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Some Remarks Concerning the Scattering Theory for the Sturm–Liouville Operator

Luca Zampogni[1]

Dedicated to the memory of Russell Johnson

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Abstract
We start to discuss some aspects of the scattering theory for the Sturm–Liouville operator $L : \frac{1}{y}\left[-D^2 + q\right]$. In particular, we pose and solve the problem of reconstructing the function q when y is fixed and when a set $\mathcal{S}$ of scattering data is given. In the meanwhile, several relations concerning the spectral properties of L and the solutions of the related eigenvalue equation are established.

Keywords Inverse problems · Sturm–Liouville operators · Scattering theory

Mathematics Subject Classification 34B24 · 34L25 · 37B55 · 35Q53 · 34L40 · 34L05

1 Introduction

In the 60's Gel'fand, Levitan, Marchenko and their collaborators drew an important procedure to reconstruct and characterize Schrödinger potentials starting from the scattering data. They discussed various aspects concerning spectral measures and characterized the Schrödinger potentials by means of a sort of orthogonalizing process [25,26,28]. All these matters are very well-known to researchers working with both inverse spectral problems and also the K-dV equation, and in fact in [10] the authors showed a beautiful procedure to build solutions of the K-dV equation having as initial data potentials $q_0(x)$ which are of "scattering type". These facts gave rive to a large amount of papers concerning the relations between Schrödinger potentials and the K-dV equation [1,8,24,25,27,29,30,33].

✉ Luca Zampogni
luca.zampogni@unipg.it

1 Dipartimento di Matematica e Informatica, Universitá degli Studi di Perugia, Perugia, Italy

On the other hand, the acoustic equation[1]

$$\varphi'' + \varphi = \lambda y \varphi \tag{1}$$

with weight function y is connected to the Camassa–Holm equation (briefly, CH) (see [3])

$$4u_t = u_{xxt} + 2uu_{xxx} + 4u_x u_{xx} - 24uu_x$$

in the following way: starting from an initial data $y_0(x) = 4u(0, x) - u_{xx}(0, x)$, it is possible in many cases to build a solution of (CH) by maintaining unchanged certain spectral properties of the associated equation (1). This has been done in many cases, such as Algebro-geometric potentials [11,20,32], some of their limits [12,21–23] and so on. Constantin et al. [4–6] constructed also solutions of (CH) via a scattering procedure, which, however, reflects the general property of the inverse problem for the Eq. (1) of being implicit in its nature. Moreover, they make use of the Liouville transform to develop the scattering theory (see [4] for more information). Recently new hierarchies of evolution equations, involving the more general Sturm–Liouville operator

$$L = \frac{1}{y}\left[-pD^2 + q\right], \quad D = \frac{d}{dx}$$

have been introduced [18,21–23]. These hierarchies produce evolution equations which naturally include both the K-dV and the (CH) equations, and others which have applications in the fields of magnetic fluids, quantum theory, hydro-thermal particles and others [21–23].

Motivated by these facts, and from the evidence, as far as we know, that a scattering theory for the Sturm–Liouville operator has never been investigated in detail, in this paper we start to discuss the scattering theory for the operator

$$L := \frac{1}{y}\left[-D^2 + q\right], \quad D = \frac{d}{dx}, \tag{2}$$

depending on the parameters $q, y \in C_0(\mathbb{R})$ with strictly positive weight function y. We will focus our attention to the reconstruction of the potentials in terms of the scattering data, and vice-versa, leaving out the aspects concerning the characterization of the potentials by means of the spectral measure (a matter which, however, will be considered in forthcoming papers [34,35]).

In this paper, however, a special case of the above problem is studied: we begin our investigation by considering the operator

$$L = \frac{1}{y}\left[-D^2 + q\right],$$

where y is fixed. We will first find a condition for which the associated spectral problem has solution of scattering type, and indeed we will find a scattering hypothesis on the function q. Then, we will focus our attention to the relation between the scattering coefficients and the potential q, providing the necessary background for the inverse spectral problem to be solved.

The case when q is fixed and y is, let us say, the unknown function, will be studied in a forthcoming paper, since it uses some instruments which are in some sense different from those used in this paper (see [34,35]). The paper is organized as follows. In Sect. 2, we review some important spectral properties of the Sturm–Liouville operator: we will not spend too

[1] The term acoustic is due to the fact that (1) appears in the separation of variables method for solving the wave equation.

much time in reviewing these facts, as the reader can find them in several papers [13–19]. We will mainly describe some of the properties of the so-called Weyl m-function, which will give us the hint to set the scattering problem consistently. Section 3 deals with the extension of the Gel'fand–Levitan theory for the Sturm–Liouville operator. We will derive the basic equations for relating the solutions of two different spectral problems via a kernel function $K(x, t)$. In Sect. 4 we will prove the existence of the kernel function $K(x, t)$ and state some relations which will be useful for the reconstruction of the function q once the scattering data are given. Finally, in Sect. 5, we will start to study the scattering problem, by proving the relations occurring between the transmission-reflection coefficients, and the relation between these coefficients and the potential q.

The author wishes to finish this introductory part by remembering one of the most important men in his life: Russell Johnson has been my guide in many occasions, he helped me many times in the work, and in the life. Besides being one of the most brilliant mathematicians I had the pleasure to meet, he was also a great, honest, kind man. We started our collaboration (and our friendship) during my Ph.D. studies, since he was the advisor of my Ph.D. thesis. After that, we started a long beautiful collaboration, and he never stopped teaching me mathematics. It has been a privilege to work with him, and it was special to speak with him. I and Russell Johnson started together to think at and to write down the ideas of this paper some years ago. I only hope he has been sometimes proud of my work, I miss him a lot.

2 Preliminaries

Let us denote by

$$\mathcal{E}_2 = \{a = (q, y) : \mathbb{R} \to \mathbb{R}^2 \mid a \in C^1(\mathbb{R}),\ a \text{ is uniformly continuous and } \delta < y < \Delta\},$$

equipped with the topology of uniform convergence on compact subsets. For $a \in \mathcal{E}_2$, consider the Sturm–Liouville operator

$$L_a : \mathcal{D} \to L^2(\mathbb{R}, dx) : \varphi \mapsto L\varphi = \frac{1}{y}(-\varphi'' + q\varphi),$$

where $\mathcal{D} = \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \in L^2(\mathbb{R}, ydx),\ \ \varphi$ is absolutely continuous and $\varphi'' \in L^2(\mathbb{R}, ydx)\}$. It is well-known that L admits a self-adjoint extension to all $L^2(\mathbb{R}, ydx)$ which we still denote by L. The associated eigenvalue equation can be written as

$$-\varphi'' + q\varphi = \lambda y\varphi.$$

In matrix form, it reads

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(t) - \lambda y(t) & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{3}$$

Let $a = (q, y) \in \mathcal{E}_2$. Let us denote by $\{\tau_s\}_{s\in\mathbb{R}}$ the translation flow, i.e., if a is as above, $\tau_s(a) = a(s + \cdot)$. If $a_0 = (q_0, y_0)$ is fixed, then we set

$$\mathcal{A} = cls\ Hull(a_0) = cls\ \{\tau_s(a_0) \mid s \in \mathbb{R}\}.$$

Then $\mathcal{A}$ is a compact invariant subset of $\mathcal{E}_2$. It now makes sense to write $a(x,\lambda) = \begin{pmatrix} 0 & 1 \\ q(x)-\lambda y(x) & 0 \end{pmatrix}$, and study the family of equations

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}' = \tau_x(a) \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}, \qquad a \in \mathcal{A}. \tag{4}$$

This dynamical approach has revealed to be very useful in studying the spectral properties of the Sturm–Liouville operator, see [9,13,14,16–19]. One of the most important concepts which are related to the family of equations above is that of the exponential dichotomy. Let $\Phi_a(x)$ be the fundamental matrix solution of the family (4).

Definition 2.1 The family (4) is said to have an *exponential dichotomy* (briefly, E.D.) over $\mathcal{A}$ if there are positive constants η, ρ, together with a continuous, projection valued function $P : \mathcal{A} \to \mathbb{M}_2(\mathbb{C})$ such that the following estimates holds:

(i) $|\Phi_a(x)P(a)\Phi_a(s)^{-1}| \le \eta e^{-\rho(x-s)}, \qquad x \ge s,$
(ii) $|\Phi_a(x)(I-P(a))\Phi_a(s)^{-1}| \le \eta e^{\rho(x-s)}, \qquad x \le s.$

Here, $\mathbb{M}_2(\mathbb{C})$ denotes the set of 2×2 complex valued matrices. One has the following fundamental result (see [9,13,15–17])

Theorem 2.2 *Let $\mathcal{A}$ be obtained by the construction above. Consider the family (4). If $a \in \mathcal{A}$ has dense orbit, then the spectrum Σ_a of the operator L_a equals the set*

$$\Sigma_{ed} := \{\lambda \in \mathbb{C} \mid the family \ \ (4) \ \ does \ \textbf{not} \ admit\, an\, E.D.\, over\ \mathcal{A}\}.$$

It is known that, if $\Im\lambda \neq 0$, then the family (4) admits an exponential dichotomy over $\mathcal{A}$, hence $\Sigma_a \subset \mathbb{R}$ [9]. Another consequence of the above theorem is that if $a \in \mathcal{E}_2$ and $\mathcal{A} = cls\ Hull(a)$ then the spectrum of L_a and that of all the operators $L_{\tau_x(a)}$ coincide, i.e., $\Sigma_a = \Sigma_{\tau_x(a)} = \Sigma_{ed}$ for every $x \in \mathbb{R}$ [9].

Since $a \in \mathcal{A}$ and $\det \Phi_a(x) = 1$ for every $x \in \mathbb{R}$, both $\ker P(a)$ and $\operatorname{Im} P(a)$ are complex lines in $\mathbb{C}^2$, which can be parametrized by

$$\operatorname{Im} P(a) = Span \begin{pmatrix} 1 \\ m_+(a,\lambda) \end{pmatrix} \qquad \ker P(a) = Span \begin{pmatrix} 1 \\ m_-(a,\lambda) \end{pmatrix}.$$

The numbers $m_\pm(a,\lambda)$ are called the Weyl m-functions at $\pm\infty$. Acting by translation, we can define the Weyl m-functions

$$m_\pm(x,\lambda) = m_\pm(\tau_x(a),\lambda),$$

which satisfy the Riccati equation

$$m' + m^2 = q - \lambda y.$$

We now make the following fundamental assumption.

