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Geometric Singular Perturbation Theory for Systems with Symmetry

Pedro Toniol Cardin[1] · **Marco Antonio Teixeira**[2]

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Abstract

In this paper we focus on a class of symmetric vector fields in the context of singularly perturbed fast-slow dynamical systems. Our main question is to know how symmetry properties of a dynamical system are affected by singular perturbations. In addition, our approach uses tools in geometric singular perturbation theory [8], which address the persistence of normally hyperbolic compact manifolds. We analyse the persistence of such symmetry properties when the singular perturbation parameter ε is positive and small enough, and study the existing relations between symmetries of the singularly perturbed system and symmetries of the limiting systems, which are obtained from the limit $\varepsilon \to 0$ in the fast and slow time scales. This approach is applied to a number of examples.

Keywords Fast-slow systems · Reversible vector fields · Symmetries

Mathematics Subject Classification 34D15 · 37C80 · 34C45

1 Introduction

1.1 Prelude

In this paper we consider systems of singularly perturbed (fast-slow) ordinary differential equations of the form

$$x' = f(x, y, \varepsilon), \qquad y' = \varepsilon g(x, y, \varepsilon), \tag{1}$$

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✉ Pedro Toniol Cardin
pedro.cardin@unesp.br

Marco Antonio Teixeira
teixeira@ime.unicamp.br

1 Faculdade de Engenharia, Universidade Estadual Paulista (UNESP), Ilha Solteira, São Paulo, Brazil

2 Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas (UNICAMP), Campinas, São Paulo, Brazil

where $(x, y) \in U \subseteq \mathbb{R}^m \times \mathbb{R}^n$, U is an open set with compact closure, and ε is a small parameter ($0 < \varepsilon \ll 1$). The functions $f : U \times I \longrightarrow \mathbb{R}^m$ and $g : U \times I \longrightarrow \mathbb{R}^n$ are assumed to be sufficiently smooth (typically C^∞ or C^r with r big enough for our purposes) on the set $U \times I$, where I is an open interval of the form $[0, \delta)$. The space of all C^r vector fields on U is endowed with the C^r-topology.

We shall deal with the concept of symmetries in the context of systems (1). Moreover, our approach uses geometric singular perturbation theory [8] (see also [6,13]).

It is well known that symmetry is a fundamental topic in many areas of physics and mathematics, in particular in the context of nonlinear dynamical systems. In fact, the symmetries of a given system of nonlinear ordinary differential equations can be used to analyze and understand many general mechanisms of pattern formation (see, e.g., [11]).

In what follows we shall give a brief mathematical description of symmetries and reversing symmetries in the setting of autonomous ordinary differential equations. We shall also recall the main dynamical consequences of systems possessing symmetries and reversing symmetries.

1.2 Symmetries and Reversing Symmetries

Let M be a k-dimensional manifold (e.g. $M = \mathbb{R}^k$) and $X : M \to TM$ be a smooth vector field on M. We recall the definition of a reversible system on M (see [7,14,19,20]). The differential system

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = X(\mathbf{x}), \qquad \mathbf{x} \in M \tag{2}$$

is said to be *reversible* if there exists an involutive diffeomorphism $\varphi : M \to M$ (i.e. $\varphi \circ \varphi = Id_M$) such that the equality

$$\frac{d}{dt}(\varphi(\mathbf{x})) = -(X \circ \varphi)(\mathbf{x}) \tag{3}$$

holds, for all $\mathbf{x} \in M$. In this case, we also say that system (2) (or the vector field X) is *φ-reversible* and that φ is a *reversing symmetry* of (2).

It follows from Eq. (3) that if $\gamma(t)$ is a trajectory of X, then $\varphi(\gamma(-t))$ is also a trajectory of X. In other words, a reversing symmetry φ of system (2) maps trajectories onto trajectories of X with the direction of time being reversed.

Reversible systems often appear in applications. In fact, time-reversal symmetry is one of the fundamental symmetries discussed in many branches of physics. We refer to [14] (see also the references quoted therein) for a survey on reversibility in dynamical systems (both for the time continuous case and for the time discrete case).

Similarly, there may be a transformation that maps trajectories onto trajectories of X with the direction of time being preserved. In this case, such a transformation is said to be a *symmetry* of system (2). More specificialy, an involutive diffeomorphism $\varphi : M \to M$ is a *symmetry* of system (2) if

$$\frac{d}{dt}(\varphi(\mathbf{x})) = (X \circ \varphi)(\mathbf{x}) \tag{4}$$

for all $\mathbf{x} \in M$. It follows from Eq. (4) that if $\gamma(t)$ is a trajectory of X, then $\varphi(\gamma(t))$ is also a trajectory of X.

Let $\mathbf{x} = \mathbf{x}(t)$ be a solution of Eq. (2) with maximal interval I, and let $\gamma = \{\mathbf{x}(t) : t \in I\}$ be the associated orbit of the equation. We say that the orbit γ is *symmetric* with respect to a symmetry (or a reversing symmetry) φ if $\varphi(\gamma) = \gamma$.

In what follows we recall some classical properties of systems possessing symmetries and reversing symmetries. First, let φ be a reversing symmetry of system (2), and let $\Sigma = \{\mathbf{x} \in M : \varphi(\mathbf{x}) = \mathbf{x}\}$ be the fixed point set of φ. The following properties are valid:

(1) If $\mathbf{p}$ is an equilibrium point of system (2) then so is $\varphi(\mathbf{p})$.
(2) An orbit γ is symmetric if and only if $\gamma \cap \Sigma \neq \emptyset$. In particular, every equilibrium point $\mathbf{p}$ of system (2) on Σ is symmetric.
(3) If γ is a periodic orbit of system (2) and $\gamma \cap \Sigma = \emptyset$ then so is $\varphi(\gamma)$.
(4) If a regular orbit γ of system (2) intersects Σ in two distinct points, then γ is periodic. In fact, an orbit γ is symmetric and periodic if and only if $\gamma \cap \Sigma = \{\mathbf{x}_0, \mathbf{x}_1\}$ for two distinct points $\mathbf{x}_0 \neq \mathbf{x}_1$.
(5) Symmetric equilibria and symmetric periodic orbits cannot be neither attractor nor repeller.

