not immediately follow. In this section, we show that in the case of finite lattices, the Priestley space can indeed be represented as a poset consisting of elements in I, but the issue is more subtle in the infinite case.

8.1 Spectral Representations

Let L be a finite distributive lattice. Then the convexity semilattice $\mathsf{B}^{\updownarrow}(\mathsf{L})$ is the lattice of all convex sets $\mathsf{Co}(\Sigma(\mathsf{L}))$ in the spectrum $\Sigma(\mathsf{L})$. The lattice $\mathsf{Co}(\Sigma(\mathsf{L}))$ is a (complete) atomic lattice, which is not distributive in general, cf. [7]. The anti-chain of atoms in $\mathsf{Co}(\Sigma(\mathsf{L}))$ contains exactly the sets $\{I\}$ where $I \in \Sigma(\mathsf{L})$ ignoring the order structure of $\Sigma(\mathsf{L})$. Let $\mathsf{C}\colon \mathsf{L}\times\mathsf{L} \to \mathsf{I}$ be a Conley form on L, then $\mathsf{B}^{\updownarrow}(\mathsf{L}) \cong \mathsf{C}(\mathsf{L}\times\mathsf{L}) = \mathsf{I}_{\mathsf{C}}$, and C determines the injective semilattice homomorphism $\gamma\colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}_{\mathsf{C}}$ given by

$$\gamma(A \smallsetminus B) = \mathsf{C}(j^{-1}(A), j^{-1}(B)) \quad \text{for } A, B \in \mathsf{B}^{\downarrow}(\mathsf{L}).$$

Writing $\{I\}$ as $\{I\} = \downarrow I \smallsetminus (\downarrow I \smallsetminus \{I\})$ gives

$$\gamma(\{I\}) = \mathsf{C}\big(j^{-1}\big(\downarrow I\big), j^{-1}\big(\downarrow I \smallsetminus \{I\}\big)\big).$$

Since the join irreducible elements of $\mathsf{B}^{\downarrow}(\mathsf{L})$ are exactly those of the form $\downarrow I$ for $I \in \Sigma(\mathsf{L})$, the join irreducible elements of L are exactly $a = j^{-1}(\downarrow I)$ for $I \in \Sigma(\mathsf{L})$, since j is an isomorphism, cf. Sect. 2. Consequently, $\{\mathsf{C}(a, \overleftarrow{a}) \mid a \in \mathsf{J}(\mathsf{L})\} = \{\gamma(\{I\}) \mid I \in \Sigma(\mathsf{L})\}$ is a representation of $\Sigma(\mathsf{L})$ in I. This motivates the following definition.

Definition 6 Let L be a finite, distributive lattice and let $\mathsf{C}\colon \mathsf{L}\times\mathsf{L} \to \mathsf{I}$ be a representation of the Conley form in a meet semilattice I and $\gamma\colon \mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}$ be the injective semilattice homomorphism given by C. The *spectral representation of* $\Sigma(\mathsf{L})$ *in* I is defined to be the poset $\big(\mathsf{M}(\mathsf{L}), \leqslant\big)$ where

$$\mathsf{M}(\mathsf{L}) := \big\{\gamma\big(\{I\}\big) \mid I \in \Sigma(\mathsf{L})\big\} = \big\{a - \overleftarrow{a} \mid a \in \mathsf{J}(\mathsf{L})\big\}$$

and

$$\begin{aligned} \gamma(\{I\}) \leqslant \gamma(\{I'\}) \quad &\Longleftrightarrow \quad I \subset I', \quad \text{or equivalently} \\ a - \overleftarrow{a} \leqslant a' - \overleftarrow{a}' \quad &\Longleftrightarrow \quad a \leqslant a'. \end{aligned} \tag{32}$$

Since γ is an isomorphism, it follows that

$$\begin{aligned} \boldsymbol{\mu}\colon \big(\Sigma(\mathsf{L}), \subset\big) &\to \big(\mathsf{M}(\mathsf{L}), \leqslant\big) \\ I &\mapsto \gamma\big(\{I\}\big) = a - \overleftarrow{a} \quad \text{for } a = j^{-1}(\downarrow I) \in \mathsf{J}(\mathsf{L}), \end{aligned}$$

is an order isomorphism.

Lemma 12 *It holds that* $\gamma(\{I\}) \neq 0$ *for all* $I \in \Sigma(\mathsf{L})$ *and* $\gamma(\{I\}) \wedge \gamma(\{I'\}) = 0$ *for all* $I \neq I'$.

Proof Since $\overleftarrow{a} < a$ Proposition 3(iv) implies that $a - \overleftarrow{a} \neq 0$. Note that elements of $\mathsf{M}(\mathsf{L})$ are pairwise disjoint since $\gamma(\{I\}) \wedge \gamma(\{I'\}) = \gamma(\{I\} \cap \{I'\}) = 0$ and γ is an isomorphism $\mathsf{B}^{\updownarrow}(\mathsf{L}) \to \mathsf{I}_{\mathsf{C}}$. □

The above construction implies that in the finite case, the (clopen) singleton, convex sets in $\mathsf{B}^{\updownarrow}(\mathsf{L})$ are in one-to-one correspondence with $\mathsf{J}(\mathsf{L})$ and thus P, cf. Fig. 3[left/middle] and $\Sigma(\mathsf{L})$ which is used to construct spectral representations. The semilattice of all convex sets $\mathsf{Co}(\Sigma(\mathsf{L}))$ is isomorphic to the convex sets in P and is given in Fig. 3[right]. In general, when L is infinite, $\mathsf{B}^{\updownarrow}(\mathsf{L})$ is only a subsemilattice of $\mathsf{Co}(\Sigma(\mathsf{L}))$, [7]. Indeed, there are infinite, bounded, distributive lattices that possess clopen, singleton, convex sets that are not associated to a join irreducible elements. In this case, representing the spectrum is more subtle and will not be addressed in this paper, cf. [18]. For dynamics, the finite case is often sufficient as we are interested in Morse representations which are finite.

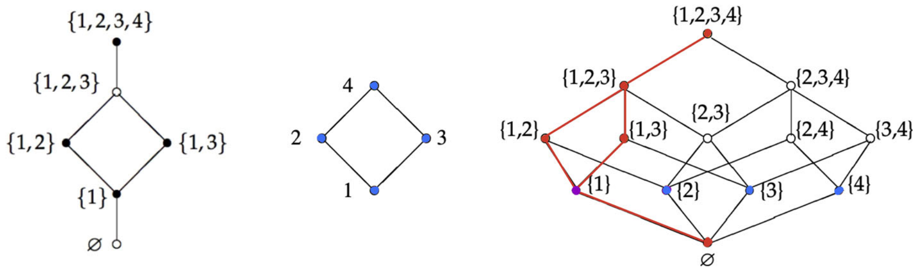


Fig. 3 A finite lattice $\mathsf{L} = \mathsf{O}(\mathsf{P})$ [left], the poset P representing the spectrum [middle] and the convexity semilattice [right]. The subset in red [right] gives an embedding of L into the convexity semilattice, cf. Remark 5

Since $\mathsf{M}(\mathsf{L})$ is order-isomorphic to $\Sigma(\mathsf{L})$, Birkhoff's Representation Theorem implies that $\mathsf{O}(\mathsf{M}) \cong \mathsf{L}$. We denote this isomorphism by $\boldsymbol{\nu}\colon \mathsf{O}(\mathsf{M}) \to \mathsf{L}$. Let C denote the Conley form on L represented in I. This gives the diagram:

$$\begin{array}{ccc} \mathsf{O}(\mathsf{M}) \times \mathsf{O}(\mathsf{M}) & \xrightarrow{\mathsf{C}^{\sigma}} & \mathsf{B}^{\updownarrow}(\mathsf{O}(\mathsf{M})) \\ {\scriptstyle \boldsymbol{\nu}\times\boldsymbol{\nu}}\downarrow \cong & & {\scriptstyle \theta}\downarrow \cong \\ \mathsf{L} \times \mathsf{L} & \xrightarrow{\mathsf{C}} & \mathsf{I}_{\mathsf{C}} \end{array} \tag{33}$$

where by commutativity the final isomorphism is given by

$$\theta(\alpha \setminus \beta) = \boldsymbol{\nu}(\alpha) - \boldsymbol{\nu}(\beta) \text{ for all } \alpha, \beta \in \mathsf{O}(\mathsf{M}),$$

and consequently $\theta(\{M\}) = M$ for all $M \in \mathsf{M}(\mathsf{L})$. If we consider a lattice homomorphism $h\colon \mathsf{K} \to \mathsf{L}$ have the following commutative diagrams:

$$\begin{array}{ccccc} \mathsf{K} \times \mathsf{K} & \twoheadrightarrow & \mathsf{I}_{\mathsf{C}} & \rightarrowtail & \mathsf{I} \\ {\scriptstyle h\times h}\downarrow & & {\scriptstyle \theta}\downarrow & {\scriptstyle \theta}\searrow & \\ \mathsf{L} \times \mathsf{L} & \twoheadrightarrow & \mathsf{J}_{\mathsf{C}} & \rightarrowtail & \mathsf{J} \end{array} \qquad \begin{array}{ccc} \Sigma(\mathsf{L}) & \xrightarrow{h^{-1}} & \Sigma(\mathsf{K}) \\ {\scriptstyle (\uparrow a)^c}\uparrow \cong & & \cong \downarrow {\scriptstyle I \mapsto \min I^c} \\ \mathsf{J}(\mathsf{L}) & \xrightarrow{\mathsf{J}(h)} & \mathsf{J}(\mathsf{K}) \end{array}$$

which yields $\mathsf{J}(h)(a) := \min h^{-1}(\uparrow a)$. This construction induces the map:

$$\mathsf{M}(h)\colon \mathsf{M}(\mathsf{L}) \to \mathsf{M}(\mathsf{K}), \quad a - \overleftarrow{a} \mapsto \mathsf{J}(h)(a) - \overleftarrow{\mathsf{J}(h)(a)} \tag{34}$$

