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Asymptotic Self-Similarity in Diffusion Equations with Nonconstant Radial Limits at Infinity

Thierry Gallay[1] · **Romain Joly**[1] · **Geneviève Raugel**[2]

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Abstract
We study the long-time behavior of localized solutions to linear or semilinear parabolic equations in the whole space $\mathbb{R}^n$, where $n \geq 2$, assuming that the diffusion matrix depends on the space variable x and has a finite limit along any ray as $|x| \to \infty$. Under suitable smallness conditions in the nonlinear case, we prove convergence to a self-similar solution whose profile is entirely determined by the asymptotic diffusion matrix. Examples are given which show that the profile can be a rather general Gaussian-like function, and that the approach to the self-similar solution can be arbitrarily slow depending on the continuity and coercivity properties of the asymptotic matrix. The proof of our results relies on appropriate energy estimates for the diffusion equation in self-similar variables. The new ingredient consists in estimating not only the difference w between the solution and the self-similar profile, but also an antiderivative W obtained by solving a linear elliptic problem which involves w as a source term. Hence, a good part of our analysis is devoted to the study of linear elliptic equations whose coefficients are homogeneous of degree zero.

Keywords Diffusion equations · Inhomogeneous media · Long-time asymptotics · Self-similar solutions

1 Introduction

We consider semilinear parabolic equations of the form

$$\partial_t u(x,t) = \operatorname{div}\big(A(x)\nabla u(x,t)\big) + N(u(x,t)), \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{1.1}$$

Geneviève Raugel: († May 10, 2019).

✉ Thierry Gallay
Thierry.Gallay@univ-grenoble-alpes.fr

Romain Joly
Romain.Joly@univ-grenoble-alpes.fr

[1] Institut Fourier, Université Grenoble Alpes et CNRS, 100 rue des Maths, 38610 Gières, France

[2] Département de Mathématiques, CNRS et Université Paris-Saclay, 91405 Orsay, France

which describe the evolution of a scalar quantity $u(x,t) \in \mathbb{R}$ under the action of inhomogeneous diffusion and nonlinear self-interaction. We assume that the diffusion matrix $A(x)$ in (1.1) is symmetric, Lipschitz continuous as a function of $x \in \mathbb{R}^n$, and satisfies the following uniform ellipticity condition: there exist positive constants λ_1, λ_2 such that

$$\lambda_1|\xi|^2 \le \big(A(x)\xi,\xi\big) \le \lambda_2|\xi|^2, \qquad \text{for all } x \in \mathbb{R}^n \text{ and all } \xi \in \mathbb{R}^n, \tag{1.2}$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product in $\mathbb{R}^n$. As for the nonlinearity, we suppose that N is globally Lipschitz, that $N(0) = 0$, and that $N(u) = \mathcal{O}(|u|^\sigma)$ as $u \to 0$ for some $\sigma > 1+2/n$. Our goal is to investigate the long-time behavior of all solutions of (1.1) starting from sufficiently small and localized initial data.

Even in the linear case where $N = 0$, it is necessary to make further assumptions on the diffusion matrix $A(x)$ to obtain accurate results on the long-time behavior of solutions of (1.1). In fact, two classical situations are well understood: the *asymptotically flat* case, and the *periodic* case. More precisely, if $A(x)$ converges to the identity matrix as $|x| \to \infty$, it is possible to show that all solutions of the diffusion equation $\partial_t u = \mathrm{div}(A(x)\nabla u)$ in $L^1(\mathbb{R}^n)$ behave asymptotically like the solutions of the heat equation $\partial_t u = \Delta u$ with the same initial data, see e.g. [11]. On the other hand, if $A(x)$ is a periodic function of x with respect to a lattice of $\mathbb{R}^n$, the relevant asymptotic equation is $\partial_t u = \mathrm{div}(\bar{A}\,\nabla u)$, where $\bar{A} \in \mathcal{M}_n(\mathbb{R})$ is a *homogenized matrix* which is determined by solving an elliptic problem in a cell of the lattice [12]. These results can be extended to a class of semilinear equations as well [10–12].

In this paper, we consider a different situation which is apparently less studied in the literature: we assume that the diffusion matrix $A(x)$ has *radial limits* at infinity in all directions. This means that, for all $x \in \mathbb{R}^n$, the following limit exists:

$$A_\infty(x) := \lim_{r\to+\infty} A(rx). \tag{1.3}$$

It is clear that the limiting matrix $A_\infty(x)$ is symmetric, homogeneous of degree zero with respect to $x \in \mathbb{R}^n$, and uniformly elliptic in the sense of (1.2). We also suppose that the restriction of A_∞ to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is Lipschitz continuous, and that the limit in (1.3) is reached uniformly on $\mathbb{S}^{n-1}$ at some rate $\nu > 0$:

$$\sup_{x\in\mathbb{R}^n} |x|^\nu \, \|A(x) - A_\infty(x)\| < \infty. \tag{1.4}$$

Following [14,31], to investigate the long-time behavior of solutions to (1.1), we introduce forward *self-similar variables* defined by $y = x/\sqrt{1+t}$ and $\tau = \log(1+t)$. More precisely, we look for solutions of (1.1) in the form

$$u(x,t) = \frac{1}{(1+t)^{n/2}}\, v\Big(\frac{x}{\sqrt{1+t}},\, \log(1+t)\Big), \qquad x \in \mathbb{R}^n, \quad t \ge 0. \tag{1.5}$$

Note that the change of variables (1.5) reduces to identity at initial time, so that $u(x,0) = v(x,0)$. The new function $v(y,\tau)$ satisfies the rescaled equation

$$\partial_\tau v = \mathrm{div}\Big(A\big(ye^{\tau/2}\big)\nabla v\Big) + \frac{1}{2}\, y\cdot\nabla v + \frac{n}{2}\, v + \mathcal{N}(\tau, v), \qquad y \in \mathbb{R}^n, \quad \tau > 0, \tag{1.6}$$

where

$$\mathcal{N}(\tau, v) = e^{(1+\frac{n}{2})\tau}\, N\big(e^{-n\tau/2} v\big). \tag{1.7}$$

Equation (1.6) is non-autonomous, but has (at least formally) a well-defined limit as $\tau \to +\infty$. Indeed, using (1.3) and the assumption that $N(u) = \mathcal{O}(|u|^\sigma)$ as $u \to 0$ for some

$\sigma > 1 + 2/n$, we arrive at the limiting equation

$$\partial_\tau v \;=\; \mathrm{div}\big(A_\infty(y)\nabla v\big) + \frac{1}{2}\, y\cdot\nabla v + \frac{n}{2}\, v, \qquad y\in\mathbb{R}^n, \quad \tau > 0. \tag{1.8}$$

In what follows we denote by L the differential operator in the right-hand side of (1.8).

Our main results show that, under appropriate assumptions, the solutions of (1.6) indeed converge to solutions of (1.8) as $\tau\to\infty$, so that the long-time asymptotics are determined by the linear equation (1.8). We first observe that the limiting equation has a unique steady state:

Proposition 1.1 *There exists a unique solution $\varphi\in H^1(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ of the elliptic equation*

$$L\varphi(y) \;\equiv\; \mathrm{div}\big(A_\infty(y)\nabla\varphi(y)\big) + \frac{1}{2}\, y\cdot\nabla\,\varphi(y) + \frac{n}{2}\,\varphi(y) \;=\; 0, \qquad y\in\mathbb{R}^n, \tag{1.9}$$

satisfying the normalization condition $\int_{\mathbb{R}^n}\varphi(y)\,\mathrm{d}y = 1$. Moreover φ is Hölder continuous, and there exists a constant $C\geq 1$ such that

$$C^{-1}\, e^{-C|y|^2} \;\leq\; \varphi(y) \;\leq\; C\, e^{-|y|^2/C}, \qquad \text{for all } y\in\mathbb{R}^n. \tag{1.10}$$

Remark 1.2 If we suppose that $A_\infty(y) = \mathbb{1}$ (the identity matrix), or more generally that $A_\infty(y)y = y$ for all $y\in\mathbb{R}^n$, the "principal eigenfunction" φ defined in Proposition 1.1 is given by the explicit formula $\varphi(y) = (4\pi)^{-n/2}\, e^{-|y|^2/4}$. In contrast, we show in Remark 3.11 below that, if $B(y)$ is a symmetric matrix that is homogeneous of degree zero and uniformly elliptic, the Gaussian-like function $\varphi(y) = \exp\big(-\frac{1}{4}(B(y)y,y)\big)$ satisfies (1.9) for some appropriate choice of the limiting matrix A_∞, provided the oscillations of $B(y)$ are not too rapid. This indicates that the profile φ given by Proposition 1.1 can be a pretty general function satisfying the Gaussian bounds (1.10).

We next consider solutions of (1.6) in the weighted L^2 space

$$L^2(m) \;=\; \Big\{ v\in L^2_{\mathrm{loc}}(\mathbb{R}^n) \,\Big|\, \|v\|_{L^2(m)} < \infty\Big\}, \quad \|v\|^2_{L^2(m)} \;=\; \int_{\mathbb{R}^n}(1+|y|^2)^m|v(y)|^2\,\mathrm{d}y, \tag{1.11}$$

which was used in a similar context in [15]. The parameter $m\in\mathbb{R}$ specifies the behavior of the solutions at infinity. In particular, we observe that $L^2(m)\hookrightarrow L^1(\mathbb{R}^n)$ when $m > n/2$, as a consequence of Hölder's inequality.

We are now ready to state our main result in the linear case where $\mathcal{N} = 0$.

Theorem 1.3 (Asymptotics in the linear case) *Assume that $n\geq 2$ and that the diffusion matrix $A(x)$ satisfies hypotheses (1.2)–(1.4). For all $m > n/2$ and all initial data $v_0\in L^2(m)$, the rescaled equation (1.6) with $\mathcal{N} = 0$ has a unique global solution $v\in C^0([0,+\infty), L^2(m))$ such that $v(0) = v_0$. Moreover, for any μ satisfying*

$$0 \;<\; \mu \;<\; \frac{1}{2}\,\min\Big(m - \frac{n}{2},\; \nu,\; \beta\Big), \tag{1.12}$$

where $\nu > 0$ is as in (1.4) and $\beta\in(0,1]$ is the exponent in (1.16) below, there exists a positive constant C (independent of v_0) such that

$$\|v(\cdot,\tau) - \alpha\varphi\|_{L^2(m)} \;\leq\; C\,\|v_0\|_{L^2(m)}\, e^{-\mu\tau}, \qquad \text{for all } \tau\geq 0, \tag{1.13}$$

where $\alpha = \int_{\mathbb{R}^n} v_0(y)\,\mathrm{d}y$ and φ is given by Proposition 1.1.

Remark 1.4 In terms of the original variables, the convergence result (1.13) implies in particular that, in the linear case $N = 0$, the solution $u(x,t)$ of (1.1) with initial data $u_0 \in L^2(m)$ satisfies

$$\int_{\mathbb{R}^n} \Big| u(x,t) - \frac{\alpha}{(1+t)^{n/2}}\,\varphi\Big(\frac{x}{\sqrt{1+t}}\Big)\Big|\,\mathrm{d}x \;=\; \mathcal{O}(t^{-\mu}), \qquad \text{as} \quad t \to +\infty, \tag{1.14}$$

where $\alpha = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x$. Using parabolic regularity, it is possible to prove convergence in higher L^p norms too, as in [11].

Remark 1.5 Theorem 1.3 holds true in all space dimensions $n \ge 1$, but the proof we propose only works for $n \ge 2$ and depends on n in a nontrivial way. In fact, as we shall see in Sect. 4 below, the number of energy functionals we need increases with n, so that our method becomes cumbersome in high dimensions. For simplicity we concentrate on the most relevant cases $n = 2$ and $n = 3$, for which we provide a complete proof, but we also give a pretty detailed sketch of the argument when $4 \le n \le 7$, see Sect. 4.5. On the other hand, the one-dimensional case, which is substantially simpler for several reasons, is completely solved in our previous work [14], where damped hyperbolic equations are also considered. In many respects, the present paper can be viewed as a (rather nontrivial) extension of the method of [14] to higher dimensions.

Before considering semilinear equations, we comment on the formula (1.12) for the convergence rate μ, which is quite instructive. We first recall that, for any measurable matrix $A(x)$ satisfying the ellipticity conditions (1.2), the solutions of the linear equation $\partial_t u = \mathrm{div}\big(A\nabla u\big)$ with localized initial data satisfy $\|u(\cdot,t)\|_{L^\infty} = \mathcal{O}(t^{-n/2})$ as $t \to +\infty$, see for instance [13]. The purpose of Theorem 1.3 is to exhibit the leading-order term in the asymptotic expansion of $u(x,t)$, and to estimate the rate μ at which the leading term is approached by the solutions. As can be seen from the simple example of the heat equation, where $A = \mathbb{1}$, the convergence rate μ depends on how fast the initial data decay as $|x| \to \infty$. More precisely, it is known in that example that Theorem 1.3 holds for any $\mu \le 1/2$ such that $2\mu < m - \frac{n}{2}$ [15]. This result is sharp and the constraints on μ are determined by the spectral properties of the differential operator L in (1.8), considered as acting on the weighted space $L^2(m)$. If $m > n/2$, so that $L^2(m) \hookrightarrow L^1(\mathbb{R}^n)$, the origin $\lambda = 0$ is a simple isolated eigenvalue, with Gaussian eigenfunction φ as in Remark 1.2. The convergence rate μ is determined by the *spectral gap* between the origin and the rest of the spectrum of L, see Fig. 1 in Sect. 3.

In more general situations, the convergence rate μ obviously depends on how fast the limits in (1.3) are reached. This effect can be studied using the techniques of [11] if we assume that $A(x) = \mathbb{1} + B(x)$, where $\|B(x)\| = \mathcal{O}(|x|^{-\nu})$ as $|x| \to \infty$. In that case, the solutions of the linear equation $\partial_t u = \mathrm{div}\big(A\nabla u\big)$ in $L^2(m)$ behave asymptotically like the solutions of the heat equation $\partial_t u = \Delta u$ with the same initial data, but the convergence rate in (1.13) or (1.14) is further constrained by the relation $\mu \le \nu/2$, which appears to be sharp. As can be expected, we thus have $\mu \to 0$ as $\nu \to 0$.

Finally, it is important to realize that the convergence rate μ also depends on the properties of the limiting matrix $A_\infty(x)$ itself, and cannot be arbitrarily large even if $A = A_\infty$ and $m \gg n/2$. We have already seen that $\mu \le 1/2$ when $A_\infty = \mathbb{1}$, due to the presence of an isolated eigenvalue $\lambda = -1/2$ in the spectrum of L if $m > 1 + n/2$, see Fig. 1. For a more general matrix $A_\infty(x)$, the principal eigenvalue of the corresponding operator L is fixed at the origin, as asserted by Proposition 1.1, but the next eigenvalue can be pretty arbitrary, and this determines the width of the spectral gap. In Sect. 3.2, we study an instructive example for which

$$A_\infty(x) \;=\; b\,\mathbb{1} + (1-b)\frac{x \otimes x}{|x|^2}, \tag{1.15}$$

where $b > 0$ is a free parameter. In that case, we can compute explicitly all eigenvalues and eigenfunctions of the linear operator L in (1.8), and we observe that the spectral gap shrinks to zero as $b \to 0$, see Fig. 2.

The example (1.15) is already considered in classical papers by Meyers [25] and Serrin [29], where uniqueness and regularity properties are studied for the solutions of the linear elliptic equation $-\text{div}\big(A_\infty(x)\nabla u\big) = f$ in $\mathbb{R}^n$. It turns out that this equation plays a crucial role in our analysis because, as we shall see in Sect. 4, the convergence result (1.13) is obtained using energy estimates not only for the difference $w = v - \alpha\varphi$, but also for the "antiderivative" W defined by $-\text{div}\big(A_\infty(x)\nabla W\big) = w$. It is important to keep in mind that the matrix $A_\infty(x)$, being homogeneous of degree zero, is not smooth at the origin unless it is constant. So we do not expect that the solutions of the elliptic equation above are smooth, even if f is, but the celebrated De Giorgi–Nash theory asserts that all weak solutions in $H^1_{\text{loc}}(\mathbb{R}^n)$ are at least Hölder continuous with exponent β, for some $\beta \in (0, 1)$. This exponent is the third quantity that appears in the formula (1.12) for the convergence rate. Consequently, Theorem 1.3 draws an original connection between the regularity properties of the elliptic problem and the long-time behavior of the solutions of the evolution equation.

To study the elliptic problem, we consider the associated Green function $G(x, y)$, which is uniquely defined at least if $n \ge 3$. For the reasons mentioned above, that function is Hölder continuous with exponent β, but not more regular unless A_∞ is constant. However, using the assumption that A_∞ is homogeneous of degree zero and Lipschitz outside the origin, it is possible to establish the following gradient estimate

$$|\nabla_x G(x, y)| \le C\left(\frac{1}{|x-y|^{n-1}} + \frac{1}{|x|^{1-\beta}|x-y|^{n-2+\beta}}\right), \qquad x \ne y, \quad x \ne 0, \tag{1.16}$$

where the second term in the right-hand side describes the precise nature of the singularity at the origin. As is well known, the Green function of the Laplace operator satisfies (1.16) with $\beta = 1$, but for nonconstant homogeneous matrices $A_\infty(x)$ we have $\beta < 1$ in general. Estimate (1.16) is apparently new and plays an important role in our analysis of the elliptic problem, hence in the proof of Theorem 1.3.

Although we only considered linear equations so far, the techniques we use in the proof of Theorem 1.3 are genuinely nonlinear, and were originally developed to handle semilinear problems, see [14,31]. To illustrate the scope of our method, we also treat the full Eq. (1.1) with a nonlinearity N that is "irrelevant" for the long-time asymptotics of small and localized solutions, according to the terminology introduced in [5]. For simplicity, we make here rather strong assumptions on N, which could be relaxed at the expense of using additional energy functionals in the proof. We suppose that there exist two constants $C > 0$ and $\sigma > 1 + 2/n$ such that

$$\big|N(u)\big| \le C|u|^\sigma \quad \text{and} \quad \big|N(u) - N(\tilde{u})\big| \le C|u - \tilde{u}|, \qquad \text{for all } u, \tilde{u} \in \mathbb{R}. \tag{1.17}$$

Our second main result is the following:

Theorem 1.6 (Asymptotics in the semilinear case) *Assume that $n \ge 2$, that the diffusion matrix $A(x)$ satisfies hypotheses* (1.2)–(1.4)*, and that conditions* (1.17) *are fulfilled by the nonlinearity N. Given any $m > n/2$, there exist a positive constant ϵ_0 such that, for all initial data $v_0 \in L^2(m)$ with $\|v_0\|_{L^2(m)} \le \epsilon_0$, the rescaled equation* (1.6) *has a unique global solution $v \in C^0([0, +\infty), L^2(m))$ such that $v(0) = v_0$. Moreover, there exists some $\alpha_* \in \mathbb{R}$ and, for all μ satisfying*

$$0 < \mu < \frac{1}{2}\min\Big(m - \frac{n}{2},\ \nu,\ \beta,\ 2\eta\Big), \qquad \textit{where} \quad \eta = \frac{n}{2}\big(\sigma - 1\big) - 1, \tag{1.18}$$

there exists a positive constant C (independent of v_0) such that

$$\|v(\cdot,\tau)-\alpha_*\varphi\|_{L^2(m)} \le C\,\|v_0\|_{L^2(m)}\,e^{-\mu\tau}, \qquad \textit{for all } \tau \ge 0, \tag{1.19}$$

where φ is given by Proposition 1.1.

Remark 1.7 The integral of u is not preserved under the nonlinear evolution defined by (1.1), and this explains why there is no formula for the asymptotic mass α_* in Theorem 1.6. However, the proof shows that $\alpha_* = \int_{\mathbb{R}^n} v_0\,\mathrm{d}y + \mathcal{O}(\|v_0\|^{\sigma}_{L^2(m)})$, where σ is as in (1.17). It is important to observe that the convergence rate μ in (1.18) is also affected by the nonlinearity, through the value of the parameter σ. In particular μ converges to zero as σ approaches from above the critical value $1+2/n$, and no convergence at all is expected if $\sigma \le 1+2/n$.

Remark 1.8 As in the linear case, our strategy to prove Theorem 1.6 becomes complicated in large space dimensions. For simplicity we provide a complete proof only if $n=2$, or if $n=3$ and $\mu<1/4$. The other cases can be treated using the hierarchy of energy functionals introduced in Sect. 4.5.

The rest of this paper is organized as follows. In Sect. 2, we study in some detail the elliptic equation $-\mathrm{div}\big(A_\infty\nabla u\big) = f$ under the assumption that the matrix $A_\infty(x)$ is homogeneous of degree zero and uniformly elliptic. In particular, we derive estimates for the associated Green function, and we apply them to bound the solution u in terms of the data f in weighted L^2 spaces. In this process we use a general result on integral operators with homogeneous kernels, which is essentially due to Karapetiants and Samko [19]. In Sect. 3, we investigate the spectral properties of the linear operator defined by the right-hand side of (1.8); in particular, we prove Proposition 1.1 and we establish a few additional properties of the principal eigenfunction φ. We also study in detail the particular case where the matrix A_∞ is given by (1.15). Section 4 is devoted to the proof of Theorem 1.3, using weighted energy estimates for the perturbation $w = v-\alpha\varphi$. As was already mentioned, the main original idea is to introduce the "antiderivative" W, which is defined as the solution of the elliptic equation $-\mathrm{div}\big(A_\infty\nabla W\big) = w$. It turns out that weighted L^2 estimates for both W and w are sufficient to establish the convergence result (1.13) if $n=2$, or if $n=3$ and $\mu<1/4$, whereas additional energy functionals are needed in the other cases. The same strategy works in the nonlinear case too, under suitable assumptions on the function N, and the details are worked out in Sect. 5. The final Sect. 6 is an appendix where a few auxiliary results are collected for easy reference.

2 The Diffusion Operator with Homogeneous Coefficients

In this section, we study the elliptic operator H on $L^2(\mathbb{R}^n)$ formally defined by

$$Hu = -\mathrm{div}\big(A_\infty(x)\nabla u\big), \qquad u \in L^2(\mathbb{R}^n), \tag{2.1}$$

where the matrix-valued coefficient $A_\infty(x)$ satisfies the following assumptions:

1) The $n\times n$ matrix $A_\infty(x)$ is symmetric for all $x\in\mathbb{R}^n$, and the operator H is uniformly elliptic in the sense of (1.2);
2) The map $A_\infty : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ is homogeneous of degree zero: $A_\infty(\lambda x) = A_\infty(x)$ for all $x\in\mathbb{R}^n$ and all $\lambda>0$;
3) The restriction of A_∞ to the unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$ is a Lipschitz continuous function.

