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Long Time Behavior of Random and Nonautonomous Fisher–KPP Equations: Part I—Stability of Equilibria and Spreading Speeds

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Abstract

In the current series of two papers, we study the long time behavior of nonnegative solutions to the following random Fisher–KPP equation,

$$u_t = u_{xx} + a(\theta_t \omega)u(1-u), \quad x \in \mathbb{R}, \tag{1}$$

where $\omega \in \Omega$, $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space, θ_t is an ergodic metric dynamical system on Ω, and $a(\omega) > 0$ for every $\omega \in \Omega$. We also study the long time behavior of nonnegative solutions to the following nonautonomous Fisher–KPP equation,

$$u_t = u_{xx} + a_0(t)u(1-u), \quad x \in \mathbb{R}, \tag{2}$$

where $a_0(t)$ is a positive locally Hölder continuous function. In this first part of the series, we investigate the stability of positive equilibria and the spreading speeds. Under some proper assumption on $a(\omega)$, we show that the constant solution $u = 1$ of (1) is asymptotically stable with respect to strictly positive perturbations and show that (1) has a deterministic spreading speed interval $[2\sqrt{\underline{a}}, 2\sqrt{\bar{a}}]$, where $\underline{a}$ and $\bar{a}$ are the least and the greatest means of $a(\cdot)$, respectively, and hence the spreading speed interval is linearly determinate. It is shown that the solution of (1) with a nonnegative initial function which is bounded away from 0 for $x \ll -1$ and is 0 for $x \gg 1$ propagates at the speed $2\sqrt{\hat{a}}$, where $\hat{a}$ is the mean of $a(\cdot)$. Under some assumption on $a_0(\cdot)$, we also show that the constant solution $u = 1$ of (2) is asymptotically stably and (2) admits a bounded spreading speed interval. It is not assumed that $a(\omega)$ and $a_0(t)$ are bounded above and below by some positive constants. The results obtained in this part are new and extend the existing results in literature on spreading speeds of Fisher–KPP equations. In the second part of the series, we will study the existence and stability of transition fronts of (1) and (2).

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1 Introduction and Statements of the Main Results

The current series of two papers is concerned with the long time behavior of nonnegative solutions to the following random Fisher–KPP equation,

$$u_t = u_{xx} + a(\theta_t\omega)u(1-u), \quad x \in \mathbb{R}, \tag{1.1}$$

where $\omega \in \Omega$, $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is an ergodic metric dynamical system on Ω, $a : \Omega \to (0, \infty)$ is measurable, and $a^\omega(t) := a(\theta_t\omega)$ is locally Hölder continuous for every $\omega \in \Omega$. It also considers the long time behavior of nonnegative solutions to the following nonautonomous Fisher–KPP equation,

$$u_t = u_{xx} + a_0(t)u(1-u), x \in \mathbb{R}, \tag{1.2}$$

where $a_0 : \mathbb{R} \to (0, \infty)$ is locally Hölder continuous. Among others, (1.1) and (1.2) are used to model the population growth of a species in biology. In such case, $u(t, x)$ denotes the population density of the species. Thanks to the biological reason, we are only interested in nonnegative solutions of (1.1) and (1.2).

Observe that (1.1) [resp. (1.2)] with $a(\omega) \equiv 1$ (resp. with $a_0(t) \equiv 1$) becomes

$$u_t = u_{xx} + u(1-u), \quad x \in \mathbb{R}. \tag{1.3}$$

Equation (1.3) is called in literature Fisher–KPP equation due to the pioneering works of Fisher [13] and Kolmogorov et al. [25] on traveling wave solutions and take-over properties of (1.3). It is clear that the constant solution $u = 1$ of (1.3) is asymptotically stable with respect to strictly positive perturbations. Fisher in [13] found traveling wave solutions $u(t, x) = \phi(x - ct)$ of (1.3) ($\phi(-\infty) = 1, \phi(\infty) = 0, \phi(s) > 0$) of all speeds $c \geq 2$ and showed that there are no such traveling wave solutions of slower speed. He conjectured that the take-over occurs at the asymptotic speed 2. This conjecture was proved in [25] for some special initial distribution and was proved in [3] for general initial distributions. More precisely, it is proved in [25] that for the nonnegative solution $u(t, x)$ of (1.3) with $u(0, x) = 1$ for $x < 0$ and $u(0, x) = 0$ for $x > 0$, $\lim_{t\to\infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$. It is proved in [3] that for any nonnegative solution $u(t, x)$ of (1.3), if at time $t = 0$, u is 1 near $-\infty$ and 0 near ∞, then $\lim_{t\to\infty} u(t, ct)$ is 0 if $c > 2$ and 1 if $c < 2$. In literature, $c^* = 2$ is called the *spreading speed* for (1.3).

A huge amount of research has been carried out toward various extensions of traveling wave solutions and take-over properties of (1.3) to general time and space independent as well as time and/or space dependent Fisher–KPP type equations. See, for example, [2,3,11, 15,24,41,48], etc., for the extension to general time and space independent Fisher–KPP type equations; see [4,5,7,14,22,26–29,31,37,38,49,50], and references therein for the extension to time and/or space periodic Fisher–KPP type equations; and see [5,8–10,16,21,30,32–36,43–47,51,52], and references therein for the extension to quite general time and/or space dependent Fisher–KPP type equations. The reader is referred to [12,17,53], etc. for the study of Fisher–KPP reaction diffusion equations with time delay.

All the existing works on (1.1) [resp. (1.2)] assumed $\inf_{t\in\mathbb{R}} a^{\omega}(t) > 0$ and $a^{\omega}(\cdot) \in L^{\infty}(\mathbb{R})$ (resp. $\inf_{t\in\mathbb{R}} a_0(t) > 0$ and $\sup_{t\in\mathbb{R}} a_0(t) < \infty$). The objective of the current series of two papers is to study the long time behavior, in particular, the stability of positive constant solutions, the spreading speeds, and the transition fronts of (1.1) [resp. (1.2)] without the assumption $\inf_{t\in\mathbb{R}} a^{\omega}(t) > 0$ and $a^{\omega}(\cdot) \in L^{\infty}(\mathbb{R})$ (resp. without the assumption $\inf_{t\in\mathbb{R}} a_0(t) > 0$ and $\sup_{t\in\mathbb{R}} a_0(t) < \infty$). It will also discuss the applications of the results established for (1.1) to Fisher–KPP equations whose growth rate and/or carrying capacity are perturbed by real noises.

In this first part of the series, we investigate the stability of positive constant solutions and the spreading speeds of (1.1) and (1.2). We first consider the stability of positive constant solutions and spreading speeds of (1.1) and then consider the stability of positive constant solutions and spreading speeds of (1.2). In the second part of the series, we will study the existence and stability of transition fronts of (1.1) and (1.2).

In the following, we state the main results of the current paper. Let

$$C^b_{\rm unif}(\mathbb{R}) = \{u \in C(\mathbb{R}) \,|\, u \text{ is bounded and uniformly continuous}\}$$

with norm $\|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)|$ for $u \in C^b_{\rm unif}(\mathbb{R})$. For given $u_0 \in X := C^b_{\rm unif}(\mathbb{R})$ and $\omega \in \Omega$, let $u(t, x; u_0, \omega)$ be the solution of (1.1) with $u(0, x; u_0, \omega) = u_0(x)$. Note that, for $u_0 \in X$ with $u_0 \geq 0$, $u(t, x; u_0, \omega)$ exists for $t \in [0, \infty)$ and $u(t, x; u_0, \omega) \geq 0$ for all $t \geq 0$. Note also that $u \equiv 0$ and $u \equiv 1$ are two constant solutions of (1.1). Let

$$\hat{a}_{\rm inf}(\omega) = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau := \lim_{r\to\infty} \inf_{t-s\geq r} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau \tag{1.4}$$

and

$$\hat{a}_{\rm sup}(\omega) = \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau := \lim_{r\to\infty} \sup_{t-s\geq r} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau. \tag{1.5}$$

Observe that

$$\hat{a}_{\rm inf}(\theta_t\omega) = \hat{a}_{\rm inf}(\omega) \quad \text{and} \quad \hat{a}_{\rm sup}(\theta_t\omega) = \hat{a}_{\rm sup}(\omega), \ \ \forall\, t \in \mathbb{R}, \tag{1.6}$$

and that

$$\hat{a}_{\rm inf}(\omega) = \liminf_{t,s\in\mathbb{Q},t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau)d\tau \quad \text{and} \quad \hat{a}_{\rm sup}(\omega) = \liminf_{t,s\in\mathbb{Q},t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau)d\tau.$$

Then by the countability of the set $\mathbb{Q}$ of rational numbers, both $\hat{a}_{\rm inf}(\omega)$ and $\hat{a}_{\rm sup}(\omega)$ are measurable in ω.

Throughout this paper, we assume that the following standing assumption holds.

(H1) $0 < \hat{a}_{\rm inf}(\omega) \leq \hat{a}_{\rm sup}(\omega) < \infty$ *for a.e.* $\omega \in \Omega$.

Note that **(H1)** implies that $\hat{a}_{\rm inf}(\cdot), a(\cdot), \hat{a}_{\rm sup}(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and that there are $\hat{a}, \underline{a}, \bar{a} \in \mathbb{R}^+$ and a measurable subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\begin{cases} \theta_t\Omega_0 = \Omega_0 \quad \forall\, t \in \mathbb{R}, \\ \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t a(\theta_\tau\omega)d\tau = \hat{a} \quad \forall\, \omega \in \Omega_0, \\ \hat{a}_{\rm inf}(\omega) = \underline{a} \quad \forall\, \omega \in \Omega_0, \\ \hat{a}_{\rm sup}(\omega) = \bar{a} \quad \forall\, \omega \in \Omega_0 \end{cases} \tag{1.7}$$

(see Lemma 2.1). Throughout this paper, $\hat{a}$ is referred to as the *mean* or *average* of $a(\cdot)$, and $\underline{a}$ and $\bar{a}$ are referred to as the *least mean* and the *greatest mean* of $a(\cdot)$, respectively.

Our main result on the stability of the constant solution $u \equiv 1$ of (1.1) reads as follows.

Theorem 1.1 *For every $u_0 \in C^b_{\rm uinf}(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} u_0(x) > 0$ and for every $\omega \in \Omega$, we have that*

$$\|u(t,\cdot;u_0,\omega)-1\|_\infty \le M(u_0)e^{-\int_0^t a(\theta_s\omega)ds}, \tag{1.8}$$

where $M(u_0) := \max\{1, \|u_0\|_\infty\} \cdot \max\left\{\left|1-\frac{1}{\min\{1,\inf_{x\in\mathbb{R}} u_0(x)\}}\right|, \left|1-\frac{1}{\max\{1,\sup_{x\in\mathbb{R}} u_0(x)\}}\right|\right\}$. *Hence if $\int_0^\infty a(\theta_s\omega)ds = \infty$, then*

$$\lim_{t\to\infty} \|u(t,\cdot;u_0,\omega)-1\|_\infty = 0.$$

In particular, if **(H1)** *holds, then for every $0 < \tilde{a} < \underline{a}$, every $u_0 \in C^b_{\rm uinf}(\mathbb{R})$ with $\inf_x u_0(x) > 0$, and almost all $\omega \in \Omega$, there is positive constant $M > 0$ such that*

$$\|u(t,\cdot;u_0,\theta_{t_0}\omega)-1\|_\infty \le Me^{-\tilde{a}t}, \quad \forall\, t \ge 0,\ t_0 \in \mathbb{R}.$$

If $a(\theta.\omega) \in L^1(0,\infty)$, then the constant equilibrium solution, $u \equiv 1$, of (1.1) is not asymptotically stable.

To state our main results on the spreading speeds of (1.1), let

$$\underline{c}^* = 2\sqrt{\underline{a}}, \quad \hat{c}^* = 2\sqrt{\hat{a}}, \quad \text{and} \quad \bar{c}^* = 2\sqrt{\bar{a}}. \tag{1.9}$$

Let

$$X_c^+ = \{u \in C^b_{\rm unif}(\mathbb{R}) \,|\, u \ge 0,\ \text{supp}(u) \ \text{is bounded and not empty}\}.$$

Definition 1.1 For given $\omega \in \Omega$, let

$$C_{\sup}(\omega) = \{c \in \mathbb{R}^+ \,|\, \limsup_{t\to\infty} \sup_{s\in\mathbb{R},|x|\ge ct} u(t,x;u_0,\theta_s\omega) = 0 \quad \forall\, u_0 \in X_c^+\}$$

and

$$C_{\inf}(\omega) = \{c \in \mathbb{R}^+ \,|\, \limsup_{t\to\infty} \sup_{s\in\mathbb{R},|x|\le ct} |u(t,x;u_0,\theta_s\omega)-1| = 0 \quad \forall\, u_0 \in X_c^+\}.$$

Let

$$c^*_{\sup}(\omega) = \inf\{c \,|\, c \in C_{\sup}(\omega)\}, \quad c^*_{\inf}(\omega) = \sup\{c \,|\, c \in C_{\inf}(\omega)\}.$$

$[c^*_{\inf}(\omega), c^*_{\sup}(\omega)]$ is called the spreading speed interval of (1.1) with respect to compactly supported initial functions.

The following theorem shows that the spreading speed interval of (1.1) with respect to compactly supported initial functions is deterministic and is linearly determinate, that is, $[c^*_{\inf}(\omega), c^*_{\sup}(\omega)] = [\underline{c}^*, \bar{c}^*]$ for all $\omega \in \Omega_0$.

Theorem 1.2 *Assume that* **(H1)** *holds. Then the following hold.*

(i) *For any $\omega \in \Omega_0$, $c^*_{\sup}(\omega) = \bar{c}^*$.*
(ii) *For any $\omega \in \Omega_0$, $c^*_{\inf}(\omega) = \underline{c}^*$.*

The above theorem concerns the spreading speeds of solutions of (1.1) with compactly supported nonnegative initial functions. To consider the spreading speeds of solutions of (1.1) with front-like initial functions, let

$$\tilde{X}_c^+ = \{u \in C^b_{\rm unif}(\mathbb{R}) \,|\, u \ge 0,\ \liminf_{x\to-\infty} u_0(x) > 0,\ u_0(x) = 0 \text{ for } x \gg 1\}.$$

Definition 1.2 For given $\omega \in \Omega$, let

$$\tilde{C}_{\sup}(\omega) = \{c \in \mathbb{R}^+ \mid \limsup_{t\to\infty} \sup_{s\in\mathbb{R}, x\geq ct} u(t, x; u_0, \theta_s\omega) = 0 \quad \forall\, u_0 \in \tilde{X}_c^+\}$$

and

$$\tilde{C}_{\inf}(\omega) = \{c \in \mathbb{R}^+ \mid \limsup_{t\to\infty} \sup_{s\in\mathbb{R}, x\leq ct} |u(t, x; u_0, \theta_s\omega) - 1| = 0 \quad \forall\, u_0 \in \tilde{X}_c^+\}.$$

Let

$$\tilde{c}^*_{\sup}(\omega) = \inf\{c \mid c \in \tilde{C}_{\sup}(\omega)\}, \quad \tilde{c}^*_{\inf}(\omega) = \sup\{c \mid c \in \tilde{C}_{\inf}(\omega)\}.$$

$[\tilde{c}^*_{\inf}(\omega), \tilde{c}^*_{\sup}(\omega)]$ is called the spreading speed interval of (1.1) with respect to front-like initial functions.

We have the following theorem on the spreading speeds of the solutions with front-like initial functions.

Theorem 1.3 *Assume that* **(H1)** *holds. Then the following hold.*

(i) *For any* $\omega \in \Omega_0$, $\tilde{c}^*_{\sup}(\omega) = \bar{c}^*$.
(ii) *For any* $\omega \in \Omega_0$, $\tilde{c}^*_{\inf}(\omega) = \underline{c}^*$.

We also have the following theorem on the take-over property of the solutions of (1.1) with front-like initial functions and with the initial function $u_0^*(x) = 1$ for $x < 0$ and $u_0^*(x) = 0$ for $x > 0$. Note that $u(t, x; u_0^*, \omega)$ exists for all $t > 0$ (see [25, Theorem 1]).

Theorem 1.4 (i) *For a.e.* $\omega \in \Omega$,

$$\lim_{t\to\infty} \frac{x(t,\omega)}{t} = \hat{c}^*, \tag{1.10}$$

where $x(t, \omega)$ *is such that* $u(t, x(t, \omega); u_0^*, \omega) = \frac{1}{2}$. *Moreover,*

$$\lim_{t\to\infty} \sup_{x\geq(\hat{c}^*+h)t} u(t, x; u_0^*, \omega) = 0, \forall\, h > 0, \ a.e \ \omega \tag{1.11}$$

and

$$\lim_{t\to\infty} \inf_{x\leq(\hat{c}^*-h)t} u(t, x; u_0^*, \omega) = 1, \forall\, h > 0, \ a.e \ \omega. \tag{1.12}$$

(ii) *For any* $u_0 \in \tilde{X}_c^+$, *it holds that*

$$\lim_{t\to\infty} \sup_{x\geq(\hat{c}^*+h)t} u(t, x; u_0, \omega) = 0, \forall\, h > 0, \ a.e \ \omega \tag{1.13}$$

and

$$\lim_{t\to\infty} \inf_{x\leq(\hat{c}^*-h)t} u(t, x; u_0, \omega) = 1, \forall\, h > 0, \ a.e \ \omega. \tag{1.14}$$

Consider now (1.2). Define $\underline{a}_0$ and $\overline{a}_0$ by

$$\underline{a}_0 = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a_0(\tau)d\tau, \quad \overline{a}_0 = \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t a_0(\tau)d\tau. \tag{1.15}$$

Let (H2) be the following standing assumption.