When we apply the spectral representation in the dynamical setting using $\mathsf{C}_{\mathsf{Att}}$, we use the terminology of a *Morse representation* in place of spectral representation. For example, consider $\mathsf{L} = \mathsf{Att}(\varphi)$ and the Conley form

$$\mathsf{C}_{\mathsf{Att}}(A, A') = A - A' := A \cap A'^{*}.$$

Following Definition 6 the Morse representation subordinate to a finite sublattice $\mathsf{A} \subset \mathsf{Att}(\varphi)$ is defined to be

$$\mathsf{M}(\mathsf{A}) := \{A - \overleftarrow{A} \mid A \in \mathsf{J}(\mathsf{A})\}.$$

In the combinatorial setting, the Morse representation subordinate to a sublattice A of $\mathsf{Att}(\mathcal{F})$ is

$$\mathsf{M}(\mathsf{A}) := \{\mathcal{A} - \overleftarrow{\mathcal{A}} \mid \mathcal{A} \in \mathsf{J}(\mathsf{A})\}.$$

In both cases the Morse representation is a poset isomorphic to $\mathsf{J}(\mathsf{A})$ which is a partially ordered set via set-inclusion.

8.2 Stable and Unstable Set Representations

Consider the maps $W^u\colon \mathsf{Morse}(\varphi) \to \mathsf{Invset}(\varphi)$ and $W^s\colon \mathsf{Morse}(\varphi) \to \mathsf{Invset}^{\pm}(\varphi)$ where the latter is an injective semilattice homomorphism, cf. App. 1[Lemma 21]. These maps induce a lattice form ρ^u and a Conley form C^s given by

$$\rho^u := W^u \circ \mathsf{C}_{\mathsf{Att}} \quad \text{and} \quad \mathsf{C}^s := W^s \circ \mathsf{C}_{\mathsf{Rep}}.$$

From the fundamental theorem of attractor-repeller pairs [25, Theorem 3.19] we have that $A \setminus \overleftarrow{A} \subset \rho^u(A, \overleftarrow{A})$, and therefore ρ^u satisfies the additivity property in Proposition 6. Let $\boldsymbol{\nu}\colon \mathsf{O}(\mathsf{M}) \to \mathsf{Att}(\varphi)$ be the injective lattice homomorphism with range A from Birkhoff's Representation Theorem. Then, for $\alpha \in \mathsf{O}(\mathsf{M})$, we have $\alpha = \bigcup_{\substack{\alpha' \subset \alpha \\ \alpha' \in \mathsf{J}(\mathsf{O}(\mathsf{M}))}} \alpha'$ and consequently, $A = \bigcup_{\substack{A' \subset A \\ A' \in \mathsf{J}(\mathsf{A})}} A'$ where $A' = \boldsymbol{\nu}(\alpha')$. Proposition 6 implies

$$\boldsymbol{\nu}(\alpha) = \bigcup_{\substack{A' \subset A \\ A' \in \mathsf{J}(\mathsf{A})}} \rho^u(A', \overleftarrow{A}') = \bigcup_{M \in \alpha} W^u(M), \tag{35}$$

which provides an explicit expression for $\boldsymbol{\nu}(\alpha)$. Similarly, since also C^s satisfies the additivity property in Proposition 6, a representation for repellers can be obtained via the homomorphism $\boldsymbol{\nu}^*\colon \mathsf{U}(\mathsf{M}) \to \mathsf{Rep}(\varphi)$ given by

$$\boldsymbol{\nu}^*(\beta) = \bigcup_{M \in \beta} W^s(M). \tag{36}$$

From Remark 5 we have the following commutative diagram

$$\begin{array}{ccc} \mathsf{O}(\mathsf{M}) & \overset{c}{\longleftrightarrow} & \mathsf{U}(\mathsf{M})) \\ \boldsymbol{\nu}\Big\downarrow\cong & & \cong\Big\downarrow\boldsymbol{\nu}^* \\ \mathsf{A} & \overset{*}{\longleftrightarrow} & \mathsf{A}^* \end{array} \tag{37}$$

where we have identified $\mathsf{C}_{\mathsf{Att}}(\omega(X), A) = \omega(X) \cap A^*$ with A^*, since $\omega(X) \cap A^* \mapsto W^s\big(\omega(X) \cap A^*\big) = A^*$ is injective, cf. Lemma 21. This yields the correspondence

$$A = \nu(\alpha) = \bigcup_{M \in \alpha} W^u(M) \Longleftrightarrow A^* = \boldsymbol{\nu}(\alpha)^* = \boldsymbol{\nu}^*(\alpha^c) = \bigcup_{M \in \alpha^c} W^s(M). \tag{38}$$

Remark 12 We leave it to the reader to verify that the homomorphisms θ and θ^* induced by $\boldsymbol{\nu}$ and $\boldsymbol{\nu}^*$ respectively coincide, i.e. $\theta(\alpha \setminus \beta) = \theta^*(\alpha \setminus \beta)$.

We can use the above decompositions to obtain a decomposition in terms of connecting orbits. We now show that the decompositions in (35) and (36) can be utilized to relate the partial order on $\mathsf{M}(\mathsf{A})$ to the dynamics of φ. Theorem 5 below provides a dynamical description of this order, which serves as an extension of an attractor-repeller pair and can be used as a dynamical definition of a Morse representation.

Theorem 5 *Let* $(\mathsf{M}(\mathsf{A}), \leqslant)$[4] *be the Morse representation subordinate to a finite sublattice* $\mathsf{A} \subset \mathsf{Att}(\varphi)$. *Then, the sets* $M \in \mathsf{M}(\mathsf{A})$ *are compact, nonempty, pairwise disjoint, invariant sets in* X, *and for every* $x \in X$ *there exists* $M \in \mathsf{M}$ *such that* $\omega(x) \subset M$. *Moreover, for every complete orbit* γ_x *with* $x \notin \bigcup_{M\in\mathsf{M}} M$ *there exist* $M, M' \in \mathsf{M}$ *with* $M < M'$ *such that* $\omega(x) \subset M$ *and* $\alpha_o(\gamma_x^-) \subset M'$.

Proof By definition $M = A - \overleftarrow{A} = A \cap \overleftarrow{A}^* \in \mathsf{M}(\mathsf{A}) \subset \mathsf{Invset}(\varphi)$. Since M is the intersection of an attractor and repeller, it is compact and isolated, cf. [25] and Lemma 8. Finally, by Lemma 12 Morse sets M are nonempty. Furthermore, by Lemma 12, $M \wedge M' = \varnothing$ for all $M \neq M'$. This implies that $M \cap M' = \varnothing$ for all $M \neq M'$. Indeed, the intersection $M \cap M'$ is compact and forward invariant and thus $M \wedge M' = \mathrm{Inv}(M \cap M') = \omega(M \cap M')$ is nonempty unless $M \cap M' = \varnothing$.

The decompositions in (35) and (36) imply that

$$X = \bigcup_{M\in\mathsf{M}} W^s(M) \quad \text{and} \quad \omega(X) = \bigcup_{M\in\mathsf{M}} W^u(M), \tag{39}$$

so that for each $x \in X$ there exists M such that $x \in W^s(M)$ and thus $\omega(x) \subset M$. Let γ_x be a complete orbit with $x \in X \smallsetminus \big(\bigcup_{M\in\mathsf{M}} M\big)$. Then, by the decompositions in (39) we have that $x \in W^s(M) \cap W^u(M')$. By definition $\omega(x) \subset M$ and $\alpha_o(\gamma_x^-) \subset M'$. It remains to show that $M < M'$. Suppose $M > M'$ or $M \| M'$ and write the singleton convex set $\{M\}$ in $\mathsf{O}(\mathsf{M})$ as $\{M\} = \alpha \smallsetminus \beta$ with $\alpha = \downarrow M \in \mathsf{O}(\mathsf{M})$ and $\beta = (\uparrow M)^c \in \mathsf{O}(\mathsf{M})$, and likewise $\{M'\} = \alpha' \smallsetminus \beta'$ with $\alpha' = \downarrow M'$ and $\beta' = (\uparrow M')^c$. Then,

$$\begin{aligned} W^s(M) \cap W^u(M') &= \mathsf{C}^s\big(\boldsymbol{\nu}(\alpha), \boldsymbol{\nu}(\beta)\big) \cap \rho^u\big(\boldsymbol{\nu}(\alpha'), \boldsymbol{\nu}(\beta')\big) \\ &= W^s\big(\boldsymbol{\nu}(\alpha) \cap \boldsymbol{\nu}(\beta)^*\big) \cap W^u\big(\boldsymbol{\nu}(\alpha') \cap \boldsymbol{\nu}(\beta')^*\big) \\ &\subset W^s(\boldsymbol{\nu}(\beta)^*) \cap W^u(\boldsymbol{\nu}(\alpha')) = \boldsymbol{\nu}(\beta)^* \cap \boldsymbol{\nu}(\alpha') = \boldsymbol{\nu}(\alpha') - \boldsymbol{\nu}(\beta) \end{aligned}$$

By the mapping property of the Conley form we have

$$\boldsymbol{\nu}(\alpha') - \boldsymbol{\nu}(\beta) = \theta(\alpha' \smallsetminus \beta) = \varnothing,$$

since $\alpha' \smallsetminus \beta = \downarrow M' \cap \uparrow M = \varnothing$ by the assumptions on M and M', which proves that $M < M'$. □

8.3 Reconstruction of Attractor Lattices

Theorem 5 establishes dynamical properties of a Morse representation. The next result shows that the characterization in Theorem 5 can be used as a dynamical definition of Morse representations.

