Journal of Dynamics and Differential Equations (2020) 32:1291–1310
https://doi.org/10.1007/s10884-019-09778-7

The Dynamics of a Predator–Prey Model with Diffusion and Indirect Prey-Taxis

Jianping Wang[1] · Mingxin Wang[1]

Received: 4 July 2018 / Revised: 29 June 2019 / Published online: 17 July 2019

Abstract
This paper concerns with a reaction–diffusion system modeling the population dynamics of the predator and prey, in which the predator moves toward the gradient of concentration of some chemical released by prey instead of moving directly toward the higher density of prey. The first objective is to investigate the global existence and boundedness of the unique classical solution. Then we study the asymptotic stabilities of nonnegative spatially homogeneous equilibria. Moreover, the convergence rates are also studied.

Keywords Diffusive predator–prey model · Indirect prey-taxis · Global existence and boundedness · Global stability and convergence rate

Mathematics Subject Classification 35A01 · 35B40 · 35K57 · 35Q92 · 92D25

1 Introduction

In this paper, we consider the following predator–prey model with nonlinear “indirect prey-taxis”:

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(u\chi(w)\nabla w) + bug(v) - uh(u), & x\in\Omega,\ t>0,\\ w_t = d_2\Delta w - \mu w + rv, & x\in\Omega,\ t>0,\\ v_t = d_3\Delta v + f(v) - ug(v), & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu w = \partial_\nu v = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ w(x,0)=w_0(x),\ v(x,0)=v_0(x), & x\in\Omega. \end{cases} \tag{1.1}$$

In this model, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\partial_\nu = \frac{\partial}{\partial\nu}$ and ν is the unit outward normal vector of $\partial\Omega$. Functions u and v are, respectively, population densities of the predator and prey, and w is the concentration of chemoattractant released by the prey. Here d_1, d_2, d_3, b, μ, r are positive constants. The decay rate of the chemical w is μ, and the parameter r is the production rate. The term $\chi(w)$ is the chemotactic sensitivity which

This work was supported by NSFC Grant 11771110.

✉ Mingxin Wang
mxwang@hit.edu.cn

[1] School of Mathematics, Harbin Institute of Technology, Harbin 150001, People’s Republic of China

depends only upon w. The term $uh(u)$ describes the population kinetic of the predator u. Function $g(v)$ is the functional response accounting for the intake rate of the predator as a function of prey density. And $f(v)$ is the growth function of prey.

The system (1.1), which was recently proposed by Tello and Wrzosek [17], describes "indirect prey-taxis" in the sense that the predator moves following the gradient of some chemicals which indicate the presence of prey instead of moving directly toward the higher density of prey. The substance released by the prey, such as pheromones, chemical alarm cues, sexual signals, can be viewed as the chemoattractant for the foraging predator. The known example is that the wolf spider Pardosa milvina responses to chemical cues left by the prey [7]. For the detailed biological background, please refer to [17] and the references therein. For the special case $b = d_3 = 0$, $h(u) = 0$ and $ug(v)$ is replaced by $vF_0(u)$, where F_0 is positive, bounded, smooth function and satisfies

$$F_0(0) = 0, \quad \lim_{z\to\infty} F_0(z) = F_m$$

with positive constant F_m, the global existence of solutions, linearized stability and asymptotic behavior of steady states in two dimensional case for (1.1) were established. It was proved in [17] that the positive constant steady state may be unstable if chemotactic sensitivity or the rate of release of the chemoattractant is big enough. However, to our best knowledge, no other results are available. Studies concerning the model (1.1) with general functional responses and nonlinear indirect prey-taxis are required.

In order to better understand the system (1.1), it is worth mentioning some studies for the prey-taxis system in which the movement of the predator is determined by the prey density gradient. In the spatial predator–prey interaction, in addition to the random diffusion of predator and prey, the predator has the tendency to move towards the area with higher density of prey population. Kareiva and Odell [10] first derived a prey-taxis model to describe the predator aggregation in high prey density areas. Since then, various reaction–diffusion models have been proposed to interpret the prey-taxis phenomenon [1,4,15]. The general predator–prey model with prey-taxis reads as follows

$$\begin{cases} u_t = d_1\Delta u - \chi_0\nabla\cdot(u\nabla v) + bug(v) - uh(u), & x\in\Omega,\ t>0,\\ v_t = d_3\Delta v + f(v) - ug(v), & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu v = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where the constant $\chi_0 > 0$ and the term $\chi_0\nabla\cdot(u\nabla v)$ describes the tendency of the predator moving towards the increasing prey gradient direction. This system has been studied by many authors. Lee et al. [13] studied the pattern formation of (1.2), they showed that prey-taxis in most cases tends to stabilize predator–prey interactions, which is an opposite result to the case of Keller-Segel chemotaxis system (the chemotaxis may lead to the formation of aggregates or inhomogeneous space patterns [3]). In [12], Lee et al. studied the continuous traveling waves for (1.2) and they showed that prey-taxis can reduce the likelihood of effective biocontrol. Wu et al. [27] investigated the global existence and boundedness of solutions of (1.2) under a smallness assumption on χ_0. Jin and Wang [9] proved the global boundedness of solution and stabilities of nonnegative spatially homogeneous equilibria of (1.2) in the two-dimensional case. Recently, It was shown in [24] that the prey-taxis destabilizes predator–prey homogeneity when prey repulsion is present (i.e. $\chi_0 < 0$). Moreover, the nonconstant positive steady states of a wide class of prey-taxis systems with general functional responses over 1-D domain were obtained in [24]. For more related works, we refer the readers to [5,16,18,19,25].

In the present paper, the initial data u_0, w_0, v_0 are supposed to satisfy

$$u_0,\ w_0,\ v_0 \geq, \not\equiv 0 \quad \text{and} \quad u_0,\ w_0,\ v_0 \in W^{1,\infty}(\Omega).$$

And we suppose that $\chi,\ h,\ f$ and g satisfy the following hypotheses [9,19,27]:

(A1) The function $\chi \in C^2([0,\infty)),\ \chi \geq 0$. The well known examples are

$$\text{(i) } \chi(s) = \chi_1, \quad \text{(ii) } \chi(s) = \frac{\chi_1}{s+\varepsilon}, \quad \text{(iii) } \chi(s) = \frac{\chi_1}{(s+\varepsilon)^2}$$

with positive constants χ_1, ε.

(A2) The function $g \in C^2([0,\infty)),\ g(0) = 0,\ g(s) > 0$ in $(0,\infty)$. The typical examples are

$$\text{(type I) } g(s) = \gamma s, \quad \text{(type II) } g(s) = \frac{\gamma s}{l+s},$$

$$\text{(type III) } g(s) = \frac{\gamma s^\kappa}{l^\kappa + s^\kappa}, \quad \text{(Ivlev type) } g(s) = \gamma(1 - e^{-ls}),$$

where γ, l, κ are positive constants and $\kappa > 1$.

(A3) The function $h \in C^2([0,\infty))$ and there exist two constants $a > 0$ and $\theta \geq 0$ such that $h(s) \geq a$ and $h'(s) \geq \theta$ in $[0,\infty)$. In some sense, the constant a can be regarded as the minimal death rate of the predator. The typical example is $h(s) = a + \theta s$.

(A4) The function $f \in C^2([0,\infty))$ satisfying $f(0) = 0$, and there exist two positive constants η, K such that $f(s) \leq \eta s$ for $s \geq 0$, $f(K) = 0$ and $f(s) < 0$ for $s > K$. Some examples are

$$\text{(logistic) } f(s) = \eta s\left(1 - \frac{s}{K}\right), \quad \text{(Allee effect) } f(s) = \eta' s\left(1 - \frac{s}{K}\right)\left(\frac{s}{N} - 1\right)$$

with $0 < N < K$ and $\eta' = \frac{4KN}{(K-N)^2}\eta$.

Throughout this paper we denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, and use C and C_i to denote the generic positive constants.

In contrast to the prey-taxis system (1.2), the model (1.1) involves chemoattractant which is released by the prey and attracts the predator. A natural question is: Does the chemoattractant affect the dynamical properties of the predator and prey? Our conclusions show that, in "most situations", the chemoattractant does not affect the dynamical properties of the predator and prey.

The first result of this paper asserts that the solution of the prey-taxis system (1.1) exists globally and maintains bounded. This property is the same as that of the classical problem of predator–prey model without prey-taxis:

$$\begin{cases} u_t = d_1\Delta u + bug(v) - uh(u), & x \in \Omega,\ t > 0, \\ v_t = d_3\Delta v + f(v) - ug(v), & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

Theorem 1.1 *Let $n \geq 1$ and the hypotheses* (A1)–(A4) *hold. Then* (1.1) *has a unique non-negative and bounded global solution* (u, w, v)*, and*

$$u, w, v \in C(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)).$$

Remark 1.1 We note that the solution of (1.2) exists globally in two-dimensional case ([9, Theorem 1.1]). In the higher dimensional case ($n \geq 3$), if χ_0 is small and $g(v) \leq c$ for some

$c > 0$, then (1.2) admits a unique nonnegative global bounded solution ([27, Theorem 1.1]). It remains unknown whether or not the solution of (1.2) blows up in higher dimensional case when χ_0 is large. However, for the system (1.1), Theorem 1.1 claims the global existence and boundedness of solution of (1.1). This also shows that, compared to the prey-taxis, the indirect prey-taxis will prevent the growth of the predator to ensure the global existence and boundedness of the solution.

