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Limiting Solutions of Nonlocal Dispersal Problem in Inhomogeneous Media

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Abstract
This paper is concerned with the nonlocal dispersal problem in inhomogeneous media. Our goal is to show the limiting behavior of perturbation equation with parameters. By analyzing the asymptotic behavior of solutions when the parameter is small, we find that *convection* appears in inhomogeneous media. Moreover, if the effect of inhomogeneous media changes, then we prove a convergence result that *convection* disappears in nonlocal dispersal problems.

Keywords Evolution equation · Inhomogeneous · Convection

Mathematics Subject Classification 35B40 · 35K57 · 92D25

1 Introduction

Let $K : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative function such that $\int_{\mathbb{R}^N} K(x)\,dx = 1$. Nonlocal dispersal equation

$$u_t(x,t) = K * u(x,t) - u(x,t) = \int_{\mathbb{R}^N} K(x-y)u(y,t)\,dy - u(x,t), \tag{1.1}$$

and variations of it have been widely used to model diffusion process (see e.g. [3,16]). As stated in [1,12], if $u(x,t)$ is thought as a density at position x at time t and the probability distribution that individuals jump from y to x is given by $K(x-y)$, then $\int_{\mathbb{R}^N} K(x-y)u(y,t)\,dy$ denotes the rate at which individuals are arriving to position x from all other places and $u(x,t) = \int_{\mathbb{R}^N} K(y-x)u(x,t)\,dy$ is the rate at which they are leaving position x to all other places. This consideration, in the absence of external sources, leads immediately to that u satisfies (1.1). For recent references on nonlocal dispersal equations, see [2,5,9,15,27–30] and references therein.

Since the natural environments are generally heterogeneous and the habitat fragmentation makes a basic change on the spreading and diffusion of species [4,19,31]. In this case, the

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inhomogeneous media plays a great role in the study of evolution problems. We study the nonlocal dispersal problems in inhomogeneous media and consider the following nonlocal dispersal equation

$$u_t = \int_{\mathbb{R}^N} k_1(x, y)u(y, t)\, dy - \int_{\mathbb{R}^N} k_2(x, y)u(x, t)\, dy, \tag{1.2}$$

here k_1, k_2 are nonnegative dispersal kernel functions. In the present paper, we shall investigate the effect of spatial fragmentation on the solutions of (1.2). By employing a parameter in (1.2) and analyzing its limiting behavior of solutions, we provide an implicit understanding on the effect of heterogeneous environment. At this respect, we refer the reader to the seminal works of Cortazar et al. [7,8], Molino and Rossi [18], Shen and Xie [26] on the study of approximation problems for nonlocal dispersal problems.

In this paper, we study two forms of kernel functions k_1, k_2. We first assume that

$$k_1(x, y) = g(x)J(x-y) \text{ and } k_2(x, y) = g(y)J(x-y),$$

here J is a nonnegative kernel function and g is a nonnegative weight function. Then we have the nonlocal dispersal equation

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)[g(x)u(y,t) - g(y)u(x,t)]\, dy & \text{in } \bar{\Omega} \times (0,\infty), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \backslash \bar{\Omega} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \bar{\Omega}, \end{cases} \tag{1.3}$$

where Ω a bounded smooth domain of $\mathbb{R}^N$ and $u_0(x)$ is the given initial value. In (1.3), let $u(x, t)$ be the density of population at position x and time t, and the probability distribution that individual jump from position y to x be given by $J(x - y)$. We assume the rate that individuals arrive at x is affected by the inhomogeneous media with weight function $g(x)$. Then the rate that individuals arrive at x is given by

$$\int_{\mathbb{R}^N} g(x)J(x-y)u(y,t)\, dy$$

and the rate that individuals are leaving position x is given by

$$\int_{\mathbb{R}^N} g(y)J(y-x)u(x,t)\, dy.$$

In fact, here we assume that the probability of population is affected by the inhomogeneous media with weight function at the location where they are going. Hence there has a nonsymmetric effect of inhomogeneous media. The case of symmetric effect is investigated in the second part below. Also in (1.3), the individuals live in Ω and there is no individual outside $\bar{\Omega}$. This is called nonlocal Dirichlet boundary condition, see [6,14]. Throughout this paper, we make the following assumptions.

- $(A1)$ $J : \mathbb{R}^N \to \mathbb{R}$ is nonnegative, radial, continuous with unit integral, J is strictly positive in $B(0, 1)$ and vanishes in $\mathbb{R}^N \backslash B(0, 1)$.
- $(A2)$ The functions $g(x)$, $u_0(x)$ are smooth in $\bar{\Omega}$ and $g(x) > 0$ for $x \in \bar{\Omega}$.

The existence and uniqueness of solutions to (1.3) will be stated in Sect. 2. We show that there exists a unique solution $u(x, t)$ to (1.3) such that

$$u \in C([0,\infty); C(\bar{\Omega})) \cap C^1((0,\infty); C(\bar{\Omega})).$$

Our aim is to investigate the effect of inhomogeneous media on the the nonlocal evolution problem (1.3). Then we study the role of heterogeneous weight function $g(x)$ by employing a parameter and investigate the limiting behavior of solutions when the parameter is small. So we consider the nonlocal dispersal equation

$$\begin{cases} u_t^\varepsilon(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J^\varepsilon(x-y)[g(x)u^\varepsilon(y,t) - g(y)u^\varepsilon(x,t)]\,dy & \text{in } \bar{\Omega}\times(0,\infty),\\ u^\varepsilon(x,t) = 0 & \text{in } \mathbb{R}^N\backslash\bar{\Omega}\times(0,\infty),\\ u^\varepsilon(x,0) = u_0(x) & \text{in } \bar{\Omega}, \end{cases} \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter and the kernel function $J^\varepsilon(\cdot)$ is given by