Theorem 6 *Let* $(\mathsf{M}, \leqslant)$ *be a finite poset of nonempty, pairwise disjoint, compact, invariant sets in* X. *Then* M *is a Morse representation subordinate to a finite sublattice* $\mathsf{A}(\mathsf{M}) \subset \mathsf{Att}(\varphi)$ *if and only if for every* $x \in X$ *there exists* $M \in \mathsf{M}$ *such that* $\omega(x) \subset M$, *and for each complete orbit* γ_x *with* $x \notin \bigcup_{M\in\mathsf{M}} M$ *there exists* $M < M'$ *such that* $\omega(x) \subset M$ *and* $\alpha_o(\gamma_x^-) \subset M'$. *The associated lattice* $\mathsf{A}(\mathsf{M})$ *is the image of the injective lattice homomorphism* $\boldsymbol{\nu}\colon \mathsf{O}(\mathsf{M}) \to \mathsf{Invset}(\varphi)$ *given by*

$$\alpha \mapsto \boldsymbol{\nu}(\alpha) = \bigcup_{M\in\alpha} W^u(M) \subset \mathsf{Att}(\varphi), \tag{40}$$

[4] The poset $(\mathsf{M}(\mathsf{A}), \leqslant)$ is a lattice induced and is defined in Definition 6.

and $\mathsf{M} = \mathsf{M}\big(\mathsf{A}(\mathsf{M})\big)$.

The "only if" direction is Theorem 5, so the proof of Theorem 6 is divided into two lemmas in which we assume the second set of conditions stated in the theorem.

Lemma 13 *Let $M_0 \in \mathsf{M}$ be a minimal element. Then, M_0 is an attractor.*

Proof The set $X' = \omega(X)$ is a compact metric space, and the restriction $\varphi' = \varphi|_{X'}$ is a surjective dynamical system on X'. Due to the invariance of both X' and the sets $M \in \mathsf{M}$ we have that for every $x \in X' \smallsetminus (\bigcup_{M\in\mathsf{M}} M)$, there exists $M < M'$ such that $\omega(x) \subset M$ and $\alpha_o(\gamma_x^-) \subset M'$. By assumption we can choose a compact neighborhood $N \supset M_0$ with $N \cap M = \varnothing$ for all $M \neq M_0$. For $x \in N \smallsetminus M_0$ the assumptions imply that $\alpha_o(\gamma_x^-) \subset M \neq M_0$ for all backward orbits γ_x^-. Consequently, there are no backward orbits $\gamma_x^- \colon \mathbb{T}^- \to N$ for all $x \in N \smallsetminus M_0$. By [25, Lemma 3.11] the set M_0 is an attractor for φ'. By [25, Proposition 3.7], M_0 is also an attractor for φ. □

Lemma 14 *Let $M_0 \in \mathsf{M}$ be a minimal element. Then*

$$R = \bigcup_{M\neq M_0} W^s(M)$$

is the repeller dual to $A = M_0$.

Proof By Proposition 3.16 in [25], since M_0 is an attractor, cf. Lemma 13, the dual attractor of M_0 is characterized by $M_0^* = \{x \in X \mid \omega(x) \cap M_0 = \varnothing\}$. Suppose $x \in R$, then $\omega(x) \subset M$ for some $M \neq M_0$, and therefore $R \subset M_0^*$. Conversely, if $x \in M_0^*$, then $\omega(x) \subset M$ with $M \neq M_0$, which implies $M_0^* \subset R$, and thus $M_0^* = R$. □

Proof of Theorem 6 Since the sets $W^s(M) \in \mathsf{Invset}^{\pm}(\varphi)$ for $M \in \mathsf{M}$ are mutually disjoint sets in $\mathsf{I} = \mathsf{Invset}^{\pm}(\varphi)$, the map

$$\begin{aligned} \boldsymbol{\nu}^* \colon \mathsf{U}(\mathsf{M}) &\rightarrowtail \mathsf{Invset}^{\pm}(\varphi) \\ \beta &\mapsto \bigcup_{M\in\beta} W^s(M) \end{aligned} \tag{41}$$

defines a injective lattice homomorphism and the range is denoted by $\mathsf{A}^*(\mathsf{M})$, cf. Lemma 21.

Lemmas 13 and 14 show that $R = \bigcup_{M\neq M_0} W^s(M) = \boldsymbol{\nu}^*\big((\downarrow M_0)^c\big)$ are repellers for all minimal elements in $M_0 \in \mathsf{M}$. Let X' be the intersection of these repellers, which is again a repeller, and let M' be the poset obtained from M by removing all minimal elements. Then, M' satisfies the conditions of Theorem 6 in X'. Repeat the above lemmas in X'. By [25, Proposition 3.28] repellers in X' are repellers in X, and thus by exhausting the poset M we establish that all elements of the form $\boldsymbol{\nu}^*\big((\downarrow M)^c\big)$, $M \in \mathsf{M}$, are repellers in X. Since the elements $\boldsymbol{\nu}^*\big((\downarrow M)^c\big)$ are meet-irreducible, all elements in $\mathsf{A}^*(\mathsf{M})$, except for $\boldsymbol{\nu}^*(\mathsf{M})$, are meets of meet-irreducible repellers. By (38) $\boldsymbol{\nu}^*(\mathsf{M}) = X$, a repeller, which establishes $\boldsymbol{\nu}^* \colon \mathsf{U}(\mathsf{M}) \to \mathsf{Rep}(\varphi)$ as a injective lattice homomorphism. Consequently, $\alpha \mapsto \boldsymbol{\nu}^*(\alpha^c)^* \in \mathsf{Att}(\varphi)$ is a injective lattice homomorphism $\boldsymbol{\nu} \colon \mathsf{O}(\mathsf{M}) \to \mathsf{Att}(\varphi)$ by Diagram (37).

Moreover, for $\alpha \smallsetminus \beta = \{M\}$ we have

$$\begin{aligned} \boldsymbol{\nu}(\alpha) - \boldsymbol{\nu}(\beta) &= \boldsymbol{\nu}^*(\alpha^c)^* - \boldsymbol{\nu}^*(\beta^c)^* \\ &= \boldsymbol{\nu}^*(\alpha^c)^* \cap \boldsymbol{\nu}^*(\beta^c) = \boldsymbol{\nu}^*(\beta^c) - \boldsymbol{\nu}^*(\alpha^c) \\ &= \theta^*(\beta^c \smallsetminus \alpha^c) = \theta^*(\alpha \smallsetminus \beta) = M, \end{aligned}$$

which implies, by (38), that $\boldsymbol{\nu}$ is given by (40) completing the proof. □

Given a finite attractor lattice $\mathsf{A} \subset \mathsf{Att}(\varphi)$, then $\mathsf{M}(\mathsf{A})$ satisfies the hypotheses of Theorem 6 and the associated attractor lattice $\mathsf{A}(\mathsf{M})$ in (40) is isomorphic to A due to Birkhoff's Representation Theorem and the attractors coincide, which shows that $\mathsf{A}(\mathsf{M}(\mathsf{A})) = \mathsf{A}$. We conclude

$$\mathsf{A} \circ \mathsf{M} = \mathrm{id} \quad \text{and} \quad \mathsf{M} \circ \mathsf{A} = \mathrm{id}\,. \tag{42}$$

Remark 13 The above characterization and construction of Morse representations can also be implemented for finite binary relations $\mathcal{F} \subset \mathcal{X} \times \mathcal{X}$. In [24, Defn. 3.9] a dynamical definition of Morse representation is given. The results in Theorem 5 and Theorem 6 also hold in this setting.