Elliptic operators of the form (2.1) are of course well known, and were extensively studied in the literature, see for instance [7,16]. For the reader's convenience we recall here a few basic properties, paying special attention to the homogeneity assumption 2), which will play an important role in our analysis. As a consequence of homogeneity, the function $x \mapsto A_\infty(x)$ is necessarily discontinuous at $x = 0$, unless it is identically constant. Moreover, in view of 2) and 3), there exists a constant $C > 0$ such that $\|A_\infty(x)\| \le C$ for all $x \in \mathbb{R}^n$ and

$$\|\nabla A_\infty(x)\| \le \frac{C}{|x|}, \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

2.1 Definition and Domain

To give a rigorous definition of the operator H, the easiest way is to consider the corresponding quadratic form and to use the classical representation theorem, see e.g. [20, Section VI.2]. Let $\mathcal{B}$ be the bilinear form on $L^2(\mathbb{R}^n)$ defined by $D(\mathcal{B}) = H^1(\mathbb{R}^n)$ and

$$\mathcal{B}(u_1, u_2) = \int_{\mathbb{R}^n} \big(A_\infty(x)\nabla u_1(x), \nabla u_2(x)\big)\,\mathrm{d}x, \qquad u_1, u_2 \in D(\mathcal{B}).$$

Under our assumptions on the matrix $A_\infty(x)$, it is easily verified that the form $\mathcal{B}$ is symmetric, closed, and nonnegative. Applying the representation theorem, we thus obtain:

Proposition 2.1 *There exists a (unique) nonnegative selfadjoint operator $H : D(H) \to L^2(\mathbb{R}^n)$ such that $D(H) \subset D(\mathcal{B})$ and $\mathcal{B}(u_1, u_2) = (Hu_1, u_2)$ for all $u_1 \in D(H)$ and all $u_2 \in D(\mathcal{B})$. In addition $D(H) = \{u \in H^1(\mathbb{R}^n) \,|\, \mathrm{div}(A_\infty \nabla u) \in L^2(\mathbb{R}^n)\}$ where the divergence is understood in the sense of distributions.*

If H has constant coefficients, namely if the matrix A_∞ does not depend on x, it is clear that $D(H) = H^2(\mathbb{R}^n)$. However, this is not true in the general case, as can be seen from the example of the Meyers–Serrin matrix (1.15) where $D(H)$ contains functions u that are not H^2 in a neighborhood of the origin, see Sect. 3.2. As a matter of fact, it does not seem obvious to determine exactly the domain $D(H)$ under our assumptions on the diffusion matrix A_∞, but the following (elementary) observations can nevertheless be made.

Remark 2.2 (On the domain of H)

1. Since A_∞ is Lipschitz outside the origin, the elliptic regularity theory [16, Section 8.4] asserts that $D(H) \subset H^1(\mathbb{R}^n) \cap H^2(\mathbb{R}^n \setminus B_r)$ for any $r > 0$, where we denote $B_r = \{x \in \mathbb{R}^n \,|\, |x| \le r\}$.
2. If $n \ge 3$, then $D(H) \supset H^2(\mathbb{R}^n)$. Indeed, if $u \in H^2(\mathbb{R}^n)$, we have by Leibniz's rule

$$Hu = -\sum_{i,j=1}^{n} \Big(A_\infty(x)_{ij}\partial^2_{x_i x_j} u + \partial_{x_i}(A_\infty(x)_{ij})\partial_{x_j} u\Big).$$

The first term in the right-hand side obviously belongs to $L^2(\mathbb{R}^n)$, and so does the second one due to estimate (2.2) and Hardy's inequality

$$\Big\| \frac{v}{|x|} \Big\|_{L^2(\mathbb{R}^n)} \le \frac{2}{n-2}\, \|\nabla v\|_{L^2(\mathbb{R}^n)}, \qquad v \in H^1(\mathbb{R}^n), \quad n \ge 3, \tag{2.3}$$

see e.g. [28, Section 2.1]. Thus $Hu \in L^2(\mathbb{R}^n)$, hence $u \in D(H)$.

3. If $n \geq 3$ and $A_\infty(x) = \mathbb{1} + \epsilon B(x)$, where B is homogeneous of degree zero and Lipschitz continuous on the sphere $\mathbb{S}^{n-1}$, then $D(H) = H^2(\mathbb{R}^n)$ for all sufficiently small $\epsilon \in \mathbb{R}$. Indeed, in that case, the argument above shows that H is a small perturbation of $-\Delta$ in $\mathcal{L}(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$, the space of bounded linear maps from $H^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Since $\mathbf{1} - \Delta \in \mathcal{L}(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ is invertible, the same property remains true for $\mathbf{1} + H$ if ϵ is sufficiently small, and this implies that $D(H) = H^2(\mathbb{R}^n)$.

2.2 Semigroup and Fundamental Solution

We next consider the evolution equation $\partial_t u + Hu = 0$, namely the linear diffusion equation

$$\partial_t u(x,t) \,=\, \mathrm{div}\big(A_\infty(x)\nabla u(x,t)\big), \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{2.4}$$

which is the analogue of (1.8) in the original variables. Since the operator H is selfadjoint and nonnegative, it is well known that $-H$ generates an analytic semigroup e^{-tH} in $L^2(\mathbb{R}^n)$ which satisfies the contraction property $\|e^{-tH}u\|_{L^2} \leq \|u\|_{L^2}$ for all $t \geq 0$, see e.g. [27, Chapter 1]. In particular, the Cauchy problem for equation (2.4) is well posed for all initial data $u_0 \in L^2(\mathbb{R}^n)$, the solution being $u(t) = e^{-tH}u_0$ for all $t \geq 0$.

On the other hand, using the fact that the matrix A_∞ satisfies the uniform ellipticity condition (1.2), one can show that the semigroup generated by $-H$ is hypercontractive [7, Section 2], which means that e^{-Ht} is a bounded operator from $L^2(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ for any $t > 0$, and also from $L^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ by duality. By the semigroup property, it follows that e^{-Ht} is also a bounded operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, and this implies that there exists a unique integral kernel $\Gamma(x,y,t)$ such that, for any $u \in L^1(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$,

$$\big(e^{-tH}u\big)(x) \,=\, \int_{\mathbb{R}^n} \Gamma(x,y,t)u(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{2.5}$$

see Remark 2.3 below. The kernel $\Gamma(x,y,t)$ is usually called the *fundamental solution* of the parabolic equation (2.4).

From the pioneering work of De Giorgi [8] and Nash [26], we know that Γ is a Hölder continuous function of its three arguments, and the strong maximum principle [16, Section 8.7] implies that Γ is strictly positive. The following additional properties will be used later on:

a) Since H is selfadjoint, we have $\Gamma(x,y,t) = \Gamma(y,x,t)$ for all $x, y \in \mathbb{R}^n$ and all $t > 0$.
b) For all $x, y \in \mathbb{R}^n$ and all $t > 0$, the following identities hold

$$\int_{\mathbb{R}^n} \Gamma(x,y,t)\,\mathrm{d}x \,=\, \int_{\mathbb{R}^n} \Gamma(x,y,t)\,\mathrm{d}y \,=\, 1. \tag{2.6}$$

c) There exists a constant $C > 1$ such that, for all $x, y \in \mathbb{R}^n$ and all $t > 0$,

$$\frac{1}{Ct^{n/2}}\, e^{-C|x-y|^2/t} \,\leq\, \Gamma(x,y,t) \,\leq\, \frac{C}{t^{n/2}}\, e^{-|x-y|^2/(Ct)}. \tag{2.7}$$

Such Gaussian bounds were first established by Aronson [2,3], see also [7, Chap. 3].
d) Since A_∞ is homogeneous of degree zero, we have

$$\lambda^n\, \Gamma(\lambda x, \lambda y, \lambda^2 t) \,=\, \Gamma(x,y,t), \tag{2.8}$$

for all $x, y \in \mathbb{R}^n$ and all $t > 0$.

Remark 2.3 That an integral kernel can be associated to any bounded linear operator from $L^p(\Omega)$ to $L^q(\Omega)$ with $q > p$ is a “classical” result, which is however rather difficult to locate precisely in the literature. According to [32], this result is due to Dunford in the particular case where $\Omega = [0, 1]$, and to Buhvalov [6] in more general situations.

2.3 The Green Function in Dimension $n \geq 3$

We next consider the elliptic equation $Hu = f$, namely

$$-\operatorname{div}\big(A_\infty(x)\nabla u(x)\big) \;=\; f(x), \qquad x \in \mathbb{R}^n, \tag{2.9}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given and $u : \mathbb{R}^n \to \mathbb{R}$ is the unknown function. If $n \geq 3$ and f is, for instance, a continuous function with compact support, it is well known that equation (2.9) has a unique solution u that vanishes at infinity. In fact, uniqueness is a consequence of the maximum principle for the uniformly elliptic operator H, see [16, Chapter 3], and existence follows from the integral representation

$$u(x) \;=\; \int_{\mathbb{R}^n} G(x, y) f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{2.10}$$

where $G(x, y)$ is the *Green function* defined by

$$G(x, y) \;=\; \int_0^\infty \Gamma(x, y, t)\,\mathrm{d}t \;>\; 0, \qquad \text{for all } x, y, \in \mathbb{R}^n, \quad x \neq y. \tag{2.11}$$

The following elementary properties are direct consequences of the corresponding assertions for the fundamental solution Γ:

a) The Green function G is symmetric: $G(x, y) = G(y, x)$ for all $x \neq y$.
b) There exists a constant $C > 1$ such that

$$\frac{C^{-1}}{|x-y|^{n-2}} \;\leq\; G(x, y) \;\leq\; \frac{C}{|x-y|^{n-2}}, \qquad \text{for all } x \neq y. \tag{2.12}$$

c) The Green function is homogeneous of degree $2 - n$: $\lambda^{n-2} G(\lambda x, \lambda y) = G(x, y)$ for all $x \neq y$ and all $\lambda > 0$.
d) For any $y \in \mathbb{R}^n$ and any test function $v \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \big(A_\infty(x)\nabla_x G(x, y), \nabla v(x)\big)\,\mathrm{d}x \;=\; v(y). \tag{2.13}$$

The last property implies that $-\operatorname{div}_x\big(A_\infty(x)\nabla_x G(x, y)\big) = \delta(x-y)$ in the sense of distributions, so that $G(x, y)$ can be considered as the fundamental solution of the elliptic equation (2.9). The main statement in this section is the following proposition, which gives accurate Hölder and gradient estimates for G under our assumptions on the diffusion matrix A_∞.

Proposition 2.4 *Assume that $n \geq 3$, and let G be the Green function associated with the elliptic problem (2.9), where the diffusion matrix is symmetric, uniformly elliptic, and homogeneous of degree zero. There exist constants $C > 0$ and $\beta \in (0, 1)$ such that*

$$|G(x_1, y) - G(x_2, y)| \;\leq\; C|x_1 - x_2|^\beta \bigg(\frac{1}{|x_1 - y|^{n-2+\beta}} + \frac{1}{|x_2 - y|^{n-2+\beta}}\bigg), \tag{2.14}$$

for all $x_1, x_2, y \in \mathbb{R}^n$ with $x_1 \neq y$ and $x_2 \neq y$. Moreover

$$|\nabla_x G(x, y)| \;\leq\; C\bigg(\frac{1}{|x-y|^{n-1}} + \frac{1}{|x|^{1-\beta}|x-y|^{n-2+\beta}}\bigg), \tag{2.15}$$

for all $x, y \in \mathbb{R}^n$ *with* $x \neq y$ *and* $x \neq 0$.

Proof The Hölder estimate (2.14) is explicitly stated in [17, Theorem 1.9], but in that classical reference the elliptic equation (2.9) is considered in a bounded domain $\Omega \subset \mathbb{R}^n$ with homogeneous Dirichlet conditions at the boundary $\partial\Omega$. The more recent work [18] studies a class of strongly elliptic systems that includes the scalar equation (2.9). In the whole space $\mathbb{R}^n$, the following estimate is stated in [18, Section 3.6]: there exist $C > 0$ and $0 < \beta < 1$ such that

$$|G(x_1, y) - G(x_2, y)| \le C|x_1 - x_2|^\beta |x_1 - y|^{2-n-\beta}, \quad \text{if} \quad |x_1 - x_2| < |x_1 - y|/2. \tag{2.16}$$

Exchanging the roles of x_1 and x_2, we deduce

$$|G(x_1, y) - G(x_2, y)| \le C|x_1 - x_2|^\beta |x_2 - y|^{2-n-\beta}, \quad \text{if} \quad |x_1 - x_2| < |x_2 - y|/2. \tag{2.17}$$

In the intermediate region where $x_j \neq y$ and $|x_1 - x_2| \ge |x_j - y|/2$ for $j = 1, 2$, we have by (2.12)

$$|G(x_j, y)| \le C|x_j - y|^{2-n} \le C|x_1 - x_2|^\beta |x_j - y|^{2-n-\beta}, \qquad j = 1, 2,$$

hence

$$|G(x_1, y) - G(x_2, y)| \le G(x_1, y) + G(x_2, y) \le C\left(\frac{|x_1 - x_2|^\beta}{|x_1 - y|^{n-2+\beta}} + \frac{|x_1 - x_2|^\beta}{|x_2 - y|^{n-2+\beta}}\right). \tag{2.18}$$

Combining (2.16)–(2.18), we obtain (2.14) in all cases.

We now prove the gradient estimate (2.15), which takes into account the fact that the diffusion matrix in (2.9) is homogeneous of degree zero. We use the following auxiliary result.

Lemma 2.5 [17] *Assume that* u *is a bounded solution of the elliptic equation* $Hu = 0$ *in the domain* $\Omega = \{x \in \mathbb{R}^n \,|\, |x - x_0| < r\}$, *where* $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, *and* $0 < r \le |x_0|/2$. *Then*

$$|\nabla u(x_0)| \le \frac{C}{r} \sup_{x \in \Omega} |u(x)|, \tag{2.19}$$

where $C > 0$ *depends only on* n, *on* λ_1, λ_2 *in* (1.2), *and on the constant in* (2.2).

Estimate (2.19) follows immediately from Lemma 3.1 in [17] and its proof, if we use the fact that the matrix $A_\infty(x)$ in (2.9) satisfies the Lipschitz estimate

$$\|A_\infty(x) - A_\infty(y)\| \le \frac{C}{|x_0|} |x - y|, \qquad \text{for all } x, y \in \Omega.$$

We now come back to the proof of estimate (2.15). Fix $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, and take $y \in \mathbb{R}^n$, $y \neq x_0$. If $|x_0| \le |x_0 - y|/2$, we apply Lemma 2.5 with $r = |x_0|/2$ and $u(x) = G(x, y) - G(x_0, y)$. We know from (2.14) that $|u(x)| \le C|x - x_0|^\beta |x_0 - y|^{2-n-\beta}$ for $x \in \Omega = B(x_0, r)$, and we deduce from (2.19) that

$$|\nabla u(x_0)| = |\nabla G(x_0, y)| \le \frac{C}{|x_0|^{1-\beta} |x_0 - y|^{n-2+\beta}}. \tag{2.20}$$

In the converse case where $|x_0| > |x_0 - y|/2$, we apply Lemma 2.5 with $r = |x_0 - y|/4$ and $u(x) = G(x, y)$. As $|u(x)| \le C|x - y|^{2-n}$, we deduce from (2.19) that

$$|\nabla u(x_0)| = |\nabla G(x_0, y)| \le \frac{C}{|x_0 - y|^{n-1}}. \tag{2.21}$$

Combining (2.20), (2.21), we obtain estimate (2.15) in all cases. The proof of Proposition 2.4 is now complete. □

2.4 The Green Functions in Dimension $n = 2$

In the two-dimensional case, the integral in (2.11) does not converge anymore, and it is no longer possible to solve the elliptic problem (2.9) using a positive Green function that decays to zero at infinity. However, as is shown in the Appendix of [21], see also [9,30], it is still possible to define a Green function $G(x, y)$ with the following properties:

i) G is symmetric: $G(x, y) = G(y, x)$ for all $x, y \in \mathbb{R}^2$ with $x \neq y$.
ii) G is Hölder continuous for $x \neq y$, and there exists a constant $C > 0$ such that

$$|G(x, y)| \le C\Big(1 + \big|\log|x-y|\big|\Big), \qquad x \neq y. \tag{2.22}$$

iii) For any $f \in C_c^0(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f(y)\,\mathrm{d}y = 0$, the unique solution of the elliptic equation (2.9) such that $u(x) \to 0$ as $|x| \to \infty$ is given by

$$u(x) = \int_{\mathbb{R}^2} G(x, y) f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^2. \tag{2.23}$$

iv) Equality (2.13) with $n = 2$ holds for all $y \in \mathbb{R}^2$ and all test functions $v \in C_c^\infty(\mathbb{R}^2)$.

The Green function with these properties is unique up to an additive constant. In the particular case $A_\infty = \mathbb{1}$, we have the explicit expression $G(x, y) = -(2\pi)^{-1}\log|x-y|$. As is clear from that example, the Green function is not homogeneous. However, using the fact that $A_\infty(x)$ is homogeneous of degree zero, it is easy to verify that, if $G(x, y)$ is a Green function, so is $G(\lambda x, \lambda y)$ for any $\lambda > 0$. Thus $G(\lambda x, \lambda y) - G(x, y)$ must be equal to a constant $c(\lambda)$, which depends continuously only on λ. As $c(\lambda_1\lambda_2) = c(\lambda_1) + c(\lambda_2)$ for all $\lambda_1, \lambda_2 > 0$ by construction, we conclude that there exists a (positive) real number c_0 such that

$$G(\lambda x, \lambda y) = G(x, y) + c_0 \log\frac{1}{\lambda}, \tag{2.24}$$

for all $x \neq y$ and all $\lambda > 0$.

The analogue of Proposition 2.4 in the present case is:

Proposition 2.6 *Assume that $n = 2$, and let G be a Green function associated with the elliptic problem (2.9), where the diffusion matrix is symmetric, uniformly elliptic, and homogeneous of degree zero. There exist constants $C > 0$ and $\beta \in (0, 1)$ such that estimates (2.14), (2.15) hold with $n = 2$.*

Proof For a class of elliptic systems that includes the scalar equation (2.9), a Green function in the whole plane $\mathbb{R}^2$ is constructed in [30, Section 6], and is shown to satisfy the Hölder estimate

$$|G(x_1, y) - G(x_2, y)| \le C\,\frac{|x_1 - x_2|^\beta}{|x_1 - y|^\beta}, \quad \text{if} \quad |x_1 - x_2| < |x_1 - y|/2,$$

which is the exact analogue of (2.16) when $n = 2$. Exchanging the roles x_1 and x_2, we also have

$$|G(x_1, y) - G(x_2, y)| \le C\,\frac{|x_1 - x_2|^\beta}{|x_2 - y|^\beta}, \quad \text{if} \quad |x_1 - x_2| < |x_2 - y|/2.$$

In the intermediate region where $x_j \neq y$ and $|x_1 - x_2| \ge |x_j - y|/2$ for $j = 1, 2$, we use the fact that the function $(x_1, x_2, y) \mapsto G(x_1, y) - G(x_2, y)$ is homogeneous of degree zero, as a consequence of (2.24). We can thus assume that $|x_1 - x_2| = 1$, and using (2.22) we easily find

$$|G(x_1, y) - G(x_2, y)| \le C\biggl(\frac{|x_1 - x_2|^\beta}{|x_1 - y|^\beta} + \frac{|x_1 - x_2|^\beta}{|x_2 - y|^\beta}\biggr), \tag{2.25}$$

which completes the proof of (2.14) when $n = 2$.

To establish the gradient estimate (2.15) for $n = 2$, we use again Lemma 2.5, which is valid in all space dimensions. Proceeding as in the proof of Proposition 2.4, we fix $x_0 \in \mathbb{R}^2$, $x_0 \neq 0$, and take $y \in \mathbb{R}^2$, $y \neq x_0$. If $|x_0| \le |x_0 - y|/2$, we apply Lemma 2.5 with $r = |x_0|/2$ and $u(x) = G(x, y) - G(x_0, y)$. From (2.25) we know that $|u(x)| \le C|x - x_0|^\beta |x_0 - y|^{-\beta}$ for $x \in \Omega = B(x_0, r)$, and we deduce from (2.19) that

$$|\nabla u(x_0)| = |\nabla G(x_0, y)| \le \frac{C}{|x_0|^{1-\beta}|x_0 - y|^\beta}.$$

In the converse case where $|x_0| > |x_0 - y|/2$, we apply Lemma 2.5 with $r = |x_0 - y|/4$ and again $u(x) = G(x, y) - G(x_0, y)$. As $|u(x)| \le C$ by (2.25), we deduce from (2.19) that

$$|\nabla u(x_0)| = |\nabla G(x_0, y)| \le \frac{C}{|x_0 - y|}.$$

This completes the proof of estimate (2.15) in the two-dimensional case. □

2.5 Weighted Estimates for the Elliptic Equation

The aim of this section is to derive estimates on the integral operator K formally defined by

$$K[f](x) = \int_{\mathbb{R}^n} G(x, y) f(y) \,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{2.26}$$

where G is the Green function introduced in Sects. 2.3 or 2.4. In the two-dimensional case, the Green function is only defined up to an additive constant, but we always assume that f is integrable and $\int_{\mathbb{R}^2} f(y)\,\mathrm{d}y = 0$, so that there is no ambiguity in definition (2.26).

If $n \ge 3$, we know from (2.12) that $G(x, y) \le C|x - y|^{2-n}$ for all $x \neq y$. Using the classical Hardy–Littlewood–Sobolev inequality [23], we deduce the useful estimate

$$\bigl\|K[f]\bigr\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \qquad \text{if} \quad 1 < p < \frac{n}{2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{2}{n}. \tag{2.27}$$

However, the bound (2.27) is not sufficient for our purposes, first because the case $n = 2$ is excluded, and also because we need estimates in the weighted spaces. These improved bounds will be obtained using the following general result, which concerns integral operators of the form

$$\mathcal{K}[f](x) = \int_{\mathbb{R}^n} k(x, y) f(y) \,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \tag{2.28}$$

where the integral kernel $k(x, y)$ satisfies the following assumptions:

1) The measurable function $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $-d$, where $d \in (0, n]$:

$$k(\lambda x, \lambda y) = \lambda^{-d} k(x, y), \qquad x, y \in \mathbb{R}^n, \quad \lambda > 0. \tag{2.29}$$

2) The function k is invariant under simultaneous rotations of both arguments:

$$k(Sx, Sy) = k(x, y), \qquad x, y \in \mathbb{R}^n, \quad S \in SO(n). \tag{2.30}$$

3) There exists $p \in [1, +\infty]$ with $(n-d)p \le n$ such that, for $x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$,

$$\kappa_1 := \int_{\mathbb{R}^n} |k(x, y)|^{n/d} \, |y|^{-n^2/(dq)} \, \mathrm{d}y < \infty, \qquad \text{where} \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{d}{n}. \tag{2.31}$$

As a consequence of (2.30), the quantity κ_1 does not depend on the choice of $x \in \mathbb{S}^{n-1}$.

Proposition 2.7 *Assume that the integral kernel $k(x, y)$ satisfies assumptions (2.29)–(2.31) above. Then the operator $\mathcal{K}$ defined by (2.28) is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and*

$$\big\|\mathcal{K}[f]\big\|_{L^q(\mathbb{R}^n)} \le \kappa_1^{d/n} \, \|f\|_{L^p(\mathbb{R}^n)}, \qquad \textit{for all } f \in L^p(\mathbb{R}^n). \tag{2.32}$$

Remark 2.8 Proposition 2.7 can be seen as a clever, but relatively straightforward generalization of the classical Young inequality for convolution operators. In the particular case where $d = n$, so that $q = p$, the result is apparently due to L. G. Mikhailov, N. K. Karapetiants, and S. G. Samko, see [19, Section 6] and [24]. For the reader's convenience, we give a proof of the general case in Sect. 6.1. As is explained in [24], many classical inequalities, including Hilbert's inequality and various forms of Hardy's inequality, can be deduced from Proposition 2.7 by an appropriate choice of the integral kernel k. We add to this list the Stein–Weiss inequality [22], which corresponds to the kernel

$$k(x, y) = \frac{1}{|x|^a} \frac{1}{|x-y|^\lambda} \frac{1}{|y|^b}, \qquad x \ne y,$$

where $0 < \lambda < n$, $d := a + b + \lambda \in [\lambda, n]$, and $a < n/q$, $b < n(1 - 1/p)$ with p, q as in (2.31). As is easily verified, we can apply Proposition 2.7 to that example under the additional assumption that $a + b > 0$. In particular the limiting case $a = b = 0$, which corresponds to the classical HLS inequality, cannot be obtained in this way.

As a first application of Proposition 2.7, we establish the following estimate for the linear operator (2.26) in the weighted spaces $L^2(m)$ defined in (1.11).

