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# Whiskered Tori for Forced Beam Equations with Multi-dimensional Liouvillean Frequency

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**Abstract**

In this paper we develop a Kolmogorov–Arnold–Moser (KAM) theory close to two fixed points for quasi-periodically forced nonlinear beam equation

$$y_{tt} + my + y_{xxxx} = y^3 + \varepsilon f(\omega t, x, y), \quad x \in [0, \ \pi],$$

where the forcing frequency $\omega$ is a small dilatation of a fixed vector $\overline{\omega}$, i.e., $\omega = \xi\overline{\omega} \in \mathbb{R}^d$ with $\xi \in \mathcal{O} := [1, 2]$. We will prove the existence of real analytic quasi-periodic solutions of the above equations under the hypothesis that the frequency $\overline{\omega}$ is Liouvillean. The quasi-periodic solutions we obtain are around the equilibria, $y(t, x) \equiv \pm\sqrt{m}$, $\forall (t, x) \in \mathbb{R} \times [0, \ \pi]$, of the system

$$y_{tt} + my + y_{xxxx} = y^3,$$

and are whiskered, that is the linearized equation around $\pm\sqrt{m}$ owns the hyperbolic directions (the hyperbolic directions are finitely many, depending on $m$, and the elliptic directions are infinitely many in our case). The proof is based on a modified KAM iteration for infinite dimensional systems with finitely many hyperbolic directions, infinitely many elliptic directions and Liouvillean forcing frequency. We believe that the approach in this paper can be applied also to other integrable PDEs. For example, the same strategy should work for the non-linear wave equations and the non-linear Schrödinger equations.

**Keywords** Whiskered tori · KAM theorem · Liouvillean frequency · Real analytic solutions

**Mathematics Subject Classification** 35R25 · 37L10 · 35Q56 · 34D35

## 1 Introduction and Main Result

Since the pioneering works for the existence of quasi-periodic solutions of one-dimensional nonlinear wave and Schrödinger equations proved by Kuksin [16] and Wayne [25], many

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progresses have been made concerning KAM theory for nonlinear Hamiltonian PDEs, one may refer to [4–7,10,12,14,15,17,18,20,21] and see also [8,9] for further developments. With the development of the KAM theory for PDEs, great attention has been paid to the study of quasi-periodic solutions for quasi-periodically forced nonlinear Hamiltonian PDEs by KAM and Nash–Moser theory in the last years. For example, see [19,22,23,30] for one-dimensional case. In particular, for higher dimensional case, Berti and Bolle [3] and [2] proved the existence of quasi-periodic solutions for quasi-periodically forced Hamiltonian wave equations

$$y_{tt} - \Delta y + V(x)y = \varepsilon f(\omega t, x, y), \ \ x \in \mathbb{T}^d,$$

and Schrödinger equations

$$iu_t - \Delta u + V(x)u = \varepsilon \partial_{\bar{u}} H(\omega t, x, u), \ \ x \in \mathbb{T}^d.$$

We note that the above works on forced systems need assume that the forced frequency meets some Diophantine conditions. That is, there exist $\gamma > 0$ and $\tau > d$ such that

$$|\langle k, \omega\rangle| \geq \frac{\gamma}{|k|^\tau}, \quad k \in \mathbb{Z}^d \setminus \{0\}$$

or assume that the forced frequency $\omega$ is a small dilatation of a fixed Diophantine vector $\overline{\omega} \in \mathbb{R}^d$, namely

$$\omega = \xi\overline{\omega}, \quad \xi \in \mathcal{O} := [1, 2],$$

where for some $\gamma > 0$ and $\tau > d$,

$$|\langle k, \overline{\omega}\rangle| \geq \frac{\gamma}{|k|^\tau}, \quad k \in \mathbb{Z}^d \setminus \{0\}.$$

Slightly weaker than the Diophantine conditions can be often required as Brjuno conditions, which was defined by

$$\sum_{n\geq 0} \frac{1}{2^n} \max_{0<|k|\leq 2^n,\ k\in\mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega\rangle|} < \infty.$$

However, if the forced frequency $\omega$ is ‘too close’ to rational vector, the resonance effects in general destroy the persistence of the invariant torus. Actually, it is difficult to obtain the persistence of the invariant torus of PDEs even in space one-dimensional cases beyond Brjuno conditions. If the frequency $\omega$ is not Brjuno, we call it is Liouvillean. Recently, Wang–You–Zhou [28] considered the quasi-periodically forced harmonic oscillators with Liouvillean frequency vector $\omega = (1, \alpha)$ :

$$\ddot{x} + \lambda^2 x = \varepsilon f(\omega t, x), \tag{1.1}$$

where $\lambda \in \mathcal{O}$, a closed real interval not containing 0, the forcing term $f$ is real analytic in $(\theta, x) \in \mathbb{T}^2 \times \mathbb{R}$. They proved the existence of response solution of (1.1) by using the CD-bridge given by Avila–Fayad–Krikorian [1].

The question is that whether it is possible to obtain the persistence of the invariant torus with Liouvillean frequency for the infinite-dimensional dynamical system with quasi-periodic forcing ?

By applying the ideas in [28], Xu et al. [29] constructed the quasi-periodic solutions of forced nonlinear Schrödinger equations with the frequency vector $\omega = (\widetilde{\omega}, \overline{\omega})$ and $\widetilde{\omega} = \xi(1, \alpha)$ satisfying

$$\begin{cases} \beta(\alpha) := \limsup_{n>0} \frac{\ln \ln q_{n+1}}{\ln q_n} < \infty, \\ |\langle k, \widetilde{\omega}\rangle| + |\langle l, \overline{\omega}\rangle| \geq \frac{\gamma}{(|k|+|l|)^\tau}, \quad \text{for } k \in \mathbb{Z}^2,\ l \in \mathbb{Z}^d \setminus \{0\}, \end{cases}$$

where $\frac{p_n}{q_n}$ is the continued fraction approximates to $\alpha$. In this paper, we consider the $d$-dimensional frequency vector $\omega \in \mathbb{R}^d$ satisfying, for $a \in (0, 1]$ and any $K > 1$,

$$\max_{0<|k|\leq K,\ k\in\mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega\rangle|} \leq |K|(\ln |K|)^{-a}. \tag{1.2}$$

It is worth pointing out that the result in [29] only allows $\widetilde{\omega}$ to be Liouvillean, not to all $\omega$. Thus, if we allow $d > 2$, our frequency, defined by (1.2), is much weaker than the frequency $\omega$ in [29]. However, if we restrict our frequency to be 2-dimensional vector, i.e., $d = 2$, then the frequency $\widetilde{\omega} = \xi(1, \alpha)$ in [29] is weaker than the frequency in our paper.

To verify that the frequencies defined in (1.2) include the Liouvillean, we, for example, can impose the following restriction on the forced frequency

$$\max_{K_1<|k|\leq K_2,\ k\in\mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega\rangle|} \geq |K_1|(\ln |K_1|)^{-a}, \tag{1.3}$$

where $K_2 > K_1 > 1$. It is easy to see that such $\omega$ is Liouvillean. In fact, for any $n \in \mathbb{N}$, (1.3) implies that there exists $k \in \mathbb{Z}^d \setminus \{0\}$ with $2^n \leq |k| \leq 2^{n+1}$ such that

$$\max_{2^{n-1}<|k|\leq 2^n,\ k_n\in\mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega\rangle|} \geq 2^{n-1}(\ln 2^{n-1})^{-a},\ \ a \in (0, 1].$$

It follows that

$$\begin{aligned} \sum_{n\geq 0} \frac{1}{2^n} \max_{0<|k|\leq 2^n, k\in\mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega\rangle|} &\geq \sum_{n\geq 0} \frac{1}{2^n} \max_{2^{n-1}<|k|\leq 2^n, k\in\mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega\rangle|} \\ &> \sum_{n\geq 0} \frac{1}{2^n} 2^{n-1}(\ln 2^{n-1})^{-a} \\ &= 2^{-1}(\ln 2)^{-a} \sum_{n\geq 1} (n-1)^{-a} = \infty,\ \ a \in (0, 1], \end{aligned}$$

which shows $\omega$ defined in (1.2) includes some Liouvillean frequencies.

The goal of this paper is to develop a KAM theory for whiskered tori (i.e., the tori own the hyperbolic directions) of nonlinear beam equation *with Liouvillean frequency vector*

$$y_{tt} + my + y_{xxxx} = y^3 + \varepsilon f(\omega t, x, y), \quad x \in [0,\ \pi]. \tag{1.4}$$

We consider the main part of (1.4)

$$y_{tt} + my + y_{xxxx} = y^3. \tag{1.5}$$

The system (1.5) has three equilibria, $y(t, x) \equiv 0, \forall(t, x) \in \mathbb{R} \times [0,\ \pi]$, and $y(t, x) \equiv \sqrt{m},\ y(t, x) \equiv -\sqrt{m}, \forall(t, x) \in \mathbb{R} \times [0,\ \pi]$. The main result of the paper will prove that two solutions $u(t, x) \equiv \pm\sqrt{m}$ of Eq. (1.5) can be continued to solutions of the Eq. (1.4). More precisely, we look for quasi-periodic solutions

$$y(t, x) = \varepsilon^{\frac{1}{2}} u(t, x) \pm \sqrt{m},$$

of (1.4), where $u(t, x)$ is a quasi-periodic solution of the following nonlinear beam equations

$$u_{tt} - 2mu + u_{xxxx} = \varepsilon u^3 \pm 3\varepsilon^{\frac{1}{2}} \sqrt{m} u^2 + \varepsilon^{\frac{1}{2}} f(\omega t, x, \varepsilon^{\frac{1}{2}} u \pm \sqrt{m}). \tag{1.6}$$

Throughout this paper, we always assume the following:

**(H)**: $f : \mathbb{T}^d \times [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is a real analytic function and even on $x$, and $\omega$ is a small dilatation of a fixed vector $\overline{\omega}$, namely,

$$\omega = \xi\overline{\omega} \text{ with } \xi \in \mathcal{O} := [1, 2],$$

$m > 0,\ (2m)^{\frac{1}{4}} - [(2m)^{\frac{1}{4}}] \in [\frac{1}{100}, \frac{1}{2}]$, here $[\cdot]$ denotes the integer part of a real number and $\overline{\omega}$ satisfies the hypothesis (1.2).

