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Existence of Attractors for a Nonlinear Timoshenko System with Delay

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Abstract
This paper deals with Timoshenko's classic model for beams vibrations. Regarding the linear model of Timoshenko, there are several known results on exponential decay, controllability and numerical approximation, but there are few results that deal with the nonlinear case or even the linear case with delay type damping. In this paper, we will establish the existence of global and exponential attractors for a semilinear Timoshenko system with delay in the rotation angle equation and a friction-type damping in the transverse displacement equation. Since the damping acts on the two equations of the system, we should not assume the well-known velocity equality.

Keywords Nonlinear Timoshenko system · Time delay · Quasi-stability · Global attractor · Long-time dynamics

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1 Introduction

In 1921 Stephen Timoshenko [1] described a model of beams which takes into account the effects of shear deformation on transverse vibrations. The model is based on D'Alembert's principle for the equilibrium dynamics [2], from which the following coupled equations of evolution

$$\rho A\varphi_{tt} - S_x = 0, \tag{1.1}$$

$$\rho I\psi_{tt} - M_x + S = 0, \tag{1.2}$$

with $S = k'AG(\varphi_x + \psi)$ denoting the transverse shear force and $M = E\psi_x$ the moment of flexion. The functions φ and ψ denote the vertical displacement of the beam centerline and the rotation of the vertical filament in the beam. The positive constants ρ, A, I, E, G, k' denote, respectively, the mass density of the material, the cross-sectional area, the moment of inertia of the cross section, the Young's modulus, the stiffness modulus and the shear factor. The Tymoshenko system is usually studied in coupled form

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \tag{1.3}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \tag{1.4}$$

where $\rho_1 = \rho A$, $\kappa = k'AG$, $\rho_2 = \rho I$ and $b = EI$, but can also be represented in the decoupled form [3],

$$\frac{\rho_1\rho_2}{\kappa}\varphi_{tttt} + \rho_1\varphi_{tt} - b\left(\frac{\rho_1}{\kappa} + \frac{\rho_2}{b}\right)\varphi_{xxtt} + b\varphi_{xxxx} = 0, \tag{1.5}$$

eliminating the variable ψ in both equations. It is important to note that in the (1.3)–(1.4) the curvature effects, vertical displacement, shear deformation and rotational inertia are present.

One of the first works to study the stabilization of the Timoshenko system belongs to Kim and Renardy [4]. They considered the Eqs. (1.3)–(1.4) with boundary conditions

$$\varphi(0, t) = \psi(0, t) = 0, \tag{1.6}$$

$$\kappa\big(\varphi_x(L, t) + \psi(L, t)\big) + \alpha\varphi_t(L, t) = 0, \tag{1.7}$$

$$b\psi_x(L, t) + \beta\psi_t(L, t) = 0, \tag{1.8}$$

and showed that the system can be uniformly stabilized by means of control (1.6) in the boundary. In addition, a numerical study on the spectrum is presented. On the other hand, in the case of internal control, Soufyane [5] considered the system of Tymoshenko

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.9}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + b(x)\psi_t = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.10}$$

with the variable coefficient satisfying the relation $0 < b_0 \leq b(x) \leq b_1$ and homogeneous Dirichlet boundary conditions. He proved that the system decays exponentially if and only if the equality between the velocities

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}, \tag{1.11}$$

is satisfied. In [6], Soufyane and Wehbe considered again the same system, but with damping located satisfying $0 < \bar{b} \leq b(x)$ in $[b_0, b_1] \subset [0, L]$ and once again proved that the equality relationship between velocities is a necessary and sufficient condition to establish the exponential decay of the system. In [7], Rivera and Racke have proven a similar result to that of Soufyane [5], where the damping function $b(x) \in L^\infty(0, L)$ can change signal, but must satisfy the conditions $\bar{a} = (1/L) \int_0^L b(x)dx > 0$ and $||a - \bar{a}||_{L^2} < \epsilon$ with ϵ small enough. Since then, several studies have emerged considering damping in a single equation, among them we can cite [8–10], in all of them, the relationship was used (1.11).

Many researchers [11–16] have studied the asymptotic behavior of the Timoshenko system under the action of several dissipative mechanisms. Among them, we highlight the dynamics of systems with a time delay [17,18], quite widespread in the 1970s. The effects of time lag are in many cases a source of instability, however for some systems, the presence of delay may have a stabilizing effect. For example, in the wave equation the time delay in the feedback term (internal or at the boundary) can destabilize the system, depending on the weight of each term [19,20]. In this context, Said-Houari and Laskri [21] studied the stability of the system

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0 \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.12}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(\cdot, t - \tau) = 0 \quad \text{in} \quad (0, 1) \times (0, \infty). \tag{1.13}$$

They have proved that if $\mu_2 \leq \mu_1$ and the relationship (1.11) is satisfied, then the system is exponentially stable. On the other hand, Feng and Yang [22] based on [21] obtained the existence of global attractors with finite fractal dimension as well as the existence of exponential attractor for the following nonlinear system of Timoshenko with delay

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = h \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.14}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \mu_1\psi_t + \mu_2\psi_t(\cdot, t - \tau) + f(\psi) = g \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.15}$$

under the hypothesis $\mu_2 \leq \mu_1$ and (1.11), where $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq k_0(|x|^\theta + |y|^\theta)|x - y|, \quad \forall x, y \in \mathbb{R}, \tag{1.16}$$

with $k_0, \theta > 0$ and

$$-k_1 \leq \hat{f}(x) \leq f(x)x, \quad \forall x \in \mathbb{R}, \quad \hat{f}(y) = \int_0^y f(s)ds, \quad \text{for some} \quad k_1 > 0. \tag{1.17}$$

In [23] Fatori et al. studied the following nonlinear Timoshenko system

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + f_1(\varphi, \psi) + \varphi_t = h_1 \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.18}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + f_2(\varphi, \psi) + \psi_t = h_2 \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.19}$$

where f_1 and f_2 are nonlinear source terms representing the elastic foundation and h_1, h_2 are external forces. They obtained the existence of global and exponential attractor and without assuming the well-known equal wave speeds condition (1.11).

There are also works that consider localized nonlinear damping. For example, in [24] Guesmia and Messaoudi studied the system

$$\rho_1\varphi_{tt} - \kappa_1(\varphi_x + \psi)_x = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.20}$$

$$\rho_2\psi_{tt} - \kappa_2\psi_{xx} + \int_0^t g(t-\tau)\big(a(x)\psi_x(\tau)\big)_x d\tau + \kappa_1(\varphi_x + \psi) + b(x)h(\psi_t) = 0$$
$$\text{in} \quad (0, L) \times (0, \infty), \tag{1.21}$$

with Dirichlet boundary conditions and initial data where a, b, g and h are specific functions and $\rho_1, \rho_2, \kappa_1, \kappa_2$ and L are given positive constants. They establish a general stability estimate using the multiplier method and some properties of convex functions. Without imposing any growth condition on h at the origin, they showed that the energy of the system is bounded above by a quantity, depending on g and h, which tends to zero as time goes to infinity. In [25] Cavalcanti et al. considered the system

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \alpha_1(x)g_1(\varphi_t) = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.22}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \alpha_2(x)g_2(\psi_t) = 0 \quad \text{in} \quad (0, L) \times (0, \infty), \tag{1.23}$$

where the functions α_1 and α_2 are supposed to be smooth and nonnegative, while the nonlinear functions g_1 and g_2 are continuous, monotone increasing, and zero at the origin. Using a method introduced in Daloutli et al. [26], they proved that the damping placed on an arbitrarily small support, unquantitized at the origin and without assuming equal speeds of propagation of waves, leads to uniform decay rates (asymptotic in time) for the energy function.

A common point in both localized damping papers is that they do not take into account the presence of external forces or the action of nonlinear source terms. According to Malatkar [27], nonlinear terms arise in a system whenever there are products of dependent variables and their derivatives in equations of motion, boundary conditions, and / or constitutive laws, and whenever there is any kind of discontinuity or jump in the system. In the literature (e.g. [28–30]) we find various types of nonlinearities, such as damping nonlinearities, geometric nonlinearities (caused by large deformations) and others.

In [31], Zhong and Guo, studied a Timoshenko beam systems, taking into account various nonlinear effects. But precisely, they considered Hooke's law and the nonlinear relations given by

$$\varepsilon := u_x + \frac{1}{2}w_x^2, \quad \kappa := \frac{\theta_x}{\sqrt{1+w_x^2}} \approx \theta_x\left(1 - \frac{1}{2}w_x^2 + \frac{3}{8}w_x^4\right) \text{ and}$$
$$\gamma := \tan^{-1} w_x - \theta \approx w_x - \frac{1}{3}w_x^3 - \theta, \tag{1.24}$$

where $u, w, \theta, \varepsilon, \kappa$ and γ represent the axial displacement, the deflection, the cross-section rotation, the membrane strain, the bending curvature, and the shear strain, respectively. Keeping in mind the Lagrangian function $L = K - U$, with

$$K := \frac{1}{2}\int_0^L \rho A w_t^2 dx + \frac{1}{2}\int_0^L \rho I \theta_t^2 dx \quad \text{and} \quad U := \frac{1}{2}\int_0^L \left(EA\varepsilon^2 + EI\kappa^2 + kGA\gamma^2\right)dx, \tag{1.25}$$

and applying the Hamilton Principle

$$\delta\int_{t_1}^{t_2} L\, dt = 0, \tag{1.26}$$

the authors obtained the Timoshenko system given by

$$\rho A w_{tt} - kGA(w_x - \theta)_x + f_1(w, \theta) = 0, \tag{1.27}$$

$$\rho I \theta_{tt} - EI\theta_{xx} - kGA(w_x - \theta) + f_2(w, \theta) = 0, \tag{1.28}$$

with nonlinear source terms

$$f_1(w, \theta) := EI\frac{\partial}{\partial x}\Big[\theta_x^2(w_x - 2w_x^2)\Big] + \frac{1}{3}kGA\frac{\partial}{\partial x}\Big[w_x^3 + 3(w_x - \theta)w_x^2\Big], \tag{1.29}$$

$$f_2(w, \theta) := EI\frac{\partial}{\partial x}\Big[(w_x^2 - w_x^4)\theta_x^2\Big] + \frac{1}{3}kGAw_x^3. \tag{1.30}$$

The interesting thing about studying such models is that many physical phenomena such as jumps, saturation, subharmonic, superharmonic, combination resonances, self-excited oscillations, modal interactions and chaos are present only in nonlinear systems. In fact, no physical system is strictly linear and linear constraints are only applied to very small amplitude vibrations [27]. Therefore, to accurately study and understand the dynamic behavior of structural systems under general loading conditions, it is essential that we consider more general source terms $f_i(\cdot, \cdot)$ $(i = 1, 2)$, given due to its intrinsic mathematical properties.