(H2) $0 < \underline{a}_0 \le \bar{a}_0 < \infty$.

The assumption **(H2)** is the analogue of **(H1)**. We will give some example for $a_0(\cdot)$ satisfying **(H2)** in Sect. 5. Assume **(H2)**. Let

$$\bar{c}_0^* = 2\sqrt{\bar{a}_0} \quad \text{and} \quad \underline{c}_0^* = 2\sqrt{\underline{a}_0}. \tag{1.16}$$

For given $u_0 \in C^b_{\rm unif}(\mathbb{R})$ with $u_0 \ge 0$ and $s \in \mathbb{R}$, let $u(t,x;u_0,\sigma_s a_0)$ be the solution of

$$u_t = u_{xx} + \sigma_s a_0(t) u(1-u), \quad x \in \mathbb{R},\ t > 0,$$

with $u(0,x;u_0,\sigma_s a_0) = u_0(x)$, where $\sigma_s a_0(t) = a_0(s+t)$.

We have the following theorem on the spreading speeds of (1.2).

Theorem 1.5 *Assume* **(H2)**. *Then for every* $u_0 \in X_c^+$,

$$\liminf_{t\to\infty} \sup_{s\in\mathbb{R}, |x|\le ct} |u(t,x;u_0,\sigma_s a_0) - 1| = 0, \quad \forall\, 0 < c < \underline{c}_0^* := 2\sqrt{\underline{a}_0} \tag{1.17}$$

and

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R}, |x|\ge ct} u(t,x;u_0,\sigma_s a_0) = 0, \quad \forall\, c > \bar{c}_0^* := 2\sqrt{\bar{a}_0}. \tag{1.18}$$

We conclude the introduction with the following four remarks.

First, the results in Theorems 1.2–1.5 are new. If $a_0(t)$ is periodic with period T, then $\underline{a}_0 = \bar{a}_0 = \hat{a}_0 := \frac{1}{T}\int_0^T a_0(\tau)d\tau$ and hence $\underline{c}_0^* = \bar{c}_0^* = 2\sqrt{\hat{a}_0}$. More generally, if $a_0(t)$ in globally Hölder continuous and is uniquely ergodic in the sense that the space $H(a_0)$ is compact and the flow $(H(a_0), \sigma_t)$ is uniquely ergodic, where $H(a_0) = {\rm cl}\{\sigma_s a_0 \,|\, s \in \mathbb{R}\}$ with open compact topology and $\sigma_s a_0(\cdot) = a_0(\cdot + s)$, then $\underline{a}_0 = \bar{a}_0 = \hat{a}_0 := \lim_{T\to\infty} \frac{1}{T}\int_0^T a_0(\tau)d\tau$ and hence $\underline{c}_0^* = \bar{c}_0^* = 2\sqrt{\hat{a}_0}$. Therefore the existing results on spreading speeds of (1.2) in the time periodic and time almost periodic cases are recovered. The current paper provides a new and simpler proof in these special cases.

Second, by Theorems 1.2 and 1.3 ,

$$[c^*_{\rm inf}(\omega), c^*_{\rm sup}(\omega)] = [\tilde{c}^*_{\rm inf}(\omega), \tilde{c}^*_{\rm sup}(\omega)] = [\underline{c}^*, \bar{c}^*]$$

for any $\omega \in \Omega_0$. Hence $[\underline{c}^*, \bar{c}^*]$ is called the *spreading speed interval* of (1.1), which is deterministic and is determined by the linearized equation of (1.1) at $u \equiv 0$. Theorem 1.4 is an extension of the take-over property proved in [3] and [25] for (1.3). In order to prove Theorem 1.4 we are first led to prove that $x(t,\omega)$ is a subadditive process (see Lemma 5.4 for more detail). The fact that $x(t,\omega)$ is a subadditive process is interesting. Its proof relies on comparison between various translation of the solution and on a zero-number argument enabling to bound the width of the interface. It is our belief that this result will open the way to other applications in the future.

Third, the results established for (1.1) and (1.2) can be applied to the following general random Fisher–KPP equation,

$$u_t = u_{xx} + u(r(\theta_t\omega) - \beta(\theta_t\omega)u), \tag{1.19}$$

where $r : \omega \to (-\infty, \infty)$ and $\beta : \Omega \to (0, \infty)$ are measurable with locally Hölder continuous sample paths $r^\omega(t) := r(\theta_t\omega)$ and $\beta^\omega(t) := \beta(\theta_t\omega)$, and to the following nonautonomous Fisher–KPP equation,

$$u_t = u_{xx} + u(r_0(t) - \beta_0(t)u), \tag{1.20}$$

where $r_0 : \mathbb{R} \to \mathbb{R}$ and $\beta_0 : \mathbb{R} \to (0, \infty)$ are locally Hölder continuous. Note that (1.19) models the population growth of a species with random perturbations on its growth rate and carrying capacity, and (1.20) models the population growth of a species with deterministic time dependent perturbations on its growth rate and carrying capacity.

In fact, under some assumptions on $r(\omega)$ and $\beta(\omega)$, it can be proved that

$$u(t;\omega) := Y(\theta_t\omega) = \frac{1}{\int_{-\infty}^0 e^{-\int_s^0 r(\theta_{\tau+t}\omega)d\tau}\beta(\theta_{s+t}\omega)ds}$$

is an random equilibrium of (1.19). Let $\tilde{u} = \frac{u}{Y(\theta_t\omega)}$ and drop the tilde, (1.19) becomes (1.1) with $a(\theta_t\omega) = \beta(\theta_t\omega) \cdot Y(\theta_t\omega)$, and then the results established for (1.1) can be applied to (1.19). For example, consider the following random Fisher–KPP equation,

$$u_t = u_{xx} + u(1 + \xi(\theta_t\omega) - u), \quad x \in \mathbb{R}, \tag{1.21}$$

where $\omega \in \Omega$, $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is an ergodic metric dynamical system, $\xi : \Omega \to \mathbb{R}$ is measurable, and $\xi_t(\omega) := \xi(\theta_t\omega)$ is locally Hölder continuous (ξ_t denotes a real noise or a colored noise). Let $\hat{\xi}_{\inf}(\omega)$ and $\hat{\xi}_{\sup}(\omega)$ be defined as in (1.4) and (1.5) with $a(\cdot)$ being replaced by $\xi(\cdot)$, respectively. Assume that $\xi_t(\cdot)$ satisfies the following **(H3)**.

(H3) *$\xi : \Omega \to \mathbb{R}$ is measurable; $\int_\Omega |\xi(\omega)|d\mathbb{P}(\omega) < \infty$ and $\int_\Omega \xi(\omega)d\mathbb{P}(\omega) = 0$; $-1 < \hat{\xi}_{\inf}(\omega) \le \hat{\xi}_{\sup}(\omega) < \infty$ and $\inf_{t\in\mathbb{R}} \xi(\theta_t\omega) > -\infty$ for a.e. $\omega \in \Omega$; and $\xi^\omega(t) := \xi(\theta_t\omega)$ is locally Hölder continuous.*

Assume **(H3)**. By the arguments of Lemma 2.1, there are $\underline{\xi}, \overline{\xi} \in \mathbb{R}$ such that $\hat{\xi}_{\inf}(\omega) = \underline{\xi}$ and $\hat{\xi}_{\sup}(\omega) = \overline{\xi}$ for a.e. $\omega \in \Omega$. It can be proved that

$$Y(\omega) = \frac{1}{\int_{-\infty}^0 e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau}ds} \tag{1.22}$$

is a spatially homogeneous asymptotically stable random equilibrium of (1.21) (see Theorem 3.2 and Corollary 3.1). It can also be proved that for any $u_0 \in X_c^+$,

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R}, |x|\le ct} \left|\frac{u(t,x;u_0,\theta_s\omega)}{Y(\theta_{t+s}\omega)} - 1\right| = 0, \quad \forall\, 0 < c < 2\sqrt{1+\underline{\xi}}$$

and

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R}, |x|\ge ct} \frac{u(t,x;u_0,\theta_s\omega)}{Y(\theta_{t+s}\omega)} = 0, \quad \forall\, c > 2\sqrt{1+\bar{\xi}},$$

for a.e. $\omega \in \Omega$. where $u(t,x;u_0,\theta_s\omega)$ is the solution of (1.21) with ω being replaced by $\theta_s\omega$ and $u(0,x;u_0,\theta_s\omega) = u_0(x)$ (see Corollary 4.1).

Fourth, it is interesting to study the spreading properties of (1.1) with **(H1)** being replaced by the following weaker assumption,

(H1)$'$ $0 < \hat{a} := \int_\Omega a(\omega)d\mathbb{P}(\omega) < \infty$.

We plan to study this general case somewhere else, which would have applications to the study of the spreading properties of the following stochastic Fisher–KPP equation,

$$du = (u_{xx} + u(1-u))dt + \sigma u dW_t, \quad x \in \mathbb{R}, \tag{1.23}$$

where W_t denotes the standard two-sided Brownian motion (dW_t is then the white noise). In fact, let $\Omega := \{\omega \in C(\mathbb{R}, \mathbb{R}) \mid \omega(0) = 0\}$ equipped with the open compact topology, $\mathcal{F}$ be the Borel $\sigma-$field and $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathcal{F})$. Let W_t be the one dimensional Brownian motion on the Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by $W_t(\omega) = \omega(t)$. Let $\theta_t\omega$ be the canonical Brownian shift: $(\theta_t\omega)(\cdot) = \omega(t+\cdot) - \omega(t)$ on Ω. It is easy to see that $W_t(\theta_s\omega) = W_{t+s}(\omega) - W_s(\omega)$. If $\frac{\sigma^2}{2} < 1$, then it can be proved that

$$Y(\omega) = \frac{1}{\int_{-\infty}^{0} e^{(1-\frac{\sigma^2}{2})s+\sigma W_s(\omega)ds}} \tag{1.24}$$

is a spatially homogeneous stationary solution process of (1.23). Let $\tilde{u} = \frac{u}{Y(\theta_t\omega)}$ and drop the tilde, (1.23) becomes (1.1) with $a(\theta_t\omega) = Y(\theta_t\omega)$. The reader is referred to [18–20,23,39,40] for some study on the front propagation dynamics of (1.24). Note that Theorem 1.4 (i) is an analogue of [20, Theorem 1].

It is important to note that the authors of the work [10] studied the asymptotic spreading speeds for space-time heterogeneous equations of the form

$$u_t = \sum_{i,j=1}^{N} a_{i,j}(t,x)u_{x_ix_j} + \sum_{i=1}^{N} q_i(t,x)u_{x_i} + f(t,x,u), \quad x \in \mathbb{R}^N, \tag{1.25}$$

where $f(t,x,0) = f(t,x,1) = 0$. We note that Theorem 1.5 improves [10, Proposition 3.9], since $\inf_{t\in\mathbb{R}} a_0(t) = 0$ and $\sup_{t\in\mathbb{R}} a_0(t) = +\infty$ are allowed here. Moreover, the techniques developed in the current work are different from the ones in [10]. Certainly, it should be mentioned that (1.25) is more general than (1.2).

The rest of the paper is organized as follows. In Sect. 2, we present some preliminary lemmas, which will be used in the proofs of main results of the current paper in later sections. In Sect. 3, we establish some results about the stability of the positive constant equilibrium solution $u \equiv 1$ of (1.1) (resp. (1.2)) and prove Theorem 1.1. In Sect. 4, we study the spreading properties of solutions of (1.1) with nonnegative and compactly supported initial functions or front like initial functions and prove Theorems 1.2 and 1.3 . We investigate in Sect. 4 the take-over property of (1.1) and prove Theorem 1.4. We consider spreading properties of (1.2) in Sect. 5.

2 Preliminary Lemmas

In this section, we present some preliminary lemmas to be used in later sections of this paper as well as in the second part of the series.

Lemma 2.1 **(H1)** *implies that* $\hat{a}_{\inf}(\cdot), a(\cdot), \hat{a}_{\sup}(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *and that there are* $\underline{a}, \bar{a}, \hat{a} \in \mathbb{R}^+$ *and a measurable subset* $\Omega_0 \subset \Omega$ *with* $\mathbb{P}(\Omega_0) = 1$ *such that* $\theta_t\Omega_0 = \Omega_0$ *for all* $t \in \mathbb{R}$, $\hat{a}_{\inf}(\omega) = \underline{a}$ *and* $\hat{a}_{\sup}(\omega) = \bar{a}$ *for all* $\omega \in \Omega_0$, *and* $\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t a(\theta_\tau\omega)d\tau = \hat{a}$ *for all* $\omega \in \Omega_0$.

Proof First, let

$$\Omega_n = \{\omega \in \Omega \,|\, \hat{a}_{\sup}(\omega) \le n\} \quad \forall n \in \mathbb{N},$$

and

$$\Omega_\infty = \{\omega \in \Omega \,|\, \hat{a}_{\sup}(\omega) = \infty\}.$$

Then $\Omega_\infty \cup \cup_{n=1}^{\infty} \Omega_n = \Omega$. By **(H1)**, there is $\bar{n} \in \mathbb{N}$ such that $\mathbb{P}(\Omega_{\bar{n}}) > 0$. By (1.6), $\theta_t\Omega_n = \Omega_n$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Then by the ergodicity of the metric dynamical system

$(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, we have $\mathbb{P}(\Omega_{\bar{n}}) = 1$. This implies that $\hat{a}_{\sup}(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and then $\hat{a}_{\inf}(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, by (1.6),

$$\hat{a}_{\inf}(\omega) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \hat{a}_{\inf}(\theta_\tau\omega)d\tau = \int_\Omega \hat{a}_{\inf}(\omega)d\mathbb{P}(\omega) \quad \text{for} \quad a.e.\ \omega \in \Omega,$$

and

$$\hat{a}_{\sup}(\omega) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \hat{a}_{\sup}(\theta_\tau\omega)d\tau = \int_\Omega \hat{a}_{\sup}(\omega)d\mathbb{P}(\omega) \quad \text{for} \quad a.e.\ \omega \in \Omega.$$

It then follows that there are $\underline{a}, \bar{a} \in \mathbb{R}$ and a measurable subset $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that $\theta_t\Omega_1 = \Omega_1$ for all $t \in \mathbb{R}$, and $\hat{a}_{\inf}(\omega) = \underline{a}$ and $\hat{a}_{\sup}(\omega) = \bar{a}$ for all $\omega \in \Omega_1$.

