Dziobek Equilibrium Configurations on a Sphere

Shuqiang Zhu[1]

Abstract

We investigate the n-body problem on a sphere with a general interaction potential that depends on the mutual distances. We focus on the equilibrium configurations, especially on the Dziobek equilibrium configurations, which is an analogy of Dziobek central configurations of the classical n-body problem. We obtain a criterion and then reduce it to two sets of equations. Then we apply these equations to the curved n-body problem in $\mathbb{S}^3$. In the end, we find the derivative of the Cayley-Menger determinant.

Keywords Curved n-body problem · Dziobek configurations · Equilibrium configurations · Stability · Cayley-Menger determinant

1 Introduction

The classical n-body problem has been generalized in many ways, for example, under the potential $\sum \frac{m_i m_j}{r_{ij}^\alpha}$, or in higher dimensional Euclidean space. In particular, the *curved n-body problem*, which generalizes the classical n-body problem to surfaces of constant curvature has received lot of attentions in the last decade (cf [2,6,10,13] and the references therein).

Motivated by those work, we study the generalization of the n-body problem to unit sphere of the Euclidean space. We assume that the potential depends on the shortest geodesic distance. We also assume that the potential is attractive (repulsive) in most cases. We only specify the potential in the last section.

One major distinction between the generalization and the classical n-body problem is the existence of equilibrium configurations, due to the compactness of spheres. This paper is devoted to the study of equilibrium configurations on spheres.

In particular, we consider the equilibrium configurations formed by N bodies on some $(N-2)$-dimensional sphere. We call them the *Dziobek equilibrium configurations*. In the classical n-body problem, Otto Dziobek [9] first introduced a set of equations for non collinear

✉ Shuqiang Zhu
  zhusq@swufe.edu.cn

[1] School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

four-body central configurations on $\mathbb{R}^2$, an approach proved fruitful in the study of four-body central configurations (cf [1,11] and the references therein).

We obtain a criterion similar to that of Otto Dziobek for the Dziobek equilibrium configurations in the n-body problem on a sphere. If the potential is attractive (repulsive), there is an obstacle for the equilibrium configurations, namely, the particles could not lie on one hemisphere. By this property, we can further separate the criterion into two sets of equations. One set of the equations can be used to determine the manifold in the configuration space that admits equilibrium configurations, then the other set of equations can be used to determine the corresponding masses.

The paper is organized as follows. In Sect. 2, we discuss the basic setting of the n-body problem on a sphere and the equilibrium configurations. In Sect. 3, we define the Dziobek equilibrium configurations and obtain a criterion. Then we separate the criterion into two sets. In Sect. 4, we turn to the *curved n-body problem* in $\mathbb{S}^3$. We apply the criterion to equilibrium configurations of three- four- and five-body in $\mathbb{S}^3$ and discuss the stability of associated equilibria. We discuss the derivative of the Cayley-Menger determinant in the "Appendix".

2 The Equilibrium Configurations for Mechanical System on a Sphere

Let $\mathbb{S}^n$ be the unit sphere of the Euclidean space $\mathbb{R}^{n+1}$. Let us consider N points of positive mass m_i on $\mathbb{S}^n$ that interacting mutually by a potential depending on the shortest geodesic distance between the points. The position vector of the i-th point is $\mathbf{q}_i = (x_{i1}, ..., x_{i,n+1})^T$ with $x_{i1}^2 + ... + x_{i,n+1}^2 = 1$, $i = 1, ..., N$. Denote the configuration by $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$. The configuration space is

$$Q = (\mathbb{S}^n)^N \setminus \{\text{collisions and configurations where the vector field is undefined}\}.$$

The mechanical system is given by the Lagrangian $L : TQ \to \mathbb{R}$

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T_{\mathbf{q}}(\dot{\mathbf{q}}) - V(\mathbf{q}) \tag{1}$$

where T is a Riemannian metric on the configuration space and $V(\mathbf{q})$ is the interaction potential. Denote the distance between two points $\mathbf{q}_i, \mathbf{q}_j$ by d_{ij}. Then $\cos d_{ij} = \mathbf{q}_i \cdot \mathbf{q}_j$. Assume that the potential V is

$$V(\mathbf{q}) = \sum_{1 \leq i < j \leq N} m_i m_j G(d_{ij}), \tag{2}$$

where $G : (0, \pi) \to \mathbb{R}$ is some given smooth function.

Definition 1 ([13]) A potential V as given by (2) is called attractive (repulsive) if the binary potential G, is such that $G'(x) > 0 (G'(x) < 0)$ for all $x \in (0, \pi)$.

The *equilibrium motion*, or simply *equilibrium*, is solution in the form of $\mathbf{q}(t) = \mathbf{q}(0)$. The configuration $\mathbf{q}(0)$, called a *equilibrium configuration*, is a critical point of V. The derivative of V is

$$\nabla_{\mathbf{q}_i} V(\mathbf{q}) = \sum_{j=1, j \neq i}^{N} m_i m_j G'(d_{ij}) \nabla_{\mathbf{q}_i} d_{ij} = \sum_{j=1, j \neq i}^{N} m_i m_j G'(d_{ij}) \nabla_{\mathbf{q}_i} \cos^{-1} \mathbf{q}_i \cdot \mathbf{q}_j$$

$$= \sum_{j=1, j \neq i}^{N} \frac{-m_i m_j G'(d_{ij})}{\sin d_{ij}} \nabla_{\mathbf{q}_i} \mathbf{q}_i \cdot \mathbf{q}_j.$$

By extending $\mathbf{q}_i \cdot \mathbf{q}_j$ into a homogeneous function of degree zero in $\mathbb{R}^{2(n+1)} \setminus \{0\}$, i.e., $\frac{\mathbf{q}_i}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i}} \cdot \frac{\mathbf{q}_j}{\sqrt{\mathbf{q}_j \cdot \mathbf{q}_j}}$, we obtain

$$\nabla_{\mathbf{q}_i} V(\mathbf{q}) = \sum_{j=1, j \neq i}^{N} \frac{-m_i m_j G'(d_{ij})}{\sin d_{ij}} [\mathbf{q}_j - \cos d_{ij} \mathbf{q}_i] \quad i = 1, ..., N.$$

Hence, a configuration $\mathbf{q} \in Q$ is an equilibrium configuration if $\mathbf{q}$ satisfies the following system

$$\sum_{j=1, j \neq i}^{N} \frac{m_i m_j G'(d_{ij})}{\sin d_{ij}} [\mathbf{q}_j - \cos d_{ij} \mathbf{q}_i] = 0, \quad i = 1, ..., N. \tag{3}$$

Remark 1 The Lyapunov stability of such equilibrium is related with the second variation of V. In particular, the well-known Lagrange-Dirichlet Theorem says it is stable if the configuration is an isolated minimum. The converses of this theorem is widely discussed (cf. [12,14] and the references therein). If the potential is analytic, then it is unstable if it is not a minimum. The equilibrium configurations also lead to relative equilibria of the system [16].