Now, let φ be a symmetry of system (2). Then the following properties are valid:

(1) If $\mathbf{p}$ is an equilibrium point of system (2) then so is $\varphi(\mathbf{p})$.
(2) The set $\Sigma = \{\mathbf{x} \in M : \varphi(\mathbf{x}) = \mathbf{x}\}$ is invariant under the flow of system (2). So orbits lie entirely in Σ or entirely outside of Σ.

1.3 A Brief Introduction to the Fenichel Theory of Fast-Slow Systems

In this paper, we are interested in studying how symmetry properties of a dynamical system may be affected by singular perturbations. Our approach follows the Fenichel's [8] working definition. In what follows we present some basic ideas of the techniques which lead to a geometric analysis of singularly perturbed systems.

Let τ denote the independent variable in (1). It is referred to as *fast* time scale, so that system (1) is the *fast system*. The *slow* time scale is defined by $t := \varepsilon\tau$. By switching the system (1) to the slow time variable t we obtain the *slow system*

$$\varepsilon\dot{x} = f(x, y, \varepsilon), \qquad \dot{y} = g(x, y, \varepsilon). \tag{5}$$

Note that, for $\varepsilon > 0$ systems (1) and (5) are equivalent. On the other hand, by letting $\varepsilon \to 0$ in (1) and (5) we obtain two systems with dynamics essentially different: the *layer problem*

$$x' = f(x, y, 0), \qquad y' = 0, \tag{6}$$

and the *reduced problem*

$$f(x, y, 0) = 0, \qquad \dot{y} = g(x, y, 0). \tag{7}$$

The reduced problem (7) is a differential-algebraic system that describes the evolution of the slow variables $y \in \mathbb{R}^n$ constrained to the set $\mathcal{S} = \{(x, y) \in U : f(x, y, 0) = 0\}$, while the layer problem (6) is a differential system that describes the evolution of the fast variables $x \in \mathbb{R}^m$ sufficiently far away from $\mathcal{S}$. From the Fenichel theory [8] we can obtain information on the dynamics of the full system (1) (or (5)) for positive and small values of ε by combining results on the dynamics of the limiting problems (6) and (7). Clearly the set $\mathcal{S}$ plays a special role in the theory. It is referred to as *critical manifold* since it consists of equilibria of system (6).

Among other things, Fenichel theory [8] guarantees the persistence of a normally hyperbolic compact submanifold $\mathcal{S}_0 \subseteq \mathcal{S}$ for ε positive and small enough as a locally invariant slow manifold $\mathcal{S}_\varepsilon$ of system (1), $\mathcal{O}(\varepsilon)$-close and diffeomorphic to $\mathcal{S}_0$. Moreover, the flow of the slow system (5) when restricted to $\mathcal{S}_\varepsilon$ approaches the flow of the reduced problem

(7) as $\varepsilon \to 0$. Normal hyperbolicity of $\mathcal{S}_0$ means that the eigenvalues of the $m \times m$ matrix $D_x f(x, y, 0)$, for $(x, y) \in \mathcal{S}_0$, have nonzero real part.

Note that, under the normal hyperbolicity condition, it is a natural assumption, at least locally, to assume that $\mathcal{S}_0$ is given by a graph of a function. In fact, since the matrix $D_x f(x, y, 0)$ is invertible for any $(x, y) \in \mathcal{S}_0$, Implicit Function Theorem guarantees that the equation $f(x, y, 0) = 0$ can locally be solved for x in terms of y. In order to simplify the mathematical structure of the paper, let us assume that such a solution can be taken globally over $\mathcal{S}_0$. That is, we assume that there is a smooth function $h : K \to \mathbb{R}^m$, being $K \subset \mathbb{R}^n$ a compact set, such that

$$\mathcal{S}_0 = \{(h(y), y) \in \mathbb{R}^{m+n} : y \in K\}.$$

In this case, the reduced problem (7) takes the simpler form

$$\dot{y} = g(h(y), y, 0), \tag{8}$$

and the perturbed manifold $\mathcal{S}_\varepsilon$ is described by a perturbation h_ε of h, namely $\mathcal{S}_\varepsilon = \{(h_\varepsilon(y), y) : y \in K\}$, and the flow on $\mathcal{S}_\varepsilon$ is given by

$$\dot{y} = g(h_\varepsilon(y), y, \varepsilon). \tag{9}$$

1.4 Main Result

The first question we will explore is the following one: Do symmetry properties of the reduced problem (8) on $\mathcal{S}_0$ persist for the full system (1) on $\mathcal{S}_\varepsilon$, for $\varepsilon > 0$ small enough? As we will illustrate below, since the flow on $\mathcal{S}_\varepsilon$ can be an arbitrary perturbation of the flow on $\mathcal{S}_0$, the answer to this question may be negative. For example, consider the following class of fast-slow systems

$$\varepsilon\dot{x} = -x, \qquad \dot{y} = p(y) + \varepsilon q(y, \varepsilon),$$

where p and q are smooth functions. The critical manifold $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : x = 0\} = y$-axis is normally hyperbolic and attracting, and the flow on $\mathcal{S}$ is given by $\dot{y} = p(y)$. On the other hand, for the above system, the persistent slow manifold $\mathcal{S}_\varepsilon$ is also the y-axis, and the flow on $\mathcal{S}_\varepsilon$ (see Eq. (9)) is given by $\dot{y} = p(y) + \varepsilon q(y, \varepsilon)$. This shows that the flow on $\mathcal{S}_\varepsilon$ can be any perturbation of the flow on $\mathcal{S}$.

