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Hopf Bifurcation in a Reaction–Diffusion–Advection Two Species Model with Nonlocal Delay Effect

Zhenzhen Li[1] · **Binxiang Dai**[1] · **Renji Han**[2]

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Abstract
The dynamics of a general reaction–diffusion–advection two species model with nonlocal delay effect and Dirichlet boundary condition is investigated in this paper. The existence and stability of the positive spatially nonhomogeneous steady state solution are studied. Then by regarding the time delay τ as the bifurcation parameter, we show that Hopf bifurcation occurs near the steady state solution at the critical values $\tau_n (n = 0, 1, 2, \ldots)$. Moreover, the Hopf bifurcation is forward and the bifurcated periodic solutions are stable on the center manifold. The general results are applied to a Lotka–Volterra competition–diffusion–advection model with nonlocal delay.

Keywords Reaction–diffusion–advection two species model · Nonlocal delay · Stability · Hopf bifurcation

Mathematics Subject Classification 35R10 · 35K57 · 35B32 · 35B35 · 35B10 · 92D40

1 Introduction

In the past few decades, the dynamical models in the form of reaction-diffusion equations have been frequently used to solve problems related to spatial ecology and evolution, see [5,22,23,25,27]. In the real world, due to reproductive maturity or other time lags in biological processes, historical information may have a significant impact on the dynamics of population

✉ Binxiang Dai
bxdai@csu.edu.cn

Zhenzhen Li
zhzhli@csu.edu.cn

Renji Han
rjhan@zust.edu.cn

1 School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, Hunan, People's Republic of China

2 School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, Zhejiang, People's Republic of China

systems, so many delayed reaction-diffusion equations are used to describe the evolution of population distribution [1,4,6,15,30].

The dispersal by random diffusion is one of the most basic dispersal strategies. In reality, the movements of species may be a combination of both random and biased ones. Since species are intelligent, many organisms can sense their environment and pay attention to moving in a direction that is favorable to them. Based on this observation, Belgacem and Cosner [2] assumed that the population can exhibit a taxis in the direction of increasing environmental favorability, and studied the reaction–diffusion–advection logistic model

$$\begin{cases} u_t = \nabla \cdot [d\nabla u - au\nabla m] + m(x)u - cu^2, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $u(x, t)$ denotes the species density at location x and time t. In model (1.1), the term $-d\nabla u$ accounts for random diffusion, $au\nabla m(x)$ represents the migration of species along the gradient direction of resource function $m(x)$. The results of Belgacem and Cosner [2] and the subsequent literature [9] show that for single species, migration along the gradient direction of the food distribution of the species will generally contribute to the survival of the species.

We would like to know what spatiotemporal patterns can be indeced by the joint effect of time delays, spatial diffusion, advection, heterogeneous environment and population interaction. In a reaction–diffusion–advection model with time delay effect, the effects of dispersal and time delays are not independent of each other, and an individual that was previously at location x may now not be at the same point in space [3,6,12]. Therefore, it is more reasonable to consider the model with nonlocal time delay. Recently, Jin and Yuan [19] investigated the following general delayed reaction–diffusion–advection equation

$$\begin{cases} u_t = \nabla \cdot [d\nabla u - au\nabla m] + u(x,t) f\left(x, u(x,t), \int_\Omega k(x,y)u(y,t-\tau)\mathrm{d}y\right), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.2}$$

where $f(x, 0, 0) = m(x)$ and the term $\int_\Omega k(x, y)u(y, t-\tau)\mathrm{d}y$ is called the nonlocal delayed term, which represents the spatial weighted time delays according to distance from the original position. In [19], Jin and Yuan showed the existence of spatially nonhomogeneous steady-state solutions of (1.2) and investigated whether time delay τ can induce Hopf bifurcation near the steady-state solution. They also showed the influence of the advection rate a on Hopf bifurcation.

The dynamics of the two-species model has been extensively studied, including the global stability of (non-)constant steady states [7,10,17,20,24,26] or Hopf bifurcations induced by time delays at the (non-)constant steady states [13,18,29,31,33]. For instance, in [14,15], the authors considered the diffusive two-species model with nonlocal delay effect and investigated the stability of spatially nonhomogeneous positive steady state and the corresponding Hopf bifurcation problem. Recently, Li and Dai [21] have studied the following Lotka–Volterra competition–diffusion–advection model with time delay effect:

$$\begin{cases} u_t = \nabla \cdot [d\nabla u - au\nabla m] + u\left[m(x) - a_{11}u(x,t-\tau) - a_{12}v(x,t-\tau)\right], & x \in \Omega, t > 0, \\ v_t = \nabla \cdot [d\nabla v - av\nabla m] + v\left[m(x) - a_{21}u(x,t-\tau) - a_{22}v(x,t-\tau)\right], & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{1.3}$$

They obtained the existence of spatially nonhomogeneous positive steady state and showed that this positive steady state loses its stability for a large delay τ and a Hopf bifurcation occurs such that system (1.3) exhibits oscillatory pattern.

Motivated by [6,14,19], we can assume that in a two-species model, the per-capita growth rates of two species do not depend on its density at the current positions and time but on all positions in region Ω and previous time τ. Hence, the localized density-dependent per capita growth rates $m(x)-a_{11}u(x,t-\tau)-a_{12}v(x,t-\tau)$ and $m(x)-a_{21}u(x,t-\tau)-a_{22}v(x,t-\tau)$ in (1.3) are not realistic. Instead, it is more reasonable to consider the following general reaction–diffusion–advection two species model with nonlocal delay effect as follows:

$$\begin{cases} u_t = \nabla\cdot[d\nabla u - au\nabla m] + u(x,t)f_1\big(x,(k_{11}*u)(x,t-\tau),(k_{12}*v)(x,t-\tau)\big), & x\in\Omega, t>0,\\ v_t = \nabla\cdot[d\nabla v - av\nabla m] + v(x,t)f_2\big(x,(k_{21}*u)(x,t-\tau),(k_{22}*v)(x,t-\tau)\big), & x\in\Omega, t>0,\\ u(x,t)=v(x,t)=0, & x\in\partial\Omega, t>0. \end{cases} \tag{1.4}$$

Here $u(x,t)$ and $v(x,t)$ denote the species densities at time t and location x, respectively; the two species have the same diffusion rate $d>0$ and the same advection rate $a>0$; Ω is a bounded domain in $\mathbb{R}^n(1\le n\le 3)$ with smooth boundary $\partial\Omega$; τ is the time delay representing the maturation time; $k_{ij}(i,j=1,2)$ are continuous kernel functions on $\Omega\times\Omega$ which describe the dispersal behavior of the populations and

$$(k_{i1}*u)(x,t)=\int_\Omega k_{i1}(x,y)u(y,t)\mathrm{d}y,\ (k_{i2}*v)(x,t)=\int_\Omega k_{i1}(x,y)v(y,t)\mathrm{d}y,\ i=1,2;$$

the nonlinear smooth functions $f_i(x,s_1,s_2)(i=1,2):\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are called the general per capita growth rates and satisfy the condition

($\mathbf{H_1}$) $f_1(x,0,0)=f_2(x,0,0)=m(x)$, where $m(x)\in C^2(\overline{\Omega})$ and $\max_{\overline{\Omega}} m(x)>0$.

The Dirichlet boundary conditions imply that the exterior environment is hostile and the two species cannot move across the boundary of environment. We consider model (1.4) with the following initial condition:

$$u(x,s)=\varphi_1(x,s),\ v(x,0)=\varphi_2(x,s),\ \ x\in\Omega, s\in[-\tau,0],$$

where the initial data $\varphi_1,\varphi_2\in\mathscr{C}\triangleq C([-\tau,0],\mathbb{Y})$ with $\mathbb{Y}=L^2(\Omega)$.

For the convenience of analysis, we first make a variable transformation. Letting $\tilde{u}=e^{(-a/d)m(x)}u$, $\tilde{v}=e^{(-a/d)m(x)}v$, $t=\tilde{t}/d$, denoting $\lambda=1/d$, $\alpha=a/d$, $\tau=\tilde{\tau}/d$, and dropping the tilde sign, model (1.4) can be transformed as follows:

$$\begin{cases} u_t = e^{-\alpha m(x)}\nabla\cdot\big[e^{\alpha m(x)}\nabla u\big] + \lambda u(x,t)f_1\big(x,(K_{11}*u)(x,t-\tau),(K_{12}*v)(x,t-\tau)\big), & x\in\Omega, t>0,\\ v_t = e^{-\alpha m(x)}\nabla\cdot\big[e^{\alpha m(x)}\nabla v\big] + \lambda v(x,t)f_2\big(x,(K_{21}*u)(x,t-\tau),(K_{22}*v)(x,t-\tau)\big), & x\in\Omega, t>0,\\ u(x,t)=v(x,t)=0, & x\in\partial\Omega, t>0, \end{cases} \tag{1.5}$$

where

$$(K_{ij}*g)(x,t)=\int_\Omega K_{ij}(x,y)g(y,t)\mathrm{d}y,$$

with $K_{ij}(x,y)=k_{ij}(x,y)e^{\alpha m(y)}$ for $i,j=1,2$.

For the simplification of calculation, denote

$$r_1^i(x)=\frac{\partial f_i}{\partial s_1}(x,0,0),\ \ r_2^i(x)=\frac{\partial f_i}{\partial s_2}(x,0,0),$$

and

$$\kappa_{ij} = \int_{\Omega} \int_{\Omega} e^{\alpha m(x)} r_j^i(x) K_{ij}(x, y) \phi^2(x) \phi(y) \mathrm{d}y \mathrm{d}x \neq 0$$

for $i, j = 1, 2$. Suppose further the following two assumptions hold:

($\mathbf{H_2}$) $(\kappa_{21} - \kappa_{11})(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}) > 0$ and $(\kappa_{12} - \kappa_{22})(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21}) > 0$;
($\mathbf{H_3}$) $\kappa_{11}\kappa_{22} \geq 0$ and $\kappa_{12}\kappa_{21} \geq 0$.

It follows from [2,5,23] that, under the assumption ($\mathbf{H_1}$), the following eigenvalue problem

$$\begin{cases} -e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla u] = \lambda m(x) u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

has a positive principal eigenvalue λ_* and the corresponding eigenfunction $\phi \in C^{1+\delta}(\overline{\Omega})$ can be chosen strictly positive in Ω, where $\delta \in (0, 1)$. In this paper our main results are in the spirit of [21] for the local growth rate case: under the assumptions ($\mathbf{H_1}$) $-$ ($\mathbf{H_3}$), there exists a λ^* with $0 < \lambda^* - \lambda_* \ll 1$, such that for any $\lambda \in (\lambda_*, \lambda^*]$, system (1.5) admits a spatially nonhomogeneous positive steady state (u_λ, v_λ) and there exists a sequence of values $\{\tau_n(\lambda)\}_{n=0}^{\infty}$ such that (u_λ, v_λ) is locally asymptotically stable when $\tau \in [0, \tau_0(\lambda))$, unstable when $\tau \in (\tau_0(\lambda), \infty)$, and for system (1.5), a forward Hopf bifurcation occurs at $\tau_n(\lambda)$ from the positive steady state (u_λ, v_λ). Moreover, Hopf bifurcation is more likely to occur when adding a term describing advection along the environmental gradients for the diffusive Lotka–Volterra competition model with nonlocal delay. Here, the assumption ($\mathbf{H_2}$) is used to guarantee the existence of positive steady-state solutions. ($\mathbf{H_3}$) is imposed to make sure the simplicity of pure imaginary eigenvalue and is actually satisfied by many population biological models.

The rest part of this paper is organized as follows. In Sect. 2, we establish the existence of the positive steady state of model (1.5). Sections 3 and 4 are devoted to the stability and Hopf bifurcation of the positive steady state through analyzing the corresponding eigenvalue problem. Then the normal form of Hopf bifurcation is derived in Sect. 5 to determine the bifurcation direction and stability of the bifurcating periodic solutions. In Sect. 6, the general results are applied to a competition–diffusion–advection model with nonlocal delay effect.

Notice that the elliptic operator in (1.6) is not self-adjoint because of advection term, which causes some technical difficulties. In view of these difficulties we introduce some weighted spaces. The weighted space plays a vital role in the Hopf bifurcation analysis of system (1.5). Throughout the paper, we use the following notations. Denote by $L_w^2(\Omega)$ the weighted L^2 space with a weighted norm

$$\|u\|_{L_w^2(\Omega)} = \left(\int_{\Omega} e^{\alpha m(x)} |u(x)|^2 \mathrm{d}x \right)^{1/2}.$$

Let $H_w^k(\Omega)(k \geq 0)$ be the weighted Sobolev space of the L_w^2-function $u(x)$ defined on Ω, and the norm of space $H_w^k(\Omega)(k \geq 0)$ is defined by

$$\|u\|_{H_w^k(\Omega)} = \left(\sum_{|j| \leq k} \int_{\Omega} e^{\alpha m(x)} |\partial^j u|^2 \mathrm{d}x \right)^{1/2}.$$

Define the space $\mathbb{X} = H_w^2(\Omega) \cap H_{0,w}^1(\Omega)$ and $\mathbb{Y} = L_w^2(\Omega)$, where

$$H_{0,w}^1(\Omega) = \left\{ u \in H_w^1(\Omega) | u(x) = 0, \forall x \in \partial\Omega \right\}.$$

For a space Z, we also define the complexification of Z to be $Z_{\mathbb{C}} := Z \oplus \mathrm{i}Z = \{x_1 + \mathrm{i}x_2 | x_1, x_2 \in Z\}$. Let $\mathcal{C} = C([-\tau, 0], \mathbb{Y})$ be the Banach space of continuous mapping from $[-\tau, 0]$ into $\mathbb{Y}$, and $\langle \cdot, \cdot \rangle_w$ be the L^2_w inner product on complex-valued Hilbert space $\mathbb{Y}_{\mathbb{C}}$ or $\mathbb{Y}^2_{\mathbb{C}}$, defined as

$$\langle u, v \rangle_w = \int_\Omega e^{\alpha m(x)} \overline{u}(x)^T v(x) \mathrm{d}x. \tag{1.7}$$

2 Existence of Positive Steady State

This section is devoted to the the existence of the positive steady state of model (1.5), which satisfies the following boundary value equation:

$$\begin{cases} e^{-\alpha m(x)} \nabla \cdot \left[e^{\alpha m(x)} \nabla u \right] + \lambda u(x) f_1\big(x, (K_{11} * u)(x), (K_{12} * v)(x)\big) = 0, & x \in \Omega, \\ e^{-\alpha m(x)} \nabla \cdot \left[e^{\alpha m(x)} \nabla v \right] + \lambda v(x) f_2\big(x, (K_{21} * u)(x), (K_{22} * v)(x)\big) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

To solve problem (2.1), define $F : \mathbb{X}^2 \times R^+ \to \mathbb{Y}^2$ as

$$F(U, \lambda) = \begin{bmatrix} e^{-\alpha m} \nabla \cdot \left[e^{\alpha m} \nabla u \right] + \lambda u f_1\big(x, K_{11} * u, K_{12} * v\big) \\ e^{-\alpha m} \nabla \cdot \left[e^{\alpha m} \nabla v \right] + \lambda v f_2\big(x, K_{21} * u, K_{22} * v\big) \end{bmatrix}$$

for all $U = (u, v)^T$. At first, for any fixed $\lambda \in \mathbb{R}^+$, $F(U, \lambda)$ always has a trivial steady state $(0, 0)$. Denote

$$\mathcal{L} := \begin{pmatrix} e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla] + \lambda_* m(x) & 0 \\ 0 & e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla] + \lambda_* m(x) \end{pmatrix}, \tag{2.2}$$

which is the Frechét derivative of F with respect to U at $(0, \lambda_*)$. It is easy to check that $\mathcal{L}$ is a self-adjoint operator in the sense of weighted inner product, and $\mathscr{N}(\mathcal{L}) = \mathrm{span}\{q_1, q_2\}$, where $q_1 = (\phi, 0)^T$ and $q_2 = (0, \phi)^T$. Then operator $\mathcal{L} : \mathbb{X}^2 \to \mathbb{Y}^2$ is Fredholm with index zero. Clearly, the Crandall-Rabinowitz bifurcation theorem cannot be applied here to show the existence of positive solution of (2.1) since $\dim \mathscr{N}(\mathcal{L}) = 2$. Now, we deal with this situation by implicit function theorem. For later discussion, we decompose the spaces $\mathbb{X}^2$, $\mathbb{Y}^2$:

$$\mathbb{X}^2 = \mathscr{N}(\mathcal{L}) \oplus X_1^2, \quad \mathbb{Y}^2 = \mathscr{N}(\mathcal{L}) \oplus Y_1^2,$$

where

$$X_1 = \{ y \in \mathbb{X} : \langle \phi, y \rangle_w = 0 \}, \ Y_1 = \{ y \in \mathbb{Y} : \langle \phi, y \rangle_w = 0 \}.$$

Then we have the following result on the existence of positive steady states for system (1.5).

Theorem 1 *Suppose that* $(\mathbf{H_1}) - (\mathbf{H_3})$ *hold. Then there exist* $\lambda^* > \lambda_*$ *and a continuously differential mapping* $\lambda \mapsto (\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda)$ *from* $[\lambda_*, \lambda^*]$ *to* $X_1^2 \times \mathbb{R}^+ \times \mathbb{R}^+$ *such that, for* $\lambda \in (\lambda_*, \lambda^*]$, *system* (1.5) *has a positive steady state* $(u_\lambda(x), v_\lambda(x))$, *where*

$$\begin{cases} u_\lambda = \beta_\lambda (\lambda - \lambda_*) [\phi + (\lambda - \lambda_*) \xi_\lambda], \\ v_\lambda = c_\lambda (\lambda - \lambda_*) [\phi + (\lambda - \lambda_*) \eta_\lambda]. \end{cases} \tag{2.3}$$

Moreover, for $\lambda = \lambda_*$,

$$\begin{cases} \beta_{\lambda_*} = \dfrac{(\kappa_{12} - \kappa_{22}) \int_\Omega m(x) e^{\alpha m(x)} \phi^2(x) dx}{\lambda_* (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})}, \\ c_{\lambda_*} = \dfrac{(\kappa_{21} - \kappa_{11}) \int_\Omega m(x) e^{\alpha m(x)} \phi^2(x) dx}{\lambda_* (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})}, \end{cases} \tag{2.4}$$

and $(\xi_{\lambda_*}, \eta_{\lambda_*})^T \in X_1^2$ *is the unique solution of the following equation*

$$\mathcal{L}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} m(x) + \lambda_* \beta_{\lambda_*} r_1^1(x) \int_\Omega K_{11}(x,y)\phi(y)\mathrm{d}y + \lambda_* c_{\lambda_*} r_2^1(x) \int_\Omega K_{12}(x,y)\phi(y)\mathrm{d}y \\ m(x) + \lambda_* \beta_{\lambda_*} r_1^2(x) \int_\Omega K_{21}(x,y)\phi(y)\mathrm{d}y + \lambda_* c_{\lambda_*} r_2^2(x) \int_\Omega K_{22}(x,y)\phi(y)\mathrm{d}y \end{pmatrix} \phi = 0. \tag{2.5}$$

Proof One can easily have

$$\lambda_* \int_\Omega m(x) e^{\alpha m(x)} \phi^2(x) dx = \int_\Omega e^{\alpha m(x)} |\nabla \phi(x)|^2 dx > 0,$$

and then β_{λ_*} and c_{λ_*} are well defined and positive. From (2.4), we see that

$$\begin{cases} \left\langle \phi, \left(m(x) + \lambda_* \beta_{\lambda_*} r_1^1(x) \int_\Omega K_{11}(x,y)\phi(y)\mathrm{d}y + \lambda_* c_{\lambda_*} r_2^1(x) \int_\Omega K_{12}(x,y)\phi(y)\mathrm{d}y \right)\phi \right\rangle_w = 0, \\ \left\langle \phi, \left(m(x) + \lambda_* \beta_{\lambda_*} r_1^2(x) \int_\Omega K_{21}(x,y)\phi(y)\mathrm{d}y + \lambda_* c_{\lambda_*} r_2^2(x) \int_\Omega K_{22}(x,y)\phi(y)\mathrm{d}y \right)\phi \right\rangle_w = 0, \end{cases}$$

and hence ξ_{λ_*} and η_{λ_*} are also well defined. Now, setting $u = \beta(\lambda - \lambda_*)[\phi + (\lambda - \lambda_*)\xi]$ and $v = c(\lambda - \lambda_*)[\phi + (\lambda - \lambda_*)\eta]$ into $F(U, \lambda) = 0$, we obtain that (β, c, ξ, η) satisfies

$$\mathcal{F}(\xi, \eta, \beta, c, \lambda) = \mathcal{L}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + m(x)\begin{pmatrix} \phi + (\lambda - \lambda_*)\xi \\ \phi + (\lambda - \lambda_*)\eta \end{pmatrix} + \lambda \begin{pmatrix} [\phi + (\lambda - \lambda_*)\xi] h_1(\xi, \eta, \beta, c, \lambda) \\ [\phi + (\lambda - \lambda_*)\eta] h_2(\xi, \eta, \beta, c, \lambda) \end{pmatrix} = 0,$$

where

$$h_i(\xi, \eta, \beta, c, \lambda) = \begin{cases} \dfrac{f_i(x, K_{i1} * u, K_{i2} * v) - m(x)}{\lambda - \lambda_*}, & \lambda \neq \lambda_*, \\ \beta r_1^i(x) \widetilde{k}_{i1}(x) + c r_2^i(x) \widetilde{k}_{i2}(x), & \lambda = \lambda_* \end{cases} \tag{2.6}$$

with

$$\widetilde{k}_{ij}(x) = \int_\Omega K_{ij}(x,y)\phi(y)\mathrm{d}y \ \text{ for } \ i, j = 1, 2. \tag{2.7}$$