Proposition 1 *The i-th equation of system* (3) *holds if and only if there is a constant θ_i such that*

$$\sum_{j \neq i, j=1}^{N} \frac{m_j m_i G'(d_{ij}) \mathbf{q}_j}{\sin d_{ij}} + \theta_i \mathbf{q}_i = 0. \tag{4}$$

Proof Assume that Eq. (4) holds. Multiply $\mathbf{q}_i$ to the both sides of Eq. (4). Since $\mathbf{q}_i \cdot \mathbf{q}_j = \cos d_{ij}$ and $\mathbf{q}_i \cdot \mathbf{q}_i = 1$, we obtain $\theta_i = -\sum_{j \neq i, j=1}^{N} \frac{m_j m_i G'(d_{ij}) \cos d_{ij}}{\sin d_{ij}}$. Thus Eq. (4) is equivalent to the i-th equation of (3). □

The following result generalizes one result of Diacu [6] for the curved n-body problem, see Sect. 4.

Theorem 1 *Assume that the potential is attractive (repulsive). There is no equilibrium configuration for any positive masses in any closed hemisphere of $\mathbb{S}^n$ (i.e. a hemisphere that contains its boundary), as long as at least one body does not lie on the boundary.*

Proof Let $\mathbf{q}$ be a configuration that lies in a closed hemisphere of $\mathbb{S}^n$ and that there is at least one body not on the boundary. Then there is some point $\mathbf{v} \in \mathbb{S}^n$ such that $\mathbf{v} \cdot \mathbf{q}_i \geq 0$ for all i and at least one of them is strictly positive. Assume that $\mathbf{v} \cdot \mathbf{q}_1$ is the smallest. Then $\nabla_{\mathbf{q}_1} V = 0$ implies

$$\sum_{j \neq 1}^{N} \frac{m_j G'(d_{1j})}{\sin d_{1j}} [\mathbf{v} \cdot \mathbf{q}_j - \cos d_{1j} \mathbf{v} \cdot \mathbf{q}_1] = 0.$$

Since we have assumed $G'(d_{1j})$ is of the same sign for all j, this is a contradiction. □

We end this section by several examples of equilibrium configurations for equal masses. The examples extend those constructed by Diacu in [6] for the curved n-body problem (Sect. 4). Denote the standard bases of $\mathbb{R}^{n+1}$ by $e_1, ..., e_{n+1}$. Denote the unit sphere in $span\{e_1, ..., e_{k+1}\}$ by $\mathbb{S}^k$. We assume that the configurations constructed below are not those where $G'(d_{ij})$ is undefined.

Example 1 (regular simplex with equal masses) Consider a regular k-simplex. Place one unit mass at each of the vertices. The configuration obtained is an equilibrium configuration. It is enough to check that Eq. (4) holds for $i = 1$. Since $d_{ij} = d_{12}$ for any pair of $\{i, j\}$, we find that

$$\sum_{i=2}^{k+1} \frac{m_i G'(d_{i1}) \mathbf{q}_i}{\sin d_{i1}} = \frac{G'(d_{21})}{\sin d_{21}} \left(\sum_{i=2}^{k+1} \mathbf{q}_i \right) = -\frac{G'(d_{21})}{\sin d_{21}} \mathbf{q}_1.$$

Example 2 (regular polygon with equal masses) Consider a regular $2k+1$-gon located on the unit circle of $span\{e_1, e_2\}$. Place one unit mass at each of the vertices. By complex number notation, the position vectors are $\mathbf{q}_j = e^{i\phi j}$, $j = 1, ..., 2k+1$, $\phi = \frac{2\pi}{2k+1}$. let us check equation (4) for $j = 2k+1$. Since $d_{2k+1,j} = d_{2k+1,2k+1-j}$, we have

$$\frac{m_j G'(d_{2k+1,j}) e^{i\phi j}}{\sin d_{2k+1,j}} + \frac{m_{2k+1-j} G'(d_{2k+1,2k+1-j}) e^{i\phi(2k+i-j)}}{\sin d_{2k+1,2k+1-j}}$$
$$= 2 \frac{G'(d_{2k+1,j})}{\sin d_{2k+1,j}} \cos j\phi e^{i\phi(2k+1)}.$$

Then it follows that Eq. (4) holds for $i = 2k+1$, then for all i by symmetry.

Similarly, the regular polygon of even vertices with equal masses is also an equilibrium configuration.

Example 3 (two regular polygons with equal masses on two complementary circles) Consider one regular n_1-polygon located on the unit circle of $span\{e_1, e_2\}$ and another regular n_2-polygon located on the unit circle of $span\{e_3, e_4\}$. Place a unit mass at each of the vertices. Note that the distance between the particles from different polygons is always $\frac{\pi}{2}$ since

$$\cos d_{ij} = \mathbf{q}_i \cdot \mathbf{q}_j = 0, \quad 1 \le i \le n_1, \ n_1 + 1 \le j \le n_1 + n_2.$$

Let us check that equation (4) holds for any $1 \le i \le n_1 + n_2$, say $i = 1$.

$$\sum_{i=2}^{n_1+n_2} \frac{m_i G'(d_{i1}) \mathbf{q}_i}{\sin^3 d_{i1}} = \sum_{i=2}^{n_1} \frac{G'(d_{i1})}{\sin d_{i1}} \mathbf{q}_i + G'(\frac{\pi}{2}) \sum_{i=n_1+1}^{n_1+n_2} \mathbf{q}_i.$$

The first part is collinear with $\mathbf{q}_1$ by the above example, and the second part is zero. Thus this configuration is one equilibrium configuration.

3 Dziobek Equilibrium Configurations

In this section, we consider equilibrium configurations where N masses span an $(N-2)$-sphere. We obtain a criterion, then separate it into two sets of equations, the shape equations and the mass equations. In the classical n-body problem, a central configuration of N bodies that span an $(N-2)$-dimensional affine plane are called *Dziobek central configurations* [9,11]. For equilibrium configurations on sphere, equation (4) implies that the N position vectors are always dependent, so $1 \le \text{rank}(\mathbf{q}_1, ..., \mathbf{q}_N) \le N-1$.