In this paper we consider the following nonlinear Timoshenko system with delay term

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + f_1(\varphi, \psi) + \varphi_t = h_1 \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.31}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + f_2(\varphi, \psi) + \mu_1\psi_t + \mu_2\psi_t(\cdot, t - \tau) = h_2 \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.32}$$

where, $\varphi = \varphi(x, t)$ and $\psi = \psi(x, t)$, represent the transverse displacement and the is the rotation angle of the filament of beam, respectively. Positive constants ρ_1, ρ_2, κ and b represent physical properties of the beam material and τ is time delay. The functions $f_1(\varphi, \psi)$ and $f_2(\varphi, \psi)$ are nonlinear source terms, whereas h_1 and h_2 represent external forces. This system is subjected to the following initial conditions

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x),\ \psi(x, 0) = \psi_0(x),\ \psi_t(x, 0) = \psi_1(x), \quad x \in (0, 1), \tag{1.33}$$

and boundary conditions

$$\varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \quad t \geq 0. \tag{1.34}$$

We believe that this work is the first to study the dynamics of attractors in the system (1.31)–(1.34) and our main results refer to the existence of global and exponential attractors without the need for equality (1.11) between speeds.

The plan of this paper is as follows: In Sect. 2, we present our assumptions and state the results on existence and global well-posedness to the system (1.31)–(1.34). In Sect. 3 we consider the corresponding dynamical system and state our main result concerning long-time dynamics and without assuming the equal wave speeds condition. Finally, Sect. 4 is dedicated to prove the existence of exponential attractor with finite fractal dimension in the generalized space.

2 Well-Posedness

In order to obtain the well-posedness of the problem (1.31)–(1.34), consider the following change of variable as found in [21,32],

$$z(x, y, t) = \psi_t(x, t - \tau y) \text{ in } (0, 1) \times (0, 1) \times (0, \infty), \tag{2.1}$$

so we have readily

$$\tau z_t(x, y, t) + z_y(x, y, t) = 0 \text{ in } (0, 1) \times (0, 1) \times (0, \infty). \tag{2.2}$$

Therefore, the system (1.31)–(1.34) takes the following form

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + f_1(\varphi, \psi) + \varphi_t = h_1 \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{2.3}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + f_2(\varphi, \psi) + \mu_1\psi_t + \mu_2 z(x, 1, t) = h_2 \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{2.4}$$

$$\tau z_t(x, y, t) + z_y(x, y, t) = 0 \quad \text{in} \quad (0, 1) \times (0, 1) \times (0, \infty), \tag{2.5}$$

with initial conditions

$$\varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x),\ \psi(x, 0) = \psi_0(x),\ \psi_t(x, 0) = \psi_1(x), \quad x \in (0, 1), \tag{2.6}$$

$$z(x, y, 0) = f_0(x, -\tau y), \quad (x, y) \in (0, 1) \times (0, 1), \tag{2.7}$$

and boundary conditions

$$\varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \quad t > 0, \tag{2.8}$$

$$z(x, 0, t) = \psi_t(x, t), \quad x \in (0, 1), \quad t > 0. \tag{2.9}$$

From now on, we will use notation z_1 to represent z when $y = 1$. In this way, we can write the system (2.3)–(2.9) in the form of an abstract nonlinear initial value problem in the unknown $U(t) = (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t), z_1(t))^T$

$$U_t(t) = \mathcal{A}U(t) + \mathcal{F}(U(t)), \quad t > 0, \tag{2.10}$$

$$U(0) = U_0 \in \mathcal{H}, \tag{2.11}$$

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, f_0(\cdot, -\tau))$, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ given by

$$\mathcal{A}W = \begin{pmatrix} u \\ \frac{\kappa}{\rho_1}(\varphi_x + \psi)_x - \frac{1}{\rho_1}u \\ v \\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi) - \frac{\mu_1}{\rho_2}v - \frac{\mu_2}{\rho_2}z_1 \\ -\frac{1}{\tau}z_y \end{pmatrix} \quad \text{and}$$

$$\mathcal{F}(W) = \begin{pmatrix} 0 \\ \frac{1}{\rho_1}[h_1 - f_1(\varphi, \psi)] \\ 0 \\ \frac{1}{\rho_2}[h_2 - f_2(\varphi, \psi)] \\ 0 \end{pmatrix}, \tag{2.12}$$

with domain

$$D(\mathcal{A}) := \Big\{W = (\varphi, u, \psi, v, z) \in \mathscr{H};\ v = z(\cdot, 0) \text{ in } (0, 1)\Big\}, \tag{2.13}$$

where

$$\mathscr{H} := (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \times L^2(0,1; H_0^1(0,1)). \tag{2.14}$$

The energy space is given by

$$\mathcal{H} := H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2\big((0,1)\times(0,1)\big). \tag{2.15}$$

We consider ξ a positive constat satisfying

$$\tau\mu_2 \le \xi \le \tau(2\mu_1 - \mu_2). \tag{2.16}$$

We define in $\mathcal{H}$ the following inner produto and norm

$$\left(W, \hat{W}\right)_{\mathcal{H}} = \rho_1(u,\hat{u}) + \rho_2(v,\hat{v}) + b(\psi_x,\hat{\psi}_x) + \kappa(\varphi_x + \psi, \hat{\varphi}_x + \hat{\psi}) + \xi \int_0^1 \int_0^1 z(x,y)\hat{z}(x,y)dxdy, \tag{2.17}$$

$$\|W\|_{\mathcal{H}}^2 = \rho_1\|u\|_2^2 + \rho_2\|v\|_2^2 + b\|\psi_x\|_2^2 + \kappa\|\varphi_x + \psi\|_2^2 + \xi \int_0^1 \int_0^1 z^2(x,y)dxdy, \tag{2.18}$$

for any $W = (\varphi, u, \psi, v, z)$ and $\hat{W} = (\hat{\varphi}, \hat{u}, \hat{\psi}, \hat{v}, \hat{z})$ in $\mathcal{H}$, where $(\cdot,\cdot)$ and $\|\cdot\|_2$ are inner product and norm in $L^2(0,1)$, respectively.

2.1 Existence and Uniqueness

The question of the existence and uniqueness of the solution of problem (2.11) will be considered in this subsection. Firstly, let us remember the following concepts:

- A function $U : [0,T) \to \mathcal{H}$, with $T > 0$, is a *strong solution* of (2.11), if U is continuous on $[0,T)$, continuously differentiable on $(0,T)$, with $U(t) \in D(\mathcal{A})$ for all $t \in (0,T)$ and satisfies (2.11) on $[0,T)$ almost everywhere.
- A function $U \in C([0,T), \mathcal{H})$, $T > 0$, satisfying the integral equation

$$U(t) = e^{\mathcal{A}t}U_0 + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{F}(U(s))ds, \quad t \in [0,T), \tag{2.19}$$

  is called a *mild solution* of initial value problem (2.11).

In order to obtain well-posedness, consider the following assumptions on f_i and h_i for $i = 1, 2$.