That is, the existence problem of positive solution of (2.1) is reduced to solving $\mathcal{F}(\xi, \eta, \beta, c, \lambda) = 0$. Seeing that Ω is a bounded domain in $\mathbb{R}^n (1 \leq n \leq 3)$ with smooth boundary, we can deduce that X_1^2 is compactly imbedded into $C^\delta(\overline{\Omega}) \times C^\delta(\overline{\Omega})$ for some $\delta \in (0, 1)$. Then $\mathcal{F}(\xi, \eta, \beta, c, \lambda)$ is a function from $X_1^2 \times \mathbb{R}^3$ to $\mathbb{Y}^2$. From the definition of $\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}$, it holds that $\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*) = 0$. Let $D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)[\chi, \kappa, \rho, \varsigma]$ be the Fréchet derivative of $\mathcal{F}$ with respect to (ξ, η, β, c) evaluated at $(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)$. Then a calculation gives that

$$\begin{aligned} & D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)[\chi, \kappa, \rho, \varsigma] \\ & = \mathcal{L}\begin{pmatrix} \chi \\ \kappa \end{pmatrix} + \lambda_* \phi \begin{pmatrix} \rho r_1^1(x) \widetilde{k}_{11}(x) + \varsigma r_2^1(x) \widetilde{k}_{12}(x) \\ \rho r_1^2(x) \widetilde{k}_{21}(x) + \varsigma r_2^2(x) \widetilde{k}_{22}(x) \end{pmatrix}. \end{aligned}$$

For applying the implitic function theorem, we will verify that $D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)$ is a bijection from $X_1^2 \times \mathbb{R}^2$ to $\mathbb{Y}^2$. Since $\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} \neq 0$ due to assumption $(\mathbf{H_2})$, one can deduce that

$$(\xi, \eta, \beta, c) = (0, 0, 0, 0) \ \text{ if } \ D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)[\chi, \kappa, \rho, \varsigma] = 0.$$

That is, $D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)$ is injective. Next, we show that it is a surjection. For any $(\hat{u}, \hat{v})^T \in \mathbb{Y}^2$, we have the following decomposition

$$\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix} = \begin{pmatrix}u_1\\ v_1\end{pmatrix} + \begin{pmatrix}u_2\\ v_2\end{pmatrix}, \quad \text{where } \begin{pmatrix}u_1\\ v_1\end{pmatrix} \in \mathscr{N}(\mathcal{L}), \begin{pmatrix}u_2\\ v_2\end{pmatrix} \in Y_1^2.$$

By choosing

$$\rho_0 = \frac{\kappa_{22}\langle\phi, u_1\rangle_w - \kappa_{12}\langle\phi, v_1\rangle_w}{\lambda_*(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})} \text{ and } \varsigma_0 = \frac{\kappa_{11}\langle\phi, u_1\rangle_w - \kappa_{21}\langle\phi, v_1\rangle_w}{\lambda_*(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})},$$

there holds that

$$-\lambda_*\phi\begin{pmatrix}\rho_0 r_1^1(x)\widetilde{k}_{11}(x) + \varsigma_0 r_2^1(x)\widetilde{k}_{12}(x)\\ \rho_0 r_1^2(x)\widetilde{k}_{21}(x) + \varsigma_0 r_2^2(x)\widetilde{k}_{22}(x)\end{pmatrix} + \begin{pmatrix}u_1\\ v_1\end{pmatrix} \in \mathscr{R}(\mathcal{L}).$$

Notice that $(u_2, v_2)^T \in Y_1^2 = \mathscr{R}(\mathcal{L})$ and $\mathcal{L} : X_1^2 \to Y_1^2$ is a bijection, then there is $(\chi_0, \kappa_0) \in X_1^2$ such that

$$D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)[\chi_0, \kappa_0, \rho_0, \varsigma_0] = (\hat{u}, \hat{v})^T.$$

Consequently, $D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*}, \eta_{\lambda_*}, \beta_{\lambda_*}, c_{\lambda_*}, \lambda_*)$ is bijective from $X_1^2 \times \mathbb{R}^2$ to $\mathbb{Y}^2$. Then from implicit function theorem, there exist $\lambda^* > \lambda_*$ and a continuously differential mapping $\lambda \mapsto (\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_1^2 \times \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\mathcal{F}(\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda, \lambda) = 0, \quad \lambda \in [\lambda_*, \lambda^*].$$

Therefore, (u_λ, v_λ) is a positive solution of Eq. (2.1). The proof is completed. □

3 Eigenvalue Problems

In this section, we will study the eigenvalue problem associated with the positive steady state $U_\lambda = (u_\lambda, v_\lambda)^T$ defined in Theorem 1. Unless otherwise specified, we always assume that $\lambda \in [\lambda_*, \lambda^*]$ with $0 < \lambda^* - \lambda_* \ll 1$, and $(\mathbf{H_1}) - (\mathbf{H_3})$ hold. Linearizing system (1.5) at U_λ, we obtain

$$\begin{cases} \dfrac{\partial u}{\partial t} = e^{-\alpha m(x)}\nabla\cdot\left[e^{\alpha m(x)}\nabla u\right] + \lambda f_1\big(x, K_{11}*u_\lambda, K_{12}*v_\lambda\big)u(x,t) \\ \qquad +\lambda u_\lambda\big[r_1^{1\lambda}(x)K_{11}*u(x,t-\tau) + r_2^{1\lambda}(x)K_{12}*v(x,t-\tau)\big], \quad x\in\Omega, t>0, \\ \dfrac{\partial v}{\partial t} = e^{-\alpha m(x)}\nabla\cdot\left[e^{\alpha m(x)}\nabla v\right] + \lambda f_2\big(x, K_{21}*u_\lambda, K_{22}*v_\lambda\big)v(x,t) \\ \qquad +\lambda v_\lambda\big[r_1^{2\lambda}(x)K_{21}*u(x,t-\tau) + r_2^{2\lambda}(x)K_{22}*v(x,t-\tau)\big], \quad x\in\Omega, t>0, \\ u(x,t) = v(x,t) = 0 \qquad x\in\partial\Omega, t>0, \end{cases} \tag{3.1}$$

where

$$r_1^{i\lambda}(x) = \frac{\partial f_i}{\partial s_1}\big(x, K_{i1}*u_\lambda, K_{i2}*v_\lambda\big), \quad r_2^{i\lambda}(x) = \frac{\partial f_i}{\partial s_2}\big(x, K_{i1}*u_\lambda, K_{i2}*v_\lambda\big)$$

are the Fréchet derivative of $f_i (i = 1, 2)$ about the second term and the third term respectively. Define two linear operators $A_\lambda : \mathbb{X}_\mathbb{C}^2 \to \mathbb{Y}_\mathbb{C}^2$ and $B_\lambda : \mathbb{Y}_\mathbb{C}^2 \to \mathbb{Y}_\mathbb{C}^2$ by

$$A_\lambda := \begin{pmatrix} e^{-\alpha m(x)}\nabla\cdot\left[e^{\alpha m(x)}\nabla\right] & 0 \\ \quad +\lambda f_1\big(x, K_{11}*u_\lambda, K_{12}*v_\lambda\big) & \\ 0 & e^{-\alpha m(x)}\nabla\cdot\left[e^{\alpha m(x)}\nabla\right] \\ & \quad +\lambda f_2\big(x, K_{21}*u_\lambda, K_{22}*v_\lambda\big) \end{pmatrix}$$

and

$$B_\lambda \psi := \begin{pmatrix} \lambda u_\lambda (r_1^{1\lambda}(x) K_{11} * \psi_1 + r_2^{1\lambda}(x) K_{12} * \psi_2) \\ \lambda v_\lambda (r_1^{2\lambda}(x) K_{21} * \psi_1 + r_2^{2\lambda}(x) K_{22} * \psi_2) \end{pmatrix}$$

for $\psi = (\psi_1, \psi_2)^T \in \mathbb{Y}^2_{\mathbb{C}}$. Then A_λ is an infinitesimal generator of a compact C_0 semigroup [28]. From [32], the solution semigroup of Eq. (3.1) has an infinitesimal generator $A_{\tau,\lambda}$ defined by

$$A_{\tau,\lambda}\psi = \dot{\psi} \tag{3.2}$$

with the domain

$$\mathscr{D}(A_{\tau,\lambda}) = \left\{\psi \in C_{\mathbb{C}} \cap C^1_{\mathbb{C}} : \psi(0) \in \mathbb{X}^2_{\mathbb{C}}, \dot{\psi}(0) = A_\lambda \psi(0) + B_\lambda \psi(-\tau)\right\},$$

where $C^1_{\mathbb{C}} = C^1([-\tau, 0], \mathbb{X}^2_{\mathbb{C}})$. Now the stability of (u_λ, v_λ) is determined by the point spectrum of $A_{\tau,\lambda}$, which is

$$\sigma_p(A_{\tau,\lambda}) = \left\{\mu \in \mathbb{C} : \Lambda(\lambda, \mu, \tau)\psi = 0 \text{ for some } \psi \in \mathbb{X}^2_{\mathbb{C}} \setminus \{(0,0)\}\right\},$$

with

$$\Lambda(\lambda, \mu, \tau)\psi = A_\lambda \psi + B_\lambda \psi e^{-\mu\tau} - \mu\psi. \tag{3.3}$$

As in [14], we also need to consider the adjoint operator $A^*_{\tau,\lambda}$ of $A_{\tau,\lambda}$ in the sense of weighted inner product, which is defined as

$$A^*_{\tau,\lambda}\psi = -\dot{\psi} \tag{3.4}$$

with the domain

$$\mathscr{D}(A^*_{\tau,\lambda}) = \left\{\psi = (\psi_1, \psi_2)^T \in C^1([0, \tau], \mathbb{X}^2_{\mathbb{C}}) : \psi(0) \in \mathbb{X}^2_{\mathbb{C}}, -\dot{\psi}(0) = A_\lambda \psi(0) + B^*_\lambda \psi(\tau)\right\},$$

where $B^*_\lambda : \mathbb{Y}^2 \to \mathbb{Y}^2$ is given by

$$B^*_\lambda \psi = \lambda \begin{pmatrix} \int_\Omega r_1^{1\lambda}(x) \widetilde{K}_{11}(x, \cdot)\psi_1(x) u_\lambda(x) \mathrm{d}x + \int_\Omega r_1^{2\lambda}(x) \widetilde{K}_{21}(x, \cdot)\psi_2(x) v_\lambda(x) \mathrm{d}x \\ \int_\Omega r_2^{1\lambda}(x) \widetilde{K}_{12}(x, \cdot)\psi_1(x) u_\lambda(x) \mathrm{d}x + \int_\Omega r_2^{2\lambda}(x) \widetilde{K}_{22}(x, \cdot)\psi_2(x) v_\lambda(x) \mathrm{d}x \end{pmatrix}$$

for $\psi = (\psi_1, \psi_2)^T \in \mathbb{Y}^2$ with $\widetilde{K}_{ij}(x, \cdot) = k_{ij}(x, \cdot) e^{\alpha m(x)}$ for $i, j = 1, 2$. The spectral set of $A^*_{\tau,\lambda}$ is

$$\sigma(A^*_{\tau,\lambda}) = \left\{\mu \in \mathbb{C} : \tilde{\Lambda}(\lambda, \mu, \tau)\psi = 0 \text{ for some } \psi \in \mathbb{X}^2_{\mathbb{C}} \setminus \{(0,0)\}\right\},$$

with

$$\tilde{\Lambda}(\lambda, \mu, \tau)\psi = A_\lambda \psi + B^*_\lambda \psi e^{-\mu\tau} - \mu\psi. \tag{3.5}$$

One can easily check that

$$\left\langle \tilde{\psi}, \Lambda(\lambda, \mu, \tau)\psi \right\rangle_w = \left\langle \tilde{\Lambda}(\lambda, \bar{\mu}, \tau)\tilde{\psi}, \psi \right\rangle_w. \tag{3.6}$$

In the sense of weighted inner product, $\tilde{\Lambda}(\lambda, \bar{\mu}, \tau)$ is the adjoint operator of $\Lambda(\lambda, \mu, \tau)$ and they has the same point spectrum, i.e.

$$\sigma(\Lambda(\lambda, \mu, \tau)) = \sigma(\tilde{\Lambda}(\lambda, \bar{\mu}, \tau)),$$

which means $\mu \in \sigma(A_{\tau,\lambda})$ if and only if $\bar{\mu} \in \sigma(A^*_{\tau,\lambda})$.

Theorem 2 *Under assumptions* $(\mathbf{H_1}) - (\mathbf{H_3})$*, for* $\lambda \in [\lambda_*, \lambda^*]$ *and* $\tau \geq 0$*, 0 is not an eigenvalue of* $A_{\tau,\lambda}$*.*

Proof Suppose to the contrary that 0 is an eigenvalue of $A_{\tau,\lambda}$, then there exists $\psi \in \mathbb{X}^2_{\mathbb{C}} \setminus \{(0,0)\}$ such that

$$\Lambda(\lambda, 0, \tau)\psi = 0. \tag{3.7}$$

By the decomposition $\mathbb{X}^2 = \mathscr{N}(\mathcal{L}_{\lambda_*}) \oplus X_1^2$, ψ takes the following form

$$\psi(x) = \tilde{a}(\lambda)\phi(x) + (\lambda - \lambda_*)b(\lambda, x) \tag{3.8}$$

where $\tilde{a}(\lambda) \in \mathbb{R}^2$ and $b(\lambda, \cdot) \in X_1^2$. Then substituting (3.8) to (3.7) gives

$$\Lambda(\lambda, 0, \tau)\tilde{a}(\lambda)\phi + (\lambda - \lambda_*)\Lambda(\lambda, 0, \tau)b(\lambda, \cdot) = 0. \tag{3.9}$$

Since $(u_\lambda, v_\lambda) \to (0, 0)$ as $\lambda \to \lambda_*$, then $r_j^{i\lambda}(x) \to r_j^i(x)$ uniformly in Ω as $\lambda \to \lambda_*$. A straightforward calculation yields

$$\lim_{\lambda \to \lambda_*} \frac{\Lambda(\lambda, 0, \tau)\tilde{a}(\lambda)\phi}{\lambda - \lambda_*} = m(x)\phi\tilde{a}(\lambda_*) + \lambda_*\phi\overline{\mathcal{K}}\tilde{a}(\lambda_*) + \lambda_*\phi\varrho\widetilde{\mathcal{K}}\tilde{a}(\lambda_*),$$

where

$$\overline{\mathcal{K}} = \begin{pmatrix} \beta_{\lambda_*} r_1^1 \widetilde{k}_{11} + c_{\lambda_*} r_2^1 \widetilde{k}_{12} & 0 \\ 0 & \beta_{\lambda_*} r_1^2 \widetilde{k}_{21} + c_{\lambda_*} r_2^2 \widetilde{k}_{22} \end{pmatrix}, \ \varrho = \begin{pmatrix} \beta_{\lambda_*} & 0 \\ 0 & c_{\lambda_*} \end{pmatrix}, \ \widetilde{\mathcal{K}} = \begin{pmatrix} r_1^1 \widetilde{k}_{11} & r_2^1 \widetilde{k}_{12} \\ r_1^2 \widetilde{k}_{21} & r_2^2 \widetilde{k}_{22} \end{pmatrix},$$

in which $\widetilde{k}_{ij}$ is defined as in (2.7). By expanding $\tilde{a}(\lambda)$ and $b(\lambda, x)$ near λ_*, we obtain

$$\tilde{a}(\lambda) = \sum_{i=0}^{\infty} \tilde{a}_i (\lambda - \lambda_*)^i, \quad b(\lambda, x) = \sum_{i=0}^{\infty} b_i(x)(\lambda - \lambda_*)^i. \tag{3.10}$$

Note that $\Lambda(\lambda_*, 0, \tau) = \mathcal{L}$, where $\mathcal{L}$ is defined as (2.2). Then from (3.9) and (3.10) we see that

$$\mathcal{L}b_0(\cdot) = -m(x)\phi\tilde{a}_0 - \lambda_*\phi\overline{\mathcal{K}}\tilde{a}_0 - \lambda_*\phi\varrho\widetilde{\mathcal{K}}\tilde{a}_0.$$

Calculating the weighted inner product of above equation with ϕ, we have $\varrho\widetilde{\mathcal{K}}\tilde{a}_0 = 0$. Due to the positivity of $\beta_{\lambda_*}, c_{\lambda_*}$ and the assumption $(\mathbf{H_2})$, we can deduce that $\tilde{a}_0 = 0$. Since $\mathcal{L}_{\lambda_*}|_{X_1^2} : X_1^2 \to Y_1^2$ is invertible, then $b_0(x) = 0$ for all x. Likewise, considering the term of $(\lambda - \lambda_*)^i (i \geq 1)$, we still have that $\tilde{a}_i = 0$ and $b_i(x) = 0$ for all x. Consequently, $\psi = 0$ is the unique solution of $\Lambda(\lambda, 0, \tau)\psi = 0$, a contradiction with $\psi \in \mathbb{X}^2_{\mathbb{C}} \setminus \{(0,0)\}$. The proof is completed. □

In the following, we show the situation when $A_{\tau,\lambda}$ has a pair of purely imaginary eigenvalues $\mu = \pm i\omega (\omega > 0)$ for some $\tau > 0$. From previous argument, $\mu = i\omega \in \sigma_p(A_{\tau,\lambda})$ for some $\tau > 0$ if and only if

$$A_\lambda\psi + B_\lambda\psi e^{-i\theta} - i\omega\psi = 0 \tag{3.11}$$

is solvable for some value of $\omega > 0$, $\theta \in [0, 2\pi)$ and $\psi \in \mathbb{X}^2_{\mathbb{C}} \setminus \{(0,0)\}$, where $\theta := \omega\tau$. We first show the following lemma for further discussion.

Lemma 1 *Recall that* λ_* *is the principal eigenvalue of problem* (1.6)*, the following results hold:*

(i) *if* $z \in \mathbb{X}_{\mathbb{C}}$ *and* $\langle \phi, z\rangle_w = 0$, *then* $|\langle Lz, z\rangle_w| \geq \lambda_2 \|z\|^2_{\mathbb{Y}_{\mathbb{C}}}$, *where the operator* $L : \mathbb{X} \to \mathbb{Y}$ *is defined by*

$$L = e^{-\alpha m(x)} \nabla \cdot [e^{\alpha m(x)} \nabla] + \lambda_* m(x)$$

and λ_2 *is the second eigenvalue of operator* $-L$;

(ii) *if* (ω, θ, ψ) *is a solution of Eq.* (3.11) *with* $\omega > 0, \theta \in [0, 2\pi)$ *and* $\psi \in \mathbb{X}^2_{\mathbb{C}} \setminus \{(0, 0)\}$, *then* $\dfrac{\omega}{\lambda - \lambda_*}$ *is bounded for* $\lambda \in (\lambda_*, \lambda^*]$.