Now we are ready to state our main result.

**Theorem 1.1** *Assume that the hypothesis* **(H)** *holds, then for any* $0 < \gamma \ll 1$, *there exist* $\varepsilon_* > 0$ *and* $\mathcal{O}_\gamma \subset \mathcal{O}$ *with* meas$\mathcal{O}_\gamma > 1 - c\gamma$, *such that for any* $\xi \in \mathcal{O}_\gamma$ *the Eq.* (1.4) *possesses quasi-periodic solutions of the form* $y(t, x) = \pm\sqrt{m} + \varepsilon^{\frac{1}{2}} u(\omega t, x)$ *provided that* $\varepsilon \le \varepsilon_*^2$, *where* $u(\omega t, x)$ *is the quasi-periodic solution of* (1.6).

Let us conclude the introduction with some comments on the result.

1. The reason why we choose beam equation as our object of study in this work is that it is one of the most important equation in mathematical physics besides this equation interesting by itself. A lot of important works have been done on the study the existence of quasi-periodic solution with *Diophantine frequency* for beam equation. We refer to [11–13] and [24,26]. More recently, a groundbreaking work was made by Eliasson–Grebert–Kuksin [8], who proved a KAM theorem for beam equation with $x \in \mathbb{T}^d$ under the Diophantine conditions. The system we're considering here is a system with multi-dimensional Liouvillean forced frequency. Furthermore, since the solutions we obtain are around $\pm\sqrt{m}$, the linearization operator will possess the hyperbolic spectrum (eigenvalue), the motion equation, Hamiltonian and the symplectic form are different from the ones in the papers mentioned above. To avoid the multiply spectrum we will restrict ourself to the space that is even about the spacial variable $x$ and take the $\sqrt{m}$ as the example.
2. Note that we have to exclude some $m$ such that the spectrum is discrete. Thus we will introduce another parameter $\xi$, which belongs to a compact set. This parameter is the dilatation of the fixed vector $\overline{\omega}$. We will dig out some bad parameters from this compact set at each KAM iteration to make sure that the first and second Melnikov's conditions are satisfied.
3. Digging out the parameter $\xi$ will make homological equations along the hyperbolic direction complicated. Since the solution we construct is around the equilibrium $\sqrt{m}$, the linear operator of the linearized equation possesses the hyperbolic spectrum (the spectrum whose real part is not zero). Generally, there will be no small divisor in the homological equation along the hyperbolic directions, so that one solves the equation by applying the implicit function theorem. However, in this paper, the parameter $\xi$ makes the implicit function theorem invalid. See the Remark 3.1 for details. Thus we will not separate the equations along the elliptic directions and the hyperbolic directions.
4. Similar to [27,29], we will also solve, in the KAM iteration, the variable coefficients homological equation

$$\partial_\omega u(\theta, \xi) + \mathrm{i}(\Omega + B(\theta, \xi))u(\theta, \xi) = f(\theta, \xi),\ 0 \ne \Omega \in \mathbb{R},\ \partial_\omega = \sum_{l=1}^{d} \omega_l \partial_{\theta_l}.$$

Follows the famous Kuksin's Lemma, we will kill the function $B(\theta, \xi)$ by making a change $\widetilde{u} = e^{-\mathrm{i}\mathcal{B}} u$ and $\widetilde{f} = e^{-\mathrm{i}\mathcal{B}} f$, where

$$\mathrm{i}\partial_\omega \mathcal{B}(\theta, \xi) = -B(\theta, \xi) + \left[B(\theta, \xi)\right]_\theta.$$

In [27,29], the authors used the technique of CD-bridge to make sure that $\mathcal{B}$ is controllable. However, the frequency defined in (1.2) is more complicated, for example, there is no CD-bridge in our work. We will separate the equation above into a series of equations by using the special structure of $B$. By carefully choosing the iteration parameters we can guarantee that $\mathcal{B}$ is controllable.

5. If the smallness of the perturbation does not depend on the Diophantine constants of the frequency $\omega$, we say the result is non-perturbative. We stress that the smallness of the perturbation in our results is dependent on the Diophantine constants of the frequency $\omega$. For example, one of the hypotheses on $\epsilon_*$ is $\ln \epsilon_*^{\frac{-1}{40(2\tau+1)}} \geq 3\exp\left\{(e^{-4}s_0(48)^{-1})^{\frac{-1}{a}}\right\}$, where $a$ is the ones in (1.2) and $0 < s_0 < 1$. Thus our result is perturbative, not like the result in [27].

## 2 Preliminaries

In this section, we first give some notations which will be used in the sequel.

### 2.1 Some Notations

Denote the set $\mathbb{Z}_1 = \left\{j \in \mathbb{Z} \ : \ 0 \leq j \leq [(2m)^{\frac{1}{4}}]\right\}$ and $\mathcal{J} = \mathbb{N} \setminus \mathbb{Z}_1$. Let $\ell_{a,p} = \{q = (q_j)_{j\in\mathcal{J}} \ : \ q_j \in \mathbb{C}\}$ with $a \geq 0$, $p \geq \frac{1}{2}$ be the space of complex sequences with inner product

$$\langle q, \widetilde{q}\rangle := \sum_{j\in\mathcal{J}} e^{2a|j|}|j|^{2p} q_j \overline{\widetilde{q}}_j$$

for any $q, \widetilde{q} \in \ell_{a,p}$. Then $\left(\ell_{a,p}, \langle\cdot,\cdot\rangle\right)$ is a Hilbert space. Let $\|q\|_{a,p} = \sqrt{\langle q, q\rangle}$. Similarly, we also define $\widetilde{\ell}_{a,p} = \{q = (q_j)_{j\in\mathbb{Z}_1} : q_j \in \mathbb{C}\}$ with $a \geq 0$, $p \geq \frac{1}{2}$ be the space of complex sequences with inner product

$$\langle q, \widetilde{q}\rangle := \sum_{0\neq j\in\mathbb{Z}_1} e^{2a|j|}|j|^{2p} q_j \overline{\widetilde{q}}_j + q_0\overline{\widetilde{q}}_0$$

for any $q, \widetilde{q} \in \widetilde{\ell}_{a,p}$. Obviously, $\left(\widetilde{\ell}_{a,p}, \langle\cdot,\cdot\rangle\right)$ is also a Hilbert space. Denote $\|q\|_{a,p} = \sqrt{\langle q, q\rangle}$.

Let $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ ($\mathbb{T}^d_c = \mathbb{C}^d/2\pi\mathbb{Z}^d$) be the standard $d$-dimensional real (complex) torus and define

$$u(s) = \left\{\theta : |Im\theta| < s\right\}, \quad \mathcal{O} = [1, 2],$$

where $|\cdot|$ denotes the supremum norm for the finite-dimensional vectors. Denote the complex neighborhood of $\mathbb{T}^d \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\}$ by

$$\begin{aligned} D(s,r) &= \{(\theta, I, z, \overline{z}, \rho, \overline{\rho}) : |Im\theta| < s, |I| < r^2, \|z\|_{a,p}, \ \|\overline{z}\|_{a,p}, \|\rho\|_{a,p}, \|\overline{\rho}\|_{a,p} < r\} \\ &\subset \mathbb{C}^d \times \mathbb{C}^d \times \ell_{a,p} \times \ell_{a,p} \times \widetilde{\ell}_{a,p} \times \widetilde{\ell}_{a,p} := \mathcal{P}_{a,p}. \end{aligned}$$

For the function $f(\theta, \xi)$ defined on $u(s) \times \mathcal{O}$ with the Fourier expansion

$$f(\theta, \xi) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k, \xi) e^{\mathrm{i}\langle k,\theta\rangle},$$

we define the norms $\|f\|^*_{s,\mathcal{O}}$ and $\|f\|^L_{s,\mathcal{O}}$ as

$$\|f\|^*_{s,\mathcal{O}} = \sum_{k\in\mathbb{Z}^d} \|\widehat{f}(k)\|^*_{\mathcal{O}} e^{|k|s}, \quad \|f\|^L_{s,\mathcal{O}} = \sum_{k\in\mathbb{Z}^d} \|\widehat{f}(k)\|^L_{\mathcal{O}} e^{|k|s},$$

with

$$\|\widehat{f}(k)\|^*_{\mathcal{O}} = \sup_{\xi\in\mathcal{O}} |\widehat{f}(k,\xi)|, \quad \|\widehat{f}(k)\|^L_{\mathcal{O}} = \sup_{\xi_1\neq\xi_2,\xi_1,\xi_2\in\mathcal{O}} \frac{|\widehat{f}(k,\xi_1) - \widehat{f}(k,\xi_2)|}{|\xi_1 - \xi_2|},$$

we also define the norms $\|\widehat{f}(k)\|_{\mathcal{O}}$ and $\|f\|_{s,\mathcal{O}}$ as

$$\|\widehat{f}(k)\|_{\mathcal{O}} = \|\widehat{f}(k)\|^*_{\mathcal{O}} + \|\widehat{f}(k)\|^L_{\mathcal{O}},$$
$$\|f\|_{s,\mathcal{O}} = \|f\|^*_{s,\mathcal{O}} + \|f\|^L_{s,\mathcal{O}} = \sum_{k\in\mathbb{Z}^d} \|\widehat{f}(k)\|_{\mathcal{O}} e^{|k|s}.$$

Moreover, we define the truncation operator $\mathcal{T}_K$ and projection operator $\mathcal{R}_K$ as

$$\mathcal{T}_K f(\theta,\xi) = \sum_{|k|\leq K} \widehat{f}(k,\xi) e^{\mathrm{i}\langle k,\theta\rangle}, \quad \mathcal{R}_K f(\theta,\xi) = \sum_{|k|>K} \widehat{f}(k,\xi) e^{\mathrm{i}\langle k,\theta\rangle},$$

and the average of $f(\theta,\xi)$ in $\theta$ by

$$[f(\theta,\xi)]_\theta = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta,\xi) d\theta = \widehat{f}(0,\xi).$$