If we take, for example, $p(y) = y^2$ and $q(y, \varepsilon) = y$, then the flow on $\mathcal{S}$ is given by $\dot{y} = y^2$, which has the reversing symmetry $\psi : \mathcal{S} \to \mathcal{S}$ given by $\psi(0, y) = (0, -y)$. On the other hand, note that ψ is not a reversing symmetry for the flow on $\mathcal{S}_\varepsilon$ for $\varepsilon > 0$, which is given by $\dot{y} = y^2 + \varepsilon y$. Therefore, a reversing symmetry of a dynamical system need not be preserved by singular perturbations.

It is also easy to obtain an example where a symmetry of the reduced problem is not preserved on $\mathcal{S}_\varepsilon$. Take, for example, $p(y) = -y$ and $q(y, \varepsilon) = y^2$. Then, $\psi(0, y) = (0, -y)$ is a symmetry for the flow on $\mathcal{S}$, which is given by $\dot{y} = -y$, but ψ is not a symmetry for the flow on $\mathcal{S}_\varepsilon$ for $\varepsilon > 0$, which is given by $\dot{y} = -y + \varepsilon y^2$.

Remark 1 It is worthwhile to note that the previous discussion does not imply that the flow on $\mathcal{S}_\varepsilon$ has no (reversing) symmetry. It just shows that a (reversing) symmetry ψ of the reduced problem may not be preserved for the flow on $\mathcal{S}_\varepsilon$ for $\varepsilon > 0$. The flow on $\mathcal{S}_\varepsilon$ could have some other (reversing) symmetry ϕ, which could vary even with the parameter ε. Identifying such a (reversing) symmetry, if it exists, can be a very difficult (perhaps impossible) task, even for the simplest examples.

Remark 2 Of course, there may be situations where a (reversing) symmetry of the reduced problem on $\mathcal{S}_0$ persists (or perturbs) on $\mathcal{S}_\varepsilon$, for $\varepsilon > 0$ small enough. For example, this always happens if the flow on $\mathcal{S}_0$ and the flow on $\mathcal{S}_\varepsilon$ are differentially conjugate. In fact, if $\psi : \mathcal{S}_0 \to \mathcal{S}_0$ is a (reversing) symmetry for the reduced problem (8) and if $K_\varepsilon : \mathcal{S}_0 \to \mathcal{S}_\varepsilon$ is a conjugacy between the solutions of systems (8) and (9), then one can show that $\psi_\varepsilon : \mathcal{S}_\varepsilon \to \mathcal{S}_\varepsilon$ defined by $\psi_\varepsilon = K_\varepsilon \circ \psi \circ K_\varepsilon^{-1}$ is a (reversing) symmetry for the flow on $\mathcal{S}_\varepsilon$. Moreover, $\psi_\varepsilon \to \psi$ when $\varepsilon \to 0$, since the flow on $\mathcal{S}_\varepsilon$ converges for the flow on $\mathcal{S}_0$ when $\varepsilon \to 0$ (the diffeomorphisms K_ε and K_ε^{-1} converge to the identity map when $\varepsilon \to 0$). In such a situation the (reversing) symmetry ψ_ε can vary with the parameter ε. Obviously, the assumption above (that is, the flows on $\mathcal{S}_0$ and $\mathcal{S}_\varepsilon$ to be differentially conjugate) is very restrictive and difficult to verify in practice.

Remark 3 The (reversing) symmetries considered in this article will not vary with the parameter ε.

In what follows we present the main result of this paper.

Theorem A. *Consider a C^∞ family like (1). Assume that $\varphi : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is a symmetry (respectively, a reversing symmetry) for system (1), for all $\varepsilon > 0$ sufficiently small. Then:*

(i) φ is a symmetry (respectively, a reversing symmetry) for the layer problem (6).
(ii) The critical manifold $\mathcal{S}$ is symmetric with respect to $\Sigma = Fix\,\varphi$.
(iii) If the functions f and g in (1) do not depend on ε, then $\phi := \varphi_{|\mathcal{S}} : \mathcal{S} \to \mathcal{S}$, the restriction of φ to $\mathcal{S}$, is a symmetry (respectively, a reversing symmetry) for the reduced problem (7).
(iv) If $\mathcal{S}_0 \subseteq \mathcal{S}$ is a normally hyperbolic compact manifold and $\mathcal{S}_\varepsilon$ is a persistent slow manifold of system (1), then $\phi_\varepsilon := \varphi_{|\mathcal{S}_\varepsilon} : \mathcal{S}_\varepsilon \to \mathcal{S}_\varepsilon$, the restriction of φ to $\mathcal{S}_\varepsilon$, is a symmetry (respectively, a reversing symmetry) for system (1) on $\mathcal{S}_\varepsilon$. Moreover, if the functions f and g in (1) do not depend on ε, then the sets $\Sigma_\varepsilon = Fix\,\phi_\varepsilon$ and $\Sigma_0 = Fix\,\phi$ are diffeomorphic and $\Sigma_\varepsilon \to \Sigma_0$ when $\varepsilon \to 0$ according to Hausdorff distance.
(v) If φ is a reversing symmetry and m is odd, then the $m \times m$ matrix $D_x f(p, 0)$ has at least one zero eigenvalue, for all $p \in \mathcal{S} \cap \Sigma$.
(vi) If φ is a reversing symmetry and m is odd, then $\mathcal{S}$ is not normally hyperbolic at any point in $\mathcal{S} \cap \Sigma$.