Hypotheses 2.3 *Let $L_a = \frac{1}{y}(-D^2+q)$ be the self-adjoint operator defined on $L^2(\mathbb{R}, y dx)$, where $a = (q, y) \in \mathcal{E}_2$. We assume that $a = (p, q)$ is chosen in such a way the spectrum Σ_a of L_a contains a (positive) half-line $[\lambda^*, \infty)$ in which the Weyl m-functions extend holomorphically from $\mathbb{C}^+$ to $\mathbb{C}^-$.*

This assumption has an important consequence: namely, let $z^2 = -\lambda$ be a local parameter at $\lambda = \infty$. Then it can be proved that (see [9,16])

$$m_+(q, z) = \sqrt{y}z - \frac{y'}{4y} + \sum_{n=1}^{\infty} \alpha_{-n} z^{-n}$$

and

$$m_-(q, z) = -\sqrt{y}z - \frac{y'}{4y} + \sum_{n=1}^{\infty} (-1)^n \alpha_{-n} z^{-n}.$$

The coefficients α_{-j} $(j \geq 1)$ can be determined recursively by using the Riccati equation

$$m' + m^2 = q - \lambda y$$

which both $m_\pm$ satisfy when $\Im\lambda \neq 0$, and which however retains validity also in the half-line $[\lambda^*, \infty)$ in view of Hypotheses 2.3 (see [9,15,16]). In particular, setting $\alpha_0 = -\dfrac{y'}{4y}$, then

$$\alpha_{-1} = \frac{1}{2}\left[\frac{q - \alpha_0' - \alpha_0^2}{\sqrt{y}}\right],$$
$$\alpha_{-1}' + 2[\alpha_0\alpha_{-1} - \sqrt{y}\alpha_{-2}] = 0,$$

and so on.

From now on, we fix $y : \mathbb{R} \to \mathbb{R}$ and determine a particular function $\tilde{q}$ (and hence an operator $\tilde{L} := L_{\tilde{q}}$) which will play a crucial role in the following. This function will be simply chosen in such a way that the corresponding operator $\tilde{L}$ has sine-cosine solution whenever $\lambda > 0$.

Once y is fixed, let $\alpha_0 = -\dfrac{y'}{4y}$, and set

$$\tilde{q} = \alpha_0' + \alpha_0^2,$$

so that $\alpha_{-1} = 0$ and all the coefficients α_{-j} vanish as well. Writing down things explicitely,

$$\tilde{q} := \alpha_0' + \alpha_0^2 = -\left(\frac{y'}{4y}\right)' + \left(\frac{y'}{4y}\right)^2$$

and

$$m_\pm(\tilde{q}, z) = \pm\sqrt{y}z - \frac{y'}{4y} \qquad (z^2 = -\lambda).$$

The operator $\tilde{L} := \dfrac{1}{y}(-D^2 + \tilde{q})$ defined on $L^2(\mathbb{R}, ydx)$ has general solutions

$$f(x, \lambda) = A \exp\left(\int_0^x \sqrt{-\lambda}\sqrt{y(s)} - \frac{y'(s)}{4y(s)}\right) ds + B \exp\left(\int_0^x -\sqrt{-\lambda}\sqrt{y(s)} - \frac{y'(s)}{4y(s)}\right)$$

which, for $\lambda \geq 0$ may be written simply as

$$f(x, \lambda) = y(x)^{-1/4}\left(A_1 \sin\left(\int_0^x \sqrt{\lambda y(s)} ds\right) + B_1 \cos\left(\int_0^x \sqrt{\lambda y(s)} ds\right)\right).$$

We now use the notation

$$\mathcal{I}(x) := \int_0^x \sqrt{y(s)}ds,$$

so that

$$f(x,\lambda) = y(x)^{-1/4}\left(A_1 \sin\sqrt{\lambda}\mathcal{I}(x) + B_1 \cos\sqrt{\lambda}\mathcal{I}(x)\right).$$

The operator $\tilde{L}$ has only absolutely continuous spectrum $\tilde{\Sigma} = [0,\infty)$, and no isolated (negative) eigenvalues: it is the analogous of the operator $L_0 = -D^2$ in the Schrödinger case.

3 Transformation Operators

Let $(\tilde{q}, y) \in \mathcal{E}_2$, where

$$\tilde{q} = -\left(\frac{y'}{4y}\right)' + \left(\frac{y'}{4y}\right)^2.$$

We now discuss a procedure which allows to move from solutions of $\tilde{L}$ to solutions of certain operators L_q. Consider the operator

$$\tilde{L} := \frac{1}{y}(-D^2 + \tilde{q}),$$

where $\tilde{q}$ is constructed as above. Then, choose another function $q : \mathbb{R} \to \mathbb{R}$ such that $(q, y) \in \mathcal{E}_2$, and consider the corresponding operator

$$L_q := \frac{1}{y}(-D^2 + q).$$

We move our attention to the half-line restricted operators: by $\tilde{L}^{\pm}$ and $L_q^{\pm}$ and define the self-adjoint operators defined on $L^2(\mathbb{R}^{\pm}, ydx)$ respectively. They give rise to the eigenvalue equations

$$-\psi_{\pm}'' + \tilde{q}\psi_{\pm} = \lambda y \psi_{\pm}$$

and

$$-\varphi_{\pm}'' + q\varphi_{\pm} = \lambda y \varphi_{\pm}.$$

Focus firstly on $\tilde{L}^+$ and L_q^+. For simplicity, let us write $\psi,\ \varphi$ instead of $\psi_+,\ \varphi_+$. Let

$$\psi(x) = y(x)^{-1/4} e^{i\sqrt{\lambda}\mathcal{I}(x)}.$$

Note that $\psi(x)$ is a solution of $\tilde{L}^+$. Let us impose the condition

$$\varphi(x,\lambda) \sim y(x)^{-1/4} e^{i\sqrt{\lambda}\mathcal{I}(x)} \quad \text{as} \quad x \to \infty.$$

We want to relate φ and ψ by means of

$$\varphi(x) = \psi(x) + \int_x^{\infty} K_+(x,t)y(t)\psi(t)dt, \tag{5}$$

where $K_+(x,t)$ is defined on $\mathcal{D}$ and satisfies

$$\lim_{t+x\to\infty} K_+(x,t) = \lim_{t+x\to\infty} K_{+,x}(x,t) = 0.$$

It is at first sight clear that the existence of such $K_+(x,t)$ must depend on the choice of the potential q. We abuse notation and write for the moment K instead of K_+.

Now,

$$\varphi'(x) = \psi'(x) - K(x,x)\psi(x)y(x) + \int_x^\infty \left[\frac{\partial}{\partial x}K(x,t)\right] y(t)\psi(t)dt,$$

and

$$\varphi''(x) = \psi''(x) - \frac{d}{dx}\left[K(x,x)\psi(x)y(x)\right] - \left[\frac{\partial}{\partial x}K(x,x)\right] y(x)\psi(x) + \int_x^\infty \left[\frac{\partial^2}{\partial x^2}K(x,t)\right] y(t)\psi(t)dt.$$

Thus

$$\begin{aligned} L_q\varphi = \frac{1}{y}(-D^2+q)\varphi &= -\frac{1}{y(x)}\psi''(x) + \frac{1}{y(x)}\frac{d}{dx}\left[K(x,x)\psi(x)y(x)\right] \\ &+ \frac{1}{y(x)}\left[\frac{\partial}{\partial x}K(x,x)\right] y(x)\psi(x) - \frac{1}{y(x)}\int_x^\infty \frac{\partial^2}{\partial x^2}K(x,t)y(t)\psi(t)dt \\ &+ \frac{q(x)}{y(x)}\psi(x) + \frac{q(x)}{y(x)}\int_x^\infty K(x,t)y(t)\psi(t)dt. \end{aligned} \tag{6}$$

On the other hand, $-\frac{1}{y}\psi''(x) + \frac{q(x)}{y(x)}\psi(x) = \tilde{L}\psi$, and solving $\tilde{L}\psi = 0$, substituting into (6) and integrating twice, we obtain

$$\frac{1}{y(x)}\frac{d}{dx}[K(x,x)y(x)] + \frac{\partial}{\partial x}K(x,x) + \frac{q(x)}{y(x)} = \frac{\tilde{q}(x)}{y(x)} - \frac{\partial}{\partial x}K(x,x) \tag{7}$$

and

$$\begin{aligned} &-\frac{1}{y(x)}\int_x^\infty \left[\frac{\partial^2}{\partial x^2}K(x,t)\right] y(t)\psi(t)dt + \frac{q(x)}{y(x)}\int_x^\infty K(x,t)y(t)\psi(t)dt \\ &= -\int_x^\infty \left[\frac{\partial^2}{\partial t^2}K(x,t)\right]\psi(t)dt + \int_x^\infty K(x,t)\tilde{q}(t)\psi(t)dt. \end{aligned} \tag{8}$$

From Eq. (8) we obtain (simplifying the notation)

$$\frac{1}{y(x)}K_{xx}(x,t)y(t) + \frac{q(x)}{y(x)}K(x,t)y(t) = -K_{tt}(x,t) + K(x,t)\tilde{q}(t), \tag{9}$$

and finally

$$\frac{1}{y(x)}K_{xx}(x,t) - \frac{1}{y(t)}K_{tt}(x,t) = \left[\frac{q(x)}{y(x)} - \frac{\tilde{q}(t)}{y(t)}\right] K(x,t) \qquad (t \geq x > 0). \tag{10}$$

Condition (7) is an initial condition: indeed, we obtain

$$2y(x)\frac{d}{dx}K(x,x) + K(x,x)\frac{d}{dx}y(x) = \tilde{q}(x) - q(x), \tag{11}$$

or, equivalently,

$$\frac{d}{dx}\left[\sqrt{y(x)}K(x,x)\right] = \frac{\tilde{q}(x) - q(x)}{2\sqrt{y(x)}}.$$

All the reasoning above can be summarized as follows: if a kernel $K(x,t)$, defined in the domain $\mathcal{D} = \{(x,t) \in \mathbb{R}^2 \mid t \geq x > 0\}$, exists and satisfies

$$(i) \qquad \lim_{x+t\to\infty} \max\{K(x,t), K_x(x,t), K_t(x,t)\} = 0,$$

$$(ii) \qquad K(x,\cdot), K_x(x,\cdot), K_t(x,\cdot), K_{xx}(x,\cdot), K_{tt}(x,\cdot) \in L^1(\mathbb{R}^+),$$

then it is a solution of

$$\begin{cases} \dfrac{1}{y(x)}K_{xx}(x,t) - \dfrac{1}{y(t)}K_{tt}(x,t) = \left[\dfrac{q(x)}{y(x)} - \dfrac{\tilde{q}(t)}{y(t)}\right]K(x,t) \\ \\ 2y(x)\dfrac{d}{dx}K(x,x) + K(x,x)\dfrac{d}{dx}y(x) = \tilde{q}(x) - q(x) \end{cases} \tag{12}$$

in the domain $\mathcal{D} = \{(x,t) \in \mathbb{R}^2 \mid t \geq x > 0\}$. It is clear that the existence of a solution of (12) depends on the potential $q(x)$ (not on $\tilde{q}(x)$, which is fixed once $y(x)$ is chosen).