Definition 2 A Dziobek configuration of N bodies on sphere is one such that $\text{rank}(\mathbf{q}_1, ..., \mathbf{q}_N) = N - 1$.

Let $\{\mathbf{q}_1, \cdots, \mathbf{q}_N\}$ be a collection of vectors in $\mathbb{R}^{N-1}$. Assume the rank of these N vectors is $N - 1$. Consider the $(N - 1) \times N$ matrix:

$$X = [\mathbf{q}_1, \cdots, \mathbf{q}_N].$$

Since the rank of X is $N - 1$, dim ker $X = 1$. The kernel can be found as follows. Let X_k be the $(N - 1) \times (N - 1)$ matrix obtained from X by deleting the k-th column and let $|X_k|$ denote its determinant.

Lemma 1 *Let*

$$\Delta = (\Delta_1, ... \Delta_N) = (|X_1|, -|X_2|, ..., (-1)^{k+1}|X_k|, ...). \tag{5}$$

Then Δ^T *is the base of* ker X. *In other words,* $\Delta \neq 0$ *and* $\Delta_1\mathbf{q}_1 + \cdots + \Delta_N\mathbf{q}_N = 0$.

Proof Assume that $\Delta_N = (-1)^{N+1}|X_N| \neq 0$. Consider the linear system in $X_N u = \mathbf{q}_N$, $u = (u_1, ..., u_{N-1})^T$. By Cramer's rule, we obtain $u_k = \frac{-\Delta_k}{\Delta_N}$, $k = 1, ..., N - 1$.
   Then it follows that $\Delta_1\mathbf{q}_1 + \cdots + \Delta_N\mathbf{q}_N = 0$. □

Proposition 2 *Consider a Dziobek configuration of N bodies on $\mathbb{S}^{N-2}$. Then the configuration is not on a hemisphere if and only if all Δ_i are of the same sign.*

Proof We only prove that if not all Δ_i are of the same sign the Dziobek configuration lies on a hemisphere. There are two cases.
   If there is some $\Delta_i = 0$, say Δ_1, then rank$\{\mathbf{q}_2, ..., \mathbf{q}_N\} = N - 2$. Let Π be the hyperplane spanned by $\{\mathbf{q}_2, ..., \mathbf{q}_N\}$ and $\vec{n}$ be the normal of Π in $\mathbb{R}^{N-1}$ with the property $\vec{n} \cdot \mathbf{q}_1 > 0$. Then we have

$$\vec{n} \cdot \mathbf{q}_i \geq 0, \ i = 1, ..., N,$$

which implies that the Dziobek configuration lies on a hemisphere.
   If all Δ_i are nonzero, there are two consecutive elements of Δ that are of different sign, say $\Delta_1 > 0$, $\Delta_2 < 0$. Then

$$|X_1| = \det(\mathbf{q}_2, \mathbf{q}_3, ..., \mathbf{q}_N) > 0, \ |X_2| = \det(\mathbf{q}_1, \mathbf{q}_3, ..., \mathbf{q}_N) > 0.$$

Let $\tilde{\Pi}$ be the $(N - 2)$-dimensional hyperplane spanned by $\{\mathbf{q}_3, ..., \mathbf{q}_N\}$ and $\vec{m}$ be the normal of $\tilde{\Pi}$ in $\mathbb{R}^{N-1}$ with the property $\vec{m} \cdot \mathbf{q}_1 > 0$. Assume that $\mathbf{q}_2 = \lambda_1\mathbf{q}_1 + \sum_{i=3}^{N} \lambda_i\mathbf{q}_i$. Then

$$|X_1| = \det(\lambda_1\mathbf{q}_1 + \sum_{i=3}^{N} \lambda_i\mathbf{q}_i, \mathbf{q}_3, ..., \mathbf{q}_N) = \lambda_1|X_2|.$$

Then $\lambda_1 > 0$. Hence we have

$$\vec{m} \cdot \mathbf{q}_i \geq 0, \ i = 1, ..., N,$$

which implies that the Dziobek configuration lies on a hemisphere. □

Denote the quantity $\frac{G'(d_{ij})}{\sin d_{ij}}$ by S_{ij}. Then equation (4) becomes

$$\sum_{j \neq i} m_j S_{ij}\mathbf{q}_j + \theta_i\mathbf{q}_i = 0, \ 1 \leq i \leq N.$$

Theorem 2 *Assume that the potential is attractive (repulsive) and that* $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$ *is a Dziobek configuration in* $\mathbb{S}^{N-2}$. *Then the configuration* $\mathbf{q}$ *is an equilibrium configuration if and only if there is a nonzero real number* p *such that*

$$m_i m_j S_{ij} = p \Delta_i \Delta_j \text{ for any } i \neq j. \tag{6}$$

Proof The proof of the sufficient conditions: Since $\mathbf{q}$ is an equilibrium configuration, Eq. (4) holds. That is, there is some nonzero real number p_j such that

$$(m_1 S_{1j}, m_2 S_{2j}, ..., \theta_j, ..., m_N S_{Nj}) = p_j (\Delta_1, ..., \Delta_N) \text{ for all } j = 1, ..., N. \tag{7}$$

by Lemma 1. System (7) is equivalent to

$$S = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix} \left(\frac{\Delta_1}{m_1}, ..., \frac{\Delta_N}{m_N}\right), \quad \text{where } S = \begin{bmatrix} \theta_1 & S_{12} & \cdots & S_{1N} \\ S_{21} & \theta_2 & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & \theta_N \end{bmatrix}$$

Since the left matrix S is symmetric, we see that $p_j \frac{\Delta_1}{m_1} = p_1 \frac{\Delta_j}{m_j}$, or, by Proposition 2,

$$(p_1 m_1, ..., p_N m_N) = \frac{p_1 m_1}{\Delta_1} (\Delta_1, ..., \Delta_N).$$

Let $M = \text{diag}\{m_1, ..., m_N\}$. We have

$$MSM = \frac{p_1 m_1}{\Delta_1} (\Delta_1, ..., \Delta_N)^T (\Delta_1, ..., \Delta_N),$$

which gives (6).

The proof of the sufficient conditions: Let $(p_1 m_1, ..., p_N m_N) = p \Delta$. The system (6) implies system (7), so the condition is also sufficient. □

The system (6) can be obtained in another way, see "Appendix". It implies that all Δ_i $(i \geq 1)$ are of the same sign, which agrees with Proposition 2. Eliminating the constant p, we get a system of $\frac{N(N-1)}{2} - 1$ equations from (6). The system can be written in a form with the property that most of the equations are just constraints on the shapes, or, independent of the masses.