The second goal of this paper is to understand the role of the indirect prey-taxis in the global stabilities of nonnegative spatially homogeneous equilibria of (1.1). The global stability of the prey-taxis system (1.2) has been studied in [9]. Therefore, we are able to compare the stability results of (1.1) with that of (1.2).

Let $\varphi(v) = f(v)/g(v)$. In order to achieve our aim, we shall need other assumptions [9]:

(A2)$'$ Function $g \in C^2([0,\infty))$, $g(0) = 0$, $g(s) > 0$ in $(0,\infty)$, and $g'(s) > 0$ in $[0,\infty)$.
(A5) Function $\varphi \in C^1([0,\infty))$, $\varphi(0) > 0$ and $\varphi'(s) < 0$ in $[0,\infty)$.

Remark 1.2 The Holling type I, type II and Ivlev type response functions satisfy the condition (A2)$'$. Moreover, if f is of logistic type and g is of Holling type I or type II with $l > K$, then (A5) is fulfilled. We should mention that (A5) can not be satisfied by the bistable function $f(v)$ or the Holling type III response function $g(v)$ (see [9]).

Let us first note that the possible homogeneous steady states of the system (1.1) are given by

$$(u_s, w_s, v_s) = \begin{cases} (0,\ 0,\ 0),\ (0,\ rK/\mu,\ K) & \text{if } bg(K) \le a, \\ (0,\ 0,\ 0),\ (0,\ rK/\mu,\ K),\ (u_*,\ rv_*/\mu,\ v_*) & \text{if } bg(K) > a, \end{cases}$$

where the positive constants $u_*,\ v_*$ are determined by

$$\begin{cases} bu_*g(v_*) - u_*h(u_*) = 0, \\ f(v_*) - u_*g(v_*) = 0. \end{cases} \tag{1.4}$$

It is easy to deduce that, if $g,\ h$ and f take biological meaningful forms like some of those given in (A2)$'$, (A3)–(A5), then (u_*, v_*) is uniquely determined and can be explicitly found. Hence, in what follows, we shall suppose that (1.4) has a unique positive solution (u_*, v_*). Moreover, if f and g satisfy the assumptions (A2)$'$ and (A4), then by the second equation of (1.4) we have $v_* < K$, and hence $m = \max\{\|v_0\|_\infty,\ K\} > v_*$.

In the case of $bg(K) > a$, we shall show that if the chemotactic coefficient $\chi(w)$ is small or one of the diffusion coefficients of the predator and chemical is large then the positive spatially homogeneous equilibrium $(u_*, rv_*/\mu, v_*)$ is globally asymptotically stable.

Theorem 1.2 *Assume $bg(K) > a$ and the hypotheses* (A1), (A2)$'$,(A3)–(A5) *are satisfied. Let (u, w, v) be the unique bounded global solution of* (1.1)*, which is given by Theorem* 1.1*. Set*

$$m = \max\{\|v_0\|_\infty,\ K\}, \quad M = \max\{\|w_0\|_\infty,\ rm/\mu\},$$

and

$$\hat{\chi} = \sup_{z\in[0,M]} \chi(z), \quad k_1 = \inf_{z\in[0,m]} g'(z), \quad k_2 = \inf_{z\in[0,m]} |\varphi'(z)|.$$

If

$$\frac{\hat{\chi}^2}{d_1 d_2} < \frac{16\mu b k_1 k_2}{r^2 u_*}, \tag{1.5}$$

then $(u_*, rv_*/\mu, v_*)$ *is globally asymptotically stable. Furthermore, if we further assume* $\theta > 0$*, then* $(u_*, rv_*/\mu, v_*)$ *is exponentially stable, i.e., there exist constants* $C, \lambda > 0$ *such that*

$$\|u - u_*\|_\infty + \|w - rv_*/\mu\|_\infty + \|v - v_*\|_\infty \le Ce^{-\lambda t}, \ \ \forall\, t > 0. \tag{1.6}$$

In the case of $bg(K) \le a$, the following theorem asserts that the semi-trivial spatially homogeneous equilibrium $(0,\ rK/\mu,\ K)$ is globally asymptotically stable.

Theorem 1.3 *Let the hypotheses* (A1), (A2)′,(A3)–(A5) *be satisfied and* (u, w, v) *be the unique bounded global solution of* (1.1)*, which is given by Theorem 1.1.*

(i) *If* $bg(K) < a$*, then* $(0,\ rK/\mu,\ K)$ *is globally asymptotically stable with exponential rate, i.e., there exist constants* $C, \lambda > 0$ *such that*

$$\|u\|_\infty + \|w - rK/\mu\|_\infty + \|v - K\|_\infty \le Ce^{-\lambda t}, \ \ \forall\, t > 0. \tag{1.7}$$

(ii) *If* $bg(K) = a$*, then* $(0,\ rK/\mu,\ K)$ *is globally asymptotically stable. Furthermore, if* $\theta > 0$*, then* $(0,\ rK/\mu,\ K)$ *is algebraically stable, i.e., there exist constants* $C, \lambda > 0$ *such that*

$$\|u\|_\infty + \|w - rK/\mu\|_\infty + \|v - K\|_\infty \le C(t+1)^{-\lambda}, \ \ \forall\, t > 0. \tag{1.8}$$

In the conditions (A3) and (A4), constants a and K can be considered as the minimal death rate of predator and carrying capacity of prey, respectively. Hence, the maximal value of the predation is $g(K)$. The cases $g(K) > a/b$ and $g(K) \le a/b$ can be regarded as the strong and weak predation, respectively.

In the strong predation case ($g(K) > a/b$), under our assumptions, the problem (1.1) has a unique positive constant steady state $(u_*,\ rv_*/\mu,\ v_*)$ and it is globally asymptotically stable. Furthermore, if $\theta > 0$, then $(u_*,\ rv_*/\mu,\ v_*)$ is also exponentially stable (Theorem 1.2).

Noticing that the condition (1.5) involves the coefficients d_2, μ, r. Hence, the chemoattractant plays an important role in the stability of $(u_*,\ rv_*/\mu,\ v_*)$. It is observed that the value of k_2 also affects the stability of $(u_*,\ rv_*/\mu,\ v_*)$. Moreover, from the condition (1.5) we discover that the diffusion rate of the prey does not influence the long time behavior of solution of (1.1). Since the predator responses to the chemoattractant released by prey rather than the prey itself, the diffusion of prey may be negligible in this case.

In the weak predation case ($g(K) \le a/b$), the problem (1.1) has no positive constant steady state and the semi-trivial constant steady state $(0,\ rK/\mu,\ K)$ is globally asymptotically stable (Theorem 1.3). This shows that, in the weak predation case, the presence of the chemoattractant does not influence the steady states and their stabilities for the problem (1.1). In contrast to the prey-taxis system (1.2) in such a case, please refer to [9, Theorem 1.3 (1)].

For the asymptotic behavior of solution, in contrast to the classical predator–prey model (1.3), we have the following assertions:

(i) in the weak predation case, the asymptotic dynamical properties of (1.1) are the same as those of (1.3).
(ii) in the strong predation case, under the assumption (1.5), the asymptotic dynamical properties of (1.1) are the same as those of (1.3).

The proofs of Theorems 1.2 and 1.3 rely on two Lyapunov functionals. The constructions of these Lyapunov functionals are inspired by [9]. However, the arguments leading to Theorems 1.2 and 1.3 are different from that of [9] which are based on LaSalle's invariant principle. Our method depends on an important lemma (see Lemma 3.1) and some basic arguments which seems friendlier to the readers.

The methods in the proofs of Theorems 1.2 and 1.3 can be applied to the model (1.2). The case $bg(K) = a$ and $\theta = 0$ was not considered in [9] for the problem (1.2). Using the method in the proof of Theorem 1.3 (ii), we can show that the semi-trivial spatially homogeneous equilibrium $(0, K)$ is globally asymptotically stable for the problem (1.2) in this case.

The article is organized as follows. Section 2 provides the uniqueness, global existence and boundedness of the classical solution of (1.1). Section 3 is devoted to proving the global stability results in Theorems 1.2 and 1.3. In the last section, we present two examples.

2 Existence, Uniqueness, Boundedness and Uniform Estimates of Global Solution

2.1 Existence and Uniqueness of Local Solution, Some Preliminaries

We first give a claim concerning the local-in-time existence of classical solution to (1.1).