Before proving the existence of a solution $K(x,t)$ of (12), we observe that the converse of the above reasoning is true as well. In fact, suppose that we have fixed a function $y(x)$ such that, if

$$\tilde{q}(x) = -\left(\frac{y'}{4y}\right)' + \left(\frac{y'}{4y}\right)^2,$$

then $(\tilde{q}, y) \in \mathcal{E}_2$ as above. Suppose that $(q, y) \in \mathcal{E}_2$ is such that (12) admits a solution $K(x,t)$ satisfying (i) and (ii) above. Let $\psi(x)$ be a solution of the equation

$$-\psi'' + \tilde{q}\psi = \lambda y \psi.$$

Define

$$\varphi(x) = \psi(x) + \int_x^\infty K(x,t)\psi(t)y(t)dt.$$

Then ψ is a solution of the eigenvalue equation

$$-\varphi'' + q\varphi = \lambda y \varphi,$$

where q is defined via the second equation in (12).

As we already pointed out, the existence of a (unique) solution of (12) or, equivalently, of a kernel of the transformation $\tilde{L} \mapsto L_q$, depends on the choice of q, and in fact on its behaviour compared to that of $\tilde{q}$. We discuss now this matter. There are various ways to prove the existence of a unique solution of (12) and (i) and (ii). One is that of applying the Riemann's method (see [7]) (which however does not give further information on the properties of K), and another is based on successive approximation, by first transforming (12) into a more convenient form. We will follow none of these two ways, but we will focus on a method which uses basic properties of the Sturm–Liouville operators $\tilde{L}^{\pm}$ and $L_q^{\pm}$, in particular the behaviour of their solutions for complex values of the parameter λ when $\Im\lambda > 0$.

Before entering into the topic of the existence of a solution of (12), we find useful to make some observations: (1) Although the spectral problem depends on the parameter $\lambda \in \mathbb{C}$, the equation (12), and hence its solution, does not. The $\lambda-$dependence in the equation (5) lays in the functions $\varphi(x)$ and $\psi(x)$;

(2) As we will see, however, $K(x,t)$ depends on the spectrum of L_q. According to the spectrum of L_q, the kernel $K(x,t)$ may or may not have exponential parts due to the (isolated) eigenvalues of L_q. We will prove that indeed there exists an expression for $K(x,t)$ which contains two main parts: one is related to the absolutely continuous part of the spectrum (it will be the half-line $[0,\infty)$, as we will prove shortly...) and not connected to the isolated eigenvalues of L_q. The other part instead is only related to the eigenvalues of L_q.

(3) Note that the second equation in (12) allows to express $q(x)$ by means of $K(x,t)$ and $y(x)$. Indeed, one has

$$q(x) = \tilde{q}(x) - 2y(x)\frac{d}{dx}K(x,x) - K(x,x)\frac{d}{dx}y(x), \tag{13}$$

or its more appealing form

$$\frac{d}{dx}\left[\sqrt{y(x)}K(x,x)\right] = \frac{\tilde{q}(x) - q(x)}{2\sqrt{y(x)}}. \tag{14}$$

We now move our attention to the operators $\tilde{L}^-$ and L_q^-. To distinguish the cases in a precise manner, we reformulate the fundamental relations (5) and (12) by using the correct signs. If we consider the operators $\tilde{L}^+$ and L_q^+, defined on $L^2(\mathbb{R}^+, ydx)$, one introduces the eigenvalue equations

$$\tilde{L}^+\psi_+ = \lambda\psi_+$$

and

$$L_q^+\varphi_+ = \lambda\varphi_+$$

together with the boundary condition

$$\varphi_+(x,\lambda) \sim y(x)^{-1/4}e^{i\sqrt{\lambda}\mathcal{I}(x)} \quad \text{as} \quad x \to \infty.$$

In this case, one looks for a kernel $K_+(x,t)$, defined in $\mathcal{D}_+ = \{(x,t) \in \mathbb{R}^2 \mid t > x \geq 0\}$, such that

$$\varphi_+(x,\lambda) = \psi_+(x,\lambda) + \int_x^\infty K_+(x,t)y(t)\psi_+(t)dt.$$

The above procedure leads to the equations

$$\begin{cases} \dfrac{1}{y(x)}K_{+,xx}(x,t) - \dfrac{1}{y(t)}K_{+,tt}(x,t) = \left[\dfrac{q(x)}{y(x)} - \dfrac{\tilde{q}(t)}{y(t)}\right]K_+(x,t) \\[2ex] 2y(x)\dfrac{d}{dx}K_+(x,x) + K_+(x,x)\dfrac{d}{dx}y(x) = \tilde{q}(x) - q(x), \end{cases} \tag{15}$$

which are coupled with the conditions

(i) $\lim_{x+t\to\infty} \max\{K_+(x,t), K_{+,x}(x,t), K_{+,t}(x,t)\} = 0$,

(ii) $K_+(x,\cdot), K_{+,x}(x,\cdot), K_{+,t}(x,\cdot), K_{+,xx}(x,\cdot), K_{+,tt}(x,\cdot) \in L^1(\mathbb{R}^+)$.

In a completely analogous way, we can consider the operators $\tilde{L}^-$ and L_q^- defined on $L^2(\mathbb{R}^-, ydx)$, introduce the eigenvalue equations

$$\tilde{L}^- \psi_- = \lambda \psi_-$$

and

$$L_q^- \varphi_- = \lambda \varphi_-,$$

together with the boundary condition

$$\varphi_-(x, \lambda) \sim y(x)^{-1/4} e^{-i\sqrt{\lambda}\mathcal{I}(x)} \quad \text{as} \quad x \to -\infty.$$

In this case, one looks for a kernel $K_-(x, t)$, defined on $\mathcal{D}_- = \{(x, t) \in \mathbb{R} \mid 0 \geq x > t\}$, such that

$$\varphi_-(x, \lambda) = \psi_-(x, \lambda) + \int_{-\infty}^{x} K_-(x, t) y(t) \psi_-(\lambda, t) dt.$$

We can now repeat the argument used before, and conclude that $K_-(x, t)$ satisfies the system

$$\begin{cases} \dfrac{1}{y(x)} K_{-,xx}(x, t) - \dfrac{1}{y(t)} K_{-,tt}(x, t) = \left[\dfrac{q(x)}{y(x)} - \dfrac{\tilde{q}(t)}{y(t)} \right] K_-(x, t) \\ \\ -2y(x) \dfrac{d}{dx} K_-(x, x) - K_-(x, x) \dfrac{d}{dx} y(x) = \tilde{q}(x) - q(x), \end{cases} \tag{16}$$

which are coupled with the conditions

(i) $\lim\limits_{x+t \to -\infty} \max\{K_-(x, t), K_{-,x}(x, t), K_{-,t}(x, t)\} = 0,$

(ii) $K_-(x, \cdot), K_{+,-}(x, \cdot), K_{-,t}(x, \cdot), K_{-,xx}(x, \cdot), K_{-,tt}(x, \cdot) \in L^1(\mathbb{R}^-).$

4 Existence of the Kernel Function

As we already pointed out, we choose to solve equation (12) with a procedure which makes use of basic facts concerning the differential equation

$$-\varphi'' + q\varphi = \lambda y \varphi.$$

Let us thus consider the eigenvalue problem

$$-\varphi_+''(x) + q(x)\varphi_+(x) = \lambda y(x) \varphi_+(x) \tag{17}$$

on $L^2(\mathbb{R}^+, ydx)$. Recall the notation $\mathcal{I}(x) := \int_0^x \sqrt{y(s)} ds$. It will be convenient from now on to write $\lambda = k^2$ $(k \in \mathbb{C})$. A suitable decay condition will be fixed now: we require that

$$\varphi_+(x) \sim y^{-1/4}(x) e^{ik\mathcal{I}(x)} \quad \text{as} \quad x \to \infty. \tag{18}$$

For a fixed $y \in C^1(\mathbb{R})$ which is uniformly continuous and such that $\delta < y(x) < \Delta$, we can define a potential $\tilde{q}(x) = -\left(\dfrac{y'}{4y}\right)' + \left(\dfrac{y'}{4y}\right)^2$, in such a way that $(\tilde{q}, y) \in \mathcal{E}_2$ (see the previous section). We can write the eigenvalue equation (17) as

$$-\varphi_+''(x) + [\tilde{q}(x) - k^2 y(x)]\varphi_+(x) = [\tilde{q}(x) - q(x)]\varphi_+(x). \tag{19}$$

By using the variation of constants formula, a solution of (19) which satisfies (18) can be written as

$$f_+(k,x) = y^{-1/4}(x)\Big[e^{ik\mathcal{I}(x)} + \int_x^\infty \frac{\sin k(\mathcal{I}(t)-\mathcal{I}(x))}{k} \cdot \\ \cdot [\tilde{q}(t)-q(t)]y^{-1/4}(t)f_+(t)dt\Big]. \tag{20}$$

The integral equation (20) is of Volterra-type, and can be solved by successive approximations, as follows:

$$f_+^{(0)}(x) = e^{ik\mathcal{I}(x)}y^{-1/4}(x),$$
$$\vdots$$
$$f_+^{(n)}(x) = y^{-1/4}(x)\int_x^\infty \frac{\sin k(\mathcal{I}(t)-\mathcal{I}(x))}{k}[\tilde{q}(t)-q(t)]y(t)^{-1/4}f_+^{(n-1)}(t)dt$$
$$\vdots$$

If $\Im k > 0$, using the bound $\left|\frac{\sin k}{k}\right| \le \frac{e^{|\Im k|}}{1+|k|}$, we have that the series $\sum_{n=0}^\infty |f_+^{(n)}|$ is bounded by the series

$$Cy^{-1/4}(x)e^{-|\Im k|\mathcal{I}(x)}\sum_{n=0}^\infty \frac{1}{n!}\left[\int_x^\infty \frac{\mathcal{I}(t)|\tilde{q}(t)-q(t)|}{1+|k|\mathcal{I}(t)}dt\right]^n \qquad \Im k > 0, \tag{21}$$

where C is a constant. The series (21) converges absolutely if

$$\int_x^\infty \mathcal{I}(t)|\tilde{q}(t)-q(t)|dt < \infty, \quad \text{for every } x > 0. \tag{22}$$

The Eq. (22) is the *scattering hypotheses* for the problem in the half-line $[0,\infty)$. It expresses a restriction on the potentials $q(x)$ for which a solution of the orthogonalizing Eq. (5) [or, equivalently, of the system (12)] exists.