Denote $\delta = \{\delta_j, \ \ j \in \mathcal{J}\}$, $\beta = \{\beta_j, \ \ j \in \mathcal{J}\}$, and $\alpha = \{\alpha_j, \ \ j \in \mathbb{Z}_1\}$, $\eta = \{\eta_j, \ \ j \in \mathbb{Z}_1\}$ with finitely many non-zero components $\delta_j, \beta_j, \alpha_j, \eta_j \in \mathbb{N}$. For the function $P : D(s,r) \times \mathcal{O} \to \mathbb{C}$, which is analytic in variables $(\theta, I, z, \overline{z}, \rho, \overline{\rho})$ and Lipschitz on the parameter $\xi$, we take the following Taylor–Fourier expansion

$$\begin{aligned} P(\theta, I, z, \overline{z}, \rho, \overline{\rho}, \xi) &= \sum_{\delta,\beta,\alpha,\eta} P_{\delta,\beta,\alpha,\eta}(\theta, I, \xi) z^\delta \overline{z}^\beta \rho^\alpha \overline{\rho}^\eta \\ &= \sum_{\mu,\delta,\beta,\alpha,\eta,k} \widehat{P}_{\delta,\beta,\alpha,\eta,\mu}(k,\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^\mu z^\delta \overline{z}^\beta \rho^\alpha \overline{\rho}^\eta, \end{aligned}$$

where $z^\delta \overline{z}^\beta = \Pi_{j\in\mathcal{J}} z_j^{\delta_j} \overline{z}_j^{\beta_j}$ and $\rho^\alpha \overline{\rho}^\eta = \Pi_{j\in\mathbb{Z}_1} \rho_j^{\alpha_j} \overline{\rho}_j^{\eta_j}$. We define the norm of $P$ by

$$\|P\|_{D(s,r),\mathcal{O}} = \sup_{\|z\|_{a,p},\|\overline{z}\|_{a,p},\|\rho\|_{a,p},\|\overline{\rho}\|_{a,p}\leq r} \sum_{\delta,\beta,\alpha,\eta} \|P_{\delta,\beta,\alpha,\eta}\| |z^\delta| |\overline{z}^\beta| |\rho^\alpha| |\overline{\rho}^\eta|,$$

where

$$\begin{aligned} \|P_{\delta,\beta,\alpha,\eta}\| &= \sum_{k,\mu} \|\widehat{P}_{\delta,\beta,\alpha,\eta,\mu}(k)\|_{\mathcal{O}} e^{|k|s} r^{2|\mu|} \\ &= \sum_{\mu} \|P_{\delta,\beta,\alpha,\eta,\mu}\|_{s,\mathcal{O}} r^{2|\mu|}. \end{aligned}$$

Moreover, for the function $P$ above we associate a Hamiltonian vector field defined by

$$X_P = \left(P_I, \ -P_\theta, \ \mathrm{i}P_{\overline{z}}, \ -\mathrm{i}P_z, \ P_{\overline{\rho}}, \ -P_\rho\right)^T.$$

For the vector $P_Y$, $(Y = z, \overline{z})$ we define

$$\|P_Y\|_{a,p,D(s,r),\mathcal{O}} = \left\{ \sum_{j\in\mathcal{J}} \left(\|P_{Y_j}\|_{D(s,r),\mathcal{O}}\right)^2 e^{2aj} j^{2p} \right\}^{\frac{1}{2}},$$

and for $P_Y$, $(Y = \rho, \overline{\rho})$

$$\|P_Y\|_{a,p,D(s,r),\mathcal{O}} = \left\{ \sum_{0\neq j\in\mathbb{Z}_1} \left(\|P_{Y_j}\|_{D(s,r),\mathcal{O}}\right)^2 e^{2aj} j^{2p} + \|P_{Y_0}\|_{D(s,r),\mathcal{O}}\right)^2 \right\}^{\frac{1}{2}}.$$

We also define the weighted norm

$$\begin{aligned}\|X_P\|_{r,s,r,\mathcal{O}} = \left\|P_I\right\|_{D(s,r),\mathcal{O}} + \frac{1}{r^2}\left\|P_\theta\right\|_{D(s,r),\mathcal{O}} + \frac{1}{r}\big(\|\mathrm{i}P_{\overline{z}}\|_{a,p,D(s,r),\mathcal{O}} \\ + \|\mathrm{i}P_z\|_{a,p,D(s,r),\mathcal{O}} + \|P_{\overline{\rho}}\|_{a,p,D(s,r),\mathcal{O}} + \|P_\rho\|_{a,p,D(s,r),\mathcal{O}}\big).\end{aligned}$$

In this paper, for $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, we denote

$$\langle k\rangle = \max\{1, |k|\}, \ |k| := |k_1| + \cdots + |k_d|.$$

## 2.2 An Infinite Dimensional KAM Theorem

In this section, we develop an abstract KAM theorem for a general infinite-dimensional quasi-periodically forced system. As an application of the theorem we can prove Theorem 1.1 immediately. We consider a more general infinite-dimensional Hamiltonian system

$$H = \langle\omega, I\rangle + \langle\Omega z, \overline{z}\rangle - \langle\Lambda\rho, \overline{\rho}\rangle + P(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi) \tag{2.1}$$

endowed with symplectic structure $d\theta \wedge dI + \mathrm{i}dz \wedge d\overline{z} + d\rho \wedge d\overline{\rho}$, where $P$ is real analytic in the variables $(\theta, z, \overline{z}, \rho, \overline{\rho})$ and Lipschitz in parameters $\xi$. Denote

$$\begin{aligned}&\Omega = \mathrm{diag}(\Omega_j, \ j \in \mathcal{J}), \ |\Omega_j| \geq j^2, \ \ |\Omega_j \pm \Omega_i| \geq |j^2 \pm i^2|, \\ &\Lambda = \mathrm{diag}(\Lambda_j, \ j \in \mathbb{Z}_1), \ 1 \leq |\Lambda_j| \leq 2, \ \ |\Lambda_j \pm \Lambda_i| \geq 1.\end{aligned} \tag{2.2}$$

We also identify the above two diagonal matrices as the vectors $\Omega = (\Omega_j, \ \ j \in \mathcal{J})^T$ and $\Lambda = (\Lambda_j, \ \ j \in \mathbb{Z}_1)^T$. The same notions are also for the diagonal matrices $B(\theta, \xi)$, $W(\theta, \xi)$ and $b(\theta, \xi)$, $w(\theta, \xi)$, which will be given later.

**Theorem 2.1** *Let $\omega = \xi\overline{\omega}$ with $\overline{\omega} \in \mathbb{R}^d$ satisfying (1.2), and $s, r > 0$, $\tau > d + 2$. Consider the real-analytic Hamiltonian $H$ defined in (2.1). Then there exists a $\varepsilon_*(\overline{\omega}, \gamma, s, r, \tau) > 0$, for every real analytic perturbation $P$ with*

$$\varepsilon = \|X_P\|_{r,s,r,\mathcal{O}} \leq \varepsilon_*(\overline{\omega}, \gamma, s, r, \tau),$$

*there exists a nonempty subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with meas$\mathcal{O}_\gamma > 1 - c\gamma$, and for every $\xi \in \mathcal{O}_\gamma$, there is a real analytic symplectic map $\Phi: \mathbb{T}^d \times \mathcal{O}_\gamma \to \mathcal{P}_{a,p}$, such that $\Phi$ casts the Hamiltonian $H$ defined by (2.1) into*

$$H \circ \Phi = N_* + P_*,$$

*where*

$$N_*(\theta, I, z, \overline{z}, \rho, \overline{\rho}, \xi) = E_*(\theta, \xi) + \langle \omega, I\rangle + \langle (\Omega + B_*(\theta, \xi))z, \overline{z}\rangle \\ - \langle (\Lambda - W_*(\theta, \xi))\rho, \overline{\rho}\rangle,$$

$$P_*(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi) = \sum_{|\delta+\beta|+|\alpha+\eta|\geq 3} P^*_{\delta,\beta,\alpha,\eta}(\theta, \xi) z^{\delta}\overline{z}^{\beta}\rho^{\alpha}\overline{\rho}^{\eta},$$

*with*

$$\|X_{E_*}\|_{r_*,s_*,r_*,\mathcal{O}_\gamma} < 4\varepsilon^{\frac{1}{2}}, \quad \|B_*\|_{s_*,\mathcal{O}_\gamma} \leq 4\varepsilon^{\frac{1}{2}}, \quad \|W_*\|_{s_*,\mathcal{O}_\gamma} \leq 4\varepsilon^{\frac{1}{2}}. \tag{2.3}$$

## 3 Proof of Theorem 2.1

### 3.1 Main Ideas of the Proof

An essential idea of Theorem 2.1 is to construct a simplifying transformation, consisting of infinitely many successive steps (referred to as KAM steps) of iterations, so that after each step the new perturbation terms of the transformed system are much smaller than the ones in the previous system. As all KAM steps can be carried out inductively, below, we only describe one step of KAM iteration in more detail. In this work, one step of iteration will be finished by a family of subiterations. Moreover, the steps of the subiteration will go to $\infty$.

Note that the KAM theory is a small divisor problem, since we assume the forcing frequency is Liouvillean, there will be new difficulties appear compared with the classical KAM theory. The main difficulty is that we can not kill the terms whose small divisor is $\langle k, \omega\rangle$, $k \neq 0$, such as $P(\theta, \xi)$, $\sum_i P_{i,i}(\theta, \xi)z_i\overline{z}_i$ and $R(\theta, \xi)I$. We overcome this problem by putting the first two terms into the normal from. Moreover, we assume that the variable $\theta$ comes form the forcing such that the perturbation will not depend on the action variable $I$. Thus, the normal form in our work will be variable coefficients, take the $n$-step Hamiltonian for example,

$$H_n = E_n(\theta, \xi) + \langle \omega, I\rangle + \left\langle \left[\Omega + B_n(\theta, \xi)\right]z, \bar{z}\right\rangle \\ - \left\langle \left[\Lambda - W_n(\theta, \xi)\right]\rho, \overline{\rho}\right\rangle + P_n(\theta, z, \bar{z}, \rho, \overline{\rho}, \xi),$$

where the functions $B_n(\theta, \xi)$ and $W_n(\theta, \xi)$ own the special structure. Thus the homological equations in this paper is variable coefficients. The special structure of $B_n(\theta, \xi)$ and $W_n(\theta, \xi)$ is one of the key conditions when we try to eliminate the effect taken by $B_n(\theta, \xi)$ and $W_n(\theta, \xi)$. Moreover, the perturbation $P_n(\theta, z, \bar{z}, \rho, \overline{\rho}, \xi)$ is of size $\epsilon_n$.