Proposition 3 *Let* $A = (a_{ij})$ *be a symmetric matrix and* $b = (b_1, ..., b_n)$. *Assume that* $A = b^T b$ *and* $b_1 b_2 b_n \neq 0$. *Consider the system consisting of the* $\frac{n(n-1)}{2}$ *equations*

$$a_{ij} = b_i b_j, \ i = 1, ..., n-1, \ j = i+1, ..., n.$$

The system of equations is equivalent to

$$a_{1n} = b_1 b_n, \ b_k = b_n \frac{a_{1k}}{a_{1n}} \ k = 2, ..., n-1; \tag{8}$$

$$b_1 = b_n \frac{a_{12}}{a_{2n}}, \ a_{2k} a_{1n} = a_{1k} a_{2n}, \ k = 3, ..., n-1; \tag{9}$$

$$a_{jn} a_{12} = a_{1j} a_{2n}, \ a_{jk} a_{1n} = a_{1k} a_{jn}, \ j = 3, ..., n-2, \ k = j+1, ..., n-1; \tag{10}$$

$$a_{n-1,n} a_{12} = a_{1n-1} a_{2n}. \tag{11}$$

Proof From the first system, we see $a_{ij} a_{kl} = b_i b_j b_k b_l = a_{ik} a_{jl}$ holds for any 4-tuple $\{i, j, k, l\}$ of $\{1, ..., n\}$. Then we derive the second system from the first one.

Let us derive the first system from the second one. For convenience, put the first system in an upper triangular form

$$
E = \begin{bmatrix}
a_{12} = b_1 b_2 & a_{13} = b_1 b_3 & \cdots & a_{1n} = b_1 b_n \\
& a_{23} = b_2 b_3 & \cdots & a_{2n} = b_2 b_n \\
& & \ddots & \vdots & \vdots \\
& & & a_{n-1,n} = b_{n-1} b_n
\end{bmatrix}
$$

By (8), we can recover the first row of E. By (9) and the first row of E, we see

$$
a_{2n} = a_{12} \frac{b_n}{b_1} = b_2 b_n, \; a_{2k} = \frac{a_{1k} a_{2n}}{a_{1n}} = \frac{b_k b_2 b_n}{b_n} = b_2 b_k, \, k = 3, ..., n-1.
$$

Hence the second row of E is recovered. Similarly, the j-th row can be obtained by

$$
a_{jn} = a_{1j} \frac{b_n}{b_1} = b_j b_n, \; a_{2k} = \frac{a_{1k} a_{jn}}{a_{1n}} = \frac{b_k b_j b_n}{b_n} = b_j b_k, \, k = j+1, ..., n-1.
$$

Thus, we obtain all equations of E. This completes the proof. □

Applying the above result to the system (6), where $A = S$ and $b = \sqrt{p} \Delta M^{-1}$, we get

Theorem 3 *Assume that the potential is attractive (repulsive) and that $\mathbf{q} = (\mathbf{q}_1, ..., \mathbf{q}_N)$ is a Dziobek configuration in $\mathbb{S}^{N-2}$ with masses $m_1, ..., m_N$. Then $\mathbf{q}$ is an equilibrium configuration if and only if the following system of equations are satisfied*

$$
\begin{cases}
m_2 = \frac{S_{1N} \Delta_2}{S_{12} \Delta_N} m_N, \; ..., m_{N-1} = \frac{S_{1N} \Delta_{N-1}}{S_{1,N-1} \Delta_N} m_N, \, m_1 = \frac{S_{2N} \Delta_1}{S_{12} \Delta_N} m_N, \\
S_{1j} S_{2N} = S_{12} S_{jN}, \, j = 3, ..., N-1; \\
S_{jk} S_{1N} = S_{1k} S_{jN}, \, k = j+1, ..., N-1, \; j = 2, ..., N-2.
\end{cases}
\tag{12}
$$

Note that the first $N-1$ equations are involved with the masses, while the remaining equations are not. Let us call the first $N-1$ equations the *mass equations*, and the others $\frac{N(N-3)}{2}$ equations the *shape equations*.

The shape equations alone can not determine the configurations. Indeed, there are configurations that satisfies the shape equations, but the configuration lies on a hemisphere (see Remark 4 in Sect. 4). Generally speaking, because of the $\frac{N(N-3)}{2}$ shape constraints and the $SO(N-1)$ symmetry, the set of Dziobek equilibrium configurations forms a $(N-1)$-dimensional manifold. Denote by $\mathcal{D}$ the set of configurations with the $\frac{N(N-3)}{2}$ shape equations and the N inequality constraints $\Delta_1 > 0, ..., \Delta_N > 0$ satisfied. Assume $m_N = 1$. Then the mass equations define a map

$$
\mathcal{M} : \mathcal{D} \to (0, \infty)^{N-1}, \mathbf{q} \mapsto (m_1, ..., m_{N-1}).
$$

Remark 2 For Dziobek central configurations of the classical n-body problem [11], one can only be certain that at least two elements of Δ are nonzero if $N > 4$. Hence, we can get a system similar to (12) there, which is necessary but may not be sufficient. Furthermore, S_{ij} is involved with the multiplier λ. To get the constraints on configurations, we need to eliminate the multiplier λ from the $\frac{N(N-3)}{2}$ equations. The resulting $\frac{N(N-3)}{2} - 1$ constraints looks more complex, see Corbera et al. [5] for $N = 4$.

Remark 3 If we allow the masses to be negative, then the inequality constraints $\Delta_1 > 0, ..., \Delta_N > 0$ are not necessary.

4 Example: The Curved N-body Problem in $\mathbb{S}^3$

In this section we consider the problem in $\mathbb{S}^3$ with the gravitational interaction. The potential is defined as spherical-symmetric solutions of the Laplace equation on $\mathbb{S}^3$. This is the *curved n-body problem* in $\mathbb{S}^3$. For more on this problem, see [2,6,16]. For any two points $\mathbf{q}_i$ and $\mathbf{q}_j$, the binary potential and the potential are

$$G(d_{ij}) = -m_i m_j \cot d_{ij}, \quad V(\mathbf{q}) = \sum_{1 \leq i < j \leq N} -m_i m_j \cot d_{ij}$$

respectively. Since $G(x) = -\cot x$, $G'(x) = \frac{1}{\sin^2 x} > 0$, the potential is attractive and $S_{ij} = \frac{G'(d_{ij})}{\sin d_{ij}} = \frac{1}{\sin^3 d_{ij}}$.