Theorems A is proved in Sect. 3.

Remark 4 Similar result to items (v) and (vi) of Theorem A. are not valid if m is even. In fact, for $m = 2$ consider the fast-slow vector field $X_\varepsilon(x, y) = (x_2, x_1, \varepsilon y^2)$, where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y \in \mathbb{R}$. It is not difficult to check that $\varphi(x, y) = (x_1, -x_2, -y)$ is a reversing symmetry of X_ε, for all $\varepsilon \geq 0$. The fixed point set of φ is $\Sigma = \{(x_1, 0, 0) : x_1 \in \mathbb{R}\} = x_1$-axis and the critical manifold is $\mathcal{S} = \{(0, 0, y) \in \mathbb{R}^3 : y \in \mathbb{R}\} = y$-axis. Note that $\mathcal{S}$ is normally hyperbolic and it is of saddle type, since the matrix

$$J = D_x f(x_1, x_2, y) = D_x[(x_2, x_1)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues ± 1. In particular, $\mathcal{S}$ is normally hyperbolic in $\mathcal{S} \cap \Sigma = \{(0, 0, 0)\}$. If $m = 2k$ is any even number, we can generalize the previous example by considering the fast-slow vector field

$$Y_\varepsilon(x, y) = (x_2, x_1, x_4, x_3, \ldots, x_{2k}, x_{2k-1}, \varepsilon y^2),$$

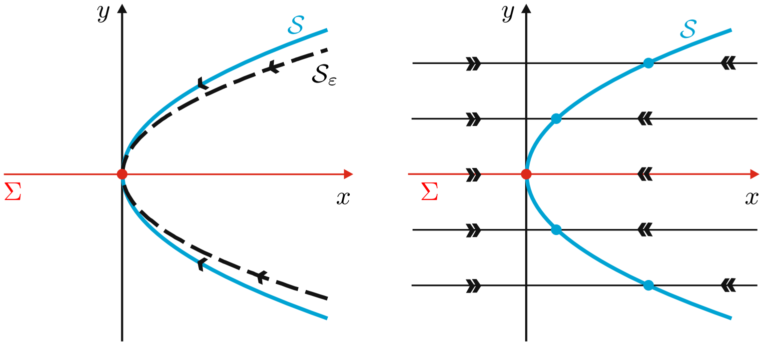

Fig. 1 Phase portraits of the reduced and layer problems from Example 1, respectively, the persistent slow manifold $\mathcal{S}_\varepsilon$ (dashed black), and the flow on $\mathcal{S}_\varepsilon$

where $x = (x_1, x_2, \ldots, x_{2k}) \in \mathbb{R}^{2k}$ and $y \in \mathbb{R}$. Then, one can check that $\varphi(x, y) = (x_1, -x_2, x_3, -x_4, \ldots, x_{2k-1}, -x_{2k}, -y)$ is a reversing symmetry of Y_ε, for all $\varepsilon \geq 0$. Moreover, note that $D_x f(x, y) = \text{diag}(J, J, \ldots, J)$, where J is like above. Thus, ± 1 are the (multiplicity k) eigenvalues of $D_x f(x, y)$. Therefore, the critical manifold $\mathcal{S} = \{(0, y) \in \mathbb{R}^{2k+1} : y \in \mathbb{R}\}$ is normally hyperbolic. In particular, $\mathcal{S}$ is normally hyperbolic in $\mathcal{S} \cap \Sigma = \{(0, 0, 0)\}$.

2 Examples

In this section we present several examples illustrating the main result of the paper. We consider theoretical examples as well as models of practical relevance. We begin with one of the simplest system where it is possible to compute the slow manifold $\mathcal{S}_\varepsilon$ analytically.

Example 1 Consider the following fast-slow system in $\mathbb{R}^2$:

$$x' = y^2 - x, \qquad y' = -\varepsilon y. \tag{10}$$

The critical manifold is $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : x = y^2\}$, which is normally hyperbolic and attracting, since $D_x f(x, y, 0) = -1$ for all $(x, y) \in \mathcal{S}$, where $f(x, y, \varepsilon) = y^2 - x$. The corresponding limiting problems are, respectively, the layer problem $x' = y^2 - x$, $y' = 0$, and the reduced problem $x = y^2$, $\dot{y} = -y$.

It is easy to check that system (10), for all $\varepsilon > 0$, and the layer problem have the symmetry $\varphi(x, y) = (x, -y)$. Moreover, note that the critical manifold $\mathcal{S}$ is symmetric with respect to $\Sigma = \text{Fix}\,\varphi = \{(x, y) \in \mathbb{R}^2 : y = 0\} = x$-axis, and the reduced problem has the symmetry $\phi : \mathcal{S} \to \mathcal{S}$ given by $\phi(y^2, y) = (y^2, -y)$ (that is the restriction of φ to $\mathcal{S}$). We can also parameterize the reduced problem using the variable x. In fact, differentiating the equation $x = y^2$ with respect to t, we obtain that the slow flow on the critical manifold $\mathcal{S}$ is given by $\dot{x} = -2x$, which has the symmetry $x \mapsto -x$.