9 Morse Decompositions

Let P be a finite poset. A lattice homomorphism $\mathsf{O}(\mathsf{P}) \to \mathsf{L}$ can be factored through its range $\mathsf{A} \subset \mathsf{L}$, i.e. $\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{A} \rightarrowtail \mathsf{L}$. This yields the factorization $\Sigma(\mathsf{L}) \twoheadrightarrow \Sigma(\mathsf{A}) \hookrightarrow \Sigma(\mathsf{O}(\mathsf{P}))$. Given a spectral representation $\mathsf{M}(\mathsf{A})$ via a Conley form on A and Birkhoff's Representation Theorem, we obtain

$$\Sigma(\mathsf{L}) \twoheadrightarrow \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{P}.$$

In the context of dynamics, we make the following definition.

Definition 7 Let P be a finite poset and A be the image of a lattice homomorphism $\mathsf{O}(\mathsf{P}) \to \mathsf{Att}$. The order-embedding $\pi \colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{P}$ is called the *Morse decomposition* dual to the lattice epimorphism $\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{A}$.

The term Morse decomposition was first defined in Conley theory in the setting of continuous time dynamical systems via labelings of collections of invariant sets by a poset whose order is consistent with the dynamics, cf. [10]. By reformulating this concept in terms of embeddings of posets we obtain a formulation of Morse decomposition consistent with the algebraic theory developed in this paper. Here we emphasize the algebraic nature of a Morse decomposition as an order-embedding from a Morse representation into a poset. The importance of the role of the poset P and the information it provides about the dynamical system becomes most apparent in computations where the poset P is the computable object, cf. Sect. 9.2. Generally, we refer to a Morse decomposition without mentioning the dual lattice homomorphism.

9.1 Tessellated Morse Decompositions

In this section we present a dynamically meaningful choice of poset P in a Morse decomposition. Let $\mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(\varphi)$ be a finite sublattice of regular closed attracting blocks, and consider the Conley form given in Sect. 7.3 in the setting of regular closed attracting neighborhoods. From 3 and Diagram (27) we derive the commutative diagrams

$$\begin{array}{ccc} \mathsf{N} & \xleftrightarrow{\;\#\;} & \mathsf{N}^{\#} \\ \omega\big\downarrow & & \big\downarrow\alpha \\ \mathsf{A} & \xleftrightarrow{\;*\;} & \mathsf{A}^{*} \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{N} \times \mathsf{N} & \xrightarrow{\;\mathsf{C}^{b}\;} & \mathsf{MTile}_{\mathscr{R}}(\mathsf{N}) \\ \omega\times\omega\big\downarrow & & \big\downarrow\mathrm{Inv} \\ \mathsf{A} \times \mathsf{A} & \xrightarrow{\;\mathsf{C}_{\mathsf{Att}}\;} & \mathsf{Morse}(\mathsf{A}) \end{array} \tag{43}$$

where $\mathsf{MTile}_{\mathscr{R}}(\mathsf{N}) \subset \mathsf{MTile}(\varphi)$ denotes the image of C^b restricted to $\mathsf{N} \times \mathsf{N}$ and $\mathsf{Morse}(\mathsf{A}) \subset \mathsf{Morse}(\varphi)$ denotes the image of $\mathsf{C}_{\mathsf{Att}}$ restricted to $\mathsf{A} \times \mathsf{A}$. As in Sect. 8.1, we obtain a spectral representation in $\mathsf{MTile}_{\mathscr{R}}(\mathsf{N})$ that is called a *Morse tessellation*, see Example 3, and denoted by

$$\mathsf{T}(\mathsf{N}) = \big\{T = N - \overleftarrow{N} \mid N \in \mathsf{J}(\mathsf{N})\big\}$$

where $N - \overleftarrow{N} = \mathrm{cl}\,(N \smallsetminus \overleftarrow{N})$, and $N - \overleftarrow{N} \leqslant N' - \overleftarrow{N}'$ if and only if $N \subset N'$. Hence $\mathsf{T} = \mathsf{T}(\mathsf{N})$ is a poset, and the map $\mathsf{J}(\mathsf{N}) \to \mathsf{T}(\mathsf{N})$ given by $N \mapsto N - \overleftarrow{N}$ is an order-isomorphism. Moreover, the functoriality of Birkhoff's Representation Theorem yields the order-embedding $\pi : \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$ subordinate to the lattice surjection $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$, where π is explicitly given by (34):

$$\begin{aligned} M = A - \overleftarrow{A} &\mapsto \pi(M) = N - \overleftarrow{N} \\ \text{with} \quad N &= \mathsf{J}(\omega)(A) = \min \omega^{-1}(\uparrow A) \in \mathsf{J}(\mathsf{N}). \end{aligned} \tag{44}$$

Definition 8 Let $\mathsf{T}(\mathsf{N})$ be a Morse tessellation of regular closed sets subordinate to the sublattice $\mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(\varphi)$. Then, the homomorphism $\pi : \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$ is called a *tessellated Morse decomposition* subordinate to $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$.

For a given tessellated Morse decomposition $\pi : \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$, the Morse tessellation $\mathsf{T}(\mathsf{N})$ plays the role of the poset P in the definition of Morse decomposition.

Remark 14 Observe that by Corollary 2, $\theta\colon \mathsf{C}^b\big(\mathsf{O}(\mathsf{T}(\mathsf{N})) \times \mathsf{O}(\mathsf{T}(\mathsf{N}))\big) \to \mathsf{C}_{\mathsf{Att}}(\mathsf{A} \times \mathsf{A})$ is a semilattice homomorphism. Furthermore, by Lemma 8, $\theta = \mathrm{Inv}$. The map $\pi : \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$ is a order-embedding. If we identify the elements of $\mathsf{T}(\mathsf{N})$ with the singleton sets in $\mathsf{C}^b\big(\mathsf{O}(\mathsf{T}(\mathsf{N})) \times \mathsf{O}(\mathsf{T}(\mathsf{N}))\big)$, then

$$\mathrm{Inv} \circ \pi = \theta \circ \pi = \mathrm{id}_{\mathsf{M}(\mathsf{A})},$$

and $\theta = \mathrm{Inv}$ acts as a left-inverse for π. Therefore, the Morse sets can be recovered as the maximal invariant sets within the Morse tiles. Observe that since $\varnothing$ is in the range of θ, we capture the possibility that the maximal invariant set in a Morse tile may be empty.

Remark 15 One can also define tessellated Morse decomposition via $\mathsf{ANbhd}(\varphi)$ or $\mathsf{ANbhd}_{\mathscr{R}}(\varphi)$.

9.2 Spans and Combinatorial Models

For a given dynamical system φ in this paper we have constructed the following *span* in the category of bounded distributive lattices

$$\mathscr{R}(X) \xleftarrow{\ \supset\ } \mathsf{ABlock}_{\mathscr{R}}(\varphi) \xrightarrow{\ \omega\ } \mathsf{Att}(\varphi)\ .$$

Spans can be used to define equivalence classes of dynamical systems based on their gradient behavior. Two dynamical systems (X, φ) and (Y, ψ) are *span equivalent* if there exist isomorphisms such that following diagram commutes

$$\begin{array}{ccccc} \mathscr{R}(X) & \xleftarrow{\ \supset\ } & \mathsf{ABlock}_{\mathscr{R}}(\varphi) & \xrightarrow{\ \omega\ } & \mathsf{Att}(\varphi) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathscr{R}(Y) & \xleftarrow{\ \supset\ } & \mathsf{ABlock}_{\mathscr{R}}(\psi) & \xrightarrow{\ \omega\ } & \mathsf{Att}(\psi). \end{array}$$

The analogue of a span in the category of finite distributive lattices is given by

$$\mathsf{Set}(\mathcal{X}) \xleftarrow{\ \iota\ } \mathsf{O}(\mathsf{P}) \xrightarrow{\ h\ } \mathsf{O}(\mathsf{Q}), \tag{45}$$

where $\mathcal{X}$ is a finite set and P and Q are finite posets from Birkhoff's Representation Theorem. Spans in the category of finite distributive lattices can be equivalently described through finite binary relations $\mathcal{F} \subset \mathcal{X} \times \mathcal{X}$. To be more precise the extension of Birkhoff's Representation Theorem in [23] yields the following representation of a finite span in terms of a binary relation $\mathcal{F}$, i.e. (45) can be equivalently described by

$$\mathsf{Set}(\mathcal{X}) \xleftarrow{\ \supset\ } \mathsf{Invset}^+(\mathcal{F}) \xrightarrow{\ \boldsymbol{\omega}\ } \mathsf{Att}(\mathcal{F}),$$

where $\boldsymbol{\omega}$ is the omega limit set in the setting of binary relations, cf. Eqn. (50) and [26]. We emphasize that the choice of $\mathcal{F}$ is not unique, cf. [23]. The next step is to consider diagrams of the form

$$\begin{array}{ccccc}
\mathscr{R}(X) & \xleftarrow{\ \supset\ } & \mathsf{ABlock}_{\mathscr{R}}(\varphi) & \xrightarrow{\ \omega\ } & \mathsf{Att}(\varphi) \\
\uparrow{\scriptstyle |\cdot|} & & \uparrow{\scriptstyle |\cdot|} & & \uparrow{\scriptstyle c} \\
\mathsf{Set}(\mathcal{X}) & \xleftarrow{\ \supset\ } & \mathsf{Invset}^+(\mathcal{F}) & \xrightarrow{\ \boldsymbol{\omega}\ } & \mathsf{Att}(\mathcal{F}),
\end{array} \tag{46}$$

where the second homomorphism is a restriction of the first. We show that if the third homomorphism exists and the diagram commutes, then it is uniquely defined by $c = \omega(|\cdot|)$.