In the following, we will construct a near-identity symplectic change of variables $\Phi_{n+1}$, such that the new Hamiltonian system $H_{n+1} = H_n \circ \Phi_{n+1}$ possesses the same formula and satisfies the same estimate as the ones of new Hamiltonian system $H_n$ with $(n+1)$ in place of $n$. Note that one of the step of iteration will be finished by another a family of subiterations, that is the Hamiltonian function $H_{n+1}$ is the one in the final step of the subiteration. Thus, the size of the new perturbation $P_{n+1}$ is much smaller than $\epsilon_n^{\frac{5}{4}}$.

### 3.2 Homological Equation and Its Approximate Solution

For functions $F(\theta, I, z, \overline{z}, \rho, \overline{\rho}, \xi)$ and $G(\theta, I, z, \overline{z}, \rho, \overline{\rho}, \xi)$ with Taylor–Fourier expansion

$$F(\theta, I, z, \bar{z}, \rho, \bar{\rho}, \xi) = \sum_{\mu,\delta,\beta,\alpha,\eta,k} \widehat{F}_{\delta,\beta,\alpha,\eta,\mu}(k,\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^{\mu} z^{\delta} \bar{z}^{\beta} \rho^{\alpha} \bar{\rho}^{\eta},$$

$$G(\theta, I, z, \bar{z}, \rho, \bar{\rho}, \xi) = \sum_{\mu,\delta,\beta,\alpha,\eta,k} \widehat{G}_{\delta,\beta,\alpha,\eta,\mu}(k,\xi) e^{\mathrm{i}\langle k,\theta\rangle} I^{\mu} z^{\delta} \bar{z}^{\beta} \rho^{\alpha} \bar{\rho}^{\eta},$$

which are defined on $D(s,r) \times \mathcal{O}$, we define the Poisson bracket

$$\{G, F\} = \frac{\partial G}{\partial \theta}\frac{\partial F}{\partial I} - \frac{\partial G}{\partial I}\frac{\partial F}{\partial \theta} + \mathrm{i}\frac{\partial G}{\partial z}\frac{\partial F}{\partial \bar{z}} - \mathrm{i}\frac{\partial G}{\partial \bar{z}}\frac{\partial F}{\partial z} + \frac{\partial G}{\partial \rho}\frac{\partial F}{\partial \bar{\rho}} - \frac{\partial G}{\partial \bar{\rho}}\frac{\partial F}{\partial \rho}.$$

For fixed $0 < \varepsilon_0 < 1,\ 0 < s_0 < 1$ and $\tau > d + 2$, we denote the initial parameter $K_{-1}$ by

$$K_{-1}^{\frac{1}{2}} = \ln \varepsilon_0^{\frac{-1}{40(2\tau+1)}},\ i.e.,\ \varepsilon_0 = \exp\left\{-40(2\tau+1)K_{-1}^{\frac{1}{2}}\right\}.$$

We assume that $\varepsilon_0$ is small enough such that

$$K_{-1}^{\frac{1}{2}} = \ln \varepsilon_0^{\frac{-1}{40(2\tau+1)}} \geq 3\exp\left\{((48)^{-1}e^{-4}s_0)^{\frac{-1}{a}}\right\}.$$

From the above inequality we can see that the smallness of perturbation, $\varepsilon_0$, is related to $a$, and thus to the Liouvillean frequency $\omega$ [see (1.2)]. Then we define the iteration sequences for $i \geq 0$ :

$$\begin{aligned} \varsigma_i &= (i+2)^{-2}, \\ K_i &= \exp\left\{K_{i-1}^{\frac{1}{2}}\right\},\ \varepsilon_{i+1} = \exp\left\{-40(2\tau+1)K_i^{\frac{1}{2}}\right\},\ \widetilde{\varepsilon}_{i,j} = \varepsilon_i^{\left(\frac{5}{4}\right)^j}, \\ s_{i+1} &= s_0\Pi_{j=0}^{i}(1-\varsigma_j)^2,\ \sigma_{i,j} = 5^{-1}\varsigma_i\varsigma_j s_i,\ T_{i,j} = \sigma_{i,j}^{-1}\ln\widetilde{\varepsilon}_{i,j}^{-1}, \end{aligned} \tag{3.1}$$

where $j = 0, \ldots, \mathcal{N}_i - 1$, and $\mathcal{N}_i$ is the smallest integer number such that $\widetilde{\varepsilon}_{i,\mathcal{N}_i} \leq \varepsilon_{i+1}$, that is $\widetilde{\varepsilon}_{i,\mathcal{N}_i} \leq \varepsilon_{i+1} < \widetilde{\varepsilon}_{i,\mathcal{N}_i-1}$.

Obviously, for any $n \geq 0$, we know that

$$e^{-4}s_0 < s_n \leq s_0.$$

The inequality above is used in many places, we will not stress the reference about it. Moreover, we will also assume $\varepsilon_0$ is small enough such that for the sequence $\{K_j\}_{j\geq -1}$ defined above, the inequalities like $K_{n+1} > K_n^3,\ K_{n+1} > 20K_n,\ \ln K_n < K_n^{\frac{1}{16}}, n \geq 0$, hold.

**Lemma 3.1** *For the sequences defined above we have*

$$\exp\{T_{n-3,\mathcal{N}_{n-3}-1}\} < \ln \varepsilon_n^{-1},\ n \geq 3 \tag{3.2}$$

*and*

$$(\ln\ln K_n^{\frac{1}{2}})^{-a} < 3^{-1}e^{-4}s_0\varsigma_{n+1},\ a \in (0,1],\ n \geq 0. \tag{3.3}$$

***Proof*** Since $\widetilde{\varepsilon}_{n,\mathcal{N}_n} \leq \varepsilon_{n+1} < \widetilde{\varepsilon}_{n,\mathcal{N}_n-1}$, we know that $\varepsilon_n^{-\left(\frac{5}{4}\right)^{\mathcal{N}_n}} \geq \varepsilon_{n+1}^{-1} > \varepsilon_n^{-\left(\frac{5}{4}\right)^{\mathcal{N}_n-1}}$, which implies

$$\left(\frac{5}{4}\right)^{\mathcal{N}_n} \geq \frac{\ln \varepsilon_{n+1}^{-1}}{\ln \varepsilon_n^{-1}} = \frac{40(2\tau+1)K_n^{\frac{1}{2}}}{40(2\tau+1)K_{n-1}^{\frac{1}{2}}} > \left(\frac{5}{4}\right)^{\mathcal{N}_n-1}.$$

Note that $\left(\frac{5}{4}\right)^5 > e > \left(\frac{5}{4}\right)^4$, by the inequalities above we have

$$e^{\frac{\mathcal{N}_n}{4}} > \left(\frac{5}{4}\right)^{\mathcal{N}_n} \geq \frac{40(2\tau+1)K_n^{\frac{1}{2}}}{40(2\tau+1)K_{n-1}^{\frac{1}{2}}} > K_n^{\frac{1}{3}},$$

$$e^{\frac{\mathcal{N}_n-1}{5}} < \left(\frac{5}{4}\right)^{\mathcal{N}_n-1} < \frac{40(2\tau+1)K_n^{\frac{1}{2}}}{40(2\tau+1)K_{n-1}^{\frac{1}{2}}} < K_n^{\frac{1}{2}},$$

that is

$$\frac{4}{3}\ln K_n < \mathcal{N}_n < \frac{5}{2}\ln K_n + 1. \tag{3.4}$$

Then by (3.1) we have

$$\begin{aligned} T_{n,\mathcal{N}_n-1} &= \sigma^{-1}_{n,\mathcal{N}_n-1}\ln\widetilde{\varepsilon}^{-1}_{n,\mathcal{N}_n-1} < \sigma^{-1}_{n,\mathcal{N}_n-1}\ln\varepsilon^{-1}_{n+1} \\ &= 5(n+2)^2(\mathcal{N}_n+1)^2 s_n^{-1} 40(2\tau+1)K_n^{\frac{1}{2}} \\ &< 1800(2\tau+1)e^4 s_0^{-1}(n+2)^2(\ln K_n)^2 K_n^{\frac{1}{2}} \\ &< K_n^{\frac{2}{3}}\left(=\exp\left\{\frac{2}{3}K_{n-1}^{\frac{1}{2}}\right\}\right), \end{aligned} \tag{3.5}$$

the second inequality is from the inequality in the right side of (3.4), i.e., $\mathcal{N}_n+1 < \frac{5}{2}\ln K_n + 2 < 3\ln K_n$. Then we have

$$\begin{aligned} \ln\varepsilon_n^{-1} &= 40(2\tau+1)K_{n-1}^{\frac{1}{2}} = 40(2\tau+1)\exp\left\{\frac{1}{2}K_{n-2}^{\frac{1}{2}}\right\} \\ &> \exp\left\{K_{n-3}^{\frac{2}{3}}\right\} > \exp\left\{T_{n-3,\mathcal{N}_{n-3}-1}\right\},\ n \geq 3, \end{aligned}$$

the last inequality is from (3.5) with $(n-3)$ in place of $n$. This is the proof of (3.2).

Now we turn to (3.3). We will use the iteration technique to prove this inequality.