Note that the potential is undefined at $d_{ij} = \pi$, so we must exclude those configurations with points diametrically opposite in the examples considered in Sect. 2, for instance, the regular polygons with even vertices. Moreover, there is no equilibrium configuration for two masses. Otherwise, the Eq. (4) implies that d_{12} is 0 or π. The equilibrium configurations are also called *special central configurations* in the curved n-body problem in $\mathbb{S}^3$ [16].

4.1 Criteria for Dziobek Equilibrium Configurations of Three, Four and Five Bodies

By Theorem 3, we obtain the following criteria for Dziobek equilibrium configurations of 3, 4 and 5 bodies respectively. The regular 2, 3, and 4-simplex with equal masses (see Example 1) satisfies the following criteria respectively.

Corollary 1 $(N = 3, \mathbb{S}^1)$ *Consider one configuration* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ *on* $\mathbb{S}^1$. *Then* $\mathbf{q}$ *is a Dziobek equilibrium configuration if and only if the configuration and the masses satisfy the following equations*

$$(m_1, m_2, m_3) = \left(\frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{23} \Delta_3}, \frac{\sin^3 d_{12} \Delta_2}{\sin^3 d_{13} \Delta_3}, 1 \right) m_3.$$

Corollary 2 $(N = 4, \mathbb{S}^2)$ *Consider one configuration* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4)$ *on* $\mathbb{S}^2$. *Then* $\mathbf{q}$ *is a Dziobek equilibrium configuration if and only if the configuration and the masses satisfy the following equations*

$$\sin d_{12} \sin d_{34} = \sin d_{13} \sin d_{24} = \sin d_{14} \sin d_{23}$$

$$(m_1, m_2, m_3, m_4) = \left(\frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{24} \Delta_4}, \frac{\sin^3 d_{12} \Delta_2}{\sin^3 d_{14} \Delta_4}, \frac{\sin^3 d_{13} \Delta_3}{\sin^3 d_{14} \Delta_4}, 1 \right) m_4.$$

Corollary 3 $(N = 5, \mathbb{S}^1)$ *Consider one configuration* $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5)$ *in* $\mathbb{S}^1$. *Then* $\mathbf{q}$ *is a Dziobek equilibrium configuration if and only if the configuration and the masses satisfy the following equations*

$$\sin d_{13} \sin d_{25} = \sin d_{12} \sin d_{35} = \sin d_{23} \sin d_{15},$$
$$\sin d_{34} \sin d_{15} = \sin d_{14} \sin d_{35},$$
$$\sin d_{14} \sin d_{25} = \sin d_{12} \sin d_{45} = \sin d_{24} \sin d_{15},$$
$$(m_1, ..., m_5) = \left(\frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{25} \Delta_5}, \frac{\sin^3 d_{12} \Delta_2}{\sin^3 d_{15} \Delta_5}, \frac{\sin^3 d_{13} \Delta_3}{\sin^3 d_{15} \Delta_5}, \frac{\sin^3 d_{14} \Delta_4}{\sin^3 d_{15} \Delta_5}, 1 \right) m_5.$$

Fig. 1 An acute triangle configuration
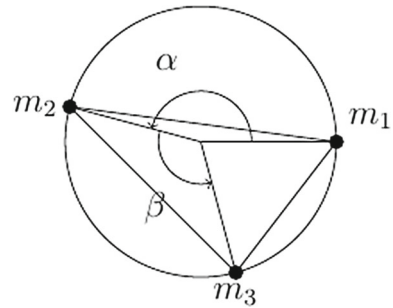
For the case of $N = 3$, the constraint on the shape is only the positiveness of Δ_1, Δ_2, and Δ_3, which implies that the configuration is not in one semicircle, in other words, $\varphi_{i+1} - \varphi_i < \pi$ for $i = 1, 2, 3$ with $\mathbf{q}_i = (\cos\varphi_i, \sin\varphi_i)$. Then all angles are acute, and the configuration forms an acute triangle, see Fig. 1. Let $d_{12} = \alpha$, $d_{23} = \beta$. Then $d_{13} = 2\pi - (\alpha + \beta)$ and

$$0 < \alpha < \pi, \quad 0 < \beta < \pi, \quad \pi < \alpha + \beta < 2\pi.$$

Note that $\Delta_1 = |\mathbf{q}_2, \mathbf{q}_3| = \sin d_{23} = \sin\beta$, $\Delta_2 = -|\mathbf{q}_1, \mathbf{q}_3| = \sin d_{13} = |\sin(\alpha + \beta)|$, $\Delta_3 = |\mathbf{q}_1, \mathbf{q}_2| = \sin d_{12} = \sin\alpha$. Thus the masses satisfy

$$\frac{m_2}{\sin^2\alpha} = \frac{m_3}{\sin^2(\alpha + \beta)}, \quad \frac{m_1}{\sin^2\alpha} = \frac{m_3}{\sin^2\beta}, \quad \frac{m_2}{\sin^2\beta} = \frac{m_1}{\sin^2(\alpha + \beta)}.$$

The above system gives all Dziobek equilibrium configurations for three masses and it has been obtained by direct computations in [7]. Generally, the map $\mathcal{M}$ defined in the previous section is not onto, which is different from that of the classical case. For instance, if $m_2 = m_3 = 1$, then $2\alpha + \beta = 2\pi$, which implies

$$m_1 = \frac{\sin^2\alpha}{\sin^2\beta} = \frac{1}{4\cos^2\alpha} > \frac{1}{4}.$$

The constraint of the masses is found explicitly as follows

$$m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2 - 2m_1 m_2 m_3 < 0,$$

if we assume that $\sum_{i=1}^{3} m_i = 1$. It is easy to see that all such configurations are local minima of the potential V restricted on $\mathbb{S}^1$. These equilibria are stable on $\mathbb{S}^1$ (Remark 1), see [7] and the generalization in [15].

For the case of $N = 4$, the system is not trivial and an equivalent system has been obtained by direct computation in [3]. We do not know much besides the regular tetrahedron equilibrium configuration on $\mathbb{S}^2$ with four equal masses. Now we present a family of 4-body Dziobek equilibrium configurations which contains the regular tetrahedron. Consider a tetrahedron configuration of four masses with position vectors

$$\mathbf{q}_1 = (1, 0, 0)^T \qquad\qquad\qquad \mathbf{q}_2 = (-c, r, 0)^T$$

$$\mathbf{q}_3 = (-c, -\frac{1}{2}r, \frac{\sqrt{3}}{2}r)^T \qquad\qquad \mathbf{q}_4 = (-c, -\frac{1}{2}r, -\frac{\sqrt{3}}{2}r)^T$$

where $m_2 = m_3 = m_4$, and $c \in (0, 1), r^2 + c^2 = 1$, see Fig. 2. Denote such a configuration by $\mathbf{q}_c$.