Proof Part (i) can be proved as in [8, Lemma 2.3]. We only discuss part (ii). By calculating the weighted inner product of Eq. (3.11) with ψ, one can obtain

$$\left\langle \psi, A_\lambda \psi + B_\lambda \psi e^{-i\theta} - i\omega\psi \right\rangle_w = 0. \tag{3.12}$$

Choose some $\theta_0 \in [0, 2\pi)$ such that

$$\left\langle \psi, B_\lambda \psi e^{-i\theta} \right\rangle_w = \left| \left\langle \psi, B_\lambda \psi \right\rangle_w \right| e^{i(\theta_0 - \theta)}.$$

Note that A_λ is self-adjoint in the sense of weighted inner product, then $\left\langle \psi, A_\lambda \psi \right\rangle_w$ is real. Separating the real and imaginary parts of (3.12) gives that

$$\omega \left\langle \psi, \psi \right\rangle_w = \left| \left\langle \psi, B_\lambda \psi \right\rangle_w \right| \sin(\theta_0 - \theta).$$

Therefore,

$$\begin{aligned} \left| \frac{\omega}{\lambda - \lambda_*} \right| &= \frac{\left| \lambda \sin(\theta_0 - \theta) \left\langle \psi, \begin{pmatrix} \beta_\lambda[\phi + (\lambda - \lambda_*)\xi_\lambda](r_1^{1\lambda}(x) K_{11} * \psi_1 + r_2^{1\lambda}(x) K_{12} * \psi_2) \\ c_\lambda[\phi + (\lambda - \lambda_*)\eta_\lambda](r_1^{2\lambda}(x) K_{21} * \psi_1 + r_2^{2\lambda}(x) K_{22} * \psi_2) \end{pmatrix} \right\rangle_w \right|}{\langle \psi, \psi \rangle_w} \\ &\leq 2\lambda M_1 M_2 e^{\max_\Omega (\alpha m(x))} \{|K_{11}|, |K_{12}|, |K_{21}|, |K_{22}|\} |\Omega|, \end{aligned}$$

where

$$\begin{aligned} M_1 &= \max\{\beta_\lambda \left[\|\phi\|_\infty + (\lambda - \lambda_*)\|\xi_\lambda\|_\infty\right], c_\lambda \left[\|\phi\|_\infty + (\lambda - \lambda_*)\|\eta_\lambda\|_\infty\right]\}, \\ M_2 &= \max\{\|r_1^{1\lambda}(x)\|_\infty, \|r_2^{1\lambda}(x)\|_\infty, \|r_1^{2\lambda}(x)\|_\infty, \|r_2^{2\lambda}(x)\|_\infty\}. \end{aligned}$$

Since (u_λ, v_λ) is bounded for $\lambda \in (\lambda_*, \lambda^*]$, then there is a constant $M > 0$ such that $\|r_j^{i\lambda}\|_\infty < M (i, j = 1, 2)$. Now, the boundedness of $\dfrac{\omega}{\lambda - \lambda_*}$ for $\lambda \in (\lambda_*, \lambda^*]$ can be obtained from the continuity of $\lambda \mapsto (\beta_\lambda, c_\lambda, \|\xi_\lambda\|_\infty, \|\eta_\lambda\|_\infty)$. The proof is finished. □

By the decomposition $\mathbb{X}^2 = \mathscr{N}(\mathcal{L}_{\lambda_*}) \oplus X_1^2$, for $\lambda \in (\lambda_*, \lambda^*]$, ignoring a scalar factor, we can rewrite $\psi = (\psi_1, \psi_2)^T$ in (3.11) as the form

$$\begin{aligned} &\psi_1 = \phi + (\lambda - \lambda_*) z_1, \quad \langle \phi, z_1 \rangle_w = 0, \\ &\psi_2 = (p_1 + i p_2)\phi + (\lambda - \lambda_*) z_2, \quad \langle \phi, z_2 \rangle_w = 0, \;\; p_1 > 0. \end{aligned} \tag{3.13}$$

Setting $h := \omega/(\lambda - \lambda_*)$, and substituting (2.3), (3.13) and $\omega = (\lambda - \lambda_*)h$ into Eq. (3.11), then Eq. (3.11) can be transformed as the following equivalent system:

$$\begin{aligned} g_1(z_1, z_2, p_1, p_2, h, \theta, \lambda) :=\ & Lz_1 + \Big\{m(x) - ih + \lambda h_1(\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda, \lambda)\Big\} \cdot \big[\phi + (\lambda - \lambda_*)z_1\big] \\ & + \Big\{r_1^{1\lambda}(x)K_{11} * \big[\phi + (\lambda - \lambda_*)z_1\big] \\ & + r_2^{1\lambda}(x)K_{12} * \big[(p_1 + ip_2)\phi + (\lambda - \lambda_*)z_2\big]\Big\} \\ & \cdot \lambda\beta_\lambda[\phi + (\lambda - \lambda_*)\xi_\lambda]e^{-i\theta} = 0, \\ g_2(z_1, z_2, p_1, p_2, h, \theta, \lambda) :=\ & Lz_2 + \Big\{m(x) - ih + \lambda h_2(\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda, \lambda)\Big\} \\ & \cdot \big[(p_1 + ip_2)\phi + (\lambda - \lambda_*)z_2\big] \\ & + \Big\{r_1^{2\lambda}(x)K_{21} * \big[\phi + (\lambda - \lambda_*)z_1\big] \\ & + r_2^{2\lambda}(x)K_{22} * \big[(p_1 + ip_2)\phi + (\lambda - \lambda_*)z_2\big]\Big\} \\ & \cdot \lambda c_\lambda\big[\phi + (\lambda - \lambda_*)\eta_\lambda\big]e^{-i\theta} = 0, \end{aligned} \tag{3.14}$$

where $h_i(\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda, \lambda)$ is defined in (2.6), and the operator L is defined as in Lemma 1. Define $G : X_1^2 \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{Y}_{\mathbb{C}}^2$ by

$$G(z_1, z_2, p_1, p_2, h, \theta, \lambda) := (g_1, g_2).$$

Now, we show that $G(z_1, z_2, p_1, p_2, h, \theta, \lambda) = 0$ is uniquely solvable when $\lambda = \lambda_*$.

Lemma 2 *Under assumptions* $(\mathbf{H_1}) - (\mathbf{H_3})$, *the equation*

$$\begin{cases} G(z_1, z_2, p_1, p_2, h, \theta, \lambda_*) = 0, \\ z_1, z_2 \in X_{1\mathbb{C}},\ h \geq 0,\ p_1 \geq 0,\ \theta \in [0, 2\pi) \end{cases}$$

has a unique solution $(z_{1\lambda_*}, z_{2\lambda_*}, p_{1\lambda_*}, p_{2\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$ *satisfying that* $p_{2\lambda_*} = 0, \theta_{\lambda_*} = \dfrac{\pi}{2}$, $p_{1\lambda_*}$ *is the positive root of the following equation*

$$\kappa_{12}(\kappa_{12} - \kappa_{22})p^2 + (\kappa_{11}\kappa_{12} - \kappa_{22}\kappa_{21})p - \kappa_{21}(\kappa_{21} - \kappa_{11}) = 0,$$

and

$$\begin{aligned} h_{\lambda_*} &= \frac{(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12}p_{1\lambda_*})\displaystyle\int_\Omega m(x)e^{\alpha m(x)}\phi^2 dx}{(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\displaystyle\int_\Omega e^{\alpha m(x)}\phi^2 dx} \\ &= \frac{(\kappa_{11} - \kappa_{21})(\kappa_{21} + \kappa_{22}p_{1\lambda_*})\displaystyle\int_\Omega m(x)e^{\alpha m(x)}\phi^2 dx}{p_{1\lambda_*}(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\displaystyle\int_\Omega e^{\alpha m(x)}\phi^2 dx}, \end{aligned}$$

and $(z_{1\lambda_*}, z_{2\lambda_*})^T \in X_{1\mathbb{C}}^2$ *is the unique solution of*

$$\begin{aligned} \mathcal{L}\begin{pmatrix} z_{1\lambda_*} \\ z_{2\lambda_*} \end{pmatrix} &= -m(x)\phi\begin{pmatrix} 1 \\ p_{1\lambda_*} \end{pmatrix} - \lambda_*\phi\begin{pmatrix} \beta_{\lambda_*}r_1^1(x)\widetilde{k}_{11}(x) + r_2^1(x)\widetilde{k}_{12}(x) \\ p_{1\lambda_*}\big(\beta_{\lambda_*}r_1^2(x)\widetilde{k}_{21}(x) + r_2^2(x)\widetilde{k}_{22}(x)\big) \end{pmatrix} \\ &\quad + i\lambda_*\phi\begin{pmatrix} \beta_{\lambda_*}\big(r_1^1(x)\widetilde{k}_{11}(x) + p_{1\lambda_*}r_2^1(x)\widetilde{k}_{12}(x)\big) \\ c_{\lambda_*}\big(r_1^2(x)\widetilde{k}_{21}(x) + p_{1\lambda_*}r_2^2(x)\widetilde{k}_{22}(x)\big) \end{pmatrix} + ih_{\lambda_*}\phi\begin{pmatrix} 1 \\ p_{1\lambda_*} \end{pmatrix}, \end{aligned}$$

where $\mathcal{L}$ is defined as (2.2)

Proof When $\lambda = \lambda_*$, we have

$$g_1(z_1, z_2, p_1, p_2, h, \theta, \lambda_*) := Lz_1 + \Big\{m(x) - ih + \lambda_*\big(\beta_{\lambda_*} r_1^1(x)\widetilde{k}_{11}(x) + r_2^1(x)\widetilde{k}_{12}(x)\big)\Big\} \cdot \phi + \Big\{r_1^1(x)\widetilde{k}_{11}(x) + (p_1 + ip_2)r_2^1(x)\widetilde{k}_{12}(x)\Big\} \cdot \lambda_*\beta_{\lambda_*}\phi e^{-i\theta} = 0,$$

$$g_2(z_1, z_2, p_1, p_2, h, \theta, \lambda_*) := Lz_2 + \Big\{m(x) - ih + \lambda_*\big(\beta_{\lambda_*} r_1^2(x)\widetilde{k}_{21}(x) + r_2^2(x)\widetilde{k}_{22}(x)\big)\Big\} \cdot (p_1 + ip_2)\phi + \Big\{r_1^2(x)\widetilde{k}_{21}(x) + (p_1 + ip_2)r_2^2(x)\widetilde{k}_{22}(x)\Big\} \cdot \lambda_* c_{\lambda_*}\phi e^{-i\theta} = 0.$$

Then

$$\begin{cases} G(z_1, z_2, p_1, p_2, h, \theta, \lambda_*) = 0, \\ z_1, z_2 \in X_{1\mathbb{C}}, \ h \geq 0, \ p_1 \geq 0, \ \theta \in [0, 2\pi) \end{cases}$$

is solvable if and only if

$$\Big\langle \phi, \Big\{m(x) - ih + \lambda_*\big(\beta_{\lambda_*} r_1^1(x)\widetilde{k}_{11}(x) + r_2^1(x)\widetilde{k}_{12}(x)\big)\Big\} \cdot \phi + \Big\{r_1^1(x)\widetilde{k}_{11}(x) + (p_1 + ip_2)r_2^1(x)\widetilde{k}_{12}(x)\Big\} \cdot \lambda_*\beta_{\lambda_*}\phi e^{-i\theta}\Big\rangle_w = 0,$$

$$\Big\langle \phi, \Big\{m(x) - ih + \lambda_*\big(\beta_{\lambda_*} r_1^2(x)\widetilde{k}_{21}(x) + r_2^2(x)\widetilde{k}_{22}(x)\big)\Big\} \cdot (p_1 + ip_2)\phi + \Big\{r_1^2(x)\widetilde{k}_{21}(x) + (p_1 + ip_2)r_2^2(x)\widetilde{k}_{22}(x)\Big\} \cdot \lambda_* c_{\lambda_*}\phi e^{-i\theta}\Big\rangle_w = 0.$$

That is,

$$\begin{aligned} ih\int_\Omega e^{\alpha m(x)}\phi^2 dx &= (\cos\theta - i\sin\theta)\lambda_*\beta_{\lambda_*}\big(\kappa_{11} + \kappa_{12}(p_1 + ip_2)\big), \\ ih(p_1 + ip_2)\int_\Omega e^{\alpha m(x)}\phi^2 dx &= (\cos\theta - i\sin\theta)\lambda_* c_{\lambda_*}\big(\kappa_{21} + \kappa_{22}(p_1 + ip_2)\big). \end{aligned} \tag{3.15}$$

From (3.15), we see that $p_1 + ip_2$ is a root of

$$\kappa_{12}(\kappa_{12} - \kappa_{22})p^2 + (\kappa_{11}\kappa_{12} - \kappa_{22}\kappa_{21})p - \kappa_{21}(\kappa_{21} - \kappa_{11}) = 0.$$

Due to assumptions $(\mathbf{H_2})$, $(\mathbf{H_3})$ and $p_1 \geq 0$, there holds that $p_2 = p_{2\lambda_*} = 0$ and $p_1 = p_{1\lambda_*}$ is the unique positive root of above quadratic equation. Now, it can be derived from (3.15) that $\theta = \theta_{\lambda_*} = \frac{\pi}{2}$ and

$$\begin{aligned} h = h_{\lambda_*} &= \frac{(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12}p_{1\lambda_*})\displaystyle\int_\Omega m(x)e^{\alpha m(x)}\phi^2 dx}{(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\displaystyle\int_\Omega e^{\alpha m(x)}\phi^2 dx} \\ &= \frac{(\kappa_{11} - \kappa_{21})(\kappa_{21} + \kappa_{22}p_{1\lambda_*})\displaystyle\int_\Omega m(x)e^{\alpha m(x)}\phi^2 dx}{p_{1\lambda_*}(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\displaystyle\int_\Omega e^{\alpha m(x)}\phi^2 dx}. \end{aligned}$$

The proof is finished. □

In what follows, we will provide the solvability result of the equation $G = 0$ for λ near λ_* by applying the implicit function theorem.

Theorem 3 *Assume that the assumptions* $(\mathbf{H_1})-(\mathbf{H_3})$ *hold, then the following statements are true:*

(i) *there exist* $\tilde{\lambda}^*>\lambda_*$ *and a continuously differentiable mapping* $\lambda\mapsto(z_{1\lambda},z_{2\lambda},p_{1\lambda},p_{2\lambda},h_\lambda,\theta_\lambda)$ *from* $[\lambda_*,\tilde{\lambda}^*]$ *to* $X_{1\mathbb{C}}^2\times\mathbb{R}^4$ *such that* $G(z_{1\lambda},z_{2\lambda},p_{1\lambda},p_{2\lambda},h_\lambda,\theta_\lambda,\lambda)=0$*;*
(ii) *if* $(z_1^\lambda,z_2^\lambda,r_1^\lambda,r_2^\lambda,h^\lambda,\theta^\lambda)$ *with* $h^\lambda>0,\theta^\lambda>0$ *solves* $G(\cdot,\lambda)=0$ *for* $\lambda\in[\lambda_*,\tilde{\lambda}^*]$*, then* $(z_1^\lambda,z_2^\lambda,r_1^\lambda,r_2^\lambda,h^\lambda,\theta^\lambda)=(z_{1\lambda},z_{2\lambda},r_{1\lambda},r_{2\lambda},h_\lambda,\theta_\lambda)$.

Proof Denote by $T=(T_1,T_2):X_{1\mathbb{C}}^2\times\mathbb{R}^4\to\mathbb{Y}_{\mathbb{C}}^2$ the Fréchet derivative of G with respect to $(z_1,z_2,p_1,p_2,h,\theta)$ evaluated at $(z_{1\lambda_*},z_{2\lambda_*},p_{1\lambda_*},p_{2\lambda_*},h_{\lambda_*},\theta_{\lambda_*},\lambda_*)$. A direct calculation gives

$$\begin{aligned}T_1(\chi_1,\chi_2,\nu_1,\nu_2,\epsilon,\vartheta)&=L\chi_1-i\epsilon\phi-\vartheta\lambda_*\beta_{\lambda_*}\phi\Big[r_1^1(x)\widetilde{k}_{11}(x)+p_{1\lambda_*}r_2^1(x)\widetilde{k}_{12}(x)\Big]\\&\quad-(i\nu_1-\nu_2)\lambda_*\beta_{\lambda_*}\phi r_2^1(x)\widetilde{k}_{12}(x),\\T_2(\chi_1,\chi_2,\nu_1,\nu_2,\epsilon,\vartheta)&=L\chi_2-i\epsilon p_{1\lambda_*}\phi-\vartheta\lambda_*c_{\lambda_*}\phi\Big[r_1^2(x)\widetilde{k}_{21}(x)+p_{1\lambda_*}r_2^2(x)\widetilde{k}_{22}(x)\Big]\\&\quad+(\nu_1+i\nu_2)\phi\Big[m(x)-ih_{\lambda_*}+\lambda_*\big(\beta_{\lambda_*}r_1^2(x)\widetilde{k}_{21}(x)\\&\quad+c_{\lambda_*}r_2^2(x)\widetilde{k}_{22}(x)\big)-i\lambda_*c_{\lambda_*}r_2^2(x)\widetilde{k}_{22}(x)\Big].\end{aligned}$$

Next, we show that T is a bijection from $X_{1\mathbb{C}}^2\times\mathbb{R}^4$ to $\mathbb{Y}_{\mathbb{C}}^2$. We first prove T is injective. If $T(\chi_1,\chi_2,\nu_1,\nu_2,\epsilon,\vartheta)=0$, then

$$\begin{aligned}&-i\epsilon\int_\Omega e^{\alpha m(x)}\phi^2\mathrm{d}x-\vartheta\lambda_*\beta_{\lambda_*}(\kappa_{11}+p_{1\lambda_*}\kappa_{12})=(i\nu_1-\nu_2)\lambda_*\beta_{\lambda_*}\kappa_{12},\\&\quad-i\epsilon p_{1\lambda_*}\int_\Omega e^{\alpha m(x)}\phi^2\mathrm{d}x-\vartheta\lambda_*c_{\lambda_*}(\kappa_{21}+p_{1\lambda_*}\kappa_{22})\\&=(i\nu_1-\nu_2)\Big(h_{\lambda_*}\int_\Omega e^{\alpha m(x)}\phi^2\mathrm{d}x+\lambda_*c_{\lambda_*}\kappa_{22}\Big).\end{aligned}\tag{3.16}$$

It can be seen from (3.15) that

$$-p_{1\lambda_*}h_{\lambda_*}\int_\Omega e^{\alpha m(x)}\phi^2\mathrm{d}x=p_{1\lambda_*}\lambda_*\beta_{\lambda_*}(\kappa_{11}+p_{1\lambda_*}\kappa_{12})=\lambda_*c_{\lambda_*}(\kappa_{21}+p_{1\lambda_*}\kappa_{22}).\tag{3.17}$$

This result combined with (3.16) leads to that

$$(i\nu_1-\nu_2)\left[h_{\lambda_*}\int_\Omega e^{\alpha m(x)}\phi^2\mathrm{d}x+\lambda_*c_{\lambda_*}\kappa_{22}-p_{1\lambda_*}\lambda_*\beta_{\lambda_*}\kappa_{12}\right]=0.$$

Then from assumption $(\mathbf{H_2})$, $(\mathbf{H_3})$, (3.17) and the definitions of β_{λ_*}, c_{λ_*} and h_{λ_*}, we obtain that

$$\begin{aligned}&h_{\lambda_*}\int_\Omega e^{\alpha m(x)}\phi^2\mathrm{d}x+\lambda_*c_{\lambda_*}\kappa_{22}-p_{1\lambda_*}\lambda_*\beta_{\lambda_*}\kappa_{12}\\&\quad=\frac{\kappa_{12}(\kappa_{22}-\kappa_{12})p_{1\lambda_*}^2+\kappa_{21}(\kappa_{11}-\kappa_{21})}{p_{1\lambda_*}(\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21})}\int_\Omega m(x)e^{\alpha m(x)}\phi^2\mathrm{d}x\neq0,\end{aligned}$$

which implies that $i\nu_1-\nu_2=0$, i.e. $\nu_1=\nu_2=0$. Substituting $i\nu_1-\nu_2=0$ into (3.16), we must have $\epsilon=0$ and $\vartheta=0$. Consequently, $\chi_1=\chi_2=0$. So, T is injective from $X_{1\mathbb{C}}^2\times\mathbb{R}^4$ to $\mathbb{Y}_{\mathbb{C}}^2$. By a similar manner to the proof of $D_{(\xi,\eta,\beta,c)}\mathcal{F}(\xi_{\lambda_*},\eta_{\lambda_*},\beta_{\lambda_*},c_{\lambda_*},\lambda_*)$ in Theorem 1, we can also prove T is surjective. Now, we have prove T is bijective. Therefore, part (i) can be obtained from the implicit function theorem immediately.