Next, for given $n \in \mathbb{N}$, let

$$a_n(\omega) = \min\{a(\omega), n\}.$$

Then $a_n(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $0 < a_1(\omega) \le a_2(\omega) \le \cdots$, and $\lim_{n\to\infty} a_n(\omega) = a(\omega)$. By the ergodicity of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, we have that for a.e. $\omega \in \Omega$,

$$\int_\Omega a_n(\omega)d\mathbb{P}(\Omega) = \lim_{t\to\infty} \frac{1}{t} \int_0^t a_n(\theta_\tau\omega)d\tau \le \hat{a}_{\sup}(\omega) = \int_\Omega \hat{a}_{\sup}(\omega)d\mathbb{P}(\omega).$$

This together with the Monotone Convergence Theorem implies that

$$\int_\Omega a(\omega)d\mathbb{P}(\omega) = \lim_{n\to\infty} \int_\Omega a_n(\omega)d\mathbb{P}(\omega) \le \int_\Omega \hat{a}_{\sup}(\omega)d\mathbb{P}(\omega).$$

Therefore, $a(\cdot) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and moreover, by the ergodicity of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, there are $\hat{a} \in \mathbb{R}$ and a measurable subset $\Omega_2 \subset \Omega$ with $\mathbb{P}(\Omega_2) = 1$ such that $\theta_t\Omega_2 = \Omega_2$ for all $t \in \mathbb{R}$, and

$$\hat{a} = \lim_{t\to\infty} \frac{1}{t} \int_0^t a(\theta_\tau\omega)d\tau = \lim_{t\to\infty} \frac{1}{t} \int_{-t}^0 a(\theta_\tau\omega)d\tau = \int_\Omega a(\omega)d\mathbb{P}(\omega) \quad \text{for} \quad a.e.\ \omega \in \Omega.$$

The lemma thus follows with $\Omega_0 = \Omega_1 \cap \Omega_2$. □

Lemma 2.2 *Suppose that* $b \in C(\mathbb{R}, (0, \infty))$ *and that* $0 < \underline{b} \le \overline{b} < \infty$*, where*

$$\underline{b} = \liminf_{t-s\to\infty} \frac{1}{t-s} \int_s^t b(\tau)d\tau, \quad \overline{b} = \limsup_{t-s\to\infty} \frac{1}{t-s} \int_s^t b(\tau)d\tau.$$

Then

$$\underline{b} = \sup_{B\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})\cap L^\infty(\mathbb{R})} \operatorname{essinf}_{\tau\in\mathbb{R}}(b(\tau) - B'(\tau)). \tag{2.1}$$

Proof The proof of this lemma follows from a proper modification of the proof of [33, Lemma 3.2]. For the sake of completeness we give a proof here. Let $0 < \gamma < \underline{b}$. By $\overline{b} < \infty$, there is $T > 0$ such that

$$\gamma < \frac{1}{T} \int_s^{s+T} b(\tau)d\tau < 2\overline{b}, \qquad \forall s \in \mathbb{R}. \tag{2.2}$$

Define

$$B(t) = \int_{kT}^t \Big(b(\tau) - \varepsilon_k\Big)d\tau, \quad \forall t \in [kT, (k+1)T] \quad \text{where} \quad \varepsilon_k := \frac{1}{T} \int_{kT}^{(k+1)T} b(\tau)ds, \quad \forall\, k \in \mathbb{Z}.$$

It is clear that $B \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with

$$\varepsilon_k = b(t) - B'(t) \quad \text{for } \ t \in (kT, (k+1)T). \tag{2.3}$$

Furthermore, it follows from (2.2) that $\|B\|_\infty \le 2T\bar{b}$ and that $\gamma < \varepsilon_k$ for every $k \in \mathbb{Z}$. Hence (2.3) implies that

$$\gamma \le \sup_{B \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})} \operatorname{essinf}_{t\in\mathbb{R}} (b(t) - B'(t)).$$

Since γ is arbitrarily chosen less than $\underline{b}$ we deduce that

$$\underline{b} \le \sup_{B \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})} \operatorname{essinf}_{t\in\mathbb{R}} (b(t) - B'(t)).$$

On the other hand for each given $B \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $t > s$ we have

$$\begin{aligned}\frac{1}{t-s}\int_s^t b(\tau)d\tau &\ge \operatorname{essinf}_{\tau\in\mathbb{R}}(b(\tau) - B'(\tau)) + \frac{(B(t)-B(s))}{t-s}\\ &\ge \operatorname{essinf}_{\tau\in\mathbb{R}}(b(\tau) - B'(\tau)) - \frac{2\|B\|_\infty}{t-s}.\end{aligned}$$

Hence

$$\underline{b} = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t b(\tau)d\tau \ge \operatorname{essinf}_{\tau\in\mathbb{R}}(b(\tau) - B'(\tau)) \quad \forall B \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

This completes the proof of the lemma. □

In the following, let $b \in C(\mathbb{R}, (0,\infty))$ be given and satisfy that $0 < \underline{b} \le \bar{b} < \infty$. Consider

$$u_t = u_{xx} + b(t)u(1-u), \quad x \in \mathbb{R}. \tag{2.4}$$

For given $u_0 \in C^b_{\rm unif}(\mathbb{R})$ with $u_0 \ge 0$, let $u(t, x; u_0, b)$ be the solution of (2.4) with $u(0, x; u_0, b) = u_0(x)$.

For every $0 < \mu < \underline{\mu}^* := \sqrt{\underline{b}}$, $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $\omega \in \Omega$, let

$$c(t; b, \mu) = \frac{\mu^2 + b(t)}{\mu}, \quad C(t; b, \mu) = \int_0^t c(\tau; b, \mu)d\tau, \tag{2.5}$$

and

$$\phi^\mu(t, x; b) = e^{-\mu(x - C(t;b,\mu))}. \tag{2.6}$$

Then the function ϕ^μ satisfies

$$\phi^\mu_t = \phi^\mu_{xx} + b(t)\phi^\mu, \quad x \in \mathbb{R}. \tag{2.7}$$

Lemma 2.3 *Let*

$$\phi^\mu_+(t, x; b) = \min\{1, \phi^\mu(t, x; b)\}.$$

Then

$$u(t, x; \phi^\mu_+(0, \cdot; b), b) \le \phi^\mu_+(t, x; b) \quad \forall\ t > 0, \ x \in \mathbb{R}.$$

Proof It follows directly from the comparison principle for parabolic equations. □

Lemma 2.4 *For every* μ *with* $0 < \mu < \tilde{\mu} < \min\{2\mu, \underline{\mu}^*\}$, *there exist* $\{t_k\}_{k\in\mathbb{Z}}$ *with* $t_k < t_{k+1}$ *and* $\lim_{k\to\pm\infty} t_k = \pm\infty$, $B_b \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *with* $B_b(\cdot) \in C^1((t_k, t_{k+1}))$ *for* $k \in \mathbb{Z}$, *and a positive real number* d_b *such that for every* $d \geq d_b$ *the function*

$$\phi^{\mu,d,B_b}(t,x) := e^{-\mu(x-C(t;b,\mu))} - de^{\left(\frac{\tilde{\mu}}{\mu}-1\right)B_b(t)-\tilde{\mu}(x-C(t;b,\mu))}$$

satisfies

$$\phi^{\mu,d,B_b}_t \leq \phi^{\mu,d,B_b}_{xx} + b(t)\phi^{\mu,d,B_b}(1-\phi^{\mu,d,B_b})$$

for $t \in (t_k, t_{k+1})$, $x \geq C(t,b,\mu) + \frac{\ln d}{\tilde{\mu}-\mu} + \frac{B_b(t)}{\mu}$, $k \in \mathbb{Z}$.

Proof First of all, for given $0 < \mu < \tilde{\mu} < \min\{2\mu, \underline{\mu}^*\}$, let $0 < \delta \ll 1$ such that $(1-\delta)\underline{b} > \tilde{\mu}\mu$. It then follows from the arguments of Lemma 2.2 that there exist $T > 0$ and $B_b \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $B_b \in C^1((t_k, t_{k+1}))$, where $t_k = kT$ for $k \in \mathbb{Z}$, and

$$\tilde{\mu}\mu \leq (1-\delta)b(t) + B_b'(t) \quad \text{for all } t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}.$$

Next, fix the above $\delta > 0$ and $B_b(t)$. Let $d > 1$ to be determined later. Let $\xi(t,x) = x - C(t;b,\mu)$. We have

$$\begin{aligned}
&\phi^{\mu,d,B_b}_t - \left(\phi^{\mu,d,B_b}_{xx} + b(t)\phi^{\mu,d,B_b}(1-\phi^{\mu,d,B_b})\right)\\
&= d\Big[-(\frac{\tilde{\mu}}{\mu}-1)B_b'(t) + \tilde{\mu}^2 - \tilde{\mu}c(t;b,\mu) + b(t)\Big]e^{(\frac{\tilde{\mu}}{\mu}-1)B_b(t)-\tilde{\mu}\xi(t,x)}\\
&\quad + b(t)\Big[e^{-2\mu\xi(t,x)} - 2de^{(\frac{\tilde{\mu}}{\mu}-1)B_b(t)-(\mu+\tilde{\mu})\xi(t,x)} + d^2e^{2(\frac{\tilde{\mu}}{\mu}-1)B_b(t)-2\tilde{\mu}\xi(t,x)}\Big]\\
&= d\Big(\frac{\tilde{\mu}}{\mu}-1\Big)\Big[\tilde{\mu}\mu - b(t) - B_b'(t)\Big]e^{(\frac{\tilde{\mu}}{\mu}-1)B_b(t)-\tilde{\mu}\xi(t,x)} + b(t)e^{-2\mu\xi(t,x)}\\
&\quad - d\Big[2e^{-\mu\xi(t,x)} - de^{\left(\frac{\tilde{\mu}}{\mu}-1\right)B_b(t)-\tilde{\mu}\xi(t,x)}\Big]e^{\left(\frac{\tilde{\mu}}{\mu}-1\right)B_b(t)-\tilde{\mu}\xi(t,x)}\\
&= d\Big(\frac{\tilde{\mu}}{\mu}-1\Big)\Big[\tilde{\mu}\mu - (1-\delta)b(t) - B_b'(t)\Big]e^{(\frac{\tilde{\mu}}{\mu}-1)B_b(t)-\tilde{\mu}\xi(t,x)}\\
&\quad + \Big[e^{-(2\mu-\tilde{\mu})\xi(t,x)} - d\delta\Big(\frac{\tilde{\mu}}{\mu}-1\Big)e^{\left(\frac{\tilde{\mu}}{\mu}-1\right)B_b(t)}\Big]a(\theta_t\omega)e^{-\tilde{\mu}\xi(t,x)}\\
&\quad + d\Big[-2e^{-\mu\xi(t,x)} + de^{\left(\frac{\tilde{\mu}}{\mu}-1\right)B_b(t)-\tilde{\mu}\xi(t,x)}\Big]e^{\left(\frac{\tilde{\mu}}{\mu}-1\right)B_b(t)-\tilde{\mu}\xi(t,x)}
\end{aligned} \tag{2.8}$$

for $t \in (t_k, t_{k+1})$.

Observe now that

$$d\delta\Big(\frac{\tilde{\mu}}{\mu}-1\Big)e^{\left(\frac{\tilde{\mu}}{\mu}-1\right)B_b(t)} \geq 1, \qquad \forall\, d \geq \max\left\{\frac{e^{-\left(\frac{\tilde{\mu}}{\mu}-1\right)\|B_b\|_\infty}}{\delta\left(\frac{\tilde{\mu}}{\mu}-1\right)}, e^{\left(\frac{\tilde{\mu}}{\mu}-1\right)\|B_b\|_\infty}\right\}.$$

For this choice of d, if $\phi^{\mu,d,B_b}(t,x) \geq 0$, which is equivalent to $\xi(t,x) = x - C(t;b,\mu) \geq \frac{\ln d}{\tilde{\mu}-\mu} + \frac{B_b(t)}{\mu}$, then $\xi(t,x) \geq 0$ and each term in the expression at the right hand side of (2.8) is less or equal to zero. The lemma thus follows. □

Recall that $u_0^*(x) = 1$ for $x < 0$ and $u_0^*(x) = 0$ for $x > 0$. By [25, Theorem 1], the solution of (2.4) with initial function u_0^*, denoted by $u(t,x;u_0^*,b)$, exists for $t > 0$.

Lemma 2.5 *Suppose that $u_\epsilon \in C^b_{\text{unif}}(\mathbb{R})$ with $u_\epsilon \geq 0$ and $\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}|u_\epsilon(x)-u_0^*(x)|dx = 0$. Then for each $t>0$,*

$$\lim_{\epsilon\to 0}\|u(t,\cdot;u_\epsilon,b)-u(t,\cdot;u_0^*,b)\|_\infty = 0.$$

Proof See [25, Theorem 8]. □

Lemma 2.6 *For given $u_i \in C^b_{\text{unif}}(\mathbb{R})$ with $u_i \geq 0$ $(i=1,2)$, if $u_1(x)-u_2(x)$ has exactly one simple zero x_0 and $u_1(x)>u_2(x)$ for $x<x_0$ and $u_1(x)<u_2(x)$ for $x>x_0$, then for any $t>0$, there is $\xi(t)\in[-\infty,\infty]$ such that*

$$u(t,x;u_1,b) = \begin{cases} > u(t,x;u_2,b) & x<\xi(t) \\ < u(t,x;u_2,b) & x>\xi(t). \end{cases}$$

Proof Let $v(t,x)=u(t,x;u_1,b)-u(t,x;u_2,b)$. Then $v(t,x)$ satisfies

$$v_t = v_{xx}+q(t,x)v, \quad x\in\mathbb{R},$$

where $q(t,x)=b(t)-b(t)(u(t,x;u_1,b)+u(t,x;u_2,b))$. Note that $v(0,x)$ has exactly one simple zero x_0 and $v(0,x)>0$ for $x<x_0$, $v(x)<0$ for $x>x_0$. The lemma then follows from [1, Theorems A,B]. □

Let $x(t,b)$ and $x_+(t,b)$ be such that

$$u(t,x(t,b);u_0^*,b)=\frac{1}{2} \quad \text{and} \quad u(t,x_+(t,b);\phi_+^\mu(0,\cdot;b),b)=\frac{1}{2}.$$

Lemma 2.7 *For any $t>0$, there holds*

$$u(t,x+x(t,b);u_0^*,b)) \begin{cases} \geq u(t,x+x_+(t,b);\phi_+^\mu(0,\cdot;b),b) & x<0 \\ \leq u(t,x+x_+(t,b);\phi_+^\mu(0,\cdot;b),b) & x>0. \end{cases} \tag{2.9}$$

Proof First, let $\phi_n(x)=\min\{1-\frac{1}{n},\phi^\mu(0,x;b)\}$. Then $\lim_{n\to\infty}\phi_n(x)=\phi_+^\mu(0,x;b)$ uniformly in $x\in\mathbb{R}$. Then for any given $t>0$,

$$u(t,x;\phi_+^\mu(0,\cdot;b),b)=\lim_{n\to\infty}u(t,x;\phi_n,b)$$

uniformly in $x\in\mathbb{R}$. Let $x_+^n(t,b)$ be such that $u(t,x_+^n(t,b);\phi_n,b)=\frac{1}{2}$. We have

$$\lim_{n\to\infty}x_+^n(t,b)=x_+(t,b).$$

Next, for given $n\geq 1$, let $u_\epsilon^*(x)$ be a nonincreasing function such that $u_\epsilon^*\in C^b_{\text{unif}}(\mathbb{R})$; $u_\epsilon^*(x)=1$ for $x\ll -1$ and $u_\epsilon^*(x)=0$ for $x\gg 0$; $u_\epsilon^*(x)-\phi_n(x+h)$ has exactly one simple zero for any $h\in\mathbb{R}$; and

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}|u_\epsilon^*(x)-u_0^*(x)|dx=0.$$

Let $x_\epsilon(t,b)$ be such that

$$u(t,x;u_\epsilon^*,b)=\frac{1}{2}.$$

By Lemma 2.6, for any $t>0$,

$$u(t,x+x_\epsilon(t,b),b) \begin{cases} > u(t,x+x_+^n(t,b);\phi_n,b) & x<0 \\ < u(t,x+x_+^n(t,b);\phi_n,b) & x>0. \end{cases}$$

By Lemma 2.5, for any $t > 0$,

$$\lim_{\epsilon\to 0} \|u(t,\cdot;u_\epsilon^*,b) - u(t,\cdot;u_0^*,b)\|_\infty = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} x_\epsilon(t,b) = x(t,b).$$

Letting $\epsilon \to 0$, we get

$$u(t,x+x(t,b);u_0^*,b) \begin{cases} \geq u(t,x+x_+^n(t,b);\phi_n,b) & x<0 \\ \leq u(t,x+x_+^n(t,b);\phi_n,b) & x>0. \end{cases}$$

Letting $n\to\infty$, the lemma follows. □

Lemma 2.8 *Let $F:\mathbb{R}\times\Omega\to\mathbb{R}$ be measurable in $\omega\in\Omega$ and continuous hemicompact in $x\in\mathbb{R}$ (i.e for every $\omega\in\Omega$, $F(\cdot,\omega)$ is continuous in x and any sequence $\{x_n\}_{n\geq 1}\subset\mathbb{R}$ with $|x_n - F(x_n,\omega)|\to 0$ as $n\to\infty$ has a convergent subsequence). Then F has a deterministic fixed point (i.e there is $X:\Omega\to\mathbb{R}$ such that $F(X(\omega),\omega)=X(\omega)$) if and only if F has random fixed point (i.e there is a measurable function $X:\Omega\to\mathbb{R}$ such that $F(X(\omega),\omega)=X(\omega)$).*

Proof See [42, Lemma 4.7] □

Lemma 2.9 *Let $f:\mathbb{R}\times\Omega\to(0,1)$ be a measurable function such that for every $\omega\in\Omega$ the function $f^\omega := f(\cdot,\omega):\mathbb{R}\to(0,1)$ is continuously differentiable and strictly decreasing. Assume that $\lim_{x\to-\infty} f^\omega(x)=1$ and $\lim_{x\to\infty} f^\omega(x)=0$ for every $\omega\in\Omega$. Then for every $a\in(0,1)$ the function $\Omega\ni\omega\mapsto f^{\omega,-1}(a)\in\mathbb{R}$ is measurable, where $f^{\omega,-1}$ denotes the inverse function of f^ω.*

Proof Let $a\in(0,1)$ be given. Note that for every $\omega\in\Omega$, we have that $f^{\omega,-1}(a)$ is the unique fixed point of the function

$$\mathbb{R}\ni x\mapsto F(x,\omega) := f(x,w)+x-a.$$

Note that

$$|x_n - F(x_n,\omega)| = |f(x_n,w)-a| \to 0 \text{ as } n\to\infty \Rightarrow |x_n - f^{\omega,-1}(a)|\to 0 \text{ as } n\to\infty.$$

Hence the function $F(x,\omega)$ is hemicompact in x. By Lemma 2.8, the function $\Omega\ni\omega\mapsto f^{\omega,-1}(a)$ is measurable. The lemma is thus proved. □

3 Stability of Positive Random Equilibrium Solutions

In this section, we establish some results about the stability of the positive constant equilibrium solution $u\equiv 1$ of (1.1) (resp. (1.2)). We also study the existence and stability of positive random equilibria of (1.21). The results obtained in this section will play a role in later sections for the investigation of spreading speeds and take-over property of solutions of (1.1) [resp. (1.2)].