$$J^\varepsilon(\xi) = \frac{1}{d\varepsilon^N g(x)} J\left(\frac{\xi}{\varepsilon}\right) \tag{1.5}$$

with the constant

$$d = \frac{1}{2N}\int_{\mathbb{R}^N} J(y)|y|^2 dy. \tag{1.6}$$

In this paper we obtain that the solution of (1.4) converges to the solution of the classical reaction–diffusion equation

$$\begin{cases} u_t = \Delta u + p(x)u & \text{in } \Omega\times(0,\infty),\\ u(x,t) = 0 & \text{in } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.7}$$

as $\varepsilon \to 0$, here the coefficient $p(x)$ is given by

$$p(x) = \frac{1}{dg(x)}\sum_{i=1}^N \frac{\partial^2 g(x)}{\partial x_i \partial x_i}. \tag{1.8}$$

Note that the regularity of solution $u(x,t)$ to (1.7) is related to the initial value $u_0(x)$ and the coefficient $p(x)$, see [10,11]. Let $u(x,t)$ be the unique solution to (1.7) such that

$$u \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times[0,T])$$

for some $0 < \alpha < 1$. We are ready to state the main result.

Theorem 1.1 *Assume that $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ is the solution of (1.7) and $u^\varepsilon(x,t)$ is the solution of (1.4) for $\varepsilon > 0$, respectively. Then there exists $C = C(T)$ such that*

$$\max_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{C(\bar{\Omega})} \le C\varepsilon^\alpha \to 0 \text{ as } \varepsilon \to 0.$$

From Theorem 1.1 we know that the inhomogeneous media may provide a linear increase (or decrease) on the nonlocal dispersal system (1.3) provided $p(x)$ is positive (or negative).

Now we consider the second case that

$$k_1(x,y) = k_2(x,y) = g(y)J(x-y)$$

in (1.2). In this case, we can see that the effect of inhomogenous media is related to the same position and there appears a symmetric effect of heterogeneous environment. So we may assume the rates that individuals arrive at x or departure form x are all affected by the inhomogeneous media with weight function $g(y)$. We then have the following nonlocal dispersal equation

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^N} g(y)J(x-y)[u(y,t)-u(x,t)]\,dy & \text{in } \bar{\Omega}\times(0,\infty), \\ u(x,t)=0 & \text{in } \mathbb{R}^N\backslash\bar{\Omega}\times(0,\infty), \\ u(x,0)=u_0(x) & \text{in } \bar{\Omega}. \end{cases} \tag{1.9}$$

In order to study the effect of heterogeneous environment on the problem (1.9), we consider the following problem

$$\begin{cases} u_t^\varepsilon(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} g(y)J^\varepsilon(x-y)[u^\varepsilon(y,t)-u^\varepsilon(x,t)]\,dy & \text{in } \bar{\Omega}\times(0,\infty), \\ u^\varepsilon(x,t)=0 & \text{in } \mathbb{R}^N\backslash\bar{\Omega}\times(0,\infty), \\ u^\varepsilon(x,0)=u_0(x) & \text{in } \bar{\Omega}, \end{cases} \tag{1.10}$$

here $\varepsilon > 0$ and the kernel function J^ε is given by (1.5). We prove that the limiting behavior of nonlocal dispersal Eq. (1.9) is similar to the convection–diffusion equation

$$\begin{cases} u_t = \Delta u + q(x)\cdot\nabla u & \text{in } \Omega\times(0,\infty), \\ u(x,t)=0 & \text{in } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases} \tag{1.11}$$

here

$$\begin{aligned} \nabla u &= \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots, \frac{\partial u(x)}{\partial x_N}\right), \\ q(x) &= \frac{2\nabla g(x)}{dg(x)} = \frac{2}{dg(x)}\left(\frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \ldots, \frac{\partial g(x)}{\partial x_N}\right), \end{aligned} \tag{1.12}$$

and d is given by (1.6).

We have the following result for nonlocal problem (1.10).

Theorem 1.2 *Assume that $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ is the solution of (1.11) and $u^\varepsilon(x,t)$ is the solution of (1.10) for $\varepsilon > 0$, respectively. Then we have*

$$\max_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{C(\bar{\Omega})} \to 0 \text{ as } \varepsilon \to 0.$$

The conclusion of Theorems 1.1–1.2 reveals different effects of inhomogeneous media on the nonlocal dispersal systems. In the later case (1.9), we can see that the convection appears in inhomogeneous media.

On the other hand, since the natural environment is typically periodic, it is interesting to consider the periodic evolution problems, see [20,22–24]. This paper also deals with the periodic nonlocal dispersal equation

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)[g(x)u(y,t)-g(y)u(x,t)]\,dy + a(x)u(x,t) & \text{in } \mathbb{R}^N\times(0,\infty), \\ u(x,t)=u(x+p_j\mathbf{e}_j,t) & \text{in } \mathbb{R}^N\times(0,\infty), \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.13}$$

and

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^N} g(y)J(x-y)[u(y,t)-u(x,t)]\,dy + a(x)u(x,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,t) = u(x+p_j\mathbf{e}_j,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.14}$$

where $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N)$ is the unit vector of $\mathbb{R}^N$ and $p_j > 0$ for $j = 1, 2, \ldots, N$, $g, a, u_0 \in C(\mathbb{R}^N)$ are periodic functions such that

$$g(x) = g(x+p_j\mathbf{e}_j),\ \ a(x) = a(x+p_j\mathbf{e}_j),\ \ u_0(x) = u_0(x+p_j\mathbf{e}_j). \tag{1.15}$$