I): $n = 0$ or 1. Note that

$$\ln\ln K_n^{\frac{1}{2}} = \ln\left(\frac{1}{2}K_{n-1}^{\frac{1}{2}}\right) \geq \ln\left(\frac{1}{2}K_{-1}^{\frac{1}{2}}\right) \geq (3^{-1}e^{-4}s_0\varsigma_2)^{\frac{-1}{a}} \geq (3^{-1}e^{-4}s_0\varsigma_{n+1})^{\frac{-1}{a}},$$

where the inequality above is from $K_{-1}^{\frac{1}{2}} > 3\exp\{(3^{-1}e^{-4}s_0\varsigma_2)^{\frac{-1}{a}}\}$. The inequality above yields

$$\left(\ln\ln K_n^{\frac{1}{2}}\right)^{-a} \leq 3^{-1}e^{-4}s_0\varsigma_{n+1},\quad n = 0 \text{ or } 1.$$

II): $n \geq 2$. Assume that $n = j \geq 2$, the inequality in (3.3) holds, that is

$$\left(\ln\ln K_j^{\frac{1}{2}}\right)^{-a} \leq 3^{-1}e^{-4}s_0\varsigma_{j+1},$$

which implies

$$\ln\ln K_j^{\frac{1}{2}} \geq \left(3^{-1}e^{-4}s_0\varsigma_{j+1}\right)^{-\frac{1}{a}}.$$

Now we let $n = j + 1$. Note that $K_{j+1} = \exp\{K_j^{\frac{1}{2}}\}$, then we have (the first inequality below is by the inequality above)

$$\begin{aligned}\ln\ln K_{j+1}^{\frac{1}{2}} &= \ln K_j^{\frac{1}{2}} - \ln 2 > \exp\left\{\left(3^{-1}e^{-4}s_0\varsigma_{j+1}\right)^{-\frac{1}{a}}\right\} - \ln 2 \\ &\geq \frac{1}{2}\left(3^{-1}e^{-4}s_0\varsigma_{j+1}\right)^{-\frac{2}{a}} - \ln 2 \geq \frac{1}{2}\left(3^{-1}e^{-4}s_0\right)^{-\frac{2}{a}}\varsigma_{j+2}^{-\frac{1}{a}} - \ln 2 \\ &\geq \left(e^{-4}s_0\right)^{-\frac{2}{a}}\left(3^{-1}\varsigma_{j+2}\right)^{-\frac{1}{a}} - \ln 2 > \left(3^{-1}e^{-4}s_0\varsigma_{j+2}\right)^{-\frac{1}{a}},\end{aligned}$$

which implies

$$\left(\ln\ln K_{j+1}^{\frac{1}{2}}\right)^{-a} \leq 3^{-1}e^{-4}s_0\varsigma_{j+2},$$

that is the inequality in (3.3) holds when $n = j + 1$.

By the discussions above we know that the inequality in (3.3) holds for all $n \geq 0$.

With the similar calculations in (3.5) we also have

$$T_{n,\mathcal{N}_n-1} = \sigma_{n,\mathcal{N}_n-1}^{-1}\ln\widetilde{\varepsilon}_{n,\mathcal{N}_n-1}^{-1} = \frac{4}{5}\sigma_{n,\mathcal{N}_n-1}^{-1}\ln\widetilde{\varepsilon}_{n,\mathcal{N}_n}^{-1} > \ln\varepsilon_{n+1}^{-1} > K_n^{\frac{1}{2}}. \tag{3.6}$$

□

Denote $B_n(\theta,\xi) = (B_n^l(\theta,\xi) : l \in \mathcal{J})^T$ be the real analytic vector valued function defined on $u(s_n) \times \mathcal{O}$. Assume that $B_n^l$ $(l \in \mathcal{J})$ have the following splitting

$$B_n^l(\theta,\xi) = \sum_{i=0}^{n} b_i^l(\theta,\xi) = \sum_{i=0}^{n}\sum_{j=0}^{\mathcal{N}_{i-1}} b_{i,j}^l(\theta,\xi),\ n \geq 0.$$

Thus $B_n(\theta,\xi)$[1] also possesses the splitting

$$B_n(\theta,\xi) = \sum_{i=0}^{n} b_i(\theta,\xi) = \sum_{i=0}^{n}\sum_{j=0}^{\mathcal{N}_{i-1}} b_{i,j}(\theta,\xi),\ n \geq 0, \tag{3.7}$$

where $b_i(\theta,\xi) = (b_i^l(\theta,\xi) : l \in \mathcal{J})^T$ and $b_{i,j}(\theta,\xi) = (b_{i,j}^l(\theta,\xi) : l \in \mathcal{J})^T$, $j = 0,\ldots,\mathcal{N}_{i-1}$, $i = 0,\ldots,n$. Moreover, we also assume

$$b_{i,j}(\theta,\xi) = \sum_{|k|\leq T_{i-1,j-1}} \widehat{b}_{i,j}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle},\ \ \|b_{i,j}\|_{s_n,\mathcal{O}} \leq \widetilde{\varepsilon}_{i-1,j-1}. \tag{3.8}$$

Note that there are no functions $B_0 := b_0 = \sum_{j=0}^{\mathcal{N}_{-1}} b_{0,j}$ and $b_{n+1,0}$ in the system (2.1) and (3.42), thus we set, in (3.7), $b_{0,j}(\theta,\xi) = 0$, $j = 0,\ldots,\mathcal{N}_{-1}$ and $b_{i,0}(\theta,\xi) = 0$, $i = 0,\ldots,n$.

For the sequences $\{T_{i-1,j-1}\}$, $j = 0,\ldots,\mathcal{N}_{i-1}$, $i = n, n-1$, defined in (3.1) we let $Q_{i,j}^n$ be the smallest integer number such that $\exp\{3^{-Q_{i,j}^n}T_{i-1,j-1}\} \leq \ln\varepsilon_n^{-1}$, that is

$$\exp\{3^{-Q_{i,j}^n}T_{i-1,j-1}\} \leq \ln\varepsilon_n^{-1} < \exp\{3^{-(Q_{i,j}^n-1)}T_{i-1,j-1}\}. \tag{3.9}$$

[1] Note that in our work one of the step of iteration will be finished by a series of sub-iteration. Take the $i$-th step iteration and $j$-th step sub-iteration for example, there will a term, which we denote as $\langle b_{i+1,j+1}z,\bar{z}\rangle$, been put into the normal form. Thus the function $B_n(\theta,\xi)$ possesses this special structure.

Furthermore, we denote

$$\widetilde{B}_{i,j}^{(l)}(\theta,\xi)=\sum_{3^{-(l+1)}T_{i-1,j-1}<|k|\le 3^{-l}T_{i-1,j-1}}\widehat{b}_{i,j}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle},$$
$$l=0,\ldots,\ Q_{i,j}^n-1,\ j=0,\ldots,\mathcal{N}_{i-1},\ i=n,n-1,$$
$$\widetilde{B}_{i,j}^{(Q_{i,j}^n)}(\theta,\xi)=\sum_{|k|\le 3^{-Q_{i,j}^n}T_{i-1,j-1}}\widehat{b}_{i,j}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle},$$
$$j=0,\ldots,\mathcal{N}_{i-1},\ i=n,n-1.$$

By the discussions above we can rewrite $B_n(\theta,\xi)$ as

$$B_n(\theta,\xi)=\sum_{i=n,n-1}\sum_{j=0}^{\mathcal{N}_{i-1}}\sum_{l=0}^{Q_{i,j}^n}\widetilde{B}_{i,j}^{(l)}(\theta,\xi)+\sum_{i=0}^{n-2}\sum_{j=0}^{\mathcal{N}_{i-1}}\widetilde{B}_{i,j}(\theta,\xi),\tag{3.10}$$

with $\widetilde{B}_{i,j}(\theta,\xi)=b_{i,j}(\theta,\xi),\ j=0,\ldots,\ \mathcal{N}_{i-1},\ i=0,\ldots,n-2$.

The reason that we do not separate $b_{i,j}(\theta,\xi)$ ($j=0,\ldots,\mathcal{N}_{i-1},\ i=0,\ldots,n-2$), into a sum of a sequence of functions as what we have did with $b_{i,j}(\theta,\xi)$, $j=0,\ldots,\mathcal{N}_{i-1},\ i=n,n-1$, is that the inequality in (3.2) guarantees that the solutions to the equations about these $\widetilde{B}_{i,j}(\theta,\xi)$ are controllable. See the discussions in the Proposition 3.1 for the details. Moreover, if $n\le 2$, we know that there is no $\widetilde{B}_{i,j}(\theta,\xi),\ j=0,\ldots,\mathcal{N}_i,\ i\le n-2$ ($\widetilde{B}_{0,j}(\theta,\xi)=0$), so when we consider these terms we means $n\ge 3$.

**Lemma 3.2** *Assume that $B_n(\theta,\xi)$ is the one defined by* (3.7) *with the estimate* (3.8). *Then for the homological equation*

$$\partial_\omega\mathcal{B}(\theta,\xi)=-B_n(\theta,\xi)+[B_n(\theta,\xi)]_\theta,\tag{3.11}$$

*there is a unique solution $\mathcal{B}$ satisfying*

$$\|\mathcal{B}\|_{\widehat{s},\mathcal{O}}<(480)^{-1}\ln\varepsilon_n^{-1}.\tag{3.12}$$

*Moreover, the function $W_n(\theta,\xi)=(W_n^l(\theta,\xi):\ l\in\mathbb{Z}_1)^T$ has the same decomposition in* (3.7) *and satisfies the same estimate in* (3.8). *Then there is a unique solution to the equation*

$$\partial_\omega\mathcal{W}(\theta,\xi)=-W_n(\theta,\xi)+[W_n(\theta,\xi)]_\theta$$

*satisfying*

$$\|\mathcal{W}\|_{\widehat{s},\mathcal{O}}<(480)^{-1}\ln\varepsilon_n^{-1}.$$

***Proof*** Rewrite the function $B$ as the one in (3.10). Assume that the functions $\mathcal{B}_{i,j}^{(l)}(\theta,\xi)$ solve

$$\partial_\omega\mathcal{B}_{i,j}^{(l)}(\theta,\xi)=-\widetilde{B}_{i,j}^{(l)}(\theta,\xi)+[\widetilde{B}_{i,j}^{(l)}(\theta,\xi)]_\theta\tag{3.13}$$

with $l=0,\ldots,\ Q_{i,j}^n,\ j=0,\ldots,\mathcal{N}_{i-1},\ i=n,n-1$, and $\mathcal{B}_{i,j}$ solve

$$\partial_\omega\mathcal{B}_{i,j}(\theta,\xi)=-\widetilde{B}_{i,j}(\theta,\xi)+[\widetilde{B}_{i,j}(\theta,\xi)]_\theta\tag{3.14}$$

with $j=0,\ldots,\mathcal{N}_{i-1},\ i=0,\ldots,\ n-2$. Then

$$\mathcal{B}(\theta,\xi)=\sum_{i=n,n-1}\sum_{j=0}^{\mathcal{N}_{i-1}}\sum_{l=0}^{Q_{i,j}^n}\mathcal{B}_{i,j}^{(l)}(\theta,\xi)+\sum_{i=0}^{n-2}\sum_{j=0}^{\mathcal{N}_{i-1}}\mathcal{B}_{i,j}(\theta,\xi)$$

solves (3.11). By comparing the Fourier coefficients of (3.11) we have

$$\widehat{\mathcal{B}}(k,\xi)=\frac{\widehat{\widehat{B}}(k,\xi)}{\mathrm{i}\langle k,\omega\rangle},\ k\neq 0.$$