To show part (ii), we will check that if $G(z_1^\lambda, z_2^\lambda, p_1^\lambda, p_2^\lambda, h^\lambda, \theta^\lambda, \lambda) = 0$ for $h^\lambda > 0, \theta^\lambda \in [0, 2\pi)$, then

$$(z_1^\lambda, z_2^\lambda, p_1^\lambda, p_2^\lambda, h^\lambda, \theta^\lambda) \to (z_{1\lambda_*}, z_{2\lambda_*}, p_{1\lambda_*}, p_{2\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$$

as $\lambda \to \lambda_*$ in the norm of $X_{1\mathbb{C}}^2 \times \mathbb{R}^4$. First, it follows from Lemma 1 (ii) that $\{h^\lambda\}$ is bounded. As in Theorem 2.4 of [4], due to the boundedness of $\{h^\lambda\}, \{\theta^\lambda\}, \{\beta_\lambda\}, \{c_\lambda\}, \{\xi_\lambda\}, \{\eta_\lambda\}$, there are $M_1, M_2, M_3 > 0$ such that

$$\begin{aligned}\lambda_2 \|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}}^2 &\le \left|\langle L z_1^\lambda, z_1^\lambda\rangle_w\right| \\ &\le M_1\|\phi\|_{\mathbb{Y}_\mathbb{C}}\|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}} + \big[M_2(|p_1^\lambda| + |p_2^\lambda|) \\ &\quad + M_3(\lambda - \lambda_*)(\|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}} + \|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}})\big]\|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}}, \\ \lambda_2 \|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}}^2 &\le \left|\langle L z_2^\lambda, z_2^\lambda\rangle_w\right| \\ &\le M_1\|\phi\|_{\mathbb{Y}_\mathbb{C}}\|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}} + \big[M_2(|p_1^\lambda| + |p_2^\lambda|) \\ &\quad + M_3(\lambda - \lambda_*)(\|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}} + \|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}})\big]\|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}},\end{aligned} \tag{3.18}$$

where λ_2 is defined in Lemma 1 (i). By setting $0 < \tilde{\lambda}^* - \lambda \ll 1$, we see that

$$\|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}} + \|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}} \le M_4\|\phi\|_{\mathbb{Y}_\mathbb{C}} + M_5(|p_1^\lambda| + |p_2^\lambda|) \tag{3.19}$$

for some constants $M_4, M_5 > 0$. On the other hand, note that $\langle \phi, L z_1^\lambda\rangle_w = 0$, then it follows from the first equation of (3.14) that

$$\Big\langle \phi, \big[\phi + (\lambda - \lambda_*)z_1\big] \cdot \Big\{m(x) - ih + \lambda h_1(\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda, \lambda)\Big\} + \lambda\beta_\lambda[\phi + (\lambda - \lambda_*)\xi_\lambda]e^{-i\theta}$$
$$\cdot \Big\{r_1^{1\lambda}(x)K_{11} * \big[\phi + (\lambda - \lambda_*)z_1\big] + r_2^{1\lambda}(x)K_{12} * \big[(p_1 + ip_2)\phi + (\lambda - \lambda_*)z_2\big]\Big\}\Big\rangle_w = 0.$$

By separating the real and imaginary parts of the above identity, we have

$$\begin{aligned}|p_1^\lambda| &\le M_6\|\phi\|_{\mathbb{Y}_\mathbb{C}} + M_7(\lambda - \lambda_*)(\|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}} + \|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}}), \\ |p_2^\lambda| &\le M_6\|\phi\|_{\mathbb{Y}_\mathbb{C}} + M_7(\lambda - \lambda_*)(\|z_1^\lambda\|_{\mathbb{Y}_\mathbb{C}} + \|z_2^\lambda\|_{\mathbb{Y}_\mathbb{C}})\end{aligned} \tag{3.20}$$

for some constants $M_6, M_7 > 0$. Since $0 < \tilde{\lambda}^* - \lambda_* \ll 1$, then from (3.19) and (3.20) we get the boundedness of $\{z_1^\lambda\}, \{z_2^\lambda\}, \{p_1^\lambda\}, \{p_2^\lambda\}$. Recall that the operator L has a bounded inverse from $X_{1\mathbb{C}}$ to $Y_{1\mathbb{C}}$. By acting L^{-1} on $g_1\left(z_1^\lambda, z_2^\lambda, p_1^\lambda, p_2^\lambda, h^\lambda, \theta^\lambda, \lambda\right) = 0$ and $g_2\left(z_1^\lambda, z_2^\lambda, p_1^\lambda, p_2^\lambda, h^\lambda, \theta^\lambda, \lambda\right) = 0$, we obtain that $\{z_1^\lambda\}, \{z_2^\lambda\}$ are also bounded in $X_{1\mathbb{C}}$ and hence $\left\{\left(z_1^\lambda, z_2^\lambda, p_1^\lambda, p_2^\lambda, h^\lambda, \theta^\lambda\right) : \lambda \in (\lambda_*, \tilde{\lambda}^*]\right\}$ is precompact in $\mathbb{Y}_\mathbb{C}^2 \times \mathbb{R}^4$ due to the embedding theorem. Let $\left\{\left(z_1^{\lambda^n}, z_2^{\lambda^n}, p_1^{\lambda^n}, p_2^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n}\right)\right\}$ be any convergent subsequence satisfying

$$\left(z_1^{\lambda^n}, z_2^{\lambda^n}, p_1^{\lambda^n}, p_2^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n}\right) \to \left(z_1^{\lambda_*}, z_2^{\lambda_*}, p_1^{\lambda_*}, p_2^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}\right) \text{ in } \mathbb{Y}_\mathbb{C}^2 \times \mathbb{R}^4,$$
$$\lambda^n \to \lambda_* \text{ as } n \to \infty.$$

By taking the limit of equations

$$L^{-1}g_1\left(z_1^{\lambda^n}, z_2^{\lambda^n}, r_1^{\lambda^n}, r_2^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n}, \lambda^n\right) = 0$$

and

$$L^{-1}g_2\left(z_1^{\lambda^n}, z_2^{\lambda^n}, r_1^{\lambda^n}, r_2^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n}, \lambda^n\right) = 0$$

as $n \to \infty$, there holds that

$$\left(z_1^{\lambda^n}, z_2^{\lambda^n}, p_1^{\lambda^n}, p_2^{\lambda^n}, h^{\lambda^n}, \theta^{\lambda^n}\right) \to \left(z_1^{\lambda_*}, z_2^{\lambda_*}, p_1^{\lambda_*}, p_2^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}\right) \text{ in } X_{1\mathbb{C}}^2 \times \mathbb{R}^4,$$
$$\lambda^n \to \lambda_* \text{ as } n \to \infty,$$

and $G(z_1^{\lambda_*}, z_2^{\lambda_*}, p_1^{\lambda_*}, p_2^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}, \lambda_*) = 0$. Now we can oatain from the unique solvability of $G(\cdot, \lambda_*) = 0$ in Lemma 2 that

$$(z_1^{\lambda_*}, z_2^{\lambda_*}, p_1^{\lambda_*}, p_2^{\lambda_*}, h^{\lambda_*}, \theta^{\lambda_*}) = (z_{1\lambda_*}, z_{2\lambda_*}, p_{1\lambda_*}, p_{2\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}).$$

This completes part (ii). □

Now, Theorem 3 implies the following theorem immediately.

Theorem 4 *Under assumptions* $(\mathbf{H_1}) - (\mathbf{H_3})$*, for* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$*, the eigenvalue problem*

$$\Lambda(\lambda, i\omega, \tau)\psi = 0, \ \omega > 0, \ \tau \geq 0, \ \psi \in \mathbb{X}_{\mathbb{C}}^2 \setminus \{(0,0)\}$$

has a solution (ω, τ, ψ)*, or equivalently,* $i\omega \in \sigma(A_{\tau,\lambda})$ *if and only if*

$$\begin{aligned} \omega &= \omega_\lambda = (\lambda - \lambda_*)h_\lambda, \quad \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\omega_\lambda}, \ n = 0, 1, 2, \ldots, \\ \psi &= e\psi_\lambda = e\begin{pmatrix} \phi + (\lambda - \lambda_*)z_{1\lambda} \\ (p_{1\lambda} + ip_{2\lambda})\phi + (\lambda - \lambda_*)z_{2\lambda} \end{pmatrix}, \end{aligned} \tag{3.21}$$

where e is a nonzero constant and $(z_{1\lambda}, z_{2\lambda}, p_{1\lambda}, p_{2\lambda}, h_\lambda, \theta_\lambda)$ *is defined as in Theorem 3.*

From Theorem 4, we know that $i\omega_\lambda \in \sigma(A_{\tau_n,\lambda})$ with the associated eigenvector $\psi_\lambda e^{i\omega_\lambda(\cdot)}$, which implies that $-i\omega_\lambda \in \sigma(A^*_{\tau_n,\lambda})$ and the corresponding adjoint equation

$$\tilde{\Lambda}(\lambda, -i\omega_\lambda, \tau_n)\tilde{\psi} = 0,$$

or equivalently,

$$A_\lambda\tilde{\psi} + B_\lambda^*\tilde{\psi}e^{i\theta_\lambda} + i\omega_\lambda\tilde{\psi} = 0 \tag{3.22}$$

is solvable for $\tilde{\psi} \in \mathbb{X}_{\mathbb{C}}^2 \setminus \{(0,0)\}$. Similarly, ignoring a scalar factor, $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2)^T$ can also be taken as the form

$$\begin{aligned} \tilde{\psi}_1 &= \phi + (\lambda - \lambda_*)\tilde{z}_1, \ \ \langle \phi, \tilde{z}_1 \rangle_w = 0, \\ \tilde{\psi}_2 &= (\tilde{p}_1 + i\tilde{p}_2)\phi + (\lambda - \lambda_*)\tilde{z}_2, \ \ \langle \phi, \tilde{z}_2 \rangle_w = 0, \ \tilde{p}_1 > 0. \end{aligned} \tag{3.23}$$

Substituting (2.3), (3.23) and $\omega_\lambda = (\lambda - \lambda_*)h_\lambda$ into Eq. (3.22), we have the following equivalent system to Eq. (3.22):

$$\begin{aligned}
\tilde{g}_1(\tilde{z}_1, \tilde{z}_2, \tilde{p}_1, \tilde{p}_2, \lambda) := L\tilde{z}_1 &+ \Big\{m(x) + ih_\lambda + \lambda h_1(\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda, \lambda)\Big\} \cdot \big[\phi + (\lambda - \lambda_*)\tilde{z}_1\big] \\
&+ \lambda\beta_\lambda e^{i\theta_\lambda} \int_\Omega r_1^{1\lambda}(x)\widetilde{K}_{11}(x,\cdot)\big[\phi(x) + (\lambda - \lambda_*)\tilde{z}_1(x)\big] \\
&\big[\phi(x) + (\lambda - \lambda_*)\xi_\lambda(x)\big]\mathrm{d}x \\
&+ \lambda c_\lambda e^{i\theta_\lambda} \int_\Omega r_1^{2\lambda}(x)\widetilde{K}_{21}(x,\cdot)\big[(\tilde{p}_1 + i\tilde{p}_2)\phi(x) + (\lambda - \lambda_*)\tilde{z}_2(x)\big] \\
&\times \big[\phi(x) + (\lambda - \lambda_*)\eta_\lambda(x)\big]\mathrm{d}x = 0, \\
\tilde{g}_2(\tilde{z}_1, \tilde{z}_2, \tilde{p}_1, \tilde{p}_2, \lambda) := L\tilde{z}_2 &+ \Big\{m(x) + ih_\lambda + \lambda h_2(\xi_\lambda, \eta_\lambda, \beta_\lambda, c_\lambda, \lambda)\Big\} \\
&\cdot \big[(\tilde{p}_1 + i\tilde{p}_2)\phi + (\lambda - \lambda_*)\tilde{z}_2\big] \\
&+ \lambda\beta_\lambda e^{i\theta_\lambda} \int_\Omega r_2^{1\lambda}(x)\widetilde{K}_{12}(x,\cdot)\big[\phi(x) + (\lambda - \lambda_*)\tilde{z}_1(x)\big] \\
&\big[\phi(x) + (\lambda - \lambda_*)\xi_\lambda(x)\big]\mathrm{d}x \\
&+ \lambda c_\lambda e^{i\theta_\lambda} \int_\Omega r_2^{2\lambda}(x)\widetilde{K}_{22}(x,\cdot)\big[(\tilde{p}_1 + i\tilde{p}_2)\phi(x) + (\lambda - \lambda_*)\tilde{z}_2(x)\big] \\
&\times \big[\phi(x) + (\lambda - \lambda_*)\eta_\lambda(x)\big]\mathrm{d}x = 0.
\end{aligned} \tag{3.24}$$

Define $\tilde{G} : \mathbb{X}_{\mathbb{C}}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{Y}_{\mathbb{C}}^2$ as

$$\tilde{G}(\tilde{z}_1, \tilde{z}_2, \tilde{p}_1, \tilde{p}_2, \lambda) := (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4, \tilde{g}_5, \tilde{g}_6).$$

By a similar argument as in Theorem 3 and 4 , we can prove the following conclusions.

Theorem 5 *Assume that* $(\mathbf{H_1}) - (\mathbf{H_3})$ *hold. Then the following statements are true:*

(i) *there exists a continuously differentiable mapping* $\lambda \mapsto (\tilde{z}_{1\lambda}, \tilde{z}_{2\lambda}, \tilde{p}_{1\lambda}, \tilde{p}_{2\lambda})$ *from* $[\lambda_*, \tilde{\lambda}^*]$ *to* $X_{1\mathbb{C}}^2 \times \mathbb{R}^2$ *such that* $\tilde{G}(\tilde{z}_{1\lambda}, \tilde{z}_{2\lambda}, \tilde{p}_{1\lambda}, \tilde{p}_{2\lambda}, \lambda) = 0$ *with* $\tilde{p}_{2\lambda_*} = 0$, $\tilde{p}_{1\lambda_*}$ *is the positive root of the following equation*

$$\kappa_{21}(\kappa_{21} - \kappa_{11})p^2 + (\kappa_{11}\kappa_{12} - \kappa_{22}\kappa_{21})p - \kappa_{12}(\kappa_{12} - \kappa_{22}) = 0,$$

and $(\tilde{z}_{1\lambda_*}, \tilde{z}_{2\lambda_*})^T \in X_{1\mathbb{C}}^2$ *is the unique solution of*

$$\begin{aligned}
\mathcal{L}\begin{pmatrix} \tilde{z}_{1\lambda_*} \\ \tilde{z}_{2\lambda_*} \end{pmatrix} &= -(m(x) + ih_{\lambda_*})\phi \begin{pmatrix} 1 \\ \tilde{p}_{1\lambda_*} \end{pmatrix} - \lambda_*\phi \begin{pmatrix} \beta_{\lambda_*} r_1^1(x)\widetilde{k}_{11}(x) + r_2^1(x)\widetilde{k}_{12}(x) \\ \tilde{p}_{1\lambda_*}\big(\beta_{\lambda_*} r_1^2(x)\widetilde{k}_{21}(x) + r_2^2(x)\widetilde{k}_{22}(x)\big) \end{pmatrix} \\
&\quad - i\lambda_* \begin{pmatrix} \beta_{\lambda_*} \int_\Omega r_1^1(x)\widetilde{K}_{11}(x,\cdot)\phi^2(x)\mathrm{d}x + \tilde{p}_{1\lambda_*}c_{\lambda_*} \int_\Omega r_1^2(x)\widetilde{K}_{21}(x,\cdot)\phi^2(x)\mathrm{d}x \\ \beta_{\lambda_*} \int_\Omega r_2^1(x)\widetilde{K}_{12}(x,\cdot)\phi^2(x)\mathrm{d}x + \tilde{p}_{1\lambda_*}c_{\lambda_*} \int_\Omega r_2^2(x)\widetilde{K}_{22}(x,\cdot)\phi^2(x)\mathrm{d}x \end{pmatrix},
\end{aligned}$$

where $\mathcal{L}$ *is defined as* (2.2)*. Moreover, if there is* $(\tilde{z}_1^\lambda, \tilde{z}_2^\lambda, \tilde{p}_1^\lambda, \tilde{p}_2^\lambda)$ *such that* $\tilde{G}(\tilde{z}_1^\lambda, \tilde{z}_2^\lambda, \tilde{p}_1^\lambda, \tilde{p}_2^\lambda, \lambda) = 0$, *then* $(\tilde{z}_1^\lambda, \tilde{z}_2^\lambda, \tilde{p}_1^\lambda, \tilde{p}_2^\lambda) = (\tilde{z}_{1\lambda}, \tilde{z}_{2\lambda}, \tilde{p}_{1\lambda}, \tilde{p}_{2\lambda})$.

(ii) *for* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, *the eigenvalue problem*

$$\tilde{\Lambda}(\lambda, -i\omega, \tau)\tilde{\psi} = 0,\ \omega > 0,\ \tau \geq 0,\ \tilde{\psi} \in \mathbb{X}_{\mathbb{C}}^2 \setminus \{(0,0)\}$$

has a solution (ω, τ, ψ)*, or equivalently,* $-i\omega \in \sigma(A^*_{\tau,\lambda})$ *if and only if*

$$\omega = \omega_\lambda = (\lambda - \lambda_*)h_\lambda, \quad \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\omega_\lambda}, \ n = 0, 1, 2, \ldots, \\ \tilde{\psi} = \tilde{e}\tilde{\psi}_\lambda = \tilde{e}\begin{pmatrix} \phi + (\lambda - \lambda_*)\tilde{z}_{1\lambda} \\ (\tilde{p}_{1\lambda} + i\tilde{p}_{2\lambda})\phi + (\lambda - \lambda_*)\tilde{z}_{2\lambda} \end{pmatrix}, \tag{3.25}$$

where $\tilde{e}$ *is a nonzero constant and* $(\tilde{z}_{1\lambda}, \tilde{z}_{2\lambda}, \tilde{p}_{1\lambda}, \tilde{p}_{2\lambda})$ *is defined in part* (i)*,* $(h_\lambda, \theta_\lambda)$ *is defined in Theorem 3.*

Remark 1 Theorem 5 shows that $-i\omega_\lambda \in \sigma(A^*_{\tau_n,\lambda})$ with the associated eigenvector $\tilde{\psi}_\lambda e^{-i\omega_\lambda(\cdot)}$.

4 Stability and Hopf Bifurcation

Notice that system (1.5) always has the steady state (0, 0). Then we first consider the stability of (0, 0). Linearizing system (1.5) at (0, 0), we obtain the linear eigenvalue problem

$$\begin{cases} Lu + (\lambda - \lambda_*)m(x)u = \sigma u, & x \in \Omega, \\ Lv + (\lambda - \lambda_*)m(x)v = \sigma v, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

It follows from [5] that $\sigma < 0$ if and only if $\lambda < \lambda_*$. Therefore, the stability result of the trivial steady state (0, 0) is as follows.

Lemma 3 *Assume that* $(\mathbf{H_1})$ *holds. Then the trivial solution* (0, 0) *of system (1.2) is locally asymptotically stable when* $\lambda < \lambda_*$ *and unstable when* $\lambda > \lambda_*$.

In the following, we pay attention to the stability and associated Hopf bifurcation of the positive steady state $U_\lambda = (u_\lambda, v_\lambda)^T$ of Eq. (1.5) by regarding the parameter τ as the bifurcation parameter. Firstly, we show the stability of U_λ for $\tau = 0$.

Theorem 6 *Assume that assumptions* $(\mathbf{H_1}) - (\mathbf{H_3})$ *hold, then for each* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$*, all the eigenvalues of* $A_{\tau,\lambda}$ *have negative real parts when* $\tau = 0$*. That is, the positive steady state* (u_λ, v_λ) *of Eq. (1.5) is locally asymptotically stable when* $\tau = 0$.