(A1) $h_i \in L^2(0,1)$;
(A2) $f_i : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz continuous on each of its arguments, namely, there exist a constant $\gamma_i \ge 1$ and a continuous function $\sigma_i : \mathbb{R} \to \mathbb{R}_+$ such that

$$|f_i(s_1,r) - f_i(s_2,r)| \le \sigma_i(|r|)(1 + |s_1|^{\gamma_i} + |s_2|^{\gamma_i})|s_1 - s_2|, \tag{2.20}$$

$$|f_i(s,r_1) - f_i(s,r_2)| \le \sigma_i(|s|)(1 + |r_1|^{\gamma_i} + |r_2|^{\gamma_i})|r_1 - r_2|, \tag{2.21}$$

for every $(s_j, r), (s, r_j) \in \mathbb{R}^2$, $j = 1, 2$;

(A3) There is a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial s}(s,\cdot) = f_1(s,\cdot) \quad \text{and} \quad \frac{\partial F}{\partial r}(\cdot,r) = f_2(\cdot,r), \tag{2.22}$$

and

$$F(s,r) \geq -\theta_2 - \alpha_1|r|^2 - \theta_1|s|^2, \quad \forall (s,r) \in \mathbb{R}^2, \tag{2.23}$$

$$F(s,r) \leq f_1(s,r)s + f_2(s,r)r + \theta_1|s|^2 + \alpha_1|r|^2 + \theta_2, \quad \forall (s,r) \in \mathbb{R}^2, \tag{2.24}$$

where θ_i and α_i, with $i = 1, 2$, are constants satisfying

$$0 \leq \theta_1 \leq \min\left\{\frac{\kappa}{8}, \frac{b}{16}\right\}, \quad 0 \leq \alpha_1 \leq \frac{b}{4} \quad \text{and} \quad \theta_2, \alpha_2 \geq 0. \tag{2.25}$$

Remark 2.1 A simple example of $f_i(s,r)$ $(i = 1, 2)$ is given by

$$f_1(s,r) = 4(s+r)^3 - 2(s+r) + 2c_1sr^2 \quad \text{and}$$
$$f_2(s,r) = 4(s+r)^3 - 2(s+r) + 2c_1s^2r, \quad c_1 > 0, \tag{2.26}$$

the primitive function is

$$F(s,r) = |s+r|^4 - |s+r|^2 + c_1|sr|^2. \tag{2.27}$$

Lemma 2.1 *Assume that $\mu_2 \leq \mu_1$, then the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined in (2.12) is a infinitesimal generator of a C_0-semigroup of contractions in $\mathcal{H}$.*

Proof Following [21] it not so difficult to prove that $R(I - \mathcal{A}) = \mathcal{H}$, where $R(I - \mathcal{A})$ stands for range of the operator $I - \mathcal{A}$, and that $\mathcal{A}$ is dissipative operator in $\mathcal{H}$, namely, for all $W = (\varphi, u, \psi, v, z) \in D(\mathcal{A})$,

$$(\mathcal{A}W, W)_{\mathcal{H}} \leq -\|u\|_2^2 - \left(\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau}\right)\|v\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|z_1\|_2^2. \tag{2.28}$$

From (2.16), we have

$$\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau} \geq 0 \quad \text{and} \quad \frac{\xi}{2\tau} - \frac{\mu_2}{2} \geq 0, \tag{2.29}$$

which implies

$$(\mathcal{A}W, W)_{\mathcal{H}} \leq -\|u\|_2^2 - \left(\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau}\right)\|v\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|z_1\|_2^2 \leq 0. \tag{2.30}$$

Therefore, from Lumer–Phillips Theorem, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$. □

It is opportune now to define the functional energy $E(t)$ of a solution $U = (\varphi, \varphi_t, \psi, \psi_t, z)$ by the expression

$$E(t) := \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2 + \frac{\kappa}{2}\|\varphi_x + \psi\|_2^2 + \frac{\xi}{2}\int_0^1\int_0^1 z^2(x,y,t)dxdy$$
$$+ \int_0^1 F(\varphi,\psi)dx - \int_0^1 h_1\varphi dx - \int_0^1 h_2\psi dx, \quad \forall t \geq 0. \tag{2.31}$$

The prove of the following Lemma can be found in [23].

Lemma 2.2 *Suppose that **(A1)** and **(A2)** are valid, then $\mathcal{F}:\mathcal{H}\to\mathcal{H}$ defined in (2.12) is locally Lipschitz continuous operator.*

Lemma 2.3 *Assume that $\mu_2 \le \mu_1$, then energy functional $E(t)$ is non-increasing, more precisely, for any strong solution $U=(\varphi,\varphi_t,\psi,\psi_t,z)$ of (2.11), we have*

$$\frac{d}{dt}E(t) \le -\|\varphi_t\|_2^2 - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|\psi_t\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|z_1\|_2^2 \le 0, \quad \forall t \ge 0. \tag{2.32}$$

Also, there exists a constant $K_{E_1} = K_{E_1}\Big(\|h_1\|_2, \|h_2\|_2\Big) > 0$ such that

$$E(t) \ge \frac{1}{4}\|U(t)\|_{\mathcal{H}}^2 - K_{E_1}, \quad \forall t \ge 0. \tag{2.33}$$

Proof Multiplying (2.3) by φ_t and (2.4) ψ_t, integrating over $[0,1]$ with respect to x and applying Young's inequality we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2\right.\\
&\left.+\frac{\kappa}{2}\|\varphi_x+\psi\|_2^2 + \int_0^1 F(\varphi,\psi)dx - \int_0^1 (h_1\varphi + h_2\psi)dx\right\} \le -\|\varphi_t\|_2^2\\
&-\left(\mu_1 - \frac{\mu_2}{2}\right)\|\psi_t\|_2^2 + \frac{\mu_2}{2}\|z_1\|_2^2.
\end{aligned} \tag{2.34}$$

Multiplying (2.5) by $\frac{\xi}{\tau}z$ and integrating in $[0,1]\times[0,1]$ with respect to x and y, respectively, we have

$$\begin{aligned}
\frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,y,t)dydx &= -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial y}z^2(x,y,t)dydx\\
&= \frac{\xi}{2\tau}\int_0^1 (z^2(x,0,t) - z^2(x,1,t))dx.
\end{aligned} \tag{2.35}$$

Combining (2.34) and (2.35), we get (2.32).

In the other hand, it follows from (2.31) and (2.18)

$$E(t) = \frac{1}{2}\|U(t)\|_{\mathcal{H}} + \int_0^1 F(\varphi,\psi)dx - \int_0^1 h_1\varphi dx - \int_0^1 h_2\psi dx. \tag{2.36}$$

From (2.23) and by using Poincaré's inequality

$$\begin{aligned}
\int_0^1 F(\varphi,\psi)dx &\ge -\theta_2 - \alpha_1\|\psi\|_2^2 - \theta_1\|\varphi\|_2^2 \ge -\theta_2 - \alpha_1\|\psi_x\|_2^2 - \theta_1\|\varphi_x\|_2^2\\
&\ge -\theta_2 - \alpha_1\|\psi_x\|_2^2 - \theta_1\|\varphi_x+\psi\|_2^2 - \theta_1\|\psi_x\|_2^2\\
&\ge -\theta_2 - (\alpha_1 + 2\theta_1)\|\psi_x\|_2^2 - 2\theta_1\|\varphi_x+\psi\|_2^2.
\end{aligned} \tag{2.37}$$

Applying the Holder's, Poincaré's and Young's inequalities we have

$$\begin{aligned}
\int_0^1 h_1\varphi dx &\le \|h_1\|_2\|\varphi_x\|_2 \le \|h_1\|_2\|\varphi_x+\psi\|_2 + \|h_1\|_2\|\psi\|_2\\
&\le \|h_1\|_2\|\varphi_x+\psi\|_2 + \|h_1\|_2\|\psi\|_2\\
&\le \frac{1}{\kappa}\|h_1\|_2^2 + \frac{\kappa}{4}\|\varphi_x+\psi\|_2^2 + \frac{4}{b}\|h_1\|_2^2 + \frac{b}{16}\|\psi_x\|_2^2,
\end{aligned} \tag{2.38}$$

and

$$\int_0^1 h_2\psi\,dx \le \frac{4}{b}\|h_2\|_2^2 + \frac{b}{16}\|\psi_x\|_2^2. \tag{2.39}$$

Combining (2.36), (2.37), (2.38) and (2.39), we arrive

$$\frac{1}{2}\|U(t)\| - \underbrace{\left[\theta_2 + \left(\frac{1}{\kappa} + \frac{4}{b}\right)\|h_1\|_2^2 + \frac{4}{b}\|h_2\|_2^2\right]}_{2K_{E_1}}$$
$$\le E(t) + \frac{b}{2}\|\psi_x\|_2^2 + \frac{\kappa}{2}\|\varphi_x + \psi\|_2^2 \le 2E(t), \tag{2.40}$$

proving thus (2.33) which completes the prove of the Lemma 2.3. □

Theorem 2.2 (Local and Global Solution) *Suppose that $\mu_2 \le \mu_1$, **(A1)** and **(A2)** holds, we have:*

- **(i)** *If $U_0 \in \mathcal{H}$, then there exists $T_{max} > 0$ such that (2.11) has a unique mild solution $U : [0, T_{max}) \to \mathcal{H}$. In addition, if $U_0 \in D(\mathcal{A})$, then the mild solution is strong solution;*
- **(ii)** *The solution $U(t)$ is globally bounded in $\mathcal{H}$ and thus $T_{max} = +\infty$;*
- **(iii)** *If U_1 and U_2 are two mild solutions of problem (2.11), then there exists a positive constant $C_{E_1} = C_{E_1}(U_1(0), U_2(0))$ such that*

$$\|U_1(t) - U_2(t)\|_{\mathcal{H}} \le e^{C_{E_1}t}\|U_1(0) - U_2(0)\|_{\mathcal{H}}, \quad \forall t \in [0, T_{max}). \tag{2.41}$$

Proof (i) The result follows from Lemmas 2.1, 2.2 and of [33, Chap. 6, Theorems 1.4 and 1.5].
(ii) From (2.32), we have

$$E(t) \le E(0), \quad \forall t > 0, \tag{2.42}$$

and combining with (2.33), we obtain

$$\frac{1}{4}\|U(t)\|_{\mathcal{H}}^2 \le E(0) + K_{E_1}, \quad \forall t \ge 0, \tag{2.43}$$

and together with [33, Chap. 6, Theorems 1.4] results $T_{\max} = +\infty$.
(iii) Since U_1 and U_2 are mild solutions of (2.11), we have

$$\left\|U_1(t) - U_2(t)\right\|_{\mathcal{H}} = \left\|e^{\mathcal{A}t}(U_1(0) - U_2(0)) - \int_0^t e^{\mathcal{A}(t-s)}(\mathcal{F}(U_1(s)) - \mathcal{F}(U_2(s)))ds\right\|_{\mathcal{H}}. \tag{2.44}$$