Equation (21) together with the fact that $0 < \delta \le y(x) \le \Delta < \infty$, tells us that

$$|f_+(k,x)| \le De^{-|\Im k|\mathcal{I}(x)}, \tag{23}$$

where D is a constant. Using (23) in (20), we get

$$|y^{1/4}(x)f_+(k,x) - e^{ik\mathcal{I}(x)}| \le D_1 e^{-|\Im k|\mathcal{I}(x)}\int_x^\infty \mathcal{I}(t)|\tilde{q}(t)-q(t)|dt \qquad \Im k > 0,$$
$$|y^{1/4}(x)f_+(k,x) - e^{ik\mathcal{I}(x)}| \le D_1\frac{e^{-|\Im k|\mathcal{I}(x)}}{|k|}\int_x^\infty |\tilde{q}(t)-q(t)|dt, \quad k \ne 0$$

and hence

$$|(y^{-1/4}(x)f_+(k,x))' - ik\sqrt{y(x)}e^{ik\mathcal{I}(x)}| \le D_2 e^{-|\Im k|\mathcal{I}(x)}\int_x^\infty |\tilde{q}(t)-q(t)|dt,$$

when $\Im k > 0$. These estimates imply the following facts:

1. f_+ is analytic for $\Im k > 0$ and is continuos and bounded for $\Im k = 0$, for every $x \in \mathbb{R}$;

2. one has

$$f_+(k,x) = y^{-1/4}(x)e^{ik\mathcal{I}(x)} + e^{-|\Im k|\mathcal{I}(x)}o(1) \ \text{ as } \ x \to \infty$$
$$f'_+(k,x) = \left[\left(y^{-1/4}(x)\right)' + ik\sqrt{y(x)}\right]e^{ik\mathcal{I}(x)} + e^{-|\Im k|\mathcal{I}(x)}o(1) \ \text{ as } \ x \to \infty.$$

We thus can draw an important further fact concerning the solution $f_+(k,x)$, namely

Lemma 4.1 *Let y and $\tilde{q}$ be defined as usual. Choose a continuous function $q(x)$ such that the scattering Hypotheses (22) holds. Then the solution $f_+(\cdot,x)$ is square-integrable in every horizontal line the upper k-plane, except for $\Im k = 0$, and*

$$\int_{\mathbb{R}} |y^{1/4}(x)f_+(\lambda+i\varepsilon,x) - e^{i\mathcal{I}(x)(\lambda+i\varepsilon)}|^2 d\lambda = O(e^{-2|\varepsilon|(x)}). \tag{24}$$

The behavior of $f_+(k,x)$, expressed in the above Lemma, allows us to use a theorem of Titchmarsh [31]:

Theorem 4.2 (Titchmarsh) *A function $f \in L^2(\mathbb{R})$ is the limit as $y \to 0$ of a function $f(x+iy)$ which is analytic in the upper half-plane and such that*

$$\int_{\mathbb{R}} |f(x+iy)|^2 dx \le O(e^{2\alpha y}), \quad \alpha \in \mathbb{R}$$

if and only if the Fourier transform $\mathcal{F}(f)(t)$ of $f(x)$ vanishes for every $t < -\alpha$, i.e.

$$\int_{\mathbb{R}} f(x)e^{-itx}dx = 0, \quad \textit{for every } \ t < -\alpha.$$

We apply this Theorem to the function

$$h_+(k,x) = y^{1/4}(x)f_+(k,x) - e^{i\mathcal{I}(x)k}.$$

So, for $\Im k > 0$, we have

$$B_+(x,t) := \frac{1}{2\pi}\int_{\mathbb{R}} h_+(k,x)e^{-itk}dk = 0, \quad \text{for all } \ t < \mathcal{I}(x). \tag{25}$$

Inverting (25) we obtain

$$h_+(k,x) = y^{1/4}(x)f_+(k,x) - e^{i\mathcal{I}(x)k} = \int_{\mathcal{I}(x)}^{\infty} B_+(x,t)e^{ikt}dt, \tag{26}$$

or, equivalently,

$$f_+(k,x) = y^{-1/4}(x)\left[e^{ik\mathcal{I}(x)} + \int_{\mathcal{I}(x)}^{\infty} B_+(x,t)e^{ikt}dt\right]. \tag{27}$$

Equation (27) is valid for $\Im k > 0$, and moreover $B_+(x,t)$ is an L^2-function of the variable t, for every $t > \mathcal{I}(x)$. Equation (27) is very similar to the formula (5), when $\psi(x) = y^{-1/4}(x)e^{ik\mathcal{I}(x)}$. In particular, we now express (27) in such a way that it defines a kernel $K_+(x,t)$ which solves the problem (12). To do this, we only have to make a change of variables, obtaining

$$f_+(k,x) = y^{-1/4}(x)e^{ik\mathcal{I}(x)} + \int_x^{\infty} y^{-1/4}(x)B_+(x,\mathcal{I}(t))\sqrt{y(t)}e^{ik\mathcal{I}(t)}dt. \tag{28}$$

Now, we set

$$K_+(x,t) := y^{-1/4}(x)B_+(x,\mathcal{I}(t))y^{-1/4}(t), \tag{29}$$

and finally

$$f_+(k,x) = \psi(x) + \int_x^\infty K_+(x,t)y(t)\psi(t)dt, \quad \psi(x) = y^{-1/4}(x)e^{ik\mathcal{I}(x)}$$

i.e., exactly the formula in (5). Since $B_+(x,t)$ exists for every $t > \mathcal{I}(x)$, then $K_+(x,t)$ exists for $t > x$, and solves the problem (12). The technique is then the following: one looks for $B_+(x,t)$, defines $K_+(x,t)$ as in (29), then uses the second formula in (12) to retrieve $q(x)$.

Another interesting relation can be found from (27). Write it in the form (28), and define A_+ by

$$B_+(x,\mathcal{I}(t)) = \frac{1}{2}A_+\left(x, \frac{\mathcal{I}(t)-\mathcal{I}(x)}{2}\right). \tag{30}$$

Then (28) becomes

$$f_+(k,x) = e^{ik\mathcal{I}(x)}\left[y^{-1/4}(x) + \int_0^\infty A_+(x,t)e^{2ikt}dt\right]. \tag{31}$$

Putting (31) into (20), and taking the inverse Fourier transform, we have

$$\begin{aligned} A_+(x,t) = &-\int_{x+t}^\infty [\tilde{q}(s)-q(s)]y^{-1/2}(s)ds \\ &-\int_0^t dy \int_{x+t-y} [\tilde{q}(s)-q(s)]y^{-1/2}(s)A_+(s,y)ds. \end{aligned} \tag{32}$$

It follows that

$$A_+(x,0) = -\int_x^\infty [\tilde{q}(s)-q(s)]y^{-1/2}(s)ds,$$

hence

$$\frac{d}{dx}A_+(x,0) = [\tilde{q}(x)-q(x)]y^{-1/2}(x),$$

which is another way to retrieve $q(x)$. However, using (29), we have

$$K_+(x,x) = y^{-1/2}(x)B_+(x,\mathcal{I}(x)),$$

and since $A_+(x,0) = B_+(x,\mathcal{I}(x))$, we obtain

$$2y^{1/2}(x)K_+(x,x) = A_+(x,0),$$

and differentiating we have, again,

$$2y(x)\frac{d}{dx}K_+(x,x) + K_+(x,x)\frac{d}{dx}y(x) = \tilde{q}(x) - q(x),$$

i.e., the second formula in (12).

We now move our attention to the *negative* problem, i.e., to that of the operator L_q^- defined on $L^2(\mathbb{R}^-, ydx)$, together with the boundary condition

$$\varphi_-(x,\lambda) \sim \left[\frac{y(x)}{y(0)}\right]^{-1/4} e^{-i\sqrt{\lambda}\mathcal{I}(x)} \text{ as } x \to -\infty.$$

The variation of constant formula gives us the expression

$$f_-(k,t) = y^{-1/4}(x)\Big[e^{-ik\mathcal{I}(x)} + \int_{-\infty}^{x} \frac{\sin(k(\mathcal{I}(x)-\mathcal{I}(t))}{k}\cdot \\ \cdot [\tilde{q}(t)-q(t)]y^{-1/4}(t)f_-(k,t)dt\Big], \tag{33}$$

which is valid for $\Im k > 0$. In this case, estimates similar to those used for the operator L_q^+ lead us to the following condition on $q(x)$:

$$\int_{-\infty}^{x} (1+|\mathcal{I}(t))|\tilde{q}(t)-q(t)|dt < \infty, \quad \text{for every} \;\; x < 0. \tag{34}$$

Equation (34) is the *scattering hypotheses* for the problem on the negative half-line. As soon as one has to consider negative values of t, then (34) is needed. In fact, we now extend $f_+(k,x)$ (resp. $f_-(k,x)$) to negative (resp. positive) values of x. Actually both $f_\pm(k,x)$ continue to be solutions of the equation $L_q\varphi = k^2\varphi$, which now are defined for every $x \in \mathbb{R}$. We write

$$f_+(k,x) = y^{-1/4}(x)\Big[e^{ik\mathcal{I}(x)} + \int_{x}^{\infty} \frac{\sin k(\mathcal{I}(t)-\mathcal{I}(x))}{k}\cdot \\ \cdot [\tilde{q}(t)-q(t)]y^{-1/4}(t)f_+(k,t)dt\Big]$$

and

$$f_-(k,t) = y^{-1/4}(x)\Big[e^{-ik\mathcal{I}(x)} + \int_{-\infty}^{x} \frac{\sin(k(\mathcal{I}(t)-\mathcal{I}(x))}{k}\cdot \\ \cdot [\tilde{q}(t)-q(t)]y^{-1/4}(t)f_-(k,t)dt\Big],$$

which are defined for every $x \in \mathbb{R}$ when (34) holds on all $\mathbb{R}$, i.e., when

$$\int_{-\infty}^{\infty} (1+|\mathcal{I}(t))|\tilde{q}(t)-q(t)|dt < \infty. \tag{35}$$

We finish this section by noting that the Titchmarsh Theorem 4.2 can be used also in the case of the solution $f_-(k,x)$ and gives us a kernel

$$B_-(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} h_-(k,x)e^{itk}dk, \tag{36}$$

which vanishes for all values $t > \mathcal{I}(x)$: here $h_-(k,x) = y^{-1/4}(x)f_-(k,x) - e^{-i\mathcal{I}(x)k}$. Inverting the Fourier transform again, we have

$$f_-(k,x) = y^{-1/4}(x)\left[e^{-ik\mathcal{I}(x)} + \int_{-\infty}^{\mathcal{I}(x)} B_-(x,t)e^{-ikt}dt\right], \tag{37}$$

which in turn allows to recover $K_-(x,t)$ and hence $q(x)$ as above.

5 The Scattering Problem on the Whole Line

We now use the results of the previous section to discuss the scattering problem on the whole line. So, we choose a function $y(x)$ which is continous and such that $\delta < y(x) < \Delta$ for every

$x \in \mathbb{R}$, define

$$\tilde{q}(x) = -\left[\frac{y'}{4y}\right]' + \left[\frac{y'}{4y}\right]^2,$$

and consider the operator $L_q := \frac{1}{y}[-D^q + q]$ defined on $L^2(\mathbb{R}, ydx)$. The potential $q(x)$ is now chosen as to satisfy (35), i.e.,

$$\int_{-\infty}^{\infty} (1 + |\mathcal{I}(t)|)|\tilde{q}(t) - q(t)|dt < \infty,$$

where

$$\mathcal{I}(t) = \int_0^t \sqrt{y(s)}ds.$$

For the moment, we restrict ourselves to non-zero real values of k, and look for solutions $\varphi_\pm(k, x)$ of the equation

$$L_q\varphi = k^2\varphi$$

satisfying

$$\varphi_+(k, x) \sim \begin{cases} y^{-1/4}(x)\left[e^{+ik\mathcal{I}(x)} + a_{12}(k)e^{-ik\mathcal{I}(x)}\right], & x \to -\infty \\ a_{11}(k)y^{-1/4}(x)e^{ik\mathcal{I}(x)}, & x \to +\infty \end{cases} \tag{38}$$

and

$$\varphi_-(k, x) \sim \begin{cases} a_{22}(k)y^{-1/4}(x)e^{-ik\mathcal{I}(x)}, & x \to -\infty \\ y^{-1/4}(x)\left[e^{-ik\mathcal{I}(x)} + a_{21}(k)e^{ik\mathcal{I}(x)}\right], & x \to +\infty \end{cases} \tag{39}$$

In the usual terminology, the coefficients $a_{12}(k)$ and $a_{21}(k)$ are called the *(left/right) reflection coefficients*, while $a_{22}(k)$ and $a_{11}(k)$ are the *(left/right) transmission coefficients*.