Proof Suppose to the contrary that there exists a sequence $\{\lambda^n\}_{n=1}^\infty$ such that $\lambda^n > \lambda_*$ for $n \geq 1$, $\lim_{n\to\infty} \lambda^n = \lambda_*$, and for every n, the eigenvalue equation

$$\begin{cases} A_{\lambda^n}\psi + B_{\lambda^n}\psi = \mu\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega \end{cases} \tag{4.2}$$

admits an eigenvalue μ_{λ^n} with nonnegative real part, whose corresponding eigenfunction ψ_{λ^n} satisfies $\|\psi_{\lambda^n}\|_{\mathbb{Y}_\mathbb{C}} = 1$. We can take ψ_{λ^n} as $\psi_{\lambda^n} = s_{\lambda^n}U_{\lambda^n} + V_{\lambda^n}$ for each $n \geq 1$, where U^n_λ is the positive steady state of Eq. (1.5) with $\lambda = \lambda^n$, $s_{\lambda^n} \in \mathbb{C}$ and $V_{\lambda^n} = (V_{1\lambda^n}, V_{2\lambda^n})^T \in \mathbb{X}^2_\mathbb{C}$ satisfy that

$$s_{\lambda^n} = \frac{\langle U_{\lambda^n}, \psi_{\lambda^n}\rangle_w}{\langle U_{\lambda^n}, U_{\lambda^n}\rangle_w}, \quad \langle U_{\lambda^n}, V_{\lambda^n}\rangle_w = 0.$$

Notice that

$$\left\langle U_{\lambda^n}, A_{\lambda^n}V_{\lambda^n}\right\rangle_w = \left\langle A_{\lambda^n}U_{\lambda^n}, V_{\lambda^n}\right\rangle_w \text{ and } A_{\lambda^n}U_{\lambda^n} = 0,$$

then by substituting $\psi_{\lambda^n} = s_{\lambda^n} U_{\lambda^n} + V_{\lambda^n}$ and $\mu = \mu_{\lambda^n}$ into the first equation of Eq. (4.2), and computing the weighted inner product with ψ_{λ^n}, there holds that

$$\left\langle V_{\lambda^n}, A_{\lambda^n} V_{\lambda^n}\right\rangle_w + \left\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\right\rangle_w = \mu_{\lambda^n}.$$

Furthermore,

$$\begin{aligned}\left|\left\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\right\rangle_w\right| &\le \left|\lambda^n \left\langle \psi_{\lambda^n}, \begin{pmatrix} u_\lambda(r_1^{1\lambda^n}(x)K_{11} * \psi_{1\lambda^n} + r_2^{1\lambda^n}(x)K_{12} * \psi_{2\lambda^n}) \\ v_\lambda(r_1^{2\lambda^n}(x)K_{21} * \psi_{1\lambda^n} + r_2^{2\lambda^n}(x)K_{22} * \psi_{2\lambda^n}) \end{pmatrix}\right\rangle_w\right| \\ &\le 2\lambda^n(\lambda^n - \lambda_*)M_1 M_2 e^{\max_\Omega(\alpha m(x))} \max_{\Omega\times\Omega}\{|K_{11}|, |K_{12}|, |K_{21}|, |K_{22}|\}|\Omega| \\ &\to 0, \text{ as } n \to \infty,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned} M_1 &= \max\{\beta_{\lambda^n}\left[\|\phi\|_\infty + (\lambda^n - \lambda_*)\|\xi_{\lambda^n}\|_\infty\right], c_{\lambda^n}\left[\|\phi\|_\infty + (\lambda^n - \lambda_*)\|\eta_{\lambda^n}\|_\infty\right]\}, \\ M_2 &= \max\{\|r_1^{1\lambda^n}(x)\|_\infty, \|r_2^{1\lambda^n}(x)\|_\infty, \|r_1^{2\lambda^n}(x)\|_\infty, \|r_2^{2\lambda^n}(x)\|_\infty\}.\end{aligned}$$

Define the operator $A_{i\lambda} : \mathbb{X}_\mathbb{C} \to \mathbb{Y}_\mathbb{C} (i = 1, 2)$ by

$$A_{i\lambda}\varphi = e^{-\alpha m(x)}\nabla \cdot \left[e^{\alpha m(x)}\nabla\varphi\right] + \lambda f_i\big(x, K_{i1} * u_\lambda, K_{i2} * v_\lambda\big)\varphi$$

for $\varphi \in \mathbb{X}_\mathbb{C}$. Since 0 is the principal eigenvalue of $A_{1\lambda^n}$ (resp. $A_{2\lambda^n}$) with the corresponding eigenfunction u_{λ^n} (resp. v_{λ^n}), we get that

$$\left\langle V_{1\lambda^n}, A_{1\lambda^n} V_{1\lambda^n}\right\rangle_w \le 0 \text{ and } \left\langle V_{2\lambda^n}, A_{2\lambda^n} V_{2\lambda^n}\right\rangle_w \le 0.$$

This result leads to

$$0 \le \mathrm{Re}(\mu_{\lambda^n}), |\mathrm{Im}(\mu_{\lambda^n})| \le \left|\left\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\right\rangle_w\right|,$$

and hence

$$\lim_{n\to\infty} \mathrm{Re}(\mu_{\lambda^n}) = \lim_{j\to\infty} \mathrm{Im}(\mu_{\lambda^n}) = 0.$$

From the fact that $\left|\left\langle V_{i\lambda^n}, A_{i\lambda^n} V_{i\lambda^n}\right\rangle_w\right| \ge |\lambda_2^{(i)}(\lambda^n)| \cdot \|V_{i\lambda^n}\|^2_{\mathbb{Y}_\mathbb{C}}$ (the proof of this inequality is similar to that of Lemma 1 (i)), where $\lambda_2^{(i)}(\lambda^n)$ is the second eigenvalue of $A_{i\lambda^n}$, we obtain

$$\left|-\left\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\right\rangle_w\right| + |\mu_{\lambda^n}| \ge |\gamma^n| \cdot \|V_{\lambda^n}\|^2_{\mathbb{Y}_\mathbb{C}}, \tag{4.4}$$

in which $\gamma^n = \min\left\{\lambda_2^{(1)}(\lambda^n), \lambda_2^{(2)}(\lambda^n)\right\}$. In view of (4.4), using the limit $\lim\limits_{n\to\infty} \left|\left\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\right\rangle_w\right| = \lim\limits_{n\to\infty} |\mu_{\lambda^n}| = 0$, we see that $\lim\limits_{n\to\infty} \|V_{\lambda^n}\|_{\mathbb{Y}_\mathbb{C}} = 0$.

Since $\psi_{\lambda^n} = s_{\lambda^n} U_{\lambda^n} + V_{\lambda^n}$ and $\|\psi_{\lambda^n}\|_{\mathbb{Y}_\mathbb{C}} = 1$, we see that

$$\lim_{n\to\infty} |s_{\lambda^n}|^2(\lambda^n - \lambda_*)^2 \lim_{n\to\infty}\left(\left\|\frac{u_{\lambda^n}}{\lambda^n - \lambda_*}\right\|^2_{\mathbb{Y}_\mathbb{C}} + \left\|\frac{v_{\lambda^n}}{\lambda^n - \lambda_*}\right\|^2_{\mathbb{Y}_\mathbb{C}}\right) = 1,$$

which means $\lim_{n\to\infty} |s_{\lambda^n}|(\lambda^n - \lambda_*) = \dfrac{1}{\sqrt{\beta_{\lambda_*}^2 + c_{\lambda_*}^2}\|\phi\|_{\mathbb{Y}_\mathbb{C}}} > 0$, where ϕ is the eigenfunction of eigenvalue problem (1.6) with the principal eigenvalue λ_*. We now calculate that

$$\begin{aligned}
&\frac{\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\rangle_w}{\lambda^n - \lambda_*}\\
&= \frac{\lambda^n \left\langle (s_{\lambda^n}U_{\lambda^n} + V_{\lambda^n}), \begin{pmatrix} u_\lambda[r_1^{1\lambda^n}(x)K_{11} * (s_{\lambda^n}u_{\lambda^n} + V_{1\lambda^n}) + r_2^{1\lambda^n}(x)K_{12} * (s_{\lambda^n}v_{\lambda^n} + V_{2\lambda^n})] \\ v_\lambda[r_1^{2\lambda^n}(x)K_{21} * (s_{\lambda^n}u_{\lambda^n} + V_{1\lambda^n}) + r_2^{2\lambda^n}(x)K_{22} * (s_{\lambda^n}v_{\lambda^n} + V_{2\lambda^n})]\end{pmatrix}\right\rangle_w}{\lambda^n - \lambda_*}\\
&= \lambda^n \left(|s_{\lambda^n}|^2(\lambda^n - \lambda_*)^2 J_1 + s_{\lambda^n}(\lambda^n - \lambda_*)J_2 + \overline{s_{\lambda^n}}(\lambda^n - \lambda_*)J_3 + J_4\right),
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{1\lambda^n}(x)K_{11}(x,y)u_{\lambda^n}^2(x)u_{\lambda^n}(y) + r_2^{1\lambda^n}(x)K_{12}(x,y)u_{\lambda^n}^2(x)v_{\lambda^n}(y)\right]}{(\lambda^n - \lambda_*)^3}\mathrm{d}y\mathrm{d}x\\
&\quad + \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{2\lambda^n}(x)K_{21}(x,y)v_{\lambda^n}^2(x)u_{\lambda^n}(y) + r_2^{2\lambda^n}(x)K_{22}(x,y)v_{\lambda^n}^2(x)v_{\lambda^n}(y)\right]}{(\lambda^n - \lambda_*)^3}\mathrm{d}y\mathrm{d}x,\\
J_2 &= \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{1\lambda^n}(x)K_{11}(x,y)u_{\lambda^n}(x)\overline{V_{1\lambda^n}}(x)u_{\lambda^n}(y) + r_2^{1\lambda^n}(x)K_{12}(x,y)u_{\lambda^n}(x)\overline{V_{1\lambda^n}}(x)v_{\lambda^n}(y)\right]}{(\lambda^j - \lambda_*)^2}\mathrm{d}y\mathrm{d}x\\
&\quad + \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{2\lambda^n}(x)K_{21}(x,y)v_{\lambda^n}(x)\overline{V_{2\lambda^n}}(x)u_{\lambda^n}(y) + r_2^{2\lambda^n}(x)K_{22}(x,y)v_{\lambda^n}(x)\overline{V_{2\lambda^n}}(x)v_{\lambda^n}(y)\right]}{(\lambda^j - \lambda_*)^2}\mathrm{d}y\mathrm{d}x,\\
J_3 &= \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{1\lambda^n}(x)K_{11}(x,y)u_{\lambda^n}^2(x)V_{1\lambda^n}(y) + r_2^{1\lambda^n}(x)K_{12}(x,y)u_{\lambda^n}^2(x)V_{2\lambda^n}(y)\right]}{(\lambda^n - \lambda_*)^2}\mathrm{d}y\mathrm{d}x\\
&\quad + \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{2\lambda^n}(x)K_{21}(x,y)v_{\lambda^n}^2(x)V_{1\lambda^n}(y) + r_2^{2\lambda^n}(x)K_{22}(x,y)v_{\lambda^n}^2(x)V_{2\lambda^n}(y)\right]}{(\lambda^n - \lambda_*)^2}\mathrm{d}y\mathrm{d}x,\\
J_4 &= \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{1\lambda^n}(x)K_{11}(x,y)u_{\lambda^n}(x)\overline{V_{1\lambda^n}}(x)V_{1\lambda^n}(y) + r_2^{1\lambda^n}(x)K_{12}(x,y)u_{\lambda^n}(x)\overline{V_{1\lambda^n}}(x)V_{2\lambda^n}(y)\right]}{\lambda^n - \lambda_*}\mathrm{d}y\mathrm{d}x\\
&\quad + \int_\Omega\int_\Omega \frac{e^{\alpha m(x)}\left[r_1^{2\lambda^n}(x)K_{21}(x,y)v_{\lambda^n}(x)\overline{V_{2\lambda^n}}(x)V_{1\lambda^n}(y) + r_2^{2\lambda^n}(x)K_{22}(x,y)v_{\lambda^n}(x)\overline{V_{2\lambda^n}}(x)V_{2\lambda^n}(y)\right]}{\lambda^n - \lambda_*}\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

It follows from Hölder inequality that

$$\begin{aligned}
\langle u, v\rangle_w &= \int_\Omega e^{\frac{\alpha m(x)}{2}}u(x)e^{\frac{\alpha m(x)}{2}}v(x)\mathrm{d}x\\
&\le \left(\int_\Omega e^{\alpha m(x)}u^2(x)\mathrm{d}x\right)^{1/2}\left(\int_\Omega e^{\alpha m(x)}v^2(x)\mathrm{d}x\right)^{1/2} = \|u\|_{\mathbb{Y}_\mathbb{C}}\|v\|_{\mathbb{Y}_\mathbb{C}}.
\end{aligned}$$

Then by using the limit $\lim\limits_{n\to\infty}\|V_{\lambda^n}\|_{\mathbb{Y}_\mathbb{C}} = 0$, there holds that

$$\lim_{n\to\infty} J_2 = \lim_{n\to\infty} J_3 = \lim_{n\to\infty} J_4 = 0.$$

We also have that

$$\begin{aligned}
\lim_{n\to\infty} J_1 &= \kappa_{11}\beta_{\lambda_*}^3 + \kappa_{12}\beta_{\lambda_*}^2 c_{\lambda_*} + \kappa_{21}\beta_{\lambda_*}c_{\lambda_*}^2 + \kappa_{22}c_{\lambda_*}^3\\
&= -\frac{\beta_{\lambda_*}^2 + c_{\lambda_*}^2}{\lambda_*}\int_\Omega m(x)e^{\alpha m(x)}\phi^2(x)\mathrm{d}x < 0.
\end{aligned}$$

The above argument implies that there exists $N_* \in \mathbb{N}$ such that

$$\mathrm{Re}\left(\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\rangle_w\right) < 0 \text{ for each } n \geq N_*.$$

Consequently,

$$\mathrm{Re}(\mu_{\lambda^n}) = \langle V_{\lambda^n}, A_{\lambda^n}V_{\lambda^n}\rangle_w + \mathrm{Re}\left(\langle \psi_{\lambda^n}, B_{\lambda^n}\psi_{\lambda^n}\rangle_w\right) < 0,$$

which is a contradiction with that $\mathrm{Re}(\mu_{\lambda^n}) \geq 0$ for $n \geq 1$. That is to say, $A_{\tau,\lambda}$ has no eigenvalue with nonnegative real parts when $\tau = 0$. The proof is finished. □

Lemma 4 *Assume that* $(\mathbf{H_1}) - (\mathbf{H_3})$ *hold. Then for each* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, $\mu = i\omega_\lambda$ *is a simple eigenvalue of* $A_{\tau_n,\lambda}$ *for* $n = 0, 1, 2, \ldots$.

Proof From Theorem 4, we see that $\mathscr{N}[A_{\tau_n,\lambda} - i\omega_\lambda] = \mathrm{Span}[e^{i\omega_\lambda s}\psi_\lambda]$, where $s \in [-\tau_n, 0]$ and ψ_λ is defined as in Theorem 4. Suppose that $\mu = i\omega_\lambda$ is not a simple eigenvalue, then there exists $\hat{\phi} \in \mathscr{N}[A_{\tau_n,\lambda} - i\omega_\lambda]^2$, i.e.,

$$[A_{\tau_n,\lambda} - i\omega_\lambda]\hat{\phi} \in \mathscr{N}[A_{\tau_n,\lambda} - i\omega_\lambda] = \mathrm{Span}[e^{i\omega_\lambda s}\psi_\lambda].$$

Hence, we can pick a constant a such that $[A_{\tau_n,\lambda} - i\omega_\lambda]\hat{\phi} = ae^{i\omega_\lambda s}\psi_\lambda$. Then there holds that

$$\begin{aligned} &\hat{\phi}'(s) = i\omega_\lambda\hat{\phi} + ae^{i\omega_\lambda s}\psi_\lambda, \quad s \in [-\tau_n, 0], \\ &\hat{\phi}'(0) = A_\lambda\hat{\phi}(0) + B_\lambda\hat{\phi}(-\tau_n). \end{aligned} \tag{4.5}$$

In view of the first equation of Eq. (4.5), we obtain that

$$\begin{aligned} \hat{\phi}(s) &= \hat{\phi}(0)e^{i\omega_\lambda s} + ase^{i\omega_\lambda s}\psi_\lambda, \\ \hat{\phi}'(0) &= i\omega_\lambda\hat{\phi}(0) + a\psi_\lambda. \end{aligned} \tag{4.6}$$

It follows from the second equation of (4.5) and (4.6) that

$$\begin{aligned} \Lambda(\lambda, i\omega_\lambda, \tau_n)\hat{\phi}(0) &= \left[A_\lambda + B_\lambda e^{-i\theta_\lambda} - i\omega_\lambda\right]\hat{\phi}(0) \\ &= a\left(\psi_\lambda + \tau_n e^{-i\theta_\lambda}B_\lambda\psi_\lambda\right), \end{aligned}$$

where we have used the identity $\hat{\phi}(-\tau_n) = \hat{\phi}(0)e^{-i\theta_\lambda} - a\tau_n e^{-i\theta_\lambda}\psi_\lambda$. From (3.6), we have

$$\begin{aligned} 0 &= \left\langle \tilde{\Lambda}(\lambda, -i\omega_\lambda, \tau_n)\tilde{\psi}_\lambda, \hat{\phi}(0)\right\rangle_w = \left\langle \tilde{\psi}_\lambda, \Lambda(\lambda, i\omega_\lambda, \tau_n)\hat{\phi}(0)\right\rangle_w \\ &= a\left\langle \tilde{\psi}_\lambda, \psi_\lambda + \tau_n e^{-i\theta_\lambda}B_\lambda\psi_\lambda\right\rangle_w := aS_{n\lambda}. \end{aligned} \tag{4.7}$$

Let $\lambda \to \lambda_*$, then it follows that

$$\begin{aligned} \lim_{\lambda\to\lambda_*} S_{n\lambda} &= (1 + \tilde{p}_{1\lambda_*}p_{1\lambda_*})\int_\Omega e^{\alpha m(x)}\phi^2 dx \\ &\quad - i\lambda_*\frac{\beta_{\lambda_*}(\kappa_{11} + \kappa_{12}p_{1\lambda_*}) + c_{\lambda_*}\tilde{p}_{1\lambda_*}(\kappa_{21} + \kappa_{22}p_{1\lambda_*})}{h_{\lambda_*}}\left(\frac{\pi}{2} + 2n\pi\right) \\ &= \left[1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right](1 + \tilde{p}_{1\lambda_*}p_{1\lambda_*})\int_\Omega e^{\alpha m(x)}\phi^2 dx, \end{aligned}$$

which implies that $S_{n\lambda} \neq 0$ and hence $a = 0$. Therefore, $\hat{\phi} \in \mathscr{N}[A_{\tau_n,\lambda} - i\omega_\lambda]$. By induction it can be derived that

$$\mathscr{N}[A_{\tau_n,\lambda} - i\omega_\lambda]^j = \mathscr{N}[A_{\tau_n,\lambda} - i\omega_\lambda], \quad j = 2, 3, \ldots,$$

and $\mu = i\omega_\lambda$ is a simple eigenvalue of $A_{\tau_n,\lambda}$ for $n = 0, 1, 2, \ldots$. □

Now it can be inferred from the implicit function theorem that there is a neighborhood $O_n \times D_n \times H_n^2 \subset \mathbb{R} \times \mathbb{C} \times \mathbb{X}_{\mathbb{C}}^2$ of $(\tau_n, i\omega_\lambda, \psi_\lambda)$ and a continuous differential function $(\mu, \psi) : O_n \to D_n \times H_n^2$ satisfying $\mu(\tau_n) = i\omega_\lambda$ and $\psi(\tau_n) = \psi_\lambda$ such that, for each $\tau \in O_n$, $A_{\tau_n,\lambda}$ in D_n has the unique eigenvalue $\mu(\tau)$ with its associated eigenvector $\psi(\tau)e^{\mu(\tau)(\cdot)}$ and there holds that

$$\Lambda(\lambda, \mu(\tau), \tau)\psi(\tau) = \left[A_\lambda + B_\lambda e^{-\mu(\tau)\tau} - \mu(\tau)\right]\psi(\tau) = 0. \tag{4.8}$$

In the following, we verify the transversality condition for Hopf bifurcation.