It follows from [13,24,25] that (1.13) and (1.14) are related to the interesting problem whether the inhomogeneity speeds up the spreading speeds of

$$\begin{aligned} u_t(x,t) &= \int_{\mathbb{R}^N} J(x-y)[g(x)u(y,t) - g(y)u(x,t)]\,dy \\ &\quad + u(x,t)(a(x)-u(x,t)) \ \text{ in } \mathbb{R}^N \times (0,\infty), \end{aligned}$$

and

$$\begin{aligned} u_t(x,t) &= \int_{\mathbb{R}^N} g(y)J(x-y)[u(y,t) - u(x,t)]\,dy \\ &\quad + u(x,t)(a(x)-u(x,t)) \ \text{ in } \mathbb{R}^N \times (0,\infty). \end{aligned}$$

Our main goal is to examine the nonhomogeneous effects of g on the periodic nonlocal systems (1.13) and (1.14). To do this we first consider the periodic problem

$$\begin{cases} u_t^\varepsilon(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J^\varepsilon(x-y)[g(x)u^\varepsilon(y,t) - g(y)u^\varepsilon(x,t)]\,dy & \text{in } \mathbb{R}^N \times (0,\infty), \\ u^\varepsilon(x,t) = u^\varepsilon(x+p_j\mathbf{e}_j,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.16}$$

where $\varepsilon > 0$ is a small parameter, $j = 1, 2, \ldots, N$ and the kernel function $J^\varepsilon(\cdot)$ is given by (1.5).

The main result of periodic nonlocal dispersal problem (1.16) is the next theorem.

Theorem 1.3 *Assume that $g(x)$, $a(x)$, and $u_0(x)$ satisfy (1.15). Assume further that $g(x) > 0$ for $x \in \mathbb{R}^N$. Let $u \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,\infty))$ be the solution of*

$$\begin{cases} u_t = \Delta u + p(x)u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,t) = u(x+p_j\mathbf{e}_j,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.17}$$

where $p(x)$ is given by (1.8) and $j = 1, 2, \ldots, N$. Then (1.16) admits a unique solution $u^\varepsilon(x,t)$ for $\varepsilon > 0$ and there exists $C = C(T)$ such that

$$\max_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{C(\mathbb{R}^N)} \le C\varepsilon^\alpha \to 0 \text{ as } \varepsilon \to 0$$

for any $T > 0$.

At last, we consider the periodic nonlocal dispersal equation

$$\begin{cases} u_t^\varepsilon(x,t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} g(y)J^\varepsilon(x-y)[u^\varepsilon(y,t) - u^\varepsilon(x,t)]\,dy & \text{in } \mathbb{R}^N \times (0,\infty), \\ u^\varepsilon(x,t) = u^\varepsilon(x+p_j\mathbf{e}_j,t) & \text{in } \mathbb{R}^N \times (0,\infty) \\ u^\varepsilon(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.18}$$

where $\varepsilon > 0$ is a small parameter, $j = 1, 2, \ldots, N$ and the kernel function $J^\varepsilon(\cdot)$ is given by (1.5). Then we have the following theorem.

Theorem 1.4 *Assume that $g(x)$, $a(x)$, and $u_0(x)$ satisfy (1.15). Assume further that $g(x) > 0$ for $x \in \mathbb{R}^N$. Let $u \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,\infty))$ be the solution of*

$$\begin{cases} u_t = \Delta u + q(x) \cdot \nabla u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,t) = u(x + p_j \boldsymbol{e}_j, t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.19}$$

where $q(x)$ is given by (1.12) and $j = 1, 2, \ldots, N$. Then (1.18) admits a unique solution $u^\varepsilon(x,t)$ for $\varepsilon > 0$ and

$$\max_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{C(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon \to 0$$

for any $T > 0$.

The rest of the paper is organized as follows. In Sect. 2 we prove existence, uniqueness and comparison principle for general nonlocal dispersal equations. The main results Theorems 1.1 and 1.2 are proved in Sect. 3. Section 4 is devoted to the periodic nonlocal problems.

2 Existence, Uniqueness and Comparison Principles

In this section, we first present some basic results on the existence and uniqueness of solutions to nonlocal dispersal equations. To do this, we consider the following nonlocal dispersal equation

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}^N} k(x,y)u(y,t)\,dy + m(x)u(x,t) & \text{in } \bar{\Omega} \times (0,\infty), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \backslash \bar{\Omega} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \bar{\Omega}, \end{cases} \tag{2.1}$$

where $k \in C(\bar{\Omega} \times \bar{\Omega})$ is nonnegative and $m \in C(\bar{\Omega})$. Thus our problems (1.3) and (1.9) are special forms of (2.1).

Existence and uniqueness of solutions to (2.1) are followed from the classical semigroup theory (e.g., see the book of Pazy [21]). Let $X = C(\bar{\Omega})$, and $\mathcal{G} : X \to X$ be defined by

$$\mathcal{G}u(x) = \int_\Omega k(x,y)u(y)\,dy + m(x)u(x).$$

Then $\mathcal{G} : X \to X$ is a bounded linear operator. Hence for any $u_0 \in X$, (2.1) has a unique solution $u(t,x;u_0)$ with $u(0,x;u_0) = u_0(x)$ (see Theorem 1.2 in chapter 1 of [21]). In fact,

$$u(t,\cdot;u_0) = e^{\mathcal{G}t}u_0(\cdot).$$

Remark 2.1 We can see that $u \in C([0,\infty); C(\bar{\Omega})$ is a solution to (2.1) with the initial value $u_0 \in C(\bar{\Omega})$ if and only if

$$u(x,t) = e^{m(x)t}u_0(x) + \int_0^t \int_{\mathbb{R}^N} k(x,y)e^{m(x)(t-s)}u(y,s)\,dyds, \;\; (x,t) \in \bar{\Omega} \times (0,\infty),$$

and

$$u(x,t) = 0, \;\; (x,t) \in \mathbb{R}^N \backslash \bar{\Omega} \times (0,\infty).$$

We then have the following result on the existence and uniqueness of solutions to the nonlocal problem (2.1).