We have proved in the previous section that there are well-defined solutions $f_\pm(k, x)$ defined via the integral equations

$$f_+(k, x) = y^{-1/4}(x)\Big[e^{ik\mathcal{I}(x)} + \int_x^{\infty} \frac{\sin k(\mathcal{I}(t) - \mathcal{I}(x))}{k} \cdot \cdot[\tilde{q}(t) - q(t)]y^{-1/4}(t)f_+(k, t)dt\Big]$$

and

$$f_-(k, t) = y^{-1/4}(x)\Big[e^{-ik\mathcal{I}(x)} + \int_{-\infty}^{x} \frac{\sin(k(\mathcal{I}(x) - \mathcal{I}(t))}{k} \cdot \cdot[\tilde{q}(t) - q(t)]y^{-1/4}(t)f_-(k, t)dt\Big].$$

These solutions satisfy the bounds

$$|y^{1/4}(x)f_+(k, x) - e^{ik\mathcal{I}(x)}| \leq D_1\frac{e^{-|\Im k|\mathcal{I}(x)}}{1 + |k|}\int_x^{\infty} (1 + |\mathcal{I}(t)|)|\tilde{q}(t) - q(t)|dt, \tag{40}$$

$$|y^{1/4}(x)f_-(k, x) - e^{-ik\mathcal{I}(x)}| \leq D_1\frac{e^{|\Im k|\mathcal{I}(x)}}{1 + |k|}\int_{-\infty}^{x} (1 + |\mathcal{I}(t)|)|\tilde{q}(t) - q(t)|dt, \tag{41}$$

where D_1 is a constant.

The function $f_+(k,x)$ is analytic in the upper k-plane, and moreover

$$|y^{1/4}(x)f_+(k,x)-e^{ik\mathcal{I}(x)}|=O\left(\frac{1}{|k|}\right)\quad\text{as}\quad |k|\to\infty.$$

Analoguosly, $f_-(k,x)$ is analytic in the upper k-plane and

$$|y^{1/4}(x)f_-(k,x)-e^{-ik\mathcal{I}(x)}|=O\left(\frac{1}{|k|}\right)\quad\text{as}\quad |k|\to\infty.$$

Theorem 4.2 applies to both $f_\pm(k,x)$ and defines kernels $B_\pm(x,t)$ such that

$$f_+(k,x)=y^{-1/4}(x)\left[e^{ik\mathcal{I}(x)}+\int_{\mathcal{I}(x)}^{\infty}B_+(x,t)e^{ikt}dt\right],$$

and

$$f_-(k,x)=y^{-1/4}(x)\left[e^{-ik\mathcal{I}(x)}+\int_{-\infty}^{\mathcal{I}(x)}B_-(x,t)e^{-ikt}dt\right].$$

These kernels $B_\pm(x,t)$ define in turn kernels $K_\pm(x,t)$ such that

$$f_+(k,x)=y^{-1/4}(x)e^{ik\mathcal{I}(x)}+\int_x^{\infty}K_+(x,t)y(t)y^{-1/4}(t)e^{ik\mathcal{I}(t)}dt \tag{42}$$

and

$$f_-(k,x)=y^{-1/4}(x)e^{-ik\mathcal{I}(x)}+\int_{-\infty}^{x}K_-(x,t)y(t)y^{-1/4}(t)e^{-ik\mathcal{I}(t)}dt. \tag{43}$$

Our aim is to understand to what extent the coefficients $a_{ij}(k)$ $(i,j=1,2)$ determine the potential $q(x)$. The potential $q(x)$ determines uniquely the coefficients $a_{ij}(k)$, hence there must be some relation between the coefficients $a_{ij}(k)$. However, even if it is true that the potential $q(x)$ determines all the $a_{ij}(k)$, we will see that it does only when connected to the half-line operators $L_q^\pm$, and the coefficients $a_{ij}(k)$ must have some intrinsic dependence.

We study some properties of the coefficients $a_{ij}(k)$ for **non-zero real values of** k. First of all, the pairs $f_+(k,x)$, $f_+(-k,x)$ and $f_-(k,x)$, $f_-(-k,x)$ are linearly independent solutions of the equation $L_q\varphi=k^2\varphi$, and in fact

$$W[f_+(k,x),\,f_+(-k,x)]=-2ik$$

and

$$W[f_-(k,x),\,f_-(-k,x)]=2ik.$$

Thus, it is possible to write

$$\begin{cases} f_-(k,x)=b_{11}(k)f_+(k,x)+b_{12}(k)f_+(-k,x)\\[2ex] f_+(k,x)=b_{22}(k)f_-(k,x)+b_{21}(k)f_-(-k,x). \end{cases} \tag{44}$$

These last two relations imply immediately that

$$\begin{aligned} b_{11}(k)b_{22}(k)+b_{12}(-k)b_{21}(k)&=1\\ b_{12}(k)b_{22}(k)+b_{21}(k)b_{11}(-k)&=0\\ b_{11}(k)b_{22}(k)+b_{12}(k)b_{21}(-k)&=1 \end{aligned}$$

$$b_{11}(k)b_{21}(k) + b_{12}(k)b_{22}(-k) = 0.$$

Moreover, we have

$$W[f_+(k,x),\, f_-(k,x)] = -2ikb_{12}(k) = -2ikb_{21}(k),$$
$$W[f_-(k,x),\, f_+(-k,x)] = -2ib_{11}(k)$$

and

$$W[f_-(-k,x),\, f_+(k,x)] = -2ikb_{22}(k).$$

All these relations imply that (if $k \neq 0$ is a real number)

1. $b_{11}(k) = -b_{22}(-k)$
2. $b_{12}(k) = b_{21}(k)$
3. $|b_{12}(k)|^2 = 1 + |b_{11}(k)|^2 = 1 + |b_{22}(k)|^2$
4. $b_{ij}(k) = \overline{b_{ij}(-k)}$.

Now, as $x \to +\infty$,

$$\begin{aligned} f_-(k,x) &= y^{-1/4}(x)\left[e^{-ik\mathcal{I}(x)} + \int_{-\infty}^{\infty} \frac{\sin k(\mathcal{I}(x) - \mathcal{I}(t))}{k} y^{-1/4}(t)[\tilde{q}(t) - q(t)]f_-(k,t)dt\right] \\ &= y^{-1/4}(x)\left[e^{-ik\mathcal{I}(x)} + \frac{e^{ik\mathcal{I}(x)}}{2ik}\int_{-\infty}^{\infty} e^{-ik\mathcal{I}(t)}y^{-1/4}(t)[\tilde{q}(t) - q(t)]f_-(k,t)dt\right] \\ &\quad - y^{-1/4}(x)\frac{e^{-ik\mathcal{I}(x)}}{2ik}\int_{-\infty}^{\infty} e^{ik\mathcal{I}(t)}y^{-1/4}(t)[\tilde{q}(t) - q(t)]f_-(k,t)dt. \end{aligned}$$

Now, since $f_-(k,x) = b_{11}(k)f_+(k,x) + b_{12}(k)f_+(-k,x)$, we obtain

$$b_{11}(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{-ik\mathcal{I}(t)}y^{-1/4}(x)[\tilde{q}(t) - q(t)]f_-(k,t)dt$$

and

$$b_{12}(k) = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} e^{ik\mathcal{I}(t)}y^{-1/4}(t)[\tilde{q}(t) - q(t)]f_-(k,t)dt.$$

Analogously,

$$b_{21}(k) = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} e^{-ik\mathcal{I}(t)}y^{-1/4}(t)[\tilde{q}(t) - q(t)]f_+(k,t)dt$$

ands

$$b_{22}(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{ik\mathcal{I}(t)}y^{-1/4}(t)[\tilde{q}(t) - q(t)]f_+(k,t)dt.$$

This facts, together with the asymptotic behaviour of $f_\pm(k,x)$, prove the following

Lemma 5.1 *For nonzero real values of k, the coefficients $b_{ij}(k)$ satisfy the following asymptotics*

$$b_{11}(k), b_{22}(k) \sim \frac{o(1)}{k} \quad as \quad |k| \to \infty,$$

and

$$b_{12}(k) = b_{21}(k) \sim 1 + \frac{\alpha}{k} + \frac{o(1)}{k} \quad as \quad |k| \to \infty,$$

where

$$\alpha = -\frac{1}{2i}\int_{-\infty}^{\infty} y^{-1/2}(t)[\tilde{q}(t) - q(t)]dt.$$

Moreover, $b_{12}(k)$ admits an analytic extension in the upper k-plane, and $kb_{12}(k)$ is bounded and continuous on the real line.

We now obtain some results concerning the zeros of $b_{12}(k)$.