Fig. 2 Configuration $\mathbf{q}_c$ on $\mathbb{S}^2$
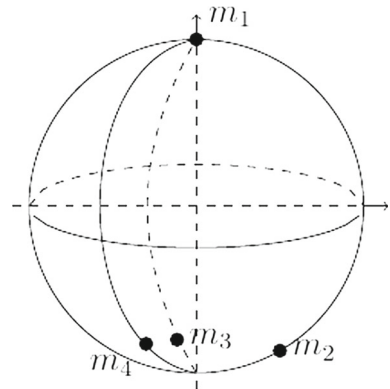
Proposition 4 *The configuration* $\mathbf{q}_c$, $c \in (0, 1)$ *is a Dziobek equilibrium configuration if*

$$\frac{m_1}{m_4} = \frac{8\sqrt{3}c}{3(1 + 3c^2)^{\frac{3}{2}}} \tag{13}$$

.

By numerical study, all such equilibrium configurations are not minima of the potential V restricted on $\mathbb{S}^2$. These equilibria are unstable on $\mathbb{S}^2$, see Remark 1.

Proof The tetrahedron is not on one hemisphere and the shape equations are satisfied since $d_{12} = d_{13} = d_{14}$, and $d_{23} = d_{24} = d_{34}$. The last two of the mass equations $\frac{m_i}{m_4} = \frac{\sin^3 d_{1i}\Delta_i}{\sin^3 d_{14}\Delta_4}$, $i = 2, 3$ are true since since $\Delta_2 = \Delta_3 = \Delta_4$. We only need to check the first mass equation.

Direct computation leads to

$$\cos d_{12} = -c, \;\; \sin^3 d_{12} = r^3, \;\; \cos d_{24} = c^2 - \frac{1}{2}r^2, \;\; \sin^3 d_{24} = 3\sqrt{3}r^3\left(\frac{1}{4} + \frac{3}{4}c^2\right)^{\frac{3}{2}},$$

and $\Delta_4 = \frac{\sqrt{3}}{2}r^2$, $\Delta_1 = 3c\Delta_4$. Thus the configuration is a Dziobek equilibrium configuration if and only if

$$\frac{m_1}{m_4} = \frac{\sin^3 d_{12}\Delta_1}{\sin^3 d_{24}\Delta_4} = \frac{r^3 3c\Delta_4}{3\sqrt{3}r^3\left(\frac{1}{4} + \frac{3}{4}c^2\right)^{\frac{3}{2}}\Delta_4} = \frac{8\sqrt{3}c}{3(1 + 3c^2)^{\frac{3}{2}}}.$$

$\square$

Remark 4 Consider the configuration with $c = -\frac{1}{3}$. Then $\sin d_{12} = \sin d_{24}$, so the shape equations are satisfied. However, the configuration is on the north hemisphere.

As $c \to 0$, we have $\frac{m_1}{m_4} \to 0$. This is intuitively clear. As $c \to 0$, the three masses m_2, m_3, m_4 tend to form an equilibrium configuration of their own on the equator. Then we may place an infinitesimal mass at $\pm(1, 0, 0)$ to form an equilibrium configuration of 4 bodies. The function $f(c) = \frac{8\sqrt{3}c}{3(1+3c^2)^{\frac{3}{2}}}$, $c \in (0, 1)$, is increasing on $(0, \frac{\sqrt{6}}{6})$ and decreasing on $(\frac{\sqrt{6}}{6}, 1)$. The maximum is $\frac{16}{9\sqrt{3}} > 1$, $\lim_{c \to 0} f(c) = 0$ and $\lim_{c \to 1} f(c) = \frac{\sqrt{3}}{3}$.

Corollary 4 *Consider four masses* $(\bar{m}, m, m, m)$ *on* $\mathbb{S}^2$. *If* $\frac{\bar{m}}{m} \in (0, \frac{16}{9\sqrt{3}}]$, *then there is at least one Dziobek equilibrium configuration. If* $\frac{\bar{m}}{m} \in (\frac{\sqrt{3}}{3}, \frac{16}{9\sqrt{3}})$, *then there are at least two Dziobek equilibrium configurations. Especially, there are at least two equilibrium configurations for four equal masses.*

For the case of $N = 5$, the system is not trivial and an equivalent system has been obtained by direct computation in [3]. We do not know much besides the regular pentatope equilibrium configuration on $\mathbb{S}^1$ with five equal masses. Nevertheless, it is easy to construct a family of 5-body Dziobek equilibrium configurations similar to the 4-body equilibrium configurations constructed above and obtain conclusions similar to Proposition 4 and Corollary 4.

4.2 Another Example

Consider Dziobek equilibrium configurations of N masses with the property that $\sum_{i=1}^{N} m_i \mathbf{q}_i = 0$. By Lemma 1, the vector $(m_1, ..., m_N)$ is a multiple of $(\Delta_1, ..., \Delta_N)$. Then equations of (6) implies that $\sin d_{ij}$ is a constant for all pairs of all $\{i, j\}$. Thus, there is some $c \in (0, \pi)$ such that $d_{ij} = c$, or $\pi - c$.

If all d_{ij} equal to c, thus the configuration is a regular simplex, which implies that $\Delta_1 = \Delta_2 = ... = \Delta_N$ and $m_1 = m_2 = ...m_N$. For instance, on $\mathbb{S}^1$, this is the only possibility. However, this is not the only case if the sphere is of higher dimension. A similar phenomenon happens in [8].

For example, consider the following Dziobek configuration on $\mathbb{S}^2$ with position vectors

$$\mathbf{q}_1 = (a, b, 0)^T \qquad\qquad \mathbf{q}_2 = (-c, r, 0)^T$$

$$\mathbf{q}_3 = (-c, -\frac{1}{2}r, \frac{\sqrt{3}}{2}r)^T \qquad\qquad \mathbf{q}_4 = (-c, -\frac{1}{2}r, -\frac{\sqrt{3}}{2}r)^T$$
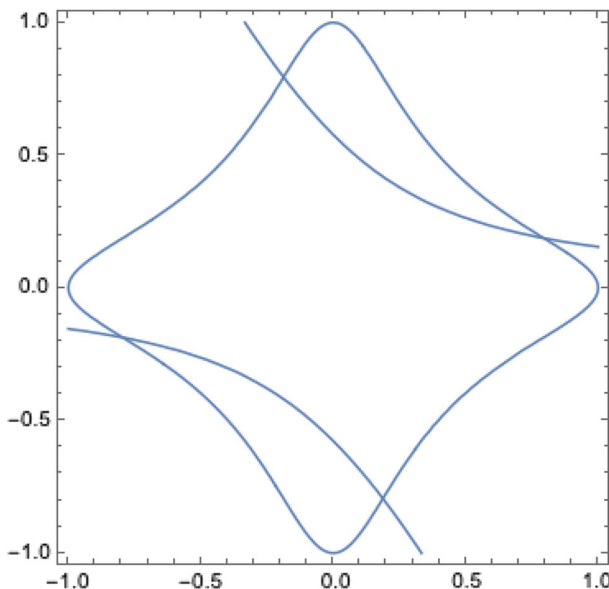