Proposition 2.9 *If $n \ge 3$ and if $m \ge 0$ satisfies $2 - n/2 < m < n/2$, the operator K defined by (2.26) is bounded from $L^2(m)$ to $L^2(m-2)$. Specifically, if $f \in L^2(m)$ and $u = K[f]$, we have the homogeneous estimate*

$$\int_{\mathbb{R}^n} |x|^{2m-4} \, |u(x)|^2 \, \mathrm{d}x \le C \int_{\mathbb{R}^n} |x|^{2m} \, |f(x)|^2 \, \mathrm{d}x < \infty, \tag{2.33}$$

for some constant $C > 0$ independent of f.

Proof If $f \in L^2(m)$ and $u = K[f]$ we have, in view of (2.26) and (2.12),

$$|x|^{m-2} \, |u(x)| \le C \int_{\mathbb{R}^n} k(x, y) \, |y|^m |f(y)| \, \mathrm{d}y, \quad \text{where} \quad k(x, y) = \frac{|x|^{m-2}}{|x-y|^{n-2}|y|^m}. \tag{2.34}$$

The integral kernel $k(x, y)$ in (2.34) is homogeneous of degree $-n$ and invariant under rotations, in the sense of (2.30). Moreover, for any $x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, we have

$$\kappa_1 = \int_{\mathbb{R}^n} k(x, y) \, |y|^{-n/2} \, \mathrm{d}y < \infty. \tag{2.35}$$

Indeed, the integral in (2.35) converges near the origin because $m+n/2 < n$, and near infinity because $n-2+m+n/2 > n$. Moreover, the singularity at $y = x$ is always integrable. So, applying Proposition 2.7 with $d = n$ and $p = q = 2$, we obtain the estimate (2.33). If $m \le 2$, this immediately implies that K is bounded from $L^2(m)$ to $L^2(m-2)$. If $m > 2$, which is only possible when $n \ge 5$, it remains to bound the L^2 norm of u on the unit ball $B = B(0, 1) \subset \mathbb{R}^n$, which is not controlled by (2.33) since $2m - 4 > 0$. This is easily done using the HLS inequality (2.27), which shows that $\|u\|_{L^2(B)} \le C\|u\|_{L^{2n/(n-4)}(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}$. □

Remark 2.10 By a similar argument, using estimate (2.15), one can show that the function $u = K[f]$ in Proposition 2.9 satisfies $\nabla u \in L^2(m-1)$ and $\nabla u(x) = \int \nabla_x G(x, y) f(y)\,\mathrm{d}y$. Thus, if we multiply equality (2.13) by $f(y)$ and integrate over $y \in \mathbb{R}^n$, we obtain the relation $\int \big(A_\infty(x)\nabla u(x), \nabla v(x)\big)\,\mathrm{d}x = \int v(x) f(x)\,\mathrm{d}x$, which is valid for all $v \in C_c^\infty(\mathbb{R}^n)$. This implies that $-\mathrm{div}(A_\infty \nabla u) = f$ in the sense of distributions on $\mathbb{R}^n$, namely $HK[f] = f$ where H is defined in (2.1).

The assumption that $m < n/2$ is essential in Proposition 2.9, even in the particular case where $A_\infty = \mathbb{1}$. As we now show, it is possible to establish estimate (2.33) for larger values of m, if we assume that the function $f \in L^2(m)$ has *zero mean*. At this point, we recall that $L^2(m) \hookrightarrow L^1(\mathbb{R}^n)$ precisely when $m > n/2$. For technical reasons that will become clear in the proof of Theorem 1.6, we formulate our next result in the more general framework of weighted L^p spaces, with $p \in [1, 2]$. Those spaces are defined in close analogy with (1.11):

$$L^p(m) = \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \,\Big|\, \|f\|_{L^p(m)} < \infty \Big\}, \quad \|f\|^p_{L^p(m)} = \int_{\mathbb{R}^n} (1+|y|)^{mp} |v(y)|^p \,\mathrm{d}y. \tag{2.36}$$

If $m > n(1-\frac{1}{p})$, we have $L^p(m) \hookrightarrow L^1(\mathbb{R}^n)$ by Hölder's inequality, and in that case we denote by $L^p_0(m)$ the closed subspace of $L^p(m)$ defined by

$$L^p_0(m) = \Big\{ f \in L^p(m) \,\Big|\, \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0 \Big\}, \qquad m > n\big(1-\tfrac{1}{p}\big). \tag{2.37}$$

Proposition 2.11 *Let $n \ge 2$ and let $\beta \in (0, 1)$ be as in (2.14). For any $m \in (n/2, n/2 + \beta)$ and any $p \in [1, 2]$ such that $p > 2n/(n+4)$, the operator K defined by (2.26) is bounded from $L^p_0(m-s)$ to $L^2(m-2)$, where $s = n/p - n/2$. Specifically, if $f \in L^p_0(m-s)$ and $u = K[f]$, we have the homogeneous estimate*

$$\int_{\mathbb{R}^n} |x|^{2m-4} |u(x)|^2 \,\mathrm{d}x \;\le\; C \left(\int_{\mathbb{R}^n} |x|^{p(m-s)} |f(x)|^p \,\mathrm{d}x \right)^{2/p} \;<\; \infty, \tag{2.38}$$

for some constant $C > 0$ independent of f.

Remark 2.12 If $p = 2$, so that $s = 0$, estimate (2.38) reduces to (2.33), and Proposition 2.11 thus shows that K is bounded from $L^2_0(m)$ to $L^2(m-2)$ if $n/2 < m < n/2 + \beta$. We believe that the upper bound on m is sharp. In the particular case were $A_\infty = \mathbb{1}$, so that $\beta = 1$, estimate (2.38) is not valid for $m > n/2+1$ unless one assumes that not only the integral but also the first order moments of f vanish. In the proof of Theorem 1.6 below, Proposition 2.11 will also be used with $p = 1$ and $s = n/2$.

Remark 2.13 If $n = 2$, or if $n = 3$ and $\beta \le 1/2$, we necessarily have $m < 2$ in Proposition 2.11, so that $2m - 4 < 0$. In that case, if f satisfies the assumptions of Proposition 2.11, the solution u of the elliptic equation (2.9) may not belong to $L^2(\mathbb{R}^n)$, because $u(x)$ decays too slowly as $|x| \to +\infty$. Explicit examples of this phenomenon can be constructed using the Meyers–Serrin matrix (1.15), see Sect. 6.3.

Proof Our assumptions on the parameters m and p obviously imply that $s \in [0, n/2]$, $s < 2$, and $m - s > n(1 - \frac{1}{p})$, so that $L^p(m-s) \hookrightarrow L^1(\mathbb{R}^n)$. If $f \in L^p_0(m-s)$ and $u = K[f]$, we thus have the representation formula

$$u(x) = \int_{\mathbb{R}^n} \Big(G(x, y) - G(x, 0)\Big) f(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^n,$$

which is equivalent to (2.26) since $\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x = 0$. We recall that the above integral uniquely defines u even if $n = 2$ because G is unique up to a constant in that case. We also note that, in any dimension $n \ge 2$, the difference $G(x, y) - G(x, 0)$ is homogeneous of degree $2 - n$, see Sects. 2.3 and 2.4. The general idea is to bound that difference using estimate (2.14) when $|y|$ is small compared to $|x|$, and estimate (2.12) or (2.22) when $|y| \ge |x|/2$. We thus introduce a smooth cut-off function $\chi : \mathbb{R}_+ \to [0, 1]$ satisfying $\chi(r) = 1$ when $r \in [0, 1/2]$ and $\chi(r) = 0$ when $r \ge 3/4$. We observe that $|u(x)| \le u_1(x) + u_2(x)$ where

$$u_1(x) = \int_{\mathbb{R}^n} \Big|G(x, y) - G(x, 0)\Big|\, \chi\Big(\tfrac{|y|}{|x|}\Big) |f(y)|\, \mathrm{d}y,$$
$$u_2(x) = \int_{\mathbb{R}^n} \Big|G(x, y) - G(x, 0)\Big| \Big(1 - \chi\Big(\tfrac{|y|}{|x|}\Big)\Big) |f(y)|\, \mathrm{d}y.$$

We shall prove that, for $j = 1, 2$, the following estimate holds:

$$|x|^{m-2} u_j(x) \le C \int_{\mathbb{R}^n} k_j(x, y)\, |y|^{m-s} |f(y)|\, \mathrm{d}y, \tag{2.39}$$

where $k_j(x, y)$ is an integral kernel which fulfills the assumptions of Proposition 2.7 with $d = n - s$ and $p = 2n/(n+2s)$. This will imply that both u_1 and u_2 satisfy estimate (2.38) with $q = 2$, which gives the desired conclusion.

We start with u_1. Using (2.14) to bound the difference $G(x, y) - G(x, 0) \equiv G(y, x) - G(0, x)$, we obtain estimate (2.39) for $j = 1$ where

$$k_1(x, y) = \frac{|x|^{m-2}}{|y|^{m-s}} \Big(\frac{|y|^\beta}{|x - y|^{n-2+\beta}} + \frac{|y|^\beta}{|x|^{n-2+\beta}}\Big) \chi\Big(\frac{|y|}{|x|}\Big).$$

The kernel $k_1(x, y)$ is obviously homogeneous of degree $-d = s - n$ and invariant under rotations. Moreover, if $|x| = 1$, we have $\chi(|y|/|x|) = \chi(|y|) = 0$ when $|y| \ge 3/4$, so that condition (2.31) becomes

$$\int_{\mathbb{R}^n} k_1(x, y)^{n/d}\, |y|^{-n^2/(2d)}\, \mathrm{d}y \equiv \int_{|y| \le 3/4} \Big(k_1(x, y)\, |y|^{-n/2}\Big)^{n/d} \mathrm{d}y < \infty, \quad \text{when } |x| = 1.$$

The only singularity of the integrand is at the origin where $k_1(x, y)\, |y|^{-n/2} \sim |y|^{\beta+s-m-n/2}$, and the assumption that $m < n/2 + \beta$ ensures that $(n/d)\big(m + n/2 - \beta - s\big) < n$. So we can apply Proposition 2.7 and conclude that the function u_1 satisfies estimate (2.38) with $q = 2$.

To estimate u_2 if $n \ge 3$, we use (2.12) and we obtain estimate (2.39) for $j = 2$, where

$$k_2(x, y) = \frac{|x|^{m-2}}{|y|^{m-s}} \Big(\frac{1}{|x - y|^{n-2}} + \frac{1}{|x|^{n-2}}\Big) \Big(1 - \chi\Big(\tfrac{|y|}{|x|}\Big)\Big), \qquad n \ge 3.$$

If $n = 2$, the difference $G(x, y) - G(x, 0)$ is homogeneous of degree zero, and it follows that $G(x, y) - G(x, 0) = G(x/|x|, y/|x|) - G(x/|x|, 0)$. Using (2.22), we thus obtain estimate (2.39) for $j = 2$, where

$$k_2(x, y) = \frac{|x|^{m-2}}{|y|^{m-s}} \Big(1 + \Big|\log \frac{|x - y|}{|x|}\Big|\Big) \Big(1 - \chi\Big(\tfrac{|y|}{|x|}\Big)\Big), \qquad n = 2.$$

In any case, the kernel $k_2(x, y)$ is homogeneous of degree $-d = s - n$, invariant under rotations, and if $|x| = 1$ we have

$$\int_{\mathbb{R}^n} k_2(x, y)^{n/d}\, |y|^{-n^2/(2d)}\, \mathrm{d}y \;\equiv\; \int_{|y|\geq 1/2} \Bigl(k_2(x, y)\, |y|^{-n/2}\Bigr)^{n/d} \mathrm{d}y \;<\; \infty.$$

Indeed, the singularity at $y = x$ is integrable provided $(n/d)(n-2) < n$, which is the case because we assumed that $s < 2$, and the convergence of the integral at infinity is guaranteed since $m > n/2$. Applying Proposition 2.7 again, we conclude that u_2 also satisfies estimate (2.38) with $q = 2$. This completes the proof of (2.38).

It is now easy to conclude the proof of Proposition 2.11. If $m \leq 2$, estimate (2.38) implies of course that $u \in L^2(m-2)$ and $\|u\|_{L^2(m-2)} \leq C\|f\|_{L^p(m-s)}$. If $m > 2$, which is possible only when $n \geq 3$, it remains to bound the L^2 norm of u on the unit ball $B = B(0, 1) \subset \mathbb{R}^n$. If $p > 1$, which is automatic when $n \geq 4$, this follows from the HLS inequality (2.27), which implies that $\|u\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ for $q = np/(n-2p) > 2$. In the particular case where $p = 1$ and $n = 3$, we can obtain the bound $\|u\|_{L^q(B)} \leq C\|f\|_{L^1(\mathbb{R}^n)}$ for all $q < 3$ using definition (2.26), estimate (2.12), and Hölder's inequality. □

We also need to estimate the function $u = K[f]$ in the particular case where $f = \mathrm{div} g$ for some vector field $g : \mathbb{R}^n \to \mathbb{R}^n$. In that situation, if we integrate by parts formally in (2.26), we obtain the relation $u = (K \circ \mathrm{div})[g]$, where the new operator $K \circ \mathrm{div}$ is defined by

$$\bigl(K \circ \mathrm{div}\bigr)[g](x) \;=\; -\int_{\mathbb{R}^n} \nabla_y G(x, y) \cdot g(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^n. \tag{2.40}$$

We first prove that this operator is well defined on $L^2(m-1)$ if $m > 2 - n/2$ and $m \geq 1$, and we next give conditions on g that ensure that $(K \circ \mathrm{div})[g] = K[\mathrm{div} g]$.

Proposition 2.14 *Let $n \geq 2$ and let $\beta \in (0, 1)$ be as in (2.14). For any $m \in (2-n/2, n/2+\beta)$ such that $m \geq 1$, the operator $K \circ \mathrm{div}$ defined by (2.40) is bounded from $L^2(m-1)^n$ to $L^2(m-2)$. Specifically, if $g \in L^2(m-1)^n$ and $u = (K \circ \mathrm{div})[g]$, we have the homogeneous estimate*

$$\int_{\mathbb{R}^n} |x|^{2m-4}\, |u(x)|^2\, \mathrm{d}x \;\leq\; C \int_{\mathbb{R}^n} |x|^{2m-2}\, |g(x)|^2\, \mathrm{d}x \;<\; \infty, \tag{2.41}$$

for some constant $C > 0$ independent of g.

Proof Let $g \in L^2(m-1)^n$ and $u = (K \circ \mathrm{div})[g]$. We estimate the integral kernel $\nabla_y G(x, y)$ in (2.40) using the bound (2.15) and keeping in mind that $\nabla_y G(x, y) = \nabla_z G(z, x)\big|_{z=y}$ by symmetry. This gives

$$|x|^{m-2}\, |u(x)| \;\leq\; C \int_{\mathbb{R}^n} k(x, y)\, |y|^{m-1} |g(y)|\, \mathrm{d}y, \tag{2.42}$$

where

$$k(x, y) \;=\; \frac{|x|^{m-2}}{|y|^{m-1}} \left(\frac{1}{|x-y|^{n-1}} + \frac{1}{|y|^{1-\beta}|x-y|^{n-2+\beta}}\right).$$

The kernel k is homogeneous of degree $-n$ and invariant under rotations. To apply Proposition 2.7 with $p = q = 2$, we need to verify that, for any $x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} k(x, y)\, |y|^{-n/2}\, \mathrm{d}y \;<\; \infty.$$

The integral converges for small $|y|$ if and only if $m - \beta + n/2 < n$, namely $m < n/2 + \beta$. At infinity, the integrability condition is $m + n - 2 + n/2 > n$, namely $m > 2 - n/2$. Thus, applying Proposition 2.7, we deduce (2.41) from (2.42).

To show that $u \in L^2(m-2)$, it remains to control the L^2 norm of u when $m > 2$. In that case, we simply observe that $2 \in (2-n/2, n/2+\beta)$, and applying the argument above (with $m = 2$) we obtain the bound $\|u\|_{L^2(\mathbb{R}^n)} \le C\|g\|_{L^2(1)} \le C\|g\|_{L^2(m-1)}$. This concludes the proof. □

Corollary 2.15 *If $g \in L^2(m-1)^n$ for some $m > n/2$ and if $f = \operatorname{div} g \in L^2(m)$, then $f \in L^2_0(m)$ and $K[f] = (K \circ \operatorname{div})[g]$.*

Proof As $m > n/2$, we have $L^2(m-1) \hookrightarrow L^p(\mathbb{R}^n)$ for some $p < n/(n-1)$, by Hölder's inequality. Thus, applying Lemma 6.2 below, we see that $\int_{\mathbb{R}^n} f \,\mathrm{d}x = 0$ if f is as in the statement. To show that $K[f] = (K \circ \operatorname{div})[g]$, we have to justify the integration by parts leading to (2.40). As in Sect. 6.2, we denote $\chi_k(x) = \chi(x/k)$, where $\chi : \mathbb{R}^n \to [0, 1]$ is a smooth cut-off function satisfying $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. We start from the identity

$$\int_{\mathbb{R}^n} \chi_k(y)\Big(G(x, y)\operatorname{div} g(y) + \nabla_y G(x, y) \cdot g(y)\Big)\,\mathrm{d}y \;=\; -\int_{\mathbb{R}^n} G(x, y)\, g(y) \cdot \nabla\chi_k(y)\,\mathrm{d}y,$$

which holds for all $k \in \mathbb{N}^*$ and almost all $x \in \mathbb{R}^n$. If $m \in (n/2, n/2 + \beta)$, the left hand-side has a limit in $L^2(m-2)$ as $k \to +\infty$, in view of Propositions 2.11 and 2.14. To prove the desired result, it is thus sufficient to show that the right-hand side converges to zero in the sense of distributions. Integrating against a test function $\psi \in C_c^\infty(\mathbb{R}^n)$ and denoting $\Psi(y) = \int_{\mathbb{R}^n} G(x, y)\psi(x)\,\mathrm{d}x$, we have to show that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n} \Psi(y)\, g(y) \cdot \nabla\chi_k(y)\,\mathrm{d}y \;\equiv\; \lim_{k\to+\infty} \frac{1}{k} \int_{k\le|y|\le 2k} \Psi(y)\, g(y) \cdot \nabla\chi(y/k)\,\mathrm{d}y \;=\; 0.$$

This in turn is an easy consequence of Hölder's inequality, if we use the facts that $g \in L^2(m-1)$ for some $m > 2 - n/2$, and $|\Psi(y)| \le C(1+|y|)^{2-n}$ if $n \ge 3$ or $|\Psi(y)| \le C\log(2+|y|)$ if $n = 2$. □

Remark 2.16 As a final comment, we mention that, if $f \in L^2_0(m)$ for some $m \in (n/2, n/2+1)$, there exists $g \in L^2(m-1)^n$ such that $\operatorname{div} g = f$, see Lemma 6.3. Thus $K[f] = (K \circ \operatorname{div})[g]$ by Corollary 2.15, and estimate (2.33) can be deduced from estimate (2.41) if $m < n/2 + \beta$.

3 The Diffusion Operator in Self-Similar Variables

In this section we study the generator L of the evolution equation (1.8), considered as an operator in the weighted space $L^2(m) \subset L^2(\mathbb{R}^n)$ for some $m \ge 0$. This operator is defined by

$$Lu \;=\; \operatorname{div}(A_\infty(x)\nabla u) + \frac{1}{2}\, x \cdot \nabla u + \frac{n}{2}\, u, \qquad u \in D(L), \tag{3.1}$$

where $D(L) \subset L^2(m)$ is the maximal domain

$$D(L) \;=\; \big\{u \in L^2(m) \cap H^1(\mathbb{R}^n) \,\big|\, \operatorname{div}(A_\infty(x)\nabla u) + \tfrac{1}{2}x \cdot \nabla u \in L^2(m)\big\}.$$

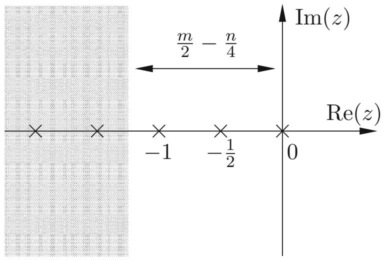

Fig. 1 When $A_\infty = \mathbb{1}$ the spectrum of the operator L in the space $L^2(m)$ consists of a sequence of eigenvalues $0, -1/2, -1, \dots$ and of essential spectrum filling the half-space $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \le \frac{n}{4} - \frac{m}{2}\}$. For any $k \in \mathbb{N}$, the eigenvalue $-k/2$ is isolated if $m > k + n/2$

3.1 The Constant Coefficient Case

In the particular case where $A_\infty = \mathbb{1}$, the operator L is studied in detail in [15, Appendix A]. It is shown there that the spectrum of L in $L^2(m)$ consists of two different parts:

a) a countable sequence of discrete eigenvalues: $\sigma_{\mathrm{disc}} = \{-k/2 \,|\, k = 0, 1, 2, \dots\}$;
b) a half-plane of essential spectrum: $\sigma_{\mathrm{ess}} = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \le \frac{n}{4} - \frac{m}{2}\}$.

The spectrum $\sigma = \sigma_{\mathrm{disc}} \cup \sigma_{\mathrm{ess}}$ is represented in Fig. 1 for a typical choice of the parameters n, m. It is worth noting that the discrete spectrum σ_{disc} does not depend on m. In fact, conjugating the operator L with the Gaussian weight $e^{-|x|^2/8}$, we obtain the useful relation

$$e^{|x|^2/8}\, L\, e^{-|x|^2/8} \,=\, \Delta - \frac{|x|^2}{16} + \frac{n}{4}, \tag{3.2}$$

where the right-hand side is the harmonic operator in $\mathbb{R}^n$, normalized so that its spectrum in $L^2(\mathbb{R}^n)$ is precisely the sequence σ_{disc}. This shows that the eigenfunctions of L associated with the discrete spectrum σ_{disc} have Gaussian decay at infinity, hence belong to $L^2(m)$ for any $m \ge 0$. Moreover we have $L\varphi = 0$, where

$$\varphi(x) \,=\, \frac{1}{(4\pi)^{n/2}}\, e^{-|x|^2/4}, \qquad x \in \mathbb{R}^n, \tag{3.3}$$

and differentiating k times the principal eigenfunction φ we obtain the kth order Hermite functions that span the kernel of $L + k/2$ if m is sufficiently large, namely $m > k + n/2$.

On the other hand, the essential spectrum σ_{ess} has a completely different origin, which is revealed by applying the Fourier transform so that L becomes a first-order differential operator acting on the Sobolev space $H^m(\mathbb{R}^n)$, see [15, Appendix A]. Using this observation, one can show that each complex point $z \notin \sigma_{\mathrm{disc}}$ is an eigenvalue of L of infinite multiplicity (if $n \ge 2$), with eigenfunctions that decay slowly, like $|x|^{2\mathrm{Re}(z)-n}$, as $|x| \to \infty$. In particular, these eigenvalues belong to $L^2(m)$ if and only if $\mathrm{Re}(z) < \frac{n}{4} - \frac{m}{2}$, which explains why the essential spectrum σ_{ess}, unlike σ_{disc}, is sensitive to the value of m.

To summarize, in the case where $A_\infty = \mathbb{1}$ the operator L has $k+1$ isolated eigenvalues if the parameter m is large enough so that $m > k+n/2$, see Fig. 1. In particular, if $m > n/2$, the zero eigenvalue is simple and isolated, and the rest of the spectrum is contained in the half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \le -\mu\}$, where $\mu = \min(1/2, m/2 - n/4)$. Note that the assumption $m > n/2$ ensures that $L^2(m) \hookrightarrow L^1(\mathbb{R}^n)$.