Lemma 5.2 *Let $b_{12}(k)$ be the analytic extension in the upper half-plane of the coefficient $b_{12}(k)$ defined in (44). Then the zeros of $b_{12}(k)$ are finite, simple and purely imaginary. Each zero of $b_{12}(k)$ is an eigenvalue of the equation $-\varphi'' + q\varphi = k^2 y\varphi$.*

Proof First of all, if $k \in \mathbb{R}$, then $b_{12}(k)$ cannot vanish, since $f_+(k, x)$ and $f_-(k, x)$ are linearly independent (or, by property 3. of the coefficients $b_{ij}(k)$). Thus, the zeros of $b_{12}(k)$ must have non-zero imaginary part. Let k_0 be such a zero: $b_{12}(k_0) = 0$ and $\Im k_0 \neq 0$. Then $f_+(k_0, x)$ and $f_-(k_0, x)$ are linearly dependent, that is, $f_+(k_0, x) = cf_-(k_0, x)$ for some constant c. But now $f_+(k_0, x)$ decays exponentially as $x \to +\infty$, while $f_-(k_0, x)$ decays exponentially as $x \to \infty$. This implies that $f_+(k_0, x)$ (and hence $f_-(k_0, x)$) is a square integrable solution of $L_q\varphi = k^2\varphi$ on all $\mathbb{R}$, hence k_0 belongs to the (point) spectrum of L_q on the line. Since L_q is self-adjoint, the spectrum of L_q is a subset of the real line, hence k_0^2 is real and k_0 has zero real part. We have proved that the zeros of $b_{12}(k)$ are purely imaginary. Next, $b_{12}(k)$ cannot vanish for large values of $|k|$, in view of its asymptotic behavior. Now, we show that these zeros are in fact finite. It suffices to show that the solutions of the equation $L_q\varphi = 0$ can have only a finite number of zeros. For, it is clear that the function $\rho(x) := y^{-1/4}(x) + \int_x^\infty K_+(x, t)y^{-1/4}(t)dt$ is a solution of $L_q\varphi = 0$. Another solution is given by

$$\eta(x) = \rho(x)\int_a^x \frac{dt}{\rho^2(t)},$$

where $a > 0$ is chosen in such a way that $\rho(x) \neq 0$ for $x > a$. Now, if $x \to +\infty$, $\rho(x) \geq \inf y^{-1/4}(x) = \delta_1 > 0$, hence

$$\eta(x) \geq \delta_1 x + o(x).$$

Since every solution φ of $L_q\varphi = 0$ is a linear combination of $\rho(x)$ and $\eta(x)$, it follows that φ itself can have only a finite number of zeros. Summarizing, the zeros (if any) of $b_{12}(k)$ are finite and purely imaginary. If k_0 is such a zero, then k_0^2 is a (real) eigenvalue λ_0 of $L_q\varphi = \lambda\varphi$. But then λ_0 must be negative, hence $k_0 = i\chi_0$, where $\chi_0 = \sqrt{|k_0|}$. It remains to prove that the zeros of $b_{12}(k)$ are simple. Assume for contradiction that $k_0 = i\chi_0$ is a multiple zero fo $b_{12}(k)$. Then $-\chi_0^2$ is a negative multiple eigenvalue of $L_q\varphi = k^2\varphi$ on the whole line. This implies that all the solutions of $L_q\varphi = \lambda\varphi$ are square integrable on all $\mathbb{R}$ for all λ. However $f_+(i\chi_0, x)$ is a solution which decays exponentially at both $\pm\infty$, and another solution of the same problem is given by

$$g(i\chi_0, x) = f_+(i\chi_0, x)\int_a^x \frac{dt}{f_+^2(i\chi_0, t)}.$$

It is then easy to show that $g(i\chi_0, x)$ increases exponentially as $|x| \to \pm\infty$, a contradiction, hence $i\chi_0$ must be a simple zero of $b_{12}(k)$, i.e., $-\chi_0^2$ is a simple eigenvalue of $L_q\varphi = k^2\varphi$ on the whole line. □

The relations occurring between the coefficients $b_{ij}(k)$ imply that actually the knowledge of $b_{11}(k)$ and $b_{12}(k)$ determines also $b_{22}(k)$ and $b_{21}(k)$ uniquely, and vice-versa (actually, we can say more, as we will see shortly). Now, we relate the coefficients $b_{ij}(k)$ to the reflection/transmission coefficients $a_{ij}(k)$. This is, however, quite easy: indeed, it suffices to look at the asymptotic behavior of $\varphi_\pm(k, x)$ and compare to that of $f_\pm(k, x)$. We have

$$f_+(k, x) \sim y^{-1/4}(x)e^{ik\mathcal{I}(x)} \text{ as } x \to +\infty$$

while

$$f_-(k, x) \sim y^{-1/4}(x)e^{-ik\mathcal{I}(x)} \text{ as } x \to -\infty.$$

This implies that

$$\varphi_+(k, x) \sim f_-(-k, x) + a_{12}(k)f_+(k, x) = a_{11}(k)\left[b_{22}(k)f_-(k, x) + b_{21}(k)f_-(-k, x)\right]$$

as $x \to \infty$, hence

$$a_{11}(k)b_{22}(k) = a_{12}(k)$$

and

$$a_{11}(k)b_{21}(k) = 1,$$

and so

$$a_{11}(k) = \frac{1}{b_{21}(k)}, \qquad a_{12}(k) = \frac{b_{22}(k)}{b_{21}(k)}.$$

We can repeat the same arguments for the function $\varphi_-(k, x)$, to the effect that

$$a_{22}(k) = \frac{1}{b_{12}(k)}, \qquad a_{21}(k) = \frac{b_{11}(k)}{b_{12}(k)}.$$

Summarizing, we have

(i)

$$a_{11}(k) = a_{22}(k) = \frac{1}{b_{12}(k)}$$

(ii)

$$a_{12}(k) = \frac{b_{22}(k)}{b_{12}(k)}$$

(iii)

$$a_{21}(k) = \frac{-b_{22}(-k)}{b_{12}(k)}.$$

(iv)

$$a_{21}(k) = -\frac{a_{12}(-k)a_{11}(k)}{a_{11}(-k)}.$$

(v) Using the property 3. of the coefficients $b_{ij}(k)$, we have

$$|a_{11}(k)| = \left[1 - |a_{12}(k)|^2\right]^{1/2},$$

hence, in particular,

$$|a_{12}(k)| < 1.$$

(vi)

$$a_{ij}(-k) = \overline{a_{ij}(k)}.$$

The last properties (v) and (vi) have a very important consequence. Since $a_{11}(k)$ is meromorphic in the upper half-plane, and $\ln a_{11}(k) \to 0$ as $|k| \to \infty$, it is possible to reconstruct $a_{11}(k)$ is the upper half-plane by the values of its absolute value in the real line and its poles, by means of the so-called Hilbert dispersion relations (see [7,31]). In this case, we obtain

$$\begin{cases} a_{11}(k) = \exp\left[\dfrac{1}{2\pi i}\displaystyle\int_{\mathbb{R}} \dfrac{\ln(1-|a_{12}(t)|^2)}{t-k} dt\right] \displaystyle\prod_{i=1}^{n} \dfrac{k+i\chi_j}{k-i\chi_j}, & \Im k > 0 \\ a_{11}(k) = \displaystyle\lim_{\varepsilon\to 0} a_{11}(k+i\varepsilon), & k \in \mathbb{R} \end{cases} \tag{45}$$

The integral above has to be intended in the sense of Cauchy principal value. The relation (45) allows to express in a similar manner the coefficient $b_{11}(k)$, once $b_{12}(k)$ is known.

Note further that $a_{11}(k)$ and $a_{12}(k)$ are well-defined for real values of k, because $b_{12}(k)$ can only have purely imaginary zeros. The above relations allow also to extend the coefficients $a_{ij}(k)$ to complex values of k with $\Im k > 0$, and all these quantities have simple poles at the zeros of $b_{12}(k)$, i.e., at those values which determine the negative eigenvalues of the equation $L_q\varphi = \lambda\varphi$ on the whole line. Once the coefficients $a_{ij}(k)$ are defined for complex values of k, also the functions $\varphi_\pm(k, x)$ are defined except at $k = 0$ and at the the points $k_j = i\chi_j$ ($j = 1, \dots, n$) which are the zeros if $b_{12}(k)$. We also notice that an argument, similar to that explained in the above lines to prove the finiteness of the eigenvalues, can be used to show that $\lambda = k^2 = 0$ is not an eigenvalue of $L_q\varphi = \lambda\varphi$.

Notice that the properties (i)–(vi) of the coefficients $a_{ij}(k)$ together with (45) imply that the coefficients $a_{ij}(k)$ are completely determined once one of then, say $a_{12}(k)$, is given. The asymptotic properties expressed in Lemma 5.1 transfer to the coefficients $a_{ij}(k)$, with the following effect:

Lemma 5.3 *The coefficients $a_{ij}(k)$ possess the following properties:*

(i) *$a_{ij}(k)$ are continuous for $k \in \mathbb{R}$, except at most at $k = 0$. However*

$$\lim_{k\to 0} a_{12}(k) = \lim_{k\to 0} a_{21}(k) = -1$$

and

$$\lim_{k\to 0} a_{11}(k) = \lim_{k\to 0} a_{22}(k) = 0$$

(ii) *The function $ka_{12}(k)$ is bounded and continuous on all $\mathbb{R}$.*
(iii) *For large values of $|k|$,*

$$a_{12}(k) = O\left(\frac{1}{k}\right), \quad a_{12}(k) = O\left(\frac{1}{k}\right),$$

and

$$a_{11}(k) = a_{22}(k) = 1 + O\left(\frac{1}{k}\right).$$

We now move our attention to the basic integral equation of the inverse problem for the scattering theory on the whole line for the operator L_q. We first recall Eqs. (42) and (43):

$$f_+(k,x) = y^{-1/4}(x)e^{ik\mathcal{I}(x)} + \int_x^\infty K_+(x,t)y^{3/4}(t)e^{ik\mathcal{I}(t)}dt$$

and

$$f_-(k,x) = y^{-1/4}(x)e^{-ik\mathcal{I}(x)} + \int_{-\infty}^x K_-(x,t)y^{3/4}(t)e^{-ik\mathcal{I}(t)}dt.$$

We express these equations in the more convenient form given by (31) and its analogous for K_-, i.e.,

$$f_+(k,x) = e^{ik\mathcal{I}(x)}\left(y^{-1/4}(x) + \int_0^\infty A_+(x,s)e^{2iks}ds\right).$$

In a completely analogous way, $f_-(k,x)$ can be represented by the formula

$$f_-(k,x) = e^{-ik\mathcal{I}(x)}\left(y^{-1/4}(x) + \int_{-\infty}^0 A_-(x,s)e^{-2iks}ds\right).$$

Moreover,

$$\frac{d}{dx}A_+(x,0) = -\frac{d}{dx}A_-(x,0) = [\tilde{q}(x) - q(x)]y^{-1/2}(x) \tag{46}$$

Now, it is easily seen that

$$a_{11}(k)f_+(k,x) = a_{12}(k)f_-(k,x) + f_-(-k,x)$$

and

$$a_{22}f_-(k,x) = a_{21}(k)f_+(k,x) + f_+(-k,x).$$

Write

$$\begin{cases} h_+(k,x) = y^{1/4}(x)e^{-ik\mathcal{I}(x)}f_+(k,x), \\ h_-(k,x) = y^{1/4}(x)e^{ik\mathcal{I}(x)}f_-(k,x), \\ g_+(k,x) = a_{22}(k)y^{1/4}(x)e^{ik\mathcal{I}(x)}f_-(k,x), \\ g_-(k,x) = a_{11}(k)y^{1/4}(x)e^{-ik\mathcal{I}(x)}f_+(k,x). \end{cases}$$

These functions are related as follows:

$$g_+(k,x) = a_{21}(k)e^{2ik\mathcal{I}(x)}h_+(k,x) + h_+(-k,x), \tag{47}$$

$$g_-(k,x) = a_{12}(k)e^{-2ik\mathcal{I}(x)}(k)h_-(k,x) + h_-(-k,x). \tag{48}$$

Now, write

$$a_{21}(k) = \int_{-\infty}^\infty C_+(t)e^{-2ikt}dt.$$

For the equation defining $h_+(k,x)$, we have

$$h_+(k,x) = 1 + y^{1/4}(x)\int_0^\infty A_+(x,s)2^{2iks}ds = 1 + y^{1/4}(x)\int_{-\infty}^\infty A_+(x,s)e^{2iks}ds,$$

where the last equality follows from the Paley–Weiner Theorem ($A_+(x,s) = 0$ for $s < 0$ because $h_+(k,x)$ is analytic in the upper k-plane). Similarly, set

$$g_+(k,x) = 1 + \int_{-\infty}^\infty \tilde{A}_+(x,s)e^{2iks}ds.$$

With these notation, the Fourier transform of (47) translates to

$$\begin{aligned} C_+(\mathcal{I}(x)+t) + y^{1/4}(x)\int_{-\infty}^\infty A_+(x,s)C_+(\mathcal{I}(x)+s+t)ds \\ + A_+(x,t) = \tilde{A}_+(x,-t). \end{aligned} \tag{49}$$