Theorem 2.2 *For every $u_0 \in C(\bar{\Omega})$, there exists a unique solution $u(x,t)$ to (2.1) and*

$$u \in C([0,\infty); C(\bar{\Omega})) \cap C^1((0,\infty); C(\bar{\Omega})). \tag{2.2}$$

Proof It follows from the semigroup theory that there exists a unique solution $u(x,t)$ of (2.1) defined in $\bar{\Omega} \times [0,\infty)$ and

$$u \in C([0,\infty); C(\bar{\Omega})). \tag{2.3}$$

For any $\delta \neq 0$ and $t > 0$, we have

$$\begin{aligned} &u(x,t+\delta) - u(x,t) \\ &\quad = [e^{m(x)(t+\delta)} - e^{m(x)t}]u_0(x) + \int_t^{t+\delta}\int_\Omega k(x,y)e^{m(x)(t-s)}u(y,s)\,dyds. \end{aligned}$$

Thus from (2.3) and Lebesgue theorem, we obtain

$$\begin{aligned} &\lim_{\delta\to 0}\frac{u(x,t+\delta)-u(x,t)}{\delta} \\ &\quad = m(x)e^{m(x)t} + e^{m(x)t}\int_\Omega k(x,y)u(y,t)\,dy \end{aligned}$$

and then (2.2) holds. □

Now we give the definition of sub-super solutions to (2.1) and the corresponding comparison principle.

Definition 2.3 A function $u \in C^1([0,T); C(\bar{\Omega}))$ is a super-solution to (2.1) if

$$\begin{cases} u_t(x,t) \geq \int_{\mathbb{R}^N} k(x,y)u(y,t)\,dy + m(x)u(x,t) & \text{in } \bar{\Omega}\times(0,\infty), \\ u(x,t) \geq 0 & \text{in } \mathbb{R}^N\backslash\bar{\Omega}\times(0,\infty), \\ u(x,0) \geq u_0(x) & \text{in } \bar{\Omega}, \end{cases}$$

The sub-solution is defined analogously by reversing the inequalities.

Lemma 2.4 *Assume that $u_0(x)$ is nonnegative. Let $u(x,t)$ be a super-solution to (2.1), then $u(x,t) \geq 0$ for $(x,t) \in \bar{\Omega}\times(0,\infty)$.*

Proof We choose

$$\theta > \max_{\bar{\Omega}}\left[m(x) + \int_\Omega k(x,y)\,dy\right] + 1$$

and define

$$v(x,t) = e^{-\theta t}u(x,t).$$

A direct computation gives that

$$v_t(x,t) \geq \int_\Omega k(x,y)v(y,t)\,dy + [m(x)-\theta]v(x,t)$$

for $x \in \bar{\Omega}$, $t > 0$.

We only need to show that $v(x,t) \geq 0$ for $(x,t) \in \bar{\Omega} \times (0,\infty)$. Assume by contradiction that $v(x,t)$ is negative at some point in $\bar{\Omega} \times (0,\infty)$. Without loss of generality, let $v(x_0,t_0)$ be the negative minimum of $v(x,t)$ for some $(x_0,t_0) \in \bar{\Omega} \times (0,\infty)$. We have $v_t(x_0,t_0) \leq 0$ and $v(x_0,t_0) < 0$. But

$$\begin{aligned} v_t(x_0,t_0) \geq & \int_{\mathbb{R}^N} k(x_0,y)[v(y,t_0) - v(x_0,t_0)]\,dy \\ & + \left[m(x_0) + \int_{\Omega} k(x_0,y)\,dy - \theta\right] v(x_0,t_0) \\ \geq & \left[m(x_0) + \int_{\Omega} k(x_0,y)\,dy - \theta\right] v(x_0,t_0) \\ > & 0, \end{aligned}$$

we get a contradiction. This completes the proof. □

Theorem 2.5 *Assume that $u(x,t)$ and $v(x,t)$ are a pair of super-sub solutions to (2.1). Then $u(x,t) \geq v(x,t)$ for $(x,t) \in \bar{\Omega} \times [0,\infty)$.*

Proof Denote $\omega(x,t) = u(x,t) - v(x,t)$, we have

$$\begin{cases} \omega_t(x,t) \geq \int_{\Omega} k(x,y)\omega(y,t)\,dy + m(x)\omega(x,t), & x \in \bar{\Omega}, t > 0, \\ \omega(x,0) \geq 0, & x \in \bar{\Omega}, \end{cases}$$

and the conclusion is followed by Lemma 2.4. □

At the end of this section, we consider the periodic nonlocal dispersal equation

$$\begin{cases} u_t(x,t) = l(x) \int_{\mathbb{R}^N} k(x,y)u(y,t)\,dy + m(x)u(x,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,t) = u(x + p_j \mathbf{e}_j, t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.4}$$

where $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N)$ is the unit vector of $\mathbb{R}^N$ and $p_j > 0$ for $j = 1, 2, \ldots, N$, $l, m, u_0 \in C(\mathbb{R}^N)$ are periodic functions such that

$$l(x) = l(x + p_j \mathbf{e}_j),\ \ m(x) = m(x + p_j \mathbf{e}_j),\ \ u_0(x) = u_0(x + p_j \mathbf{e}_j) \tag{2.5}$$

for $j = 1, 2, \ldots, N$. We then have the following results, the proof is similar to Theorems 2.2 and 2.5 .