From the equation above and note that $\mathcal{O}\subset[1,2]$ we obtain

$$\sup_{\xi\in\mathcal{O}}\left|\widehat{\mathcal{B}}(k,\xi)\right|=\sup_{\xi\in\mathcal{O}}\left|\frac{\widehat{\widehat{B}}(k,\xi)}{\mathrm{i}\langle k,\omega\rangle}\right|\leq\sup_{\xi\in\mathcal{O}}\left|\widehat{\widehat{B}}(k,\xi)\right||\langle k,\overline{\omega}\rangle|^{-1},$$

and

$$\begin{aligned}
&\sup_{\xi_1\neq\xi_2,\xi_1,\xi_2\in\mathcal{O}}\left|\frac{\widehat{\mathcal{B}}(k,\xi_1)-\widehat{\mathcal{B}}(k,\xi_2)}{\xi_1-\xi_2}\right|\\
&=\sup_{\xi_1\neq\xi_2,\xi_1,\xi_2\in\mathcal{O}}\left|\left\{\frac{\widehat{\widehat{B}}(k,\xi_1)}{\mathrm{i}\langle k,\xi_1\overline{\omega}\rangle}-\frac{\widehat{\widehat{B}}(k,\xi_2)}{\mathrm{i}\langle k,\xi_2\overline{\omega}\rangle}\right\}(\xi_1-\xi_2)^{-1}\right|\\
&\leq\sup_{\xi_1\neq\xi_2,\xi_1,\xi_2\in\mathcal{O}}\left|\left\{\frac{\widehat{\widehat{B}}(k,\xi_1)}{\mathrm{i}\langle k,\xi_1\overline{\omega}\rangle}-\frac{\widehat{\widehat{B}}(k,\xi_2)}{\mathrm{i}\langle k,\xi_1\overline{\omega}\rangle}\right\}(\xi_1-\xi_2)^{-1}\right|\\
&\quad+\sup_{\xi_1\neq\xi_2,\xi_1,\xi_2\in\mathcal{O}}\left|\left\{\frac{\widehat{\widehat{B}}(k,\xi_2)}{\mathrm{i}\langle k,\xi_1\overline{\omega}\rangle}-\frac{\widehat{\widehat{B}}(k,\xi_2)}{\mathrm{i}\langle k,\xi_2\overline{\omega}\rangle}\right\}(\xi_1-\xi_2)^{-1}\right|\\
&\leq\sup_{\xi_1\neq\xi_2,\xi_1,\xi_2\in\mathcal{O}}\left|\frac{\widehat{\widehat{B}}(k,\xi_1)-\widehat{\widehat{B}}(k,\xi_2)}{\xi_1-\xi_2}\right||\langle k,\overline{\omega}\rangle|^{-1}+\sup_{\xi_2\in\mathcal{O}}\left|\widehat{\widehat{B}}(k,\xi_2)\right||\langle k,\overline{\omega}\rangle|^{-1}.
\end{aligned}$$

So

$$\|\widehat{\mathcal{B}}(k)\|_{\mathcal{O}}\leq 2\|\widehat{\widehat{B}}(k)\|_{\mathcal{O}}|\langle k,\overline{\omega}\rangle|^{-1}. \tag{3.15}$$

For $l=0,\ldots,3^{-Q^n_{i,j}},\ j=0,\ldots,\mathcal{N}_i,\ i=n,n-1$, we know that

$$\begin{aligned}
3^{-l}T_{i-1,j-1}&\geq 3^{-Q^n_{i,j}}T_{i-1,j-1}=3^{-1}3^{-(Q^n_{i,j}-1)}T_{i-1,j-1}>3^{-1}\ln\ln\varepsilon_n^{-1}\\
&=3^{-1}\ln\left\{40(2\tau+1)K_{n-1}^{\frac{1}{2}}\right\}>3^{-1}\ln\left\{K_0^{\frac{1}{2}}\right\}\\
&>3^{-1}\exp\left\{(3^{-1}e^{-4}s_0\varsigma_2)^{\frac{-1}{a}}\right\}>\exp\left\{(e^{-4}s_0\varsigma_2)^{\frac{-1}{a}}\right\},
\end{aligned} \tag{3.16}$$

where the second inequality is by the inequality in the right side of (3.9) and the last but one inequality is from $K_0^{\frac{1}{3}}\geq\exp\exp\{(3^{-1}e^{-4}s_0\varsigma_2)^{\frac{-1}{a}}\}$. Note that the function $\Gamma(T)=\left(\ln|T|\right)^{-a}$, $a\in(0,1]$, is monotone decreasing on $[\exp\{(e^{-4}s_0\varsigma_2)^{\frac{-1}{a}}\},\ \infty)$, then by (3.16) we know that $\Gamma(3^{-l}T_{i-1,j-1})$ are well defined and $0<\Gamma(3^{-l}T_{i-1,j-1})<e^{-4}s_0\varsigma_2<1$, $l\leq 3^{-Q^n_{i,j}},\ j\leq\mathcal{N}_i,\ i=n,n-1$.

Let us consider (3.13) first.

(I) $l=0,\ldots,Q_{i,j}-1$. By (1.2) and (3.15) we obtain ($\widehat{s}=s_n(1-\varsigma_n)$),

$$\begin{aligned}
\|\mathcal{B}^{(l)}_{i,j}\|_{\widehat{s},\mathcal{O}}&=\sum_{3^{-(l+1)}T_{i-1,j-1}<|k|\leq 3^{-l}T_{i-1,j-1}}\|\widehat{\mathcal{B}}^{(l)}_{i,j}(k)\|_{\mathcal{O}}e^{|k|s_n(1-\varsigma_n)}\\
&\leq 2\exp\{3^{-l}T_{i-1,j-1}\Gamma(3^{-l}T_{i-1,j-1})\}\exp\{-3^{-(l+1)}T_{i-1,j-1}s_n\varsigma_n\}\\
&\quad\cdot\sum_{3^{-(l+1)}T_{i-1,j-1}<|k|\leq 3^{-l}T_{i-1,j-1}}\|\widehat{\widehat{B}}^{(l)}_{i,j}(k)\|_{\mathcal{O}}e^{|k|s_n}
\end{aligned}$$

$$
\begin{aligned}
&= 2\exp\{3^{-l}T_{i-1,j-1}\Gamma(3^{-l}T_{i-1,j-1})\}\exp\{-3^{-l}T_{i-1,j-1}3^{-1}s_n\varsigma_n\}\|\widetilde{B}^{(l)}_{i,j}\|_{s_n,\mathcal{O}}\\
&\le 2\exp\{3^{-l}T_{i-1,j-1}\Gamma(3^{-l}T_{i-1,j-1})\}\exp\{-3^{-l}T_{i-1,j-1}3^{-1}e^{-4}s_0\varsigma_n\}\|\widetilde{B}^{(l)}_{i,j}\|_{s_n,\mathcal{O}}\\
&\le 2\|\widetilde{B}^{(l)}_{i,j}\|_{s_n,\mathcal{O}} < 2\ln\varepsilon_n^{-1}\|\widetilde{B}^{(l)}_{i,j}\|_{s_n,\mathcal{O}},
\end{aligned}
$$

where the third inequality follows from the following: First, with the same calculations in (3.16) we obtain

$$3^{-l}T_{i-1,j-1} \ge 3^{-(Q_{i,j}-1)}T_{i-1,j-1} > \ln K_{n-1}^{\frac{1}{2}}, l \le Q_{i,j}-1, j \le \mathcal{N}_i, i = n, n-1,$$

which implies

$$\Gamma(3^{-l}T_{i-1,j-1}) < \Gamma(\ln K_{n-1}^{\frac{1}{2}}),\ l \le Q_{i,j}-1,\ j \le \mathcal{N}_i, i = n, n-1. \tag{3.17}$$

Moreover, from (3.3), (3.17) we have

$$3^{-1}e^{-4}s_0\varsigma_n > \Gamma\left(\ln K_{n-1}^{\frac{1}{2}}\right) > \Gamma(3^{-l}T_{i-1,j-1}),\ l \le Q_{i,j}-1,$$

which implies $\exp\{3^{-l}T_{i-1,j-1}3^{-1}e^{-4}s_0\varsigma_n\} > \exp\{3^{-l}T_{i-1,j-1}\Gamma(3^{-l}T_{i-1,j-1})\}$, that is

$$\exp\{3^{-l}T_{i-1,j-1}\Gamma(3^{-l}T_{i-1,j-1})\}\exp\{-3^{-l}T_{i-1,j-1}3^{-1}e^{-4}s_0\varsigma_n\} < 1.$$

(II) $l = Q^n_{i,j}$. Similarly, we have the following, note that $\Gamma(3^{-Q^n_{i,j}}T_{i-1,j-1}) < 1$,

$$
\begin{aligned}
\|\mathcal{B}^{(Q^n_{i,j})}_{i,j}\|_{\widehat{s},\mathcal{O}} &= \sum_{|k|\le 3^{-Q^n_{i,j}}T_{i-1,j-1}} \|\widehat{\mathcal{B}}^{(Q^n_{i,j})}_{i,j}(k)\|_{\mathcal{O}}e^{|k|s_n(1-\varsigma_n)}\\
&\le \sum_{|k|\le 3^{-Q^n_{i,j}}T_{i-1,j-1}} 2\exp\{3^{-Q^n_{i,j}}T_{i-1,j-1}\}\|\widehat{\widetilde{B}}^{(Q^n_{i,j})}_{i,j}(k)\|_{\mathcal{O}}e^{|k|s_n(1-\varsigma_n)}\\
&= 2\exp\{3^{-Q^n_{i,j}}T_{i-1,j-1}\}\|\widetilde{B}^{(Q^n_{i,j})}_{i,j}\|_{\widehat{s},\mathcal{O}} \le 2\ln\varepsilon_n^{-1}\|\widetilde{B}^{(Q^n_{i,j})}_{i,j}\|_{s_n,\mathcal{O}},
\end{aligned}
$$

where the last inequality is by the inequality in the left side of (3.9).