Lemma 2.1 *There exists a $\hat{T} \in (0, \infty]$ and a unique nonnegative solution (u, w, v) of (1.1) defined in $[0, \hat{T})$ and satisfies*

$$u, w, v \in C(\bar{\Omega} \times [0, \hat{T})) \cap C^{2,1}(\bar{\Omega} \times (0, \hat{T})),$$

and

$$u,\ w > 0, \quad 0 < v \le m := \max\{\|v_0\|_\infty,\ K\} \ \text{ in } \ \Omega \times (0, \hat{T}). \tag{2.1}$$

Moreover, the "existence time $\hat{T}$" can be chosen maximal: either $\hat{T} = \infty$, or $\hat{T} < \infty$ and

$$\limsup_{t \to \hat{T}} (\|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty) = \infty.$$

Proof The local-in-time existence and uniqueness of classical solution to the problem (1.1) follow from Amann's theorem [2, Theorem 7.3 and Corollary 9.3] (cf. [27, Lemma 2.1]). The estimates (2.1) can be derived by the maximum principle.

Lemma 2.2 *The solution component w of (1.1) satisfies*

$$\|w(\cdot, t)\|_\infty \le M, \quad \forall\, t \in (0, \hat{T}), \tag{2.2}$$

where $M = \max\{\|w_0\|_\infty,\ rm/\mu\}$. And for any $p \in [2, \infty)$, there is $K_p = K(p) > 0$ such that

$$\|\nabla w(\cdot, t)\|_p \le K_p, \quad \forall\, t \in (0, \hat{T}). \tag{2.3}$$

Moreover, there exists a positive constant C such that the solution component u of (1.1) satisfies

$$\|u(\cdot, t)\|_1 < C, \quad \forall\, t \in (0, \hat{T}). \tag{2.4}$$

Proof By using (2.1) and the maximum principle, one can deduce from the w-equation in (1.1) that

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \le \max\{\|w_0\|_\infty,\ rm/\mu\} =: M, \quad \forall\, t \in (0, \hat{T}).$$

In view of the variation-of-constants formula, it yields

$$w(\cdot,t)=e^{t(d_2\Delta-\mu)}w_0+r\int_0^t e^{(t-s)(d_2\Delta-\mu)}v(\cdot,s)\mathrm{d}s,\quad t\in(0,\hat{T}).$$

Making use of (2.1) and the well-known semigroup estimates [6,8,26] we have that, for some $\lambda_1,\ C_i>0,\ i=1,...,5$,

$$\begin{aligned}\|\nabla w(\cdot,t)\|_p &\le \|\nabla e^{t(d_2\Delta-\mu)}w_0\|_p+r\int_0^t\|\nabla e^{(t-s)(d_2\Delta-\mu)}v(\cdot,s)\|_p\mathrm{d}s\\ &\le C_1e^{-\lambda_1 t}\|\nabla w_0\|_p+rC_2\int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\frac{1}{2}}\|v(\cdot,s)\|_p\mathrm{d}s\\ &\le C_3\|w_0\|_{W^{1,\infty}(\Omega)}+C_4\int_0^t e^{-\lambda_1(t-s)}(t-s)^{-\frac{1}{2}}\mathrm{d}s\\ &\le C_3\|w_0\|_{W^{1,\infty}(\Omega)}+C_5,\quad t\in(0,\hat{T}).\end{aligned}$$

This implies (2.3).

We next prove (2.4). It follows from the first and third equation in (1.1) that

$$\frac{d}{dt}\left(\int_\Omega u\mathrm{d}x+b\int_\Omega v\mathrm{d}x\right)+\int_\Omega uh(u)\mathrm{d}x=b\int_\Omega f(v)\mathrm{d}x,\quad t\in(0,\hat{T}).$$

Let $N_0=\sup_{z\in[0,m]}|f(z)|$. Recall the assumption (A3) and the estimate for v in (2.1), it yields

$$\frac{d}{dt}\left(\int_\Omega u\mathrm{d}x+b\int_\Omega v\mathrm{d}x\right)+a\int_\Omega u\mathrm{d}x+\int_\Omega v\mathrm{d}x\le C_6,\quad t\in(0,\hat{T}),\tag{2.5}$$

where $C_6=(bN_0+m)|\Omega|$. Applying the Gronwall's inequality to (2.5) we have (2.4). □

Next we provide a lemma which claims that the global existence and L^∞-boundedness of u can be reduced to proving its L^p-boundedness for $p>n/2$ and $p\ge 1$.

Lemma 2.3 *Let $n\ge 1$ and (u,w,v) be the unique solution of* (1.1) *in $\Omega\times(0,\hat{T})$. Suppose that there exists a number $p\ge 1$ and $p>n/2$ for which*

$$\sup_{t\in(0,\hat{T})}\|u(\cdot,t)\|_p<\infty.\tag{2.6}$$

Then $\hat{T}=\infty$ and

$$\sup_{t>0}\|u(\cdot,t)\|_\infty<\infty.\tag{2.7}$$

Proof The estimate (2.2) implies

$$|\chi(w)|\le\|\chi\|_{L^\infty(0,M)},\quad\forall\, t\in(0,\hat{T}).$$

Note that $(bug(v)-uh(u))_+\le bug(v)$ and

$$b\|ug(v)\|_p\le bN\|u\|_p,\quad t\in(0,\hat{T}),$$

where $N=\sup_{z\in[0,m]}g(z)$. Thanks to (2.1), (2.3) and (2.4), similar to the proof of [9, Lemma 3.1] (see also [3, Lemma 3.2]), one can deduce that $\hat{T}=\infty$ and (2.7) holds. □

2.2 Proof of Theorem 1.1

Let $n \geq 2$ and $p > n/2$. Clearly, $p > 1$. Note that

$$\frac{pn-n}{2-n+pn} \in (0,1).$$

Hence, we can choose $q > p$ such that

$$\beta := \frac{pn - pn/q}{2-n+pn} \in (0,1) \ \text{ and } \ \frac{q\beta}{p} \in (0,1). \tag{2.8}$$

Let

$$\hat{\chi} = \sup_{z\in[0,M]} \chi(z), \quad N = \sup_{z\in[0,m]} g(z).$$

Multiplying the first equation of (1.1) by u^{p-1} and integrating the results over Ω, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \mathrm{d}x &= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 \mathrm{d}x + (p-1)\int_\Omega u^{p-1}\chi(w)\nabla u\cdot\nabla w \mathrm{d}x \\
&\quad + b\int_\Omega u^p g(v)\mathrm{d}x - \int_\Omega u^p h(u)\mathrm{d}x \\
&\leq -\frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2\mathrm{d}x + \frac{p-1}{2}\int_\Omega u^p\chi^2(w)|\nabla w|^2\mathrm{d}x \\
&\quad + b\int_\Omega u^p g(v)\mathrm{d}x - \int_\Omega u^p h(u)\mathrm{d}x \\
&\leq -\frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2\mathrm{d}x + \frac{(p-1)\hat{\chi}^2}{2}\int_\Omega u^p|\nabla w|^2\mathrm{d}x \\
&\quad + (bN-a)\int_\Omega u^p\mathrm{d}x \\
&= -\frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2\mathrm{d}x + \frac{(p-1)\hat{\chi}^2}{2}\int_\Omega u^p|\nabla w|^2\mathrm{d}x \\
&\quad + (bN-a)\int_\Omega u^p\mathrm{d}x, \quad t\in(0,\hat{T}),
\end{aligned} \tag{2.9}$$

where we have used Young's inequality, (2.1) and (2.2) and the assumption (A3). By use of Young's inequality again and (2.3), it yields

$$\frac{(p-1)\hat{\chi}^2}{2}\int_\Omega u^p|\nabla w|^2\mathrm{d}x \leq \frac{1}{2}\int_\Omega u^q\mathrm{d}x + C_1, \quad \forall\, t\in(0,\hat{T}) \tag{2.10}$$

with some $C_1 > 0$, and there is $C_2 > 0$ such that

$$bN\int_\Omega u^p\mathrm{d}x \leq \frac{1}{2}\int_\Omega u^q\mathrm{d}x + C_2, \quad \forall\, t\in(0,\hat{T}). \tag{2.11}$$

Inserting (2.10) and (2.11) into (2.9) gives

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\mathrm{d}x + a\int_\Omega u^p\mathrm{d}x + \frac{2(p-1)}{p^2}\int_\Omega|\nabla u^{\frac{p}{2}}|^2\mathrm{d}x \leq \int_\Omega u^q\mathrm{d}x + C_3 \tag{2.12}$$

for all $t \in (0, \hat{T})$, where $C_3 = C_1 + C_2$. Note that (2.8). Taking advantage of the Gagliardo-Nirenberg inequality and (2.4) firstly, and using the Young's inequality secondly, we have

$$\begin{aligned}\int_\Omega u^q \mathrm{d}x = \|u^{p/2}\|_{2q/p}^{2q/p} &\le C_4(\|\nabla u^{p/2}\|_2^{2q\beta/p}\|u^{p/2}\|_{2/p}^{2q(1-\beta)/p} + \|u^{p/2}\|_{2/p}^{2q/p})\\ &\le C_5(\|\nabla u^{p/2}\|_2^{2q\beta/p} + 1)\\ &\le \frac{2(p-1)}{p^2}\|\nabla u^{p/2}\|_2^2 + C_6\\ &= \frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \mathrm{d}x + C_6, \ \ \forall\, t \in (0, \hat{T}).\end{aligned} \tag{2.13}$$

Combined (2.13) with (2.12) allows us to deduce

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \mathrm{d}x + a\int_\Omega u^p \mathrm{d}x \le C_7, \ \ \forall\, t \in (0, \hat{T}).$$

Thus we have, by the Gronwall inequality,

$$\|u(\cdot, t)\|_p \le C_8, \quad t \in (0, \hat{T}). \tag{2.14}$$

Using (2.4) and Lemma 2.3 with $p = 1$ when $n = 1$, and using (2.14) and Lemma 2.3 when $n \ge 2$, we can get the conclusion of Theorem 1.1 immediately.