Theorem 2.6 *Assume that $l(x) \geq 0$ for $x \in \mathbb{R}^N$. Then for every $u_0 \in C(\bar{\Omega})$, there exists a unique solution $u(x,t)$ to (2.4) and*

$$u \in C^1((0,\infty); C(\mathbb{R}^N)).$$

Theorem 2.7 *Assume that $u(x,t)$ and $v(x,t)$ are a pair of super-sub solutions to (2.4). Then $u(x,t) \geq v(x,t)$ for $(x,t) \in \mathbb{R}^N \times [0,\infty)$.*

3 Heterogeneous Nonlocal Dispersal Equation

In this section, we consider the nonlocal dispersal problems (1.3) and (1.9). We first study the case that the inhomogeneous media provides a linear increase (or decrease) on the nonlocal dispersal system.

Proof of Theorem 1.1 In (1.7), the functions $p(x)$ and $u_0(x)$ are smooth, we then can extend the solution $u(x,t)$ to the whole space $\mathbb{R}^N$, still denoted by $u(x,t)$, see [11,17]. Define

$$L_\varepsilon^1(v) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)g(x)[v(y,t)-v(x,t)]\,dy$$

and

$$L_\varepsilon^2(v) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)[g(x)-g(y)]\,dy v(x,t),$$

Then we know that $u(x,t)$ satisfies

$$\begin{cases} u_t(x,t) = L_\varepsilon^1(u)(x,t) + F_\varepsilon(x,t), & x\in\bar\Omega, t\in(0,T], \\ u(x,t) = H(x,t), & x\in\mathbb{R}^N\backslash\bar\Omega, t\in(0,T], \\ u(x,0) = u_0(x), & x\in\bar\Omega, \end{cases} \tag{3.1}$$

where

$$F_\varepsilon(x,t) = -L_\varepsilon^1(u)(x,t) + \Delta u(x,t) + p(x)u(x,t),$$

the function $H(x,t)$ is smooth and $H(x,t)=0$ for $x\in\partial\Omega$. Thus we can find $M_1>0$, such that

$$|H(x,t)| \le M_1\varepsilon \tag{3.2}$$

for $x\in\mathbb{R}^N\backslash\bar\Omega$, $t\in(0,T]$.

The existence and uniqueness of solution $u^\varepsilon(x,t)$ to (1.4) are followed by Theorem 2.2. Denote $\omega^\varepsilon(x,t) = u(x,t) - u^\varepsilon(x,t)$, then we get

$$\begin{cases} \omega_t^\varepsilon(x,t) = L_\varepsilon^1(\omega^\varepsilon)(x,t) + F_\varepsilon^1(x,t), & x\in\bar\Omega, t\in(0,T], \\ \omega^\varepsilon(x,t) = H(x,t), & x\in\mathbb{R}^N\backslash\bar\Omega, t\in(0,T], \\ \omega^\varepsilon(x,0) = 0, & x\in\bar\Omega, \end{cases}$$

where

$$F_\varepsilon^1(x,t) = F_\varepsilon(x,t) - L_\varepsilon^2(u)(x,t).$$

Note that $u\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$, we claim that there exists $M_2>0$ such that

$$\max_{t\in[0,T]}\|F_\varepsilon^1(\cdot,t)\|_{C(\bar\Omega)} \le M_2\varepsilon^\alpha. \tag{3.3}$$

In fact, we know that

$$\begin{aligned} &\Delta u(x,t) - L_\varepsilon^1(u)(x,t) \\ &= \Delta u(x,t) - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)g(x)[u(y,t)-u(x,t)]\,dy \\ &= \Delta u(x,t) - \frac{1}{d\varepsilon^{N+2}}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)[u(y,t)-u(x,t)]\,dy \\ &= \Delta u(x,t) - \frac{1}{d\varepsilon^2}\int_{\mathbb{R}^N} J(y)[u(x-\varepsilon y,t)-u(y,t)]\,dy. \end{aligned}$$

On the other hand, since $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$, we get

$$\begin{aligned}&\Delta u(x,t) - L_\varepsilon^1(u)(x,t)\\ &\quad = \Delta v(x,t) + \frac{1}{\varepsilon d}\sum_{i=1}^{N}\frac{\partial u(x,t)}{\partial x_i}\int_{\mathbb{R}^N} J(y)y_i\,dy\\ &\qquad - \frac{1}{2d}\sum_{i,j=1}^{N}\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j}\int_{\mathbb{R}^N} J(y)y_i y_j\,dz + O(\varepsilon^\alpha),\end{aligned}$$

here $y = (y_1, y_2, \ldots, y_N)$. By the assumption $(A1)$ we have

$$\int_{\mathbb{R}^N} J(y)y_i\,dz = 0$$

for $i = 1, 2\ldots, N$ and

$$\int_{\mathbb{R}^N} J(y)y_i y_j\,dz = 0$$

for $i, j = 1, 2\ldots, N$ and $i \neq j$. Accordingly,

$$\Delta u(x,t) - L_\varepsilon(u)(x,t) = O(\varepsilon^\alpha). \tag{3.4}$$

Meanwhile, we have

$$\begin{aligned}&p(x)u(x,t) - L_\varepsilon^2(u)(x,t)\\ &\quad = p(x)u(x,t) - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)[g(y)-g(x)]\,dyu(x,t)\\ &\quad = p(x)u(x,t) - \frac{1}{d\varepsilon^{N+2}g(x)}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)[g(y)-g(x)]\,dyu(x,t)\\ &\quad = p(x)u(x,t) - \frac{1}{d\varepsilon^2 g(x)}\int_{\mathbb{R}^N} J(y)[g(x-\varepsilon y,t)-g(y,t)]\,dyu(x,t)\\ &\quad = O(\varepsilon^\alpha).\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we obtain (3.3).