Now we consider the homological equation (3.14). Note that $T_{i-1,j-1} \le T_{n-3,\mathcal{N}_{n-3}-1}$, $j = 0, \ldots, \mathcal{N}_{i-1}$, $i \le n-2$, and from (3.2) we have

$$\exp\{T_{i-1,j-1}\} \le \exp\{T_{n-3,\mathcal{N}_{n-3}-1}\} < \ln\varepsilon_n^{-1},\ j = 0, \ldots, \mathcal{N}_{i-1},\ i \le n-2.$$

Moreover, by (3.3) and (3.6) we also have

$$\Gamma(T_{n-3,\mathcal{N}_{n-3}-1}) < \Gamma(T_{0,\mathcal{N}_0-1}) < \Gamma\left(\ln K_0^{\frac{1}{2}}\right) < 3^{-1}e^{-4}s_0\eta_1 < 1,\ n \ge 3.$$

Then by the two inequalities above and with the similar discussions in the case II) , we obtain

$$\|\mathcal{B}_{i,j}\|_{\widehat{s},\mathcal{O}} \le 2\ln\varepsilon_n^{-1}\|\widetilde{B}_{i,j}\|_{s_n,\mathcal{O}},\ j = 0, \ldots, \mathcal{N}_{i-1},\ i \le n-2.$$

The discussions above imply that the function $\mathcal{B}(\theta, \xi)$, the solution to (3.11), satisfies

$$\begin{aligned} \|\mathcal{B}\|_{\widehat{s},\mathcal{O}} &= \left\| \sum_{i=n,n-1} \sum_{j=0}^{\mathcal{N}_{i-1}} \sum_{l=0}^{Q^n_{i,j}} \mathcal{B}^{(l)}_{i,j} + \sum_{i=0}^{n-2} \sum_{j=0}^{\mathcal{N}_{i-1}} \mathcal{B}_{i,j} \right\|_{\widehat{s},\mathcal{O}} \\ &= \sum_{i=n,n-1} \sum_{j=0}^{\mathcal{N}_{i-1}} \sum_{l=0}^{Q^n_{i,j}} \|\mathcal{B}^{(l)}_{i,j}\|_{\widehat{s},\mathcal{O}} + \sum_{i=0}^{n-2} \sum_{j=0}^{\mathcal{N}_{i-1}} \|\mathcal{B}_{i,j}\|_{\widehat{s},\mathcal{O}} \\ &\le 2\ln \varepsilon_n^{-1} \left\{ \sum_{i=n,n-1} \sum_{j=0}^{\mathcal{N}_{i-1}} \sum_{l=0}^{Q^n_{i,j}} \|\widetilde{B}^{(l)}_{i,j}\|_{s_n,\mathcal{O}} + \sum_{i=0}^{n-2} \sum_{j=0}^{\mathcal{N}_{i-1}} \|\widetilde{B}_{i,j}\|_{s_n,\mathcal{O}} \right\} \\ &= 2\ln \varepsilon_n^{-1} \sum_{i=0}^{n} \sum_{j=0}^{\mathcal{N}_{i-1}} \|b_{i,j}\|_{s_n,\mathcal{O}} < 4\varepsilon_0 \ln \varepsilon_n^{-1} < (480)^{-1} \ln \varepsilon_n^{-1}. \end{aligned}$$

The discussions about the equation

$$\partial_\omega \mathcal{W}(\theta, \xi) = -W_n(\theta, \xi) + [W_n(\theta, \xi)]_\theta$$

are the same as the discussions above since the functions $B_n$ and $W_n$ have the same structure and satisfy the same estimate, we omit the details. □

Assume that the real analytic functions $N$ and $R$ are defined on $D(s, r) \times \mathcal{O}$ and with Taylor expansions

$$N = E(\theta, \xi) + \langle \omega, I\rangle + \langle [\Omega + B_n(\theta, \xi) + b(\theta, \xi)]z, \overline{z}\rangle - \langle [\Lambda - W_n(\theta, \xi) - w(\theta, \xi)]\rho, \overline{\rho}\rangle,$$

and

$$R(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi) = \sum_{0<|\delta+\beta|+|\alpha+\eta|\le 2, \delta\ne\beta, \alpha\ne\eta} R_{\delta,\beta,\alpha,\eta}(\theta, \xi) z^\delta \overline{z}^\beta \rho^\alpha \overline{\rho}^\eta,$$

where $B(\theta, \xi)$ is the one defined in (3.7) with the estimate (3.8), and the function $W(\theta, \xi) = (W^l_n(\theta, \xi) : \ l \in \mathbb{Z}_1)^T$ has the same decomposition in (3.7) and satisfies the same estimate in (3.8). Moreover,

$$b(\theta, \xi) = (b_j(\theta, \xi) : \ j \in \mathcal{J})^T, \ w(\theta, \xi) = (w_j(\theta, \xi) : \ j \in \mathbb{Z}_1)^T.$$

We consider the homological equation on the unknown function $F$

$$\{F, N\} = R. \tag{3.18}$$

For the homological equation above we have the following proposition.

**Proposition 3.1** *Assume that $b(\theta, \xi)$ and $w(\theta, \xi)$ are defined on $u(s) \times \mathcal{O}$ ( $e^{-4}s_0 < s < \widehat{s} := s_n(1 - \varsigma_n)$) satisfying $\|b\|_{s,\mathcal{O}}, \|w\|_{s,\mathcal{O}} \le \varepsilon_n$, and for every $\xi \in \mathcal{O}$, the vectors $\widetilde{\Omega} = \Omega + [B_n(\theta, \xi)]_\theta$ and $\widetilde{\Lambda} = \Lambda - [W_n(\theta, \xi)]_\theta$ satisfy the Melnikov's conditions*

$$\left|\langle k, \omega\rangle + \langle \zeta, \widetilde{\Omega}\rangle\right| \ge \gamma \langle k\rangle^{-\tau}, k \in \mathbb{Z}^d, \ \ 0 < |\zeta| \le 2, \tag{3.19}$$

*and*

$$\left|\langle l, \widetilde{\Lambda}\rangle\right| \ge 1, \ \ 0 < |l| \le 2, \tag{3.20}$$

*where* $0 < \gamma \ll 1$, $\tau > d+2$. *Then for the real analytic function* $R$ *defined on* $D(s,r)\times\mathcal{O}$, (3.18) *has a real analytic approximate solution* $F(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi)$ *satisfying*

$$\left\|X_F\right\|_{r,s-\sigma,r,\mathcal{O}} \leq 2^6\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\left\|X_R\right\|_{r,s,r,\mathcal{O}}.$$

*Moreover, the error term is*

$$R^{(er)}(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi) = \sum_{0<|\delta+\beta|+|\alpha+\eta|\leq 2, \delta\neq\beta, \alpha\neq\eta} R^{(er)}_{\delta,\beta,\alpha,\eta}(\theta,\xi) z^{\delta}\overline{z}^{\beta}\rho^{\alpha}\overline{\rho}^{\eta} \tag{3.21}$$

*with*

$$\begin{aligned} R^{(er)}_{\delta,\beta,\alpha,\eta}(\theta,\xi) &= e^{\mathrm{i}\langle\delta-\beta,\mathcal{B}\rangle+\langle\alpha-\eta,\mathcal{W}\rangle}\mathcal{R}_{T_{n,j}}\Big\{e^{-\mathrm{i}\langle\delta-\beta,\mathcal{B}\rangle-\langle\alpha-\eta,\mathcal{W}\rangle}R_{\delta,\beta,\alpha,\eta}(\theta,\xi) \\ &\quad - \left[\mathrm{i}\langle\delta-\beta, b(\theta,\xi)\rangle + \langle\alpha-\eta, w(\theta,\xi)\rangle\right]e^{-\mathrm{i}\langle\delta-\beta,\mathcal{B}\rangle-\langle\alpha-\eta,\mathcal{W}\rangle}F_{\delta,\beta,\alpha,\eta}(\theta,\xi)\Big\} \end{aligned}$$

*and the estimate* $(2\sigma < s,\ j = 0,\ldots,\mathcal{N}_n - 1)$

$$\left\|X_{R^{(er)}}\right\|_{r,s-2\sigma,r,\mathcal{O}} \leq 2^7\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{40}}\widetilde{\varepsilon}_{n,j}\sigma^{-1}\left\|X_R\right\|_{r,s,r,\mathcal{O}}. \tag{3.22}$$

***Proof*** We consider the case $n \geq 1$ first. For the function $R(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi)$ given above and function $F(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi)$ with the Taylor expansion

$$F(\theta, z, \overline{z}, \rho, \overline{\rho}, \xi) = \sum_{0<|\delta+\beta|+|\alpha+\eta|\leq 2, \delta\neq\beta, \alpha\neq\eta} F_{\delta,\beta,\alpha,\eta}(\theta,\xi) z^{\delta}\overline{z}^{\beta}\rho^{\alpha}\overline{\rho}^{\eta},$$

we denote

$$\begin{aligned} \widetilde{R}_{\delta,\beta,\alpha,\eta}(\theta,\xi) &= e^{-\mathrm{i}\langle\delta-\beta,\mathcal{B}(\theta,\xi)\rangle-\langle\alpha-\eta,\mathcal{W}(\theta,\xi)\rangle}R_{\delta,\beta,\alpha,\eta}(\theta,\xi), \\ \widetilde{F}_{\delta,\beta,\alpha,\eta}(\theta,\xi) &= e^{-\mathrm{i}\langle\delta-\beta,\mathcal{B}(\theta,\xi)\rangle-\langle\alpha-\eta,\mathcal{W}(\theta,\xi)\rangle}F_{\delta,\beta,\alpha,\eta}(\theta,\xi), \end{aligned} \tag{3.23}$$

where $\mathcal{B}$ and $\mathcal{W}$ are the one in Lemma 3.2. From (3.18) we obtain