2.3 Uniform Estimates of the Global Solution

Theorem 2.1 *Let (u, w, v) be the unique global bounded classical solution of (1.1), which is given by Theorem 1.1. Then for any given $0 < \alpha < 1$, there exists $C(\alpha) > 0$ such that*

$$\|u, \ w, \ v\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} \le C(\alpha). \tag{2.15}$$

Proof This proof is based on the standard parabolic regularity for parabolic equations (cf. [20, Theorem 2.1], [21, Theorem 2.1] and [23, Theorem 2.2]). For the reader's convenience, we sketch the proof here. Applying the interior L^p estimate ([14, Theorems 7.30 and 7.35]) to the equations of w and v firstly and using the Sobolev embedding theorem secondly we have

$$\|w, v\|_{W_p^{2,1}\left(\Omega\times\left[i+\frac{1}{4}, i+3\right]\right)} + \|w, v\|_{C^{1+\alpha, \frac{1+\alpha}{2}}\left(\bar{\Omega}\times\left[i+\frac{1}{4}, i+3\right]\right)} \le C_1, \quad \forall\, i \ge 0,$$

and hence

$$\|w, v\|_{C^{1+\alpha, \frac{1+\alpha}{2}}\left(\bar{\Omega}\times\left[\frac{1}{4}, \infty\right)\right)} \le C_2. \tag{2.16}$$

Note that w satisfies

$$\begin{cases} w_t - d_2\Delta w + \mu w = rv, & x \in \Omega, \ t > 0,\\ \partial_\nu w = 0, & x \in \partial\Omega, \ t > 0,\\ w(x, 0) = w_0(x), & x \in \Omega.\end{cases}$$

By use of the interior Schauder estimate [11] and (2.16),

$$\|w\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}\left(\bar{\Omega}\times\left[i+\frac{1}{3}, i+3\right]\right)} \le C_3, \quad \forall\, i \ge 0,$$

which implies

$$\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{\Omega}\times\left[\frac{1}{3},\infty\right)\right)} \le C_4. \tag{2.17}$$

Rewrite the equation of u in (1.1) as

$$\begin{cases} u_t - d_1\Delta u + \chi(w)\nabla w\cdot\nabla u = G(x,t), & x\in\Omega,\ t>0,\\ \partial_\nu u = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{2.18}$$

where

$$G(x,t) = -u\chi'(w)|\nabla w|^2 - u\chi(w)\Delta w + bug(v) - uh(u).$$

Due to (2.16), (2.17) and the boundedness of (u, w, v), we see that $\|G\|_{L^\infty(\Omega\times[i+\frac{1}{3},i+3])} \le C_5$ and $\|\chi(w)\nabla w\|_{L^\infty(\Omega\times[i+\frac{1}{3},i+3])} \le C_5$ for all $i \ge 0$. Applying the interior L^p estimate to (2.18) we have $\|u\|_{W_p^{2,1}(\Omega\times[i+\frac{1}{2},i+3])} \le C_6$ for all $i\ge 0$. Then the embedding theorem gives

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}\left(\bar{\Omega}\times\left[i+\frac{1}{2},i+3\right]\right)} \le C_7,\quad \forall\, i\ge 0. \tag{2.19}$$

It then follows that

$$\|bug(v) - uh(u)\|_{C^{\alpha,\frac{\alpha}{2}}\left(\Omega\times\left[i+\frac{1}{2},i+3\right]\right)} \le C_8,\quad \forall\, i\ge 0.$$

This combined with (2.17) yields

$$\|G\|_{C^{\alpha,\frac{\alpha}{2}}\left(\bar{\Omega}\times\left[i+\frac{1}{2},i+3\right]\right)} + \|\chi(w)\nabla w\|_{C^{\alpha,\frac{\alpha}{2}}\left(\bar{\Omega}\times\left[i+\frac{1}{2},i+3\right]\right)} \le C_9,\quad \forall\, i\ge 0.$$

Applying the interior Schauder estimate to (2.18) we have $\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[i+1,i+3])} \le C_{10}$ for all $i\ge 0$. Thus,

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} \le C_{11}. \tag{2.20}$$

Similarly, thanks to (2.16) and (2.19), we can apply the interior Schauder estimate to the equation of v and get

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} \le C_{12}. \tag{2.21}$$

Then (2.15) follows from (2.17), (2.20) and (2.21). The proof is complete. □

3 Global Stability

Throughout this section we always assume that (u, w, v) is a bounded global solution of (1.1). We shall prove Theorems 1.2 and 1.3 by constructing suitable Lyapunov functionals. Let us first recall two basic results.

Lemma 3.1 ([22, Lemma 1.1]) *Let $\tau\ge 0$, $c>0$ be constants, $\psi(t)\ge 0$, $\int_\tau^\infty \rho(t)\mathrm{d}t < \infty$. Assume that $\varphi\in C^1([\tau,\infty))$, φ is bounded from below and satisfies*

$$\varphi'(t) \le -c\psi(t) + \rho(t) \quad in\ [\tau,\infty).$$

If either $\psi\in C^1([\tau,\infty))$ and $\psi'(t)\le k$ in $[\tau,\infty)$ for some constant $k>0$, or $\psi\in C^\alpha([\tau,\infty))$ and $\|\psi\|_{C^\alpha([\tau,\infty))}\le k$ for some constants $0<\alpha<1$ and $k>0$, then $\lim_{t\to\infty}\psi(t)=0$.

Lemma 3.2 ([9, Lemma 4.1]) *Let g satisfy the conditions in* (A2)$'$ *and* (u, w, v) *be a solution of* (1.1). *Define*

$$\zeta(v) = \int_k^v \frac{g(s) - g(k)}{g(s)} \mathrm{d}s$$

for the constant $k > 0$. Then ζ is a convex function and $\zeta \geq 0$ on $[0, \infty)$. Furthermore, if $v \to k$ as $t \to \infty$, then there exists a constant $T_0 > 0$ such that for all $t \geq T_0$ there holds

$$\frac{g'(k)(v-k)^2}{4g(k)} \leq \zeta(v) \leq \frac{g'(k)(v-k)^2}{g(k)}. \tag{3.1}$$

3.1 Global Stability of $(u_*, rv_*/\mu, v_*)$: Proof of Theorem 1.2

In this subsection we always assume that $bg(K) > a$ and (1.5) holds. The constant K is given in the assumption (A4), and (u_*, v_*) is given by (1.4). For the convenience, let $w_* = rv_*/\mu$. Due to (1.5), we fix a constant δ such that

$$\frac{u_* \hat{\chi}^2}{4d_1 d_2} < \delta < \frac{4\mu b k_1 k_2}{r^2}, \tag{3.2}$$

where $\hat{\chi}, k_1, k_2$ are given by Theorem 1.2.