For this example, it is not difficult to obtain the persistent slow manifold $\mathcal{S}_\varepsilon$ analytically. In [13], the author shows how to compute $\mathcal{S}_\varepsilon$ for system (10) (see page 56 of [13]). It is given by $\mathcal{S}_\varepsilon = \{(x, y) \in \mathbb{R}^2 : x = y^2/(1 - 2\varepsilon)\}$. The flow on $\mathcal{S}_\varepsilon$ is also expressed as $\dot{y} = -y$, which has the symmetry $\phi_\varepsilon : \mathcal{S}_\varepsilon \to \mathcal{S}_\varepsilon$ given by $\phi_\varepsilon(y^2/(1 - 2\varepsilon), y) = (y^2/(1 - 2\varepsilon), -y)$

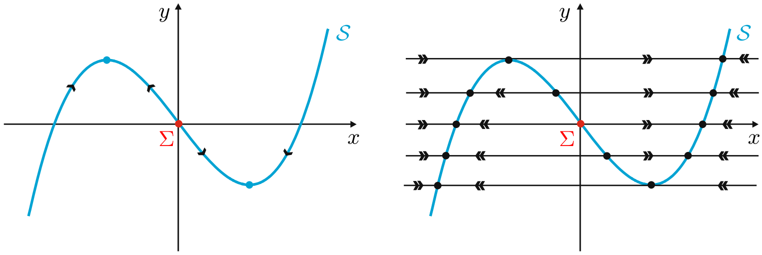

Fig. 2 Phase portraits of the reduced and layer problems from Example 2, respectively

(that is the restriction of φ to $\mathcal{S}_\varepsilon$). See Figure 1 for the phase portraits on $\mathcal{S}$ and on $\mathcal{S}_\varepsilon$, and also the phase portrait of the layer problem.

Example 2 In this example we consider the classical van der Pol equation

$$x'' + \mu(x^2 - 1)x' + x = 0, \tag{11}$$

where $\mu \in \mathbb{R}$ is sufficiently large. We can transform (11) into a fast-slow system. To do this we first consider the Liénard change $y = x'/\mu + x^3/3 - x$. This leads to system $x' = \mu(y - x^3/3 + x)$, $y' = -x/\mu$. Defining a new independent variable $t := \tau/\mu$ and setting $\varepsilon := 1/\mu^2$, we obtain the following fast-slow system

$$\varepsilon\dot{x} = y - \frac{x^3}{3} + x, \qquad \dot{y} = -x.$$

Let $X_\varepsilon(x, y) = (y - x^3/3 + x, -\varepsilon x)$ be the vector field defined by the fast system. It is not difficult to check that X_ε and the layer problem $X_0(x, y) = (y - x^3/3 + x, 0)$ have the symmetry $\varphi(x, y) = (-x, -y)$. Moreover, note that the critical manifold $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : y = x^3/3 - x\}$ is symmetric with respect to $\Sigma = \mathrm{Fix}\,\varphi = \{(0, 0)\}$ and the reduced problem $y = x^3/3 - x$, $\dot{y} = -x$ has the symmetry $\phi = \varphi_{|\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ given by $\phi(x, x^3/3 - x) = (-x, x - x^3/3)$. We can also parameterize the reduced problem using the variable x. Differentiating the equation $y = x^3/3 - x$ with respect to t, and replacing $\dot{y} = -x$, we obtain that the slow flow on $\mathcal{S}$ is given by $\dot{x} = x/(1-x^2)$, which has the symmetry $x \mapsto -x$. Figure 2 illustrates the phase portraits of the reduced and layer problems.

The van der Pol example can be generalized as follows.

Example 3 Consider the following one-parameter family of Liénard equations

$$x'' + \mu f(x)x' + g(x) = 0. \tag{12}$$

where f and g are smooth functions and $\mu \in \mathbb{R}$ is sufficiently large. Similarly to example 2, we can transform (12) into a fast-slow system given by

$$\varepsilon\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \tag{13}$$

where $F(x) = \int_0^x f(s)ds$ and $\varepsilon = 1/\mu^2$. Transforming (13) to the fast variable τ we obtain the fast system

$$x' = y - F(x), \qquad y' = -\varepsilon g(x). \tag{14}$$

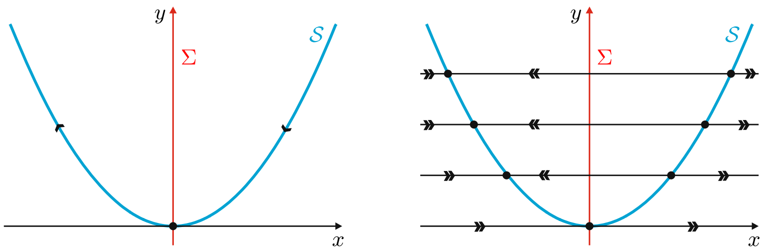

Fig. 3 Phase portraits of the reduced and layer problems from Example 3 with $F(x) = x^2$ e $g(x) = x$

The critical manifold is $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : y = F(x)\}$ and the reduced problem is given by $y = F(x)$, $\dot{y} = -g(x)$. We can also write the slow flow on $\mathcal{S}$ in terms of the fast variable x. Differentiating the equation $y = F(x)$ with respect to t yields

$$\dot{x} = -\frac{g(x)}{f(x)} \tag{15}$$

Let $X_\varepsilon(x, y) = (y - F(x), -\varepsilon g(x))$ be the vector field defined by the fast system (14), and consider $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}^2$ the (canonical) linear involutions $\varphi(x, y) = (-x, -y)$ and $\psi(x, y) = (-x, y)$. Then, one can verify that:

1. φ is a symmetry of X_ε if, and only if, the functions F and g are odd.
2. ψ is a reversing symmetry of X_ε if, and only if, the function F is even and g is odd.