3.2 A Nontrivial Example: The Meyers–Serrin Operator

We next study in detail the instructive example where the limiting matrix A_∞ is given by (1.15). It turns out that, in that case too, the eigenvalues and eigenfunctions of the linear operator (3.1) can be computed explicitly, and exhibit a nontrivial behavior when the parameter $b > 0$ is varied. In what follows we denote

$$A_b(x) \;=\; b\,\mathbb{1} + (1-b)\frac{x\otimes x}{|x|^2}, \qquad x \in \mathbb{R}^n\setminus\{0\}, \tag{3.4}$$

where $\mathbb{1}$ is the identity matrix and $(x\otimes x)_{ij} = x_i x_j$. Elliptic equations with a diffusion matrix of the form (3.4) were considered by Meyers and Serrin nearly sixty years ago. If the parameter $b > 0$ is small enough, they turn out to be useful to illustrate the optimality of general results concerning the interior regularity of solutions [25, Section 5] or the local uniqueness [4,29].

As is clear from definition (3.4), we have $A_b(x)x = x$ and $A_b(x)y = by$ for any $y \in \mathbb{R}^n$ that is orthogonal to x. If $b \neq 1$, the eigenvalues of $A_b(x)$ are thus 1 (multiplicity 1) and b (multiplicity $n-1$). For the evolution equation $\partial_t u = \mathrm{div}(A_b(x)\nabla u)$, this means that diffusion in the radial direction is unaffected by the value of b, whereas the diffusion rate is increased ($b > 1$) or decreased ($b < 1$) in the transverse directions.

We now consider the rescaled diffusion operator L_b defined by

$$L_b u \;=\; \mathrm{div}(A_b\nabla u) + \frac{1}{2}x\cdot\nabla u + \frac{n}{2}u, \qquad x\in\mathbb{R}^n. \tag{3.5}$$

Since

$$\mathrm{div}\Big(\frac{x\otimes x}{|x|^2}\,\nabla u\Big) \;=\; \mathrm{div}\Big(\frac{x}{|x|^2}\,x\cdot\nabla u\Big) \;=\; \frac{1}{|x|^2}\Big((x\cdot\nabla)^2 u + (n-2)\,x\cdot\nabla u\Big),$$

we obtain the alternative form

$$L_b u \;=\; b\Delta u + \frac{1-b}{|x|^2}\Big((x\cdot\nabla)^2 u + (n-2)\,x\cdot\nabla u\Big) + \frac{1}{2}x\cdot\nabla u + \frac{n}{2}u. \tag{3.6}$$

As is clear from (3.6), the operator L_b is invariant under rotations around the origin, and this makes it possible to compute its eigenvalues and eigenvectors by the classical method of “separation of variables”.

Indeed, let $p : \mathbb{R}^n \to \mathbb{R}$ be a *harmonic polynomial* that is *homogeneous of degree* $\ell \in \mathbb{N}$. We look for eigenfunctions of L_b of the form

$$u(x) \;=\; p(x)\,\varphi(|x|), \qquad x\in\mathbb{R}^n, \tag{3.7}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}$. As $\Delta p = 0$ and $x\cdot\nabla p = \ell p$, we easily find

$$\Delta u(x) \;=\; p(x)\Big(\varphi''(r) + \frac{n-1+2\ell}{r}\,\varphi'(r)\Big), \qquad \text{where } r = |x|.$$

Similarly

$$x\cdot\nabla u = p\big(r\varphi' + \ell\varphi\big), \qquad (x\cdot\nabla)^2 u = p\big(r^2\varphi'' + (2\ell+1)r\varphi' + \ell^2\varphi\big),$$

hence

$$\mathrm{div}\Big(\frac{x}{|x|^2}\,x\cdot\nabla u\Big) \;=\; p\Big(\varphi'' + \frac{n-1+2\ell}{r}\,\varphi' + \frac{\ell(n-2+\ell)}{r^2}\,\varphi\Big).$$

It follows that $(L_b u)(x) = p(x)(L_{b,\ell}\,\varphi)(|x|)$, where

$$L_{b,\ell}\,\varphi \;=\; \varphi'' + \frac{n-1+2\ell}{r}\,\varphi' + (1-b)\frac{\ell(n-2+\ell)}{r^2}\,\varphi + \frac{r}{2}\,\varphi' + \frac{n+\ell}{2}\,\varphi. \tag{3.8}$$

In a second step, we look for eigenfunctions of the radial operator $L_{b,\ell}$ of the following form

$$\varphi(r) = r^{\gamma} e^{-r^2/4} \psi(r^2/4), \qquad r > 0, \tag{3.9}$$

where $\gamma \in \mathbb{R}$ is a parameter that will be determined below. A direct computation shows that

$$\varphi'(r) = r^{\gamma} e^{-r^2/4} \Big(\frac{r}{2} \psi'\Big(\frac{r^2}{4}\Big) + \Big(\frac{\gamma}{r} - \frac{r}{2}\Big)\psi\Big(\frac{r^2}{4}\Big)\Big),$$

$$\varphi''(r) = r^{\gamma} e^{-r^2/4} \Big(\frac{r^2}{4} \psi'' + \Big(\gamma + \frac{1}{2} - \frac{r^2}{2}\Big)\psi' + \Big(\frac{\gamma^2 - \gamma}{r^2} - \gamma - \frac{1}{2} + \frac{r^2}{4}\Big)\psi\Big),$$

and it follows that $(L_{b,\ell}\,\varphi)(r) = r^{\gamma} e^{-r^2/4} (L_{b,\ell,\gamma}\,\psi)(r^2/4)$, where the differential operator $L_{b,\ell,\gamma}$ acts on the variable $y = r^2/4 \in \mathbb{R}_+$ and is defined in the following way. Setting

$$\alpha = \frac{n}{2} - 1 + \gamma + \ell, \qquad \delta = \gamma^2 + \gamma(n - 2 + 2\ell) + (1-b)\ell(n - 2 + \ell), \tag{3.10}$$

we have the explicit expression

$$\big(L_{b,\ell,\gamma}\,\psi\big)(y) = y\psi''(y) + (\alpha + 1 - y)\psi'(y) + \Big(\frac{\delta}{4y} - \frac{\gamma + \ell}{2}\Big)\psi(y), \qquad y > 0. \tag{3.11}$$

To find eigenfunctions, it is necessary to choose the parameter γ in such a way that the quantity δ defined in (3.10) vanishes. This leads to

$$\gamma = \frac{1}{2}\Big(-(n - 2 + 2\ell) + \sqrt{(n-2)^2 + 4b\ell(n - 2 + \ell)} \Big) \tag{3.12}$$

Note that $\gamma = 0$ if either $b = 1$ (trivial case) or $\ell = 0$ (radially symmetric solutions). In the general case, we always have $\gamma + \ell \ge 0$, which means that $p(x)|x|^{\gamma}$ is bounded near the origin.

Remark 3.1 Taking the other sign in front of the square root in (3.12) would give more singular solutions of the eigenvalue equation, for which the gradient is not square integrable near the origin; these are examples of the "pathological solutions" considered by Serrin [29].

The eigenfunctions of the operator $L_{b,\ell,\gamma}$ are easy to determine when γ is chosen so that $\delta = 0$, because for any $k \in \mathbb{N}$ the differential equation

$$y\psi''(y) + (\alpha + 1 - y)\psi'(y) + k\psi(y) = 0, \qquad y > 0,$$

has a solution of the form $\psi(y) = \mathrm{L}_k^{(\alpha)}(y)$, where $\mathrm{L}_k^{(\alpha)}$ is the k^{th} (generalized) Laguerre polynomial with parameter α, see [1, Section 22]. In particular, for $k = 0, 1, 2$, we have

$$\mathrm{L}_0^{(\alpha)}(y) = 1, \qquad \mathrm{L}_1^{(\alpha)}(y) = -y + \alpha + 1, \qquad \mathrm{L}_2^{(\alpha)}(y) = \frac{y^2}{2} - (\alpha + 2)y + \frac{(\alpha+1)(\alpha+2)}{2}.$$

Summarizing, the calculations above lead to the following statement.

Proposition 3.2 *Fix $b > 0$, $\ell \in \mathbb{N}$, $k \in \mathbb{N}$, and let*

$$\alpha = \frac{1}{2}\sqrt{(n-2)^2 + 4b\ell(n - 2 + \ell)}\Big), \qquad \gamma = -\frac{n}{2} + 1 - \ell + \alpha. \tag{3.13}$$

If $p : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial that is homogeneous of degree ℓ and if

$$u(x) = p(x)|x|^{\gamma} e^{-|x|^2/4}\, \mathrm{L}_k^{(\alpha)}(|x|^2/4), \qquad x \in \mathbb{R}^n, \tag{3.14}$$

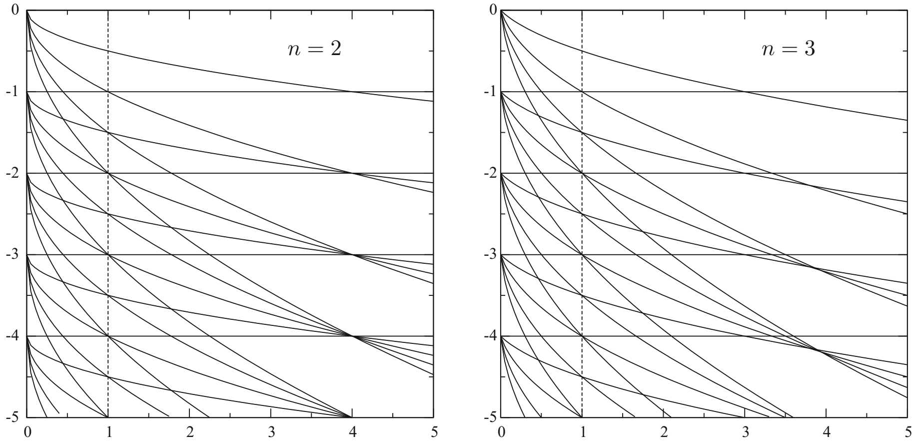

Fig. 2 The eigenvalues $\lambda = \lambda(b, \ell, k)$ of the linear operator (3.5) are represented as a function of $b \in [0, 5]$, for $\ell, k = 0, 1, 2, 3, 4$ and $n = 2$ (left) or $n = 3$ (right). The horizontal lines are eigenvalues corresponding to radially symmetric eigenfunctions ($\ell = 0$). The vertical dashed line highlights the constant coefficient case $b = 1$, where $\lambda = -\ell/2 - k$

where $\mathrm{L}_k^{(\alpha)}$ *is the* k^{th} *Laguerre polynomial with parameter* α*, then* u *is an eigenfunctions of the differential operator* L_b *defined in* (3.6) *in the sense that*

$$L_b u = \lambda u, \qquad \textit{where} \quad \lambda = -\frac{\gamma + \ell}{2} - k. \tag{3.15}$$

Remark 3.3 For all values of the parameter $b > 0$, the operator L_b is selfadjoint in the weighted L^2 space

$$X = \left\{ u \in L^2(\mathbb{R}^n) \,\middle|\, e^{|x|^2/8} u \in L^2(\mathbb{R}^n) \right\}.$$

Indeed, if $v = e^{|x|^2/8} u$, it a direct calculation shows that $\mathcal{L}_b v = e^{|x|^2/8} L_b u$ where

$$\mathcal{L}_b v = \operatorname{div}(A_b \nabla v) - \frac{|x|^2}{16} v + \frac{n}{4} v, \qquad x \in \mathbb{R}^n. \tag{3.16}$$

The operator $\mathcal{L}_b$ is obviously symmetric in $L^2(\mathbb{R}^n)$, and becomes selfadjoint when defined on its maximal domain; moreover $\mathcal{L}_b$ has compact resolvent, hence purely discrete spectrum. By conjugation, the same properties hold for the operator L_b in the weighted space X. In view of (3.14), all eigenfunctions given by Proposition 3.2 belong to X, and the method of separation of variables ensures that the corresponding eigenfunctions can be chosen so as to form an orthogonal basis of X. We conclude that all eigenvalues of L_b in X are given by expressions (3.13), (3.15). The first few of them are represented in Fig. 2, for $n = 2$ and $n = 3$.

Remark 3.4 The eigenfunction of L_b given by (3.14) satisfies $u(x) \sim |x|^{\ell+\gamma}$ as $x \to 0$. In view of (3.13), the exponent $\ell + \gamma$ vanishes if $\ell = 0$ and is an increasing function of $\ell \in \mathbb{N}$. On the other hand, using estimate (2.14) and the fact that u solves the elliptic equation (2.9) with $A_\infty = A_b$ and $f(x) = \frac{1}{2}x \cdot \nabla u + (\frac{n}{2} - \lambda)u$, it is not difficult to verify that $|u(x) - u(0)| \le C|x|^\beta$ as $|x| \to 0$. This shows that $\beta \le \ell + \gamma$ for any $\ell \ge 1$, and taking

$\ell = 1$ we obtain

$$0 < \beta \le -\frac{n}{2} + 1 + \frac{1}{2}\sqrt{(n-2)^2 + 4b(n-1)}\Big). \tag{3.17}$$

The right-hand side of (3.17) is an increasing function of b which converges to 0 as $b \to 0$ and to 1 as $b \to 1$. We conjecture that the upper bound (3.17) is optimal for $b \in (0, 1)$.

3.3 Properties of the Principal Eigenfunction: The General Case

After considering two particular examples, we now return to the general case where the matrix $A_\infty(x)$ satisfies the assumptions listed at the beginning of Sect. 2. Much less is known on the operator L in that situation, but it is still possible to prove that the kernel of L in the space $L^2(m)$ is one-dimensional if $m > n/2$, so that $L^2(m) \hookrightarrow L^1(\mathbb{R}^n)$. We claim that the kernel of L is spanned by the function $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$\varphi(x) = \Gamma(x, 0, 1), \qquad x \in \mathbb{R}^n, \tag{3.18}$$

where $\Gamma(x, y, t)$ is the fundamental solution of (2.4). We already know that φ is Hölder continuous, and the estimates (2.7) imply that φ satisfies the Gaussian bounds (1.10). Moreover the normalization condition $\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x = 1$ follows from (2.6). Finally, we observe that the definition (3.18) reduces to (3.3) in the particular case where $A_\infty = \mathbb{1}$.

Lemma 3.5 *If φ defined by (3.18), then $\varphi \in D(L)$ and $L\varphi = 0$.*

Proof In view of (1.10), we have $\varphi \in L^2(m)$ for any $m \ge 0$. Moreover, the definition (3.18) implies that $\varphi = e^{-H/2}\psi$ where $\psi(x) = \Gamma(x, 0, 1/2)$. As $\psi \in L^2(\mathbb{R}^n)$, we thus have $\varphi \in D(H) \subset H^1(\mathbb{R}^n)$. To prove that $L\varphi = 0$, we start from identity (2.8) with $(y, t) = (0, 1)$, and we set $\lambda = \sqrt{t}$ where $t > 0$ is a new parameter. This gives the useful relation

$$\varphi(x) = t^{n/2}\,\Gamma\big(x\sqrt{t}, 0, t\big), \qquad x \in \mathbb{R}^n, \quad t > 0. \tag{3.19}$$

The idea is now to differentiate both sides of (3.19) with respect to t, at point $t = 1$. Using the fact that, by definition, the fundamental solution $(x, t) \mapsto \Gamma(x, y, t)$ is a solution of the evolution equation (2.4) for any fixed $y \in \mathbb{R}^n$, we obtain after straightforward calculations:

$$0 = \frac{n}{2}\varphi(x) + \frac{1}{2}x \cdot \nabla\varphi(x) + \mathrm{div}(A_\infty(x)\nabla\varphi(x)) \equiv \big(L\varphi\big)(x), \qquad x \in \mathbb{R}^n. \tag{3.20}$$

This shows that $\varphi \in D(L)$ and $L\varphi = 0$. □

To complete the proof of Proposition 1.1, it remains to verify that the kernel of L in the space of integrable functions is one-dimensional.

Lemma 3.6 *If $\psi \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ satisfies $L\psi = 0$ and $\int_{\mathbb{R}^n} \psi\,\mathrm{d}x = 1$, then $\psi = \varphi$.*

Proof If ψ is as in the statement, we define

$$u(x, t) = \frac{1}{t^{n/2}}\,\psi\Big(\frac{x}{\sqrt{t}}\Big), \qquad x \in \mathbb{R}^n, \quad t > 0.$$

We claim that, after modifying ψ on a negligible set if needed, we have the relation

$$\psi(x) = \int_{\mathbb{R}^n} \Gamma\big(x, y, 1-t\big)u(y, t)\,\mathrm{d}y \equiv \int_{\mathbb{R}^n} \Gamma\big(x, y\sqrt{t}, 1-t\big)\psi(y)\,\mathrm{d}y, \tag{3.21}$$

for all $x \in \mathbb{R}^n$ and all $t \in (0, 1)$. Indeed, if we differentiate with respect to time the last member of (3.21), considered as a distribution on $\mathbb{R}^n$, we obtain as in Lemma 3.5

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \Gamma\big(x, y\sqrt{t}, 1-t\big)\psi(y)\,\mathrm{d}y \;=\; -\frac{1}{t}\int_{\mathbb{R}^n} \Gamma\big(x, y\sqrt{t}, 1-t\big)(L\psi)(y)\,\mathrm{d}y \;=\; 0.$$

Thus the first integral in (3.21) is independent of time, and converges to $\psi(x)$ in $L^1(\mathbb{R}^n)$ as $t \to 1$, in view of the properties (2.6), (2.7) of the fundamental solution Γ. This proves (3.21).

We next take the limit $t \to 0$ in the second member of (3.21), for a fixed $x \in \mathbb{R}^n$. As Γ is Hölder continuous and satisfies (2.7), it is clear that

$$\int_{\mathbb{R}^n}\Big(\Gamma(x, y, 1-t) - \Gamma(x, y, 1)\Big)u(y, t)\,\mathrm{d}y \;\xrightarrow[t\to 0]{}\; 0.$$

Moreover $u(\cdot, t) \rightharpoonup \delta_0$ (the Dirac measure at the origin) as $t \to 0$, so that

$$\int_{\mathbb{R}^n} \Gamma(x, y, 1)u(y, t)\,\mathrm{d}y \;\xrightarrow[t\to 0]{}\; \Gamma(x, 0, 1) \;=\; \varphi(x).$$

We conclude that $\psi(x) = \varphi(x)$ for (almost) all $x \in \mathbb{R}^n$. □

The following properties of the derivatives of φ will be useful.

Proposition 3.7 *If φ is defined by (3.18), then $|\nabla\varphi| \in L^2(m)$ for all $m \in \mathbb{N}$. In addition we have $\nabla\varphi \in L^q(\mathbb{R}^n)$ for $2 \le q < n/(1-\beta)$, where β is as in Proposition 2.4 or 2.6.*

Proof Let $\chi : \mathbb{R}^n \to [0, 1]$ be a smooth and compactly supported function such that $\chi(x) = 1$ if $|x| \le 1$. We also assume that χ is radially symmetric and satisfies $x \cdot \nabla\chi(x) \le 0$ for all $x \in \mathbb{R}^n$. Given any $m \in \mathbb{N}$, we introduce for each $k \in \mathbb{N}^*$ the truncated weight function

$$p_k(x) \;=\; |x|^{2m}\chi(x/k), \qquad x \in \mathbb{R}^n.$$

We now multiply both sides of (3.20) by $p_k\varphi$ and integrate the resulting equality over $x \in \mathbb{R}^n$. After integrating by parts, we obtain the relation

$$\begin{aligned}\int_{\mathbb{R}^n} p_k\big(\nabla\varphi, A_\infty\nabla\varphi\big)\,\mathrm{d}x + \int_{\mathbb{R}^n} \varphi\big(\nabla p_k, A_\infty\nabla\varphi\big)\,\mathrm{d}x\\ = \frac{n}{4}\int_{\mathbb{R}^n} p_k\varphi^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^n}(x\cdot\nabla p_k)\varphi^2\,\mathrm{d}x,\end{aligned}$$

and using the ellipticity assumption (1.2) we deduce that

$$\lambda_1\int_{\mathbb{R}^n} p_k|\nabla\varphi|^2\,\mathrm{d}x \;\le\; \lambda_2\int_{\mathbb{R}^n}\varphi\,|\nabla p_k||\nabla\varphi|\,\mathrm{d}x + \frac{n}{4}\int_{\mathbb{R}^n} p_k\varphi^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^n}(x\cdot\nabla p_k)\varphi^2\,\mathrm{d}x. \tag{3.22}$$

As φ satisfies the Gaussian bound (1.10), we have $\int p_k\varphi^2\,\mathrm{d}x \to \int |x|^{2m}\varphi^2\,\mathrm{d}x$ as $k \to \infty$. To control the other terms in the right-hand side of (3.22), we observe that

$$\nabla p_k(x) \;=\; 2mx|x|^{2m-2}\chi(x/k) + \frac{1}{k}\,|x|^{2m}\nabla\chi(x/k),$$

from which we infer

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi\,|\nabla p_k||\nabla\varphi|\,\mathrm{d}x &\;\xrightarrow[k\to\infty]{}\; 2m\int_{\mathbb{R}^n}|x|^{2m-1}\varphi|\nabla\varphi|\,\mathrm{d}x,\\ \int_{\mathbb{R}^n}(x\cdot\nabla p_k)\varphi^2\,\mathrm{d}x &\;\xrightarrow[k\to\infty]{}\; 2m\int_{\mathbb{R}^n}|x|^{2m}\varphi^2\,\mathrm{d}x.\end{aligned}$$

Thus taking the limit $k \to \infty$ in (3.22) and using the monotone convergence theorem, we conclude that

$$\lambda_1 \int_{\mathbb{R}^n} |x|^{2m} |\nabla\varphi|^2 \,\mathrm{d}x \;\le\; 2m\lambda_2 \int_{\mathbb{R}^n} |x|^{2m-1}\varphi|\nabla\varphi| \,\mathrm{d}x + \Big(\frac{n}{4} - \frac{m}{2}\Big)\int_{\mathbb{R}^n} |x|^{2m}\varphi^2 \,\mathrm{d}x \;<\; \infty.$$

This shows that $|x|^m|\nabla\varphi| \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$, hence $|\nabla\varphi| \in L^2(m)$ for all $m \in \mathbb{N}$.

We next prove the second assertion in Proposition 3.7. We know from (3.20) that φ satisfies the elliptic equation (2.9) with $f(x) = \frac{1}{2}x \cdot \nabla\varphi(x) + \frac{n}{2}\varphi(x) = \frac{1}{2}\mathrm{div}(x\varphi)$. We thus have the representation (2.10), which is valid even in the two-dimensional case because $\int f(x)\,\mathrm{d}x = 0$. Differentiating both sides of (2.10) we obtain

$$\nabla\varphi(x) \;=\; \int_{\mathbb{R}^n} \nabla_x G(x, y) f(y)\,\mathrm{d}y \;=\; \int_{\mathbb{R}^n} \nabla_x G(x, y)\big(\tfrac{1}{2}y \cdot \nabla\varphi(y) + \tfrac{n}{2}\varphi(y)\big)\,\mathrm{d}y, \quad (3.23)$$

for (almost) all $x \in \mathbb{R}^n$. This relation allows us to estimate $\nabla\varphi$ in $L^p(\mathbb{R}^n)$ for some $p > 2$ using the following lemma, which is proved below.

Lemma 3.8 *Let $p \in \big(1, \frac{n}{2-\beta}\big)$ where $\beta \in (0, 1)$ is as in (2.15). If $f \in L^p(\mathbb{R}^n)$, the function g defined by $g(x) = \int_{\mathbb{R}^n} \nabla_x G(x, y) f(y)\,\mathrm{d}y$ belongs to $L^q(\mathbb{R}^n)$ with q such that $\frac{1}{q} + \frac{1}{n} = \frac{1}{p}$.*

Let $p_* = n/(2-\beta)$ and $q_* = n/(1-\beta)$. We first assume that $p_* \le 2$, which means that either $n = 2$, or $n = 3$ and $\beta \le 1/2$. We know from (1.10) and from the previous step that $f \in L^2(m)$ for all $m \in \mathbb{N}$, hence $f \in L^p(\mathbb{R}^n)$ for all $p \in [1, 2]$. We can thus apply Lemma 3.8 to (3.23) for any $p \in (1, p_*)$, and we obtain that $\nabla\varphi$ belongs to $L^q(\mathbb{R}^n)$ for any $q \in (2, q_*)$, which gives the desired conclusion.