3.1 Stability of the Positive Constant Equilibrium Solution $u\equiv 1$ of (1.1)

In this subsection, we establish some results about the stability of the positive constant equilibrium solution $u\equiv 1$ of (1.1) [resp. (1.2)]. Observe that $u(t,x)=v(t,x-C(t;\omega))$ with $C(t;\omega)$ being differential in t solves (1.1) if and only if $v(t,x)$ satisfies

$$v_t = v_{xx} + c(t;\omega)v_x + a(\theta_t\omega)v(1-v), \tag{3.1}$$

where $c(t;\omega) = C'(t;\omega)$. In this subsection, we also study the stability of the positive constant equilibrium solution $u \equiv 1$ of (3.1).

We first prove Theorem 1.1.

Proof of Theorem 1.1 First, for given $u_0 \in C^b_{\rm uinf}(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} u_0(x) > 0$ and $\omega \in \Omega$, let $\underline{u}_0 := \min\{1, \inf_{x\in\mathbb{R}} u_0(x)\}$ and $\overline{u}_0 := \max\{1, \sup_{x\in\mathbb{R}} u_0(x)\}$. By the comparison principle for parabolic equations, we have that

$$\underline{u}_0 \le u(t,x;\underline{u}_0,\omega) \le \min\{1, u(t,x;u_0,\omega)\}, \quad \forall\, x\in\mathbb{R},\ \forall\, t\ge 0 \tag{3.2}$$

and

$$\max\{1, u(t,x;u_0,\omega)\} \le u(t,x;\overline{u}_0,\omega) \le \overline{u}_0, \quad \forall\, x\in\mathbb{R},\ \forall\, t\ge 0. \tag{3.3}$$

Since $\underline{u}_0$ and $\overline{u}_0$ are positive numbers, by the uniqueness of solutions of (1.1) and its corresponding ODE with a given initial function, we have that

$$u(t,x;\underline{u}_0,\omega) = u(t,0;\underline{u}_0,\omega) \quad \text{and} \quad u(t,x;\overline{u}_0,\omega) = u(t,0;\overline{u}_0,\omega) \quad \forall\, x\in\mathbb{R},\ \forall t\ge 0.$$

Next, let $\underline{u}(t) = \left(\frac{1}{u(t,0;\underline{u}_0,\omega)} - 1\right) e^{\int_0^t a(\theta_s\omega)ds}$ and $\overline{u}(t) = \left(1 - \frac{1}{u(t,0;\overline{u}_0,\omega)}\right) e^{\int_0^t a(\theta_s\omega)ds}$. It can be verified directly that

$$\frac{d}{dt}\underline{u} = \frac{d}{dt}\overline{u} = 0, \quad t>0.$$

Hence,

$$\underline{u}(t) = \underline{u}(0) \quad \text{and} \quad \overline{u}(t) = \overline{u}(0), \quad \forall\, t\ge 0,$$

which is equivalent to

$$1 - u(t,x;\underline{u}_0,\omega) = \underline{u}(0) u(t,x;\underline{u}_0,\omega) e^{-\int_0^t a(\theta_s\omega)ds} \tag{3.4}$$

and

$$u(t,x;\overline{u}_0,\omega) - 1 = \overline{u}(0) u(t,x;\overline{u}_0,\omega) e^{-\int_0^t a(\theta_s\omega)ds}. \tag{3.5}$$

Now, by (3.2)–(3.5), we have that

$$|u(t,x;u_0,\omega) - 1| \le \overline{u}_0 \max\{\overline{u}(0), \underline{u}(0)\} e^{-\int_0^t a(\theta_s\omega)ds}, \quad \forall x\in\mathbb{R},\ t\ge 0,$$

which implies that inequality (1.8) holds. Taking u_0 to be a positive constant with $0 < u_0 < 1$, it follows from (3.4) that

$$u(t,x;\underline{u}_0,\omega) = \frac{1}{1 + (\frac{1}{\underline{u}_0} - 1) e^{-\int_0^t a(\theta_s\omega)ds}}.$$

If $\|a(\theta.\omega)\|_{L^1(0,\infty)} < \infty$, then $\lim_{t\to\infty} u(t,x;\underline{u}_0,\omega) = \frac{1}{1+(\frac{1}{\underline{u}_0}-1)e^{-\|a(\theta.\omega)\|_{L^1(0,\infty)}}} < 1$, which completes the proof of the theorem. □

Remark 3.1 (1) Theorem 1.1 guarantees the exponential stability of the trivial constant equilibrium solution $u \equiv 1$ of (1.1) with respect to the solutions $u(t,x;u_0,\omega)$ with $\inf_{x\in\mathbb{R}^n} u_0(x) > 0$ provided that **(H1)** holds. This result will be useful in the later sections.

(2) Let $v(t,x;u_0,\omega)$ be the solution of (3.1) with $v(0,x;u_0,\omega) = u_0(x)$. The result in Theorem 1.1 also holds for $v(t,x;u_0,\omega)$.

Let

$$\underline{c}(\omega) = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t c(\tau;\omega)d\tau, \quad \overline{c}(\omega) = \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t c(\tau;\omega)d\tau.$$

Next, we prove the following theorem about the stability of $u \equiv 1$.

Theorem 3.1 *Assume* **(H1)**. *Suppose that $v(t, x;\omega)$ with $0 < v(t, x;\omega) < 1$, is an entire solution of (3.1) which is nonincreaing in x. For given $\omega \in \Omega$ with $0 < \underline{c}(\omega) \le \overline{c}(\omega) < \infty$, if there is $x^* \in \mathbb{R}$ such that $\inf_{t\in\mathbb{R}} v(t, x^*;\omega) > 0$, then $\lim_{x\to-\infty} v(t, x;\omega) = 1$ uniformly in $t \in \mathbb{R}$.*

To prove the above theorem, we first prove a lemma.

Lemma 3.1 *Let $u_0, u_n \in C^b_{\text{unif}}(\mathbb{R})$ be such that $0 \le u_n(x) \le u_0(x) \le 1$. Let $v(t, x; u_0, \theta_{t_0}\omega)$ (respectively $v(t, x; u_n, \theta_{t_0}\omega)$) denote the solution of (3.1) with ω being replaced by $\theta_{t_0}\omega$ and with initial function u_0 (respectively u_n). If $\lim_{n\to\infty} u_n(x) = u_0(x)$ locally uniformly in $x \in \mathbb{R}$, then for any fixed $t > 0$ with $-\infty < \inf_{t_0\in\mathbb{R}} \int_0^t c(\tau + t_0;\omega) \le \sup_{t_0\in\mathbb{R}} \int_0^t c(\tau + t_0;\omega) < \infty$, we have*

$$\lim_{n\to\infty} v(t, x; u_n, \theta_{t_0}\omega) = v(t, x; u_0, \theta_{t_0}\omega)$$

uniformly in $t_0 \in \mathbb{R}$ and locally uniformly in $x \in \mathbb{R}$.

Proof Fix $\omega \in \Omega$. For every $n \ge 1$, the function $v^n(t, x; t_0) := v(t, x; u_0, \theta_{t_0}\omega) - v(t, x; u_n, \theta_{t_0}\omega)$ is non-negative and satisfies

$$\begin{aligned} v^n_t &= v^n_{xx} + c(t + t_0;\omega)v^n_x + a(\theta_{t_0+t}\omega)(1 - (v(t, x; u_0, \theta_{t_0}\omega) + v(t, x; u_n, \theta_{t_0}\omega)))v^n \\ &\le v^n_{xx} + c(t + t_0;\omega)v^n_x + a(\theta_{t_0+t}\omega)v^n. \end{aligned}$$

It follows that, for every $n \ge 1$, $\tilde{v}^n(t, x; t_0) := v^n(t, x - \int_{t_0}^{t_0+t} c(\tau;\omega)d\tau; t_0)$ satisfies

$$\tilde{v}^n(t, x; t_0) \le \tilde{v}^n_{xx} + a(\theta_{t+t_0}\omega)\tilde{v}^n,$$

By the comparison principle for parabolic equations,

$$0 \le v^n(t, \cdot; t_0) \le e^{\int_{t_0}^{t_0+t} a(\theta_\tau\omega)d\tau} e^{t\Delta} v^n(0, \cdot + \int_0^t c(\tau + t_0)d\tau).$$

Note that $\lim_{n\to\infty} \big(e^{\int_{t_0}^{t_0+t} a(\theta_\tau\omega)d\tau} e^{t\Delta} v^n(0, \cdot + \int_0^t c(\tau + t_0)d\tau)\big)(x) = 0$ locally uniformly in $x \in \mathbb{R}$ and uniformly in $t_0 \in \mathbb{R}$. Hence $\lim_{n\to\infty} v^n(t, x; t_0) = 0$ uniformly in $t_0 \in \mathbb{R}$ and locally uniformly in $x \in \mathbb{R}$. □

We now prove Theorem 3.1.

Proof of Theorem 3.1 Fix $\omega \in \Omega$ with $-\infty < \underline{c}(\omega) \le \overline{c}(\omega) < \infty$ and assume that there is $x^* \in \mathbb{R}$ such that $\inf_{t\in\mathbb{R}} v(t, x^*;\omega) > 0$.

Consider the constant function $u_0 \equiv \inf_{t\in\mathbb{R}} v(t, x^*;\omega)$. We first note from the hypotheses of Theorem 3.1 that $u_0 > 0$. Next, let $\tilde{u}_0(\cdot)$ be uniformly continuous, $0 \le \tilde{u}_0(x) \le u_0$, $\tilde{u}_0(x) = u_0$ for $x \le x^* - 1$, and $\tilde{u}_0(x) = 0$ for $x \ge x^*$. For any $R > 0$, it holds that $\tilde{u}_0(x-n) = u_0$ for every $|x| \le R$ and $n \ge R+1+|x^*|$. This shows that $\lim_{n\to\infty} \tilde{u}_0(x-n) = u_0$ locally uniformly in $x \in \mathbb{R}$. By **(H1)** and the arguments of Theorem 1.1,

$$\lim_{t\to\infty} v(t, x; u_0, \theta_{t_0}\omega) = 1$$

uniformly in $t_0 \in \mathbb{R}$ and $x \in \mathbb{R}$. Hence, for any $\epsilon > 0$, there is $T > 0$ such that

$$-\infty < \inf_{t_0\in\mathbb{R}} \int_0^T c(\tau + t_0; \omega) \le \sup_{t_0\in\mathbb{R}} \int_0^T c(\tau + t_0; \omega) < \infty$$

and

$$1 > v(T, x; u_0, \theta_{t_0}\omega) > 1 - \epsilon \quad \forall\ t_0 \in \mathbb{R},\ \ x \in \mathbb{R}.$$

By Lemma 3.1, there is $N > 1$ such that

$$1 > v(T, 0; \tilde{u}_0(\cdot - N), \theta_{t_0}\omega) > 1 - 2\epsilon \quad \forall\ t_0 \in \mathbb{R}.$$

This implies that

$$1 > v(T, -N; \tilde{u}_0, \theta_{t_0}\omega) > 1 - 2\epsilon \quad \forall\ t_0 \in \mathbb{R}.$$

Note that

$$v(t - T, x; \omega) \ge \tilde{u}_0(x) \quad \forall\ t \in \mathbb{R},\ \ x \in \mathbb{R}$$

and

$$v(t, x; \omega) = v(T, x; v(t - T, \cdot), \theta_{t-T}\omega).$$

Hence

$$1 > v(t, x; \omega) = v(T, x; v(t - T, \cdot), \theta_{t-T}\omega) > 1 - 2\epsilon \quad \forall\ t \in \mathbb{R},\ \ x \le -N.$$

The theorem thus follows. □

3.2 Existence and Stability of Positive Random Equilibria of (1.21)

In this subsection, we first study the existence and stability of positive random equilibria of (1.21), and then show that (1.21) can be transferred to (1.1).

To this end, we consider the following corresponding ODE,

$$\dot{u} = u(1 + \xi(\theta_t\omega) - u). \tag{3.6}$$

Throughout this subsection, we assume that **(H3)** holds. For given $u_0 \in \mathbb{R}$, let $u(t; u_0, \omega)$ be the solution of (3.6) with $u(0; u_0, \omega) = u_0$. It is known that

$$u(t; u_0, \omega) = \frac{u_0 e^{t+\int_0^t \xi(\theta_\tau\omega)d\tau}}{1 + u_0 \int_0^t e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau} ds}.$$

Theorem 3.2 $Y(\omega) = \frac{1}{\int_{-\infty}^0 e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau} ds}$ *is a random equilibrium of (3.6), that is,* $u(t; Y(\omega), \omega) = Y(\theta_t\omega)$ *for* $t \in \mathbb{R}$ *and* $\omega \in \Omega$.

Proof First, we note that

$$u(t;Y(\omega),\omega)=\frac{Y(\omega)e^{t+\int_0^t \xi(\theta_\tau\omega)d\tau}}{1+Y(\omega)\int_0^t e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau}ds}$$
$$=\frac{e^{t+\int_0^t \xi(\theta_\tau\omega)d\tau}}{\int_{-\infty}^0 e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau}ds+\int_0^t e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau}ds}$$
$$=\frac{e^{t+\int_0^t \xi(\theta_\tau\omega)d\tau}}{\int_{-\infty}^t e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau}ds}.$$

Second, note that

$$Y(\theta_t\omega)=\frac{1}{\int_{-\infty}^0 e^{s+\int_0^s \xi(\theta_{t+\tau}\omega)d\tau}ds}=\frac{1}{\int_{-\infty}^t e^{(s-t)+\int_0^{s-t} \xi(\theta_{t+\tau}\omega)d\tau}ds}$$
$$=\frac{e^{t+\int_0^t \xi(\theta_\tau\omega)d\tau}}{\int_{-\infty}^t e^{s+\int_0^s \xi(\theta_\tau\omega)d\tau}ds}.$$

Hence $u(t;Y(\omega),\omega)=Y(\theta_t\omega)$ and then $Y(\omega)$ is a random equilibrium of (3.6). □

Observe that $0<Y(\omega)<\infty$. Let $\tilde u=\frac{u}{Y(\theta_t\omega)}$ and drop the tilde. We have

$$u_t=u_{xx}+Y(\theta_t\omega)u(1-u). \tag{3.7}$$

Clearly, (3.7) is of the form (1.1) with $a(\omega)=Y(\omega)$. Let $\hat{Y}_{\inf}(\omega)$ and $\hat{Y}_{\sup}(\omega)$ be defined as in (1.4) and (1.5) with $a(\cdot)$ being replaced by $Y(\cdot)$, respectively.

Lemma 3.2 *$Y(\omega)$ satisfies the following properties.*

(1) *For a.e.* $\omega\in\Omega$, $0<\inf_{t\in\mathbb{R}}Y(\theta_t\omega)\le\sup_{t\in\mathbb{R}}Y(\theta_t\omega)<\infty$.
(2) *For a.e.* $\omega\in\Omega$, $\lim_{t\to\infty}\frac{\ln Y(\theta_t\omega)}{t}=0$.
(3) *For a.e.* $\omega\in\Omega$, $\lim_{t\to\infty}\frac{\int_0^t Y(\theta_s\omega)ds}{t}=1$.
(4) $\hat{Y}_{\inf}(\omega)=1+\underline{\xi}>0$, *and* $\hat{Y}_{\sup}(\omega)=1+\overline{\xi}<\infty$ *for a.e.* $\omega\in\Omega$.