Lemma 5 *Under the assumptions* $(\mathbf{H_1}) - (\mathbf{H_3})$, *for* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$,

$$\mathrm{Re}\left(\frac{d\mu}{d\tau}(\tau_n)\right) > 0.$$

Proof Firstly, by differentiating Eq. (4.8) with respect to τ at $\tau = \tau_n$, we have

$$\frac{d\mu}{d\tau}(\tau_n)\left(\psi_\lambda + \tau_n e^{-i\theta_\lambda} B_\lambda \psi_\lambda\right) = \Lambda(\lambda, i\omega_\lambda, \tau_n)\frac{d\psi}{d\tau}(\tau_n) - i\omega_\lambda e^{-i\theta_\lambda} B_\lambda \psi_\lambda.$$

Then calculating the weighted inner product of above equation with $\tilde{\psi}_\lambda$ gives that

$$\frac{d\mu}{d\tau}(\tau_n) = -\frac{\left\langle \tilde{\psi}_\lambda, i\omega_\lambda e^{-i\theta_\lambda} B_\lambda \psi_\lambda \right\rangle_w}{S_{n\lambda}} = -\frac{I_1 + I_2}{|S_{n\lambda}|^2}.$$

where $S_{n\lambda}$ is defined as in (4.7),

$$I_1 = \left\langle \psi_\lambda, \tilde{\psi}_\lambda \right\rangle_w \left\langle \tilde{\psi}_\lambda, i\omega_\lambda e^{-i\theta_\lambda} B_\lambda \psi_\lambda \right\rangle_w,$$
$$I_2 = i\omega_\lambda \tau_n \left|\left\langle \tilde{\psi}_\lambda, B_\lambda \psi_\lambda \right\rangle_w\right|^2.$$

Direct computation yields

$$\lim_{\lambda \to \lambda_*} \left\langle \psi_\lambda, \tilde{\psi}_\lambda \right\rangle_w = (1 + \tilde{p}_{1\lambda_*} p_{1\lambda_*}) \int_\Omega e^{\alpha m(x)} \phi^2 dx,$$
$$\lim_{\lambda \to \lambda_*} \frac{1}{(\lambda - \lambda_*)^2} \left\langle \tilde{\psi}_\lambda, i\omega_\lambda e^{-i\theta_\lambda} B_\lambda \psi_\lambda \right\rangle_w = -(1 + \tilde{p}_{1\lambda_*} p_{1\lambda_*}) h_{\lambda_*}^2 \int_\Omega e^{\alpha m(x)} \phi^2 dx,$$

which implies that

$$\lim_{\lambda \to \lambda_*} \frac{1}{(\lambda - \lambda_*)^2} \mathrm{Re}\left(\frac{d\mu}{d\tau}(\tau_n)\right) = \frac{(1 + \tilde{p}_{1\lambda_*} p_{1\lambda_*})^2 h_{\lambda_*}^2}{\lim_{\lambda \to \lambda_*} |S_{n\lambda}|^2} \left(\int_\Omega e^{\alpha m(x)} \phi^2 dx\right)^2 > 0.$$

The proof is finished. □

From Theorems 4, 6 and Lemmas 4, 5, we can now conclude the stability result of the positive steady state U_λ and the associated Hopf bifurcation of Eq. (1.5) as follows.

Theorem 7 *Assume that* $(\mathbf{H_1}) - (\mathbf{H_3})$ *hold. For* $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, *the following statements are ture:*

(i) *there exists an increasing sequence* $\{\tau_n\}_{n=0}^\infty$ *such that all the eigenvalues of* $A_{\tau,\lambda}$ *have negative real parts when* $\tau \in (0, \tau_0)$, $A_{\tau,\lambda}$ *has a pair of purely imaginary eigenvalues* $\pm i\omega_\lambda$ $(\omega_\lambda > 0)$ *when* $\tau = \tau_n$, *and* $A_{\tau,\lambda}$ *has exactly* $2(n+1)$ *eigenvalues with positive real parts when* $\tau \in (\tau_n, \tau_{n+1}]$, $n = 0, 1, 2, \ldots$;

(ii) *the positive steady state* U_λ *of Eq.* (1.5) *is locally asymptotically stable when* $\tau \in [0, \tau_0)$, *and unstable when* $\tau \in (\tau_0, \infty)$;

(iii) *Hopf bifurcation occurs as the delay* τ *increasingly crosses through each* $\tau_n (n = 0, 1, 2, \ldots)$, *and there exist* $\varepsilon_0 > 0$ *and a continuous family of periodic orbits of* (1.5) *in form of*

$$\{(\tau_n(\varepsilon), u_n(x,t,\varepsilon), v_n(x,t,\varepsilon), T_n(\varepsilon)) : \varepsilon \in (0, \varepsilon_0)\},$$

where $(u_n(x,t,\varepsilon), v_n(x,t,\varepsilon))^T$ *is a* $T_n(\varepsilon)$*-periodic solution of* (1.5) *with* $\tau = \tau_n(\varepsilon)$, *and* $\tau_n(0) = \tau_n$, $\lim_{\varepsilon\to 0^+}(u_n(x,t,\varepsilon), v_n(x,t,\varepsilon))^T = (u_\lambda, v_\lambda)^T$ *and* $\lim_{\varepsilon\to 0^+} T_n(\varepsilon) = 2\pi/\omega_\lambda$.

5 The Properties of Hopf Bifurcation

In this section, we will compute the normal form of the Hopf bifurcation to determine the direction and stability of bifurcating periodic solutions emerging from (U_λ, τ_n) by applying the methods in Faria [11] and Hassard et al. [16]. At first, we let $\tilde{U}(t) = (U_1(t), U_2(t))^T = (u(\cdot,t) - u_\lambda, v(\cdot,t) - v_\lambda)^T$, $\gamma = \tau - \tau_n$ such that the steady state $U_\lambda = (u_\lambda, v_\lambda)^T$ and parameter τ is translated to the origin. Re-scale the time $\tilde{t} = t/\tau$ and drop the tilde signs for simplification of notations. Thus, $\gamma = 0$ is the Hopf bifurcation value. For the simplicity of writing, we define

$$r_{kl}^{i\lambda}(x) = \frac{\partial^{k+l} f_i}{\partial s_1^k s_2^l}(x, K_{i1} * u_\lambda, K_{i2} * v_\lambda), \quad i = 1, 2, \; k, l = 1, 2, \ldots.$$

Then we can rewrite Eq. (1.5) as the following abstract functional differential equation

$$\frac{dU(t)}{dt} = \tau_n L_0(U_t) + J(U_t, \gamma), \tag{5.1}$$

where $U_t = U(t+s) \in \mathcal{C} = C([-\tau, 0], \mathbb{Y}^2)$, and

$$\begin{aligned} L_0(U_t) &= A_\lambda U(t) + B_\lambda U(t-1), \\ J(U_t, \gamma) &= \gamma L_0(U_t) + (\gamma + \tau_n)\lambda F(U_t), \end{aligned}$$

and $F(U_t) = (F_1(U_t), F_2(U_t))^T$ is defined by

$$\begin{aligned} F_1(U_t) = {} & U_1(t)\Big[r_1^{1\lambda}(x) K_{11} * U_1(t-1) + r_2^{1\lambda}(x) K_{12} * U_2(t-1)\Big] \\ & + \frac{1}{2!}(U_1(t) + u_\lambda)\Big[r_{20}^{1\lambda}(x)(K_{11} * U_1(t-1))^2 \\ & + 2r_{11}^{1\lambda}(x)(K_{11} * U_1(t-1))(K_{12} * U_2(t-1)) \\ & + r_{02}^{1\lambda}(x)(K_{12} * U_2(t-1))^2\Big] + \frac{1}{3!}u_\lambda\Big[r_{30}^{1\lambda}(x)(K_{11} * U_1(t-1))^3 \\ & + 3r_{21}^{1\lambda}(x)(K_{11} * U_1(t-1))^2(K_{12} * U_2(t-1)) \\ & + 3r_{12}^{1\lambda}(x)(K_{11} * U_1(t-1))(K_{12} * U_2(t-1))^2 \\ & + r_{03}^{1\lambda}(x)(K_{12} * U_2(t-1))^3\Big] + h.o.t, \\ F_2(U_t) = {} & U_2(t)\Big[r_1^{2\lambda}(x) K_{21} * U_1(t-1) + r_2^{2\lambda}(x) K_{22} * U_2(t-1)\Big] \\ & + \frac{1}{2!}(U_2(t) + v_\lambda)\Big[r_{20}^{2\lambda}(x)(K_{21} * U_1(t-1))^2 \end{aligned}$$

$$
\begin{aligned}
&+2r_{11}^{2\lambda}(x)(K_{21}*U_1(t-1))(K_{22}*U_2(t-1))\\
&+r_{02}^{2\lambda}(x)(K_{22}*U_2(t-1))^2\Big]+\frac{1}{3!}v_\lambda\Big[r_{30}^{2\lambda}(x)(K_{21}*U_1(t-1))^3\\
&+3r_{21}^{2\lambda}(x)(K_{21}*U_1(t-1))^2(K_{22}*U_2(t-1))\\
&+3r_{12}^{2\lambda}(x)(K_{21}*U_1(t-1))(K_{22}*U_2(t-1))^2\\
&+r_{03}^{2\lambda}(x)(K_{22}*U_2(t-1))^3\Big]+h.o.t,
\end{aligned}
$$

in which $h.o.t$ stands for "high order terms". Denote by $\mathcal{A}_{\tau_n}$ the infinitesimal generator of the linearized equation

$$\frac{dU(t)}{dt}=\tau_n L_0(U_t). \tag{5.2}$$

Then from [32], we have

$$\mathcal{A}_{\tau_n}\Psi=\dot{\Psi},$$

with the domain

$$\mathscr{D}(A_\tau)=\left\{\Psi\in\mathcal{C}_{\mathbb{C}}\cap\mathcal{C}_{\mathbb{C}}^1:\Psi(0)\in\mathbb{X}_{\mathbb{C}}^2,\dot{\Psi}(0)=\tau_n A_\lambda\Psi(0)+\tau_n B_\lambda\Psi(-1)\right\},$$

where $\mathcal{C}_{\mathbb{C}}^1=C^1([-1,0],\mathbb{Y}_{\mathbb{C}}^2)$. So, we can rewrite Eq. (5.1) in the abstract form:

$$\frac{dU_t}{dt}=\mathcal{A}_{\tau_n}U_t+X_0J(U_t,\gamma), \tag{5.3}$$

where

$$X_0(s)=\begin{cases}0, & s\in[-1,0),\\ I, & s=0.\end{cases}$$

On the other hand, let

$$
\begin{aligned}
&\mathcal{A}_{\tau_n}^*\tilde{\Phi}=-\dot{\tilde{\Psi}},\\
&\mathscr{D}(\mathcal{A}_{\tau_n}^*)=\left\{\tilde{\Psi}\in\mathcal{C}_{\mathbb{C}}^*\cap(\mathcal{C}_{\mathbb{C}}^*)^1:\tilde{\Psi}(0)\in\mathbb{X}_{\mathbb{C}}^2,-\dot{\tilde{\Psi}}(0)=\tau_n A_\lambda\tilde{\Psi}(0)+\tau_n B_\lambda^*\tilde{\Psi}(1)\right\},
\end{aligned}
$$

where $\mathcal{C}_{\mathbb{C}}^*=C([0,1],\mathbb{Y}_{\mathbb{C}}^2)$, $(\mathcal{C}_{\mathbb{C}}^*)^1=C^1([0,1],\mathbb{Y}_{\mathbb{C}}^2)$. Define a formal duality $\langle\langle\cdot,\cdot\rangle\rangle$ by

$$\langle\langle\tilde{\Psi},\Psi\rangle\rangle=\left\langle\tilde{\Psi}(0),\Psi(0)\right\rangle_w-\tau_n\int_{-1}^0\left\langle\tilde{\Psi}(s+1),B_\lambda\Psi(s)\right\rangle_w \mathrm{d}s. \tag{5.4}$$

for $\tilde{\Psi}\in\mathscr{D}(\mathcal{A}_{\tau_n}^*)$ and $\Psi\in\mathscr{D}(\mathcal{A}_{\tau_n})$. Then $\mathcal{A}_{\tau_n}^*$ and $\mathcal{A}_{\tau_n}$ satisfy

$$\langle\langle\mathcal{A}_{\tau_n}^*\tilde{\Psi},\Psi\rangle\rangle=\langle\langle\tilde{\Psi},\mathcal{A}_{\tau_n}\Psi\rangle\rangle$$

for $\tilde{\Psi}\in\mathscr{D}(\mathcal{A}_{\tau_n}^*)$ and $\Psi\in\mathscr{D}(\mathcal{A}_{\tau_n})$. The above equality means that $\mathcal{A}_{\tau_n}^*$ and $\mathcal{A}_{\tau_n}$ are adjoint operators under the bilinear form (5.4).

It can be seen from Theorem 4 that $\mathcal{A}_{\tau_n}$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_\lambda\tau_n$. Then the eigenfunction corresponding to $i\omega_\lambda\tau_n$ (resp. $-i\omega_\lambda\tau_n$) is $p(s)=\psi_\lambda e^{i\omega_\lambda\tau_n s}$ (resp. $\overline{p}(s)=\overline{\psi}_\lambda e^{-i\omega_\lambda\tau_n s}$) for $s\in[-1,0]$, where ψ_λ is defined as in (3.21). At the same time, it follows from Theorem 5 and Remark 1 that $\pm i\omega_\lambda\tau_n$ are also a pair of simple purely imaginary eigenvalues of the operator $\mathcal{A}_{\tau_n}^*$ and the eigenfunction associated with $-i\omega_\lambda\tau_n$ (resp. $i\omega_\lambda\tau_n$) is $q(\tilde{s})=\tilde{\psi}_\lambda e^{i\omega_\lambda\tau_n\tilde{s}}$ (resp. $\overline{q}(\tilde{s})=\overline{\tilde{\psi}}_\lambda e^{-i\omega_\lambda\tau_n\tilde{s}}$) for $\tilde{s}\in[0,1]$, where $\tilde{\psi}_\lambda$

is defined in Theorem 5. Following from [32], the center subspace of Eq. (5.1) is $P = \text{span}\{p(s), \overline{p}(s)\}$, and the formal adjoint subspace of P with respect to the bilinear form (5.4) is $P^* = \text{span}\{q(\tilde{s}), \overline{q}(\tilde{s})\}$. Let $\Phi(s) = (p(s), \overline{p}(s)), \Psi(\tilde{s}) = \left(\frac{q(\tilde{s})}{\overline{S_{n\lambda}}}, \frac{\overline{q}(\tilde{s})}{S_{n\lambda}}\right)^T$. It is easy to verify that $\langle\langle \Psi, \Phi \rangle\rangle = I$, where $I \in \mathbb{R}^{2\times 2}$ is a identity matrix. Actually, we can decompose $\mathcal{C}_{\mathbb{C}}$ as $\mathcal{C}_{\mathbb{C}} = P \oplus Q$, where

$$Q = \left\{ \Psi \in \mathcal{C}_{\mathbb{C}} : \langle\langle \tilde{\Psi}, \Psi \rangle\rangle = 0 \text{ for } \tilde{\Psi} \in P^* \right\}.$$

Following the idea of Hassard et al. [16], the formulas determining the bifurcation direction and stability are all relative to $\gamma = 0$. In the remainder of this section, we take $\gamma = 0$ and define

$$z(t) = \frac{1}{S_{n\lambda}} \langle\langle q, U_t \rangle\rangle, \quad W(z(t), \overline{z}(t)) = U_t - 2\text{Re}\{z(t)p\}. \tag{5.5}$$

Then we obtain a center manifold $\mathbf{C}_0$:

$$W(z, \overline{z})(s) = W_{20}(s)\frac{z^2}{2} + W_{11}(s)z\overline{z} + W_{02}(s)\frac{\overline{z}^2}{2} + \cdots,$$

where z and $\overline{z}$ are local coordinates for the center manifold $\mathbf{C}_0$ in the direction of q and $\overline{q}$. From (5.5), for $\gamma = 0$, we see that

$$\begin{aligned} \dot{z}(t) &= \frac{1}{S_{n\lambda}} \cdot \frac{d}{dt} \langle\langle q(\tilde{s}), U_t \rangle\rangle = \frac{1}{S_{n\lambda}} \langle\langle q(\tilde{s}), \mathcal{A}_{\tau_n} U_t \rangle\rangle + \frac{1}{S_{n\lambda}} \langle\langle q(\tilde{s}), X_0 J(U_t, 0) \rangle\rangle \\ &= i\omega_\lambda \tau_n z(t) + \frac{1}{S_{n\lambda}} \langle q(0), J(2\text{Re}\{z(t)p\} + W(z(t), \overline{z}(t)), 0) \rangle_w \\ &= i\omega_\lambda \tau_n z(t) + g(z, \overline{z}). \end{aligned} \tag{5.6}$$

Then,

$$\begin{aligned} g(z, \overline{z}) &= \frac{1}{S_{n\lambda}} \langle q(0), J(2\text{Re}\{z(t)p\} + W(z(t), \overline{z}(t)), 0) \rangle_w \\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \end{aligned} \tag{5.7}$$

We now calculate that

$$\begin{aligned} g_{20} &= \frac{2\lambda\tau_n e^{-i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(\psi_\lambda, \psi_\lambda) \right\rangle_w + \frac{\lambda\tau_n e^{-2i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_2(\psi_\lambda, \psi_\lambda) \right\rangle_w, \\ g_{11} &= \frac{\lambda\tau_n e^{i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(\psi_\lambda, \overline{\psi}_\lambda) \right\rangle_w + \frac{\lambda\tau_n e^{-i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(\overline{\psi}_\lambda, \psi_\lambda) \right\rangle_w \\ &\quad + \frac{\lambda\tau_n}{2S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_2(\psi_\lambda, \overline{\psi}_\lambda) + \mathcal{T}_2(\overline{\psi}_\lambda, \psi_\lambda) \right\rangle_w, \\ g_{02} &= \frac{2\lambda\tau_n e^{i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(\overline{\psi}_\lambda, \overline{\psi}_\lambda) \right\rangle_w + \frac{\lambda\tau_n e^{2i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_2(\overline{\psi}_\lambda, \overline{\psi}_\lambda) \right\rangle_w, \\ g_{21} &= \frac{2\lambda\tau_n}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(\psi_\lambda, W_{11}(-1)) \right\rangle_w + \frac{\lambda\tau_n}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(\overline{\psi}_\lambda, W_{20}(-1)) \right\rangle_w \\ &\quad + \frac{\lambda\tau_n e^{i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(W_{20}(0), \overline{\psi}_\lambda) \right\rangle_w + \frac{2\lambda\tau_n e^{-i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_1(W_{11}(0), \psi_\lambda) \right\rangle_w \\ &\quad + \frac{\lambda\tau_n e^{-i\omega_\lambda\tau_n}}{S_{n\lambda}} \left\langle \tilde{\psi}_\lambda, \mathcal{T}_2(\psi_\lambda, W_{11}(-1)) + \mathcal{T}_2(W_{11}(-1), \psi_\lambda) \right\rangle_w \end{aligned}$$

$$+\frac{\lambda\tau_n e^{i\omega_\lambda\tau_n}}{2S_{n\lambda}}\left\langle\tilde{\psi}_\lambda, \mathcal{T}_2(W_{20}(-1), \overline{\psi}_\lambda)+\mathcal{T}_2(\overline{\psi}_\lambda, W_{20}(-1))\right\rangle_w$$
$$+\frac{\lambda\tau_n}{S_{n\lambda}}\left\langle\tilde{\psi}_\lambda, \mathcal{T}_3(\psi_\lambda,\psi_\lambda,\overline{\psi}_\lambda)+\mathcal{T}_3(\psi_\lambda,\overline{\psi}_\lambda,\psi_\lambda)\right\rangle_w+\frac{\lambda\tau_n e^{-2i\omega_\lambda\tau_n}}{S_{n\lambda}}\left\langle\tilde{\psi}_\lambda, \mathcal{T}_3(\overline{\psi}_\lambda,\psi_\lambda,\psi_\lambda)\right\rangle_w$$
$$+\frac{\lambda\tau_n e^{-i\omega_\lambda\tau_n}}{S_{n\lambda}}\left\langle\tilde{\psi}_\lambda, \mathcal{T}_4(\psi_\lambda,\psi_\lambda,\overline{\psi}_\lambda)+\mathcal{T}_4(\psi_\lambda,\overline{\psi}_\lambda,\psi_\lambda)+\mathcal{T}_4(\overline{\psi}_\lambda,\psi_\lambda,\psi_\lambda)\right\rangle_w, \tag{5.8}$$

where $\mathcal{T}_1$ is given by

$$\mathcal{T}_1(\varphi_1,\varphi_2)=\begin{pmatrix}\varphi_1^{(1)}\left(r_1^{1\lambda}(x)K_{11}*\varphi_2^{(1)}+r_2^{1\lambda}(x)K_{12}*\varphi_2^{(2)}\right)\\ \varphi_1^{(2)}\left(r_1^{2\lambda}(x)K_{21}*\varphi_2^{(1)}+r_2^{2\lambda}(x)K_{22}*\varphi_2^{(2)}\right)\end{pmatrix},$$