We next consider the case where $p_* > 2$, which requires a bootstrap argument. For any $j \in \mathbb{N}$ with $j < n/2$, we denote $p_j = 2n/(n-2j)$, and we observe that $1/p_j = 1/n + 1/p_{j+1}$. As before, we start with the knowledge that $f \in L^p(\mathbb{R}^n)$ for all $p \in [1, 2] \equiv [1, p_0]$, and a first application of Lemma 3.8 to (3.23) shows that $\nabla\varphi$ belongs to $L^q(\mathbb{R}^n)$ for all $q \in (2, p_1)$. Since we also know that $|x|^m\nabla\varphi \in L^2(\mathbb{R}^n)$ for any $m \in \mathbb{N}$, we obtain by interpolation that $y \cdot \nabla\varphi \in L^r(\mathbb{R}^n)$ for any $r \in (2, q)$; in particular, we have shown that $f \in L^p(\mathbb{R}^n)$ for all $p \in [1, p_1)$. Repeating the same argument if needed, we prove inductively that $f \in L^p(\mathbb{R}^n)$ for all $p \in [1, p_j)$ $(j = 1, 2, \dots)$, until we reach the smallest $j \in \mathbb{N}^*$ such that $p_j \ge p_*$. At this point we know that $f \in L^p(\mathbb{R}^n)$ for all $p \in [1, p_*)$, and Lemma 3.8 implies that $\nabla\varphi \in L^q(\mathbb{R}^n)$ for all $q \in (2, q_*)$. □

Proof of Lemma 3.8 In view of (2.15), we have $|g(x)| \le C\big(\psi_1(x) + \psi_2(x)\big)$ where

$$\psi_1(x) \;=\; \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}}\,|f(y)|\,\mathrm{d}y, \qquad \psi_2(x) \;=\; \frac{1}{|x|^{1-\beta}}\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2+\beta}}\,|f(y)|\,\mathrm{d}y.$$

The Hardy–Littlewood–Sobolev inequality directly yields $\psi_1 \in L^q(\mathbb{R}^n)$, see e.g. [23]. To control ψ_2, we apply Proposition 2.7 with $k(x, y) = |x|^{\beta-1}|x - y|^{2-n-\beta}$, which is a homogeneous kernel of degree $-d = -(n-1)$. As $1 + 1/q = 1/p + d/n$ and $p < n/(2-\beta) < n = n/(n-d)$, we only need to check the condition (2.31), namely

$$\int_{\mathbb{R}^n} \bigg(\frac{1}{|x-y|^{n-2+\beta}\,|y|^{n/q}}\bigg)^{n/(n-1)} \mathrm{d}y \;<\; \infty, \qquad \text{for some } x \in \mathbb{S}^{n-1}.$$

Our assumptions on p are equivalent to $\frac{n}{n-1} < q < \frac{n}{1-\beta}$, and these inequalities ensure that the integral above converges for small $|y|$ and for large $|y|$, respectively. Moreover, the singularity at $y = x$ is integrable because $\beta < 1$, so that $\psi_2 \in L^q(\mathbb{R}^n)$ by Proposition 2.7. □

Remark 3.9 Since the coefficient A_∞ of the operator L is Lipschitz outside the origin, the classical regularity theory for second order elliptic equations [16] implies that any eigenfunction of L, in particular the principal eigenfunction φ, is necessarily of class $C^{1,\alpha}$ on $\mathbb{R}^n \setminus \{0\}$ for some $\alpha > 0$. However, the example studied in Sect. 3.2 shows that $\nabla\varphi$ may have a singularity at the origin, as it is the case for the function ψ_2 in the above proof. This indicates that estimate (2.15) for the Green function cannot be substantially improved in general.

Remark 3.10 In the constant coefficient case, the relation (3.2) shows that the operator L is formally conjugated to a selfadjoint operator. Such a property is not known to hold in general, but the following observation can be made. If $\Phi : \mathbb{R}^n \to \mathbb{R}$ has bounded second-order derivatives, a direct calculation shows that

$$e^{\Phi} L\big(e^{-\Phi}u\big) = \mathrm{div}(A_\infty \nabla u) + \frac{n}{2}\, u + V_\Phi \cdot \nabla u + W_\Phi u, \tag{3.24}$$

for all $u \in C_c^2(\mathbb{R}^n)$, where the functions V_Φ and W_Φ are given by

$$V_\Phi = \frac{x}{2} - 2A_\infty \nabla\Phi, \qquad W_\Phi = \big(A_\infty \nabla\Phi, \nabla\Phi\big) - \mathrm{div}(A_\infty \nabla\Phi) - \frac{x}{2} \cdot \nabla\Phi.$$

The conjugated operator (3.24) is symmetric in $L^2(\mathbb{R}^n)$ if $V_\Phi = 0$, namely if $A_\infty \nabla\Phi = x/4$. For a general matrix $A_\infty(x)$ satisfying the assumptions listed in Sect. 2, there is no function Φ with that property. However, if we assume that $A_\infty(x)x = x$ for all $x \in \mathbb{R}^n$, which is the case for the Meyers–Serrin matrix (1.15), we can take $\Phi(x) = |x|^2/8$ and we obtain, in close analogy with (3.2),

$$e^{|x|^2/8} L\big(e^{-|x|^2/8}u\big) = \mathrm{div}(A_\infty \nabla u) - \frac{|x|^2}{16}\, u + \frac{n}{4}\, u.$$

Note that, in that situation, we also have $L\varphi = 0$ where φ is given by (3.3).

Remark 3.11 It is interesting to note that, in general, the principal eigenfunction of the operator L is not given by the explicit expression (3.3). In fact, let $B : \mathbb{R}^n \to \mathcal{M}_n(\mathbb{R})$ be a matrix valued function that is homogeneous of degree zero, smooth outside the origin, symmetric and uniformly elliptic in the sense of (1.2). We want to determine under which additional conditions the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(x) = \exp\Big(-\frac{1}{4}\big(B(x)x, x\big)\Big), \qquad x \in \mathbb{R}^n, \tag{3.25}$$

is (up to normalization) the principal eigenfunction of the operator L for some appropriate choice of the diffusion matrix A_∞. This is certainly the case if we can construct A_∞ in such a way that $A_\infty(x)\nabla\varphi(x) + \frac{x}{2}\varphi(x) = 0$ for all $x \in \mathbb{R}^n$, because the desired property $L\varphi = 0$ then follows by taking the divergence with respect to the variable x. In view of (3.25), the condition on A_∞ becomes

$$A_\infty(x)B(x)x + \frac{1}{2}A_\infty(x)\big(\nabla B(x)x, x\big) = x, \qquad x \in \mathbb{R}^n, \tag{3.26}$$

where $\big(\nabla B(x)x, x\big) \in \mathbb{R}^n$ denotes the vector with components $\big(\partial_j B(x)x, x\big)$ for $j = 1, \ldots, n$. Consider the matrix $M(x)$ and the vectors $\zeta(x), \xi(x)$ defined as follows:

$$M(x) = B(x)^{1/2} A_\infty(x) B(x)^{1/2}, \quad \zeta(x) = B(x)^{1/2} x, \quad \xi(x) = \frac{1}{2} B(x)^{-1/2}\big(\nabla B(x)x, x\big).$$

Then our condition (3.26) can be written in the equivalent form

$$M(x)\zeta(x) + M(x)\xi(x) = \zeta(x), \qquad x \in \mathbb{R}^n. \tag{3.27}$$

Moreover, we observe that

$$2\big(\zeta(x),\xi(x)\big) \,=\, \sum_{j=1}^{n} x_j\big((\partial_j B)x,x\big) \,=\, \Big(\Big(\sum_{j=1}^{n} x_j\partial_j B\Big)x,x\Big) \,=\, 0,$$

because $B(x)$ is homogeneous of degree zero; we deduce that $\zeta(x)\perp\xi(x)$ for all $x\in\mathbb{R}^n$. Now, if $M(x)$ is the symmetric matrix with components $M_{ij}(x)$ defined by

$$M_{ij}(x) \,=\, \frac{|\xi(x)|^2}{|\zeta(x)|^4}\,\zeta_i(x)\zeta_j(x) - \frac{1}{|\zeta(x)|^2}\,\big(\zeta_i(x)\xi_j(x)+\xi_i(x)\zeta_j(x)\big)+\delta_{ij}, \qquad x\in\mathbb{R}^n\setminus\{0\},$$

it is straightforward to verify that (3.27) hold for all $x\in\mathbb{R}^n$, and that the map $x\mapsto M(x)$ is homogeneous of degree zero. Moreover, the matrix $M(x)$ is positive definite if we assume that $|\xi(x)|\le\kappa|\zeta(x)|$ for some $\kappa<1$, which is the case if ∇B is sufficiently small compared to B on the unit sphere $\mathbb{S}^{n-1}$. Under that assumption, if we set $A_\infty(x)=B(x)^{-1/2}M(x)B(x)^{-1/2}$, we conclude that A_∞ satisfies the assumptions listed in Sect. 2 and that the operator L defined by (3.1) has the property that $L\varphi=0$, where φ is defined by (3.25).

4 Long-Time Asymptotics in the Linear Case

This section is devoted to the proof of Theorem 1.3. We start from the rescaled equation (1.6) with $\mathcal{N}=0$, namely

$$\partial_\tau v \,=\, \mathrm{div}\Big(A\big(ye^{\tau/2}\big)\nabla v\Big)+\frac{1}{2}\,y\cdot\nabla v+\frac{n}{2}\,v, \qquad y\in\mathbb{R}^n, \quad \tau>0, \tag{4.1}$$

and we consider it as an evolution equation in the weighted space $L^2(m)$ defined in (1.11).

Lemma 4.1 *For any $m\ge 0$, the Cauchy problem for Eq. (4.1) is globally well-posed in $L^2(m)$.*

Proof That statement, as well as all subsequent claims regarding existence and regularity of solutions to (4.1), can be justified by the following standard arguments. If we undo the change of variables (1.5), we obtain the linear diffusion equation (1.1) with $N=0$, namely

$$\partial_t u(x,t) \,=\, \mathrm{div}\big(A(x)\nabla u(x,t)\big), \qquad x\in\mathbb{R}^n, \quad t>0, \tag{4.2}$$

which is known to define an analytic evolution semigroup in the Hilbert space $L^2(\mathbb{R}^n)$, see Sect. 2 for a similar analysis. We set $u(x,t)=p(x)\tilde u(x,t)$, where $p(x)=(1+|x|^2)^{-m/2}$. The new function $\tilde u$ then satisfies the modified evolution equation

$$\partial_t\tilde u \,=\, \mathrm{div}\big(A(x)\nabla\tilde u\big)+\frac{2}{p}\,\big(\nabla p,\ A(x)\nabla\tilde u\big)+\frac{1}{p}\,\mathrm{div}\big(A(x)\nabla p\big)\tilde u, \tag{4.3}$$

which differs from (4.2) by a relatively compact perturbation, in the sense of operator theory. It follows [27, Section 3.2] that (4.3) defines an analytic semigroup in $L^2(\mathbb{R}^n)$, which amounts to saying that (4.2) defines an analytic semigroup in $L^2(m)$. In particular, given initial data $u_0\in L^2(m)$, Eq. (4.2) has a unique solution $u\in C^0([0,+\infty),L^2(m))\cap C^1((0,+\infty),L^2(m))$ such that $u(0)=u_0$. Moreover $\nabla u\in C^0((0,+\infty),L^2(m)^n)\cap L^2((0,T),L^2(m)^n)$ for any $T>0$. Applying now the change of variables (1.5), which leaves the space $L^2(m)$ invariant, we conclude in particular that, given initial data $v_0\in L^2(m)$, Eq. (4.1) has a unique global solution $v\in C^0([0,+\infty),L^2(m))$ such that $v(0)=v_0$. □

4.1 Spectral Decomposition of the Solution

We assume from now on that $m > n/2$, so that $L^2(m) \hookrightarrow L^1(\mathbb{R}^n)$. If $v \in C^0([0,+\infty), L^2(m))$ is a solution of (4.1) with initial data $v_0 \in L^2(m)$, we observe that

$$\int_{\mathbb{R}^n} v(y,\tau)\,\mathrm{d}y \;=\; \int_{\mathbb{R}^n} v_0(y)\,\mathrm{d}y, \qquad \text{for all } \tau \ge 0. \tag{4.4}$$

Indeed, if $u \in C^0([0,+\infty), L^2(m)) \cap C^1((0,+\infty), L^2(m))$ is the corresponding solution of (4.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^n} \mathrm{div}\big(A(x)\nabla u(x,t)\big)\,\mathrm{d}x \;=\; 0, \qquad \text{for all } t > 0,$$

where the last equality follows from Lemma 6.2 since $\mathrm{div}\big(A\nabla u\big) \in L^2(m)$ and $A\nabla u \in L^2(m)^n$ for any $t > 0$. It follows that the integral of $u(\cdot,t)$ does not depend on time, and the same property holds for the rescaled function $v(\cdot,\tau)$ in view of (1.5). This gives (4.4).

We also recall that, in view of (1.3) and (1.4), the diffusion matrix A can be decomposed as

$$A(x) \;=\; A_\infty(x) + B(x), \qquad x \in \mathbb{R}^n, \tag{4.5}$$

where A_∞ is homogeneous of degree zero and the remainder B satisfies

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^\nu \,\|B(x)\| \;<\; \infty, \qquad \text{for some } \nu > 0. \tag{4.6}$$

Let L be the limiting operator (3.1), and $\varphi \in L^2(m)$ be the principal eigenfunction of L given by Proposition 1.1. We decompose the solution of (4.1) in the following way:

$$v(y,\tau) \;=\; \alpha\varphi(y) + w(y,\tau), \qquad \text{where} \qquad \alpha \;=\; \int_{\mathbb{R}^n} v(y,\tau)\,\mathrm{d}y. \tag{4.7}$$

Since φ is normalized so that $\int_{\mathbb{R}^n}\varphi\,\mathrm{d}y = 1$, it follows from (4.7) that $\int_{\mathbb{R}^n} w(y,\tau)\,\mathrm{d}y = 0$ for all $\tau \ge 0$. Moreover, in view of (4.1) and (1.9), the evolution equation satisfied by w is

$$\partial_\tau w \;=\; \mathrm{div}\Big(A\big(ye^{\tau/2}\big)\nabla w\Big) + \frac{1}{2}\,y\cdot\nabla w + \frac{n}{2}\,w + r_1, \qquad y \in \mathbb{R}^n, \quad \tau > 0, \tag{4.8}$$

where

$$r_1(y,\tau) \;=\; \alpha\,\mathrm{div}\big(B(ye^{\tau/2})\nabla\varphi(y)\big). \tag{4.9}$$

Remark 4.2 As simple as it may seem, the decomposition (4.7) is an essential step in the proof of Theorem 1.3. To understand its meaning, let us assume for the moment that the solutions of (4.1) are well approximated, for large times, by those of the limiting equation (1.8); this is certainly expected in view of (4.5), (4.6). So our task is to understand the long-time behavior of the semigroup $e^{\tau L}$ generated by the limiting operator (3.1). In the weighted space $L^2(m)$ with $m > n/2$, we claim that 0 is a simple eigenvalue of L, and that the rest of the spectrum is contained in the half-plane $\{z \in \mathbb{C}\,|\,\mathrm{Re}(z) \le -\mu\}$ for some $\mu > 0$. This is in fact what Theorem 1.3 asserts in the particular case where $A = A_\infty$. As is easily verified, the spectral projection P onto the kernel of L is the map $v \mapsto Pv$ defined by

$$(Pv)(y) \;=\; \varphi(y)\int_{\mathbb{R}^n} v(y)\,\mathrm{d}y, \qquad y \in \mathbb{R}^n.$$

With this notation, the decomposition (4.7) simply reads $v = Pv + w$ where $w = (1-P)v$. To prove Theorem 1.3, our strategy is to show that the solutions of (4.8) in the invariant subspace

$L^2_0(m) \equiv (1-P)L^2(m)$ decay exponentially to zero as $\tau \to +\infty$, even though the equation for w involves the time-dependent matrix $A(ye^{\tau/2})$ instead of the limiting matrix $A_\infty(y)$. As we shall see in the rest of this section, the exponential decay of w can be established using appropriate energy estimates.

4.2 Weighted Estimates for the Perturbation

Given any solution w of (4.8) in $L^2(m)$, we consider the energy functional

$$e_{m,\delta}(\tau) \;=\; \frac{1}{2}\int_{\mathbb{R}^n}(\delta+|y|^2)^m w(y,\tau)^2\,\mathrm{d}y, \qquad \tau \ge 0, \tag{4.10}$$

where $\delta > 0$ is a parameter that will be fixed later on. This quantity is differentiable for $\tau > 0$, and using (4.8) we find

$$\begin{aligned}\partial_\tau e_{m,\delta}(\tau) \;&=\; \int(\delta+|y|^2)^m w\Big[\mathrm{div}(A(ye^{\tau/2})\nabla w)+\frac{1}{2}(y\cdot\nabla)w+\frac{n}{2}w+r_1\Big]\,\mathrm{d}y\\ &=\; -\int\big\langle\nabla\big((\delta+|y|^2)^m w\big),\,A(ye^{\tau/2})\nabla w\big\rangle\,\mathrm{d}y+\frac{1}{4}\int(\delta+|y|^2)^m y\cdot\nabla(w^2)\,\mathrm{d}y\\ &\quad+\frac{n}{2}\int(\delta+|y|^2)^m|w|^2\,\mathrm{d}y\\ &\quad-\alpha\int\big\langle\nabla\big((\delta+|y|^2)^m w\big),\,B(ye^{\tau/2})\nabla\varphi\big\rangle\,\mathrm{d}y,\end{aligned}\tag{4.11}$$

where the second equality is obtained after integrating by parts and using the definition (4.9) of the quantity r. Here and in what follows, it is understood that all integrals are taken over the whole space $\mathbb{R}^n$. In view of the elementary identities

$$\begin{aligned}\nabla\big((\delta+|y|^2)^m\big) \;&=\; 2my\,(\delta+|y|^2)^{m-1},\\ \mathrm{div}\big(y(\delta+|y|^2)^m\big) \;&=\; (n+2m)(\delta+|y|^2)^m-2m\delta(\delta+|y|^2)^{m-1},\end{aligned}$$

we can write (4.11) in the equivalent form

$$\begin{aligned}\partial_\tau e_{m,\delta}(\tau) \;&=\; -\int(\delta+|y|^2)^m\big\langle\nabla w,\,A(ye^{\tau/2})\nabla w\big\rangle\,\mathrm{d}y\\ &\quad-2m\int(\delta+|y|^2)^{m-1}w\big\langle y,\,A(ye^{\tau/2})\nabla w\big\rangle\,\mathrm{d}y\\ &\quad+\frac{n-2m}{4}\int(\delta+|y|^2)^m|w|^2\,\mathrm{d}y+\frac{m\delta}{2}\int(\delta+|y|^2)^{m-1}|w|^2\,\mathrm{d}y\\ &\quad-\alpha\int(\delta+|y|^2)^m\big\langle\nabla w,\,B(ye^{\tau/2})\nabla\varphi\big\rangle\,\mathrm{d}y\\ &\quad-2\alpha m\int(\delta+|y|^2)^{m-1}w\langle y,\,B(ye^{\tau/2})\nabla\varphi\rangle\,\mathrm{d}y.\end{aligned}\tag{4.12}$$

The second term in the right-hand side of (4.12) has no obvious sign, but applying Hölder's inequality we can estimate it as follows:

$$\begin{aligned} 2m\Big|\int(\delta+|y|^2)^{m-1}w\big\langle y, A(ye^{\tau/2})\nabla w\big\rangle\,\mathrm{d}y\Big| &\le \frac{1}{4}\int(\delta+|y|^2)^m\big\langle\nabla w, A(ye^{\tau/2})\nabla w\big\rangle\,\mathrm{d}y + 4m^2\int(\delta+|y|^2)^{m-2}|w|^2\big\langle y, A(ye^{\tau/2})y\big\rangle\,\mathrm{d}y \\ &\le \frac{1}{4}\int(\delta+|y|^2)^m\big\langle\nabla w, A(ye^{\tau/2})\nabla w\big\rangle\,\mathrm{d}y + Cm^2\int(\delta+|y|^2)^{m-1}|w|^2\,\mathrm{d}y, \end{aligned}$$

where in the last line we used the obvious fact that $(\delta+|y|^2)^{m-2}|y|^2 \le (\delta+|y|^2)^{m-1}$. Here and below, we denote by C any positive constant depending only on the properties of the matrix A. We proceed in a similar way to bound the last two lines of (4.12), and this leads to the inequality

$$\begin{aligned} \partial_\tau e_{m,\delta}(\tau) \;\le\; &-\frac{1}{2}\int(\delta+|y|^2)^m\big\langle\nabla w, A(ye^{\tau/2})\nabla w\big\rangle\,\mathrm{d}y + \frac{n-2m}{2}\,e_{m,\delta}(\tau) \\ &+ \big(m\delta + C_1 m^2\big)\,e_{m-1,\delta}(\tau) + C_2\alpha^2\int(\delta+|y|^2)^m\,\|B(ye^{\tau/2})\|^2|\nabla\varphi|^2\,\mathrm{d}y, \end{aligned} \tag{4.13}$$

for some positive constants C_1, C_2.

Remark 4.3 If we forget for the moment the last term in (4.13), assuming thus that $B \equiv 0$, we have shown that

$$\partial_\tau e_{m,\delta}(\tau) \;\le\; \frac{n-2m}{2}\,e_{m,\delta}(\tau) + \big(m\delta + C_1 m^2\big)\,e_{m-1,\delta}(\tau), \qquad \tau > 0. \tag{4.14}$$

If $m = 0$, so that $e_{0,\delta}(\tau) = \frac{1}{2}\|w(\cdot,\tau)\|_{L^2}^2$, the last term in (4.14) disappears, and we are left with the differential inequality $\partial_\tau e_{0,\delta} \le (n/2)\,e_{0,\delta}$ which allows for an exponential growth in time. This is compatible with the spectral picture in Fig. 1, where the essential spectrum of the operator L fills the half-plane $\{\mathrm{Re}(z) \le n/4\}$ if $m = 0$. Now, if we assume that $m > n/2$, the coefficient in front of $e_{m,\delta}$ in the right-hand side of (4.14) becomes negative, but then we also have the "lower order term" proportional to $e_{m-1,\delta}$ which makes it impossible to prove exponential decay using only (4.14). The obstacle we hit here is in the nature of things: we cannot prove exponential decay in time of the solution of (4.8) if we do not use the crucial fact that $\int_{\mathbb{R}^n} w\,\mathrm{d}y = 0$.

4.3 Evolution Equation for the Antiderivative

If we want to study evolutionary PDEs using just L^2 energy estimates, it is not straightforward to exploit the information, if applicable, that the solutions under consideration have zero mean. In the one-dimensional case, the following elementary observation was made in [14] and applied to the analysis of parabolic or damped hyperbolic equations: if $u : \mathbb{R} \to \mathbb{R}$ belongs to $L^2(m)$ for some $m \ge 1$ and has zero mean, the primitive function $U(x) = \int_{-\infty}^x u(y)\,\mathrm{d}y$ is square integrable and satisfies $\|U\|_{L^2} \le 2\|xu\|_{L^2}$ (this is a variant of Hardy's inequality). The idea is then to control the evolution of the primitive U using L^2 energy estimates, and it turns out that this procedure takes into account the information that the original function u has zero mean.