Remark 16 Typically in applications, $\mathcal{X}$ is a labeling of the atoms of a subalgebra of regular closed sets, ie. a grid cf. [26], and the map $|\cdot|$ is the evaluation map

$$|\mathcal{U}| = \bigcup_{\xi \in \mathcal{U}} |\xi|,$$

which is injective. Also, in the definition of span one may consider sublattices of $\mathscr{R}(X)$ and $\mathsf{Set}(\mathcal{X})$, which is useful in some applications.

We refer to the diagram in (46) as a *commutative combinatorial model* for φ, see [23]. Recall from [24–26] that a way to combinatorialize a dynamical system is to discretize both time and space. In this section we explain combinatorialization from an algebraic point of view. In order to do so we introduce two hypotheses. First, a finite binary relation $\mathcal{F}$ is called a *weak outer approximation* if

(W) $\varphi\big(t, |\xi|\big) \subset \operatorname{int} |\Gamma^+(\xi)|$ for all $t > 0$,

where $\Gamma^+(\xi)$ denotes the forward image of ξ under $\mathcal{F}$. The commutativity of the first square in (46) is equivalent to (W) by [23, Thm. 5.3]. In order to characterize commutativity of the second square in (46) we use an additional criterion for $\mathcal{F}$

(L) $\omega(|\xi|) \subset |\boldsymbol{\omega}(\xi)|$ for all $\xi \in \mathcal{X}$.

Theorem 7 *Let $\mathcal{F} \subset \mathcal{X} \times \mathcal{X}$ be a finite, binary relation. Diagram (46) commutes if and only if $\mathcal{F}$ satisfies* (W) *and* (L). *In this case $c = \omega(|\cdot|)$.*

Proof The commutativity of the first square in (46) is equivalent to $\mathcal{F}$ satisfying (W) and is proved in [23, Thm. 5.3].

If the second square in (46) commutes, then $\omega(|\mathcal{U}|) = c(\boldsymbol{\omega}(\mathcal{U}))$ for every $\mathcal{U} \in \mathsf{Invset}^+(\mathcal{F})$. In particular, each $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$ satisfies $\mathcal{A} = \boldsymbol{\omega}(\mathcal{A})$ and is an element of $\mathsf{Invset}^+(\mathcal{F})$ so that $\omega(|\mathcal{A}|) = c(\mathcal{A})$, i.e. $c = \omega(|\cdot|)$.

Let $\mathcal{U} = \Gamma^+(\xi)$ be the complete forward image of some $\xi \in \mathcal{X}$. Then,

$$\omega(|\xi|) \subset \omega(|\mathcal{U}|) = c(\boldsymbol{\omega}(\mathcal{U})) = \omega(|\boldsymbol{\omega}(\mathcal{U})|) \subset |\boldsymbol{\omega}(\mathcal{U})| = |\boldsymbol{\omega}\big(\Gamma^+(\xi)\big)| = |\boldsymbol{\omega}(\xi)|,$$

which establishes property (L).

Conversely, suppose (L) is satisfied. For $\mathcal{U} \in \mathsf{Invset}^+(\mathcal{F})$ we have that $\boldsymbol{\omega}(\mathcal{U}) \subset \mathcal{U}$ and therefore $\omega(|\boldsymbol{\omega}(\mathcal{U})|) \subset \omega(|\mathcal{U}|)$. Moreover, since ω, $\boldsymbol{\omega}$, and $|\cdot|$ are homomorphisms,

$$\omega(|\mathcal{U}|) = \bigcup_{\xi \in \mathcal{U}} \omega(|\xi)|) \subset \bigcup_{\xi \in \mathcal{U}} |\boldsymbol{\omega}(\xi)| = |\boldsymbol{\omega}(\mathcal{U})|,$$

and thus $\omega(|\mathcal{U}|) = \omega\big(\omega(|\mathcal{U}|)\big) \subset \omega(|\boldsymbol{\omega}(\mathcal{U}|)$. Combining both inclusions, $\omega(|\boldsymbol{\omega}(\mathcal{U})|) = \omega(|\mathcal{U}|)$. This establishes the commutativity of the second square in (46) when $c = \omega(|\cdot|)$. □

From this point on we assume that $|\cdot|$ is injective. If we consider the diagram in (46) by denoting the ranges of the bottom span we obtain

$$\begin{array}{ccccc} \mathscr{R}_0 & \xleftarrow{\supset} & \mathsf{N} & \xrightarrow{\omega} & \mathsf{A} \\ \updownarrow{\scriptstyle |\cdot|} & & \updownarrow{\scriptstyle |\cdot|} & & \uparrow{\scriptstyle \omega(|\cdot|)} \\ \mathsf{Set}(\mathcal{X}) & \xleftarrow{\supset} & \mathsf{Invset}^+(\mathcal{F}) & \xrightarrow{\boldsymbol{\omega}} & \mathsf{Att}(\mathcal{F}), \end{array} \tag{47}$$

where $\mathscr{R}_0$ is the algebra of grid elements, N is a finite lattice of attracting blocks, and A is a finite lattice of attractors. We now invoke the various Conley forms to dualize the above diagrams which yields the following dual diagram

$$\begin{array}{ccccc} |\mathcal{X}| & \twoheadrightarrow & \mathsf{T}(\mathsf{N}) & \xleftarrow{\pi} & \mathsf{M}(\mathsf{A}) \\ \updownarrow{\scriptstyle |\cdot|} & & \updownarrow{\scriptstyle |\cdot|} & & \downarrow \\ \mathcal{X} & \xrightarrow{\supset} & \mathsf{SC}(\mathcal{F}) & \hookleftarrow & \mathsf{RC}(\mathcal{F}), \end{array} \tag{48}$$

which provides a factorization of the tessellation $|\mathcal{X}| \twoheadrightarrow \mathsf{T}(\mathsf{N})$ and the tessellated Morse decomposition $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$. Together these define the co-span

$$|\mathcal{X}| \twoheadrightarrow \mathsf{T}(\mathsf{N}) \hookleftarrow \mathsf{M}(\mathsf{A}).$$

The posets $\mathsf{SC}(\mathcal{F})$ and $\mathsf{RC}(\mathcal{F})$ are the spectral representations of $\mathsf{Invset}^+(\mathcal{F})$ and $\mathsf{Att}(\mathcal{F})$ respectively, cf. Sect. 8.1 and [23,26]. The dual diagram shows that binary relations $\mathcal{F}$ that satisfy Hypotheses (W) and (L) give rise to tessellated Morse decompositions. This fact has been used to computationally characterize and compare global dynamics in various contexts, [2,6,8,9,11,13,17,19,23,24,26].

Given a tessellation T of X consisting of regular closed sets labeled by $\mathcal{X}$. If we choose $\mathcal{F}$ to be transitive and reflexive, then $\mathcal{F}$ is a partial order on $\mathcal{X}$ and induces a partial order on $\mathsf{T} = |\mathcal{X}|$. By Theorem 7 we can then formulate the following equivalent characterization of Morse tessellations in the spirit Theorem 6 for Morse representations.

Corollary 4 *A (finite) poset* $(\mathsf{T}, \leqslant)$ *consisting of a regular closed partition* T *of* X *is a Morse tessellation subordinate to a (finite) sublattice of attracting blocks if and only if the partial order is a weak outer approximation for* φ.

Proof Let $\mathcal{F}$ be the partial order induced on $\mathcal{X}$ by the poset T. Assume that $\mathcal{F}$ is a weak outer approximation. Since $\mathcal{F}$ is a partial order, $\boldsymbol{\omega}(\xi) = \downarrow \xi$ for every $\xi \in \mathcal{X}$. Then (W) implies $\omega(|\xi|) \subset |\downarrow \xi| = |\boldsymbol{\omega}(\xi)|$ so that (L) is satisfied. The remainder follows from Theorem 7 and fact that finite sublattices of $\mathsf{ABlock}_{\mathscr{R}}(\varphi)$ yield partially ordered partitions of regular closed sets for which the down-sets give attracting blocks by construction. □

We refer to [23] for a more detailed account of combinatorial models and applications.

Acknowledgements The authors would like to thank the referees for their very careful reading of the original version of this paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

In this section we review definitions from dynamical systems theory, from both continuous and discrete time dynamical systems as well as the dynamics of finite relations. The proofs are fairly elementary, but are included since we are unaware of any single reference for all the results presented.

Topological Dynamics

In this paper we use the following definition of a continuous dynamical system, cf. [25,26].

Definition 9 Let $\mathbb{T}$ denote either $\mathbb{Z}$ or $\mathbb{R}$. A *dynamical system* is a continuous map $\varphi\colon \mathbb{T}^+ \times X \to X$ that satisfies

(i) $\varphi(0, x) = x$ for all $x \in X$, and
(ii) for all $s, t \in \mathbb{T}^+$ and for all $x \in X$ it holds that $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$.

If $\varphi\colon \mathbb{T} \times X \to X$ satisfies (i) and (ii), then φ is called an *invertible* dynamical system.