Fig. 3 .

where $a, b, c \in (0, 1), r^2 + c^2 = 1, a^2 + b^2 = 1$. We show that there are values of a, c such that $\sin d_{ij}$ is a constant for all pairs of $\{i, j\}$. Since the configuration is not on one hemisphere, this configuration leads to a Dziobek equilibrium configuration with the property $\sum \mathbf{m}_i \mathbf{q}_i = 0$ but not a regular simplex. Indeed, we only need to solve

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = \mathbf{q}_2 \cdot \mathbf{q}_3, \ \mathbf{q}_1 \cdot \mathbf{q}_2 = -\mathbf{q}_1 \cdot \mathbf{q}_3.$$

In coordinates, the system is

$$4ac = \sqrt{(1 - a^2)(1 - c^2)}, \ 3c^2 - 1 = 6ac.$$

The two algebraic curves defined by the equations has one intersection in $(0, 1) \times (0, 1)$. Thus, there is Dziobek equilibrium configuration on $\mathbb{S}^1$ that is not regular simplex but satisfies $\sum m_i \mathbf{q}_i = 0$.

Appendix: The Derivative of the Cayley–Menger Determinant

For a Dziobek configuration of n-body in $\mathbb{S}^{n-2}$, recall the $(n-1) \times n$ matrix $X = [\mathbf{q}_1, ..., \mathbf{q}_n]$. Since rank$X = n - 1$, the corresponding Gram matrix $X^T X$ has rank $n - 1$. Then the determinant $F = 0$. We may call the quantity F the *spherical Cayley-Menger determinant*, [4]. For instance, for $n = 4$,

$$F = \begin{vmatrix} 1 & \cos d_{12} & \cos d_{13} & \cos d_{14} \\ \cos d_{12} & 1 & \cos d_{23} & \cos d_{24} \\ \cos d_{13} & \cos d_{23} & 1 & \cos d_{34} \\ \cos d_{14} & \cos d_{24} & \cos d_{34} & 1 \end{vmatrix}.$$

A by-product of equation (6) is the following. A Dziobek configuration on $\mathbb{S}^{n-2}$ can be parametrized by the C_n^2 quantities $\{\cos d_{12}, ..., \cos d_{n-1,n}\}$ with the relation $F = 0$. Then any equilibrium configuration of the system (1) is the critical point of $V + \lambda F$. Then Eq. (6) implies $\frac{\partial F}{\partial \cos d_{ij}} = \alpha \Delta_i \Delta_j$ for some α.

Proposition 5 *Let $\mathbf{q}_1, ..., \mathbf{q}_n$ be a Dziobek configuration in $\mathbb{S}^{n-2}$. Let $d_{12}, ..., d_{n-1,n}$ and F be the corresponding mutual distances and the spherical Cayley–Menger determinant. Then we have*

$$\frac{\partial F}{\partial \cos d_{ij}} = 2\Delta_i \Delta_j \text{ for any } 1 \leq i < j \leq n.$$

where Δ_i is the signed determinant defined in (5).

Proof By the symmetry of $X^T X$, we have $\frac{\partial F}{\partial \cos d_{ij}} = 2F_{ij}$, with F_{ij} being the (i, j) cofactor of matrix $X^T X$, i.e.,

$$\frac{\partial F}{\partial \cos d_{ij}} = 2(-1)^i (-1)^j |A_{ij}|,$$

where A_{ij} is the (i, j) minor of matrix $X^T X$. Let X_k be the square matrix of order $n - 1$ obtained from X by deleting the k-th column. Then $X_i^T X_j = A_{ij}$. Thus, we have $\frac{\partial F}{\partial \cos d_{ij}} = 2\Delta_i \Delta_j$. □

This derivative formula enables us to obtain Eq. (6) directly.

For a Dziobek configuration $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)$ in $\mathbb{R}^{n-2}$, the mutual distances satisfy a relation and its derivative formula is similar to the above one. Due to the translational symmetry, the appropriate Gram matrix is $\tilde{X}^T \tilde{X}$, with

$$\tilde{X} = [\mathbf{x}_2 - \mathbf{x}_1, ..., \mathbf{x}_n - \mathbf{x}_1].$$

It is easy to see that $|\tilde{X}^T \tilde{X}| = 0$. Note that the entries of $X^T \tilde{X}$ are not in terms of the mutual distances. By using the formula $(\mathbf{x}_i - \mathbf{x}_1) \cdot (\mathbf{x}_j - \mathbf{x}_1) = \frac{1}{2}(d_{1i}^2 + d_{1j}^2 - d_{ij}^2)$ and some bordering technique, [4], we can obtain another determinant

$$\Gamma = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{12}^2 & \cdots & d_{1n}^2 \\ 1 & d_{21}^2 & 0 & \cdots & d_{2n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{n1}^2 & d_{n2}^2 & \cdots & 0 \end{vmatrix}, \text{ and } \Gamma = (-1)^n 2^{n-1} |\tilde{X}^T \tilde{X}|.$$

Usually, it is Γ instead of $|\tilde{X}^T \tilde{X}|$ that is called the *Cayley–Menger determinant*. Let

$$X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}_{(n-1) \times n},$$

and X_k be the square matrix of order $n - 1$ obtained from X by deleting the k-th column. Let $\Delta_k = (-1)^{k-1} |X_k|$. For $n = 4$, Dziobek [9] observed a formula that is equivalent to

$$\frac{\partial \Gamma}{\partial d_{ij}^2} = -8 \Delta_i \Delta_j \text{ for any } 1 \leq i < j \leq 4.$$

With the technique used to relate Γ and $|\tilde{X}^T \tilde{X}|$, we have

Proposition 6 *Let $\mathbf{x}_1, ..., \mathbf{x}_n$ be a Dziobek configuration in $\mathbb{R}^{n-2}$. Let $d_{12}, ..., d_{n-1,n}$ be the corresponding mutual distances. Let Γ and Δ_i be the determinants defined above. Then we have*

$$\frac{\partial \Gamma}{\partial d_{ij}^2} = (-2)^{n-1} \Delta_i \Delta_j \text{ for any } 1 \leq i < j \leq n.$$