In the same spirit, we propose here an approach that works in dimensions two and three, and can be extended to cover the higher-dimensional cases as well (see Sect. 4.5 below). If

$m > n/2$ and $w \in L_0^2(m)$, so that $\int_{\mathbb{R}^n} w(y)\,\mathrm{d}y = 0$, the idea is to define the "antiderivative" W of w as the solution of the elliptic equation

$$-\operatorname{div}\big(A_\infty(y)\nabla W(y)\big) \;=\; w(y), \qquad y \in \mathbb{R}^n. \tag{4.15}$$

More precisely we set $W = K[w]$, where K denotes the integral operator (2.26) whose kernel is the Green function $G(x, y)$ of the differential operator in (4.15), see Sect. 2.5. We recall that, if $m \in (n/2, n/2+\beta)$ where $\beta \in (0, 1)$ is defined in Proposition 2.4, then K is a bounded linear operator from $L_0^2(m)$ to $L^2(m-2)$. Moreover, as is shown in Proposition 2.14, the operator K can be extended so as to act on first order distributions of the form $w = \operatorname{div} g$, where $g \in L^2(m-1)^n$.

The definition (4.15) has the property that the antiderivative W satisfies a nice equation if w evolves according to (4.8).

Lemma 4.4 *Assume that $m \in (n/2, n/2 + \beta)$, and that $w \in C^0([0, +\infty), L_0^2(m))$ is a solution of Eq. (4.8). If we define $W(\cdot, \tau) = K[w(\cdot, \tau)]$ for $\tau \geq 0$, then $W \in C^0([0, +\infty), L^2(m-2))$ is a solution of the evolution equation*

$$\partial_\tau W \;=\; \operatorname{div}\big(A_\infty(y)\nabla W\big) + \frac{1}{2}\, y \cdot \nabla W + \frac{n-2}{2}\, W + R_1, \tag{4.16}$$

where the remainder term $R(y, \tau)$ is given by

$$R_1(\cdot, \tau) \;=\; K\Big[\operatorname{div}\Big(B(\cdot\, e^{\tau/2})(\alpha\nabla\varphi + \nabla w)\Big)\Big], \qquad \tau \geq 0. \tag{4.17}$$

Proof We rewrite the evolution equation (4.8) in the equivalent form

$$\partial_\tau w \;=\; \operatorname{div}\Big(A_\infty(y)\nabla w\Big) + \frac{1}{2}\operatorname{div}\big(yw\big) + \tilde{r}_1, \tag{4.18}$$

where $\tilde{r}_1(y, \tau) = \operatorname{div}\big[B(ye^{\tau/2})\big(\alpha\nabla\varphi(y) + \nabla w(y, \tau)\big)\big]$, and we apply the linear operator K to both sides of (4.18). Since $W = K[w]$ and $R_1 = K[\tilde{r}_1]$ by definition, it remains to treat the first two terms in the right-hand side, which are in divergence form so that we can apply Corollary 2.15. We assume here that $\nabla w(\cdot, \tau) \in L^2(m)^n$, which is the case as soon as $\tau > 0$. We make the following observations:

1. Let $F = -(K \circ \operatorname{div})\big[A_\infty \nabla w\big]$, where $w \in L_0^2(m)$ and $\nabla w \in L^2(m)^n$. By (2.40), we have

$$F(x) \;=\; \int_{\mathbb{R}^n} \Big(\nabla_y G(x, y) \cdot \big(A_\infty(y)\nabla w(y)\big)\Big)\,\mathrm{d}y \;=\; \int_{\mathbb{R}^n} \big(A_\infty(y)\nabla_y G(x, y), \nabla w(y)\big)\,\mathrm{d}y,$$

for (almost) all $x \in \mathbb{R}^n$. If $w \in C_c^\infty(\mathbb{R}^n)$, the right-hand side is equal to $w(x)$ by (2.13), and using a density argument we deduce that $F = w$ in the general case. As $w = -\operatorname{div}\big(A_\infty \nabla W\big)$ by Remark 2.10, this gives the elegant relation $(K \circ \operatorname{div})\big[A_\infty \nabla w\big] = \operatorname{div}\big(A_\infty \nabla W\big)$.

2. As the matrix A_∞ is homogeneous of degree zero, the Green function G has the following property: there exists a constant $c_0 \in \mathbb{R}$ such that, for all $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$(n-2)G(x, y) + x \cdot \nabla_x G(x, y) + y \cdot \nabla_y G(x, y) \;=\; -c_0. \tag{4.19}$$

Indeed, if $n \geq 3$, we have $\lambda^{n-2}G(\lambda x, \lambda y) = G(x, y)$ for any $\lambda > 0$, and this implies the Euler relation (4.19) with $c_0 = 0$; when $n = 2$, we deduce (4.19) directly from (2.24).

If $w \in L^2_0(m)$ and $g(y) = yw(y)$, then $g \in L^2(m-1)^n$ and, in view of (2.40) and Proposition 2.14, we have

$$\begin{aligned}\big[K \circ \mathrm{div} g\big](x) &= -\int \big(y \cdot \nabla_y G(x,y)\big) w(y)\,\mathrm{d}y \\ &= \int \big(x \cdot \nabla_x G(x,y) + (n-2)G(x,y) + c_0\big) w(y)\,\mathrm{d}y \\ &= \int \Big(\mathrm{div}_x\big(x\,G(x,y)\big) - 2G(x,y)\Big) w(y)\,\mathrm{d}y \\ &= \Big(\mathrm{div}\big(xK[w]\big) - 2K[w]\Big)(x),\end{aligned}$$

where we used (4.19) and the fact that $\int w\,\mathrm{d}y = 0$. After changing x into y, the relation above becomes $(K \circ \mathrm{div})[yw] = \mathrm{div}(yW) - 2W = y \cdot \nabla W + (n-2)W$.

Summarizing, if apply the operator K to all terms in (4.18) and use the steps **1** and **2** above, we arrive at (4.16). □

Notice that Eq. (4.16) is very similar to (4.8), with the important difference that the "amplification factor" $n/2$ in the right-hand side of (4.8) is reduced to $(n-2)/2$ in (4.16). This makes it possible to control the evolution of the antiderivative W using energy estimates if $n \le 3$. To this end, we introduce the following additional energy functional:

$$E_{m-2,\delta}(\tau) = \frac{1}{2}\int_{\mathbb{R}^n} (\delta + |y|^2)^{m-2} |W(y,\tau)|^2\,\mathrm{d}y, \qquad \tau \ge 0. \tag{4.20}$$

Repeating the same calculations as in Sect. 4.2, we obtain in analogy with (4.13):

$$\begin{aligned}\partial_\tau E_{m-2,\delta}(\tau) \le& -\frac{1}{2}\int (\delta + |y|^2)^{m-2}\big\langle \nabla W, A_\infty \nabla W\big\rangle\,\mathrm{d}y + \frac{n-2m}{2} E_{m-2,\delta}(\tau) \\ &+ \Big((m-2)\delta + C_1(m-2)^2\Big) E_{m-3,\delta}(\tau) + \int (\delta + |y|^2)^{m-2}\, W R_1\,\mathrm{d}y.\end{aligned} \tag{4.21}$$

Remark 4.5 In the derivation of (4.21), the coefficient in front of $E_{m-2,\delta}(\tau)$ in the right-hand side is obtained through the elementary calculation

$$\frac{n-2m}{4} = \frac{n-2}{2} - \frac{n+2(m-2)}{4},$$

where we observe that the smaller "amplification factor" $(n-2)/2$ in (4.16) is exactly compensated by the fact that we estimate the antiderivative W in $L^2(m-2)$ instead of $L^2(m)$. As a result, we obtain exactly the same coefficient $(n-2m)/2$ in both estimates (4.13) and (4.21).

4.4 Exponential Decay of the Perturbation in Low Dimensions

In this section, we assume that $n = 2$ or $n = 3$, and we combine estimates (4.13), (4.21) to prove that the solutions of (4.8) in $L^2_0(m)$ converge exponentially to zero as $\tau \to +\infty$. For the moment, we assume that $m \in (n/2, n/2 + \beta)$, so that we can apply Proposition 2.11 to control the antiderivative W, and for convenience we also suppose that $m \le 2$ (note, however, that all upper bounds on m will be relaxed later). The crucial observation is that the coefficient of $E_{m-3,\delta}$ in (4.21) vanishes if $m = 2$, and becomes negative if $m < 2$ provided that the parameter $\delta > 0$ is chosen large enough. Therefore, we assume that

$$m = \frac{n}{2} + \lambda, \quad \text{where} \quad 0 < \lambda < \beta, \quad \lambda \le 2 - \frac{n}{2}, \quad \text{and} \quad \delta \ge 2C_1(2-m). \tag{4.22}$$

Under these hypotheses, inequalities (4.21), (4.13) become

$$\begin{aligned} \partial_\tau E_{m-2,\delta}(\tau) &\le -\frac{\lambda_1}{2}\int(\delta+|y|^2)^{m-2}|\nabla W|^2\,\mathrm{d}y - \lambda E_{m-2,\delta}(\tau) + \int(\delta+|y|^2)^{m-2} W R_1\,\mathrm{d}y, \\ \partial_\tau e_{m,\delta}(\tau) &\le -\frac{\lambda_1}{2}\int(\delta+|y|^2)^{m}|\nabla w|^2\,\mathrm{d}y - \lambda e_{m,\delta}(\tau) + C_3\, e_{m-1,\delta}(\tau) \\ &\quad + C_2\,\alpha^2\int(\delta+|y|^2)^m\|B(ye^{\tau/2})\|^2|\nabla\varphi|^2\,\mathrm{d}y, \end{aligned} \tag{4.23}$$

where $C_3 = m\delta + C_1 m^2$ and $\lambda_1 > 0$ is as in (1.2).

The next step is a simple interpolation argument which allows us to control the undesirable quantity $C_3\, e_{m-1,\delta}$ in (4.23) using the negative terms involving ∇w and ∇W. In view of (4.15), we have

$$\begin{aligned} 2e_{m-1,\delta} &= \int(\delta+|y|^2)^{m-1}|w|^2\,\mathrm{d}y = -\int(\delta+|y|^2)^{m-1} w\,\mathrm{div}\big(A_\infty\nabla W\big)\,\mathrm{d}y \\ &= \int(\delta+|y|^2)^{m-1}\big\langle\nabla w, A_\infty\nabla W\big\rangle\,\mathrm{d}y + 2(m-1)\int(\delta+|y|^2)^{m-2} w\big\langle y, A_\infty\nabla W\big\rangle\,\mathrm{d}y \\ &\le \epsilon_0\int(\delta+|y|^2)^m|\nabla w|^2\,\mathrm{d}y + C_{\epsilon_0,m}\int(\delta+|y|^2)^{m-2}|\nabla W|^2\,\mathrm{d}y + e_{m-1,\delta}, \end{aligned}$$

where the parameter $\epsilon_0 > 0$ can be taken arbitrarily small. In the last line, we used again the obvious inequality $(\delta+|y|^2)^{m-2}|y|^2 \le (\delta+|y|^2)^{m-1}$. Assuming that $C_3\epsilon_0 \le \lambda_1/4$, we thus obtain

$$C_3\, e_{m-1,\delta} \le \frac{\lambda_1}{4}\int(\delta+|y|^2)^m|\nabla w|^2\,\mathrm{d}y + C_4\int(\delta+|y|^2)^{m-2}|\nabla W|^2\,\mathrm{d}y, \tag{4.24}$$

for some positive constant C_4.

We now choose a constant $\kappa > 0$ large enough so that $\kappa\lambda_1 \ge 2C_4$, and we consider the combined energy functional

$$\mathcal{E}_{m,\delta}(\tau) = e_{m,\delta}(\tau) + \kappa E_{m-2,\delta}(\tau), \qquad \tau \ge 0. \tag{4.25}$$

By Proposition 2.11, we have $e_{m,\delta}(\tau) \le \mathcal{E}_{m,\delta}(\tau) \le C_5\, e_{m,\delta}(\tau)$ for some $C_5 > 0$. Moreover, it follows from (4.23) and from our choice of κ that $\mathcal{E}_{m,\delta}(\tau)$ satisfies the differential inequality

$$\partial_\tau\mathcal{E}_{m,\delta}(\tau) \le -\frac{\lambda_1}{4}\int(\delta+|y|^2)^m|\nabla w|^2\,\mathrm{d}y - \lambda\mathcal{E}_{m,\delta}(\tau) + \kappa\mathcal{F}_1(\tau) + C_2\mathcal{F}_2(\tau), \tag{4.26}$$

where

$$\mathcal{F}_1(\tau) = \int(\delta+|y|^2)^{m-2}\, W R_1\,\mathrm{d}y, \qquad \mathcal{F}_2(\tau) = \alpha^2\int(\delta+|y|^2)^m\|B(ye^{\tau/2})\|^2\,|\nabla\varphi|^2\,\mathrm{d}y.$$

Our final task is to estimate the remainder terms $\mathcal{F}_1$, $\mathcal{F}_2$ in (4.26), which involve the matrix $B(x) = A(x) - A_\infty(x)$, either explicitly or through the definition (4.17) of R_1. We recall that B satisfies the bound (4.6) for some $\nu > 0$. We start with the term $\mathcal{F}_1$, which can be bounded using Young's inequality and Proposition 2.14. For $\epsilon > 0$ arbitrarily small, we thus obtain

$$\mathcal{F}_1(\tau) \le \epsilon E_{m-2,\delta}(\tau) + C_\epsilon \int (\delta + |y|^2)^{m-2} |R_1(y,\tau)|^2 \,\mathrm{d}y$$
$$\le \epsilon E_{m-2,\delta}(\tau) + C_\epsilon \int |y|^{2m-2} \|B(ye^{\tau/2})\|^2 \big(\alpha^2 |\nabla\varphi|^2 + |\nabla w|^2\big) \,\mathrm{d}y,$$

where in the second line we used the fact that $(\delta + |y|^2)^{m-2} \le |y|^{2m-4}$ because $m \le 2$, and we applied estimate (2.41) with $u = R_1$ and $g = B(\cdot\, e^{\tau/2})(\alpha\nabla\varphi + \nabla w)$. To bound the last integral, we take $\gamma = \min(\nu, m-1) > 0$ and we observe that

$$|y|^{2m-2} \|B(ye^{\tau/2})\|^2 \le C|y|^{2\gamma} \|B(ye^{\tau/2})\|^2 (\delta + |y|^2)^m \le C\, e^{-\gamma\tau} (\delta + |y|^2)^m,$$

because $\sup_{x\in\mathbb{R}^n} |x|^\gamma \|B(x)\| < \infty$. Using in addition Proposition 3.7, we arrive at

$$\mathcal{F}_1(\tau) \le \epsilon E_{m-2,\delta}(\tau) + C_\epsilon\, e^{-\gamma\tau} \Big(\alpha^2 + \int (\delta + |y|^2)^m |\nabla w|^2 \,\mathrm{d}y\Big). \tag{4.27}$$

To control $\mathcal{F}_2$, we use the bound $(\delta + |y|^2)^m \le 2^{m-1}(\delta^m + |y|^{2m})$, and we treat the term involving $|y|^{2m}$ exactly as before. When no power of $|y|$ is available, this argument does not work, but taking $0 < \epsilon < \gamma$ we can apply Hölder's inequality with conjugate exponents

$$q = \frac{n}{2(\gamma-\epsilon)}, \qquad p = \frac{n}{n - 2(\gamma-\epsilon)}, \qquad \text{so that} \quad 1 < p < \frac{n}{2(1-\beta)}.$$

We know that $\nabla\varphi \in L^{2p}(\mathbb{R}^n)$ by Proposition 3.7, and that $x \mapsto B(x) \in L^{2q}(\mathbb{R}^n)$ in view of (4.6) because $2q = n/(\gamma-\epsilon) > n/\nu$. It follows that

$$\int \|B(ye^{\tau/2})\|^2 |\nabla\varphi|^2 \,\mathrm{d}y$$
$$\le \Big(\int \|B(ye^{\tau/2})\|^{2q} \,\mathrm{d}y\Big)^{1/q} \Big(\int |\nabla\varphi|^{2p} \,\mathrm{d}y\Big)^{1/p} \le C_\epsilon\, e^{-(\gamma-\epsilon)\tau} \|\nabla\varphi\|^2_{L^{2p}},$$

hence

$$\mathcal{F}_2(\tau) \le C_\epsilon\, \alpha^2 e^{-(\gamma-\epsilon)\tau} \Big(\|\nabla\varphi\|^2_{L^{2p}} + \|(1+|y|)^m \nabla\varphi\|^2_{L^2}\Big). \tag{4.28}$$

To summarize, it follows from (4.26), (4.27), (4.28) that

$$\partial_\tau \mathcal{E}_{m,\delta}(\tau) \le -(\lambda - \epsilon)\mathcal{E}_{m,\delta}(\tau) + \Big(C_\epsilon\, e^{-\gamma\tau} - \lambda_1\Big)\mathcal{D}_{m,\delta}(\tau) + C'_\epsilon\, \alpha^2 e^{-(\gamma-\epsilon)\tau}, \quad \tau > 0, \tag{4.29}$$

where

$$\mathcal{D}_{m,\delta}(\tau) = \frac{1}{4} \int (\delta + |y|^2)^m |\nabla w(y,\tau)|^2 \,\mathrm{d}y. \tag{4.30}$$

Here the parameter $\epsilon > 0$ can be taken arbitrarily small, and the constants $C_\epsilon, C'_\epsilon > 0$ depend only on ϵ and on the properties of the matrix A. If $\tau > 0$ is large enough, the coefficient of $\mathcal{D}_{m,\delta}(\tau)$ in the right-hand side of (4.29) becomes negative, and we obtain a differential inequality for the combined energy which implies that $\mathcal{E}_{m,\delta}(\tau)$ decays exponentially as $\tau \to +\infty$. More precisely, using inequalities (4.13) and (4.29), we obtain:

Proposition 4.6 *Assume that $n = 2$ or 3, $m \in (n/2, n/2 + \beta)$, and $m \le 2$. For any real number μ satisfying (1.12), there exists a positive constant C such that, for any $\alpha \in \mathbb{R}$ and any initial data $w_0 \in L^2_0(m)$, the solution $w \in C^0([0,+\infty), L^2_0(m))$ of (4.8) satisfies*

$$\|w(\tau)\|_{L^2(m)} \le C\big(\|w_0\|_{L^2(m)} + |\alpha|\big)\, e^{-\mu\tau}, \qquad \tau \ge 0. \tag{4.31}$$

Proof Given μ satisfying (1.12), we choose $\epsilon > 0$ small enough so that $2\mu < \min(\lambda, \gamma) - \epsilon$, where (as above) $\lambda = m - n/2$ and $\gamma = \min(\nu, m-1)$. If $\tau_* > 0$ is large enough so that $\lambda_1 e^{\gamma \tau_*} \geq C_\epsilon$, we can omit the term involving $\mathcal{D}_{m,\delta}(\tau)$ in the right-hand side of (4.29), and integrating the resulting differential inequality we obtain $\mathcal{E}_{m,\delta}(\tau) \leq C\big(\mathcal{E}_{m,\delta}(\tau_*) + \alpha^2\big)e^{-2\mu(\tau-\tau_*)}$ for $\tau \geq \tau_*$. Since the combined energy $\mathcal{E}_{m,\delta}(\tau)$ is equivalent to $\|w(\tau)\|^2_{L^2(m)}$, this gives the large time estimate

$$\|w(\tau)\|^2_{L^2(m)} \leq C\big(\|w(\tau_*)\|^2_{L^2(m)} + \alpha^2\big)\, e^{-2\mu(\tau-\tau_*)}, \qquad \tau \geq \tau_*. \tag{4.32}$$

To control the solution for intermediate times, we use the differential inequality (4.13) with $\delta = 1$, which is in fact valid regardless of the value of the parameter m. If we bound the last term in the right-hand side using (4.28), we obtain the useful inequality

$$\partial_\tau \|w(\tau)\|^2_{L^2(m')} \leq \frac{n-2m'}{2}\, \|w(\tau)\|^2_{L^2(m')} + \big(m'\delta + C_1 m'^2\big)\|w(\tau)\|^2_{L^2(m'-1)} + C_2\, \alpha^2\, e^{-\gamma'\tau}, \tag{4.33}$$

which holds for any $m' \geq 0$ and any $\gamma' \in [0, m']$ with $\gamma' < \nu$. In particular, if $m' = 0$ and $\gamma' = 0$, we have $\partial_\tau \|w(\tau)\|^2_{L^2} \leq (n/2)\|w(\tau)\|^2_{L^2} + C_2\alpha^2$, so that $\|w(\tau)\|^2_{L^2} \leq \big(\|w(0)\|^2_{L^2} + C\alpha^2\big)e^{n\tau/2}$ for all $\tau \geq 0$. Then, taking successively $m' = 1$, $m' = 2$, …we obtain in a finite number of steps the rough estimate

$$\|w(\tau)\|^2_{L^2(m)} \leq C\big(\|w_0\|^2_{L^2(m)} + \alpha^2\big)\, e^{n\tau/2}, \qquad \tau \geq 0. \tag{4.34}$$

Combining (4.34) for $\tau \leq \tau_*$ and (4.32) for $\tau \geq \tau_*$, we easily obtain (4.31). □

4.5 Higher-Order Antiderivatives

Proposition 4.6 is the main ingredient in the proof of Theorem 1.3 in low space dimensions. It is obtained, however, under the (unfortunate) assumption that $m \leq 2$, which implies first that $n \leq 3$, and also that the convergence rate μ cannot exceed the value $1/4$ if $n = 3$, even if the parameters β, ν are larger than $1/2$. To remove these artificial restrictions, we need to introduce higher-order antiderivatives, as we now explain. The reader who is satisfied with the assumptions of Proposition 4.6 can skip what follows and jump directly to Sect. 4.6.

We first recall that most of our analysis so far, including the weighted estimates in Sect. 2.5, is valid in arbitrary space dimension $n \geq 2$. In Sect. 4, the differential inequality (4.13) for the energy functional $e_{m,\delta}(\tau)$ holds for all $n \geq 2$ and any $m \geq 0$, but is not sufficient by itself to prove exponential decay of the solutions. This was precisely the reason for introducing the additional functional $E_{m-2,\delta}(\tau)$, which involves the antiderivative $W = K[w]$. The assumption that $m \leq 2$ is needed to eliminate the undesirable term involving $E_{m-3,\delta}(\tau)$ in the right-hand side of (4.21), so as to obtain exponential decay by combining (4.13) and (4.21).

We now consider the situation where $m \in (n/2, n/2 + \beta)$ and $2 < m \leq 4$, which is possible if $n = 3$ and $\beta > 1/2$, or if $4 \leq n \leq 7$. In that case, keeping in mind the conclusions of Propositions 2.11 and 2.14, which show that each application of the linear operator K decreases by two units the power m in the weight $(\delta + |y|^2)^m$, we introduce the "second antiderivative" $W^{(2)} = K[W] = K^2[w]$. We know from Remark 2.12 that $W \in L^2(m-2)$, and our current assumptions on m imply that $0 < m - 2 < n/2$. Thus we can apply Proposition 2.9 which asserts that $W^{(2)} \in L^2(m-4)$ with $\|W^{(2)}\|_{L^2(m-4)} \leq C\|W\|_{L^2(m-2)} \leq C\|w\|_{L^2(m)}$. Moreover, proceeding as in Sect. 4.3, it is straightforward to verify that $W^{(2)}(y,\tau)$ satisfies the evolution equation

$$\partial_\tau W^{(2)} \,=\, \mathrm{div}\big(A_\infty(y)\nabla W^{(2)}\big) + \frac{1}{2}\,y\cdot\nabla W^{(2)} + \frac{n-4}{2}\,W^{(2)} + K[R_1], \tag{4.35}$$

where R_1 is as in (4.17). Note that the factor $(n-2)/2$ in (4.16) becomes $(n-4)/2$ in (4.35).