Being $e^{\mathcal{A}t}$ a semigroup of contractions, we have

$$\left\|U_1(t) - U_2(t)\right\|_{\mathcal{H}} \le \left\|U_1(0) - U_2(0)\right\|_{\mathcal{H}} + \int_0^t \left\|\mathcal{F}(U_1(s)) - \mathcal{F}(U_2(s))\right\|_{\mathcal{H}} ds. \tag{2.45}$$

From Lemma 2.2 and (2.43), there exists a positive constant C_{E_1} such that, for any $T > 0$

$$\left\|U_1(t) - U_2(t)\right\|_{\mathcal{H}} \le \left\|U_1(0) - U_2(0)\right\|_{\mathcal{H}}$$
$$+C_{E_1}\int_0^t \left\|U_1(s) - U_2(s)\right\|_{\mathcal{H}} ds, \quad \forall t \in [0, T_{\max}). \tag{2.46}$$

Applying the Gronwall's inequality we get (2.41). This completes the proof of Theorem 2.2. □

Remark 2.3 It is worth emphasizing that being $D(\mathcal{A})$ dense in $\mathcal{H}$, then for every $U_0 \in \mathcal{H}$ and its respective mild solution $U : [0, \infty) \to \mathcal{H}$, it is possible to obtain a sequence (U_0^n) in $D(\mathcal{A})$ with $U_0^n \to U_0$ and a sequence $U^n \in C([0, +\infty); \mathcal{H})$ where U^n is a strong solution of

$$\frac{d}{dt}U^n(t) = \mathcal{A}U^n(t) + \mathcal{F}\big(U^n(t)\big), \quad t > 0, \tag{2.47}$$

$$U^n(0) = U_0^n, \tag{2.48}$$

with

$$U^n \to U \quad \text{in} \quad C([0, T]; \mathcal{H}), \ \forall T > 0. \tag{2.49}$$

This means that the regularity for the solutions obtained in Theorem 2.2 are sufficient to justify the calculations that will be performed in this work.

3 Long-Time Dynamics

We can from the Theorem 2.2 and Lemma 2.3 defining the dynamical system $(\mathcal{H}, S(t))$, associated with the problem (2.11), where $\mathcal{H}$ was defined in (2.15) and $S(t)$ is the semigroup (evolution operator) given by

$$S(t)U_0 = U(t), \quad \forall t \geq 0, \tag{3.1}$$

$$U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, f_0) \in \mathcal{H}, \tag{3.2}$$

where $U(t)$ is the mild solution of (2.11) with initial condition U_0.

Key concepts as well as main results related to dynamical systems can be found, among others, in [34–40].

3.1 Some Concepts and Results Related to Dynamical Systems

In this subsection, we will outline some concepts and results related to dynamical systems that will be important for this work. In the sequence, H will represent a generic Banach space and $\mathcal{S}(t)$ a strongly continuous evolution operator.

A dynamical system $(H, \mathcal{S}(t))$ is said *asymptotically smooth* if for any bounded set $\mathscr{D} \subset H$, such that $\mathcal{S}(t)\mathscr{D} \subset \mathscr{D}$ for all $t > 0$, there exists a compact set $\mathscr{K} \subset \overline{\mathscr{D}}$, where $\overline{\mathscr{D}}$ is the closure of $\mathscr{D}$, such that

$$\lim_{t \to +\infty} d_H(\mathcal{S}(t)\mathscr{D}, \mathscr{K}) = 0, \tag{3.3}$$

where d_H denotes the Hausdorff semi-distance between sets in H, that is

$$d_H(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_H \quad \text{for sets } A, B \subset H.$$

An closed and bounded set $\mathscr{A} \subset H$ is called a *global attractor* for $(H, \mathcal{S}(t))$ if $\mathscr{A}$ is an invariante set, that is $\mathcal{S}(t)\mathscr{A} = \mathscr{A}$, for all $t \geq 0$ and $\mathscr{A}$ is uniformly attracting, that is, for every bounded set $\mathscr{D} \subset H$, we have

$$\lim_{t \to +\infty} d_H(\mathcal{S}(t)\mathscr{D}, \mathscr{A}) = 0. \tag{3.4}$$

An closed and bounded set $\mathfrak{A}_{\min} \subset H$ is called a *global minimal attractor* for $(H, \mathcal{S}(t))$ if $\mathfrak{A}_{\min}$ is positively invariant, that is, $\mathcal{S}(t)\mathfrak{A}_{\min} \subseteq \mathfrak{A}_{\min}$ for all $t \geq 0$ and $\mathfrak{A}_{\min}$ attracts every point of H, that is,

$$\lim_{t\to+\infty} d_H(\mathcal{S}(t)U_0, \mathfrak{A}_{\min}) = 0, \quad \forall U_0 \in H, \tag{3.5}$$

and $\mathfrak{A}_{\min}$ is minimal, that is, $\mathfrak{A}_{\min}$ has no proper subsets satisfying these two properties.

Let $\mathscr{N}$ be the sets of stationary points of $(H, \mathcal{S}(t))$, that is,

$$\mathscr{N} := \Big\{h \in H;\ \mathcal{S}(t)h = h,\ \forall t \geq 0\Big\}, \tag{3.6}$$

an *unstable manifold* emanating from $\mathscr{N}$, represented by $\mathscr{M}^u(\mathscr{N})$, is the set of all $h \in H$ such that there is a full trajectory $\gamma = \{u(t);\ t \in \mathbb{R}\}$ satisfying

$$u(0) = h \quad \text{and} \quad \lim_{t\to-\infty} \operatorname{dist}_H(u(t), \mathscr{N}) = 0. \tag{3.7}$$

It is clear that $\mathscr{M}^u(\mathscr{N})$ is an invariant set for $(H, \mathcal{S}(t))$ and if $\mathscr{A} \subset H$ is global attractor for $(H, \mathcal{S}(t))$, then $\mathscr{M}^u(\mathscr{N}) \subset \mathscr{A}$ (cf. [34,37]).

The dynamical system $(H, \mathcal{S}(t))$ is called *gradient*, if there exists a strict Lyapunov function on H, that is, there exists a continuous function Φ such that $t \mapsto \Phi(\mathcal{S}(t)y)$ is non-increasing for any $y \in H$, and if $\Phi(\mathcal{S}(t)y_0) = \Phi(y_0)$ for all $t > 0$ and some $y_0 \in H$, then y_0 is a stationary point of $(H, \mathcal{S}(t))$.

The fractal dimension of a compact set M in H is defined by

$$\dim_f^H M := \lim_{\varepsilon\to 0} \sup \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)}, \tag{3.8}$$

where $n(M, \varepsilon)$ is the minimal number of closed balls of radius ε which covers M.

An compact set $\mathfrak{A}_{\exp} \subset H$ is called a *exponential attractor* for $(H, \mathcal{S}(t))$ if $\mathfrak{A}_{\exp}$ is a positively invariant set with finite fractal dimension in H and for any bounded set $\mathscr{D} \subset H$ there exist $t_{\mathscr{D}}, C_{\mathscr{D}}, \gamma_{\mathscr{D}} > 0$ such that

$$d_H(\mathcal{S}(t)\mathscr{D}, \mathfrak{A}_{\exp}) \leq C_{\mathscr{D}} e^{-\gamma_{\mathscr{D}}(t-t_{\mathscr{D}})}, \quad \forall t \geq t_{\mathscr{D}}.$$

The proof of the following theorem can be found in [37] p. 360.

Theorem 3.1 *Let $(H, \mathcal{S}(t))$ be a gradient and asymptotically smooth dynamical system. Assume that the Lyapunov function $\Phi(y)$ of $(H, \mathcal{S}(t))$ is bounded from above on any bounded subset of H and the set $\Phi_R = \{y;\ \Phi(y) \leq R\}$ is bounded for every R. If the set $\mathscr{N}$ of stationary points of $(H, \mathcal{S}(t))$ is bounded, then $(H, \mathcal{S}(t))$ possesses a compact global attractor $\mathscr{A} = \mathscr{M}^u(\mathscr{N})$.*

Let X, Y and Z be reflexives Banach spaces with X compactly embedded in Y. We consider the space $H = X \times Y \times Z$, with norm

$$\|h\|_H^2 := \|\pi_0\|_X^2 + \|\pi_1\|_Y^2 + \|\eta_0\|_Z^2, \quad h = (\pi_0, \pi_1, \eta_0) \in H, \tag{3.9}$$

and the dynamical system $(H, \mathcal{S}(t))$ given by an evolution operator

$$\mathcal{S}(t)h_0 = (\pi(t), \pi_t(t), \eta(t)) \quad t \geq 0, \quad h_0 = (\pi(0), \pi_t(0), \eta(0)) \in H, \tag{3.10}$$

where the functions $\pi(t)$ and $\eta(t)$ possess the properties

$$\pi \in C(\mathbb{R}_+, X) \cap C^1(\mathbb{R}_+, Y), \quad \eta \in C(\mathbb{R}_+, Z). \tag{3.11}$$

The dynamical system $(H, \mathcal{S}(t))$ is called *quasi-stable* on a set $\mathscr{B} \subset H$ if there exist a compact seminorm $\eta_X(\cdot)$ on the space X and nonnegative scalar functions $a(t)$, $b(t)$ and $c(t)$ on $\mathbb{R}_+$ such that

Q(S1) $a(t)$ and $c(t)$ are locally bounded on $[0, \infty)$;
Q(S2) $b(t) \in L^1(\mathbb{R}_+)$ possesses the property

$$\lim_{t\to\infty} b(t) = 0; \tag{3.12}$$

Q(S3) for every $h_1, h_2 \in \mathscr{B}$ and $t > 0$ the following relations

$$\|\mathcal{S}(t)h_1 - \mathcal{S}(t)h_2\|_H^2 \le a(t)\|h_1 - h_2\|_H^2 \tag{3.13}$$

and

$$\|\mathcal{S}(t)h_1 - \mathcal{S}(t)h_2\|_H^2 \le b(t)\|h_1 - h_2\|_H^2 + c(t) \sup_{0\le s\le t} [\eta_X(\pi^1(s) - \pi^2(s))]^2 \tag{3.14}$$

hold. Here we denote $\mathcal{S}(t)h_i = (\pi^i(t), \pi_t^i(t), \eta^i(t)),\ i = 1, 2.$

The following two results can be found in [37, Chapter 7], show us how strong the property of quasi-stability is for a dynamical system. The first, relates the quasi-stability to the asymptotically smooth and the second relates the quasi-stability to the fractal dimension of an attractor.