$\mathcal{T}_2=(\mathcal{T}_2^{(1)},\mathcal{T}_2^{(2)})^T$ is given by

$$\begin{aligned}\mathcal{T}_2^{(1)}(\varphi_1,\varphi_2)&=u_\lambda\Big(r_{20}^{1\lambda}(x)(K_{11}*\varphi_1^{(1)})(K_{11}*\varphi_2^{(1)})+2r_{11}^{1\lambda}(x)(K_{11}*\varphi_1^{(1)})(K_{12}*\varphi_2^{(2)})\\&\quad+r_{02}^{1\lambda}(x)(K_{12}*\varphi_1^{(2)})(K_{12}*\varphi_2^{(2)})\Big),\\ \mathcal{T}_2^{(2)}(\varphi_1,\varphi_2)&=v_\lambda\Big(r_{20}^{2\lambda}(x)(K_{21}*\varphi_1^{(1)})(K_{21}*\varphi_2^{(1)})+2r_{11}^{2\lambda}(x)(K_{21}*\varphi_1^{(1)})(K_{22}*\varphi_2^{(2)})\\&\quad+r_{02}^{2\lambda}(x)(K_{22}*\varphi_1^{(2)})(K_{22}*\varphi_2^{(2)})\Big),\end{aligned}$$

$\mathcal{T}_3=(\mathcal{T}_3^{(1)},\mathcal{T}_3^{(2)})^T$ is given by

$$\begin{aligned}\mathcal{T}_3^{(1)}(\varphi_1,\varphi_2,\varphi_3)&=\varphi_1^{(1)}\Big(r_{20}^{1\lambda}(x)(K_{11}*\varphi_2^{(1)})(K_{11}*\varphi_3^{(1)})+2r_{11}^{1\lambda}(x)(K_{11}*\varphi_2^{(1)})(K_{12}*\varphi_3^{(2)})\\&\quad+r_{02}^{1\lambda}(x)(K_{12}*\varphi_2^{(2)})(K_{12}*\varphi_3^{(2)})\Big),\\ \mathcal{T}_3^{(2)}(\varphi_1,\varphi_2,\varphi_3)&=\varphi_1^{(2)}\Big(r_{20}^{2\lambda}(x)(K_{21}*\varphi_2^{(1)})(K_{21}*\varphi_3^{(1)})+2r_{11}^{2\lambda}(x)(K_{21}*\varphi_2^{(1)})(K_{22}*\varphi_3^{(2)})\\&\quad+r_{02}^{2\lambda}(x)(K_{22}*\varphi_2^{(2)})(K_{22}*\varphi_3^{(2)})\Big),\end{aligned}$$

and $\mathcal{T}_4=(\mathcal{T}_4^{(1)},\mathcal{T}_4^{(2)})^T$ is given by

$$\begin{aligned}&\mathcal{T}_4^{(1)}(\varphi_1,\varphi_2,\varphi_3)\\&=u_\lambda\Big(r_{30}^{1\lambda}(x)(K_{11}*\varphi_1^{(1)})(K_{11}*\varphi_2^{(1)})(K_{11}*\varphi_3^{(1)})+3r_{21}^{1\lambda}(x)(K_{11}*\varphi_1^{(1)})(K_{11}*\varphi_2^{(1)})(K_{12}*\varphi_3^{(2)})\\&\quad+3r_{12}^{1\lambda}(x)(K_{11}*\varphi_1^{(1)})(K_{12}*\varphi_2^{(2)})(K_{12}*\varphi_3^{(2)}+r_{03}^{1\lambda}(x)(K_{12}*\varphi_1^{(2)})(K_{12}*\varphi_2^{(2)})(K_{12}*\varphi_3^{(2)})\Big),\\&\mathcal{T}_4^{(2)}(\varphi_1,\varphi_2,\varphi_3)\\&=v_\lambda\Big(r_{30}^{2\lambda}(x)(K_{21}*\varphi_1^{(1)})(K_{21}*\varphi_2^{(1)})(K_{21}*\varphi_3^{(1)})+3r_{21}^{2\lambda}(x)(K_{21}*\varphi_1^{(1)})(K_{21}*\varphi_2^{(1)})(K_{22}*\varphi_3^{(2)})\\&\quad+3r_{12}^{2\lambda}(x)(K_{21}*\varphi_1^{(1)})(K_{22}*\varphi_2^{(2)})(K_{22}*\varphi_3^{(2)}+r_{03}^{2\lambda}(x)(K_{22}*\varphi_1^{(2)})(K_{22}*\varphi_2^{(2)})(K_{212}*\varphi_3^{(2)})\Big),\end{aligned}$$

for $\varphi_i=(\varphi_i^{(1)},\varphi_i^{(2)})^T\in\mathbb{Y}^2$, $i=1,2,3$. From (5.8), we see that there are only $W_{20}(s)$ and $W_{11}(s)$ in g_{21} left to calculate.

It can be deduced from (5.3) and (5.5) that

$$\dot{W}=\begin{cases}\mathcal{A}_{\tau_n}W-gp(s)-\overline{gp}(s), & s\in[-1,0),\\ \mathcal{A}_{\tau_n}W-gp(0)-\overline{gp}(0)+J(2\mathrm{Re}\{z(t)p\}+W(z(t),\overline{z}(t)),0), & s=0.\end{cases} \tag{5.9}$$

Meanwhile, W also satisfies that

$$\begin{aligned}\dot{W} &= W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}\\ &= [W_{20}(s)z + W_{11}(s)\bar{z}]\,\dot{z} + [W_{11}(s)z + W_{02}(s)\bar{z}]\,\dot{\bar{z}} + \cdots\\ &= [W_{20}(s)z + W_{11}(s)\bar{z}]\,(i\theta_{n\lambda}z + g(z,\bar{z}))\\ &\quad + [W_{11}(s)z + W_{02}(s)\bar{z}]\,(-i\theta_{n\lambda}\bar{z} + \bar{g}(z,\bar{z})) + \cdots,\end{aligned}$$

on the center manifold $\mathbf{C}_0$ near the origin. Combining the above equation with Eq. (5.9), we have

$$(2i\theta_{n\lambda}I - \mathcal{A}_{\tau_n})W_{20}(s) = \begin{cases}-g_{20}p(s) - \bar{g}_{02}\bar{p}(s), & s \in [-1,0),\\ -g_{20}p(0) - \bar{g}_{02}\bar{p}(0) + 2\lambda\tau_n e^{-i\omega_\lambda\tau_n}\mathcal{T}_1(\psi_\lambda,\psi_\lambda) & \\ +\lambda\tau_n e^{-2i\omega_\lambda\tau_n}\mathcal{T}_2(\psi_\lambda,\psi_\lambda), & s = 0,\end{cases} \tag{5.10}$$

and

$$-\mathcal{A}_{\tau_n}W_{11}(s) = \begin{cases}-g_{11}p(s) - \bar{g}_{11}\bar{p}(s), & s \in [-1,0),\\ -g_{11}p(0) - \bar{g}_{11}\bar{p}(0) + \lambda\tau_n e^{-i\omega_\lambda\tau_n}\mathcal{T}_1(\overline{\psi}_\lambda,\psi_\lambda) & \\ +\lambda\tau_n e^{i\omega_\lambda\tau_n}\mathcal{T}_1(\psi_\lambda,\overline{\psi}_\lambda) + \frac{\lambda\tau_n}{2}\left(\mathcal{T}_2(\psi_\lambda,\overline{\psi}_\lambda) + \mathcal{T}_2(\overline{\psi}_\lambda,\psi_\lambda)\right), & s = 0.\end{cases} \tag{5.11}$$

To compute W_{20}, from (5.10), we have

$$W_{20}'(s) = 2i\theta_{n\lambda}W_{20}(s) + g_{20}p(s) + \bar{g}_{02}\bar{p}(s), \quad s \in [-1,0).$$

Note that $p(s) = \psi_\lambda e^{i\omega_\lambda\tau_n s}$, then there holds that

$$W_{20}(s) = \frac{ig_{20}}{\omega_\lambda\tau_n}p(s) + \frac{i\bar{g}_{02}}{3\omega_\lambda\tau_n}\bar{p}(s) + Ee^{2i\omega_\lambda\tau_n s}. \tag{5.12}$$

Especially, Eqs. (5.10) and (5.12) imply that

$$(2i\omega_\lambda\tau_n I - \mathcal{A}_{\tau_n})Ee^{2i\theta_{n\lambda}s}\Big|_{s=0} = 2\lambda\tau_n e^{-i\omega_\lambda\tau_n}\mathcal{T}_1(\psi_\lambda,\psi_\lambda) + \lambda\tau_n e^{-2i\omega_\lambda\tau_n}\mathcal{T}_2(\psi_\lambda,\psi_\lambda),$$

or equivalently,

$$\Lambda(\lambda, 2i\omega_\lambda, \tau_n)E = -2\lambda e^{-i\omega_\lambda\tau_n}\mathcal{T}_1(\psi_\lambda,\psi_\lambda) - \lambda e^{-2i\omega_\lambda\tau_n}\mathcal{T}_2(\psi_\lambda,\psi_\lambda). \tag{5.13}$$

Notice that $2i\omega_\lambda$ is not the eigenvalue of $A_{\tau_n,\lambda}$ for $\lambda \in (\lambda_*, \tilde{\lambda}^*]$. Then

$$E = -2\lambda e^{-i\omega_\lambda\tau_n}\Lambda(\lambda, 2i\omega_\lambda, \tau_n)^{-1}\mathcal{T}_1(\psi_\lambda,\psi_\lambda) - \lambda e^{-2i\omega_\lambda\tau_n}\Lambda(\lambda, 2i\omega_\lambda, \tau_n)^{-1}\mathcal{T}_2(\psi_\lambda,\psi_\lambda).$$

Similarly, we can derive from (5.11) that, for $s \in [-1,0)$,

$$W_{11}(s) = -\frac{ig_{11}}{\omega_\lambda\tau_n}p(s) + \frac{i\bar{g}_{11}}{\omega_\lambda\tau_n}\bar{p}(s) + F, \tag{5.14}$$

and when $s = 0$, F satisfies

$$-\mathcal{A}_\tau F = \lambda\tau_n e^{-i\omega_\lambda\tau_n}\mathcal{T}_1(\overline{\psi}_\lambda,\psi_\lambda) + \lambda\tau_n e^{i\omega_\lambda\tau_n}\mathcal{T}_1(\psi_\lambda,\overline{\psi}_\lambda) + \frac{\lambda\tau_n}{2}\left(\mathcal{T}_2(\psi_\lambda,\overline{\psi}_\lambda) + \mathcal{T}_2(\overline{\psi}_\lambda,\psi_\lambda)\right).$$

Thus, we obtain

$$\begin{aligned}F = -\lambda\Lambda(\lambda, 0, \tau_{n\lambda})^{-1}\Big[&e^{-i\omega_\lambda\tau_n}\mathcal{T}_1(\overline{\psi}_\lambda,\psi_\lambda)\\ &+e^{i\omega_\lambda\tau_n}\mathcal{T}_1(\psi_\lambda,\overline{\psi}_\lambda) + \frac{1}{2}\left(\mathcal{T}_2(\psi_\lambda,\overline{\psi}_\lambda) + \mathcal{T}_2(\overline{\psi}_\lambda,\psi_\lambda)\right)\Big].\end{aligned} \tag{5.15}$$

Lemma 6 *Let E and F be defined in (5.13) and (5.15), respectively. Assume that $(\mathbf{H_1})-(\mathbf{H_3})$ hold, then for $\lambda \in (\lambda_*, \tilde{\lambda}^*]$,*

$$E = \frac{1}{\lambda - \lambda_*}(\rho_\lambda U_\lambda + \varphi_\lambda), \quad F = \frac{\tilde{\varphi}_\lambda}{\lambda - \lambda_*}, \tag{5.16}$$

where $U_\lambda = (u_\lambda, v_\lambda)^T$ is defined in (2.3), φ_λ and $\tilde{\varphi}_\lambda$ satisfy

$$\langle U_\lambda, \varphi_\lambda \rangle = 0, \quad \lim_{\lambda \to \lambda_*} \|\varphi_\lambda\|_{\mathbb{Y}_\mathbb{C}} = 0, \quad \lim_{\lambda \to \lambda_*} \|\tilde{\varphi}_\lambda\|_{\mathbb{Y}_\mathbb{C}} = 0,$$

and the constant ρ_λ satisfies

$$\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*)\rho_\lambda = \frac{2i(1 + p_{1\lambda_*}^2)(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12} p_{1\lambda_*})}{(\beta_{\lambda_*}^2 + c_{\lambda_*}^2)\left[2i(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12} p_{1\lambda_*}) - (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\right]}.$$

Proof We only show the estimate for E, and that for F can be proved in a similar manner. Since $A_\lambda U_\lambda = 0$, by substituting E, defined as in (5.16), into Eq. (5.13), we obtain

$$\begin{aligned} &A_\lambda \varphi_\lambda + B_\lambda(\rho_\lambda U_\lambda + \varphi_\lambda)e^{-2i\omega_\lambda \tau_n} - 2i\omega_\lambda(\rho_\lambda U_\lambda + \varphi_\lambda) \\ &\quad = -2\lambda(\lambda - \lambda_*)e^{-i\omega_\lambda \tau_n}\mathcal{T}_1(\psi_\lambda, \psi_\lambda) - \lambda(\lambda - \lambda_*)e^{-2i\omega_\lambda \tau_n}\mathcal{T}_2(\psi_\lambda, \psi_\lambda). \end{aligned} \tag{5.17}$$

Calculating the weighted inner product of Eq. (5.17) with U_λ gives that

$$\begin{aligned} &\rho_\lambda \left[e^{-2i\omega_\lambda \tau_n} \langle U_\lambda, B_\lambda U_\lambda \rangle_w - 2i\omega_\lambda \langle U_\lambda, U_\lambda \rangle_w\right] \\ &\quad = -e^{-2i\omega_\lambda \tau_n} \langle U_\lambda, B_\lambda \varphi_\lambda \rangle_w + 2i\omega_\lambda \langle U_\lambda, \varphi_\lambda \rangle_w \\ &\qquad - 2\lambda(\lambda - \lambda_*)e^{-i\omega_\lambda \tau_n} \langle U_\lambda, \mathcal{T}_1(\psi_\lambda, \psi_\lambda) \rangle_w \\ &\qquad - \lambda(\lambda - \lambda_*)e^{-2i\omega_\lambda \tau_n} \langle U_\lambda, \mathcal{T}_2(\psi_\lambda, \psi_\lambda) \rangle_w . \end{aligned} \tag{5.18}$$

By calculating the weighted inner product of Eq. (5.17) with φ_λ, we see

$$\begin{aligned} &\langle \varphi_\lambda, A_\lambda \varphi_\lambda \rangle_w + \rho_\lambda e^{-2i\omega_\lambda \tau_n} \langle \varphi_\lambda, B_\lambda U_\lambda \rangle_w - 2i\omega_\lambda \rho_\lambda \langle \varphi_\lambda, U_\lambda \rangle_w \\ &\quad = -e^{-2i\omega_\lambda \tau_n} \langle \varphi_\lambda, B_\lambda \varphi_\lambda \rangle_w + 2i\omega_\lambda \langle \varphi_\lambda, \varphi_\lambda \rangle_w \\ &\qquad - 2\lambda(\lambda - \lambda_*)e^{-i\omega_\lambda \tau_n} \langle \varphi_\lambda, \mathcal{T}_1(\psi_\lambda, \psi_\lambda) \rangle_w \\ &\qquad - \lambda(\lambda - \lambda_*)e^{-2i\omega_\lambda \tau_n} \langle \varphi_\lambda, \mathcal{T}_2(\psi_\lambda, \psi_\lambda) \rangle_w . \end{aligned} \tag{5.19}$$

Recall that

$$\begin{aligned} &\psi_\lambda \to \begin{pmatrix} \phi \\ p_{1\lambda_*}\phi \end{pmatrix}, \quad U_\lambda/(\lambda - \lambda_*) \to \begin{pmatrix} \beta_{\lambda_*}\phi \\ c_{\lambda_*}\phi \end{pmatrix} \text{ in } C(\overline{\Omega}) \times C(\overline{\Omega}), \\ &\omega_\lambda/(\lambda - \lambda_*) \to h_{\lambda_*}, \quad \omega_\lambda \tau_n \to \frac{\pi}{2} + 2n\pi (n = 0, 1, 2, \ldots), \quad \text{as } \lambda \to \lambda_*. \end{aligned} \tag{5.20}$$

Then we have

$$\begin{aligned} &\lim_{\lambda \to \lambda_*} \frac{\langle U_\lambda, B_\lambda U_\lambda \rangle_w}{(\lambda - \lambda_*)^3} = -(\beta_{\lambda_*}^2 + c_{\lambda_*}^2) \int_\Omega m(x)e^{\alpha m(x)}\phi^2 \mathrm{d}x, \\ &\lim_{\lambda \to \lambda_*} \frac{\lambda \langle U_\lambda, \mathcal{T}_1(\psi_\lambda, \psi_\lambda) \rangle_w}{\lambda - \lambda_*} = -(1 + p_{1\lambda_*}^2)h_{\lambda_*} \int_\Omega e^{\alpha m(x)}\phi^2 \mathrm{d}x, \\ &\lim_{\lambda \to \lambda_*} \frac{\lambda \langle U_\lambda, \mathcal{T}_2(\psi_\lambda, \psi_\lambda) \rangle_w}{\lambda - \lambda_*} = 0, \\ &\lim_{\lambda \to \lambda_*} \frac{\omega_\lambda \langle U_\lambda, U_\lambda \rangle_w}{(\lambda - \lambda_*)^3} = h_{\lambda_*}(\beta_{\lambda_*}^2 + c_{\lambda_*}^2) \int_\Omega e^{\alpha m(x)}\phi^2 \mathrm{d}x. \end{aligned}$$

Hence from Eq. (5.18), there exist constants $\tilde{\lambda}^* > \lambda_*$ and $M_1, M_2 > 0$ so that for any $\lambda \in (\lambda_*, \tilde{\lambda}^*)$,

$$|(\lambda - \lambda_*)\rho_\lambda| \le M_1 \|\varphi_\lambda\|_{\mathbb{Y}^2_{\mathbb{C}}} + M_2. \tag{5.21}$$

Similar to the proof of Lemma 2.3 of [4], we have

$$|\langle \varphi_\lambda, A_\lambda \varphi_\lambda \rangle_w| \ge |\lambda_2(\lambda)| \|\varphi_\lambda\|^2_{\mathbb{Y}^2_{\mathbb{C}}},$$

where $\lambda_2(\lambda)$ is the second eigenvalue of $-A(\lambda)$. On the other hand, we can also obtain from (5.19), (5.20) and (5.21) that there exist constants $M_3, M_4 > 0$ such that for any $\lambda \in (\lambda_*, \tilde{\lambda})$,

$$|\lambda_2(\lambda)| \cdot \|\varphi_\lambda\|^2_{\mathbb{Y}^2_{\mathbb{C}}} \le (\lambda - \lambda_*) M_3 \|\varphi_\lambda\|^2_{\mathbb{Y}^2_{\mathbb{C}}} + (\lambda - \lambda_*) M_4 \|\varphi_\lambda\|_{\mathbb{Y}^2_{\mathbb{C}}}.$$

Note that $\lim_{\lambda \to \lambda_*} \lambda_2(\lambda) = \lambda_2 > 0$, where λ_2, defined as in Lemma 1 (i), is the second eigenvalue of $-L$, then $\lim_{\lambda \to \lambda_*} \|\varphi_\lambda\|_{\mathbb{Y}^2_{\mathbb{C}}} = 0$. Now, we can derive from (5.18) that

$$\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*)\rho_\lambda = \frac{2i(1 + p^2_{1\lambda_*})(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12} p_{1\lambda_*})}{(\beta^2_{\lambda_*} + c^2_{\lambda_*})\left[2i(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12} p_{1\lambda_*}) - (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\right]}.$$

The proof is finished. □

From (5.8), we see that each g_{kl} is determined by the parameters of original system (1.5). Notice that

$$\begin{aligned}
&\lim_{\lambda \to \lambda_*} S_{n\lambda} = (1 + \tilde{p}_{1\lambda_*} p_{1\lambda_*}) \left[1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right] \int_\Omega e^{\alpha m(x)} \phi^2 dx, \\
&\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*)\tau_n = \frac{\frac{\pi}{2} + 2n\pi}{h_{\lambda_*}}, \quad \lim_{\lambda \to \lambda_*} (\lambda - \lambda_*) F = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
&\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*) E = \frac{2i(1 + p^2_{1\lambda_*})(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12} p_{1\lambda_*})}{(\beta^2_{\lambda_*} + c^2_{\lambda_*})\left[2i(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12} p_{1\lambda_*}) - (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\right]} \begin{pmatrix} \beta_{\lambda_*}\phi \\ c_{\lambda_*}\phi \end{pmatrix}.
\end{aligned}$$