The natural energy functional for the new variable $W^{(2)}$ is

$$E^{(2)}_{m-4,\delta}(\tau) \,=\, \frac{1}{2}\int_{\mathbb{R}^n}(\delta+|y|^2)^{m-4}|W^{(2)}(y,\tau)|^2\,\mathrm{d}y, \qquad \tau\ge 0. \tag{4.36}$$

In analogy with (4.21) we find

$$\begin{aligned}\partial_\tau E^{(2)}_{m-4,\delta}(\tau) \,\le\, &-\frac{1}{2}\int(\delta+|y|^2)^{m-4}\big\langle\nabla W^{(2)},A_\infty\nabla W^{(2)}\big\rangle\,\mathrm{d}y \,+\, \frac{n-2m}{2}\,E^{(2)}_{m-2,\delta}(\tau)\\ &+\Big((m-4)\delta + C_1(m-4)^2\Big)E^{(2)}_{m-5,\delta}(\tau) \,+\, \int(\delta+|y|^2)^{m-4}\,W^{(2)}K[R_1]\,\mathrm{d}y.\end{aligned} \tag{4.37}$$

As in Sect. 4.4, since $m\le 4$, the coefficient of $E^{(2)}_{m-5,\delta}$ in (4.37) can be made non-positive by an appropriate choice of δ. Moreover the negative term involving $\nabla W^{(2)}$ can be used to control the undesirable quantity $\big((m-2)\delta + C_1(m-2)^2\big)E_{m-3,\delta}(\tau)$ in (4.21), in view of the interpolation inequality

$$E_{m-3,\delta} \,\le\, \varepsilon\int(\delta+|y|^2)^{m-2}|\nabla W|^2\,\mathrm{d}y + C_\varepsilon\int(\delta+|y|^2)^{m-4}|\nabla W^{(2)}|^2\,\mathrm{d}y,$$

which is established exactly as in (4.24). Finally, the remainder term in (4.37) can be estimated just as the quantity $\mathcal{F}_1$ in (4.26). Indeed, since $m-4\le 0$, Proposition 2.9 yields

$$\int(\delta+|y|^2)^{m-4}\,|K[R_1]|^2\,\mathrm{d}y \,\le\, \int|y|^{2(m-4)}\,|K[R_1]|^2\,\mathrm{d}y \,\le\, C\int|y|^{2(m-2)}\,|R_1|^2\,\mathrm{d}y.$$

The arguments above allow us to control the solution of (4.8) using the new functional

$$\mathcal{E}^{(2)}_{m,\delta}(\tau) \,=\, e_{m,\delta}(\tau) + \kappa_1 E_{m-2,\delta}(\tau) + \kappa_2 E^{(2)}_{m-4,\delta}(\tau), \qquad \tau\ge 0,$$

where κ_1,κ_2 are positive constants satisfying $\kappa_2\gg\kappa_1\gg 1$. Combining the differential inequalities (4.13), (4.21), (4.37) and proceeding as in Sect. 4.4, it is straightforward to prove the exponential decay of the energy $\mathcal{E}^{(2)}_{m,\delta}(\tau)$ as $\tau\to+\infty$.

In yet higher space dimensions, namely when $m\in(n/2,n/2+\beta)$ and $m>4$, the strategy is similar but it becomes necessary to use the iterated antiderivatives $W^{(\ell)}=K^\ell[w]$ for larger values of $\ell\in\mathbb{N}$. To give a flavor, let ℓ be the smallest integer such that $m-2\ell\le 0$. The energy functional $E^{(\ell)}_{m-2\ell,\delta}(\tau)$ is defined in close analogy with (4.36), and satisfies a differential inequality similar to (4.37) where the coefficient $(m-2\ell)\delta + C_1(m-2\ell)^2$ in front of $E^{(\ell)}_{m-2\ell-1,\delta}$ is either zero or can be made negative by an appropriate choice of δ. Moreover, the negative term involving $|\nabla W^{(\ell)}|^2$ can be used to control an undesirable quantity in the evolution equation for the next functional in the hierarchy, which is $E^{(\ell-1)}_{m-2(\ell-1),\delta}$. Exponential decay can thus be established using a combined functional of the form

$$\mathcal{E}^{(\ell)}_{m,\delta}(\tau) \,=\, e_{m,\delta}(\tau) + \kappa_1 E_{m-2,\delta}(\tau) + \kappa_2 E^{(2)}_{m-4,\delta}(\tau) + \cdots + \kappa_\ell E^{(\ell)}_{m-2\ell,\delta}(\tau),$$

for some suitable constants $\kappa_1,\ldots,\kappa_\ell$. The details are left to the reader.

Taking the above arguments for granted, we thus obtain:

Corollary 4.7 *The conclusion of Proposition 4.6 holds for all $n\ge 2$ if $m\in(n/2,n/2+\beta)$.*

4.6 End of the Proof of Theorem 1.3

We conclude here the proof of Theorem 1.3 assuming the validity of Corollary 4.7, which was carefully established at least in low dimensions, see Proposition 4.6. What remains to be done is essentially to remove the upper bound $n/2+\beta$ on the parameter m. This will not increase the convergence rate μ, as can be seen from (1.12), but estimate (1.13) will nevertheless be obtained in a stronger norm. To do that, our strategy is to introduce an auxiliary parameter $\bar{m} \le m$ such that $\bar{m} \in (n/2, n/2+\beta)$. Estimate (4.31) allows us to control the solution in the larger space $L^2(\bar{m})$, and a simple interpolation gives convergence in $L^2(m)$ too.

We now provide the details. Assume that $n \ge 2$ and take initial data $v_0 \in L^2(m)$ for some $m > n/2$. We decompose $v_0 = \alpha\varphi + w_0$, where $\alpha = \int v_0(y)\,\mathrm{d}y$, and we consider the unique solution $w \in C^0([0,+\infty), L^2_0(m))$ of equation (4.8) such that $w(0) = w_0$. Given μ satisfying (1.12), we choose $\bar{m} \le m$ such that $\bar{m} \in (n/2+2\mu, n/2+\beta)$. We start from estimate (4.33) with $m' = m$ and $\gamma' = 2\mu$, which gives

$$\partial_\tau \|w(\tau)\|^2_{L^2(m)} \le \frac{n-2m}{2}\,\|w(\tau)\|^2_{L^2(m)} + \bigl(m\delta + C_1 m^2\bigr)\|w(\tau)\|^2_{L^2(m-1)} + C_2\,\alpha^2\,e^{-2\mu\tau}.$$

We next use the elementary bound

$$\|w(\tau)\|^2_{L^2(m-1)} \le \epsilon\|w(\tau)\|^2_{L^2(m)} + C_\epsilon\|w(\tau)\|^2_{L^2(\bar{m})},$$

which is obtained by interpolation if $\bar{m} < m-1 < m$, and is completely obvious if $m-1 \le \bar{m} \le m$. Taking any λ such that $2\mu < \lambda < (n-2m)/2$ and choosing $\epsilon > 0$ small enough, we thus obtain

$$\partial_\tau \|w(\tau)\|^2_{L^2(m)} \le -\lambda\,\|w(\tau)\|^2_{L^2(m)} + C'_\epsilon\|w(\tau)\|^2_{L^2(\bar{m})} + C_2\,\alpha^2\,e^{-2\mu\tau}.$$

The second term in the right-hand side is controlled using estimate (4.31) in the space $L^2(\bar{m})$, and taking into account the fact that $\bar{m} \in (n/2+2\mu, n/2+\beta)$. This gives

$$\partial_\tau \|w(\tau)\|^2_{L^2(m)} \le -\lambda\,\|w(\tau)\|^2_{L^2(m)} + C''_\epsilon\bigl(\|w_0\|^2_{L^2(\bar{m})} + \alpha^2\bigr)e^{-2\mu\tau} + C_2\,\alpha^2\,e^{-2\mu\tau}.$$

As $\|w_0\|_{L^2(\bar{m})} \le \|w_0\|_{L^2(m)}$ and $\lambda > 2\mu$, a final application of Grönwall's lemma gives the desired estimate

$$\|w(\tau)\|_{L^2(m)} \le C\bigl(\|w_0\|_{L^2(m)} + |\alpha|\bigr)\,e^{-\mu\tau}, \qquad \tau \ge 0,$$

where the constant C depends on n, m, μ, and on the properties of the matrix A. □

5 Long-Time Asymptotics in the Semilinear Case

In this section we study the long-time behavior of small solutions to the full equation (1.6), where the nonlinearity $\mathcal{N}(\tau, v)$ is given by (1.7). As before, we concentrate on the low space dimensions $n = 2$ and $n = 3$, but using the ideas introduced in Sect. 4.5 it is possible to treat the higher-dimensional case as well. We recall that the function N in (1.7) satisfies (1.17), and we suppose without loss of generality that the exponent σ in (1.17) lies in the range

$$1 + \frac{2}{n} < \sigma \le 1 + \frac{3}{n}. \tag{5.1}$$

This means that the quantity η defined in (1.18) satisfies $0 < \eta \le 1/2$. Clearly, a larger value of σ, hence of η, would not increase the convergence exponent μ in (1.18), since $\beta < 1$.

In view of (1.7), (1.17), the nonlinearity $\mathcal{N}$ in (1.6) satisfies

$$\big|\mathcal{N}(\tau, v)\big| \le C_0\, e^{-\eta\tau}|v|^{\sigma}, \qquad \text{and} \qquad \big|\mathcal{N}(v) - \mathcal{N}(\tilde{v})\big| \le C_0\, e^{\tau}|v - \tilde{v}|, \tag{5.2}$$

for all $v, \tilde{v} \in \mathbb{R}$ and all $\tau \ge 0$, where C_0 is some positive constant. In particular, since $\mathcal{N}(\tau, v)$ is a globally Lipschitz function of v, uniformly in τ on compact intervals, it is straightforward to verify, as in Lemma 4.1, that the Cauchy problem for Eq. (1.6) is globally well-posed in the space $L^2(m)$ for any $m \ge 0$. In other words, given any initial data $v_0 \in L^2(m)$, there exists a unique global solution $v \in C^0([0, +\infty), L^2(m))$ of (1.6) such that $v(0) = v_0$. Our goal here is to compute the long-time behavior of that solution when the initial data are sufficiently small.

We assume henceforth that $m > n/2$, so that $L^2(m) \hookrightarrow L^1(\mathbb{R}^n)$. We decompose the solution as in (4.7), with the important difference that the integral of v is no longer a conserved quantity. Instead we have

$$\alpha(\tau) = \int_{\mathbb{R}^n} v(y, \tau)\, \mathrm{d}y, \qquad \text{and} \qquad \alpha'(\tau) = \int_{\mathbb{R}^n} \mathcal{N}\big(\tau, v(y, \tau)\big)\, \mathrm{d}y. \tag{5.3}$$

The equation satisfied by the perturbation $w(y, \tau) = v(y, \tau) - \alpha(\tau)\varphi(y)$ is of the form (4.8), except that the remainder term r_1 given by (4.9) is replaced by $r_1 + r_2$, where

$$r_2(y, \tau) = \mathcal{N}\big(\tau, \alpha(\tau)\varphi(y) + w(y, \tau)\big) - \alpha'(\tau)\varphi(y). \tag{5.4}$$

Similarly, the antiderivative $W(y, \tau)$ defined by (4.15) satisfies Eq. (4.16), except that the remainder term R_1 defined by (4.17) is replaced by $R_1 + R_2$, where $R_2 = K[r_2]$.

As in the previous section, our strategy is to control the solution of (4.8) or (4.16) using weighted energy estimates, where the weight is a power of $\rho(y) := (\delta + |y|^2)^{1/2}$. To treat the nonlinear terms, the following auxiliary results will be useful.

Lemma 5.1 *If $w \in L^2(m)$ and $\nabla w \in L^2(m)^n$, we have, for all $\tau \ge 0$,*

$$\begin{aligned}&\int_{\mathbb{R}^n} \rho^{2m}|w|\,\big|\mathcal{N}(\tau, \alpha\varphi{+}w)\big|\,\mathrm{d}y \\ &\quad\le C\, e^{-\eta\tau}\Big(|\alpha|^{\sigma}\|w\|_{L^2(m)} + \|w\|_{L^2(m)}^{\sigma+1} + \|\nabla w\|_{L^2(m)}^{\eta+1}\,\|w\|_{L^2(m)}^{\sigma-\eta}\Big),\end{aligned} \tag{5.5}$$

where $\eta > 0$ is as in (1.18).

Proof In view of (5.2), we have $|\mathcal{N}(\tau, \alpha\varphi{+}w)| \le C\, e^{-\eta\tau}\big(|\alpha|^{\sigma}\varphi^{\sigma} + |w|^{\sigma}\big)$, hence

$$\int_{\mathbb{R}^n} \rho^{2m}|w|\,\big|\mathcal{N}(\tau, \alpha\varphi{+}w)\big|\,\mathrm{d}y \le C\, e^{-\eta\tau}\bigg(|\alpha|^{\sigma}\|w\|_{L^2(m)} + \int_{\mathbb{R}^n} \rho^{2m}|w|^{\sigma+1}\,\mathrm{d}y\bigg),$$

where we used the Cauchy–Schwarz inequality and the fact that $\varphi^{\sigma} \in L^2(m)$, see (1.10). To bound the last integral, we observe that $\rho^{2m}|w|^{\sigma+1} \le |\omega|^{\sigma+1}$ where $\omega = \rho^m w$, and we use the interpolation inequality

$$\int_{\mathbb{R}^n} |\omega|^{\sigma+1}\,\mathrm{d}y \le C\,\|\nabla\omega\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}(\sigma-1)}\,\|\omega\|_{L^2(\mathbb{R}^n)}^{\sigma+1-\frac{n}{2}(\sigma-1)}, \tag{5.6}$$

which is valid because $(\sigma{+}1)(n{-}2) \le 2n$. Since $\|\nabla\omega\|_{L^2(\mathbb{R}^n)} \le C\big(\|\nabla w\|_{L^2(m)}{+}\|w\|_{L^2(m)}\big)$ and $(n/2)(\sigma - 1) = 1 + \eta$, we obtain (5.5). □

Remark 5.2 We can simplify somehow estimate (5.5) by applying Young's inequality to the various terms in the right-hand side. The appropriate pairs of conjugate exponents are

$p = p' = 2$ for the first two terms, and $q = 2/(1+\eta)$, $q' = 2/(1-\eta)$ for the last one. We observe that $q' > 2$ and $q'(\sigma - \eta) > 2\sigma$, hence assuming that $\|w\|_{L^2(m)} \le 1$ we obtain, for any $\epsilon > 0$,

$$\int_{\mathbb{R}^n} \rho^{2m}|w|\,\big|\mathcal{N}(\tau,\alpha\varphi+w)\big|\,\mathrm{d}y \le \epsilon\Big(\|w\|^2_{L^2(m)} + \|\nabla w\|^2_{L^2(m)}\Big) + C_\epsilon\, e^{-2\eta\tau}\Big(|\alpha|^{2\sigma} + \|w\|^{2\sigma}_{L^2(m)}\Big).$$

Lemma 5.3 *If $w \in L^2(m)$ and $\nabla w \in L^2(m)^n$, we have, for all $\tau \ge 0$,*

$$\int_{\mathbb{R}^n} \rho^{m-n/2}\big|\mathcal{N}(\tau,\alpha\varphi+w)\big|\,\mathrm{d}y \;\le\; C\,e^{-\eta\tau}\Big(|\alpha|^\sigma + \|w\|^\sigma_{L^2(m)} + \zeta_n\|\nabla w\|^{\sigma-2}_{L^2(m)}\,\|w\|^2_{L^2(m)}\Big), \tag{5.7}$$

where $\zeta_n = 0$ if $n \ge 3$ and $\zeta_n = 1$ if $n = 2$.

Proof In view of (5.2) and (1.10), we have as before

$$\int_{\mathbb{R}^n} \rho^{m-n/2}\big|\mathcal{N}(\tau,\alpha\varphi+w)\big|\,\mathrm{d}y \;\le\; C\,e^{-\eta\tau}\bigg(|\alpha|^\sigma + \int_{\mathbb{R}^n}\rho^{m-n/2}|w|^\sigma\,\mathrm{d}y\bigg).$$

If $n \ge 3$, then $\sigma \in (1, 2]$ by (5.1), and a simple application of Hölder's inequality yields

$$\int_{\mathbb{R}^n} \rho^{m-n/2}|w|^\sigma\,\mathrm{d}y \;\le\; \|\rho^{-m(\sigma-1)-n/2}\|_{L^{2/(2-\sigma)}(\mathbb{R}^n)}\|w\|^\sigma_{L^2(m)} \;\le\; C\|w\|^\sigma_{L^2(m)}.$$

We thus obtain (5.7) with $\zeta_n = 0$, for any $w \in L^2(m)$. If $n = 2$, then $\sigma > 2$ by (5.1), and we can control the term involving $|w|^\sigma$ as in the proof of Lemma 5.1. Setting $\omega = \rho^m w$ and using the interpolation inequality (5.6) with σ replaced by $\sigma - 1$, we arrive at (5.7) with $\zeta_n = 1$. □

Our main goal is to prove that the quantities $|\alpha'(\tau)|$ and $\|w(\cdot,\tau)\|_{L^2(m)}$ decay exponentially to zero as $\tau \to +\infty$, if we assume a priori that $|\alpha(\tau)| + \|w(\cdot,\tau)\|_{L^2(m)} \le 1$ for all $\tau \ge 0$. As we shall see, that condition will be fulfilled if we take sufficiently small initial data. Proceeding as in Sect. 4, our strategy is to use the differential inequalities satisfied by the energy functionals $e_{m,\delta}(\tau)$ and $\mathcal{E}_{m-2,\delta}(\tau)$ defined by (4.10), (4.25), respectively. In what follows, we fix some $\delta \ge 1$ and we denote $\rho(y) = (\delta + |y|^2)^{1/2}$.

We first control the evolution of the scalar quantity α. The derivative $\alpha'(\tau)$ given by (5.3) can be estimated using Lemma 5.3, if we disregard the factor $\rho^{m-n/2} \ge 1$ in the left-hand side of (5.7). We thus find

$$|\alpha'(\tau)| \;\le\; C\,e^{-\eta\tau}\Big(|\alpha|^\sigma + \|w\|^\sigma_{L^2(m)} + \zeta_n\|\nabla w\|^{\sigma-2}_{L^2(m)}\,\|w\|^2_{L^2(m)}\Big). \tag{5.8}$$

Next, since the function w satisfies (4.8) with remainder term $r_1 + r_2$, we obtain as in (4.23):

$$\partial_\tau e_{m,\delta}(\tau) \;\le\; -2\lambda_1\mathcal{D}_{m,\delta}(\tau) - \lambda e_{m,\delta}(\tau) + C_3\, e_{m-1,\delta}(\tau) + C_2\mathcal{F}_2(\tau) + \mathcal{F}_3(\tau), \tag{5.9}$$

where $\lambda = m - n/2$, $C_3 = m\delta + C_1 m^2$, $\mathcal{D}_{m,\delta}(\tau)$ is defined in (4.30), and $\mathcal{F}_3(\tau) = \int \rho^{2m} w\, r_2\,\mathrm{d}y$. In view of (4.28), we have $\mathcal{F}_2(\tau) \le C_\epsilon\, \alpha^2\, e^{-(\gamma-\epsilon)\tau}$ for any small $\epsilon > 0$, where $\gamma = \min(\nu, m-1)$. Moreover, the definition (5.4) of r_2 implies that

$$\mathcal{F}_3(\tau) \;\le\; \int \rho^{2m}\,|w|\,\big|\mathcal{N}(\tau,\alpha\varphi+w)\big|\,\mathrm{d}y \;+\; |\alpha'(\tau)|\int \rho^{2m}\,|w|\,\varphi\,\mathrm{d}y.$$

The first term in the right-hand side is estimated using Lemma 5.1 and Remark 5.2, whereas for the second term we use (5.8), the Cauchy–Schwarz inequality, and Young's inequality. We thus find

$$\mathcal{F}_3(\tau) \le \epsilon\Big(\|w\|^2_{L^2(m)} + \|\nabla w\|^2_{L^2(m)}\Big) + C_\epsilon\, e^{-2\eta\tau}\Big(|\alpha|^{2\sigma} + \|w\|^{2\sigma}_{L^2(m)}\Big), \tag{5.10}$$

where $\epsilon > 0$ is arbitrarily small. Replacing these estimates into (5.9), we arrive at

$$\begin{aligned}\partial_\tau e_{m,\delta}(\tau) \le &-(2\lambda_1-\epsilon)\mathcal{D}_{m,\delta}(\tau) - (\lambda-\epsilon)e_{m,\delta}(\tau) + C_3\, e_{m-1,\delta}(\tau)\\ &+ C_\epsilon\,\alpha^2\, e^{-(\gamma-\epsilon)\tau} + C_\epsilon\, e^{-2\eta\tau}\big(|\alpha|^{2\sigma} + e_{m,\delta}(\tau)^\sigma\big),\end{aligned} \tag{5.11}$$

for some sufficiently small $\epsilon > 0$.

The second important quantity we want to control is the combined energy functional (4.25), which involves both w and the antiderivative W. At this point, we have to assume as in Proposition 4.6 that $m \in (n/2, n/2+\beta)$ and $m \le 2$. We also suppose that δ satisfies (4.22). Due to the additional nonlinear terms in the evolution equations for w and W, we obtain instead of (4.26):

$$\partial_\tau \mathcal{E}_{m,\delta}(\tau) \le -\lambda_1\mathcal{D}_{m,\delta}(\tau) - \lambda\mathcal{E}_{m,\delta}(\tau) + \kappa\big(\mathcal{F}_1(\tau) + \mathcal{F}_4(\tau)\big) + C_2\mathcal{F}_2(\tau) + \mathcal{F}_3(\tau), \tag{5.12}$$

where $\mathcal{F}_1(\tau)$ satisfies (4.27) and $\mathcal{F}_4(\tau) = \int \rho^{2m-4} W R_2\,\mathrm{d}y = \int \rho^{2m-4} W K[r_2]\,\mathrm{d}y$. To estimate the new term, we first apply the Cauchy–Schwarz inequality, and then Proposition 2.11 with $p = 1$ and $s = n/2$. We thus obtain

$$|\mathcal{F}_4(\tau)| \le C\|W\|_{L^2(m-2)} \int_{\mathbb{R}^n} \rho^{m-n/2}\Big|\mathcal{N}\big(\tau, \alpha\varphi + w\big) - \alpha'(\tau)\varphi\Big|\,\mathrm{d}y,$$

where the integral in the right-hand side can be controlled using Lemma 5.3 and estimate (5.8). Using in addition Young's inequality when $n = 2$ (in which case $\zeta_n = 1$), we obtain

$$\mathcal{F}_4(\tau) \le \epsilon\Big(\|W\|^2_{L^2(m-2)} + \zeta_n\|\nabla w\|^2_{L^2(m)}\Big) + C_\epsilon\, e^{-2\eta\tau}\Big(|\alpha|^{2\sigma} + \|w\|^{2\sigma}_{L^2(m)}\Big). \tag{5.13}$$

If we bound $\mathcal{F}_1(\tau)$ by (4.27), $\mathcal{F}_4(\tau)$ by (5.13), and $\mathcal{F}_2(\tau)$, $\mathcal{F}_3(\tau)$ as in (5.11), we can write (5.12) in the form

$$\begin{aligned}\partial_\tau \mathcal{E}_{m,\delta}(\tau) \le &-(\lambda-\epsilon)\mathcal{E}_{m,\delta}(\tau) + \Big(C'_\epsilon\, e^{-\gamma\tau} - \lambda_1\Big)\mathcal{D}_{m,\delta}(\tau)\\ &+ C''_\epsilon\,\alpha^2\, e^{-(\gamma-\epsilon)\tau} + C''_\epsilon\, e^{-2\eta\tau}\Big(|\alpha|^{2\sigma} + e_{m,\delta}(\tau)^\sigma\Big),\end{aligned} \tag{5.14}$$

where $\epsilon > 0$ is small enough. Both inequalities (5.11), (5.14) are valid as long as $\|w(\tau)\|_{L^2(m)} \le 1$, and the constants C_ϵ, C'_ϵ, C''_ϵ therein depend only on ϵ and on the properties of the matrix A.