Theorem 3.2 *Let $(H, \mathcal{S}(t))$ be a dynamical system with the evolution operator of the form* (3.10)*. Assume that $(H, \mathcal{S}(t))$ is quasi-stable over bounded forward invariant set $\mathscr{B} \subset H$. Then, $(H, \mathcal{S}(t))$ is asymptotically smooth.*

Theorem 3.3 *Suppose $(H, \mathcal{S}(t))$ be a dynamical system with the evolution operator of the form* (3.10)*. Assume that $(H, \mathcal{S}(t))$ possesses a compact global attractor $\mathscr{A}$ and is quasi-stable on $\mathscr{A}$. Then the fractal dimension of $\mathscr{A}$ is finite.*

3.2 Existence of Global Attractor

In this subsection, we will study the existence of global attractor for the dynamic system $(\mathcal{H}, S(t))$ defined in (3.2), we will often refer to the inequality $\mu_2 \le \mu_1$ associated with (2.16).

Lemma 3.1 *If $\mu_2 \le \mu_1$, then the dynamical system $(\mathcal{H}, S(t))$ is gradient, that is, there exists a strict Lyapunov function Φ defined in $\mathcal{H}$. In addition,*

(a) *Φ is bounded from above on any bounded subset of $\mathcal{H}$;*
(b) *For all $R > 0$, the set $\Phi_R = \{W_0 \in \mathcal{H};\ \Phi(W_0) \le R\}$ is bounded.*

Proof Let us consider the functional energy defined in (2.31) as the Lyapunov function, that is, $\Phi \equiv E$. Thus, given $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, f_0) \in \mathcal{H}$, it follows from the Lemma (2.3) that the function $t \mapsto \Phi(S(t)U_0)$ is non-increasing and

$$\Phi(S(t)U_0) + \int_0^t \left[\|\varphi_t\|_2^2 + \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|\psi_t\|_2^2 + \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\|z(1)\|_2^2\right]ds \\ \le \Phi(U_0), \quad \forall t \ge 0, \tag{3.15}$$

with

$$\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0 \quad \text{and} \quad \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0. \tag{3.16}$$

If $\Phi(S(t)U_0) = \Phi(U_0)$ for all $t \geq 0$ then, from (3.15), we have

$$\varphi_t(t) = \psi_t(t) = z(1, t) = 0, \quad \text{a.e. in} \ (0, 1), \quad \forall t \geq 0, \tag{3.17}$$

which implies

$$\varphi(t) = \varphi_0, \ \psi(t) = \psi_0 \ \text{ and } \ z(\cdot, 1, t) = 0 \quad \forall t \geq 0. \tag{3.18}$$

This gives us $U(t) = S(t)U_0 = (\varphi_0, 0, \psi_0, 0, 0)$ for all $t \geq 0$, that is, U_0 is a stationary point of $(\mathcal{H}, S(t))$, thus proving that Φ is a strict Lyapunov function of $(\mathcal{H}, S(t))$ and therefore, the dynamical system is gradient.

It is easy to see from (3.15) that Φ is bounded from above on bounded subsets of $\mathcal{H}$, which proves (a). Given $W_0 \in \Phi_R$, consider $W(t)$ the mild solution corresponding to W_0, from the inequalities (2.33) and (3.15) we have

$$\|W(t)\|_{\mathcal{H}} \leq 4\Phi(S(t)W_0) + 4K_{E_1} \leq 4\Phi(W_0) + 4K_{E_1}, \quad t \geq 0, \tag{3.19}$$

for $t = 0$, we obtain

$$\|W_0\|_{\mathcal{H}} \leq 4R + 4K_{E_1}, \tag{3.20}$$

showing thus Φ_R is a bounded set of $\mathcal{H}$, which proves (b) and completes the proof of the Lemma 3.1. □

Lemma 3.2 *The set of stationary points $\mathcal{N}$ of the dynamical system $(\mathcal{H}, S(t))$ is bounded.*

Proof Based on the Lemma 3.1, the set $\mathcal{N}$ is given by

$$\mathcal{N} = \Big\{U = (\varphi, 0, \psi, 0, 0) \in \mathcal{H}; \ \mathcal{A}U + \mathcal{F}(U) = 0\Big\}. \tag{3.21}$$

Therefore, φ and ψ must satisfy

$$-\kappa(\varphi_x + \psi)_x + f_1(\varphi, \psi) = h_1 \quad \text{in} \quad (0, 1), \tag{3.22}$$

$$-b\psi_{xx} + \kappa(\varphi_x + \psi) + f_2(\varphi, \psi) = h_2 \quad \text{in} \quad (0, 1). \tag{3.23}$$

Multiplying (3.22) by φ and (3.23) by ψ, integrating over $(0, 1)$ and adding the results, we obtain

$$\|\varphi_x + \psi\|_2^2 + b\|\psi_x\|_2^2 + \int_0^1 [f_1(\varphi, \psi)\varphi + f_2(\varphi, \psi)\psi]dx = \int_0^1 [h_1\varphi + h_2\psi]dx. \tag{3.24}$$

From (2.23) and (2.24), we have

$$\begin{aligned} &\int_0^1 [f_1(\varphi, \psi)\varphi + f_2(\varphi, \psi)\psi]dx \\ &\geq -2\theta_2 - 2\theta_1\|\varphi\|_2^2 - \alpha_1\|\psi\|_2^2 \geq -2\theta_2 - 2\theta_1\|\varphi_x\|_2^2 - \alpha_1\|\psi_x\|_2^2 \\ &\geq -2\theta_2 - 2\theta_1\|\varphi_x + \psi\|_2^2 - (\alpha_1 + 2\theta_1)\|\psi_x\|_2^2. \end{aligned} \tag{3.25}$$

By Hölder's and Poincaré's inequalities, we have

$$\int_0^1 [h_1\varphi + h_2\psi]dx \le \|h_1\|_2\|\varphi\|_2 + \|h_2\|_2\|\psi\|_2 \le \|h_1\|_2\|\varphi_x\|_2 + \|h_2\|_2\|\psi_x\|_2$$
$$\le \frac{\kappa}{2}\|\varphi_x+\psi\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2 + \left(\frac{1}{2\kappa}+\frac{1}{b}\right)\|h_1\|_2^2 + \frac{1}{b}\|h_2\|_2^2. \tag{3.26}$$

From (3.24)–(3.26) we obtain

$$\frac{1}{4}\|U\|_{\mathcal{H}}^2 = \frac{\kappa}{8}\|\varphi_x+\psi\|_2^2 + \frac{b}{8}\|\psi_x\|_2^2 \le \left(\frac{1}{2\kappa}+\frac{1}{b}\right)\|h_1\|_2^2 + \frac{1}{b}\|h_2\|_2^2, \tag{3.27}$$

showing that $\mathcal{N}$ is bounded in $\mathcal{H}$, this completes the proof of the Lemma 3.2. □

Lemma 3.3 *Suppose that $\mu_2 \le \mu_1$ and **(A1)–(A3)** are valid. For every set bounded $\mathscr{B} \subset \mathcal{H}$, there are positive constants γ, ϑ and $C_{\mathscr{B}}$, with $C_{\mathscr{B}}$ depending on $\mathscr{B}$, such that*

$$\|S(t)U_1 - S(t)U_2\|_{\mathcal{H}}^2 \le \vartheta e^{-\gamma t}\|U_1-U_2\|_{\mathcal{H}}^2$$
$$+C_{\mathscr{B}}\int_0^t e^{-\gamma(t-s)}\Big[\|u(s)\|_2^2 + \|v(s)\|_2^2\Big]ds, \quad t\ge 0, \tag{3.28}$$

for any $U_i = (\varphi_0^i, \varphi_1^i, \psi_0^i, \psi_1^i, f_0^i) \in \mathscr{B}$, where $S(t)U_i = (\varphi^i(t), \varphi_t^i(t), \psi^i(t), \psi_t^i(t), z_1^i(t))$ is mild solution of (2.11) to $i = 1, 2$, $u = \varphi^1 - \varphi^2$ and $v = \psi^1 - \psi^2$.