Proof (1) First, note that

$$\frac{1}{Y(\theta_t\omega)}=\int_{-\infty}^{-T} e^{s-\int_s^0 \xi(\theta_{\tau+t}\omega)\tau}ds+\int_{-T}^0 e^{s-\int_s^0 \xi(\theta_{\tau+t}\omega)\tau}ds,\quad \forall T>0,\forall t\in\mathbb{R}. \tag{3.8}$$

By (H3), for every $\lambda\in(0,1)$ and a.e. $\omega\in\Omega$ there is $T_\lambda\gg 1$,

$$\lambda\underline{\xi}\le\frac{1}{T}\int_0^T \xi(\theta_{x+\tau}\omega)d\tau\le\frac{\overline{\xi}}{\lambda},\quad \forall x\in\mathbb{R},\forall T\ge T_\lambda.$$

It then follows that

$$\int_{-\infty}^{-T_\lambda} e^{(1+\frac{\overline{\xi}}{\lambda})s}ds\le\int_{-\infty}^{-T_\lambda} e^{s-\int_s^0 \xi(\theta_{\tau+t}\omega)\tau}ds\le\int_{-\infty}^{-T_\lambda} e^{(1+\lambda\underline{\xi})s}ds.$$

That is,

$$\frac{e^{-(1+\frac{\overline{\xi}}{\lambda})T_\lambda}}{(1+\frac{\overline{\xi}}{\lambda})}\le\int_{-\infty}^{-T_\lambda} e^{s-\int_s^0 \xi(\theta_{\tau+t}\omega)\tau}ds\le\frac{e^{-(1+\lambda\underline{\xi})T_\lambda}}{(1+\lambda\underline{\xi})}. \tag{3.9}$$

The first inequality of (3.9) combined with (3.8) yields that

$$\frac{1}{Y(\theta_t\omega)} \geq \frac{e^{-(1+\frac{\overline{\xi}}{\lambda})T_\lambda}}{(1+\frac{\overline{\xi}}{\lambda})}.$$

Hence

$$Y(\theta_t\omega) \leq (1+\frac{\overline{\xi}}{\lambda})e^{(1+\frac{\overline{\xi}}{\lambda})T_\lambda}, \quad \forall\, t \in \mathbb{R}. \tag{3.10}$$

Next, let $\xi_{\inf}(\omega) = \inf_{t\in\mathbb{R}} \xi(\theta_t\omega)$. Observe that

$$\int_{-T_\lambda}^{0} e^{s-\int_s^0 \xi(\theta_{\tau+t}\omega)\tau} ds \leq \int_{-T_\lambda}^{0} e^{s-\int_s^0 \xi_{\inf}(\omega)d\tau} ds = \int_{-T_\lambda}^{0} e^{s(1+\xi_{\inf}(\omega))} ds.$$

This combined with the second inequality in (3.9) yield that

$$\frac{1}{Y(\theta_t\omega)} \leq \frac{e^{-(1+\lambda\underline{\xi})T_\lambda}}{(1+\lambda\overline{\xi})} + \int_{-T_\lambda}^{0} e^{s(1+\xi_{\inf}(\omega))} ds, \quad \forall\, t \in \mathbb{R},\ a.e.\, \omega \in \Omega. \tag{3.11}$$

It easily follows from (3.10) and (3.11) that

$$0 < \inf_{t\in\mathbb{R}} Y(\theta_t\omega) \leq \sup_{t\in\mathbb{R}} Y(\theta_t\omega) < \infty, \quad a.e.\, \omega \in \Omega.$$

The result (1) then follows.

(2) It follows from (1).

(3) Note that

$$\frac{\dot{Y}(\theta_t\omega)}{Y(\theta_t\omega)} = 1 + \xi(\theta_t\omega) - Y(\theta_t\omega).$$

Integrating both sides with respect to t, we obtain that

$$\frac{1}{t-s}\int_s^t Y(\theta_\sigma\omega)d\sigma + \frac{\ln(Y(\theta_t\omega)) - \ln(Y(\theta_s\omega))}{t-s} = 1 + \frac{1}{t-s}\int_s^t \xi(\theta_\sigma\omega)d\sigma. \tag{3.12}$$

The result (3) follows from (2) and the fact that $\lim_{t\to\infty} \frac{1}{t}\int_0^t \xi(\theta_s\omega)ds = 0$ for a.e. $\omega \in \Omega$.

(4) Observe that (3.12) implies that

$$1 + \underline{\xi} \leq \hat{Y}_{\inf}(\omega) + \limsup_{t-s\to\infty} \frac{\ln(Y(\theta_t\omega)) - \ln(Y(\theta_s\omega))}{t-s}$$

and

$$1 + \underline{\xi} \geq \hat{Y}_{\inf}(\omega) + \liminf_{t-s\to\infty} \frac{\ln(Y(\theta_t\omega)) - \ln(Y(\theta_s\omega))}{t-s}$$

for a.e. $\omega \in \Omega$. It follows from (1) that

$$\liminf_{t-s\to\infty} \frac{\ln(Y(\theta_t\omega)) - \ln(Y(\theta_s\omega))}{t-s} = \limsup_{t-s\to\infty} \frac{\ln(Y(\theta_t\omega)) - \ln(Y(\theta_s\omega))}{t-s} = 0 \quad \text{for } a.e.\, \omega \in \Omega.$$

Hence we have that $\hat{Y}_{\inf}(\omega) = 1 + \underline{\xi} > 0$ for a.e. $\omega \in \Omega$. Similar arguments yield that $\hat{Y}_{\sup}(\omega) = 1 + \overline{\xi}$ for a.e. $\omega \in \Omega$. □

Corollary 3.1 *For given $u_0 \in C^b_{\rm uinf}(\mathbb{R})$ with $\inf_x u_0(x) > 0$, for a.e. $\omega \in \Omega$,*

$$\lim_{t\to\infty} \|\frac{u(t,\cdot;u_0,\theta_{t_0}\omega)}{Y(\theta_t\theta_{t_0}\omega)} - 1\|_\infty = 0$$

uniformly in $t_0 \in \mathbb{R}$, where $u(t,x;u_0,\theta_{t_0}\omega)$ is the solution of (1.21) with $u(0,x;u_0,\theta_{t_0}\omega) = u_0(x)$.

Proof It follows from Theorems 1.1, 3.2, and Lemma 3.2. □

4 Deterministic and Linearly Determinate Spreading Speed Interval

In this section, we discuss the spreading properties of solutions of (1.1) with nonempty compactly supported initials or front like initials and prove Theorems 1.2 and 1.3 .

We first prove some lemmas.

Lemma 4.1 *Let $\omega \in \Omega_0$. If there is a positive constant $c(\omega) > 0$ such that*

$$\liminf_{t\to\infty} \inf_{s\in\mathbb{R},|x|\le c(\omega)t} u(t,x;u_0,\theta_s\omega) > 0, \quad \forall\, u_0 \in X_c^+ \tag{4.1}$$

then $c^*_{\inf}(\omega) \ge c(\omega)$. Therefore it holds that

$$c^*_{\inf}(\omega) = \sup\{c \in \mathbb{R}^+ \mid \liminf_{t\to\infty} \inf_{s\in\mathbb{R},|x|\le ct} u(t,x;u_0,\theta_s\omega) > 0, \quad \forall\, u_0 \in X_c^+\}. \tag{4.2}$$

Proof Let $\omega \in \Omega_0$ and $c(\omega)$ satisfy (4.1). Let $0 < c < c(\omega)$ and $u_0 \in X_c^+$ be given. Choose $\tilde{c} \in (c, c(\omega))$. It follows from (4.1) that

$$m_{\tilde{c}} := \liminf_{t\to\infty} \inf_{s\in\mathbb{R},|x|\le\tilde{c}t} u(t,x;u_0,\theta_s\omega) > 0.$$

There is $T \gg 1$ such that

$$\frac{m_{\tilde{c}}}{2} \le \min_{|x|\le\tilde{c}t} u(t,x;u_0,\theta_s\omega), \quad \forall\, s \in \mathbb{R},\ t \ge T. \tag{4.3}$$

Suppose by contradiction that there is $(s_n,t_n,x_n) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}$ with $|x_n| \le ct_n$ for every $n \ge 1$ and $t_n \to \infty$ such that

$$0 < \delta := \inf_{n\ge1} |u(t_n,x_n;u_0,\theta_{s_n}\omega) - 1|. \tag{4.4}$$

Let $0 < \varepsilon < 1$ be fixed. By **(H1)**, Theorem 1.1 implies that there is $\tilde{T}_\varepsilon > T$ such that

$$\|u(t,\cdot;\frac{m_{\tilde{c}}}{2},\theta_s\omega) - 1\|_\infty + \|u(t,\cdot;\|u_0\|_\infty,\theta_s\omega) - 1\|_\infty \le \varepsilon, \quad \forall t \ge \tilde{T}_\varepsilon,\ \forall\, s \in \mathbb{R}. \tag{4.5}$$

Observe that $(\tilde{c}-c)(t_n-\tilde{T}_\varepsilon) - 2c\tilde{T}_\varepsilon \to \infty$ as $n \to \infty$. Then there is n_ε such that

$$(\tilde{c}-c)(t_n-\tilde{T}_\varepsilon) - 2c\tilde{T}_\varepsilon \ge T, \quad \forall\ n \ge n_\varepsilon.$$

For every $n \ge n_\varepsilon$, let $u_{0n} \in C^b_{\rm unif}(\mathbb{R})$ with $\|u_{0n}\|_\infty \le \frac{m_{\tilde{c}}}{2}$ and

$$u_{0n}(x) = \begin{cases} \frac{m_{\tilde{c}}}{2}, & |x| \le (\tilde{c}-c)(t_n-\tilde{T}_\varepsilon) - 2c\tilde{T}_\varepsilon, \\ 0, & |x| \ge (\tilde{c}-c)(t_n-\tilde{T}_\varepsilon) - c\tilde{T}_\varepsilon. \end{cases} \tag{4.6}$$

Since $|x| \le (\tilde{c}-c)(t_n - \tilde{T}_\varepsilon) - c\tilde{T}_\varepsilon$ implies that $|x + x_n| \le \tilde{c}(t_n - \tilde{T}_\varepsilon)$ for every $n \ge n_\varepsilon$, it follows from (4.3) to (4.6) that

$$u_{0n}(x) \le u(t_n - \tilde{T}_\varepsilon, x + x_n; u_0, \theta_{s_n}\omega), \quad \forall\, x \in \mathbb{R},\ \forall\, n \ge n_\varepsilon.$$

By the comparison principle for parabolic equations, we have

$$u(t, x; u_{0n}, \theta_{\tilde{s}_n}\omega) \le u(t + t_n - \tilde{T}_\varepsilon, x + x_n; u_0, \theta_{s_n}\omega), \quad \forall\, x \in \mathbb{R},\ t > 0,\ n \ge n_\varepsilon, \tag{4.7}$$

where $\tilde{s}_n = s_n + t_n - \tilde{T}_\varepsilon$.

Observe from the definition of u_{0n} that $u_{0n}(x) \to \frac{m_{\tilde{c}}}{2}$ as $n \to \infty$ locally uniformly in $x \in \mathbb{R}$. It then follows from Lemma 3.1 that for every $t > 0$,

$$|u(t, x; u_{0n}, \theta_{\tilde{s}_n}\omega) - u(t, x; \frac{m_{\tilde{c}}}{2}, \theta_{\tilde{s}_n}\omega)| \to 0 \text{ as } n \to \infty \text{ locally uniformly in } x \in \mathbb{R}. \tag{4.8}$$

By (4.5), we have that

$$1 - \varepsilon \le u(\tilde{T}_\epsilon, x; \frac{m_{\tilde{c}}}{2}, \theta_{\tilde{s}_n}\omega), \quad \forall\, x \in \mathbb{R}, \forall\, n \ge 1.$$

This combined with (4.7) and (4.8) yields that

$$1 - \varepsilon \le \liminf_{n\to\infty} u(\tilde{T}_\varepsilon, 0; u_{0n}, \theta_{\tilde{s}_n}\omega) \le \liminf_{n\to\infty} u(t_n, x_n; u_0, \theta_{s_n}\omega). \tag{4.9}$$

On the other hand, since $\|u_0(\cdot + x_n)\|_\infty = \|u_0\|_\infty$ for every $n \ge 1$, it follows from the comparison principle for parabolic equations that

$$\begin{aligned} u(t_n, x; \|u_0\|_\infty, \theta_{s_n}\omega) &\ge u(t_n, x; u_0(\cdot + x_n), \theta_{s_n}\omega), \\ &= u(t_n, x + x_n; u_0, \theta_{s_n}\omega), \quad \forall\, x \in \mathbb{R},\ t > 0,\ n \ge 1. \end{aligned}$$

This together with (4.5) implies that

$$\limsup_{n\to\infty} u(t_n, x_n; u_0, \theta_{s_n}\omega) \le \limsup_{n\to\infty} \|u(t_n, \cdot; \|u_0\|_\infty, \theta_{s_n}\omega)\|_\infty \le 1 + \varepsilon,$$

which combined with (4.9) yields that

$$1 - \varepsilon \le \limsup_{n\to\infty} u(t_n, x_n; u_0, \theta_{s_n}\omega) \le \limsup_{n\to\infty} u(t_n, x_n; u_0, \theta_{s_n}\omega) \le 1 + \varepsilon, \forall\, \varepsilon > 0.$$

Letting $\varepsilon \to 0$, we obtain that

$$\lim_{n\to\infty} |u(t_n, x_n; u_0, \theta_{s_n}\omega) - 1| = 0,$$

which contradicts to (4.4). Thus we have that

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}, |x|\le ct} |u(t, x; u_0, \theta_s\omega) - 1| = 0, \quad \forall u_0 \in X_c^+,\ \forall 0 < c < c(\omega).$$

This implies that $c_{\inf}^*(\omega) \ge c(\omega)$.

Therefore, we have that

$$c_{\inf}^*(\omega) \ge \sup\{c \in \mathbb{R}^+ \mid \liminf_{t\to\infty} \inf_{|x|\le ct, s\in\mathbb{R}} u(t, x; u_0, \theta_s\omega) > 0, \quad \forall\, u_0 \in X_c^+\}.$$

On the other hand, it is clear from the definition of $C_{\sup}^*(\omega)$ that

$$c_{\inf}^*(\omega) \le \sup\{c \in \mathbb{R}^+ \mid \liminf_{t\to\infty} \inf_{|x|\le ct, s\in\mathbb{R}} u(t, x; u_0, \theta_s\omega) > 0, \quad \forall\, u_0 \in X_c^+\}.$$

The lemma is thus proved. □

Lemma 4.2 *Let $b > 0$ be a positive number and $v_0 \in X_c^+$. Let $v(t, x; v_0, b)$ be the solution of*

$$\begin{cases} v_t = v_{xx} + bv(1 - v), & x \in \mathbb{R} \\ v(0, x) = v_0(x), & x \in \mathbb{R}. \end{cases}$$

Then

$$\lim_{t\to\infty} \min_{|x|\le ct} v(t, x; v_0, b) = 1, \quad \forall\, 0 < c < 2\sqrt{b}.$$

Proof It follows from [3, Page 66, Corollary 1]. □

Lemma 4.3 *Assume* (**H1**). *Then for every $\omega \in \Omega_0$,*

$$\liminf_{t\to\infty} \inf_{s\in\mathbb{R}, |x|\le ct} u(t, x; u_0, \theta_s\omega) > 0, \quad \forall\, 0 < c < 2\sqrt{\underline{a}},\ \forall\, u_0 \in X_c^+. \tag{4.10}$$

Therefore, $c^*_{\inf}(\omega) \ge 2\sqrt{\underline{a}}$, $\forall\, \omega \in \Omega_0$.

Proof First, fix $\omega \in \Omega_0$ and $u_0 \in X_c^+$. Let $0 < c < 2\sqrt{\underline{a}}$ be given. Choose $b > c$ and $0 < \delta < 1$ such that $c < 2\sqrt{b} < 2\sqrt{\delta \underline{a}}$. By the proof of Lemma 2.2, there are $\{t_k\}_{k\in\mathbb{Z}}$ with $t_k < t_{k+1}$, $t_k \to \pm\infty$ as $k \to \pm\infty$ and $A \in W^{1,\infty}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $A \in C^1(t_k, t_{k+1})$ for every k and

$$b \le \delta a(\theta_t\omega) - A'(t), \quad \text{for } t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}.$$

Let $\sigma = \frac{(1-\delta)e^{-\|A\|_\infty}}{\|u_0\|_\infty+1}$ and $v(t, x; b) = v(t, x; u_0, b)$. By Lemma 4.2, we have that

$$\liminf_{t\to\infty} \min_{|x|\le ct} v(t, x; b) = 1. \tag{4.11}$$

Next, for given $s \in \mathbb{R}$, let $\tilde{v}(t, x; s) = \sigma e^{A(t+s)} v(t, x; b)$. By the comparison principle for parabolic equations, we have that

$$0 < v(t, x; b) \le \max\{\|u_0\|_\infty, 1\} < \|u_0\|_\infty + 1, \quad \forall\, x \in \mathbb{R},\ t \ge 0.$$

Hence, it follows from the definition of σ that

$$0 < \tilde{v}(t, x; s) \le \sigma e^{\|A\|_\infty}(\|u_0\|_\infty + 1) = 1 - \delta, \quad \forall\, x \in \mathbb{R},\ t \ge 0,\ s \in \mathbb{R}.$$

Thus for any $s \in \mathbb{R}$,

$$\begin{aligned} \tilde{v}_t - \tilde{v}_{xx} - a(\theta_{s+t}\omega)\tilde{v}(1 - \tilde{v}) &= \left(A'(s + t) + b(1 - v) - a(\theta_{s+t}\omega)(1 - \tilde{v})\right) \tilde{v}(t, x) \\ &\le \left(A'(s + t) + b(1 - v) - \delta a(\theta_{s+t}\omega)\right) \tilde{v}(t, x) \\ &\le \left(A'(s + t) + b - \delta a(\theta_{s+t}\omega)\right) \tilde{v}(t, x) \\ &\le 0, \quad t \in (t_k, t_{k+1}) \cap [0, \infty),\ x \in \mathbb{R}. \end{aligned}$$

Note that

$$\tilde{v}(0, x; s) = \sigma e^{A(s)} u_0(x) \le u_0(x), \quad \forall\, x \in \mathbb{R}.$$

By the comparison principle for parabolic equations again, we have that

$$\sigma e^{-\|A\|_\infty} v(t, x, b) \le \tilde{v}(t, x; s) \le u(t, x; u_0, \theta_s\omega), \quad \forall\, x \in \mathbb{R},\ s \in \mathbb{R},\ t \ge 0.$$

This combined with (4.11) yields that

$$0 < \sigma e^{-\|A\|_\infty} \le \liminf_{t\to\infty} \inf_{s\in\mathbb{R}|x|\le ct} u(t,x;u_0,\theta_s\omega), \quad \forall\, 0<c<2\sqrt{\underline{a}}.$$

Hence (4.10) holds. By (4.10) and Lemma 4.1, we have $c^*_{\inf}(\omega) \ge 2\sqrt{\underline{a}}, \quad \forall\, \omega\in\Omega_0.$ □

Now, we prove Theorem 1.2.