Proof By the symmetry, we have $\frac{\partial \Gamma}{\partial d_{ij}^2} = 2(-1)^i (-1)^j |B_{ij}|$ where B_{ij} is the $(i + 1, j + 1)$ minor of Γ. On the other hand, note that

$$|X_i| = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ \vec{0} & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vec{0} & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{vmatrix}.$$

Bordering X_j in the same way without exchanging the first two row, we obtain

$$
|X_i^T X_j| = - \begin{vmatrix}
0 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & \mathbf{x}_1 \cdot \mathbf{x}_1 & \cdots & \mathbf{x}_1 \cdot \mathbf{x}_i & \cdots & \mathbf{x}_1 \cdot \mathbf{x}_{j-1} & \mathbf{x}_1 \cdot \mathbf{x}_{j+1} & \cdots & \mathbf{x}_1 \cdot \mathbf{x}_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \mathbf{x}_{i-1} \cdot \mathbf{x}_1 & \cdots & \mathbf{x}_{i-1} \cdot \mathbf{x}_i & \cdots & \mathbf{x}_{i-1} \cdot \mathbf{x}_{j-1} & \mathbf{x}_{i-1} \cdot \mathbf{x}_{j+1} & \cdots & \mathbf{x}_{i-1} \cdot \mathbf{x}_n \\
1 & \mathbf{x}_{i+1} \cdot \mathbf{x}_1 & \cdots & \mathbf{x}_{i+1} \cdot \mathbf{x}_i & \cdots & \mathbf{x}_{i+1} \cdot \mathbf{x}_{j-1} & \mathbf{x}_{i+1} \cdot \mathbf{x}_{j+1} & \cdots & \mathbf{x}_{i+1} \cdot \mathbf{x}_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \mathbf{x}_j \cdot \mathbf{x}_1 & \cdots & \mathbf{x}_j \cdot \mathbf{x}_i & \cdots & \mathbf{x}_j \cdot \mathbf{x}_{j-1} & \mathbf{x}_j \cdot \mathbf{x}_{j+1} & \cdots & \mathbf{x}_j \cdot \mathbf{x}_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \mathbf{x}_n \cdot \mathbf{x}_1 & \cdots & \mathbf{x}_n \cdot \mathbf{x}_i & \cdots & \mathbf{x}_n \cdot \mathbf{x}_{j-1} & \mathbf{x}_n \cdot \mathbf{x}_{j+1} & \cdots & \mathbf{x}_n \cdot \mathbf{x}_n
\end{vmatrix}
$$

We then replace $\mathbf{x}_i \cdot \mathbf{x}_j$ be $\frac{1}{2}(||\mathbf{x}_i||^2 + ||\mathbf{x}_j||^2 - d_{ij}^2)$, and eliminate all the $||\mathbf{x}_i||^2$ by subtracting the appropriate multiple of the first row and column from the others. We obtain

$$
\Delta_i \Delta_j = (-1)^{i+j} |X_i^T X_j| = (-1)^{i+j} 2^{2-n} (-1)^{n-1} |B_{ij}|.
$$

Hence follows the formula $\frac{\partial \Gamma}{\partial d_{ij}^2} = (-2)^{n-1} \Delta_i \Delta_j$. $\qquad \square$

Central configuration in $\mathbb{R}^n$ of dimension $n - 2$ are considered in [11]. The equations of them are derived by vectorial method there. Note that these equations follow easily from the above derivative formula.

References

1. Albouy, A., Fu, Y., Sun, S.: Symmetry of planar four-body convex central configurations. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **464**(2093), 1355–1365 (2008)
2. Borisov, A.V., Mamaev, I.S., Kilin, A.A.: Two-body problem on a sphere. Reduction stochasticity periodic orbits. Regul. Chaotic. Dyn. **9**(3), 265–279 (2004)
3. Boulter, E., Diacu, F., Zhu, S.: The n-body problem in spaces with uniformly varying curvature. J. Math. Phys. **58**(5), 052703 (2017)
4. Berger, M.: Geometry I, II. Springer-Verlag, Berlin Heidelberg, Translated by M. Cole and S. Levy (1987)
5. Corbera, M., Cors, J.M., Roberts, G.E.: Classifying four-body convex central configurations. Celestial Mech. Dynam. Astronom. (2019). https://doi.org/10.1007/s10569-019-9911-7
6. Diacu, F.: Relative equilibrium configurations of the 3-dimensional curved n-body problem, Memoirs Amer. Math. Soc. 228 , no. 1071 (2013)
7. Diacu, F., Sánchez-Cerritos,J.M., Zhu, S.: Stability of fixed points and associated relative equilibrium configurations of the 3-body problem on $\mathbb{S}^1$ and $\mathbb{S}^2$, J. Dynam. Differential Equations 30 (2018), no. 1, 209-225. Modification after publication at arXiv:1603.03339
8. Diacu, F., Zhu, S.: Almost all 3-body relative equilibrium configurations are inclined. Discrete Contin. Dyn. Syst. Ser. S. **13**(4), 1131–1143 (2020)
9. Dziobek, O.: Über einen merkwürdigen Fall des Vielkörperproblems. Astron. Nachr. **152**, 33–46 (1900)
10. Lim, C.C.: Relative equilibrium configurations of symmetric n-body problems on a sphere: inverse and direct results. Comm. Pure Appl. Math. **51**(4), 341–371 (1998)
11. Moeckel, R.: Generic finiteness for Dziobek configurations. Trans. Amer. Math. Soc. **353**(11), 4673–4686 (2001)
12. Palamodov, V.P.: On inversion of the Lagrange-Dirichlet theorem and instability of conservative systems. Russ. Math. Surv. **75**(3), 495–508 (2020)
13. Stoica, C.: On the n-body problem on surfaces of revolution. J. Diff. Equ. **264**(10), 6191–6225 (2018)
14. Ureña, A.J.: To what extent are unstable the maxima of the potential? Ann. Mat. Pura Appl. **199**, 1763–1775 (2020)
15. Yu, X., Zhu, S.: Regular polygonal equilibrium configurations on $\mathbb{S}^1$ and stability of the associated relative equilibrium configurations. J. Dynam. Diff. Equ. (2020). https://doi.org/10.1007/s10884-020-09848-1

16. Zhu, S.: Compactness and index of ordinary central configurations for the curved N-body problem, arXiv:2003.06850 [math.DS], to appear in Regul. Chaotic Dyn.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.