We use ω and α to denote the *omega* and *alpha* limit sets under the dynamics, cf. [25, Prop.'s 2.11 and 2.13]. Recall that $A \subset X$ is an *attractor* for φ if there exists an open neighborhood U of A such that the $\omega(U) = A$, and dually, $R \subset X$ is a *repeller* for φ if there exists an open neighborhood U of R such that $\alpha(U) = R$. The bounded, distributive lattice of attractors and repellers is denoted by $\mathsf{Att}(\varphi)$ and $\mathsf{Rep}(\varphi)$, respectively. The binary relations on $\mathsf{Att}(\varphi)$ are $A \wedge A' := \omega(A \cap A')$ and $A \vee A' := A \cup A'$, and on $\mathsf{Rep}(\varphi)$ the binary relations are intersection and union. In [25], it is shown that there is a natural well-defined duality anti-isomorphism $^*\colon \mathsf{Att}(\varphi) \to \mathsf{Rep}(\varphi)$ via $A = \omega(U) \mapsto \alpha(U^c) = A^*$ where ^{c} denotes complement. The pair (A, A^*) is called an *attractor-repeller pair*.

A set $U \subset X$ is an *attracting neighborhood* if $\omega(\operatorname{cl} U) \subset \operatorname{int} U$ and a *repelling neighborhood* if $\alpha(\operatorname{cl} U) \subset \operatorname{int} U$. The collection of all attracting and repelling neighborhoods form bounded distributive lattices, $\mathsf{ANbhd}(\varphi)$ and $\mathsf{RNbhd}(\varphi)$, respectively, with binary operations intersection and union. As shown in [25], the map $\omega\colon \mathsf{ANbhd}(\varphi) \twoheadrightarrow \mathsf{Att}(\varphi)$ is a surjective lattice homomorphism. Similarly, $\alpha\colon \mathsf{RNbhd}(\varphi) \twoheadrightarrow \mathsf{Rep}(\varphi)$ is a surjective lattice homomorphism. A subset $U \subset X$ is an *attracting block* for φ if

$$\varphi(t, \mathrm{cl}\, U) \subset \mathrm{int}\, U \quad \forall t > 0.$$

The set of closed attracting blocks of φ is denoted by $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$. By [25, Lemma 3.3] and the fact that intersection and union of closed sets are closed, $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$ is a bounded distributive lattice. Since the inclusion $\mathsf{ABlock}_{\mathscr{C}}(\varphi) \rightarrowtail \mathsf{Set}(X)$ is a lattice homomorphism, we can define the dual lattice

$$\mathsf{RBlock}_{\mathscr{O}}(\varphi) := \left\{ U^c \mid U \in \mathsf{ABlock}_{\mathscr{C}}(\varphi) \right\}.$$

Furthermore, by [25, Lemma 3.17], we have that $\mathsf{RBlock}_{\mathscr{O}}(\varphi) \rightarrowtail \mathsf{Set}(X)$, and complement acts as an anti-lattice isomorphism between $\mathsf{RBlock}_{\mathscr{O}}(\varphi)$ and $\mathsf{ABlock}_{\mathscr{C}}(\varphi)$. From the perspective of dynamics, $V \in \mathsf{RBlock}_{\mathscr{O}}(\varphi)$ if and only if $\varphi(t, \mathrm{cl}\, V) \subset \mathrm{int}\, V$ for all $t < 0$.

Lemma 15 *Let $\varphi\colon \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space. Then,* $\mathsf{Att}(\varphi) = \omega(\mathsf{ABlock}_{\mathscr{C}}(\varphi))$ *and* $\mathsf{Rep}(\varphi) = \alpha(\mathsf{RBlock}_{\mathscr{O}}(\varphi))$.

Proof Let $A \in \mathsf{Att}(\varphi)$. Because $\mathsf{Att}(\varphi) := \omega(\mathsf{ANbhd}(\varphi))$, there exist $U \in \mathsf{ANbhd}(\varphi)$ such that $\omega(U) = A$. By [25, Prop. 3.5] there exists a trapping region $\widehat{U} \subset U$ such that $\omega(U) = \omega(\widehat{U}) = A$. By [24, Lemmas 6.5 and 7.7] there exists a Lyapunov function $V\colon X \to [0, 1]$ such that $V^{-1}(0) = A$, $V^{-1}(1) = A^*$, and $V(\varphi(t, x)) < V(x)$ for all $t > 0$ and $x \notin A \cup A^*$, where A^* is the dual repeller to A. Due to compactness we can choose $0 < \epsilon \ll 1$ such that $N = \{x \in X \mid V(x) \leqslant \epsilon\} \subset \widehat{U} \subset U$ is a closed attracting block with $\omega(N) = \omega(\widehat{U}) = \omega(U) = A$. Therefore, $A \in \omega(\mathsf{ABlock}_{\mathscr{C}}(\varphi))$. The proof that $\mathsf{Rep}(\varphi) = \alpha(\mathsf{RBlock}_{\mathscr{O}}(\varphi))$ is similar. ◻

The following result is a Corollary of [25, Proposition 3.16].

Lemma 16 *Let $\varphi\colon \mathbb{T}^+ \times X \to X$ be a dynamical system on a compact metric space. Then, the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{ABlock}_{\mathscr{C}}(\varphi) & \xrightarrow[\cong]{\ c\ } & \mathsf{RBlock}_{\mathscr{O}}(\varphi) \\ {\scriptstyle\omega}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ \mathsf{Att}(\varphi) & \xrightarrow[\cong]{\ *\ } & \mathsf{Rep}(\varphi) \end{array} \tag{49}$$

The upper homomorphism follows from the proof of Lemma 3.17 in [25].

Definition 10 For a compact invariant set $S \subset X$ define the sets

$$W^s(S) = \{x \in X \mid \omega(x) \subset S\};$$
$$W^u(S) = \{x \in X \mid \exists \gamma_x^- \ni \alpha_o(\gamma_x^-) \subset S\},$$

which are called the *stable* and *unstable* sets of S respectively.

Lemma 17 *The stable set $W^s(S)$ is forward-backward invariant, and the unstable set $W^u(S)$ is invariant.*

Proof Let $x \in W^s(S)$. Then $\omega(\varphi(t,x)) \subset \omega(x) \subset S$ for every $t \in \mathbb{T}$. Therefore, $\omega(\varphi(t,x)) \subset S$ for all $x \in W^s(S), t \in \mathbb{T}$, which proves that $W^s(S)$ is both forward and backward invariant.

As for $W^u(S)$, we argue as follows. Let $x \in W^u(S)$. Then a complete orbit γ_x exists. Therefore, every $y \in \gamma_x$ has a backward orbit γ_y^-, and $\alpha_o(\gamma_y^-) = \alpha_o(\gamma_x^-) \subset S$, which proves that $\gamma_x \subset W^u(S)$ for all $x \in W^u(S)$ and establishes the invariance of $W^u(S)$. □

Lemma 18 *Let S, S' be compact invariant sets. Then, $W^s(S \wedge S') = W^s(S) \cap W^s(S')$.*

Proof We have $W^s(S \wedge S') \subset W^s(S) \cap W^s(S')$. Now let $x \in W^s(S) \cap W^s(S')$. Then $\omega(x) \subset S \cap S'$, and since $\omega(x)$ is invariant, $\omega(x) = \omega(\omega(x)) \subset \omega(S \cap S') = S \wedge S'$, and thus $W^s(S) \cap W^s(S') \subset W^s(S \wedge S')$. □

Remark 17 The same property with respect to union is not clear unless the invariant sets are attractors. The equivalent of Lemma 18 does not hold for $W^u(S \wedge S')$. If φ is an invertible system, i.e. a dynamical system with time $t \in \mathbb{Z}$, or $t \in \mathbb{R}$, then then we can use the proof of Lemma 18 to show that both W^s and W^u define lattice homomorphisms from the sublattice of compact invariant sets to the invariant sets.

Lemma 19 *Let $A \in \mathsf{Att}(\varphi)$. Then, the application $W^s \colon \mathsf{Att}(\varphi) \to \mathsf{Invset}^{\pm}(\varphi)$, defined by $A \mapsto W^s(A)$, is a lattice embedding.*

Proof By Theorem 3.19 in [25] we have that $W^s(A) = \{x \in X \mid \omega(x) \subset A\} = (A^*)^c$. This implies

$$W^s(A) \cup W^s(A') = (A^*)^c \cup (A'^*)^c = (A^* \cap A'^*)^c = \big((A \cup A')^*\big)^c = W^s(A \cup A').$$

Similarly,

$$W^s(A) \cap W^s(A') = (A^*)^c \cap (A'^*)^c = (A^* \cup A'^*)^c = \big((A \wedge A')^*\big)^c = W^s(A \wedge A').$$

To prove that the homomorphism is injective we argue as follows. Suppose $W^s(A) = W^s(A')$, then equivalently $(A^*)^c = (A'^*)^c$. Since both * and ^{c} are involutions, we have that $A = A'$, which completes the proof. □

The following lemma is an extension of [25, Prop. 3.21].