Proof of Theorem 1.2 (i) We first prove $c^*_{\sup}(\omega) \le 2\sqrt{\bar{a}}$ for all $\omega\in\Omega_0$.

Suppose that $\text{supp}(u_0)\subset(-R,R)$. For every $\mu>0$, let $C_\mu(t,s)=\int_s^{s+t}\frac{\mu^2+a(\theta_\tau\omega)}{\mu}d\tau$ and $\phi^\mu(x)=\|u_0\|_\infty e^{-\mu(x-R)}$ and $\tilde{\phi}^\mu_\pm(t,x;s)=\phi^\mu(\pm x - C_\mu(t,s))$ for every $x\in\mathbb{R}$ and $t\ge 0$. Then

$$\partial_t\tilde{\phi}^\mu_\pm - \partial_{xx}\tilde{\phi}^\mu_\pm - a(\theta_{s+t}\omega)\tilde{\phi}^\mu_\pm(1-\tilde{\phi}^\mu_\pm) = a(\theta_{s+t}\omega)\left(\tilde{\phi}^\mu_\pm\right)^2 \ge 0,\ x\in\mathbb{R},\ t>0.$$

and

$$u_0(x)\le\tilde{\phi}^\mu_\pm(0,x;s), \quad \forall x\in\mathbb{R},\ \forall\, s\in\mathbb{R}.$$

By the comparison principle for parabolic equations, we have

$$u(t,x;u_0,\theta_s\omega)\le\tilde{\phi}^\mu_\pm(t,x;s)=\|u_0\|_\infty e^{-\mu(\pm x-R-C_\mu(t,s))}, \quad \forall\, x,s\in\mathbb{R}, \forall t>0, \forall\mu>0.$$

This implies that

$$\limsup_{t\to\infty}\sup_{s\in\mathbb{R},|x|\ge ct} u(t,x;u_0,\theta_s\omega)=0 \quad \forall\,\mu>0,\ c>\frac{\mu^2+\bar{a}}{\mu}.$$

For any $c>\bar{c}^*=2\sqrt{\bar{a}}=\inf_{\mu>0}\frac{\mu^2+\sqrt{\bar{a}}}{\mu}$, choose $\mu>0$ such that $c>\frac{\mu^2+\sqrt{\bar{a}}}{\mu}>\bar{c}^*$. By the above arguments, we have

$$\limsup_{t\to\infty}\sup_{s\in\mathbb{R},|x|\ge ct} u(t,x;u_0,\theta_s\omega)=0.$$

Hence for any $\omega\in\Omega_0$,$c^*_{\sup}(\omega)\le 2\sqrt{\bar{a}}$.

Next, we prove that $c^*_{\sup}(\omega)\ge 2\sqrt{\overline{a}}$ for all $\omega\in\Omega_0$. We prove this by contradiction.

Assume that there is $\omega\in\Omega_0$ such that $c^*_{\sup}(\omega)<2\sqrt{\overline{a}}$. Then there is $0<\delta<1$ such that

$$c^*_{\sup}(\omega)<2\sqrt{\delta\overline{a}}.$$

Note that

$$\limsup_{t-s\to\infty}\frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau=\bar{a}>\delta\bar{a}.$$

Then there is $0<\delta^{'}<1$ and $\{t_n\}$, $\{s_n\}$ such that $\lim_{n\to\infty} t_n-s_n=\infty$ and

$$\delta^{'}\frac{1}{t_n-s_n}\int_{s_n}^{t_n} a(\theta_\tau\omega)d\tau>\delta\bar{a}. \tag{4.12}$$

Choose $c\in(c^*_{\sup}(\omega),2\sqrt{\delta\overline{a}})$. Set $L=\frac{2\pi}{\sqrt{4\bar{a}\delta-c^2}}$ and

$$w^+(x)=e^{-\frac{c}{2}x}\sin\left(\frac{\sqrt{4\bar{a}\delta-c^2}}{2}x\right).$$

Then $w^+(x)$ satisfies

$$\begin{cases} w^+_{xx} + cw^+_x + \bar{a}\delta w^+ = 0, \quad 0 < x < L, \\ w^+(0) = w^+(L) = 0, \end{cases} \tag{4.13}$$

and $0 < w^+(x) < 1$ for $0 < x < L$.

For any given $u_0 \in X_c^+$, by the assumption that $c > c^*_{\sup}(\omega)$,

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R},|x|\geq ct} u(t, x; u_0, \theta_s\omega) = 0. \tag{4.14}$$

Hence there is $T > 0$ such hat

$$u(t, x; u_0, \theta_s\omega) < 1 - \delta^{'} \quad \forall\, t \geq T,\ |x| \geq ct,\ s \in \mathbb{R},$$

and then

$$u(t, x; u_0, \theta_s\omega)(1 - u(t, x; u_0, \theta_s\omega)) > \delta^{'} u(t, x; u_0, \theta_s\omega) \quad \forall\, t \geq T,\ |x| \geq ct,\ s \in \mathbb{R}. \tag{4.15}$$

Observe that $u(t, x; u_0, \theta_s\omega) \geq u(t, x; \frac{u_0}{1+\|u_0\|_\infty}, \theta_s\omega)$ and

$$u_t(t, x; \frac{u_0}{1 + \|u_0\|_\infty}, \theta_s\omega) \geq u_{xx}(t, x; \frac{u_0}{1 + \|u_0\|_\infty}, \theta_s\omega), \quad x \in \mathbb{R}.$$

This implies that

$$\alpha := \inf_{s\in\mathbb{R},0\leq x\leq L} u(T, x + cT; u_0, \theta_s\omega) \geq \inf_{s\in\mathbb{R},0\leq x\leq L} u(T, x + cT; \frac{u_0}{1 + \|u_0\|_\infty}, \theta_s\omega) > 0. \tag{4.16}$$

Let $v(t, x; s) = u(t, x + ct; u_0, \theta_{s-T}\omega)$. By (4.15),

$$v_t \geq v_{xx} + cv_x + \delta^{'} a(\theta_{s-T+t}\omega)v, \quad t \geq T,\ x \geq 0.$$

Let $w(t, x; s) = e^{-\int_s^{s-T+t} \left(\delta^{'} a(\theta_\tau\omega)-\delta\bar{a}\right)d\tau} v(t, x; s)$. Then

$$w_t \geq w_{xx} + cw_x + \delta\bar{a}w, \quad t \geq T,\ x \geq 0.$$

By (4.16) and the comparison principle for parabolic equations, we have

$$v(t, x; s) \geq \alpha e^{\int_s^{s-T+t} \left(\delta^{'} a(\theta_\tau\omega)-\delta\bar{a}\right)d\tau} w^+(x), \quad t \geq T,\ 0 \leq x \leq L.$$

This implies that for $0 \leq x \leq L$,

$$\begin{aligned} u(t_n - s_n + T, x + c(t_n - s_n + T); u_0, \theta_{s_n-T}\omega) &\geq \alpha e^{\int_{s_n}^{t_n} \left(\delta^{'} a(\theta_\tau\omega)-\delta\bar{a}\right)d\tau} w^+(x) \\ &\geq \alpha w^+(x) \quad (\text{by } (4.12)). \end{aligned} \tag{4.17}$$

By (4.14),

$$\limsup_{n\to\infty} \sup_{0\leq x\leq L} u(t_n - s_n + T, x + c(t_n - s_n + T); u_0, \theta_{s_n-T}\omega) = 0,$$

which contradicts (4.17). Therefore, $c^*_{\sup}(\omega) \geq \bar{c}^*$ and then $c^*_{\sup}(\omega) = \bar{c}^*$ for any $\omega \in \Omega_0$. (i) thus follows.

(ii) By Lemma 4.3, $c^*_{\inf}(\omega) \ge \underline{c}^*$ for every $\omega \in \Omega_0$. It then suffices to prove that $c^*_{\inf}(\omega) \le \underline{c}^*$ for every $\omega \in \Omega_0$. We prove this by contradiction.

Assume that there is $\omega \in \Omega_0$ such that $c^*_{\inf}(\omega) > \underline{c}^*$. Choose $c \in (\underline{c}^*, c^*_{\inf}(\omega))$ and $\delta > 1$ such that $c > 2\sqrt{\delta \underline{a}}$. Then

$$\liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau \omega)d\tau < \delta \underline{a}.$$

Hence there are $\{t_n\}$ and $\{s_n\}$ such that $\lim_{n\to\infty} t_n - s_n = \infty$ and

$$\frac{1}{t_n - s_n}\int_{s_n}^{t_n} a(\theta_\tau)d\tau < \delta \underline{a} \quad \forall\, n \ge 1.$$

Let $\underline{\mu} = \sqrt{\delta \underline{a}}$. Then

$$2\sqrt{\delta \underline{a}} = \frac{\delta \underline{a} + \underline{\mu}^2}{\underline{\mu}} < c. \tag{4.18}$$

Choose $u_0 \in X_c^+$ such that

$$0 \le u_0(x) < 1, \quad u_0(x) \le e^{-\underline{\mu}x}\|u_0\|_\infty \quad \forall\, x \in \mathbb{R}.$$

By the assumption that $c < c^*_{\inf}(\omega)$, there is $T > 0$ such that for any $t \ge T$ and $s \in \mathbb{R}$,

$$\inf_{|x|\le ct} u(t, x; u_0, \theta_s \omega) \ge \|u_0\|_\infty.$$

This implies that for any $n \ge 1$ with $t_n - s_n \ge T$,

$$\inf_{|x|\le c(t_n - s_n)} u(t_n - s_n, x; u_0, \theta_{s_n}\omega) \ge \|u_0\|_\infty. \tag{4.19}$$

Observe that $u(t, x; u_0, \theta_{s_n}\omega)$ satisfies

$$u_t = u_{xx} + a(\theta_{s_n+t}\omega)u(1-u) \le u_{xx} + a(\theta_{s_n+t}\omega)u.$$

It then follows from the comparison principle for parabolic equations that

$$u(t, x; u_0, \theta_{s_n}\omega) \le e^{-\underline{\mu}\left(x - \frac{1}{\underline{\mu}}\int_{s_n}^{s_n+t}(a(\theta_\tau\omega)+\underline{\mu}^2)d\tau\right)}\|u_0\|_\infty$$

and then for $x = c(t_n - s_n)$, we have

$$\begin{aligned} u(t_n - s_n, x; u_0, \theta_{s_n}\omega) &\le e^{-\underline{\mu}\left(x - \frac{1}{\underline{\mu}}\int_{s_n}^{t_n}(a(\theta_\tau\omega)+\underline{\mu}^2)d\tau\right)}\|u_0\|_\infty \\ &\le e^{-\underline{\mu}\left(x - \frac{1}{\underline{\mu}}(\delta\underline{a}+\underline{\mu}^2)(t_n-s_n)\right)}\|u_0\|_\infty \\ &= e^{-\underline{\mu}\left(c - \frac{1}{\underline{\mu}}(\delta\underline{a}+\underline{\mu}^2)\right)(t_n-s_n)}\|u_0\|_\infty \\ &< \|u_0\|_\infty \qquad (\text{by}(4.18)), \end{aligned}$$

which contradicts to (4.19). Therefore $c^*_{\inf}(\omega) \le \underline{c}^*$ for any $\omega \in \Omega_0$ and (ii) follows. □

The following corollary follows directly from Lemma 3.2 and Theorem 1.2.

Corollary 4.1 *Assume* **(H3)**. *Let $Y(\omega)$ be the random equilibrium solution of* (1.21) *given in* (1.22). *Then for any $u_0 \in X_c^+$,*

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R},|x|\le ct} \left|\frac{u(t,x;u_0,\theta_s\omega)}{Y(\theta_{t+s}\omega)} - 1\right| = 0, \quad \forall\, 0 < c < 2\sqrt{1+\underline{\xi}}$$

and

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R},|x|\ge ct} \frac{u(t,x;u_0,\theta_s\omega)}{Y(\theta_{t+s}\omega)} = 0, \quad \forall\, c > 2\sqrt{1+\bar{\xi}}$$

for a.e. $\omega \in \Omega$. where $u(t,x;u_0,\theta_s\omega)$ is the solution of (1.21) *with ω being replaced by $\theta_s\omega$ and $u(0,x;u_0,\theta_s\omega) = u_0(x)$.*

Finally, we prove Theorem 1.3.

Proof (i) It is clear that $\tilde{c}^*_{\sup}(\omega) \ge c^*_{\sup}(\omega) = \bar{c}^*$ for any $\omega \in \Omega_0$. It then suffices to prove that $\tilde{c}^*_{\sup}(\omega) \le \bar{c}^*$ for any $\omega \in \Omega_0$.

To this end, fix $\omega \in \Omega_0$. For every $\mu > 0$, let $C_\mu(t,s) = \int_s^{s+t} \frac{\mu^2+a(\theta_\tau\omega)}{\mu} d\tau$ and $\tilde{\phi}^\mu_+(t,x;s) = e^{-\mu(x-C_\mu(t,s))}$ for every $x \in \mathbb{R}$ and $t \ge 0$. Note that for any $u_0 \in \tilde{X}_c^+$, there is $M_0 > 0$ such that

$$u_0(x) \le M_0\tilde{\phi}^\mu_+(0,x;s), \quad \forall x \in \mathbb{R},\ \forall\, s \in \mathbb{R}.$$

Note also that

$$\begin{aligned}\partial_t M_0\tilde{\phi}^\mu_+ - \partial_{xx} M_0\tilde{\phi}^\mu_+ - a(\theta_{s+t}\omega)M_0\tilde{\phi}^\mu_+(1 - M_0\tilde{\phi}^\mu_+) &= a(\theta_{s+t}\omega)M_0^2\big(\tilde{\phi}^\mu_+\big)^2 \\ &\ge 0,\ x \in \mathbb{R},\ t > 0.\end{aligned}$$

Hence, by the comparison principle for parabolic equations, we have that

$$u(t,x;u_0,\theta_s\omega) \le M_0\tilde{\phi}^\mu_+(t,x;s) = M_0 e^{-\mu(x-C_\mu(t,s))}, \quad \forall\, x, s \in \mathbb{R}, \forall t > 0, \forall \mu > 0.$$

This implies that

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R},x\ge ct} u(t,x;u_0,\theta_s\omega) = 0 \quad \forall\, \mu > 0,\ c > \frac{\mu^2+\bar{a}}{\mu}.$$

For any $c > \bar{c}^* = 2\sqrt{\bar{a}} = \inf_{\mu>0} \frac{\mu^2+\sqrt{\bar{a}}}{\mu}$, choose $\mu > 0$ such that $c > \frac{\mu^2+\sqrt{\bar{a}}}{\mu} > \bar{c}^*$. By the above arguments, we have

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R},x\ge ct} u(t,x;u_0,\theta_s\omega) = 0.$$

Hence for any $\omega \in \Omega_0$, we have $\tilde{c}^*_{\sup}(\omega) \le 2\sqrt{\bar{a}}$. (i) thus follows.

(ii) First, it is clear that $\tilde{c}^*_{\inf}(\omega) \ge c^*_{\inf}(\omega) = \underline{c}^*$. It then suffices to prove that $\tilde{c}^*_{\inf}(\omega) \le \underline{c}^*$ for any $\omega \in \Omega_0$. This can be proved by the similar arguments as those in Theorem 1.2 (ii). □

The following corollary follows directly from Lemma 3.2 and Theorem 1.3.

Corollary 4.2 *Assume* **(H3)**. *Let $Y(\omega)$ be the random equilibrium solution of* (1.21) *given in* (1.22). *Then for any $u_0 \in \tilde{X}_c^+$,*

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R},x\le ct} \left|\frac{u(t,x;u_0,\theta_s\omega)}{Y(\theta_{t+s}\omega)} - 1\right| = 0, \quad \forall\, 0 < c < 2\sqrt{1+\underline{\xi}}$$

and

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R}, x\ge ct} \frac{u(t,x;u_0,\theta_s\omega)}{Y(\theta_{t+s}\omega)} = 0, \quad \forall\, c > 2\sqrt{1+\bar{\xi}}$$

for a.e. $\omega \in \Omega$. *where* $u(t,x;u_0,\theta_s\omega)$ *is the solution of* (1.21) *with* ω *being replaced by* $\theta_s\omega$ *and* $u(0,x;u_0,\theta_s\omega) = u_0(x)$.