Now denote

$$\overline{w}(x,t) = M_2\varepsilon^\alpha t + M_1\varepsilon,$$

we have

$$\overline{w}_t(x,t) - L_\varepsilon^1(\overline{w})(x,t) = M_2\varepsilon^\alpha \geq F_\varepsilon^1(x,t) = w_t^\varepsilon(x,t) - L_\varepsilon^1(w^\varepsilon)(x,t) \tag{3.6}$$

for $x \in \bar{\Omega}$. Then from (3.2) we get

$$\overline{w}(x,t) \geq w^\varepsilon(x,t)$$

for $x \in \mathbb{R}^N \backslash \Omega$ and $t \in [0,T]$. Moreover, it is clear that

$$\overline{w}(x,0) = K_2\varepsilon \geq w^\varepsilon(x,0) = 0 \tag{3.7}$$

Thanks to (3.6)–(3.7), from the comparison principle we know that

$$w^\varepsilon(x,t) \leq \overline{w}(x,t) = K_1\varepsilon^\alpha t + K_2\varepsilon.$$

Hence we have that $\overline{w}(x,t)$ is a super-solution to (3.1).

By a similar way, we can show that

$$\underline{w} = -K_1\varepsilon^{\alpha}t - K_2\varepsilon$$

is a sub-solution and

$$w^{\varepsilon}(x,t) \geq \underline{w}(x,t) = -K_1\varepsilon^{\alpha}t - K_2\varepsilon.$$

Thus

$$\max_{t\in[0,T]} \|u(\cdot,t) - u^{\varepsilon}(\cdot,t)\|_{C(\bar{\Omega})} \leq C\varepsilon^{\alpha} \to 0 \quad as \quad \varepsilon \to 0$$

and we end the proof. □

Now we analyze the nonlocal dispersal Eq. (1.9) and the parameter Eq. (1.10). Let us first consider the second-order parabolic equation

$$\begin{cases} u_t(x,t) = \sum_{i,j=1}^{N} a_{ij}^{\varepsilon}(x)\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} q_i^{\varepsilon}(x)\frac{\partial u(x,t)}{\partial x_i} & \text{in } \Omega\times(0,\infty), \\ u(x,t) = 0 & \text{in } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.8}$$

where

$$a_{ij}^{\varepsilon}(x) = \frac{1}{2dg(x)}\int_{\mathbb{R}^N} g(x-\varepsilon y)J(y)y_i y_j\,dy,$$
$$q_i^{\varepsilon}(x) = -\frac{1}{\varepsilon dg(x)}\int_{\mathbb{R}^N} g(x-\varepsilon y)J(y)y_i\,dz$$

for $i, j = 1, 2 \dots, N$ and $\varepsilon > 0$. We know from [11,17] that (3.8) exists a unique solution

$$\hat{u}^{\varepsilon} \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}\times[0,T]).$$

We then have the following results.

Lemma 3.1 *Let $u(x,t)$ and $\hat{u}^{\varepsilon}(x,t)$ be the solutions of (1.11) and (3.8), respectively. Then we have*

$$\max_{t\in[0,T]} \|\hat{u}^{\varepsilon}(\cdot,t) - u(\cdot,t)\|_{C(\bar{\Omega})} \to 0 \text{ as } \varepsilon \to 0. \tag{3.9}$$

Proof In fact, we can see that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} g(x-\varepsilon y)J(y)y_i y_j\,dy = g(x)\int_{\mathbb{R}^N} J(y)y_i y_j\,dy \text{ uniformly in } \bar{\Omega}$$

for $i, j = 1, 2, \dots, N$ and

$$\int_{\mathbb{R}^N} J(y)y_i y_j\,dy = 0$$

for $i \neq j$. Thus we get

$$\lim_{\varepsilon\to 0} a_{ij}^{\varepsilon}(x) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Meanwhile, since

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \frac{1}{\varepsilon}g(x-\varepsilon y)J(y)y_i\,dy = -\int_{\mathbb{R}^N} \frac{\partial g(x)}{\partial x_i}J(y)y_i^2\,dy,$$

we obtain

$$\lim_{\varepsilon\to 0} q_i^\varepsilon(x) = \frac{1}{dg(x)} \frac{\partial g(x)}{\partial x_i} \text{ uniformly in } \bar{\Omega}.$$

Using (1.12) we have that (3.9) holds. □

Lemma 3.2 *Let $u^\varepsilon(x,t)$ be the solution of (1.10) and $\hat{u}^\varepsilon(x,t)$ be the solution of (3.8), respectively. Then we have*

$$\max_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - \hat{u}^\varepsilon(\cdot,t)\|_{C(\bar{\Omega})} \to 0 \text{ as } \varepsilon \to 0.$$

Proof We can extend $u^\varepsilon(x,t)$ to $\mathbb{R}^N \times [0,T]$. Denote

$$L^\varepsilon(v) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} g(y)J^\varepsilon(x-y)[v(y,t)-v(x,t)]\,dy.$$

Then $\hat{u}^\varepsilon(x,t)$ satisfies

$$\begin{cases} \hat{u}_t^\varepsilon(x,t) = L^\varepsilon(\hat{u}^\varepsilon)(x,t) + F^\varepsilon(x,t), & x\in\bar{\Omega},\ t\in(0,T],\\ \hat{u}^\varepsilon(x,t) = H(x,t), & x\in\mathbb{R}^N\backslash\bar{\Omega},\ t\in(0,T],\\ \hat{u}^\varepsilon(x,0) = u_0(x), & x\in\bar{\Omega}, \end{cases}$$

where

$$F^\varepsilon(x,t) = -L^\varepsilon(\hat{u}^\varepsilon)(x,t) + \sum_{i,j=1}^N a_{ij}^\varepsilon(x)\frac{\partial^2\hat{u}^\varepsilon(x,t)}{\partial x_i\partial x_j} + \sum_{i=1}^N q_i^\varepsilon(x)\frac{\partial\hat{u}^\varepsilon(x,t)}{\partial x_i},$$

the function $H(x,t)$ is smooth, $H(x,t)=0$ for $x\in\partial\Omega$ and there exists $M_1>0$ such that

$$|H(x,t)| \le M_1\varepsilon$$

for $x\in\mathbb{R}^N\backslash\bar{\Omega}$, $t\in(0,T]$.