Lemma 20 *Let $A \subset X$ be an attractor and let N be a compact set satisfying $A \subset N \subset W^s(A)$. Then, $\omega(N) = A$.*

Proof By definition $(W^s(A))^c = A^*$ and thus $N \cap A^* = \varnothing$ by the assumptions on N. Since compact metric spaces are normal, there exist open sets separating N and A^*, i.e. there exist open sets $U \supset N$ and $V \supset A^*$ such that $\mathrm{cl}\,(U) \cap V = \varnothing$, and thus $\mathrm{cl}\,(U) \cap A^* = \varnothing$. By [25, Prop. 3.21] we have that $\omega(U) = A$, and therefore

$$A = \omega(A) \subset \omega(N) \subset \omega(U) = A,$$

which proves $\omega(N) = A$. □

Lemma 21 *The mapping $A \cap R \mapsto W^s(A \cap R)$ for an attractor A and a repeller R is injective.*

Proof Let $U \in \mathsf{ABlock}_{\mathscr{C}}(\varphi)$ be an attracting block for A so that $U^c \in \mathsf{RBlock}_{\mathscr{O}}(\varphi)$ is a repelling block for A^*. Then, since R is forward-backward invariant, $U \cap R$ is an attracting block in R, and $U^c \cap R$ is a repelling block in R. From the properties of limit sets, cf. [25, Lemma 2.9 and Propositions 2.11, 2.13], since both U and R are forward invariant and U^c and R are backward invariant, we have

$$\omega(U \cap R) = \omega(U) \wedge \omega(R) = A \cap R \quad \text{and} \quad \alpha(U^c \cap R) = \alpha(U^c) \cap \alpha(R) = A^* \cap R.$$

Therefore $(A \cap R, A^* \cap R)$ is an attractor-repeller pair in R.

Let $S = A \cap R$ and $S' = A' \cap R'$. Suppose $W^s(S) = W^s(S')$. Then $W^s(S \wedge S') = W^s(S) = W^s(S')$ by Lemma 18. Since $S \subset W^s(S)$ and $S' \subset W^s(S')$, we have

$$S \wedge S' \subset S \cup S' \subset W^s(S \wedge S'),$$

and $S \cup S'$ is compact. Applying Lemma 20 with $N = S \cup S'$, we have that $\omega(S \cup S') = S \wedge S'$. Also $\omega(S \cup S') = \omega(S) \cup \omega(S') = S \cup S'$ by the invariance of S, S'. Therefore $S \cap S' \supset S \wedge S' = S \cup S'$ so that $S = S'$, which proves the injectivity. □

Combinatorial Dynamics

Before recalling constructions associated with combinatorial dynamics we want to emphasize that our focus is on dynamics as given by Definition 9, i.e. single valued dynamics (that may not be invertible). The combinatorial structures discussed below are only used as a computational tool in which to understand the above mentioned dynamics.

Let $\mathcal{X}$ be a finite set. A *binary relation* $\mathcal{F}$ on $\mathcal{X}$ is subset of the product space $\mathcal{X} \times \mathcal{X}$. We make use of the following concepts and structures, see [25,26] for details. In what follows $\mathcal{F}$ can be interpreted as operator acting on subsets of $\mathcal{X}$ via

$$\mathcal{F}(\mathcal{U}) = \bigcup_{\xi \in \mathcal{U}} \mathcal{F}(\xi), \quad \mathcal{F}(\xi) := \{\eta \in \mathcal{X} \mid (\xi, \eta) \in \mathcal{F}\}.$$

Let $\mathcal{F}^{-1} := \{(\xi, \eta) \mid (\eta, \xi) \in \mathcal{F}\}$, which is called the *opposite* relation. By the same token we define $\mathcal{F}^{-1}(\mathcal{U})$. In term is $\mathcal{F}$ the latter is given by

$$\mathcal{F}^{-1}(\mathcal{U}) = \bigcup_{\xi \in \mathcal{U}} \mathcal{F}^{-1}(\xi), \quad \mathcal{F}^{-1}(\xi) := \{\eta \in \mathcal{X} \mid (\eta, \xi) \in \mathcal{F}\}.$$

The *forward invariant sets* and *backward invariant sets* are given by

$$\mathsf{Invset}^+(\mathcal{F}) := \{\mathcal{U} \subset \mathcal{X} \mid \mathcal{F}(\mathcal{U}) \subset \mathcal{U}\} \text{and} \mathsf{Invset}^-(\mathcal{F}) := \{\mathcal{U} \subset \mathcal{X} \mid \mathcal{F}^{-1}(\mathcal{U}) \subset \mathcal{U}\}.$$

These sets are sublattices of the Boolean algebra $\mathsf{Set}(\mathcal{X})$, and the complement map $\mathcal{U} \mapsto \mathcal{U}^c$ is a lattice isomorphism from $\mathsf{Invset}^+(\mathcal{F})$ to $\mathsf{Invset}^-(\mathcal{F})$. A subset set $\mathcal{S} \subset \mathcal{X}$ is an *invariant set* if $\mathcal{S} \subset \mathcal{F}(\mathcal{S})$ and $\mathcal{S} \subset \mathcal{F}^{-1}(\mathcal{S})$. The invariant sets are denoted by $\mathsf{Invset}(\mathcal{F})$, which is a lattice (not necessarily distributive). As in the continuous case, $\mathrm{Inv}(\mathcal{U})$ denotes the maximal invariant set in $\mathcal{U}$.

The sets of all *attractors* and *repellers* of $\mathcal{F}$ are denoted by

$$\mathsf{Att}(\mathcal{F}) := \{\mathcal{A} \subset \mathcal{X} \mid \mathcal{F}(\mathcal{A}) = \mathcal{A}\} \text{and} \mathsf{Rep}(\mathcal{F}) := \{\mathcal{R} \subset \mathcal{X} \mid \mathcal{F}^{-1}(\mathcal{R}) = \mathcal{R}\},$$

respectively, and are finite distributive lattices. Note that attractors and repellers are not necessarily invariant sets. If $\mathcal{X}$ itself is invariant, i.e. the relation $\mathcal{F}$ is *total*, then both attractors and repellers are invariant sets.

The omega and alpha limit sets in this setting are defined as follows

$$\boldsymbol{\omega}(\mathcal{U}) := \bigcap_{k \leqslant 0} \bigcup_{n \leqslant k} \mathcal{F}^n(\mathcal{U}); \tag{50}$$

and

$$\boldsymbol{\alpha}(\mathcal{U}) := \bigcap_{k \leqslant 0} \bigcup_{n \leqslant k} \mathcal{F}^n(\mathcal{U}), \tag{51}$$

which are forward and backward invariant sets respectively.

By [26, Proposition 2.8], $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ define surjective lattice homomorphisms onto $\mathsf{Att}(\mathcal{F})$ and $\mathsf{Rep}(\mathcal{F})$ which yields the followinf commutative diagram

$$\begin{array}{ccc} \mathsf{Invset}^+(\mathcal{F}) & \overset{c}{\longleftrightarrow} & \mathsf{Invset}^-(\mathcal{F}) \\ \boldsymbol{\omega} \big\downarrow & & \big\downarrow \boldsymbol{\alpha} \\ \mathsf{Att}(\mathcal{F}) & \overset{*}{\longleftrightarrow} & \mathsf{Rep}(\mathcal{F}) \end{array} \tag{52}$$

where $\mathcal{A} \mapsto \mathcal{A}^* := \boldsymbol{\alpha}(\mathcal{A}^c)$, cf. [26, Diagram (5)].

Regular Closed Sets

For the purpose of relating combinatorial dynamics to topological dynamics it is useful to restrict the collection of sets used to discretize phase space. Let $(X, \mathscr{T})$ be a topological space. Define $U^{\#\#} := \mathrm{cl}\,\mathrm{int}\, U$, then sets satisfying $U^{\#\#} = U$ are called the *regular closed* sets in $\mathsf{Set}(X)$ which form a complete Boolean algebra $\mathscr{R}(X)$ under the operations

$$U^{\#} := \mathrm{cl}\, U^c, \quad U \vee U' := U \cup U' \quad \text{and} \quad U \wedge U' := (U \cap U')^{\#\#} = \mathrm{cl}\,(\mathrm{int}\, U \cap \mathrm{int}\, U')$$

cf. [40].

Lemma 22 *$^{\#\#}\colon \mathscr{C}(X) \to \mathscr{R}(X)$ given by $U \mapsto U^{\#\#}$ is a lattice homomorphism.*

Proof By definition $U \mapsto U^{\#\#}$ is an idempotent, order-preserving operator from $\mathsf{Set}(X) \to \mathscr{R}(X)$. A set U is closed if and only if $U = \mathrm{cl}\, U$. Let $\mathscr{C}(X)$ be the lattice of closed subsets X which is a sublattice of $\mathsf{Set}(X)$. Since $\mathrm{int}\, U \subset U$, we have that $U^{\#\#} = \mathrm{cl}\,\mathrm{int}\, U \subset \mathrm{cl}\, U = U$, which proves that $U \mapsto U^{\#\#}$ is also a contractive operator. From the order-preserving property we have that $(U \cap U')^{\#\#} \subset U^{\#\#} \cap U'^{\#\#}$. From all properties combined we have

$$(U^{\#\#} \cap U'^{\#\#})^{\#\#} \subset (U \cap U')^{\#\#} = (U \cap U')^{\#\#\#\#} \subset (U^{\#\#} \cap U'^{\#\#})^{\#\#}$$

which proves

$$(U \cap U')^{\#\#} = (U^{\#\#} \cap U'^{\#\#})^{\#\#} = U^{\#\#} \wedge U'^{\#\#}.$$

For unions

$$(U \cup U')^{\#\#} = U^{\#\#} \cup U'^{\#\#},$$

is proved in [20, Sect. 4, Lem. 4] for regular open sets. The same statement for regular closed sets follows from duality $U \mapsto U^c$. □

For regular closed sets, the notion of 'set-difference' is defined by

$$U - U' := U \wedge U'^{\#} \tag{53}$$

Set-difference in $\mathscr{R}(X)$ can be related to set-difference in $\mathsf{Set}(X)$.