Lemma 3.3 *Let δ be given by* (3.2). *Let the conditions in Theorem 1.2 hold. Then there is $\varepsilon > 0$ such that functions $E_1(t)$, $F_1(t)$ defined by*

$$E_1(t) = \int_\Omega \left[\left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \frac{\delta}{2}(w - w_*)^2 + b\int_{v_*}^v \frac{g(s) - g(v_*)}{g(s)} \mathrm{d}s\right] \mathrm{d}x,$$
$$F_1(t) = \theta \int_\Omega (u - u_*)^2 \mathrm{d}x + \varepsilon \int_\Omega \left[(v - v_*)^2 + (w - w_*)^2 + |\nabla u|^2\right] \mathrm{d}x$$

satisfy

$$E_1'(t) \leq -F_1(t), \quad t > 0. \tag{3.3}$$

Proof For the convenience, we set

$$A_1(t) = \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) \mathrm{d}x,$$
$$B_1(t) = \frac{\delta}{2} \int_\Omega (w - w_*)^2 \mathrm{d}x,$$
$$D_1(t) = b\int_\Omega \int_{v_*}^v \frac{g(s) - g(v_*)}{g(s)} \mathrm{d}s\mathrm{d}x.$$

Evidently, $A_1(t), B_1(t), D_1(t) \geq 0$. Let $\hat{\chi} = \sup_{z \in [0, M]} \chi(z)$, where M is given by (2.2). Since (u, w, v) is the global bounded solution to (1.1), there is $c > 0$ such that

$$\|u(\cdot, t)\|_\infty \leq c \quad \text{for all } t \in [0, \infty). \tag{3.4}$$

Let us recall from the assumption (A3) that $h'(s) \geq \theta$ for $s \in [0, \infty)$. The straightforward calculation gives

$$\begin{aligned}
A_1'(t) =& -d_1 u_* \int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x + u_* \int_\Omega \frac{\chi(w)\nabla u \cdot \nabla w}{u}\mathrm{d}x + b\int_\Omega (u-u_*)[g(v)-g(v_*)]\mathrm{d}x \\
& - \int_\Omega (u-u_*)(h(u)-h(u_*))\mathrm{d}x \\
\leq & -d_1 u_* \int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x + u_* \hat{\chi} \int_\Omega \left| \frac{\nabla u}{u} \cdot \nabla w \right| \mathrm{d}x + b\int_\Omega (u-u_*)[g(v)-g(v_*)]\mathrm{d}x \\
& -\theta \int_\Omega (u-u_*)^2 \mathrm{d}x,
\end{aligned}$$

and

$$B_1'(t) = -\delta d_2 \int_\Omega |\nabla w|^2 \mathrm{d}x - \delta\mu \int_\Omega (w-w_*)^2\mathrm{d}x + \delta r \int_\Omega (w-w_*)(v-v_*)\mathrm{d}x,$$

as well as

$$\begin{aligned}
D_1'(t) =& \, b\int_\Omega \frac{g(v)-g(v_*)}{g(v)} v_t \mathrm{d}x \\
=& \, b\int_\Omega \frac{g(v)-g(v_*)}{g(v)} [d_3 \Delta v + f(v) - ug(v)]\mathrm{d}x \\
=& -bd_3 g(v_*) \int_\Omega \frac{g'(v)}{g^2(v)} |\nabla v|^2 \mathrm{d}x + b\int_\Omega [g(v)-g(v_*)]\left(\frac{f(v)}{g(v)} - u\right) \\
=& -bd_3 g(v_*) \int_\Omega \frac{g'(v)}{g^2(v)} |\nabla v|^2 \mathrm{d}x - b\int_\Omega [g(v)-g(v_*)](u-u_*)\mathrm{d}x \\
& + b\int_\Omega [g(v)-g(v_*)]\left(\frac{f(v)}{g(v)} - u_*\right)\mathrm{d}x \\
\leq & -b\int_\Omega [g(v)-g(v_*)](u-u_*)\mathrm{d}x + b\int_\Omega [g(v)-g(v_*)]\left(\frac{f(v)}{g(v)} - u_*\right)\mathrm{d}x.
\end{aligned}$$

Thus we have

$$E_1'(t) \leq I_1(t) + I_2(t), \tag{3.5}$$

where

$$\begin{aligned}
I_1(t) =& -d_1 u_* \int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x + u_* \hat{\chi} \int_\Omega \left| \frac{\nabla u}{u} \cdot \nabla w \right| \mathrm{d}x - \delta d_2 \int_\Omega |\nabla w|^2 \mathrm{d}x, \\
I_2(t) =& -\theta \int_\Omega (u-u_*)^2 \mathrm{d}x - \delta\mu \int_\Omega (w-w_*)^2 \mathrm{d}x + \delta r \int_\Omega (w-w_*)(v-v_*)\mathrm{d}x \\
& + b\int_\Omega [g(v)-g(v_*)]\left(\frac{f(v)}{g(v)} - u_*\right)\mathrm{d}x.
\end{aligned} \tag{3.6}$$

We first deal with $I_1(t)$. An application of the Young inequality yields

$$u_* \hat{\chi} \int_\Omega \left| \frac{\nabla u}{u} \cdot \nabla w \right| \mathrm{d}x \leq \frac{u_*^2 \hat{\chi}^2}{4\delta d_2} \int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x + \delta d_2 \int_\Omega |\nabla w|^2 \mathrm{d}x.$$

Consequently,

$$I_1(t) \le -\left(u_* d_1 - \frac{u_*^2 \hat{\chi}^2}{4\delta d_2}\right) \int_\Omega \frac{|\nabla u|^2}{u^2} \mathrm{d}x := -\varepsilon_0 \int_\Omega \frac{|\nabla u|^2}{u^2} \mathrm{d}x.$$

It follows from (3.2) that $\varepsilon_0 > 0$, and hence by (3.4),

$$I_1(t) \le -\frac{\varepsilon_0}{c^2} \int_\Omega |\nabla u|^2 \mathrm{d}x := -\varepsilon_1 \int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{3.7}$$

We next handle $I_2(t)$. It follows from the second equation of (1.4) that $u_* = f(v_*)/g(v_*)$. Thanks to the definitions of $k_1,\ k_2$, the last term in the right hand side of (3.6) can be estimated as

$$\begin{aligned} & b \int_\Omega [g(v) - g(v_*)] \left(\frac{f(v)}{g(v)} - u_*\right) \mathrm{d}x \\ &= b \int_\Omega [g(v) - g(v_*)] \left(\frac{f(v)}{g(v)} - \frac{f(v_*)}{g(v_*)}\right) \mathrm{d}x \\ &= b \int_\Omega [g(v) - g(v_*)][\varphi(v) - \varphi(v_*)] \mathrm{d}x \\ &= b \int_\Omega g'(\xi_1)\varphi'(\xi_2)(v - v_*)^2 \mathrm{d}x \le -bk_1k_2 \int_\Omega (v - v_*)^2 \mathrm{d}x, \end{aligned} \tag{3.8}$$

where $\xi_1,\ \xi_2$ are between v and v_*, and k_1, k_2 come from Theorem 1.2. Insert (3.8) into (3.6) yields

$$\begin{aligned} I_2(t) \le & -\theta \int_\Omega (u - u_*)^2 \mathrm{d}x - \delta\mu \int_\Omega (w - w_*)^2 \mathrm{d}x + \delta r \int_\Omega (w - w_*)(v - v_*) \mathrm{d}x \\ & - bk_1k_2 \int_\Omega (v - v_*)^2 \mathrm{d}x. \end{aligned} \tag{3.9}$$

Note that $bk_1k_2 > \delta r^2/(4\mu)$ by (3.2), we can choose $\varepsilon_2 > 0$ small such that

$$\delta\mu - \varepsilon_2 > 0, \quad \varepsilon_3 := bk_1k_2 - \frac{\delta^2 r^2}{4(\delta\mu - \varepsilon_2)} > 0.$$

Again, by Young's inequality, there holds

$$\delta r \int_\Omega (w - w_*)(v - v_*) \mathrm{d}x \le (\delta\mu - \varepsilon_2) \int_\Omega (w - w_*)^2 \mathrm{d}x + \frac{\delta^2 r^2}{4(\delta\mu - \varepsilon_2)} \int_\Omega (v - v_*)^2 \mathrm{d}x.$$

This combined with (3.9) allows us to derive

$$I_2(t) \le -\theta \int_\Omega (u - u_*)^2 \mathrm{d}x - \varepsilon_2 \int_\Omega (w - w_*)^2 \mathrm{d}x - \varepsilon_3 \int_\Omega (v - v_*)^2 \mathrm{d}x. \tag{3.10}$$

Finally, according to (3.5), (3.7) and (3.10), by choosing $\varepsilon = \min\{\varepsilon_1,\ \varepsilon_2, \varepsilon_3\}$ we then get (3.3). □

Lemma 3.4 *Under the conditions of Theorem 1.2, for any $0 < \alpha < 1$, the following holds:*

$$\|u - u_*\|_{C^{2+\alpha}(\bar{\Omega})} + \|w - w_*\|_{C^{2+\alpha}(\bar{\Omega})} + \|v - v_*\|_{C^{2+\alpha}(\bar{\Omega})} \to 0 \ \text{ as } t \to \infty. \tag{3.11}$$

Proof Let $E_1(t),\ F_1(t)$ be given Lemma 3.3. Clearly, $E_1(t) \ge 0$ as $g'(s) > 0$ in $[0, \infty)$. Thanks to (2.15), it is easy to see that $F_1(t) \in C^{\alpha/2}([1, \infty))$ and $\|F_1\|_{C^{\alpha/2}([1,\infty))} \le k$ in $[1, \infty)$

for some constant $k > 0$. Recall (3.3), we can apply Lemma 3.1 to deduce $\lim_{t\to\infty} F_1(t) = 0$. That is,

$$\lim_{t\to\infty} \left(\|w - w_*\|_2 + \|v - v_*\|_2 + \|\nabla u\|_2\right) = 0,$$

and

$$\lim_{t\to\infty} \|u - u_*\|_2 = 0 \quad \text{if} \quad \theta > 0.$$

Take $0 < \alpha < \alpha' < 1$. According to Theorem 2.1, in the space $C^{2+\alpha'}(\bar{\Omega})$, $u(\cdot,t)$, $w(\cdot,t)$ and $v(\cdot,t)$ are bounded for $t \ge 1$. Using the compact arguments and uniqueness of limits we can show that (3.11) holds when $\theta > 0$, and

$$\lim_{t\to\infty} \left(\|w - w_*\|_{C^{2+\alpha}(\bar{\Omega})} + \|v - v_*\|_{C^{2+\alpha}(\bar{\Omega})}\right) = 0 \tag{3.12}$$

when $\theta = 0$.