Formula (49) is, however, a little bit implicit, in the sense that it does not show explicitely how and where the scattering data determine the kernel $A_+(x,t)$. To understand this, let us analyze first the residue of the functions $a_{11}(k)$ at one of its poles $i\chi_j$: since $a_{11}(k) = \frac{1}{b_{12}(k)}$, then $\mathrm{Res}\, a_{11}(i\chi_j) = \frac{1}{\dot{b_{12}}(i\chi_j)}$, where the symbol $\dot{}$ denotes the complex derivative. Now, $b_{12}(k) = -\frac{1}{2ik}W[f_+(k,x), f_-(k,x)]$, so that $(2ikb_{12}(k))\dot{} = (W[f_+(k,x), f_-(k,x)])\dot{}$ and

$$2ib_{12}(k) + 2ik\dot{b_{12}}(k) = -W[\dot{f_+}(k,x), f_-(k,x)] - W[f_+(k,x), \dot{f_-}(k,x)].$$

At a zero $k_j = i\chi_j$ of $b_{12}(k)$ we have

$$-2\chi_j\dot{b_{12}}(i\chi_j) = -W[\dot{f_+}(i\chi_j,x), f_-(i\chi_j,x)] - W[f_+(i\chi_j,x), f_-(i\chi_j,x)\dot{}]. \tag{50}$$

Now differentiating with respect to k the equation

$$(a) \qquad -f''_\pm + qf_\pm = k^2 y f_\pm,$$

we have

$$(b) \qquad \dot{f''_\pm} + q\dot{f_+} = 2kyf_\pm + k^2 y\dot{f_\pm}.$$

Multiplying (a) by $\dot{f_\pm}$, (b) by $f_\pm$ and subtracting the formulas, we obtain

$$-\dot{f_\pm}f''_\pm + \dot{f''_\pm}f_+ = -2kyf_\pm^2.$$

The left-hand side of the above relation equals the quantity $\frac{d}{dx}W[\dot{f_\pm}, f_\pm]$, hence

$$\frac{d}{dx}W[\dot{f_\pm}, f_\pm] = -2kyf_\pm^2.$$

Since $f_+(i\chi_j,x) = c_j f_-(i\chi_j,x)$ (c_j are constants), the relation (50) together with the above formula involving the Wronskian give

$$\begin{aligned} i\dot{b_{12}}(i\chi_j) &= -\int_{-\infty}^\infty y(x)f_+(i\chi_j,x)f_-(i\chi_j,x)dx = -c_j\int_{-\infty}^\infty y(x)f_-^2(i\chi_j,x)dx \\ &= -\frac{1}{c_j}\int_{-\infty}^\infty y(x)f_+^2(i\chi_j,x)dx \end{aligned} \tag{51}$$

We can now compute the residue of $a_{11}(h)$ at $i\chi_j$:

$$\operatorname{Res} a_{11}(i\chi_j) = \left[i \int_{-\infty}^{\infty} f_+(i\chi_j, x) f_-(i\chi_j, x) y(x) dx \right]^{-1} =: i\gamma_j. \tag{52}$$

Set

$$M_j^{\pm} = \int_{-\infty}^{\infty} f_{\pm}^2(i\chi_j, x) y(x) dx.$$

Using (52) together with the fact that $f_+(i\chi_j, x) = \beta_j f_(i\chi_j, x)$, we have

$$\gamma_j^2 = \frac{1}{M_j^+ M_j^-}, \quad j = 1, \ldots, n. \tag{53}$$

Since $g_+(k, x) = a_{22}(k) y^{1/4}(x) e^{ik\mathcal{I}(x)} f_-(k, x)$, we have

$$\operatorname{Res} g_+(i\chi_j, x) = y^{1/4}(x) e^{-\chi_j \mathcal{I}(x)} f_-(i\chi_j, x) \operatorname{Res} a_{11}(i\chi_j) = -i \frac{e^{-2\chi_j \mathcal{I}(x)} h_+(i\chi_j, x)}{M_j^+}.$$

Analogously,

$$\operatorname{Res} g_-(i\chi_j, x) = -i \frac{e^{2\chi_j \mathcal{I}(x)} h_-(i\chi_j, x)}{M_j^-}.$$

Now, the function $g_+(k, x)$ is meromorphic in the upper k-plane, having simple poles at the values $i\chi_j$ ($j = 1, \ldots, n$). It follows that, for $s < 0$,

$$\begin{aligned} \tilde{A}(x, s) &= -i \sum_{j=1}^{n} \operatorname{Res} g_+(i\chi_j, x) e^{2\chi_j s} \\ &= -\sum_{j=1}^{n} \frac{e^{-2\chi_j(\mathcal{I}(x)-s)}}{M_j^+} \left(1 + y^{1/4}(x) \int_0^{\infty} A_+(x, s) e^{2iks} ds \right). \end{aligned} \tag{54}$$

Recalling that $A_+(x, s) = 0$ for $s < 0$ the main equation (49) can be written as

$$\Omega_+(\mathcal{I}(x) + t) + y^{1/4}(x) \int_0^{\infty} A_+(x, s) \Omega_+(\mathcal{I}(x) + s + t) ds + A_+(x, t) = 0, \ (t > 0) \tag{55}$$

where

$$\Omega_+(\tau) = C_+(\tau) + \sum_{j=1}^{n} \frac{e^{-2\chi_j \tau}}{M_j^+}.$$

We can now repeat the same arguments used above for the function in (48), and obtain

$$\Omega_-(\mathcal{I}(x) + t) + y^{1/4}(x) \int_{-\infty}^{0} A_-(x, s) \Omega_-(\mathcal{I}(x) + t + s) ds + A_-(x, t) = 0, \ (t < 0) \tag{56}$$

where

$$\begin{aligned} h_-(k, x) &= 1 + y^{1/4}(x) \int_{-\infty}^{\infty} A_-(x, s) e^{-2iks} ds, \\ a_{12}(k) &= \int_{-\infty}^{\infty} C_-(t) e^{2ikt} dt, \end{aligned}$$

and

$$\Omega_-(\tau) = C_-(\tau) + \sum_{j=1}^{n} \frac{e^{2\chi_j \tau}}{M_j^-}.$$

Formulas (55) and (56) are the basic equation of the inverse scattering problem of the operator $L_q = \frac{1}{y}[-D^2 + q]$ on the whole line. Once we find a solution of (55) and (56), we can use the relation (46) to reconstruct $q(x)$. We state this problem in a precise manner.

Let $y(x)$ be a fixed function satisfying the usual hypotheses we used throughout this paper. We want to determine which and how many spectral parameters have to be assigned in order to determine uniquely $q(x)$ and the transformation operators $K_\pm(x,t)$. As we already observed before, the coefficients $a_{ij}(k)$ are uniquely determined when one of them, say $a_{12}(k)$, is given of the real line, together with a set of poles $i\chi_1, \dots, i\chi_n$ ($\chi_1, \dots, \chi_n > 0$). Moreover, to determine $q(x)$, equations of the type (55) and (56) are necessary, which also depend on the coefficients M_j^+ and M_j^-. However, we showed that once the values M_j^+ are given, then M_j^- can be recovered via the relation (53). These observations explain which is the suitable set of parameters which has to be assigned in order to solve the inverse scattering problem. We define the scattering set as

$$\mathcal{S} = \{a_{12}(k), \chi_1, \dots, \chi_n, M_j^+, \dots, M_j^-\}, \tag{57}$$

and formulate the inverse problem in the following theorem

Theorem 5.4 *Let $a_{12}(k)$ $(-\infty < k < +\infty)$ be a function satisfying*

1. $a_{12} \in L^1(\mathbb{R})$.
2. *The function $ka_{12}(k)$ is bounded and continuous on all $\mathbb{R}$.*
3. $a_{12}(k) = O\left(\frac{1}{k}\right)$ *as* $|k| \to +\infty$.
4. $|a_{12}(k)| < 1$ *for all* $k \in \mathbb{R}$ *and*

$$\lim_{k\to 0} a_{12}(k) = -1.$$

5. $a_{12}(-k) = \overline{a_{12}(k)}$ *for all* $k \in \mathbb{R}$.

Choose positive numbers $\chi_1, \dots, \chi_n$ and $M_1^+, \dots, M_n^+$. Define the scattering set

$$\mathcal{S} = \{a_{12}(k), \chi_1, \dots, \chi_n, M_1^+, \dots, M_-^n\}.$$

Assume further that the Fourier transform of $a_{12}(k)$ and its derivative are in $L^1(\mathbb{R})$. Let $a_{11}(k)$ be the meromorphic function, defined in the upper half-plane, as in (45) (the integral has to be intended in the sense of Cauchy principal value):

$$\begin{cases} a_{11}(k) = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\ln(1-|a_{12}(t)|^2)}{t-k}dt\right]\prod_{i=1}^{n}\frac{k+i\chi_j}{k-i\chi_j}, & \Im k > 0 \\ a_{11}(k) = \lim_{\varepsilon\to 0} a_{11}(k+i\varepsilon), & k \in \mathbb{R} \end{cases}$$

Let $i\gamma_1, \dots, i\gamma_n$ be the residues of $a_{11}(k)$ at the points $i\chi_1, \dots, i\chi_n$, and let

$$M_j^- = \frac{1}{\gamma_j^2 M_j^+}, \quad j = 1, \dots, n.$$

Let

$$a_{21}(k) = -\frac{a_{12}(-k)a_{11}(k)}{a_{11}(-k)}.$$

Then the Fourier transform of $a_{21}(k)$ *is well defined and is in* $L^1(\mathbb{R})$*, together with its derivative. Finally set* $a_{22}(k) = a_{11}(k)$*. Let us define* $C_\pm(t)$ *via the relations*

$$a_{21}(k) = \int_{-\infty}^{\infty} C_+(t)e^{-2ikt}dt$$

and

$$a_{12}(k) = \int_{-\infty}^{\infty} C_-(t)e^{2ikt}dt.$$

Let $y(x)$ *be a uniformly continuous functions such that* $-\infty < \delta < y(x) < \Delta < \infty$*, and set*

$$\mathcal{I}(x) = \int_0^x \sqrt{y(s)}ds, \qquad \tilde{q}(x) = -\left(\frac{y'(x)}{4y(x)}\right) + \left(\frac{y'(x)}{4y(x)}\right)^2.$$