Lemma 23 *Let* $U, U' \in \mathscr{R}(X)$. *Then* $U - U' = \mathrm{cl}\,(U \smallsetminus U')$.

Proof By definition $U \wedge U'^{\#} = \mathrm{cl}\,\mathrm{int}\,(U \cap U'^{\#}) = \mathrm{cl}\,\mathrm{int}\,(U \cap \mathrm{cl}\,U'^{c})$. Since U' is a regular closed set, the complement U'^{c} is a regular open set, and therefore $\mathrm{int}\,(\mathrm{cl}\,U'^{c}) = U'^{c}$. This yields

$$\mathrm{cl}\,\mathrm{int}\,(U \cap \mathrm{cl}\,U'^{c}) = \mathrm{cl}\,(\mathrm{int}\,U \cap \mathrm{int}\,\mathrm{cl}\,U'^{c}) = \mathrm{cl}\,(\mathrm{int}\,U \cap U'^{c}).$$

Finally, since U is a regular closed set, we have that $\mathrm{cl}\,\mathrm{int}\,U = U$ and thus $\mathrm{cl}\,(\mathrm{int}\,U \cap U'^{c}) = \mathrm{cl}\,(U \cap U'^{c})$, cf. [40, pp. 35]. Combining this with the previous we obtain

$$\mathrm{cl}\,\mathrm{int}\,(U \cap U'^{c}) = \mathrm{cl}\,(U \cap U'^{c}) = \mathrm{cl}\,(U \smallsetminus U'),$$

which proves the lemma. □

References

1. Akin, E.: The General Topology of Dynamical Systems. American Mathematical Soc, London (2010)
2. Arai, Z., Kalies, W., Kokubu, H., Mischaikow, K., Oka, H., Pilarczyk, P.: A database schema for the analysis of global dynamics of multiparameter systems. SIAM J. Appl. Dyn. Syst. **8**(3), 757–789 (2009)
3. Arnold, L.: Random Dynamical Systems. Springer Monographs in Mathematics, Springer, Berlin (1998)
4. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Modern Birkhäuser Classics, Birkhäuser Boston Inc, Boston (2009)
5. Balbes, R., Dwinger, P.: Distributive Lattices. University of Missouri Press, Columbia (1974)
6. Ban, H., Kalies, W.: A computational approach to Conleys decomposition theorem. J. Comput. Nonlinear Dyn. **1**, 312–319 (2006)
7. Birkhoff, G., Bennett, M.: The convexity lattice of a poset. Order **2**(3), 223–242 (1985)
8. Boczko, E., Kalies, W., Mischaikow, K.: Polygonal approximation of flows. Topol. Appl. **154**(13), 2501–2520 (2007)
9. Bush, J., Gameiro, M., Harker, S., Kokubu, H., Mischaikow, K., Obayashi, I., Pilarczyk, P.: Combinatorial-topological framework for the analysis of global dynamics. Chaos **22(4), 047508**, 16 (2012)
10. Conley, C.: Isolated Invariant Sets and the Morse Index. CBMS Regional Conference Series in Mathematics, vol. 38. American Mathematical Society, Providence (1978)
11. Cummins, B., Gedeon, T., Harker, S., Mischaikow, K., Mok, K.: Combinatorial representation of parameter space for switching networks. SIAM J. Appl. Dyn. Syst. **15**(4), 2176–2212 (2016)
12. Davey, B., Priestley, H.: Introduction to Lattices and Order, p. xii+298. Cambridge University Press, Cambridge (2002)
13. Day, S., Kalies, W.: Rigorous computation of the global dynamics of integrodifference equations with smooth nonlinearities. SIAM J. Numer. Anal. **51**(6), 2957–2983 (2013)
14. Fiedler, B., Mallet-Paret, J.: Connections between Morse sets for delay-differential equations. J. Reine Angew. Math. **397**, 23–41 (1989)
15. Franzosa, R.: Index filtrations and the homology index braid for partially ordered Morse decompositions. Trans. Am. Math. Soc. **298**(1), 193–213 (1986)
16. Franzosa, R.: The connection matrix theory for Morse decompositions. Trans. Am. Math. Soc. **311**(2), 561–592 (1989)
17. Gedeon, T., Harker, S., Kokubu, H., Mischaikow, K., Oka, H.: Global dynamics for steep nonlinearities in two dimensions. Phys. D **339**, 18–38 (2017)
18. Goldstein, A., Kalies, W., Vandervorst, R.: Priestley space and chain-recurrence. Preparation (2021)
19. Goullet, A., Harker, S., Mischaikow, K., Kalies, W., Kasti, D.: Efficient computation of Lyapunov functions for Morse decompositions. Discrete Contin. Dyn. Syst. Ser. B **20**(8), 2418–2451 (2015)
20. Halmos, P.: Lectures on Boolean algebras. Van Nostrand Mathematical Studies, No. 1. D. Van Nostrand Co., Inc., Princeton, N.J. (1963)

21. Harker, S., Mischaikow, K., Spendlove, K.: A computational framework for connection matrix theory. Preprint. (2019)
22. Kaczynski, T., Mrozek, M., Wanner, T.: Towards a formal tie between combinatorial and classical vector field dynamics. J. Comput. Dyn. **3**(1), 17–50 (2016)
23. Kalies, W., Kasti, D., Vandervorst, R.: An algorithmic approach to lattices and order in dynamics. SIAM J. Appl. Dyn. Syst. **17**(2), 1617–1649 (2018)
24. Kalies, W., Mischaikow, K., Vandervorst, R.: An algorithmic approach to chain recurrence. Found. Comput. Math. **5**(4), 409–449 (2005)
25. Kalies, W., Mischaikow, K., Vandervorst, R.: Lattice structures for attractors I. J. Comput. Dyn. **1**(2), 307–338 (2014)
26. Kalies, W., Mischaikow, K., Vandervorst, R.: Lattice structures for attractors II. Found. Comput. Math. **16**, 1151–1191 (2016)
27. Mallet-Paret, J.: Morse decompositions for delay-differential equations. J. Differ. Equ. **72**(2), 270–315 (1988). https://doi.org/10.1016/0022-0396(88)90157-X
28. McCord, C., Mischaikow, K.: On the global dynamics of attractors for scalar delay equations. J. Am. Math. Soc. **9**(4), 1095–1133 (1996). https://doi.org/10.1090/S0894-0347-96-00207-X
29. McGehee, R.: Attractors for closed relations on compact Hausdorff spaces. Indiana Univ. Math. J. **41**(4), 1165–1209 (1992)
30. Miraglia, F.: An Introduction to Partially Ordered Structures and Sheaves, Contemporary Logic Series, vol. 1. Polimetrica Scientific Publisher, Milan (2006)
31. Mischaikow, K.: Global asymptotic dynamics of gradient-like bistable equations. SIAM J. Math. Anal. **26**(5), 1199–1224 (1995). https://doi.org/10.1137/S0036141093250827
32. Mischaikow, K., Mrozek, M.: Conley index. In: Handbook of dynamical systems, vol. 2, pp. 393–460. North-Holland, Amsterdam (2002). https://doi.org/10.1016/S1874-575X(02)80030-3. https://doi-org.proxy.libraries.rutgers.edu/10.1016/S1874-575X(02)80030-3
33. Mischaikow, K., Spendlove, K., Vandervorst, R.: Morse pre-orders, connection matrices and state transition models. Preparation (2021)
34. Mrozek, M.: Conley-Morse-Forman theory for combinatorial multivector fields on Lefschetz complexes. Found. Comput. Math. **17**(6), 1585–1633 (2017)
35. Robbin, J., Salamon, D.: Lyapunov maps, simplicial complexes and the Stone functor. Ergod. Th. Dyn. Sys (1992)
36. Roman, S.: Lattices and Ordered Sets. Springer, New York (2008)
37. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–817 (1967)
38. Ugarcovici, I., Weiss, H.: Chaotic dynamics of a nonlinear density dependent population model. Nonlinearity **17**(5), 1689–1711 (2004). https://doi.org/10.1088/0951-7715/17/5/007
39. Vickers, S.: Topology via Logic, Cambridge Tracts in Theoretical Computer Science, vol. 5. Cambridge University Press, Cambridge (1989)
40. Walker, R.: The Stone-Cech Compactification. Springer, New York (1974)