Note also that in the first case, φ is a symmetry for the layer problem $X_0(x, y) = (y - F(x), 0)$, the critical manifold $\mathcal{S} = \{y = F(x)\}$ is symmetric with respect to Fix $\varphi = \{(0, 0)\}$, since F is odd, and the reduced problem (15) has the symmetry $x \mapsto -x$ (observe that $f = F'$ is even, since F is odd). In the second case, ψ is a reversing symmetry of X_0, $\mathcal{S}$ is symmetric with respect to Fix $\psi = \{(x, y) \in \mathbb{R}^2 : x = 0\} = y$-axis, since F is even, and (15) is reversible with respect to the involution $x \mapsto -x$ (in this case $f = F'$ is odd, since F is even). Moreover, $\mathcal{S}$ is not normally hyperbolic in $\mathcal{S} \cap \text{Fix}\,\psi = \{(0, F(0))\}$, since $D_x(y - F(x))_{|x=0} = -F'(0) = 0$. Of course, the van der Pol example (Example 2) fits in the first case. A simple choice for the second case would be $F(x) = x^2$ e $g(x) = x$. Figure 3 illustrates the phase portraits of the reduced and layer problems for that choice.

Reversible fast-slow systems also appear in some physical circumstances. The next example consider an isothermal oscillator (see, e. g., [17]).

Example 4 The following system is a model for a harmonic oscillator which is coupled to a heat bath:

$$x_1' = x_2, \quad x_2' = -x_1 - x_2 y, \quad y' = \varepsilon(x_2^2 - kT), \tag{16}$$

where kT is the product of Boltzmann's constant and the temperature and ε is a parameter. For ε positive and small enough, system (16) is a fast-slow system with two fast variables x_1 and x_2 and one slow variable y. The critical manifold is $\mathcal{S} = \{(0, 0, y) \in \mathbb{R}^3 : y \in \mathbb{R}\} = y$-axis and the corresponding limiting problems are, respectively, the layer problem $x_1' = x_2$, $x_2' = -x_1 - x_2 y$, $y' = 0$, and the reduced problem $x_1 = x_2 = 0$, $\dot{y} = -kT$. We have that the system (16), for all $\varepsilon > 0$, and the layer problem are φ-reversible, where $\varphi(x_1, x_2, y) = (x_1, -x_2, -y)$. Moreover, note that $\mathcal{S}$ is symmetric with respect to $\Sigma = \text{Fix}\,\varphi = \{(x_1, 0, 0) \in$

$\mathbb{R}^3 : x_1 \in \mathbb{R}\} = x_1$-axis and the reduced problem is reversible with respect to the involution $y \mapsto -y$. Also, note that $\mathcal{S}$ is not normally hyperbolic in $\mathcal{S} \cap \Sigma = \{(0,0,0)\}$, since

$$D_x f(0,0,0) = D_x[(x_2, -x_1 - x_2 y)](0,0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has eigenvalues $\pm i$. The set $\mathcal{S} - \{(0,0,0)\}$ is normally hyperbolic, since the matrix $D_x f(0,0,y)$ has eigenvalues $\lambda_{\pm} = (-y \pm \sqrt{y^2-4})/2$, which have nonzero real part if $y \neq 0$.

3 Proofs

In this section we prove our main result (Theorem A.).

Proof of Theorem A. Assume that φ is a reversing symmetry for system (1), for all $\varepsilon > 0$ sufficiently small. First we prove item (i). Write the involution $\varphi : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ as

$$\varphi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$$

where $\varphi_1 : \mathbb{R}^{m+n} \to \mathbb{R}^m$ and $\varphi_2 : \mathbb{R}^{m+n} \to \mathbb{R}^n$.

Then, the equality $D\varphi \cdot X_\varepsilon = -X_\varepsilon \circ \varphi$ holds for all $(x,y) \in \mathbb{R}^{m+n}$ and for all $\varepsilon > 0$ sufficiently small, where $X_\varepsilon = (f, \varepsilon g)$ is the vector field defined by (1). It is straightforward to see that this equality is equivalent to

$$\begin{aligned} (D_x\varphi_1) \cdot f + \varepsilon(D_y\varphi_1) \cdot g &= -f(\varphi_1, \varphi_2, \varepsilon), \\ (D_x\varphi_2) \cdot f + \varepsilon(D_y\varphi_2) \cdot g &= -\varepsilon g(\varphi_1, \varphi_2, \varepsilon). \end{aligned} \tag{17}$$

Since the functions f and g depend smoothly on ε at $\varepsilon = 0$, by letting $\varepsilon \to 0$ in (17), the following equalities are verified

$$\begin{aligned} (D_x\varphi_1(x,y)) \cdot f(x,y,0) &= -f(\varphi_1(x,y), \varphi_2(x,y), 0), \\ (D_x\varphi_2(x,y)) \cdot f(x,y,0) &= 0, \end{aligned} \tag{18}$$

for all $(x,y) \in \mathbb{R}^{m+n}$. Thus, the equality $D\varphi \cdot X_0 = -X_0 \circ \varphi$ holds for all $(x,y) \in \mathbb{R}^{m+n}$, where $X_0(x,y) = (f(x,y,0), 0)$ is the vector field defined by (6). Therefore, φ is a reversing symmetry for the layer problem (6).

Now we prove item (ii). From the equality $(D_x\varphi_1(x,y)) \cdot f(x,y,0) = -f(\varphi_1(x,y), \varphi_2(x,y), 0)$ obtained above, it follows that if $f(x,y,0) = 0$ then $f(\varphi_1(x,y), \varphi_2(x,y), 0) = 0$. Thus, if $(x,y) \in \mathcal{S}$ then $\varphi(x,y) \in \mathcal{S}$. We shall prove that $\varphi(\mathcal{S}) = \mathcal{S}$. First, to show that $\mathcal{S} \subset \varphi(\mathcal{S})$, let $(a,b) \in \mathcal{S}$. Then $f(a,b) = 0$. As $(a,b) \in \mathcal{S}$, it follows from the above that $(p,q) := \varphi(a,b) \in \mathcal{S}$. Moreover, $\varphi(p,q) = \varphi^2(a,b) = (a,b)$, since φ is an involution. Therefore, $(a,b) \in \varphi(\mathcal{S})$. Now, to show that $\varphi(\mathcal{S}) \subset \mathcal{S}$, let $(u,v) \in \varphi(\mathcal{S})$. Then, $(u,v) = \varphi(x,y)$ with $(x,y) \in \mathcal{S}$. As $(x,y) \in \mathcal{S}$, it follows from the above that $\varphi(x,y) \in \mathcal{S}$, that is $(u,v) \in \mathcal{S}$. Therefore, the critical manifold $\mathcal{S}$ is symmetric with respect to $\Sigma = \mathrm{Fix}\,\varphi$.