Let $A_\pm(x,t)$ *be the solutions of the integral equations*

$$A_+(x,t) + \Omega_+(\mathcal{I}(x)+t) + y^{1/4}(x)\int_0^{\infty} A_+(x,s)\Omega_+(\mathcal{I}(x)+t+s)ds = 0, \qquad (t>0)$$

and

$$A_-(x,t) + \Omega_-(\mathcal{I}(x)+t) + y^{1/4}(x)\int_{-\infty}^{0} A_-(x,s)\Omega_-(\mathcal{I}(x)+t+s)ds = 0, \qquad (t<0).$$

Define

$$K_+(x,t) = \frac{1}{2}y^{-1/4}(t)A_+\left(x, \frac{\mathcal{I}(t)-\mathcal{I}(x)}{2}\right)y^{-1/4}(x), \qquad (t>x)$$

and

$$K_-(x,t) = \frac{1}{2}y^{-1/4}(t)A_-\left(x, \frac{\mathcal{I}(t)-\mathcal{I}(x)}{2}\right)y^{-1/4}(x), \qquad (t<x).$$

Let $q_+(x)$ *be defined by*

$$\tilde{q}(x) - q_+(x) = 2y(x)\frac{d}{dx}K_+(x,x) + K_+(x,x)\frac{d}{dx}y(x)$$

and $q_-(x)$ *be defined analogously by*

$$\tilde{q}(x) - q_-(x) = -2y(x)\frac{d}{dx}K_-(x,x) - K_-(x,x)\frac{d}{dx}y(x).$$

Then the functions

$$f_\pm(x,k) = y^{-1/4}(x)e^{\pm ik\mathcal{I}(x)} \pm \int_x^{\pm\infty} K_\pm(x,t)y^{3/4}(t)e^{\pm ik\mathcal{I}(t)}dt \qquad (\Im k > 0)$$

are well defined and satisfy the differential equations

$$-f_\pm''(x,k) + q_\pm(x)f_\pm(x,k) = k^2 y(x)f_\pm(x,k).$$

Moreover, $q_+(x) = q_-(x) := q(x)$ *on all* $\mathbb{R}$ *and*

$$\int_{\mathbb{R}} (1 + |\mathcal{I}(x)|)|\tilde{q}(x) - q(x)|dx < \infty.$$

Before proving the theorem, let us make an important observation concerning the last statement of the theorem: the functions $A_\pm(x, t)$ define kernels $K_\pm(x, t)$, which in turn are used to construct two functions $q_\pm(x)$. These functions are related to Sturm–Liouville problems L_{q_+} and L_{q_-} defined on $L^2(\mathbb{R}^+, ydx)$ and $L^2(\mathbb{R}^-, ydx)$ respectively. At a first glance, although $q_+(x)$ and $q_-(x)$ are defined on all $\mathbb{R}$, it is not obvious that they coincide.

Proof Let the scattering set $\mathcal{S} = \{a_{12}(k), \chi_1, \ldots, i\chi_n, M_1^+, \ldots, M_n^-\}$ be given. It is a standard fact that the solutions $A_\pm(x, t)$ exist and are unique for all $x \in \mathbb{R}$. Moreover, $C_\pm, C'_\pm \in L^1(\mathbb{R})$.

Let us first prove that, if $q_+(x)$ is constructed via the procedure explained in the statement of the theorem, then it satisfies

$$\int_{\mathbb{R}} |\mathcal{I}(x)||\tilde{q}(x) - q_+(x)|dx < \infty.$$

Introduce the notation

$$\alpha(x) = \int_x^\infty |\Omega'_+(s)|ds.$$

Note that $\alpha(0) < \infty$ and that $|\Omega_+(x)| < \alpha(x)$. It follows that also

$$\int_0^\infty |A_+(x, t)|dt \le \rho(x),$$

where $\rho(x)$ is bounded, and

$$|A_+(x, t)| \le \alpha(\mathcal{I}(x) + t)\,[1 + C\rho(x)]\,.$$

Now, observe that

$$A_+(x, 0) = -\int_x^\infty [\tilde{q}(s) - q(x)]y^{-1/2}(s)ds.$$

Differentiating with respect to x, we can write

$$\begin{aligned} |A'_+(x, t)| \le |\Omega'_+(\mathcal{I}(x) + t)| &+ \frac{1}{4}y'y^{-3/4}(x)\int_0^\infty |A_+(x, s)||\Omega_+(\mathcal{I}(x) + t + s)|ds \\ &+ y^{1/4}(x)\int_0^\infty |A'_+(x, s)||\Omega_+(\mathcal{I}(x) + t + s)|ds \\ &+ y^{1/4}(x)\int_0^\infty |A_+(x, s)||\Omega'_+(\mathcal{I}(x) + t + s)|ds \end{aligned}$$

Using the above estimates, we can show that

$$|A'_+(x, t)| = |\tilde{q}(x) - q_+(x)|y^{-1/2}(x) \le C_1\int_x^\infty |\Omega'_+(\mathcal{I}(x))|dx + C_2\alpha^2(\mathcal{I}(x))[C_3 + C_4\rho(x)].$$

Multiplying by $\mathcal{I}(x)y^{1/2}(x)$ and integrating, we have

$$\int_0^\infty \mathcal{I}(x)|\tilde{q}(x) - q_+(x)|dx \le D\int_0^\infty \mathcal{I}(x)|\Omega'_+(\mathcal{I}(x))|dx < \infty,$$

where D is a constant. Since the integral in the right-hand side is finite, we obtain the scattering condition for q in the positive half-line. Analogously, we can prove the condition in the negative half line, hence the condition on the whole line.

The next step is to prove that $K_\pm(x,t)$ satisfy the differential equations

$$\frac{1}{y(x)}K_{\pm,xx}(x,t)-\frac{1}{y(t)}K_{\pm,tt}(x,t)=\left[\frac{q(x)}{y(x)}-\frac{\tilde{q}(t)}{y(t)}\right]K_\pm(x,t)$$

which in turn implies the remaining statements of the theorem, except for the fact that $q_+(x)=q_-(x)$, which will be the last step of the proof.

We prove that the differential equation for $K_+(x,t)$ is satisfied. The proof for that of $K_-(x,t)$ is completely equivalent. For, assume for the moment that $C_+(t)$ possesses derivatives up to the second order. The case when it does not have this property can be studied by a limit procedure, which we will sketch below. Rewrite the integral equation for A_+ in terms of K_+, obtaining (we simplify the notation)

$$\begin{aligned} y^{1/4}(x)y^{1/4}(t)K(x,t)&+\frac{1}{2}\Omega\left(\frac{\mathcal{I}(x)+\mathcal{I}(t)}{2}\right)\\ &+\frac{1}{2}\int_x^\infty K(x,s)y^{3/4}(s)\Omega\left(\frac{\mathcal{I}(t)+\mathcal{I}(s)}{2}\right)ds=0. \end{aligned}$$

Differentiate this relation twice both with respect to x and t, then subtract. We obtain

$$\begin{aligned} &\frac{1}{y(x)}K_{xx}-\frac{1}{y(t)}K_{tt}-\left[\frac{q(x)}{y(x)}-\frac{\tilde{q}(t)}{y(t)}\right]K\\ &\quad+\int_x^\infty\left[\frac{1}{y(x)}K_{xx}-\frac{1}{y(s)}K_{ss}-\left[\frac{q(x)}{y(x)}-\frac{\tilde{q}(s)}{y(s)}\right]\right]\Omega\left(\frac{\mathcal{I}(t)+\mathcal{I}(s)}{2}\right)ds=0. \end{aligned}$$

It follows that the function

$$t\mapsto\frac{1}{y(x)}K_{xx}(x,t)-\frac{1}{y(t)}K_{tt}(x,t)-\left[\frac{q(x)}{y(x)}-\frac{\tilde{q}(t)}{y(t)}\right]K(x,t),\quad(t\ge x)$$

is a solution of the homogeneous equation, and belongs to $L^1(x,+\infty)$. This implies that

$$\frac{1}{y(x)}K_{xx}(x,t)-\frac{1}{y(t)}K_{tt}(x,t)-\left[\frac{q(x)}{y(x)}-\frac{\tilde{q}(t)}{y(t)}\right]K(x,t)=0,\quad 0\le x\le t<+\infty.$$

If $C_+(t)$ has only first continuous derivative, we can argue as follows. Approximate uniformly $C_+(t)$ by functions $C_{+,n}(t)$ which are twice continuously differentiable in every finite interval in such a way that, for every $N<+\infty$,

$$\lim_{n\to+\infty}\int_0^N|C_+'(t)-C_{+,n}'(t)|dt=0.$$

These functions $C_{+,n}(t)$ define integral equations for corresponding functions $A_n(x,t)$, which are solvable for sufficiently large n. The functions $A_n(x,t)$ in turn define functions $K_n(x,t)$ and corresponding potentials $q_n(x)$, as in the above steps of the proof. Now, passing to the limit again, one finds functions $A(x,t)$, $K(x,t)$ and $q(x)$ for which the statement of the theorem is still valid (see, in another context, [25]).

Now, we still have to prove that $q_+(x)=q_-(x)$. Define

$$b_{12}(k)=b_{21}(k)=\frac{1}{a_{11}(k)},\quad b_{22}(k)=b_{12}(k)a_{12}(k),\quad b_{11}(k)=-b_{22}(-k).$$

Set

$$f_*(k,x) = b_{22}(k) f_-(k,x) + b_{21}(k) f_-(-k,x). \tag{58}$$

We claim that

$$f_*(k,x) = f_+(k,x).$$

This in turn will imply that $q_+(x) = q_-(x)$, since

$$-f_*'' = -b_{22}(k) f_-''(k,x) - b_{21}(k) f_-(-k,x) = b_{22}(k)[k^2 y(x) - q_-(x)] f_-(k,x)$$
$$+b_{21}(k)[k^2 y(x) + q_-(x)] f_-(-k,x) = [k^2 y(x) + q_-(x)] f_*(k,x),$$

and

$$-f_*''(k,x) = [k^2 y(x) + q_+(x)] f_*(k,x).$$

To prove that $f_*(k,x) = f_+(k,x)$, we first observe that from the properties of f_- and $b_{21}(k)$ and $b_{22}(k)$ we can write

$$a_{11}(k) f_*(k,x) = a_{12}(k) f_-(k,x) + f_-(-k,x).$$

Writing

$$f_*(k,x) = e^{ik\mathcal{I}(x)} \left(y^{-1/4}(x) + \int_0^\infty A_*(x,s) e^{2iks} ds \right),$$

and reasoning as in the lines after (46), we conclude that $A_*(x,t)$ satisfies

$$\Omega_+(\mathcal{I}(x)+t) + y^{1/4}(x) \int_0^\infty A_*(x,s) \Omega_+(\mathcal{I}(x)+s+t) ds + A_*(x,t) = 0, \ \ (t>0),$$

which is the same equation as that for $A_+(x,t)$. By uniqueness, it follows that $A_*(x,t) = A_+(x,t)$, hence $f_*(k,x) = f_+(k,x)$. The proof is complete. □

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