In the following we consider the case $\theta = 0$. Define $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f\mathrm{d}x$ for $f \in L^1(\Omega)$. It follows from the third equation of (1.1) that

$$\begin{aligned}
\bar{v}'(t) &= \frac{1}{|\Omega|}\int_\Omega [f(v) - ug(v)]\mathrm{d}x \\
&= \frac{1}{|\Omega|}\int_\Omega [f(v) - f(v_*)]\mathrm{d}x - \frac{1}{|\Omega|}\int_\Omega u[g(v) - g(v_*)]\mathrm{d}x \\
&\quad - \frac{g(v_*)}{|\Omega|}\int_\Omega (u - u_*)\mathrm{d}x \\
&=: J_1(t) + J_2(t) + J_3(t), \quad \forall\, t \in (0,\infty).
\end{aligned} \tag{3.13}$$

It follows from (3.12) that $\lim_{t\to\infty}[J_1(t)+J_2(t)] = 0$. Recall (2.15), we have $\|\bar{v}'\|_{C^{\alpha/2}([1,\infty))} \le k$ for some positive constant k. This combined with (3.12) yields $\bar{v}'(t) \to 0$ as $t \to \infty$. Therefore, in view of (3.13), there holds $J_3(t) \to 0$ as $t \to \infty$, i.e.,

$$\bar{u}(t) \to u_* \quad \text{as } t \to \infty. \tag{3.14}$$

Making use of the Poincaré inequality $\|u-\bar{u}\|_2 \le C\|\nabla u\|_2$ with $C > 0$, we have $\|u-\bar{u}\|_2 \to 0$ as $t \to \infty$. This combined with (3.14) implies

$$\|u - u_*\|_2 \le \|u - \bar{u}\|_2 + \|\bar{u} - u_*\|_2 \to 0 \quad \text{as } t \to \infty.$$

Similar to the above we can prove (3.11). This completes the proof. □

Proof of (1.6) Let $\theta > 0$. For the given positive constant y_*, we define $h(y) = y - y_* \ln y$ for $y > 0$. By L'Hôpital's rule, one can easily check that

$$\lim_{y\to y_*} \frac{h(y) - h(y_*)}{(y - y_*)^2} = \lim_{y\to y_*} \frac{h'(y)}{2(y - y_*)} = \frac{1}{2y_*}.$$

Remember the limit (3.11) and (3.1), it follows that there is $t_0 > 1$ such that

$$\frac{1}{4u_*}\int_\Omega (u - u_*)^2\mathrm{d}x \le \int_\Omega \left(u - u_* - u_* \ln\frac{u}{u_*}\right)\mathrm{d}x \le \frac{1}{u_*}\int_\Omega (u - u_*)^2\mathrm{d}x, \tag{3.15}$$

$$\frac{g'(v_*)}{4g(v_*)}\int_\Omega (v - v_*)^2\mathrm{d}x \le \int_\Omega \int_{v_*}^{v} \frac{g(s) - g(v_*)}{g(s)}\mathrm{d}s\mathrm{d}x \le \frac{g'(v_*)}{g(v_*)}\int_\Omega (v - v_*)^2\mathrm{d}x \tag{3.16}$$

for all $t > t_0$. Recall the definitions of $E_1(t)$ and $F_1(t)$, it follows from the right inequalities in (3.15)–(3.16) that $E_1(t) \le C_1 F_1(t)$ for all $t > t_0$ and some $C_1 > 0$. Inserting this into (3.3) we get

$$E_1'(t) \le -F_1(t) \le -\frac{1}{C_1} E_1(t) \quad \text{for} \quad t > t_0.$$

Thus, $E_1(t) \le C_2 e^{-\sigma t}$ for $t > t_0$ and some $C_2,\ \sigma > 0$. In view of the left inequalities in (3.15)–(3.16), there exist $C_3,\ C_4 > 0$ such that

$$\int_\Omega (u-u_*)^2 \mathrm{d}x + \int_\Omega (v-v_*)^2 \mathrm{d}x + \int_\Omega (w-w_*)^2 \mathrm{d}x \le C_3 E_1(t) \le C_4 e^{-\sigma t}, \quad t > t_0.$$

Recall that $u(\cdot,t)$, $w(\cdot,t)$ and $v(\cdot,t)$ are bounded in $W^{1,\infty}(\Omega)$ for $t > 1$. Thanks to the Gagliardo-Nirenberg inequality (with $C_{gn} > 0$)

$$\|\psi\|_\infty \le C_{gn} \|\psi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|\psi\|_2^{\frac{2}{n+2}}, \quad \forall\, \psi \in W^{1,\infty}(\Omega), \tag{3.17}$$

we can find $C,\ \lambda > 0$ such that

$$\|u-u_*\|_\infty + \|v-v_*\|_\infty + \|w-w_*\|_\infty \le Ce^{-\lambda t}, \quad t > t_0.$$

Thus (1.6) holds, and the proof is complete. □

3.2 Global Stability of $(0,\ rK/\mu,\ K)$: Proof of Theorem 1.3

Throughout this subsection we always assume that $bg(K) \le a$. For the convenience, we denote $\hat{K} = rK/\mu$.

Lemma 3.5 *Assume that $bg(K) \le a$. Let k_1, k_2 be as in Theorem 1.2 and $0 < \delta_1 < (2\mu b k_1 k_2)/(r^2)$. Then functions $E_2(t)$, $F_2(t)$ defined by*

$$E_2(t) = \int_\Omega \left(u + \frac{\delta_1}{2}(w-\hat{K})^2 + b\int_K^v \frac{g(s)-g(K)}{g(s)} \mathrm{d}s \right) \mathrm{d}x,$$
$$F_2(t) = (a - bg(K)) \int_\Omega u \mathrm{d}x + \theta \int_\Omega u^2 \mathrm{d}x + \varepsilon_4 \left(\int_\Omega (w-\hat{K})^2 \mathrm{d}x + \int_\Omega (v-K)^2 \mathrm{d}x \right)$$

satisfy

$$E_2'(t) \le -F_2(t), \quad t > 0, \tag{3.18}$$

where $\varepsilon_4 = \min\{\delta_1\mu/2,\ bk_1k_2 - \delta_1 r^2/(2\mu)\} > 0$.

Proof In view of the assumption (A3), we have $h(u) \ge a + \theta u$, which implies that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u \mathrm{d}x &= \int_\Omega u(bg(v) - h(u))\mathrm{d}x \\
&= b\int_\Omega u(g(v)-g(K))\mathrm{d}x + \int_\Omega u(bg(K)-a)\mathrm{d}x + \int_\Omega u(a-h(u))\mathrm{d}x \\
&\le b\int_\Omega u(g(v)-g(K))\mathrm{d}x + (bg(K)-a)\int_\Omega u\mathrm{d}x - \theta\int_\Omega u^2\mathrm{d}x.
\end{aligned}$$

Similar to the proof of Lemma 3.3, by a series of calculations we can get

$$\frac{\delta_1}{2}\frac{d}{dt}\int_\Omega (w-\hat{K})^2\mathrm{d}x = -\delta_1 d_2\int_\Omega |\nabla w|^2\mathrm{d}x - \delta_1\mu\int_\Omega (w-\hat{K})^2\mathrm{d}x + \delta_1 r\int_\Omega (w-\hat{K})(v-K)\mathrm{d}x,$$

and

$$b\frac{d}{dt}\int_\Omega\int_K^v \frac{g(s)-g(K)}{g(s)}\mathrm{d}s\mathrm{d}x \le -b\int_\Omega u(g(v)-g(K))\mathrm{d}x + b\int_\Omega [g(v)-g(K)]\left(\frac{f(v)}{g(v)}-\frac{f(K)}{g(K)}\right)\mathrm{d}x.$$