Proof Consider the representation $U(t) = S(t)U_1 - S(t)U_2 = (u(t), u_t(t), v(t), v_t(t), w(t))$, $t \ge 0$, where $w = z_1^1 - z_1^2$. Thus $U(t)$, in the sense of mild solution, solves the following system

$$\rho_1 u_{tt} - \kappa(u_x+v)_x + f_1(\varphi^1,\psi^1) - f_1(\varphi^2,\psi^2) + u_t = 0 \quad \text{in} \quad (0,1)\times(0,\infty), \tag{3.29}$$

$$\rho_2 v_{tt} - bv_{xx} + \kappa(u_x+v) + f_2(\varphi^1,\psi^1) - f_2(\varphi^2,\psi^2) + \mu_1 v_t + \mu_2 w(\cdot,1,\cdot)$$
$$= 0 \quad \text{in} \quad (0,1)\times(0,\infty), \tag{3.30}$$

$$\tau w_t(x,y,t) + w_y(x,y,t) = 0 \quad \text{in} \quad (0,1)\times(0,1)\times(0,\infty). \tag{3.31}$$

Multiplying (3.29) by u_t and (3.30) by v_t, integrating with respect to x in [0, 1] and adding the results, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|u_t\|_2^2 + \rho_2\|v_t\|_2^2 + \kappa\|u_x+v\|_2^2\right) = -\int_0^1(\Delta f_1)u_t dx - \int_0^1(\Delta f_2)v_t dx$$
$$-\|u_t\|_2^2 - \mu_1\|v_t\|_2^2 - \mu_2\int_0^1 w(x,1,t)v_t dx, \tag{3.32}$$

where

$$\Delta f_i = f_i(\varphi^1,\psi^1) - f_i(\varphi^2,\psi^2) = [f_i(\varphi^1,\psi^1) - f_i(\varphi^1,\psi^2)]$$
$$+[f_i(\varphi^1,\psi^2) - f_i(\varphi^2,\psi^2)]. \tag{3.33}$$

Multiplying (3.31) by $\frac{\xi}{\tau}w$ and integrating with respect to x and y in $[0,1]\times[0,1]$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1 \xi w^2(x,y,t)dydx &= -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial y}w^2(x,y,t)dydx\\ &= -\frac{\xi}{2\tau}\int_0^1 w^2(x,y,t)|_{y=0}^{y=1}dx\\ &= \frac{\xi}{2\tau}\|u_t\|_2^2 - \frac{\xi}{2\tau}\int_0^1 w^2(x,1,t)dx. \end{aligned} \tag{3.34}$$

Adding (3.32) and (3.34), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|u_t\|_2^2+\rho_2\|v_t\|_2^2+b\|v_x\|_2^2+\kappa\|u_x+v\|_2^2+\xi\int_0^1\int_0^1 w^2(x,y,t)dydx\right)\\ &= -\int_0^1(\Delta f_1)u_t dx\\ &\quad -\int_0^1(\Delta f_2)v_t dx - \|u_t\|_2^2 - \left(\mu_1-\frac{\xi}{2\tau}\right)\|v_t\|_2^2 - \mu_2\int_0^1 w(x,1,t)v_t dx\\ &\quad -\frac{\xi}{2\tau}\int_0^1 w^2(x,1,t)dx. \end{aligned} \tag{3.35}$$

Considering now the functional $\mathscr{L}$ given by

$$\begin{aligned}\mathscr{L}(t) &:= \rho_1\|u_t\|_2^2+\rho_2\|v_t\|_2^2+b\|v_x\|_2^2+\kappa\|u_x\\ &\quad +v\|_2^2+\xi\int_0^1\int_0^1 w^2(x,y,t)dydx \equiv \|U(t)\|_{\mathcal{H}}^2. \end{aligned} \tag{3.36}$$

Let's now estimate the right side of (3.35). Since $\mathscr{B}$ is bounded, it follows from (2.32)–(2.33) the existence of a constant $K_{\mathscr{B}_1}$ depending on $\mathscr{B}$ such that

$$\|S(t)U_1\|_{\mathcal{H}},\ \|S(t)U_2\|_{\mathcal{H}} \le K_{\mathscr{B}_1},\ \forall t\ge 0. \tag{3.37}$$

Since σ_i is continuous and $H_0^1(0,1)\hookrightarrow L^\infty(0,1)$, there exists a constant $K_{\mathscr{B}_2}>0$ depending on $\mathscr{B}$ such that

$$\sigma_i(|\varphi^j|),\ \sigma_i(|\psi^j|)\le K_{\mathscr{B}_2}\quad \text{a.e in}\quad (0,1)\times(0,\infty),\quad i,j=1,2. \tag{3.38}$$

From (2.20)–(2.21), (3.37), (3.38) and Hölder's inequality we obtain

$$\begin{aligned}\left|\int_0^1(\Delta f_1)u_t dx\right| &\le \int_0^1 \sigma_1(|\varphi^1(t)|)(1+|\psi^1(t)|^{\gamma_1}+|\psi^2(t)|^{\gamma_1})|v(t)||u_t(t)|dx\\ &\quad +\int_0^1 \sigma_1(|\psi^2(t)|)(1+|\varphi^1(t)|^{\gamma_1}+|\varphi^2(t)|^{\gamma_1})|u(t)||u_t(t)|dx\\ &\le K_{\mathscr{B}_2}(1+\|\psi^1(t)\|_\infty^{\gamma_1}+\|\psi^2(t)\|_\infty^{\gamma_1})\int_0^1|v(t)||u_t(t)|dx\\ &\quad +K_{\mathscr{B}_2}(1+\|\varphi^1(t)\|_\infty^{\gamma_1}+\|\varphi^2(t)\|_\infty^{\gamma_1})\int_0^1|u(t)||u_t(t)|dx\\ &\le K_{\mathscr{B}_3}\|v(t)\|_2\|u_t(t)\|_2+K_{\mathscr{B}_3}\|u(t)\|_2\|u_t(t)\|_2, \end{aligned} \tag{3.39}$$

for some constant $K_{\mathscr{B}_3}$ depending on $\mathscr{B}$. Applying Young's inequality with $\varepsilon = \frac{\alpha_1}{4}$, there exists a constant $K_{\mathscr{B}_4} > 0$ such that

$$\left| \int_0^1 (\Delta f_1) u_t(t) dx \right| \leq K_{\mathscr{B}_4} (\|u(t)\|_2^2 + \|v(t)\|_2^2) + \frac{\alpha_1}{2} \|u_t(t)\|_2^2. \tag{3.40}$$

In a similar way we can obtain a constant $K_{\mathscr{B}_5} > 0$ depending on $\mathscr{B}$ such that

$$\left| \int_0^1 (\Delta f_2) v_t(t) dx \right| \leq K_{\mathscr{B}_5} (\|u(t)\|_2^2 + \|v(t)\|_2^2) + \frac{\alpha_2}{2} \|v_t(t)\|_2^2. \tag{3.41}$$

From Young's inequality, we have

$$\mu_2 \int_0^1 w(x, 1, t) v_t dx \leq \frac{\mu_2}{2} \|v_t(t)\|_2^2 + \frac{\mu_2}{2} \int_0^1 w^2(x, 1, t) dx. \tag{3.42}$$

Combining (3.40)–(3.42), we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \mathcal{L}(t) \leq\ & K_{\mathscr{B}_6} (\|u(t)\|_2^2 + \|v(t)\|_2^2) - \left(1 - \frac{\alpha_1}{2}\right) \|u_t(t)\|_2^2 \\ & - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} - \frac{\alpha_2}{2}\right) \|v_t\|_2^2 - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right) \int_0^1 w^2(x, 1, t) dx. \end{aligned} \tag{3.43}$$

where $K_{\mathscr{B}_6} = K_{\mathscr{B}_4} + K_{\mathscr{B}_5}$. Considering now

$$\alpha_1 = 1 \quad \text{and} \quad \alpha_2 = \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0, \tag{3.44}$$

we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(t) \leq\ & 2K_{\mathscr{B}_6} (\|u(t)\|_2^2 + \|v(t)\|_2^2) - \|u_t(t)\|_2^2 \\ & - \left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right) \|v_t\|_2^2 - \left(\frac{\xi}{\tau} - \mu_2\right) \int_0^1 w^2(x, 1, t) dx. \end{aligned} \tag{3.45}$$

We now define the following functional

$$\mathcal{I}(t) := N\mathcal{L}(t) + \mathcal{J}(t) + \mathcal{K}(t) + M\mathcal{P}(t), \tag{3.46}$$

where N and M are positive constants to be chosen a posteriori and

$$\begin{aligned} \mathcal{J}(t) := \rho_1 \int_0^1 u_t(t) u(t) dx, \ \ \mathcal{K}(t) := \rho_2 \int_0^1 v_t(t) v(t) dx \ \text{ and } \ \mathcal{P}(t) \\ := \tau \int_0^1 \int_0^1 e^{-2\tau y} w^2(x, y, t) dy dx. \end{aligned} \tag{3.47}$$

It is not difficult to check that there exists a constant $C_{\mathcal{I}_1} > 0$ such that

$$|\mathcal{I}(t) - N\mathcal{L}(t)| \leq C_{\mathcal{I}_1} \mathcal{L}(t), \quad \forall t \geq 0. \tag{3.48}$$

Therefore, for N large enough, we obtain positive constants $C_{\mathcal{I}_2}$ and $C_{\mathcal{I}_3}$ such that

$$C_{\mathcal{I}_2} \mathcal{L}(t) \leq \mathcal{I}(t) \leq C_{\mathcal{I}_3} \mathcal{L}(t), \quad \forall t \geq 0. \tag{3.49}$$