To prove item (iii), let ϕ be the restriction of φ to $\mathcal{S}$, i.e. $\phi(x,y) = (\varphi_1(x,y), \varphi_2(x,y))$, for $(x,y) \in \mathcal{S}$. Clearly ϕ is a differentiable involution. Thus, ϕ is an involutive diffeomorphism of $\mathcal{S}$ on $\phi(\mathcal{S}) = \mathcal{S}$. To prove that the reduced problem (7) is ϕ-reversible note that, for $(x,y) \in \mathcal{S}$, the second Eq. in (17) becomes

$$(D_y\varphi_2(x,y)) \cdot g(x,y) = -g(\varphi_1(x,y), \varphi_2(x,y)),$$

since the functions f and g do not depend on ε. This last equality is exactly the reversibility condition for the reduced problem $f(x, y) = 0$, $\dot{y} = g(x, y)$ relative to the involution ϕ. Therefore, ϕ is a reversing symmetry of (7).

The first part of item (iv) of Theorem A. follows immediately when we restrict equality (17) to $\mathcal{S}_\varepsilon$. Moreover, it is obvious that the set $\Sigma_\varepsilon = \text{Fix}\,\phi_\varepsilon$ is given by $\Sigma_\varepsilon = \Sigma \cap \mathcal{S}_\varepsilon$. Also, it follows from item (iii) that the set $\Sigma_0 = \text{Fix}\,\phi$ is given by $\Sigma_0 = \Sigma \cap \mathcal{S}_0$. Thus, Σ_ε and Σ_0 are diffeomorphic and $\Sigma_\varepsilon \to \Sigma_0$ when $\varepsilon \to 0$ according to Hausdorff distance, since $\mathcal{S}_\varepsilon$ and $\mathcal{S}_0$ are diffeomorphic and $\mathcal{S}_\varepsilon \to \mathcal{S}_0$ when $\varepsilon \to 0$ according to Hausdorff distance.

The proof of items (i), (ii), (iii), and (iv) of Theorem A. in the case where φ is a symmetry of system (1) follows analogously.

Now we prove item (v). Differentiating the equality (18) with respect to x we obtain

$$
\begin{aligned}
(D_x\varphi_1)_x \cdot f + D_x\varphi_1 \cdot D_x f &= -D_x f \cdot D_x\varphi_1 - D_y f \cdot D_x\varphi_2,\\
(D_x\varphi_2)_x \cdot f + D_x\varphi_2 \cdot D_x f &= 0.
\end{aligned}
$$

Evaluating at a point $p \in \mathcal{S} \cap \Sigma$ gives

$$
\begin{aligned}
&D_x\varphi_1(p) \cdot D_x f(p, 0) = -D_x f(p, 0) \cdot D_x\varphi_1(p) - D_y f(p, 0) \cdot D_x\varphi_2(p),\\
&D_x\varphi_2(p) \cdot D_x f(p, 0) = 0.
\end{aligned} \tag{19}
$$

Now, differentiating (18) with respect to y yields

$$
\begin{aligned}
(D_x\varphi_1)_y \cdot f + D_x\varphi_1 \cdot D_y f &= -D_x f \cdot D_y\varphi_1 - D_y f \cdot D_y\varphi_2,\\
(D_x\varphi_2)_y \cdot f + D_x\varphi_2 \cdot D_y f &= 0,
\end{aligned}
$$

and evaluating at $p \in \mathcal{S} \cap \Sigma$ gives

$$
\begin{aligned}
&D_x\varphi_1(p) \cdot D_y f(p, 0) = -D_x f(p, 0) \cdot D_y\varphi_1(p) - D_y f(p, 0) \cdot D_y\varphi_2(p),\\
&D_x\varphi_2(p) \cdot D_y f(p, 0) = 0.
\end{aligned} \tag{20}
$$

We may rewrite (19) and (20) in matrix format as $AB = -BA$, where A and B are matrices of order $(m + n)$ expressed as

$$
A = \begin{pmatrix} D_x\varphi_1(p) & D_y\varphi_1(p) \\ D_x\varphi_2(p) & D_y\varphi_2(p) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} D_x f(p, 0) & D_y f(p, 0) \\ 0_{n\times m} & 0_{n\times n} \end{pmatrix}
$$

where $0_{n\times m}$ and $0_{n\times n}$ denote the zero matrices of order $n \times m$ and $n \times n$, respectively. The matrix A is invertible, since φ is an involution. Thus, $B = A^{-1}(-B)A$. Therefore, B and $-B$ are similar matrices, then B and $-B$ have the same eigenvalues. Clearly, $\lambda = 0$ is an eigenvalue of multiplicity n of B and $-B$. The other eigenvalues of B and $-B$ are the eigenvalues of $D_x f(p, 0)$ and $-D_x f(p, 0)$, respectively, so that $D_x f(p, 0)$ and $-D_x f(p, 0)$ have the same eigenvalues. Thus, $\det D_x f(p, 0) = \det(-D_x f(p, 0)) = (-1)^m \det D_x f(p, 0) = -\det D_x f(p, 0)$, since m is odd, which implies that $\det D_x f(p, 0) = 0$. Therefore, $D_x f(p, 0)$ has at least one zero eigenvalue, for all $p \in \mathcal{S} \cap \Sigma$.