Set $v^\varepsilon(x,t) = \hat{u}^\varepsilon(x,t) - u^\varepsilon(x,t)$, then we have

$$\begin{cases} v_t^\varepsilon(x,t) = L^\varepsilon(v^\varepsilon)(x,t) + F^\varepsilon(x,t), & x\in\bar{\Omega}, t\in(0,T],\\ v^\varepsilon(x,t) = H(x,t), & x\in\mathbb{R}^N\backslash\bar{\Omega}, t\in(0,T],\\ v^\varepsilon(x,0) = 0, & x\in\bar{\Omega}. \end{cases}$$

But

$$\begin{aligned} &\frac{1}{\varepsilon^{N+2}}\int_{\mathbb{R}^N} g(y)J\left(\frac{x-y}{\varepsilon}\right)[\hat{u}^\varepsilon(y,t)-\hat{u}^\varepsilon(x,t)]\,dy\\ &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} g(x-\varepsilon y)J(y)[\hat{u}^\varepsilon(x-\varepsilon y,t)-\hat{u}^\varepsilon(x,t)]\,dy\\ &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} g(x-\varepsilon y)J(y)\left[\sum_{i=1}^N \frac{\partial\hat{u}^\varepsilon(x,t)}{\partial x_i}(-\varepsilon y_i) + \frac{1}{2}\sum_{i,j=1}^N \frac{\partial^2\hat{u}^\varepsilon(x,t)}{\partial x_i\partial x_j}\varepsilon^2 y_i y_j + O(\varepsilon^{2+\alpha})\right] dy\\ &= -\int_{\mathbb{R}^N}\sum_{i=1}^N \frac{1}{\varepsilon} g(x-\varepsilon y)J(y)y_i\frac{\partial\hat{u}^\varepsilon(x,t)}{\partial x_i}\,dy\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N}\sum_{i,j=1}^N g(x-\varepsilon y)J(y)y_i y_j\frac{\partial^2\hat{u}^\varepsilon(x,t)}{\partial x_i\partial x_j}\,dz + O(\varepsilon^\alpha), \end{aligned}$$

we have

$$\begin{aligned}
&\sum_{i,j=1}^{N} a_{ij}^{\varepsilon}(x)\frac{\partial^2 \hat{u}^{\varepsilon}(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} q_i^{\varepsilon}(x)\frac{\partial \hat{u}^{\varepsilon}(x,t)}{\partial x_i} - L^{\varepsilon}(\hat{u}^{\varepsilon})(x,t) \\
&= \sum_{i,j=1}^{N} a_{ij}^{\varepsilon}(x)\frac{\partial^2 \hat{u}^{\varepsilon}(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} q_i^{\varepsilon}(x)\frac{\partial \hat{u}^{\varepsilon}(x,t)}{\partial x_i} \\
&\quad - \frac{1}{d\varepsilon^{N+2}g(x)}\int_{\mathbb{R}^N} g(y)J\left(\frac{x-y}{\varepsilon}\right)[\hat{u}^{\varepsilon}(y,t)-\hat{u}^{\varepsilon}(x,t)]\,dy \\
&= O(\varepsilon^{\alpha}).
\end{aligned}$$

Thus for $\varepsilon > 0$ is small, there exists $M_2 > 0$ such that

$$\max_{t\in[0,T]} \|F^{\varepsilon}(\cdot,t)\|_{C(\bar{\Omega})} \le M_2\varepsilon^{\alpha}. \tag{3.10}$$

Denote

$$\overline{w}(x,t) = M_2\varepsilon^{\alpha}t + M_1\varepsilon.$$

It follows from (3.10) that

$$\overline{w}(x,t) - L^{\varepsilon}(\bar{w})(x,t) = M_2\varepsilon^{\alpha} \ge F_{\varepsilon}(x,t) = (w_{\varepsilon})_t(x,t) - L^{\varepsilon}(w_{\varepsilon})(x,t)$$

for $x \in \bar{\Omega}$. We also have

$$\overline{w}(x,t) \ge M_1\varepsilon \ge |H(x,t)|$$

for $x \in \mathbb{R}^N\backslash\bar{\Omega}$ such that $dist(x,\partial\Omega) \le \varepsilon$ and $t \in [0,T]$. Moreover we have

$$\overline{w}(x,0) = M_1\varepsilon > w_{\varepsilon}(x,0) = 0.$$

Then a simple argument form comparison principle gives that

$$\max_{t\in[0,T]} ||u^{\varepsilon}(\cdot,t) - v(\cdot,t)||_{C(\bar{\Omega})} \le C\varepsilon^{\alpha}.$$

We end the proof. □

At last, let $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ be the solution of (1.11). Then we have

$$\max_{t\in[0,T]} \|u(\cdot,t)-u^{\varepsilon}(\cdot,t)\|_{C(\bar{\Omega})} \le \max_{t\in[0,T]} \|\hat{u}^{\varepsilon}(\cdot,t)-u(\cdot,t)\|_{C(\bar{\Omega})} + \max_{t\in[0,T]} \|\hat{u}^{\varepsilon}(\cdot,t)-u^{\varepsilon}(\cdot,t)\|_{C(\bar{\Omega})}.$$

It follows from Lemmas 3.1–3.2 that

$$\max_{t\in[0,T]} \|u(\cdot,t)-u^{\varepsilon}(\cdot,t)\|_{C(\bar{\Omega})} \to 0 \text{ as } \varepsilon \to 0,$$

and we end the proof of Theorem 1.2.