We will now show that there are positive constants N_1 and $N_{\mathscr{B}}$, with $N_{\mathscr{B}}$ depending on $\mathscr{B}$, such that

$$\frac{d}{dt}\mathcal{I}(t) + N_1\mathcal{L}(t) \le N_{\mathscr{B}}(\|u(t)\|_2^2 + \|v(t)\|_2^2), \quad \forall t > 0. \tag{3.50}$$

In fact, deriving $\mathcal{J}$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{J}(t) &= \rho_1\int_0^1 u_t^2dx + \rho_1\int_0^1 u_{tt}udx = \rho_1\|u_t\|_2^2 + \int_0^1[\kappa(u_x+v)_x - \Delta f_1 - u_t]udx\\ &= \rho_1\|u_t\|_2^2 - \kappa\int_0^1(u_x+v)u_xdx - \int_0^1(\Delta f_1)udx - \int_0^1 u_tudx. \end{aligned}\tag{3.51}$$

Deriving now $\mathcal{K}$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{K}(t) &= \rho_2\int_0^1 v_t^2dx + \rho_2\int_0^1 v_{tt}vdx\\ &= \rho_2\|v_t\|_2^2 + \int_0^1\Big[bv_{xx} - \kappa(u_x+v) - \Delta f_2 - \mu_1v_t - \mu_2w(x,1,t)\Big]vdx\\ &= \rho_2\|v_t\|_2^2 - b\|v_x\|_2^2 - \kappa\int_0^1(u_x+v)vdx - \int_0^1(\Delta f_2)vdx\\ &\quad - \mu_1\int_0^1 v_tvdx - \mu_2\int_0^1 w(x,1,t)vdx. \end{aligned}\tag{3.52}$$

From (3.51)–(3.52), we arrived at

$$\begin{aligned}\frac{d}{dt}\Big(\mathcal{J}(t)+\mathcal{K}(t)\Big) &= \rho_1\|u_t\|_2^2 + \rho_2\|v_t\|_2^2 - b\|v_x\|_2^2 - \kappa\|u_x+v\|_2^2\\ &\quad - \int_0^1\Big[(\Delta f_1)u + (\Delta f_2)v\Big]dx\\ &\quad - \int_0^1(u_tu + \mu_1v_tv)dx - \mu_2\int_0^1 w(x,1,t)vdx. \end{aligned}\tag{3.53}$$

By analogous arguments to (3.39), we can conclude the existence of a constant $K_{\mathscr{B}_7} > 0$ depending on $\mathscr{B}$, such that

$$\int_0^1\Big((\Delta f_1)u + (\Delta f_2)v\Big)dx \le K_{\mathscr{B}_7}\Big(\|u(t)\|_2^2 + \|v(t)\|_2^2\Big). \tag{3.54}$$

By using the Young's and Poincaré's inequalities, we have

$$\begin{aligned}\left|\int_0^1[u_tu + \mu_1v_tv]dx\right| &\le \int_0^1|u_tu|dx + \mu_1\int_0^1|v_tv|dx \le \|u_t\|_2\|u_x\|_2 + \mu_1\|v_t\|_2\|v_x\|_2\\ &\le \|u_t\|_2\|u_x+v\|_2 + \|u_t\|_2\|v_x\|_2 + \mu_1\|v_t\|_2\|v_x\|_2\\ &\le \frac{b}{3}\|v_x(t)\|_2^2 + \frac{\kappa}{2}\|u_x+v\|_2^2 + \left(\frac{1}{2\kappa}+\frac{3}{2b}\right)\|u_t\|_2^2 + \frac{3\mu_1^2}{2b}\|v_t\|_2^2, \end{aligned}\tag{3.55}$$

and

$$\mu_2\int_0^1 w(x,1,t)vdx \le \frac{b}{6}\|v_x\|_2^2 + \frac{3\mu_2^2}{2b}\int_0^1 w^2(x,1,t)dx. \tag{3.56}$$

From (3.52)–(3.56) we obtain

$$\frac{d}{dt}[\mathcal{J}(t)+\mathcal{K}(t)] \leq \left(\frac{1}{2\kappa}+\frac{3}{2b}+\rho_1\right)\|u_t\|_2^2+\left(\rho_2+\frac{3\mu_1^2}{2b}\right)\|v_t\|_2^2-\frac{b}{2}\|v_x\|_2^2$$
$$-\frac{\kappa}{2}\|u_x+v\|_2^2+K_{\mathscr{B}_7}\left(\|u(t)\|_2^2+\|v(t)\|_2^2\right)+\frac{3\mu_2^2}{2b}\int_0^1 w^2(x,1,t)dx. \tag{3.57}$$

Deriving $\mathcal{P}$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{P}(t) &= 2\tau\int_0^1\int_0^1 e^{-2\tau y}w(x,y,t)w_t(x,y,t)dx\\ &= -2\int_0^1\int_0^1 e^{-2\tau y}w(x,y,t)w_y(x,y,t)dx\\ &= -\int_0^1\int_0^1 e^{-2\tau y}\frac{\partial}{\partial y}w^2(x,y,t)dx = -\int_0^1\left[e^{-2\tau y}w^2(x,y,t)\right]_{y=0}^{y=1}dx\\ &\quad -2\tau\int_0^1\int_0^1 e^{-2\tau y}w^2(x,y,t)dx\\ &= \int_0^1 w^2(x,0,t)dx-\int_0^1 w^2(x,1,t)dx-2\tau\int_0^1\int_0^1 e^{-2\tau y}w^2(x,y,t)dx\\ &= \|v_t(t)\|_2^2-\int_0^1 w^2(x,1,t)dx-2\tau\int_0^1\int_0^1 e^{-2\tau y}w^2(x,y,t)dx.\end{aligned} \tag{3.58}$$

Therefore, combining (3.46), (3.57) and (3.58) arrived at

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(t) &\leq \left(2NK_{\mathscr{B}_6}+K_{\mathscr{B}_7}\right)(\|u(t)\|_2^2+\|v(t)\|_2^2)-\left(N-\frac{1}{2\kappa}-\frac{3}{2b}-\rho_1\right)\|u_t(t)\|_2^2\\ &\quad -\left[N\left(\mu_1-\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)-\left(\rho_2+\frac{3\mu_1^2}{2b}\right)-M\right]\|v_t\|_2^2\\ &\quad -\left[N\left(\frac{\xi}{\tau}-\mu_2\right)-\frac{3\mu_2^2}{2b}+M\right]\int_0^1 w^2(x,1,t)dx-\frac{b}{2}\|v_x\|_2^2-\frac{\kappa}{2}\|u_x+v\|_2^2\\ &\quad -2M\tau e^{-2\tau}\int_0^1\int_0^1 w^2(x,y,t)dx.\end{aligned} \tag{3.59}$$

On the other hand,

$$\begin{aligned}N_1\mathscr{L}(t) &= \rho_1N_1\|u_t\|_2^2+\rho_2N_1\|v_t\|_2^2+bN_1\|v_x\|_2^2+\kappa N_1\|u_x+v\|_2^2\\ &\quad +\xi N_1\int_0^1\int_0^1 w^2(x,y,t)dydx.\end{aligned} \tag{3.60}$$

Accordingly,

$$\begin{aligned}&\frac{d}{dt}\mathcal{I}(t)+N_1\mathscr{L}(t)\\&\le\big(2NK_{\mathscr{B}_6}+K_{\mathscr{B}_7}\big)(\|u(t)\|_2^2+\|v(t)\|_2^2)-\left(N-\frac{1}{2\kappa}-\frac{3}{2b}-\rho_1-\rho_1N_1\right)\|u_t(t)\|_2^2\\&\quad-\left[N\left(\mu_1-\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)-\left(\rho_2+\frac{3\mu_1^2}{2b}\right)-M-\rho_2N_1\right]\|v_t\|_2^2\\&\quad-\left[N\left(\frac{\xi}{\tau}-\mu_2\right)-\frac{3\mu_2^2}{2b}+M\right]\int_0^1 w^2(x,1,t)dx-\left(\frac{b}{2}-bN_1\right)\|v_x\|_2^2\\&\quad-\left(\frac{\kappa}{2}-\kappa N_1\right)\|u_x+v\|_2^2-\left(2M\tau e^{-2\tau}-\xi N_1\right)\int_0^1\int_0^1 w^2(x,y,t)dx.\end{aligned}\tag{3.61}$$

We must first consider $0<N_1<1/2$ and after that $M>\xi N_1e^{2\tau}/2\tau$. Once,

$$\mu_1-\frac{\xi}{2\tau}-\frac{\mu_2}{2}>0\quad\text{and}\quad\frac{\xi}{\tau}-\mu_2>0,\tag{3.62}$$

just take $N>0$ large enough to get (3.50).