5 Take-Over Property

In this section, we investigate the take-over property of (1.1) and prove Theorem 1.4. We first prove some lemmas.

Recall that

$$u_0^*(x) = \begin{cases} 1, & x \le 0 \\ 0, & x > 0 \end{cases}$$

and that, for $t > 0$, $x(t,\omega) \in \mathbb{R}$ is such that

$$u(t, x(t,\omega); u_0^*, \omega) = \frac{1}{2}.$$

Note that, by Lemma 2.9, for each $t > 0$, $x(t,\omega)$ is measurable in ω. Note also that for $\omega \in \Omega$, the mapping $(t, t_0) \ni (0,\infty)\times\mathbb{R} \to u(t,\cdot;u_0^*,\theta_{t_0}\omega) \in C^b_{\rm unif}(\mathbb{R})$ is continuous and hence $x(t,\theta_{t_0}\omega)$ is continuous in $(t,t_0) \in (0,\infty)\times\mathbb{R}$.

Suppose that **(H1)** holds. Let $\omega \in \Omega_0$, and $0 < \mu < \tilde{\mu} < \min\{2\mu, \underline{\mu}^*\}$ be given, where $\underline{\mu}^* = \sqrt{\underline{a}}$. Let $b(t) = a(\theta_t\omega)$. Put

$$c(t;\omega,\mu) = c(t;b,\mu), \quad C(t;\omega,\mu) = C(t;b,\mu),$$

and

$$A_\omega(t) = B_b(t), \quad d_\omega = d_b,$$

where $c(t;b,\mu)$ and $C(t;b,\mu)$ are as in (2.5), and B_b and d_b are as in Lemma 2.4. Note that we can choose $d_{\theta_{t_0}\omega} = d_\omega$ and $A_{\theta_{t_0}\omega}(t) = A_\omega(t+t_0)$ for any $t_0 \in \mathbb{R}$. Let

$$x_\omega(t) = C(t;\omega,\mu) + \frac{\ln d_\omega + \ln\tilde{\mu} - \ln\mu}{\tilde{\mu}-\mu} + \frac{A_\omega(t)}{\mu}. \tag{5.1}$$

Note that for any given $t \in \mathbb{R}$,

$$\phi^{\mu,d_\omega,A_\omega}(t, x_\omega(t)) = \sup_{x\in\mathbb{R}} \phi^{\mu,d_\omega,A_\omega}(t,x) = e^{-\mu\left(\frac{\ln d_\omega}{\tilde{\mu}-\mu} + \frac{A_\omega(t)}{\mu}\right)} e^{-\mu\frac{\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}} \left(1 - \frac{\mu}{\tilde{\mu}}\right).$$

We introduce the following function

$$\phi_-^\mu(t,x;\theta_{t_0}\omega) = \begin{cases} \phi^{\mu,d_\omega,A_{\theta_{t_0}\omega}}(t,x), & \text{if } x \ge x_{\theta_{t_0}\omega}(t), \\ \phi^{\mu,d_\omega,A_{\theta_{t_0}\omega}}(t, x_{\theta_{t_0}\omega}(t)), & \text{if } x \le x_{\theta_{t_0}\omega}(t). \end{cases} \tag{5.2}$$

It is clear from Lemma 2.4, and the comparison principle for parabolic equations, that

$$0 < \phi_-^\mu(t,x;\theta_{t_0}\omega) < u(t,x;\phi_+^\mu(\cdot,x;\theta_{t_0}\omega),\theta_{t_0}\omega) \le 1, \forall\, t\in\mathbb{R},\ x\in\mathbb{R},\ t_0\in\mathbb{R}. \tag{5.3}$$

Lemma 5.1 *For every $\omega \in \Omega_0$, $\lim_{x\to-\infty} u(t, x + C(t, \theta_{t_0}\omega, \mu); \phi_+^{\mu}(0, \cdot; \theta_{t_0}\omega), \theta_{t_0}\omega) = 1$ uniformly in $t > 0$ and $t_0 \in \mathbb{R}$, and $\lim_{x\to\infty} u(t, x+C(t, \theta_{t_0}\omega, \mu); \phi_+^{\mu}(0, \cdot; \theta_{t_0}\omega), \theta_{t_0}\omega) = 0$ uniformly in $t > 0$ and $t_0 \in \Omega$.*

Proof First, it follows from Lemma 2.3 that

$$\sup_{t>0,t_0\in\mathbb{R}} u(t, x + C(t, \theta_{t_0}\omega, \mu); \phi_+^{\mu}(0, \cdot; \theta_{t_0}\omega), \theta_{t_0}\omega) \le e^{-\mu x} \to 0 \text{ as } x \to \infty.$$

Second, define $v(t, x; \theta_{t_0}\omega) = u(t, x + C(t, \theta_{t_0}\omega, \mu); \phi_+^{\mu}(0, \cdot; \theta_{t_0}\omega), \theta_{t_0}\omega)$ and

$$x^* = \frac{\ln d_\omega + \ln\tilde{\mu} - \ln\mu}{\tilde{\mu} - \mu} - \frac{\|A_\omega\|_\infty}{\mu}.$$

It follows from (5.1) and (5.3) that

$$0 < (1 - \frac{\mu}{\tilde{\mu}})e^{-\mu\left(\frac{\ln d_\omega+\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}+\frac{\|A_\omega\|_\infty}{\mu}\right)} \le \inf_{t>0,t_0\in\mathbb{R}} v(t, x^*; \phi_+^{\mu}(0, \cdot; \theta_{t_0}\omega), \theta_{t_0}\omega).$$

Moreover, $x \mapsto v(t, x; \theta_{t_0}\omega)$ is decreasing and

$$v_t = v_{xx} + c(t; \theta_{t_0}\omega, \mu)v_x + a(\theta_t\theta_{t_0}\omega)v(1 - v),$$

where $c(t; \omega, \mu) = C'(t; \omega, \mu)$. By the arguments of Theorem 3.1, we have that

$$v(t, x; \theta_{t_0}\omega) \to 1 \text{ as } x \to -\infty$$

uniformly in $t > 0$, $t_0 \in \mathbb{R}$. □

Lemma 5.2 *For each $t > 0$, there is $m(t) \le n(t) \in \mathbb{R}$ such that*

$$m(t) \le x(t, \omega) \le n(t) \quad \text{for a.e } \omega \in \Omega,$$

and hence $x(t, \omega)$ is integrable in ω.

Proof First, let

$$u_{0n}^*(x) = u_0^*(x - n), \quad x \in \mathbb{R}, \ n \in \mathbb{N}.$$

We have that $0 \le u_{0n}^*(x) \le 1$ and $u_{0n}^*(x) \to 1$ as $n \to \infty$. By Lemma 3.1, for every $\omega \in \Omega_0$ and $t > 0$

$$u(t, x; u_{0n}^*, \theta_{t_0}\omega) \to 1 \quad \text{as } n \to \infty$$

uniformly in $t_0 \in \mathbb{R}$ and locally uniformly in $x \in \mathbb{R}$. Observe that

$$u(t, x; u_{0n}^*, \theta_{t_0}\omega) = u(t, x - n; u_0^*, \theta_{t_0}\omega)$$

and the mapping $\mathbb{R} \ni x \mapsto u(t, x; u_0^*, \theta_{t_0}\omega)$ is decreasing. Thus, there is $N(t, \omega) \in \mathbb{N}$ such that

$$u(t, x; u_0^*, \theta_{t_0}\omega) \ge \frac{3}{4}, \quad \forall x \le -N(t, \omega), \quad \forall\, t_0 \in \mathbb{R}.$$

This implies that

$$-N(t, \omega) \le \inf_{t_0\in\mathbb{R}} x(t, \theta_{t_0}\omega).$$

Let

$$m(t, \omega) := \inf_{t_0\in\mathbb{R}} x(t, \theta_{t_0}\omega) = \inf_{t_0\in\mathbb{Q}} x(t, \theta_{t_0}\omega).$$

We have that $\Omega_0 \ni \omega \mapsto m(t,\omega) \in \mathbb{R}^+$ is measurable and $m(t,\theta_\tau\omega) = m(t,\omega)$ for any $\tau \in \mathbb{R}$. By the ergodicity of the metric dynamical system $(\Omega_0, \mathcal{F}, \{\theta_t\}_{t\in\mathbb{R}})$, we have that $m(t,\omega) = m(t)$ for $a.e$ in ω.

Next, let $\tilde{u}^*_{0n}(x) = u^*_0(x+n)$. We have that $0 \le u^*_{0n}(x) \le 1$ and $\tilde{u}^*_{0n}(x) \to 0$ as $n \to \infty$. By Lemma 3.1 again, for every $\omega \in \Omega_0$ and $t > 0$,

$$u(t,x;\tilde{u}^*_{0n},\theta_{t_0}\omega) \to 0 \quad \text{as } n \to \infty$$

uniformly in $t_0 \in \mathbb{R}$ and locally uniformly in $x \in \mathbb{R}$. Observe that

$$u(t,x;\tilde{u}^*_{0n},\theta_{t_0}\omega) = u(t,x+n;u^*_0,\theta_{t_0}\omega)$$

and the mapping $\mathbb{R} \ni x \mapsto u(t,x;u^*_0,\theta_{t_0}\omega)$ is decreasing. Thus, there is $\tilde{N}(t,\omega) \in \mathbb{N}$ such that

$$u(t,x;u^*_0,\theta_{t_0}\omega) \le \frac{1}{4}, \quad \forall x \ge \tilde{N}(t,\omega), \quad \forall\, t_0 \in \mathbb{R}.$$

This implies that

$$N(t,\omega) \ge \sup_{t_0\in\mathbb{R}} x(t,\theta_{t_0}\omega) \tag{5.4}$$

Let

$$n(t,\omega) := \sup_{t_0\in\mathbb{R}} x(t,\theta_{t_0}\omega) = \sup_{t_0\in\mathbb{Q}} x(t,\theta_{t_0}\omega).$$

By (5.4), we have that $-\infty < x(t,\omega) \le n(t,\omega) \le N(t,\omega) < \infty$. Hence $\Omega_0 \ni \omega \mapsto m(t,\omega) \in \mathbb{R}^+$ is measurable and $m(t,\theta_\tau\omega) = m(t,\omega)$ for any $\tau \in \mathbb{R}$. By the ergodicity of the metric dynamical system $(\Omega_0, \mathcal{F}, \{\theta_t\}_{t\in\mathbb{R}})$, we have that $m(t,\omega) = m(t)$ for $a.e$ in ω. □

Let $x_+(t,\omega,\mu)$ be such that

$$u(t,x+x_+(t,\omega,\mu)+C(t,\omega,\mu);\phi^\mu_+(0,\cdot;\omega),\omega) = \frac{1}{2}.$$

Lemma 5.3 *For any $t > 0$, there holds*

$$u(t,x+x(t;\omega);u^*_0,\omega)) \begin{cases} \ge u(t,x+x_+(t,\omega,\mu)+C(t,\omega,\mu);\phi^\mu_+(0,\cdot;\omega),\omega) & x<0 \\ \le u(t,x+x_+(t,\omega,\mu)+C(t,\omega,\mu);\phi^\mu_+(0,\cdot;\omega),\omega) & x>0. \end{cases} \tag{5.5}$$

Proof It follows from Lemma 2.7. □

Lemma 5.4 *There is $\hat{M} > 0$ such that*

$$x(t,\omega) + x(s,\theta_t\omega) \le x(t+s,\omega) + \hat{M}$$

for all $t, s \ge 0$ and a.e. $\omega \in \Omega$.

Proof First, let $\tilde{x}(t,\omega)$ and $\tilde{x}_+(t,\omega)$ be such that

$$u(t,\tilde{x}(t,\omega);u^*_0,\omega) = \frac{1}{4} \quad \text{and} \quad u(t,\tilde{x}_+(t,\omega,\mu)+C(t,\omega,\mu);\phi^\mu_+(0,\cdot;\omega),\omega) = \frac{1}{4},$$

respectively. Since the function $x \mapsto u(t,x;u_0,\omega)$ is decreasing, we have

$$\tilde{x}(t,\omega) > x(t,\omega). \tag{5.6}$$

Moreover, for each $t > 0$, $\tilde{x}(t,\omega)$ is measurable in ω, and for each $\omega \in \Omega$, $\tilde{x}(t,\theta_{t_0}\omega)$ is continuous in $(t,t_0) \in (0,\infty)\times\mathbb{R}$. By Lemma 5.3,

$$\begin{aligned}\tilde{x}(t,\omega) - x(t,\omega) &\le (\tilde{x}_+(t,\omega,\mu) - C(t,\omega,\mu)) - (x_+(t,\omega,\mu) - C(t,\omega,\mu))\\ &= \tilde{x}_+(t,\omega,\mu) - x_+(t,\omega,\mu),\ \ \forall t > 0. \end{aligned} \tag{5.7}$$

Let

$$M(\omega) = \sup_{t>0,t_0\in\mathbb{R}} \big(\tilde{x}(t,\theta_{t_0}\omega) - x(t,\theta_{t_0}\omega)\big) = \sup_{t\in(0,\infty)\cap\mathbb{Q},t_0\in\mathbb{Q}} \big(\tilde{x}(t,\theta_{t_0}\omega) - x(t,\theta_{t_0}\omega)\big).$$

Note that

$$\frac{1}{2} = u(t, x_+(t,\theta_{t_0}\omega,\mu) + C(t,\theta_{t_0}\omega,\mu); \phi_+^{\mu}(\cdot,\cdot;\theta_{t_0}\omega), \theta_{t_0}\omega), \quad \forall\, t > 0, \forall\, t_0 \in \mathbb{R},$$

and

$$\frac{1}{4} = u(t, \tilde{x}_+(t,\theta_{t_0}\omega,\mu) + C(t,\theta_{t_0}\omega,\mu); \phi_+^{\mu}(\cdot,\cdot;\theta_{t_0}\omega), \theta_{t_0}\omega), \quad \forall\, t > 0, \forall\, t_0 \in \mathbb{R}.$$

By Lemma 5.1, there is a positive constant $K(\omega)$ such that

$$|x_+(t,\theta_{t_0}\omega,\mu)| \le K(\omega) \quad \text{and} \quad |\tilde{x}_+(t,\theta_{t_0}\omega,\mu)| \le K(\omega), \quad \forall\, t > 0, \forall\, t_0 \in \mathbb{R}. \tag{5.8}$$

This combined with (5.7) implies that $M(\omega) < \infty$.

Note that the function $\Omega_0 \ni \omega \mapsto M(\omega) \in \mathbb{R}^+$ is measurable and invariant. By the ergodicity of the metric dynamical system $(\Omega_0, \mathcal{F}, \{\theta_t\}_{t\in\mathbb{R}})$, we have that there are an invariant measurable set $\tilde{\Omega}$ with $\mathbb{P}(\tilde{\Omega}) = 1$ and a positive constant $\hat{M}$ such that

$$M(\omega) = \hat{M}, \quad \forall\, \omega \in \tilde{\Omega}. \tag{5.9}$$

Second, note that

$$u_0^*(x) \le 2u(t, x + x(t,\omega); u_0^*, \omega)$$

Hence,

$$\begin{aligned} u(s,x;u_0^*,\theta_t\omega) &\le u(s,x; 2u(t,\cdot + x(t,\omega); u_0^*,\omega), \theta_t\omega)\\ &\le 2u(s,x; u(t,\cdot + x(t,\omega); u_0^*,\omega), \theta_t\omega)\\ &= 2u(s, x + x(t,\omega); u_0^*, \omega). \end{aligned}$$

This implies that

$$u(s, x(s,\theta_t\omega) + x(t,\omega); u_0^*, \omega) \ge \frac{1}{4}.$$

It then follows from (5.9) that

$$x(s,\theta_t\omega) + x(t,\omega) \le \tilde{x}(t+s,\omega) \le x(t+s,\omega) + \hat{M}.$$

The lemma follows. □

We now prove Theorem 1.4.

Proof of Theorem 1.4 (i) We first prove that there is c^* such that (1.10) holds with $\hat{c}^*$ being replaced by c^*. To this end, let $y(t,\omega) = -x(t,\omega) + \hat{M}$ where $\hat{M}$ is given by Lemma 5.4. Then, by Lemma 5.4

$$y(t+s,\omega) = -x(t+s,\omega) + \hat{M} \le -x(t,\omega) - x(s,\theta_t\omega) + 2\hat{M} = y(t,\omega) + y(s,\theta_t\omega)$$

a.e in ω. By Lemma 5.2, $y(t,\cdot) \in L^1(\Omega)$. It then follows from the subadditive ergodic theorem that there is $c^* \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{y(t,\omega)}{t} = c^* \quad \text{for } a.e.\ \omega \in \Omega.$$

Next, we claim that (1.11) and (1.12) hold with $\hat{c}^*$ being replaced by c^*. In fact, by (5.5), (5.8), and Lemma 5.1,

$$\begin{aligned}0 \le \sup_{x\ge(c^*+h)t} u(t,x;u_0^*,\omega) \le & u(t,(c^*+h)t;u_0*,\omega)\\ & \le u(t,(c^*+h)t - x(t;\omega) + x_+(t,\omega,\mu)\\ & \quad + C(t,\omega,\mu);\phi_+^{\mu}(0,\cdot;\omega),\omega)\\ & \to 0 \ \text{ as } \ t\to\infty,\ \forall h>0,\end{aligned}$$

and

$$\begin{aligned}1 \ge \inf_{x\le(c^*-h)t} u(t,x;u_0,\omega) \ge & u(t,(c^*-h)t;u_0,\omega)\\ & \ge u(t,(c^*-h)t - x(t;\omega)\\ & \quad + x_+(t,\omega,\mu) + C(t,\omega,\mu);\phi_+^{\mu}(0,\cdot;\omega),\omega)\\ & \to 1 \ \text{ as } \ t\to\infty,\ \forall h>0.\end{aligned}$$

Therefore, (1.11) and (1.12) hold with $\hat{c}^*$ being replaced by c^*.