4 The Periodic Nonlocal Boundary Problems

In this section, we consider the periodic nonlocal boundary problems (1.13) and (1.14). To do this, we first investigate the effect of spatial homogeneity on the periodic problem (1.13) and so we shall analyze the limiting behavior of solutions of (1.16).

It follows from Theorem 2.6 that there exists a unique periodic solution $u^\varepsilon(x,t)$ to (1.16). Now let $u \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,\infty))$ be the solution of

$$\begin{cases} u_t = \Delta u + p(x)u & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,t) = u(x+p_j\mathbf{e}_j,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $j = 1, 2, \ldots, N$ and $p(x)$ is the periodic coefficient given by

$$p(x) = \frac{1}{dg(x)} \sum_{i=1}^N \frac{\partial^2 g(x)}{\partial x_i \partial x_i}.$$

We are ready to prove the first result of periodic boundary problem.

Proof of Theorem 1.3 Set

$$L_\varepsilon^1(v) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J^\varepsilon(x-y)g(x)[v(y,t)-v(x,t)]\,dy$$

and

$$L_\varepsilon^2(v) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J^\varepsilon(x-y)[g(x)-g(y)]\,dy v(x,t),$$

Then we know that the unique solution $u(x,t)$ of (4.1) satisfies

$$\begin{cases} u_t(x,t) = L_\varepsilon^1(u)(x,t) + F_\varepsilon(x,t), & x \in \mathbb{R}^N, t \in (0,T], \\ u(x,t) = u(x+p_j\mathbf{e}_j,t), & x \in \mathbb{R}^N, t \in (0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $j = 1, 2, \ldots, N$ and

$$F_\varepsilon(x,t) = -L_\varepsilon^1(u)(x,t) + \Delta u(x,t) + p(x)u(x,t).$$

Denote $\omega^\varepsilon(x,t) = u(x,t) - u^\varepsilon(x,t)$, then we get

$$\begin{cases} \omega_t^\varepsilon(x,t) = L_\varepsilon^1(\omega^\varepsilon)(x,t) + F_\varepsilon^1(x,t), & x \in \mathbb{R}^N, t \in (0,T], \\ \omega^\varepsilon(x,t) = \omega^\varepsilon(x+p_j\mathbf{e}_j,t), & x \in \mathbb{R}^N, t \in (0,T], \\ \omega^\varepsilon(x,0) = 0, & x \in \mathbb{R}^N, \end{cases}$$

where $j = 1, 2, \ldots, N$ and

$$F_\varepsilon^1(x,t) = F_\varepsilon(x,t) - L_\varepsilon^2(u)(x,t).$$

Since $u \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$ is periodic, we can find $M > 0$ such that

$$\max_{[0,T]} \|F_\varepsilon^1(\cdot,t)\|_{C(\mathbb{R}^N)} \le M\varepsilon^\alpha.$$

Let

$$\omega(x,t) = M\varepsilon^\alpha t,$$

a similar argument as in the proof of Theorem 1.1 gives that

$$\max_{[0,T]} \|u(\cdot,t) - u^\varepsilon(\cdot,t)\|_{C(\mathbb{R}^N)} \le \max_{[0,T]} |\omega(x,t)| \le C\varepsilon^\alpha \to 0 \quad as \quad \varepsilon \to 0$$

and we end the proof. □

At the end of this section, we study the periodic nonlocal problem (1.18). So we consider the periodic parabolic equation

$$\begin{cases} u_t(x,t) = \sum_{i,j=1}^{N} a_{ij}^{\varepsilon}(x)\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} q_i^{\varepsilon}(x)\frac{\partial u(x,t)}{\partial x_i} & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,t) = u(x + p_j \mathbf{e}_j, t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{4.2}$$

where the coefficients

$$a_{ij}^{\varepsilon}(x) = \frac{1}{2dg(x)} \int_{\mathbb{R}^N} g(x-\varepsilon y)J(y)y_i y_j \, dy,$$
$$q_i^{\varepsilon}(x) = -\frac{1}{\varepsilon dg(x)} \int_{\mathbb{R}^N} g(x-\varepsilon y)J(y)y_i \, dz$$

for $i, j = 1, 2 \ldots, N$ and $\varepsilon > 0$.

Then we have the following results, we omit the proof here.

Lemma 4.1 *Let $u(x,t)$ and $\hat{u}^{\varepsilon}(x,t)$ be the solutions of (1.19) and (4.2), respectively. Then we have*

$$\max_{t\in[0,T]} \|\hat{u}^{\varepsilon}(\cdot,t) - u(\cdot,t)\|_{C(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon \to 0.$$

Lemma 4.2 *Let $u^{\varepsilon}(x,t)$ be the solution of (1.18) and $\hat{u}^{\varepsilon}(x,t)$ be the solution of (4.2), respectively. Then we have*

$$\max_{t\in[0,T]} \|u^{\varepsilon}(\cdot,t) - \hat{u}^{\varepsilon}(\cdot,t)\|_{C(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon \to 0.$$

Theorem 1.4 is followed by Lemmas 4.1–4.2 and a similar argument as in the proof of Theorem 1.2.

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