Finally, combining (3.49) and (3.50) and using Gronwall's inequality, we arrived at

$$\mathcal{L}(t)\le\vartheta e^{-\gamma t}\mathcal{L}(0)+C_{\mathscr{B}}\int_0^t e^{-\gamma(t-s)}\Big(\|u(s)\|_2^2+\|v(s)\|_2^2\Big)ds,\quad t>0.\tag{3.63}$$

Recalling that

$$\mathcal{L}(t)=\|U(t)\|_{\mathcal{H}}^2=\|S(t)U_1-S(t)U_2\|_{\mathcal{H}}^2,\quad t\ge0.\tag{3.64}$$

The proof of Lemma 3.3 is complete. □

Remark 3.4 Since the embedded $H_0^1(0,1)\times H_0^1(0,1)\hookrightarrow L^2(0,1)\times L^2(0,1)$ is compact, in order to obtain the quasi-stability for the dynamical system $(\mathcal{H},S(t))$, we will consider the isomorphism $\mathcal{H}\cong\widetilde{\mathcal{H}}$, where

$$\widetilde{\mathcal{H}}:=(H_0^1(0,1)\times H_0^1(0,1))\times(L^2(0,1)\times L^2(0,1))\times L^2((0,1)\times(0,1)).\tag{3.65}$$

We will make the following identification

$$(\varphi,u,\psi,v,z)\in\mathcal{H}\Longleftrightarrow(\varphi,\psi,u,v,z)\in\widetilde{\mathcal{H}}.\tag{3.66}$$

The inner product and norm in $\widetilde{\mathcal{H}}$ are the same as in (2.17)–(2.18). The trajectory of the solutions will be given by $(\varphi(t),\psi(t),\varphi_t(t),\psi_t(t),z_1(t))$. When there is no danger of confusion, we will write $\mathcal{H}$ instead of $\widetilde{\mathcal{H}}$.

Theorem 3.5 *Suppose that $\mu_2\le\mu_1$ and **(A1)–(A3)** are valid. Then the dynamical system $(\mathcal{H},S(t))$ is quasi-stable on any bounded positively invariant subset of $\mathcal{H}$.*

Proof Let $\mathscr{B}\subset\mathcal{H}$ be a limited and positively invariant set of $(\mathcal{H},S(t))$ and consider $U_1,U_2\in\mathscr{B}$. As already mentioned, we denote to $i=1,2$

$$S(t)U_i=(\varphi^i(t),\psi^i(t),\varphi_t^i(t),\psi_t^i(t),z_1^i(t)),\quad\text{and}\quad(u,v)=(\varphi^1-\varphi^2,\psi^1-\psi^2).\tag{3.67}$$

From the Theorem 2.2 (ii), we obtain $a(t) = e^{C_{E_1}t} > 0$ which is locally bounded in $[0, \infty)$. We also consider the seminorm $\eta(\cdot)$ in $X = H_0^1(0,1) \times H_0^1(0,1)$ given by

$$\eta(u, v) = \|u\|_2 + \|v\|_2, \tag{3.68}$$

which is compact in X, since the embedding $X \hookrightarrow L^2(0,1) \times L^2(0,1)$ is compact. It follows from Lemma 3.3 that

$$\|S(t)U_1 - S(t)U_2\|_H^2 \le b(t)\|U_1 - U_2\|_H^2 + c(t) \sup_{0\le s\le t} [\eta_X(u(s), v(s))]^2, \tag{3.69}$$

where

$$b(t) = \vartheta e^{-\gamma t} \quad \text{and} \quad c(t) = C_{\mathscr{B}} \int_0^t e^{-\gamma(t-s)} ds, \quad t \ge 0. \tag{3.70}$$

Thus we have $b(t) \in L^1(\mathbb{R}_+)$, with $\lim_{t\to\infty} b(t) = 0$ and $c_\infty = \sup_{t\in\mathbb{R}_+} c(t) \le \frac{C_{\mathscr{B}}}{\gamma} < \infty$. Hence **(QS1)–(QS3)** are satisfied and the $(\mathcal{H}, S(t))$ is quasi-stable over any positively invariant limited set and the Theorem 3.5 is proved. □

Theorem 3.6 *Suppose that $\mu_2 \le \mu_1$ and **(A1)–(A2)** are valid. Then the dynamical system $(\mathcal{H}, S(t))$ possesses a unique compact global attractor $\mathfrak{A} \subset \mathcal{H}$, with finite fractal dimension. Moreover, the global attractor $\mathfrak{A}$ is characterized by*

$$\mathfrak{A} := \mathscr{M}^u(\mathcal{N}), \tag{3.71}$$

where $\mathcal{N}$ is the set of stationary point of $(\mathcal{H}, S(t))$ and $\mathscr{M}^u(\mathcal{N})$ is the unstable manifold of $\mathcal{N}$.

Proof It follows from Lemma 3.1 and Theorems 3.3 and 3.5 that $(\mathcal{H}, S(t))$ is gradient and asymptotically smooth. Thus, the result is readily established by properties (a) and (b) of Lemma 3.1, Theorems 3.1 and 3.3. □

Corollary 3.1 *Suppose that $\mu_1 \le \mu_2$ and **(A1)–(A3)** are valid. Then every trajectory stabilizes to the set $\mathcal{N}$, namely, for any $U \in \mathcal{H}$ one has*

$$\lim_{t\to+\infty} dist_{\mathcal{H}}(S(t)U, \mathcal{N}) = 0.$$

In particular, there exists a global minimal attractor $\mathfrak{A}_{min}$ given by $\mathfrak{A}_{min} = \mathcal{N}$.

Proof The result follows from Theorem 3.6 and [37, Theorem 7.5.10]. □

4 Regularity and Exponential Attractors

Theorem 4.1 *Suppose that $\mu_2 \le \mu_1$ and assumptions **(A1)–(A3)** are valid. Then any full trajectory*

$$\Big(\varphi(t), \psi(t), \varphi_t(t), \psi_t(t), z(t)\Big) \quad in \quad \mathfrak{A}, \tag{4.1}$$

has further regularity

$$\Big(\varphi_t, \psi_t, \varphi_{tt}, \psi_{tt}, z_t\Big) \in L^\infty(\mathbb{R}, \mathcal{H}). \tag{4.2}$$

Moreover, there exists $R > 0$ *such that*

$$\|(\varphi_t(t), \psi_t(t))\|^2_{(H^1_0(0,L))^2} + \|(u_{tt}(t), \psi_{tt}(t))\|^2_{(L^2(0,L))^2} + \|z_t(t)\|^2_{L^2((0,1)\times(0,1))} \leq R, \quad \forall t \in \mathbb{R}. \tag{4.3}$$

Proof Since we have shown that $(\mathcal{H}, S(t))$ is quasi-stable on the global attractor $\mathfrak{A}$ with $c_\infty = \sup_{t\in\mathbb{R}^+} c(t) < \infty$, then the regularity properties (4.2) and (4.3) follows by [37, Theorem 7.9.8]. The proof is complete. □

Theorem 4.2 *Assume that* $\mu_2 \leq \mu_1$ *and* ***(A1)–(A3)*** *hold, then the dynamical system* $(\mathcal{H}, S(t))$ *possesses a generalized exponential attractor representing* $\mathfrak{A}_{exp} \subset \mathcal{H}$ *with finite dimension in the extended space*

$$\mathcal{H}_{-1} := L^2(0,1) \times H^{-1}(0,1) \times L^2(0,1) \times H^{-1}(0,1) \times L^2((0,1)\times(0,1)), \tag{4.4}$$

which is isomorphic to space $L^2(0,1) \times L^2(0,1) \times H^{-1}(0,1) \times H^{-1}(0,1) \times L^2((0,1) \times (0,1))$. *In addition, from the interpolation theorem, for all* $0 < \delta < 1$ *there exists a generalized fractal exponential attractor whose fractal dimension is finite in the extended space* $\mathcal{H}_{-\delta}$*, where*

$$\mathcal{H}_0 := \mathcal{H}, \quad \textit{and} \quad \mathcal{H} \subset H_{-\delta} \subset \mathcal{H}_{-1}. \tag{4.5}$$

Proof Let Φ be the functional of Lyapunov considered in Lemma 3.1, let us take

$$\mathfrak{B} := \{U;\ \Phi(U) \leq R\}. \tag{4.6}$$

It is clear that for R large enough, the set $\mathfrak{B}$ is absorbent and positively invariant, thus $(\mathcal{H}, S(t))$ is quasi-stable on $\mathfrak{B}$. In another hand, for strong solutions $U(t)$ with initial data $U_0 \in \mathfrak{B}$, from (2.11) and the positive invariance of $\mathfrak{B}$, we get $C_{\mathfrak{B},T} > 0$ such that for any $0 \leq t \leq T$,

$$\|U_t(t)\|_{\mathcal{H}_{-1}} \leq \|\mathcal{A}U(t)\|_{\mathcal{H}} + \|\mathcal{F}(U(t))\|_{\mathcal{H}} \leq C_{\mathfrak{B},T}. \tag{4.7}$$

Consequently,

$$\|S(t_1)U_0 - S(t_2)U_0\|_{\mathcal{H}_{-1}} \leq \int_{t_1}^{t_2} \|U_t(s)\|_{\mathcal{H}_{-1}} ds \leq C_{\mathfrak{B}}|t_1 - t_2|, \quad 0 \leq t_1 \leq t_2 \leq T. \tag{4.8}$$

Therefore, the application $t \mapsto S(t)U_0$ is Hölder continuous on space extending $\mathcal{H}_{-1}$ with exponent $\delta = 1$ for every $U_0 \in \mathfrak{B}$. Thus, based on [37, Theorem 7.9.9] the system $(\mathcal{H}, S(t))$ possesses a generalized exponential attractor with finite fractal dimension in generalized space $\widetilde{\mathcal{H}}_{-1}$.

Using an analogous argument to that found in [41,42] we can show the existence of exponential attractor with finite fractal dimension in the generalized space $\mathcal{H}_{-\delta}$ with $\delta \in (0, 1)$, thus concluding the proof of the Theorem 4.2. □

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