Now, we prove that $c^* = \hat{c}^*$. By the comparison principle for parabolic equations,

$$u(t,x;u_0^*,\omega) \le e^{-\mu(x-\frac{1}{\mu}\int_0^t(\mu^2+a(\theta_\tau\omega)d\tau))}, \forall\, t,\mu>0, \forall\, x\in\mathbb{R}.$$

Hence

$$\frac{1}{2} \le e^{-\mu(x(t,\omega)-\frac{1}{\mu}\int_0^t(\mu^2+a(\theta_\tau\omega)d\tau))},\ \forall\, t,\mu>0.$$

This implies that

$$\frac{x(t,\omega)}{t} - \frac{\ln(2)}{t\mu} \le \frac{1}{t\mu}\int_0^t (\mu^2 + a(\theta_\tau\omega)d\tau).$$

Letting $t\to\infty$, we obtain that

$$c^* \le \frac{\mu^2+\hat{a}}{\mu}, \quad \forall\, \mu>0.$$

Taking $\mu = \sqrt{\hat{a}}$, we obtain that

$$c^* \le \hat{c}^* = 2\sqrt{\hat{a}}.$$

It then remains to prove that

$$c^* \ge \hat{c}^* = 2\sqrt{\hat{a}}.$$

We prove this by contradiction.

Assume that $c^* < \hat{c}^* = 2\sqrt{\hat{a}}$. Then there are $h>0$ and $0<\delta<1$ such that

$$c^* < c := c^* + h < 2\sqrt{\delta\hat{a}}.$$

By (1.11), for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} \sup_{x\ge ct} u(t,x;u_0^*,\omega) = 0.$$

Fix such ω. There are $0 < \delta^{'} < 1$ and $T > 0$ such that

$$\delta^{'} \frac{1}{t}\int_0^t a(\theta_\tau\omega)d\tau > \delta\hat{a}$$

and

$$u(t,x;u_0^*,\omega) \le 1-\delta^{'} \quad \forall\, t \ge T,\ \ x \ge ct.$$

As in the proof of Theorem 1.2(i), let $L = \frac{2\pi}{\sqrt{4\hat{a}\delta - c^2}}$ and

$$w^+(x) = e^{-\frac{c}{2}x}\sin\Big(\frac{\sqrt{4\hat{a}\delta - c^2}}{2}x\Big).$$

By the similar arguments as those in Theorem 1.2(i), we have

$$\begin{aligned} u(t,x+ct;u_0^*,\omega) &\ge \alpha e^{\int_T^t(\delta^{'} a(\theta_\tau\omega)-\delta\hat{a})d\tau} w^+(x)\\ &= \alpha e^{-\int_0^T(\delta^{'} a(\theta_\tau\omega)-\delta\hat{a})d\tau} e^{\int_0^t(\delta^{'} a(\theta_\tau\omega)-\delta\hat{a})d\tau} w^+(x)\\ &\ge \alpha e^{-\int_0^T(\delta^{'} a(\theta_\tau\omega)-\delta\hat{a})d\tau} w^+(x) \end{aligned}$$

for $0 \le x \le L$ and $t \ge T$, where $\alpha = \sup_{0\le x\le L} u(T,x+cT;u_0^*,\omega)$. This implies that

$$\lim_{t\to\infty} \sup_{x\ge ct} u(t,x;u_0^*,\omega) > 0,$$

which is a contradiction. Hence $c^* = \hat{c}^* = 2\sqrt{\hat{a}}$.

(ii) For any given $u_0 \in \tilde{X}_c^+$, there are $0 < \alpha \le 1 \le \beta$ and $x_- < x_+$ such that

$$\alpha u_0^*(x+x_+) \le u_0(x) \le \beta u_0^*(x+x_-) \quad \forall\ x \in \mathbb{R}.$$

By the comparison principle for parabolic equations, we have

$$\alpha u(t,x;u_0^*(\cdot+x_+),\omega) \le u(t,x;u_0,\omega) \le \beta u(t,x;u_0^*(\cdot+x_-),\omega) \quad \forall t \ge 0,\ \ x \in \mathbb{R}.$$

This together with (1.11) implies that there is a measurable set $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that

$$\lim_{t\to\infty} \sup_{x\ge(\hat{c}^*+h)t} u(t,x;u_0,\omega) = 0, \quad \omega \in \Omega_1,\ \forall\, h > 0,$$

and

$$\liminf_{t\to\infty} \inf_{x\le(\hat{c}^*-h)t} u(t,x;u_0,\omega) \ge \alpha, \qquad \omega \in \Omega_1,\ \forall\, h > 0. \tag{5.10}$$

We claim that

$$\liminf_{t\to\infty} \inf_{x\le(c^*-h)t} u(t,x;u_0,\omega) = 1 \quad \text{for } \ \omega \in \Omega_1,\ \forall\, h > 0. \tag{5.11}$$

Indeed, let $\omega \in \Omega_1$ and $h > 0$ be fixed. Let $\{x_n\}$ and $\{t_n\}$ with $t_n \to \infty$ and $x_n \le (\hat{c}^* - h)t_n$ be such that

$$\liminf_{t\to\infty} \inf_{x\le(\hat{c}^*-h)t} u(t,x;u_0,\omega) = \lim_{n\to\infty} u(t_n,x_n;u_0,\omega). \tag{5.12}$$

For every $0 < \varepsilon \ll \frac{1}{2}$, Theorem 1.1 implies that there is $T_\varepsilon > 0$ such that

$$1 - \varepsilon \le u(t, x; \frac{\alpha}{2}, \theta_s\omega), \quad \forall\, x \in \mathbb{R},\ s \in \mathbb{R},\ t \ge T_\varepsilon. \tag{5.13}$$

Consider a sequence of $u_{0n} \in C^b_{\rm unif}(\mathbb{R})$ satisfying that

$$u_{0n}(x) = \begin{cases} \frac{\alpha}{2}, & x \le \frac{1}{2}ht_n - 2(\hat{c}^* - \frac{1}{2}h)T_\varepsilon \\ 0, & x \ge \frac{1}{2}ht_n - (\hat{c}^* - \frac{1}{2}h)T_\varepsilon. \end{cases}$$

Note that

$$x \le \frac{1}{2}ht_n - \left(\hat{c}^* - \frac{1}{2}h\right)T_\varepsilon \Rightarrow x + x_n \le \left(\hat{c}^* - \frac{1}{2}h)(t_n - T_\varepsilon\right).$$

By (5.10), there is $N_1 \gg 1$ such that

$$u(t_n - T_\varepsilon, x + x_n; u_0, \omega) \ge u_{0n}(x), \quad \forall\, x \in \mathbb{R}, n \ge N_1.$$

By the comparison principle for parabolic equations, we then have that

$$u(t + t_n - T_\varepsilon, x + x_n; u_0, \omega) \ge u(t, x; u_{0n}, \theta_{t_n - T_\varepsilon}\omega), \quad \forall x \in \mathbb{R}, \forall\, t \ge 0.$$

In particular, taking $t = T_\varepsilon$ and $x = 0$, we obtain

$$u(t_n, x_n; u_0, \omega) \ge u(T_\varepsilon, x; u_{0n}, \theta_{t_n - T_\varepsilon}\omega). \tag{5.14}$$

Note that $u_{0n}(x) \to \frac{\alpha}{2}$ as $n \to \infty$. Letting $t \to \infty$ in (5.14), it follows from (5.13) and Lemma 3.1 that

$$\lim_{n\to\infty} u(t_n, x_n; u_0, \omega) \ge 1 - \varepsilon.$$

Letting $\varepsilon \to 0$ in the last inequality, it follows from (5.12) that

$$\liminf_{t\to\infty} \inf_{x \le (\hat{c}^* - h)t} u(t, x; u_0, \omega) \ge 1, \quad \text{for } \omega \in \Omega_1,\ \forall\, h > 0.$$

It is clear that

$$\liminf_{t\to\infty} \inf_{x \le (\hat{c}^* - h)t} u(t, x; u_0, \omega) \le 1, \quad \text{for } \omega \in \Omega_1,\ \forall\, h > 0.$$

The Claim thus follows and (ii) is proved. □

The following corollary follows directly from Lemma 3.2 and Theorem 1.4.

Corollary 5.1 *Assume* **(H3)**. *Let $Y(\omega)$ be the random equilibrium solution of* (1.21) *given in* (1.22) *and let $U_0^*(x;\omega) = Y(\omega)$ for $x < 0$ and $U_0^*(x;\omega) = 0$ for $x > 0$. Then,*

$$\lim_{t\to\infty} \frac{X(t,\omega)}{t} = 2 \quad \text{for a.e } \omega \in \Omega,$$

where $X(t,\omega)$ is such that $u(t, X(t,\omega); U_0^*(\cdot;\omega), \omega) = \frac{1}{2}Y(\omega)$, and

$$\lim_{t\to\infty} \sup_{x \ge (2+h)t} \frac{u(t, x; U_0^*(\cdot;\omega), \omega)}{Y(\theta_t\omega)} = 0, \quad \forall\, h > 0,\ a.e\ \omega \in \Omega,$$

and

$$\lim_{t\to\infty} \inf_{x \le (2-h)t} \frac{u(t, x; U_0^*(\cdot;\omega), \omega)}{Y(\theta_t\omega)} = 1, \quad \forall\, h > 0,\ a.e\ \omega \in \Omega,$$

where $u(t, x; U_0^*(\cdot;\omega), \omega)$ is the solution of (1.21) *with $u(0, x; U_0^*(\cdot;\omega), \omega) = U_0^*(x;\omega)$.*

6 Spreading Speeds of Nonautonomous Fisher–KPP Equations

In this section we consider the nonautonomous Fisher–KPP equation (1.2) and prove Theorem 1.5.

Proof of Theorem 1.5 First, we prove (1.17). To this end, for given $0 < c < 2\sqrt{\underline{a}_0}$, choose $b > c$ and $0 < \delta < 1$ such that $c < 2\sqrt{b} < 2\sqrt{\delta \underline{a}_0}$. By the proof of Lemma 2.2, there are $\{t_k\}_{k\in\mathbb{Z}}$ with $t_k < t_{k+1}$, $t_k \to \pm\infty$ as $k \to \pm\infty$ and $A \in W^{1,\infty}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $A \in C^1(t_k, t_{k+1})$ for every k and

$$b \le \delta a_0(t) - A'(t), \quad \text{for } t \in (t_k, t_{k+1}),\ k \in \mathbb{Z}.$$

Let $\sigma = \frac{(1-\delta)e^{-\|A\|_\infty}}{\|u_0\|_\infty + 1}$ and $v(t, x; b)$ be the solution of the PDE

$$\begin{cases} v_t = v_{xx} + bv(1-v), & x \in \mathbb{R}, t > 0,\\ v(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

By Lemma 4.2, we have that

$$\liminf_{t\to\infty} \min_{|x|\le ct} v(t, x; b) = 1. \tag{6.1}$$

For given $s \in \mathbb{R}$, let $\tilde{v}(t, x; s) = \sigma e^{A(t+s)} v(t, x; b)$. By the similar arguments to those in Lemma 4.3, it can be proved that

$$\sigma e^{-\|A\|_\infty} v(t, x, b) \le \tilde{v}(t, x; s) \le u(t, x; u_0, \sigma_s a_0), \quad \forall\, x \in \mathbb{R},\ s \in \mathbb{R},\ t \ge 0.$$

This combined with (6.1) yields that

$$0 < \sigma e^{-\|A\|_\infty} \le \liminf_{t\to\infty} \inf_{s\in\mathbb{R}, |x|\le ct} u(t, x; u_0, \sigma_s a_0), \quad \forall\, 0 < c < 2\sqrt{\underline{a}_0}.$$

By the arguments in Lemma 4.1, it can be proved that

$$\lim_{t\to\infty} \inf_{s\in\mathbb{R}, |x|\le ct} |u(t, x; u_0, \sigma_s a_0) - 1| = 0, \quad \forall u_0 \in X_c^+,\ \forall 0 < c < 2\sqrt{\underline{a}_0}.$$

(1.17) then follows.

Next, we prove (1.18). To this end, for any given $u_0 \in X_c^+$, suppose that $\text{supp}(u_0) \subset (-R, R)$. For every $\mu > 0$, let $C_\mu(t, s) = \int_s^{s+t} \frac{\mu^2 + a_0(\tau\omega)}{\mu} d\tau$ and $\phi^\mu(x) = \|u_0\|_\infty e^{-\mu(x-R)}$ and $\tilde{\phi}^\mu_\pm(t, x; s) = \phi^\mu_\pm(\pm x - C_\mu(t, s))$ for every $x \in \mathbb{R}$ and $t \ge 0$. It is not difficult to see that

$$\partial_t \tilde{\phi}^\mu_\pm - \partial_{xx}\tilde{\phi}^\mu_\pm - a_0(s+t)\tilde{\phi}^\mu_\pm(1 - \tilde{\phi}^\mu_\pm) = a_0(s+t)\left(\tilde{\phi}^\mu_\pm\right)^2 \ge 0,\ x \in \mathbb{R},\ t > 0,$$

and

$$u_0(x) \le \tilde{\phi}^\mu_\pm(0, x; s), \quad \forall x \in \mathbb{R},\ \forall\, s \in \mathbb{R}.$$

By the comparison principle for parabolic equations, we then have that

$$u(t, x; u_0, \sigma_s a_0) \le \tilde{\phi}^\mu_\pm(t, x; s) = \|u_0\|_\infty e^{-\mu(\pm x - R \mp C_\mu(t,s))}, \quad \forall\, x, s \in \mathbb{R}, \forall t > 0, \forall \mu > 0.$$

This implies that

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R}, |x|\ge ct} u(t, x; u_0, \sigma_s a_0) = 0 \quad \forall\, \mu > 0,\ c > \frac{\mu^2 + \bar{a}_0}{\mu}.$$

For any $c > 2\sqrt{\bar{a}_0} = \inf_{\mu>0} \frac{\mu^2+\sqrt{\bar{a}_0}}{\mu}$, choose $\mu > 0$ such that $c > \frac{\mu^2+\sqrt{\bar{a}_0}}{\mu} > \bar{c}^*$, we have

$$\limsup_{t\to\infty} \sup_{s\in\mathbb{R}, |x|\geq ct} u(t, x; u_0, \theta_s\omega) = 0.$$

(1.18) then follows. □

We conclude this section with some example of explicit function $a_0(t)$ satisfying **(H2)**. Define the sequences $\{l_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ inductively by

$$l_0 = 0, \quad L_n = l_n + \frac{1}{2^{2(n+1)}}, \quad l_{n+1} = L_n + n + 1, \quad n \geq 0. \tag{6.2}$$

Define $a_0(t)$ such that $a_0(-t) = a_0(t)$ for $t \in \mathbb{R}$ and

$$a_0(t) = \begin{cases} f_n(t) & \text{if } t \in [l_n, L_n] \\ g_n(t) & \text{if } t \in [L_n, l_{n+1}] \end{cases} \tag{6.3}$$

for $n \geq 0$, where $g_{2n}(t) = 1$ and $g_{2n+1}(t) = 2$ for $n \geq 0$, and $f_0(t) = 1$, for $n \geq 1$, f_n is Hölder's continuous on $[l_n, L_n]$, $f_n(l_n) = g_n(l_n)$, $f_n(L_n) = g_n(L_n)$, and satisfies

$$1 \leq f_{2n}(t) \leq 2^n, \quad \max_{t\in[l_{2n}, L_{2n}]} f_{2n}(t) = 2^n, \; f_{2n}(t)dt = \frac{1}{2^{n+3}}$$

and

$$\frac{1}{2^{n+1}} \leq f_{2n+1}(t) \leq 2, \quad \min_{t\in[l_{2n+1}, L_{2n+1}]} f_{2n+1}(t) = 2^{-(n+1)}.$$

It is clear that $a_0(t)$ is locally Hölder's continuous, $\inf_{t\in\mathbb{R}} a_0(t) = 0$, and $\sup_{t\in\mathbb{R}} a_0(t) = \infty$. Moreover, it can be verified that

$$\underline{a}_0 = 1 \quad \text{and} \quad \overline{a}_0 = 2.$$

Hence $a_0(t)$ satisfies **(H2)**.

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