Then we can compute that

$$\begin{aligned}
&\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*) g_{20} = \frac{2i(c_{\lambda_*} + \beta_{\lambda_*} \tilde{p}_{1\lambda_*} p^2_{1\lambda_*})(\pi + 4n\pi)}{\beta_{\lambda_*} c_{\lambda_*} (1 + \tilde{p}_{1\lambda_*} p_{1\lambda_*})(2 + i(\pi + 4n\pi))}, \\
&\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*) g_{11} = 0, \\
&\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*) g_{02} = -\frac{2i(c_{\lambda_*} + \beta_{\lambda_*} \tilde{p}_{1\lambda_*} p^2_{1\lambda_*})(\pi + 4n\pi)}{\beta_{\lambda_*} c_{\lambda_*} (1 + \tilde{p}_{1\lambda_*} p_{1\lambda_*})(2 + i(\pi + 4n\pi))},
\end{aligned} \tag{5.22}$$

which combined with (5.12) and (5.14) yields

$$\begin{aligned}
\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*)^2 g_{21} = &\frac{8i(\pi + 4n\pi)}{3(1 + \tilde{p}_{1\lambda_*} p_{1\lambda_*})^2 |2 + i(\pi + 4n\pi)|^2} \left[\frac{1}{\beta_{\lambda_*}} + \frac{\tilde{p}_{1\lambda_*} p^2_{1\lambda_*}}{c_{\lambda_*}}\right]^2 \\
&+ \frac{2(1 + i)(1 + p^2_{1\lambda_*})(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})}{(\beta^2_{\lambda_*} + c^2_{\lambda_*})\left[2i(\kappa_{22} - \kappa_{12})(\kappa_{11} + \kappa_{12} p_{1\lambda_*}) - (\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})\right]} \\
&\times \frac{\pi + 4n\pi}{2 + i(\pi + 4n\pi)}.
\end{aligned} \tag{5.23}$$

Consequently, we have obtained the normal form (5.6) restricted on the center manifold $\mathbf{C}_0$ by computing the coefficients g_{20}, g_{11}, g_{02} and g_{21}. Denote

$$C_1(0) = \frac{i}{2\theta_{n\lambda}}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2}.$$

Then, we have

$$\begin{aligned}\mu_2 &= -\frac{\mathrm{Re}(C_1(0))}{\mathrm{Re}(\mu'(\tau_n))},\\ \beta_2 &= 2\mathrm{Re}(C_1(0)),\\ T_2 &= -\frac{\mathrm{Im}(C_1(0)) + \mu_2 \mathrm{Im}(\mu'(\tau_n))}{\tau_n},\end{aligned}$$

which determine the properties of bifurcating periodic solutions at critical value τ_n, that is,

(i) μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0(< 0)$, then the direction of the Hopf bifurcation is forward (backward) and the bifurcating periodic solutions exist for $\tau > \tau_n(\tau < \tau_n)$;
(ii) β_2 determines the stability of the bifurcating periodic solutions: if $\beta_2 < 0(> 0)$, then the bifurcating periodic solutions are orbitally asymptotically stable (unstable) on the center manifold;
(iii) T_2 determines the period of bifurcating periodic solutions: if $T_2 > 0(< 0)$, then the period of the bifurcating periodic solutions increases (decreases).

Under assumptions ($\mathbf{H_2}$) and ($\mathbf{H_3}$), there holds that

$$\lim_{\lambda\to\lambda_*} \mathrm{Re}((\lambda - \lambda_*)^2 g_{21}) < 0, \quad \lim_{\lambda\to\lambda_*} \mathrm{Re}((\lambda - \lambda_*)^2 C_1(0)) < 0.$$

Now, the following result is obtained.

Theorem 8 *Assume that* ($\mathbf{H_1}$) $-$ ($\mathbf{H_3}$) *hold, and* $\lambda \in (\lambda_*, \lambda^*]$ *with* $0 < \lambda^* - \lambda_* \ll 1$. *Let* $\tau_n(\lambda)$ *given as in* (3.21) *be the Hopf bifurcation points for Eq.* (1.5) *where spatially nonhomogeneous periodic orbits of Eq.* (1.5) *emerge from* $(\tau_n, u_\lambda, v_\lambda)$. *Then for* $n \in \mathbb{N}\cup\{0\}$, *the direction of the Hopf bifurcation at* $\tau = \tau_n$ *is forward and the bifurcating periodic solutions are orbitally asymptotically stable on the center manifold. Especially, there exist* $\varepsilon_0 > 0$ *such that* (1.5) *has a locally asymptotically stable spatially nonhomogeneous periodic solution for* $\tau \in (\tau_0, \tau_0 + \varepsilon_0)$.

6 A Lotka–Volterra Competition–Diffusion–Advection Model with Nonlocal Delay

In this section we choose the following Lotka–Volterra competitive system as examples

$$\begin{cases} u_t = \left[du_x - aum_x\right]_x + u(x,t) \\ \qquad \times\left[m(x) - a_{11}\int_0^\pi k(x,y)u(y,t-\tau)\mathrm{d}y - a_{12}\int_0^\pi k(x,y)v(y,t-\tau)\mathrm{d}y\right], & x\in(0,\pi), t>0, \\ v_t = \left[dv_x - avm_x\right]_x + v(x,t) \\ \qquad \times\left[m(x) - a_{21}\int_0^\pi k(x,y)u(y,t-\tau)\mathrm{d}y - a_{22}\int_0^\pi k(x,y)v(y,t-\tau)\mathrm{d}y\right], & x\in(0,\pi), t>0, \\ u(x,t) = v(x,t) = 0, & x = 0,\pi,\ t>0, \end{cases} \tag{6.1}$$

where $a_{11}, a_{22}, a_{12}, a_{21} > 0$ and $k(x, y)$ is a continuous nonnegative function on $\Omega \times \Omega$. Then (6.1) is a competitive system. Similar to Eq. (1.5), we can also obtain an equivalent model of Eq. (6.1) as follows:

$$\begin{cases} u_t = e^{-\alpha m(x)}\nabla \cdot \left[e^{\alpha m(x)}\nabla u\right] + \lambda u(x,t) \\ \qquad \times\left[m(x) - a_{11}\int_0^\pi K(x,y)u(y,t-\tau)\mathrm{d}y - a_{12}\int_0^\pi K(x,y)v(y,t-\tau)\mathrm{d}y\right], & x \in (0,\pi), t > 0, \\ v_t = e^{-\alpha m(x)}\nabla \cdot \left[e^{\alpha m(x)}\nabla v\right] + \lambda v(x,t) \\ \qquad \times\left[m(x) - a_{21}\int_0^\pi K(x,y)u(y,t-\tau)\mathrm{d}y - a_{22}\int_0^\pi K(x,y)v(y,t-\tau)\mathrm{d}y\right], & x \in (0,\pi), t > 0, \\ u(x,t) = v(x,t) = 0, & x = 0, \pi,\ t > 0, \end{cases} \tag{6.2}$$

where $\lambda = 1/d, \alpha = a/d$ and $K(x, y) = k(x, y)e^{\alpha m(y)}$. Here $m(x)$ satisfies the assumption $(\mathbf{H_1})$.

Suppose that (u, v) is a positive steady state of (6.2) satisfying

$$\begin{cases} e^{-\alpha m(x)}\nabla \cdot \left[e^{\alpha m(x)}\nabla u\right] + \lambda u(x)\left[m(x) - a_{11}\int_0^\pi K(x,y)u(y)\mathrm{d}y - a_{12}\int_0^\pi K(x,y)v(y)\mathrm{d}y\right] = 0, & x \in (0,\pi), \\ e^{-\alpha m(x)}\nabla \cdot \left[e^{\alpha m(x)}\nabla v\right] + \lambda v(x)\left[m(x) - a_{21}\int_0^\pi K(x,y)u(y)\mathrm{d}y - a_{22}\int_0^\pi K(x,y)v(y)\mathrm{d}y\right] = 0, & x \in (0,\pi), \\ u(x) = v(x) = 0, & x = 0, \pi. \end{cases} \tag{6.3}$$

Multiplying the first equation of (6.3) by $e^{\alpha m(x)}\phi$ and integrating the result over Ω, we have

$$\begin{aligned} \lambda_* \int_0^\pi m(x)e^{\alpha m(x)}\phi u\mathrm{d}x &= -\int_0^\pi u\nabla \cdot [e^{\alpha m(x)}\nabla\phi]\mathrm{d}x = -\int_0^\pi \phi\nabla \cdot [e^{\alpha m(x)}\nabla u]\mathrm{d}x \\ &= \lambda \int_0^\pi \phi e^{\alpha m(x)} u\left[m(x) - a_{11}\int_0^\pi K(x,y)u(y)\right. \\ &\qquad \left. - a_{12}\int_0^\pi K(x,y)v(y)\right]\mathrm{d}x \\ &\le \lambda \int_0^\pi m(x)e^{\alpha m(x)}\phi u\mathrm{d}x. \end{aligned}$$

Therefore, the problem (6.2) has no positive steady state if $\lambda < \lambda_*$.

Let

$$\begin{aligned} f_1(x, K_{11} * u, K_{12} * v) &= m(x) - a_{11}\int_0^\pi K(x,y)u\mathrm{d}y - a_{12}\int_0^\pi K(x,y)v\mathrm{d}y, \\ f_2(x, K_{21} * u, K_{22} * v) &= m(x) - a_{21}\int_0^\pi K(x,y)u\mathrm{d}y - a_{22}\int_0^\pi K(x,y)v\mathrm{d}y. \end{aligned}$$

Suppose that $\frac{a_{11}}{a_{21}} > 1 > \frac{a_{12}}{a_{22}}$. Then one can easily check that $(\mathbf{H_2})$ and $(\mathbf{H_3})$ are satisfied. According to Lemma 3, Theorems 1, 7 and 8 , we obtain the following conclusions for (6.2) which is competitive:

(i) When $0 < \lambda < \lambda_*$, the trivial steady state (0, 0) is the unique nonnegative steady state of (6.2), which is locally asymptotically stable;
(ii) When $\lambda \in (\lambda_*, \tilde{\lambda}^*]$ with $0 < \tilde{\lambda}^* - \lambda_* \ll 1$, system (6.2) admits a spatially nonhomogeneous positive steady state (u_λ, v_λ);
(iii) For $\lambda \in (\lambda_*, \tilde{\lambda}^*]$, there exists a critical point τ_0 such that the positive steady state (u_λ, v_λ) is locally asymptotically stable if $\tau \in [0, \tau_0)$, and unstable if $\tau \in (\tau_0, \infty)$;
(iv) System (6.2) undergoes a supercritical Hopf bifurcation at the positive steady state (u_λ, v_λ) when $\tau = \tau_0$, and there exists a locally stable spatially nonhomogeneous time-periodic solution for $\tau \in (\tau_0, \tau_0 + \varepsilon_0)$, where $\varepsilon_0 > 0$ is small.

It follows from Lemma 2 and Theorem 4 that

$$\theta_{\lambda_*(\alpha)}(\alpha) = \frac{\pi}{2}, \;\; h_{\lambda_*(\alpha)}(\alpha) = \frac{(a_{22} - a_{12})(a_{11} + a_{12} p_{1\lambda_*}) \int_{\Omega} m(x) e^{\alpha m(x)} \phi^2 dx}{(a_{11}a_{22} - a_{12}a_{21}) \int_{\Omega} e^{\alpha m(x)} \phi^2 dx},$$

where $p_{1\lambda_*}$ is the positive root of the equation

$$a_{12}(a_{12} - a_{22})p^2 + (a_{11}a_{12} - a_{22}a_{21})p - a_{21}(a_{21} - a_{11}) = 0,$$

and the first Hopf bifurcation value satisfies

$$\tau_0(\lambda, \alpha) = \frac{\theta_\lambda(\alpha)}{(\lambda - \lambda_*(\alpha))h_\lambda(\alpha)}.$$

Since $\frac{a_{11}}{a_{21}} > 1 > \frac{a_{12}}{a_{22}}$, we have $\frac{(a_{22}-a_{12})(a_{11}+a_{12}p_{1\lambda_*})}{(a_{11}a_{22}-a_{12}a_{21})} > 0$. By the similar argument to [8, Proposition 4.7], we can derive that $h_{\lambda_*(\alpha)}(\alpha)$ is strictly increasing with respect to $\alpha \in [0, \infty)$. Then we show how the advection rate affects the Hopf bifurcation value with respect to sufficiently small $\alpha > 0$.

Proposition 1 *Assume that* $\frac{a_{11}}{a_{21}} > 1 > \frac{a_{12}}{a_{22}}$ *and the non-constant function* $m(x)$ *satisfies* $(\mathbf{H_1})$ *and*

$$m(x) > 0 \;\text{ and }\; \Delta m < 0 \;\text{ for }\; x \in \Omega.$$

Then there exist $\delta_1, \delta_2 > 0$ *such that* $\tau_0(\lambda, \alpha)$ *is strictly decreasing with respect to* $\alpha \in [0, \delta_1)$ *for* $\lambda \in (\lambda_*(0), \lambda_*(0) + \delta_2)$.

Remark 2 Proposition 1 implies that Hopf bifurcation is more likely to occur when adding a term describing advection along the environmental gradients for the diffusive Lotka–Volterra competition model with nonlocal delay. One can use an argument similar to [19, Theorem 5.1] to prove Proposition 1.

In the following we give some numerical simulations to verify our analysis results. In order to maintain the real time scale, we will simulate the original competitive system (6.1), then the critical value for stability switch is τ_0/d. We take $\Omega = (0, \pi)$, the space step as $\pi/50$ and the time step as 0.001. In numerical simulations, different types of patterns are observed and we have found that the distribution of species u and v is always of the same type. For the sake of simplicity, only the patterns of the distribution of species u are given here for instance. Choose the following parameter set:

$$(P) \;\; k(x, y) \equiv 1, a_{11} = 0.4, a_{12} = 0.1, a_{21} = 0.1, a_{22} = 0.4, m(x) = \sin x, x \in (0, \pi)$$

and initial condition:

$$(IC) \;\; u(x, t) = v(x, t) = 0.1 \sin x, \;\; x \in \overline{\Omega}, t \in [-\tau, 0].$$

It follows from previous argument that system (6.1) admits no positive steady state if $d > 1/\lambda_*$. Then we choose $d = 1$ in Fig. 1, and observe that the solution of (6.1) converges to trivial steady state (0, 0) both when $\tau = 0$ and $\tau = 2$.

The influence of the time delay τ on the solution of (6.1) can be observed clearly in Fig. 2. We first set $d = 0.06$, $a = 0.01$. According to our theoretical analysis, when $\tau < \tau_0/d$, the positive steady state $(u_d, v_d) = (e^{am(x)/d}\tilde{u}_\lambda, e^{am(x)/d}\tilde{v}_\lambda)$ of (6.1) is locally asymptotically stable, while if $\tau > \tau_0/d$ a forward Hopf bifurcation occurs, the positive steady state (u_d, v_d)

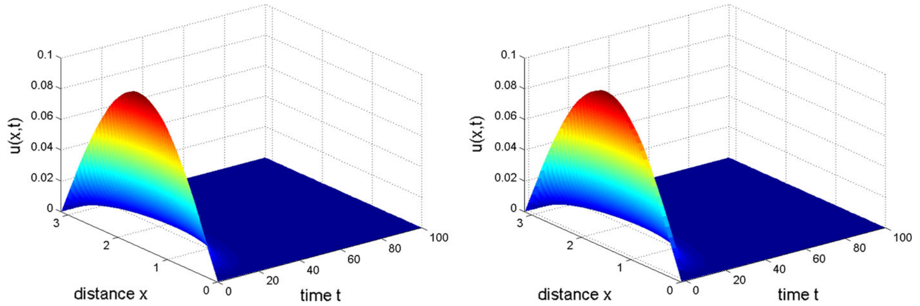

Fig. 1 Numerical simulations of (6.1) for $d = 1$, $a = 0.01$ with parameter set (P) and initial condition (IC). Left: $\tau = 0$; Right: $\tau = 2$

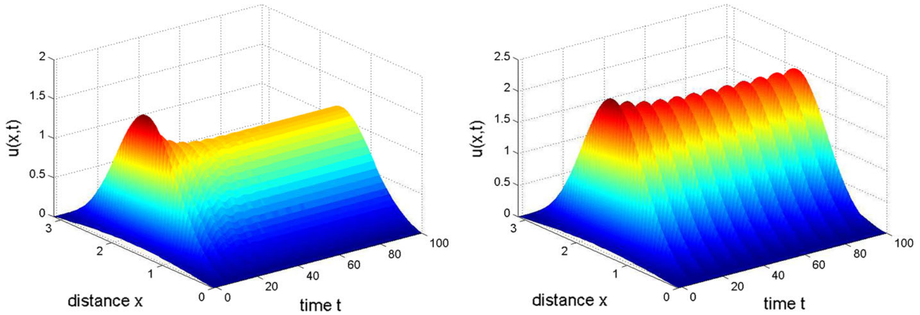

Fig. 2 Numerical simulations of (6.1) for $d = 0.06$, $a = 0.01$ with parameter set (P) and initial condition (IC). Left: $\tau = 1.5$; Right: $\tau = 2$

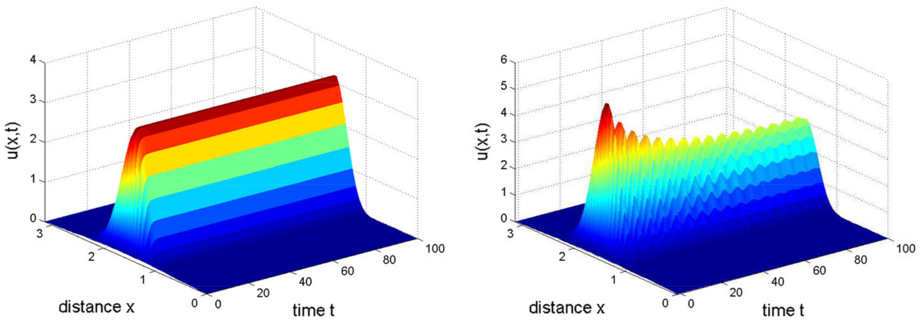

Fig. 3 Numerical simulations of (6.1) for $d = 0.06$, $a = 1$ with parameter set (P) and initial condition (IC). Left: $\tau = 0.2$; Right: $\tau = 1.5$

loses its stability and the bifurcation periodic solution is stable. The left graph in Fig. 2 show the existence of stable nonhomogeneous positive steady state and the right graph in Fig. 2 depicts the occurrence of stable periodic solutions with obvious oscillation. In Fig. 3, letting $a = 1$, we see that the solution of (6.1) converges to a positive steady state when $\tau = 0.2$; when $\tau = 1.5$, the solution of (6.1) converges to a time-periodic solution. Then Figs. 2, 3 show that, the critical Hopf bifurcation value for stability switch decreases as the advection rate a increasing.

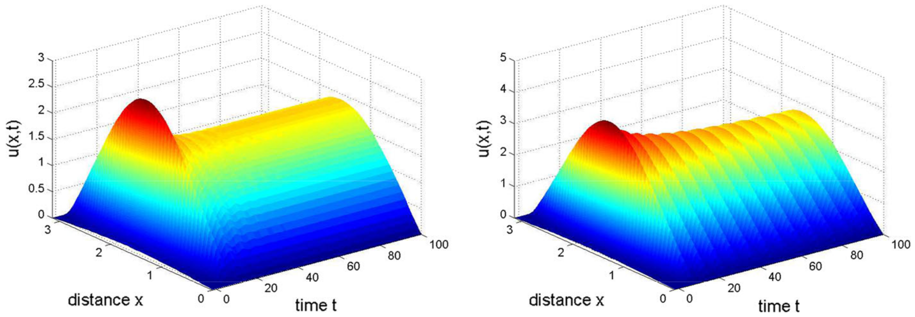

Fig. 4 Numerical simulations of (6.1) for discrete delay case with $d = 0.06$, parameter set (P) and initial condition (IC). Here $a = 0.01$. Left: $\tau = 1.5$; Right: $\tau = 2$

In Fig. 4, we show the simulation for the discrete delay case that $k(x, y) = \delta(x - y)$ in (6.1), which has been studied in [21]. By comparing Figs. 2 and 4, we see that the nonlocal delay makes the positive steady state and time-periodic solution smaller. In biology, this means that the nonlocal delay causes the intraspecific and interspecific competitions more fiercely.

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