Hence, there holds

$$E_2'(t) \le (bg(K)-a)\int_\Omega u\mathrm{d}x - \theta\int_\Omega u^2\mathrm{d}x + I_3(t), \tag{3.19}$$

where

$$I_3(t) = -\delta_1\mu\int_\Omega (w-\hat{K})^2\mathrm{d}x + \delta_1 r\int_\Omega (w-\hat{K})(v-K)\mathrm{d}x + b\int_\Omega [g(v)-g(K)][\varphi(v)-\varphi(K)]\mathrm{d}x \tag{3.20}$$

with $\varphi(v) = f(v)/g(v)$ and $\varphi(K) = f(K)/g(K)$. The last term in the right hand side of (3.20) can be estimated as

$$b\int_\Omega [g(v)-g(K)][\varphi(v)-\varphi(K)]\mathrm{d}x = b\int_\Omega g'(\xi_3)\varphi'(\xi_4)(v-K)^2\mathrm{d}x \le -bk_1k_2\int_\Omega (v-K)^2\mathrm{d}x, \tag{3.21}$$

where ξ_3 and ξ_4 are between v and K, and k_1, k_2 come from Theorem 1.2. Inserting (3.21) into (3.20) and applying the Young's inequality to derive that

$$\begin{aligned} I_3(t) &\le -\delta_1\mu\int_\Omega (w-\hat{K})^2\mathrm{d}x + \delta_1 r\int_\Omega (w-\hat{K})(v-K)\mathrm{d}x - bk_1k_2\int_\Omega (v-K)^2\mathrm{d}x \\ &\le -\frac{\delta_1\mu}{2}\int_\Omega (w-\hat{K})^2\mathrm{d}x - \left(bk_1k_2 - \frac{\delta_1 r^2}{2\mu}\right)\int_\Omega (v-K)^2\mathrm{d}x \\ &:= -\varepsilon_4\left(\int_\Omega (w-\hat{K})^2\mathrm{d}x + \int_\Omega (v-K)^2\mathrm{d}x\right), \end{aligned}$$

where $\varepsilon_4 = \min\{\delta_1\mu/2,\ bk_1k_2 - \delta_1 r^2/(2\mu)\} > 0$. This combined with (3.19) gives (3.18).
□

Proof of Theorem 1.3 (i) Assume that $bg(K) < a$. Let $E_2(t)$ and $F_2(t)$ be given in Lemma 3.5, then $E_2'(t) \le -F_2(t)$. Clearly, $F_2(t) \ge 0$. Similar to the arguments in the proof of Lemma 3.4, one can deduce that, for any $0 < \alpha < 1$,

$$\|u\|_{C^{2+\alpha}(\bar{\Omega})} + \|w-\hat{K}\|_{C^{2+\alpha}(\bar{\Omega})} + \|v-K\|_{C^{2+\alpha}(\bar{\Omega})} \to 0 \ \text{ as } t\to\infty. \tag{3.22}$$

□

According to (3.22) and Lemma 3.2, there exists $t_0 > 1$ such that

$$\frac{g'(K)}{4g(K)}\int_\Omega (v-K)^2\mathrm{d}x \le \int_\Omega\int_K^v \frac{g(s)-g(K)}{g(s)}\mathrm{d}s\mathrm{d}x \le \frac{g'(K)}{g(K)}\int_\Omega (v-K)^2\mathrm{d}x,\quad t>t_0. \tag{3.23}$$

In view of the definitions of $E_2(t)$, $F_2(t)$ and the right inequality in (3.23), we get

$$E_2(t) \le C_1F_2(t),\quad t>t_0.$$

It follows that

$$E_2'(t) \le -F_2(t) \le -\frac{E_2(t)}{C_1},\quad t>t_0.$$

This implies that there exist $C_2, \sigma > 0$ such that $E_2(t) \le C_2e^{-\sigma t}$ for $t > t_0$. By the left inequality in (3.23) we have

$$\int_\Omega u\mathrm{d}x + \int_\Omega (w-\hat{K})^2\mathrm{d}x + \int_\Omega (v-K)^2\mathrm{d}x \le C_3E_2(t) \le C_4e^{-\sigma t},\quad t>t_0. \tag{3.24}$$

In light of (3.22), in the space $W^{1,\infty}(\Omega)$, $u(\cdot,t)$, $w(\cdot,t)$ and $v(\cdot,t)$ are bounded for $t > 1$. Making use of the Gagliardo–Nirenberg inequality

$$\|\psi\|_\infty \le C_5\|\psi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+1}}\|\psi\|_1^{\frac{1}{n+1}},\quad \forall\, \psi\in W^{1,\infty}(\Omega)$$

and (3.24), we have

$$\|u\|_\infty \le C_5\|u\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+1}}\|u\|_1^{\frac{1}{n+1}} \le C_6e^{-\frac{\sigma t}{n+1}},\quad t>t_0. \tag{3.25}$$

Similarly, it follows from the Gagliardo–Nirenberg inequality (3.17) and (3.24) that

$$\|w-\hat{K}\|_\infty \le C_{gn}\|w-\hat{K}\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|w-\hat{K}\|_2^{\frac{2}{n+2}} \le C_7e^{-\frac{\sigma t}{n+2}},\quad t>t_0, \tag{3.26}$$

$$\|v-K\|_\infty \le C_{gn}\|v-K\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|v-K\|_2^{\frac{2}{n+2}} \le C_8e^{-\frac{\sigma t}{n+2}},\quad t>t_0. \tag{3.27}$$

Thanks to (3.25)–(3.27), the statement in Theorem 1.3 (i) is followed immediately.

Proof of Theorem 1.3 (ii) We first consider the case $bg(K) = a$ and $\theta > 0$. Let $E_2(t)$ and $F_2(t)$ be given in Lemma 3.5, then $E_2'(t) \le -F_2(t)$. Clearly, $F_2(t) \ge 0$. In the present situation,

$$F_2(t) = \theta\int_\Omega u^2\mathrm{d}x + \varepsilon_4\int_\Omega (w-\hat{K})^2\mathrm{d}x + \varepsilon_4\int_\Omega (v-K)^2\mathrm{d}x,\quad t>0.$$

Similarly to the above, we can show that (3.22) holds. Let $t_0 > 1$ be as in the proof of Theorem 1.3 (i). Using (3.23), the Cauchy-Schwarz inequality and boundedness of (u, w, v) we can find $C_9 > 0$ such that

$$\begin{aligned} E_2(t) &\le \int_\Omega u \mathrm{d}x + \frac{\delta_1}{2}\int_\Omega (w-\hat{K})^2 \mathrm{d}x + \frac{g'(K)}{g(K)}\int_\Omega (v-K)^2 \mathrm{d}x \\ &\le C_9\left(\int_\Omega u^2 \mathrm{d}x\right)^{1/2} + C_9\left(\int_\Omega (w-\hat{K})^2 \mathrm{d}x\right)^{1/2} + C_9\left(\int_\Omega (v-K)^2 \mathrm{d}x\right)^{1/2} \\ &\le \sqrt{3}C_9\left(\int_\Omega \left[u^2 + (w-\hat{K})^2 + (v-K)^2\right]\mathrm{d}x\right)^{1/2} \\ &= C_9\sqrt{3F_2(t)}, \quad t > t_0. \end{aligned}$$

□

This combined with $E_2'(t) \le -F_2(t)$ leads us to $E_2'(t) \le -C_{10}E_2^2(t)$ for $t > t_0$. Thus, $E_2(t) \le \frac{C_{11}}{t+1}$ for $t > t_0$. Recall the definition of $E_2(t)$ and the left inequality in (3.23), we can have

$$\int_\Omega \left[u + (w-\hat{K})^2 + (v-K)^2\right]\mathrm{d}x \le C_{12}E_2(t) \le \frac{C_{13}}{t+1}, \quad t > t_0.$$

By the similar arguments in the proof of Theorem 1.3 (i), there exist $C > 0$ and $\lambda > 0$ such that

$$\|u\|_\infty + \|w-\hat{K}\|_\infty + \|v-K\|_\infty \le C(t+1)^{-\lambda}, \quad t > t_0.$$

This implies (1.8).

Now we consider the case $bg(K) = a$ and $\theta = 0$. In this case,

$$F_2(t) = \varepsilon_4\int_\Omega (w-\hat{K})^2 \mathrm{d}x + \varepsilon_4\int_\Omega (v-K)^2 \mathrm{d}x, \quad t > 0.$$

Similarly to the above it can be shown that

$$\|w-\hat{K}\|_{C^{2+\alpha}(\bar{\Omega})} + \|v-K\|_{C^{2+\alpha}(\bar{\Omega})} \to 0 \ \text{ as } t \to \infty. \tag{3.28}$$

Integrating the equation of v in (1.1) we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega v \mathrm{d}x &= \int_\Omega f(v)\mathrm{d}x - \int_\Omega ug(v)\mathrm{d}x \\ &= \int_\Omega f(v)\mathrm{d}x + \int_\Omega u(g(K)-g(v))\mathrm{d}x - g(K)\int_\Omega u\mathrm{d}x. \end{aligned} \tag{3.29}$$

Noticing $f(K) = 0$, the limit (3.28) implies $\int_\Omega f(v)\mathrm{d}x + \int_\Omega u(g(K)-g(v))\mathrm{d}x \to 0$ as $t \to \infty$. We have known $\lim_{t\to\infty} \frac{d}{dt}\int_\Omega v\mathrm{d}x = 0$ (see the proof of Lemma 3.4). It follows from (3.29) that $\lim_{t\to\infty} \|u\|_1 = 0$. Similarly to the above (compact arguments and uniqueness of limits), we can show (3.22), which implies the globally asymptotically stability of $(0, rK/\mu, K)$. Theorem 1.3 (ii) is proved.