Obviously, item (vi) of Theorem A. follows from item (v). □

4 Conclusions

In this work we have studied the topic of symmetries in the context of singular perturbation problems with two time scales (fast and slow). First we have seen that a symmetry (or a reversing symmetry) of the reduced problem (7) on a normally hyperbolic compact submanifold $\mathcal{S}_0$ of the critical manifold $\mathcal{S}$ need not be preserved for the full system (1) on a persistent slow manifold $\mathcal{S}_\varepsilon$, for $\varepsilon > 0$ small enough. We have also explored the existing relations between symmetries of the full system (1) and symmetries of the limiting problems (6) and (7) and of the critical manifold $\mathcal{S}$, besides analyzing the normal hyperbolicity condition at points $p \in \mathcal{S} \cap \Sigma$ in the reversible case (Theorem A.).

In what follows we briefly sketch some ideas for future research mathematical problems that can be raised from this article. For instance, in the context of fast-slow systems (1), it would be interesting to study the same issues addressed in this article considering weak symmetries (or weak reversing symmetries). We recall that the term "weak" in the previous sentence means a system possessing a non-involutory (reversing) symmetry ([19]). It is worthwhile to mention that many results for reversible systems actually also hold for weakly reversible systems. See, e.g., [2,4].

We know that the fundamental hypothesis in order to apply Fenichel theory [8] is the normal hyperbolicity of the manifolds. However, it would be interesting to study the persistence of certain symmetry properties (and symmetric invariant sets of the reduced problem (7)) when the normal hyperbolicity condition is not satisfied. For instance, a situation of particular interest which leads to consider *reversible fast-slow Hamiltonian systems* with non-normally hyperbolic critical manifolds is the study of fourth-order ordinary differential equations of the form

$$\varepsilon^2 \frac{d^4u}{dt^4} + \frac{d^2u}{dt^2} + u^2 - u = 0. \tag{21}$$

Equation (21) was originally introduced in the theory of surface water waves when a weak surface tension is taken into account ([1,12]). It belongs to the class of higher Euler–Lagrange–Poisson equations (see, e.g., [3]).

We consider the more general equation

$$\varepsilon^2 \frac{d^4u}{dt^4} + \frac{d^2u}{dt^2} + q(u) = 0, \tag{22}$$

where q is a smooth function. Applying the change of variables

$$x_1 = \dot{u}, \quad x_2 = \varepsilon \ddot{u}, \quad y_1 = u, \quad y_2 = -\dot{u} - \varepsilon^2 \dddot{u},$$

Equation (22) is transformed into the following fast-slow system

$$\varepsilon \dot{x}_1 = x_2, \quad \varepsilon \dot{x}_2 = -x_1 - y_2, \quad \dot{y}_1 = x_1, \quad \dot{y}_2 = q(y_1). \tag{23}$$

For ε positive and small enough, system (23) is a fast-slow system with two fast variables x_1 and x_2 and two slow variables y_1 and y_2. In fact, system (23) is a fast-slow Hamiltonian system of the form

$$\varepsilon \dot{x}_1 = \frac{\partial H}{\partial x_2}, \quad \varepsilon \dot{x}_2 = -\frac{\partial H}{\partial x_1}, \quad \dot{y}_1 = \frac{\partial H}{\partial y_2}, \quad \dot{y}_2 = -\frac{\partial H}{\partial y_1}$$

with Hamiltonian function $H(x_1, x_2, y_1, y_2) = (x_1^2 + x_2^2)/2 + x_1 y_2 - Q(y_1)$, where $Q'(y_1) = q(y_1)$. Moreover, it is not difficult to check that system (23) is φ-reversible, where $\varphi(x_1, x_2, y_1, y_2) = (-x_1, x_2, y_1, -y_2)$. Also, we note that the critical manifold $\mathcal{S} = \{(-y_2, 0, y_1, y_2) \in \mathbb{R}^4 : y_1, y_2 \in \mathbb{R}\}$ is symmetric with respect to $\Sigma = \text{Fix}\,\varphi =$

$\{(0, x_2, y_1, 0) \in \mathbb{R}^4 : x_2, y_1 \in \mathbb{R}\}$, the layer problem $x_1' = x_2$, $x_2' = -x_1 - y_2$, $y_1' = 0$, $y_2' = 0$ is φ-reversible, and the reduced problem $x_2 = 0$, $x_1 = -y_2$, $\dot{y}_1 = -y_2$, $\dot{y}_2 = q(y_1)$ is ϕ-reversible, where $\phi : \mathcal{S} \to \mathcal{S}$ is given by $\phi(-y_2, 0, y_1, y_2) = (y_2, 0, y_1, -y_2)$ (that is the restriction of φ to $\mathcal{S}$). Also, note that the normal hyperbolicity condition is not satisfied at any point of the critical manifold $\mathcal{S}$, since

$$D_x f(p) = D_x[(x_2, -x_1 - y_2)](p) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has eigenvalues $\pm i$, for all $p \in \mathcal{S}$. So that Theorem A cannot be applied.

More generally, the study of singularly perturbed Hamiltonian systems with non-normally hyperbolic critical manifolds has recently been considered by some authors (see, e.g., [9,10,15,16,18]). In particular, when the critical manifold of the fast-slow Hamiltonian system is normally elliptic, persistence issues have been studied in [9,10,15].

Reversible singularly perturbed systems with non-normally hyperbolic critical manifolds were also considered in [5], where conditions for the existence of infinitely many periodic orbits and heteroclinic cycles converging to singular orbits with respect to the Hausdorff distance were given.

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