End of the proof of Theorem 1.6 We now show how to deduce the conclusion of Theorem 1.6 from estimates (5.8), (5.11), and (5.14), assuming for simplicity that either $n = 2$ or $n = 3$ and $\mu < 1/4$. The arguments here are pretty standard, and we only indicate the main steps. Throughout the proof, we assume that v is the solution of (1.6) with initial data $v_0 \in L^2(m)$ satisfying $\|v_0\|_{L^2(m)} \le \epsilon_0$, for some sufficiently small $\epsilon_0 > 0$. We decompose this solution as $v(y,\tau) = \alpha(\tau)\varphi(y) + w(y,\tau)$ where $\alpha(\tau)$ is defined by (5.3).

Step 1. (*Short-time estimate*) We claim that there exist constants $C_9 > 1$ and $\theta > n/2$ such that

$$\alpha(\tau)^2 + e_{m,\delta}(\tau) \le C_9\, e^{\theta\tau}\Big(\alpha(0)^2 + e_{m,\delta}(0)\Big), \qquad \tau \ge 0, \tag{5.15}$$

as long as the right-hand side is smaller than or equal to 1. To prove (5.15), we start from the differential inequality (5.11), which is valid for any $m > n/2$. Using the rough estimate $e_{m-1,\delta}(\tau) \le e_{m,\delta}(\tau)$ and assuming that $\alpha(\tau)^2 + e_{m,\delta}(\tau) \le 1$, we deduce from (5.11) that

$$\partial_\tau e_{m,\delta}(\tau) \;\le\; -c\mathcal{D}_{m,\delta}(\tau) \;+\; \theta e_{m,\delta}(\tau) + C\, e^{-2\mu\tau}\Big(\alpha(\tau)^2 \;+\; e_{m,\delta}(\tau)\Big), \tag{5.16}$$

where $c = 2\lambda_1 - \epsilon$ and $\theta = C_3 - \lambda + \epsilon$. Under the same assumptions, it follows from (5.8) and Young's inequality that

$$2\alpha(\tau)\alpha'(\tau) \;\le\; \epsilon\Big(\alpha(\tau)^2 + \zeta_n D_{m,\delta}(\tau)\Big) + C_\epsilon\, e^{-2\eta\tau}\Big(\alpha(\tau)^2 \;+\; e_{m,\delta}(\tau)\Big), \tag{5.17}$$

where $\epsilon > 0$ is arbitrarily small. Combining (5.16), (5.17) we obtain a differential inequality for the quadratic quantity $\alpha(\tau)^2 + e_{m,\delta}(\tau)$, which can be integrated to give (5.15).

Step 2. (*Exponential decay for large times*) We assume for the time being that $m \le 2$ and $m \in (n/2+2\mu, n/2+\beta)$, so that estimate (5.14) is valid. We take $\tau_1 > 0$ large enough so that $C'_\epsilon\, e^{-\gamma\tau_1} \le \lambda_1/2$, where C'_ϵ is as in (5.14), and we assume that $\epsilon_1^2 := \alpha(\tau_1)^2 + e_{m,\delta}(\tau_1) \ll 1$. In view of (5.15), this condition is fulfilled if the initial data are sufficiently small. For $\tau \ge \tau_1$, as long as the solution satisfies $\alpha(\tau)^2 + e_{m,\delta}(\tau) \le M^2\epsilon_1^2 \le 1$, for some fixed constant $M > 1$, we can integrate the differential inequality (5.14) to obtain

$$\mathcal{E}_{m,\delta}(\tau) + \frac{\lambda_1}{2}\int_{\tau_1}^{\tau} e^{-\lambda'(\tau-s)}\mathcal{D}_{m,\delta}(s)\,\mathrm{d}s \;\le\; e^{-\lambda'(\tau-\tau_1)}\,\mathcal{E}_{m,\delta}(\tau_1) + CM^2\epsilon_1^2\, e^{-2\mu\tau}, \tag{5.18}$$

where $\lambda' = \lambda - \epsilon$. Under the same assumptions, integrating (5.8), we obtain for $\tau_1 \le \tau_2 \le \tau$:

$$\big|\alpha(\tau) - \alpha(\tau_2)\big| \;\le\; \int_{\tau_2}^{\tau}\big|\alpha'(s)\big|\,\mathrm{d}s \;\le\; CM^\sigma\epsilon_1^\sigma\, e^{-\eta\tau_2}. \tag{5.19}$$

Estimate (5.19) is straightforward to obtain when $n \ge 3$, but in the two-dimensional case we must use the integral term in the left-hand side of (5.18) to control the quantity involving $\|\nabla w\|_{L^2(m)}$ in the expression (5.8) of $\alpha'(\tau)$. In any case, it follows from (5.18), (5.19) that

$$\alpha(\tau)^2 + e_{m,\delta}(\tau) \;\le\; C_{10}\Big(\epsilon_1^2 + M^2\epsilon_1^2\, e^{-2\mu\tau_1}\Big), \qquad \tau \ge \tau_1, \tag{5.20}$$

as long as $\alpha(\tau)^2 + e_{m,\delta}(\tau) \le M^2\epsilon_1^2$. Here the constant C_{10} does not depend on M nor on τ_1. Thus we can choose M large enough so that $M^2 > 2C_{10}$, and then τ_1 large enough so that $e^{2\mu\tau_1} \ge M^2$. Under these assumptions, we deduce from (5.20) that $\alpha(\tau)^2 + e_{m,\delta}(\tau) \le M^2\epsilon_1^2 \le 1$ for all $\tau \ge \tau_1$, and this in turn implies that estimates (5.18), (5.19) hold for all $\tau \ge 0$. In particular, we have $e_{m,\delta}(\tau) \le \mathcal{E}_{m,\delta}(\tau) \le C\epsilon_1^2\, e^{-2\mu\tau}$, and there exists $\alpha_* \in \mathbb{R}$ such that $|\alpha(\tau) - \alpha_*| \le C\epsilon_1\, e^{-\eta\tau}$ for all $\tau \ge \tau_1$. Together with the short time estimate (5.15), this proves (1.19) in the case where $m \in (n/2 + 2\mu, n/2 + \beta)$ and $m \le 2$.

The final step consists in proving the exponential decay for large times in the general case where $m > n/2$. This can be done using the previous result and a simple interpolation argument as in the proof of Theorem 1.3. We omit the details.

Remark 5.4 It is possible to relax considerably our assumptions (1.17) on the nonlinearity N and to strengthen our convergence result (1.19) by using additional functionals that control derivatives of the solution $v(y, \tau)$. In view of (4.13), it is natural to consider the functional

$$D_{m,\delta}(\tau) \;=\; \frac{1}{2}\int(\delta + |y|^2)^m\big\langle\nabla w(y,\tau),\, A(ye^{\tau/2})\nabla w(y,\tau)\big\rangle\,\mathrm{d}y, \tag{5.21}$$

which is equivalent to $\mathcal{D}_{m,\delta}(\tau)$ in (4.30). However, controlling the time evolution of $D_{m,\delta}(\tau)$ requires the additional hypothesis that the matrix $A(x)$ in (1.1) satisfies $x \cdot \nabla A \in L^\infty(\mathbb{R}^n)$.

Such an assumption is quite natural in our problem, but is not required in Theorems 1.3 and 1.6.

6 Appendix

6.1 A generalized Young inequality

In this section, following [24], we give a short proof of Proposition 2.7.

Lemma 6.1 *Under the assumptions of Proposition 2.7, we define, for any $y \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$,*

$$\kappa_2 \;=\; \int_{\mathbb{R}^n} |k(x,y)|^{n/d}\, |x|^{-n^2/(dp')}\, \mathrm{d}x, \qquad \textit{where} \quad \frac{1}{p} + \frac{1}{p'} \;=\; 1. \tag{6.1}$$

Then $\kappa_2 = \kappa_1$, where κ_1 is given by (2.31).

Proof Proceeding as in [24, Lemma 1], we write $x = r\sigma$ and $y = \rho\theta$, where $r = |x|$, $\rho = |y|$, and $\sigma, \theta \in \mathbb{S}^{n-1}$. By rotation invariance, the definition (2.31) does not depend on the choice of $x \equiv \sigma \in \mathbb{S}^{n-1}$. Thus, averaging over σ, we obtain

$$\kappa_1 \;=\; \frac{1}{s_n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty |k(\sigma,\rho\theta)|^{n/d}\, \rho^{n-1-n^2/(dq)}\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\sigma,$$

where $s_n = 2\pi^{n/2}\,\Gamma(n/2)^{-1}$ is the measure of $\mathbb{S}^{n-1}$. We perform the change of variable $\rho = 1/r$ in the inner integral, and use the fact that $|k(x,y)|^{n/d}$ is homogeneous of degree $-n$. This gives

$$\kappa_1 \;=\; \frac{1}{s_n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty |k(r\sigma,\theta)|^{n/d}\, r^{n-1-n^2/(dp')}\, \mathrm{d}\rho\, \mathrm{d}\sigma\, \mathrm{d}\theta, \tag{6.2}$$

because $n^2/(dq) = n - n^2/(dp')$ in view of (2.31). The right-hand side of (6.2) is the average over $\theta \in \mathbb{S}^{n-1}$ of the quantity (6.1), which does not depend on the choice of $y \equiv \theta \in \mathbb{S}^{n-1}$. This yields the desired equality $\kappa_1 = \kappa_2$. □

Proof of Proposition 2.7 We assume for definiteness that $1 < p < q < \infty$, which is the most interesting situation. The other cases, where some of the inequalities above are not strict, can be established by similar (or simpler) arguments. If $f \in L^p(\mathbb{R}^n)$ and $g = \mathcal{K}[f]$, we have

$$|g(x)| \;\le\; \int_{\mathbb{R}^n} \Bigl(|k(x,y)|^a |y|^{-b}\Bigr)\Bigl(|k(x,y)|^{1-a}|y|^b|f(y)|^{p/q}\Bigr)\,|f(y)|^{1-p/q}\,\mathrm{d}y, \quad x\in\mathbb{R}^n,$$

where $a = n/(dp')$ and $b = n^2/(dqp')$. We apply to the right-hand side the trilinear Hölder inequality with exponents p', q, and $r := pq/(q-p)$, which satisfy $1/p' + 1/q + 1/r = 1$. This gives

$$|g(x)|^q \;\le\; I(x)\left(\int_{\mathbb{R}^n} |k(x,y)|^{(1-a)q}|y|^{bq}|f(y)|^p\,\mathrm{d}y\right)\,\|f\|_{L^p(\mathbb{R}^n)}^{q-p}, \tag{6.3}$$

where

$$I(x)^{p'/q} \;=\; \int_{\mathbb{R}^n} |k(x,y)|^{ap'}|y|^{-bp'}\,\mathrm{d}y \;=\; \int_{\mathbb{R}^n} |k(x,y)|^{n/d}|y|^{-n^2/(dq)}\,\mathrm{d}y.$$

Applying the change of variables $y = |x|z$ in the last integral and using the assumption that the expression $|k(x, y)|^{n/d}$ is homogeneous of degree $-n$, we obtain

$$I(x)^{p'/q} = |x|^{n-n^2/dq} \int_{\mathbb{R}^n} \big|k(x, |x|z)\big|^{n/d} |z|^{-n^2/(dq)} \, dz = \kappa_1 |x|^{-n^2/(dq)}.$$

We now replace this expression into (6.3) and integrate over $x \in \mathbb{R}^n$, using Fubini's theorem to exchange the integrals. Since $(1-a)q = n/d$ and $bq = n^2/(dp')$, this gives

$$\|g\|^q_{L^q(\mathbb{R}^n)} \le \kappa_1^{q/p'} \|f\|^{q-p}_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^n} J(y)|y|^{n^2/(dp')}|f(y)|^p \, dy, \tag{6.4}$$

where

$$J(y) = \int_{\mathbb{R}^n} |k(x, y)|^{n/d} |x|^{-n^2/(dp')} \, dx.$$

As for the computation of I, we use the homogeneity of k and the change of variable $z = x/|y|$ to obtain

$$J(y) = \int_{\mathbb{R}^n} \big|k(z, y/|y|)\big|^{n/d} |z|^{-n^2/(dp')} |y|^{-n^2/(dp')} \, dz = \kappa_2 |y|^{-n^2/(dp')}.$$

Using Lemma 6.1, we conclude that $\|g\|_{L^q(\mathbb{R}^n)} \le \kappa_1^{1/p'} \kappa_2^{1/q} \|f\|_{L^p(\mathbb{R}^n)} = \kappa_1^{d/n} \|f\|_{L^p(\mathbb{R}^n)}$. □

6.2 On the Divergence of Localized Vector Fields

Let $\chi : \mathbb{R}^n \to [0, 1]$ be any smooth, compactly supported function such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Given any $k \in \mathbb{N}^*$, we denote $\chi_k(x) = \chi(x/k)$.

Lemma 6.2 *Assume that $g \in L^p(\mathbb{R}^n)^n$ for some $p \in [1, \infty)$ such that $(n-1)p < n$. Then we have $\langle \mathrm{div} g, \chi_k \rangle \to 0$ as $k \to +\infty$. In particular, if $\mathrm{div} g \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \mathrm{div} g \, dx = 0$.*

Proof For any $k \ge 1$, we have

$$\langle \mathrm{div} g, \chi_k \rangle = -\langle g, \nabla \chi_k \rangle = -\frac{1}{k} \int_{\mathbb{R}^n} g(x) \cdot \nabla \chi(x/k) \, dx. \tag{6.5}$$

The integral in the right-hand side is easily estimated using Hölder's inequality:

$$\Big| \int_{\mathbb{R}^n} g(x) \cdot \nabla \chi(x/k) \, dx \Big| \le C \int_{|k| \le |x| \le 2|k|} |g(x)| \, dx \le C \|g\|_{L^p} (k^n)^{1-\frac{1}{p}}.$$

Our assumption on p ensures that $n(1-1/p) < 1$, hence the last member of (6.5) converges to zero as $k \to \infty$. Finally, if $\mathrm{div} g \in L^1(\mathbb{R}^n)$, the first member of (6.5) converges to $\int_{\mathbb{R}^n} \mathrm{div} g \, dx$ by Lebesgue's dominated convergence theorem. □

Lemma 6.3 *Let $n \ge 2$, $m \in (n/2, n/2+1)$, and assume that $f \in L^2(m)$ satisfies $\int_{\mathbb{R}^n} f \, dx = 0$. Then there exists $g \in L^2(m-1)^n$ such that $\mathrm{div} g = f$ and $\|g\|_{L^2(m-1)} \le C \|f\|_{L^2(m)}$.*

Proof Under our assumptions on f, it is known that the elliptic equation $\Delta u = f$ has a unique solution $u : \mathbb{R}^n \to \mathbb{R}$ that decays to zero at infinity [16]. We take $g = \nabla u$. Using the explicit form of the fundamental solution of the Laplace equation in $\mathbb{R}^n$, we obtain the representations

$$g(x) = \frac{1}{s_n} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} f(y) \, dy = \frac{1}{s_n} \int_{\mathbb{R}^n} \Big(\frac{x-y}{|x-y|^n} - \frac{x}{|x|^n} \Big) f(y) \, dy, \tag{6.6}$$

where s_n is again the measure of the unit sphere $\mathbb{S}^{n-1}$. Since $f \in L^2(\mathbb{R}^n)$, we can apply the Hardy–Littlewood–Sobolev inequality to the first expression of g in (6.6), and we deduce that $g \in L^p(\mathbb{R}^n)$ for $p = 2n/(n-2)$ when $n \geq 3$. If $n \geq 2$, using the fact that $L^2(m) \hookrightarrow L^q(\mathbb{R}^2)$ for $q \in (1, 2)$, we obtain that $g \in L^p(\mathbb{R}^2)$ for $p \in (2, \infty)$. In particular, we have in all cases

$$\int_{|x|\leq 1} |g(x)|^2 \,\mathrm{d}x \;\leq\; C\|f\|^2_{L^2(m)}. \tag{6.7}$$

We next exploit the second expression of g in (6.6). We claim that

$$\left|\frac{x-y}{|x-y|^n} - \frac{x}{|x|^n}\right| \;\leq\; \frac{C|y|}{|x||x-y|}\left(\frac{1}{|x-y|^{n-2}} + \frac{1}{|x|^{n-2}}\right), \tag{6.8}$$

for all $x, y \in \mathbb{R}^n$ with $x \neq 0$ and $x \neq y$. Equivalently,

$$\Big||x|^n(x-y) - x\,|x-y|^n\Big| \;\leq\; C|x|\,|y|\,|x-y|\,\Big(|x-y|^{n-2} + |x|^{n-2}\Big), \tag{6.9}$$

for all $x, y \in \mathbb{R}^n$. To establish (6.9), we decompose

$$|x|^n(x-y) - x\,|x-y|^n \;=\; |x|^{n-1}\Big(|x|(x-y) - x\,|x-y|\Big) + x\,|x-y|\Big(|x|^{n-1} - |x-y|^{n-1}\Big), \tag{6.10}$$

and we use the following two elementary observations:

1. For any $x, z \in \mathbb{R}^n$ we have $\big||x|z - x|z|\big| \leq 2|z||x-z|$. This can be proved by taking the square of both sides and considering two cases according to whether $|x| \leq 4|z|$ or $|x| \geq 4|z|$.
2. For any $x, z \in \mathbb{R}^n$, we have

$$\Big||x|^{n-1} - |z|^{n-1}\Big| \;\leq\; \frac{n-1}{2}\,\big||x| - |z|\big|\,\Big(|x|^{n-2} + |z|^{n-2}\Big).$$

Indeed the map $t \mapsto h(t) = (n-1)t^{n-2}$ is convex on $\mathbb{R}_+$, so that for all $b \geq a \geq 0$ we have $\int_a^b h(t)\,\mathrm{d}t \;\leq\; (b-a)(h(a) + h(b))/2$, which gives the result if $a = \min(|x|, |z|)$, $b = \max(|x|, |z|)$.
Applying these elementary estimates with $z = x - y$, we can bound both terms in the right-hand side of (6.10), and we arrive at (6.9).

Now, in view of (6.6), (6.8), we have $|x|^{m-1}|g(x)| \leq C\int_{\mathbb{R}^n} k(x, y)|y|^m|f(y)|\,\mathrm{d}y$, where

$$k(x, y) \;=\; \frac{|x|^{m-2}}{|x-y|\,|y|^{m-1}}\left(\frac{1}{|x-y|^{n-2}} + \frac{1}{|x|^{n-2}}\right).$$

The kernel $k(x, y)$ is homogeneous of degree $-n$ and invariant under rotations in $\mathbb{R}^n$. Moreover, if $|x| = 1$, the assumption that $m \in (n/2, n/2+1)$ ensures that $\int_{\mathbb{R}^n} k(x, y)|y|^{-n/2}\,\mathrm{d}y < \infty$. Applying Proposition 2.7 with $d = n$ and $p = q = 2$, we deduce that

$$\int_{\mathbb{R}^n} |x|^{2m-2}|g(x)|^2\,\mathrm{d}x \;\leq\; C\int_{\mathbb{R}^n} |y|^{2m}|f(y)|^2\,\mathrm{d}y,$$

and combining this estimate with (6.7) we conclude that $\|g\|_{L^2(m-1)} \leq C\|f\|_{L^2(m)}$. □

6.3 On the Optimality of Proposition 2.11

We show here using an explicit example that the assumption $m < n/2+\beta$ in Proposition 2.11 cannot be relaxed. Given $a, b > 0$, we consider the functions $u, f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u(x) = \frac{x_1}{(1+|x|^2)^a}, \qquad f(x) = -\mathrm{div}\big(A_b(x)\nabla u(x)\big), \tag{6.11}$$

where A_b is the Meyers–Serrin matrix (3.4). We have

$$\nabla u(x) = \frac{e_1}{(1+|x|^2)^a} - \frac{2ax_1 x}{(1+|x|^2)^{a+1}}, \qquad \text{where} \quad e_1 = (1, 0, \dots, 0),$$

and since $A_b(x)x = x$ we find

$$A_b(x)\nabla u(x) = \frac{1}{(1+|x|^2)^a}\Big(be_1 + (1-b)\frac{x_1 x}{|x|^2}\Big) - \frac{2ax_1 x}{(1+|x|^2)^{a+1}}.$$

Taking the divergence with respect to x, we arrive at

$$f(x) = -\frac{(1-b)(n-1)}{(1+|x|^2)^a}\,\frac{x_1}{|x|^2} + \frac{2a(n+2)x_1}{(1+|x|^2)^{a+1}} - \frac{4a(a+1)x_1|x|^2}{(1+|x|^2)^{a+2}}, \qquad x \in \mathbb{R}^n.$$

For simplicity, we assume henceforth that $n \ge 3$, so that $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. As $|x| \to \infty$, we have

$$f(x) = x_1\Big(\frac{c}{|x|^{2a+2}} + \mathcal{O}\Big(\frac{1}{|x|^{2a+4}}\Big)\Big), \qquad \text{as} \quad |x| \to +\infty, \tag{6.12}$$

where $c = -(1-b)(n-1) + 2an - 4a^2$. The idea is now to choose the parameters a, b so that $c = 0$, in order to maximize the decay of f. For instance, we can take

$$a = \frac{1}{4}\Big(n + \sqrt{n^2 - 4(1-b)(n-1)}\Big) = \frac{1}{4}\Big(n + \sqrt{(n-2)^2 + 4b(n-1)}\Big). \tag{6.13}$$

With this choice, given $m > n/2$, it follows from (6.11), (6.12) that

$$\begin{aligned} |x|^m f \in L^2(\mathbb{R}^n) \quad &\text{if and only if} \quad 2a > n/2 + m - 3, \\ |x|^{m-2} u \in L^2(\mathbb{R}^n) \quad &\text{if and only if} \quad 2a > n/2 + m - 1. \end{aligned} \tag{6.14}$$

Under the first condition in (6.14), we also have $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0$ since f is odd, hence $f \in L^2_0(m)$.

According to (6.14), the pair (u, f) violates inequality (2.38) with $p = 2, s = 0$ provided $m > n/2$ and

$$n/2 + m - 3 < 2a < n/2 + m - 1. \tag{6.15}$$

For instance, if $n = 3$ and $m = 2$, we have $1/2 < 2a < 5/2$ by (6.13) if $b > 0$ is sufficiently small, and it follows that $f \in L^2(m)$, $\int_{\mathbb{R}^3} f(x)\,\mathrm{d}x = 0$, and yet $u \notin L^2(\mathbb{R}^3)$. The explanation is that the Hölder exponent β in Proposition 2.4 tends to zero as $b \to 0$ in the case of the Meyers–Serrin operator, see Remark 3.4, and that the value $m = 2$ is not allowed in Proposition 2.11 if $n = 3$ and $\beta < 1/2$. More generally, if $n \ge 3$ and $n/2 < m < n/2 + 1$, we can choose $b > 0$ small enough so that inequalities (6.15) hold, which implies the failure of estimate (2.38) with $p = 2, s = 0$; but it follows from (6.13) and (3.17) that $2a \ge n - 1 + \beta$, hence the second inequality in (6.15) implies that $m > n/2 + \beta$. This shows that the assumption $m < n/2 + \beta$ in Proposition 2.11 is sharp in the case of the Meyers–Serrin matrix (3.4), at least if the quantity β is understood as given by the right-hand side of (3.17).

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