4 Two Examples

To better understand our stability results, we shall use Theorems 1.2 and 1.3 to study two examples which are of biologically meaningful.

Let us first consider the Lotka-Volterra predator–prey system with indirect prey-taxis, i.e.,

$$\chi(w) = \chi_0, \quad h(u) = a + \theta u, \quad g(v) = v, \quad f(v) = qv(1 - v/K),$$

where the constants $\chi_0,\ a,\ \eta,\ K > 0$ and $\theta \geq 0$. Note that $g'(v) = 1,\ g(K) = K$ and

$$\varphi(v) = f(v)/g(v) = \eta(1 - v/K), \quad \varphi(0) = \eta > 0, \quad \varphi'(v) = -\eta/K < 0.$$

It is easy to see that (A5) is satisfied, and if $bK > a$ then the positive constant steady state reads

$$(u_*, rv_*/\mu, v_*) = \left(\frac{\eta(bK - a)}{bK + \theta\eta}, \frac{rK(a + \theta\eta)}{\mu(bK + \theta\eta)}, \frac{K(a + \theta\eta)}{bK + \theta\eta}\right).$$

According to Theorems 1.2 and 1.3, we have

- If $bK > a$ and
$$\frac{\chi_0^2}{d_1 d_2} < \frac{16\mu b(bK + \theta\eta)}{r^2 K(bK - a)},$$
then (1.1) admits a unique positive steady state $(u_*, rv_*/\mu, v_*)$.
- If $bK \leq a$, then the steady state $(0, rK/\mu, K)$ is globally asymptotically stable. This implies that the problem (1.1) has no positive steady state.

We next study the Rosenzweig-MacArthur predator–prey system with indirect prey-taxis, i.e.,

$$\chi(w) = \chi_0, \quad h(u) = a, \quad g(v) = v/(L + v), \quad f(v) = qv\left(1 - \frac{v}{K}\right),$$

where the constants $\chi_0,\ a,\ L,\ \eta,\ K > 0$ and $L > K$. Note that $g'(v) = L/(L+v)^2,\ g(K) = K/(L + K)$ and

$$\varphi(v) = \frac{f(v)}{g(v)} = \eta(L + v)(1 - v/K), \quad \varphi(0) = qL > 0, \quad \varphi'(v) = \eta(1 - L/K - 2v/K) < 0.$$

Then,

$$k_1 = \inf_{z\in[0,m]} g'(z) = \frac{L}{(L + m)^2}, \quad k_2 = \inf_{z\in[0,m]} |\varphi'(z)| = \eta\left(\frac{L}{K} - 1\right),$$

where $m = \max\{\|v_0\|_\infty,\ K\}$. It is easy to see that (A5) is satisfied and if $bK/(L + K) > a$, then

$$(u_*, rv_*/\mu, v_*) = \left(\frac{bqL[(b - a)K - aL]}{K(b - a)^2}, \frac{raL}{\mu(b - a)}, \frac{aL}{b - a}\right).$$

Thanks to Theorems 1.2 and 1.3, we have

- If $bK/(L + K) > a$, $L > K$ and
$$\frac{\chi_0^2}{d_1 d_2} < \frac{16\mu(L - K)(b - a)^2}{r^2(L + m)^2[(b - a)K - aL]},$$
then (1.1) admits a unique positive steady state $(u_*,\ rv_*/\mu,\ v_*)$.
- If $bK/(L + K) \leq a$, then the steady state $(0,\ rK/\mu,\ K)$ is globally asymptotically stable. This shows that the problem (1.1) has no positive steady state.

Acknowledgements The authors would like to thank the anonymous referees for their helpful comments and suggestions.

References

1. Ainseba, B.E., Bendahmane, M., Noussair, A.: A reaction–diffusion system modeling predator-prey with prey-taxis. Nonlinear Anal. Real World Appl. **9**, 2086–2105 (2008)
2. Amann, H.: Dynamic theory of quasilinear parabolic equations. II. Reaction–diffusion systems. Differ. Integral Equ. **3**(1), 13–75 (1990)
3. Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. Math. Models Methods Appl. Sci. **25**, 1663–1763 (2015)
4. Grünbaum, D.: Advection–diffusion equations for generalized tactic searching behaviours. J. Math. Biol. **38**(2), 169–194 (1999)
5. He, X., Zheng, S.: Global boundedness of solutions in a reaction–diffusion system of predator–prey model with prey-taxis. Appl. Math. Lett. **49**, 73–77 (2015)
6. Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Springer, Berlin (1981)
7. Hoefler, C.D., Taylor, M., Jakob, E.M.: Chemosensory response to prey in *Phidippus audax* and *Pardosa milvina*. J. Arachnol. **30**, 155–158 (2002)
8. Horstmann, D., Winkler, M.: Boundedness vs. blow-up in a chemotaxis system. J. Differ. Equ. **215**(1), 52–107 (2005)
9. Jin, H.Y., Wang, Z.A.: Global stability of prey-taxis systems. J. Differ. Equ. **262**, 1257–1290 (2017)
10. Kareiva, P., Odell, G.: Swarms of predators exhibit “preytaxis” if individual predators use area-restricted search. Am. Nat. **130**, 233–270 (1987)
11. Ladyzenskaja, O.A., Solonnikov, V.A., Ural’ceva, N.N.: Linear and Quasi-linear Equations of Parabolic Type. Academic Press, New York (1968)
12. Lee, J.M., Hillen, T., Lewis, M.A.: Continuous traveling waves for prey-taxis. Bull. Math. Biol. **70**, 654–676 (2008)
13. Lee, J.M., Hillen, T., Lewis, M.A.: Pattern formation in prey-taxis systems. J. Biol. Dyn. **3**(6), 551–573 (2009)
14. Lieberman, G.M.: Second Order Parabolic Differential Equations. World Scientific Publishing Co., Inc., River Edge (1996)
15. Sapoukhina, N., Tyutyunov, Y., Arditi, R.: The role of prey taxis in biological control: a spatial theoretical model. Am. Nat. **162**, 61–76 (2003)
16. Tao, Y.S.: Global existence of classical solutions to a predator–prey model with nonlinear prey-taxis. Nonlinear Anal. Real World Appl. **11**(3), 2056–2064 (2010)
17. Tello, J.I., Wrzosek, D.: Predator–prey model with diffusion and indirect prey-taxis. Math. Models Methods Appl. Sci. **26**(11), 2129–2162 (2016)
18. Wang, J.P., Wang, M.X.: The diffusive Beddington-DeAngelis predator–prey model with nonlinear prey-taxis and free boundary. Math. Methods Appl. Sci. **41**, 6741–6762 (2018)
19. Wang, J.P., Wang, M.X.: Boundedness and global stability of the two-predator and one-prey models with nonlinear prey-taxis. Z. Angew. Math. Phys. **69**(3), 63(1-24) (2018)
20. Wang, M.X.: The diffusive logistic equation with a free boundary and sign-changing coefficient. J. Differ. Equ. **258**, 1252–1266 (2015)
21. Wang, M.X.: A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment. J. Funct. Anal. **270**(2), 483–508 (2016)
22. Wang, M.X.: Note on the Lyapunov functional method. Appl. Math. Lett. **75**, 102–107 (2018)
23. Wang, M.X., Zhang, Y.: Dynamics for a diffusive prey–predator model with different free boundaries. J. Differ. Equ. **264**, 3527–3558 (2018)
24. Wang, Q., Song, Y., Shao, L.J.: Nonconstant positive steady states and pattern formation of 1D prey-taxis systems. J. Nonlinear Sci. **27**, 71–97 (2017)
25. Wang, X.L., Wang, W.D., Zhang, G.H.: Global bifurcation of solutions for a predator–prey model with prey-taxis. Math. Methods Appl. Sci. **38**, 431–443 (2015)
26. Winkler, M.: Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model. J. Differ. Equ. **248**(12), 2889–2905 (2010)
27. Wu, S.N., Shi, J.P., Wu, B.Y.: Global existence of solutions and uniform persistence of a diffusive predator–prey model with prey-taxis. J. Differ. Equ. **260**, 5847–5874 (2016)