$$\begin{aligned} &\partial_\omega \widetilde{F}_{\delta,\beta,\alpha,\eta}(\theta,\xi) + \big\{\mathrm{i}\langle\delta-\beta, \widetilde{\Omega} + b(\theta,\xi)\rangle \\ &\quad - \langle\alpha-\eta, \widetilde{\Lambda} - w(\theta,\xi)\rangle\big\}\widetilde{F}_{\delta,\beta,\alpha,\eta}(\theta,\xi) = \widetilde{R}_{\delta,\beta,\alpha,\eta}(\theta,\xi), \end{aligned} \tag{3.24}$$

where $\widetilde{\Omega} = \Omega + \left[B_n(\theta,\xi)\right]_\theta$ and $\widetilde{\Lambda} = \Lambda - \left[W_n(\theta,\xi)\right]_\theta$.
(I) $\delta = (\cdots, 1, \cdots)$, $\beta = (\cdots, 1, \cdots)$, where 1 is the $i$-th ($l$-th) component of the vectors $\delta$ ($\beta$), $i \neq l$, and "$\cdots$" stands for zeros, and $\alpha = \eta = 0$. Denote

$$\begin{aligned} R^{(1)}(\theta, z, \overline{z}, \xi) &= \sum_{\zeta=\delta-\beta} R_\zeta(\theta,\xi) z^{\delta}\overline{z}^{\beta} = \sum_{i,l\in\mathcal{J}} R_{i,l}(\theta,\xi) z_i\overline{z}_l, \\ F^{(1)}(\theta, z, \overline{z}, \xi) &= \sum_{\zeta=\delta-\beta} F_\zeta(\theta,\xi) z^{\delta}\overline{z}^{\beta} = \sum_{i,l\in\mathcal{J}} R_{i,l}(\theta,\xi) z_i\overline{z}_l, \end{aligned}$$

where

$$R_{i,l}(\theta,\xi) = R_\zeta(\theta,\xi) = R_{\delta,\beta,\alpha,\eta}(\theta,\xi),\ \ F_{i,l}(\theta,\xi) = F_\zeta(\theta,\xi) = F_{\delta,\beta,\alpha,\eta}(\theta,\xi).$$

Then by the Eq. (3.24) we obtain

$$\partial_\omega \widetilde{F}_\zeta(\theta,\xi) + \mathrm{i}\langle\zeta, \widetilde{\Omega} + b(\theta,\xi)\rangle\widetilde{F}_\zeta(\theta,\xi) = \widetilde{R}_\zeta(\theta,\xi). \tag{3.25}$$

We solve the truncated system of (3.25), i.e.,

$$\mathcal{T}_{T_{n,j}}\partial_\omega\widetilde{F}_\zeta + \mathcal{T}_{T_{n,j}}\left\{\mathrm{i}\langle\zeta, \widetilde{\Omega} + b(\theta,\xi)\rangle\widetilde{F}_\zeta\right\} = \mathcal{T}_{T_{n,j}}\widetilde{R}_\zeta,\ \mathcal{T}_{T_{n,j}}\widetilde{F}_\zeta = \widetilde{F}_\zeta,$$

which is equivalent to, for any $|k| \le T_{n,j}$ (by comparing the Fourier coefficients)

$$\mathrm{i}\big[\langle k, \omega\rangle + \langle \zeta, \widetilde{\Omega}\rangle\big]\widehat{\widetilde{F}}_{\zeta}(k,\xi) + \mathrm{i}\sum_{|k_1|\le T_{n,j}} \langle \zeta, \widehat{b}(k-k_1,\xi)\rangle \widehat{\widetilde{F}}_{\zeta}(k_1,\xi) = \widehat{\widetilde{R}}_{\zeta}(k,\xi). \tag{3.26}$$

Rewrite (3.26) as

$$\big(\widehat{E} + \Xi_s \widehat{D}\Xi_s^{-1}\big)\Xi_s \mathcal{F}_\zeta = \Xi_s \mathcal{R}_\zeta,$$

where

$$\widehat{E} = diag(\cdots, \mathrm{i}(\langle k,\omega\rangle + \langle \zeta, \widetilde{\Omega}\rangle, \cdots)_{|k|\le T_{n,j}},$$
$$\widehat{D} = \mathrm{i}(\langle \zeta, \widehat{b}(k-k_1,\xi)\rangle)_{|k_1|,|k|\le T_{n,j}}, \quad \Xi_s = diag(\cdots, e^{|k|s}, \cdots)_{|k|\le T_{n,j}},$$
$$\mathcal{F}_\zeta = \mathcal{F}_\zeta(\xi) = (\widehat{\widetilde{F}}_\zeta(k,\xi))^T_{|k|\le T_{n,j}}, \quad \mathcal{R}_\zeta = \mathcal{R}_\zeta(\xi) = (\widehat{\widetilde{R}}_\zeta(k,\xi))^T_{|k|\le T_{n,j}}.$$

From (3.19) we have

$$\|\widehat{E}^{-1}\|_{op(l^1)} \le \gamma^{-1}T^\tau_{n,j},$$

where $op(l^1)$ denotes the operator norm associate to the $l^1$-norm, which is defined by $|u|_{l^1} = \sum_{|k|\le T_{n,j}} |u(k)|$, for the vector $u = (u(k))^T_{|k|\le T_{n,j}}$. Since [the second inequality below is by (3.5)]

$$T_{n,j} \le T_{n,\mathcal{N}_n-1} < \exp\left\{\frac{2}{3}K_{n-1}^{\frac{1}{2}}\right\} \le \exp\left\{K_{n-1}^{\frac{1}{2}}\right\} = \varepsilon_n^{\frac{-1}{40(2\tau+1)}},$$

we know that

$$\|\widehat{E}^{-1}\|_{op(l^1)} \le \gamma^{-1}T^\tau_{n,j} < 4^{-1}\varepsilon_n^{\frac{-1}{40}}.$$

By direct calculations we have

$$\|\Xi_s\widehat{D}\Xi_s^{-1}\|_{op(l^1)} \le 2\|b\|_{s,\mathcal{O}} < 2\varepsilon_n.$$

The above two inequalities yield

$$\|\widehat{E}^{-1}\Xi_s\widehat{D}\Xi_s^{-1}\|_{op(l^1)} \le \frac{1}{2},$$

which implies that $\widehat{E} + \Xi_s\widehat{D}\Xi_s^{-1}$ has a bounded inverse. The above three inequalities yield

$$\begin{aligned}\|(\widehat{E} + \Xi_s\widehat{D}\Xi_s^{-1})^{-1}\|_{op(l^1)} &\le \|(Id + \widehat{E}^{-1}\Xi_s\widehat{D}\Xi_s^{-1})^{-1}\|_{op(l^1)}\|\widehat{E}^{-1}\|_{op(l^1)}\\ &\le 2\gamma^{-1}T^\tau_{n,j}.\end{aligned}$$

It follows that

$$\begin{aligned}\|\widetilde{F}_\zeta\|^*_{s,\mathcal{O}} &= \sum_{|k|\le T_{n,j}} \|\widehat{\widetilde{F}}_\zeta(k)\|^*_{\mathcal{O}}e^{|k|s} = \|\Xi_s\mathcal{F}_\zeta\|^*_{\mathcal{O}}\\ &\le \|(\widehat{E} + \Xi_s\widehat{D}\Xi_s^{-1})^{-1}\|_{op(l^1)}\|\Xi_s\mathcal{R}_\zeta\|^*_{\mathcal{O}}\\ &= \|(\widehat{E} + \Xi_s\widehat{D}\Xi_s^{-1})^{-1}\|_{op(l^1)}\|\widetilde{R}_\zeta\|^*_{s,\mathcal{O}}\\ &\le 2\gamma^{-1}T^\tau_{n,j}\|\widetilde{R}_\zeta\|^*_{s,\mathcal{O}}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|F_\zeta\|^*_{s,\mathcal{O}} &\le e^{2\|\mathcal{B}\|_{s,\mathcal{O}}}\|\widetilde{F}_\zeta\|^*_{s,\mathcal{O}} \le e^{2\|\mathcal{B}\|_{\widehat{s},\mathcal{O}}} 2\gamma^{-1}T^\tau_{n,j}\|\widetilde{R}_\zeta\|^*_{s,\mathcal{O}}\\ &\le 2\gamma^{-1}T^\tau_{n,j}e^{4\|\mathcal{B}\|_{\widehat{s},\mathcal{O}}}\|R_\zeta\|^*_{s,\mathcal{O}}\\ &\le 2\gamma^{-1}T^\tau_{n,j}\varepsilon_n^{\frac{-1}{120}}\|R_\zeta\|^*_{s,\mathcal{O}},\end{aligned} \tag{3.27}$$

where the last inequality is from (3.12).

Next we give the estimate of the Lipschitz semi-norm. Denote $\Delta_{\xi_1,\xi_2}Q = Q(\cdot,\xi_1) - Q(\cdot,\xi_2)$. From (3.26) we have

$$\begin{aligned}&\mathrm{i}\Big(\langle k,\xi_1\overline{\omega}\rangle + \langle\zeta,\widetilde{\Omega}(\xi_1)\rangle\Big)\Delta_{\xi_1,\xi_2}\widehat{\widetilde{F}}_\zeta(k) + \mathrm{i}\sum_{|k_1|\le T_{n,j}}\langle\zeta,\widehat{b}(k-k_1,\xi_1)\rangle\Delta_{\xi_1,\xi_2}\widehat{\widetilde{F}}_\zeta(k_1)\\ &\quad + \mathrm{i}\Delta_{\xi_1,\xi_2}\Big(\langle k,\omega\rangle + \langle\zeta,\widetilde{\Omega}\rangle\Big)\widehat{\widetilde{F}}_\zeta(k,\xi_2) + \mathrm{i}\sum_{|k_1|\le T_{n,j}}\langle\zeta,\Delta_{\xi_1,\xi_2}\widehat{b}(k-k_1)\rangle\widehat{\widetilde{F}}_\zeta(k_1,\xi_2)\\ &= \Delta_{\xi_1,\xi_2}\widehat{\widetilde{R}}_\zeta(k),\quad |k|\le T_{n,j}.\end{aligned}$$

In the similar way to get (3.27) we obtain,

$$\|\Delta_{\xi_1,\xi_2}F_\zeta\|^*_{s,\mathcal{O}} \le 2\gamma^{-1}T^\tau_{n,j}\varepsilon_n^{\frac{-1}{120}}\Big\{\big|\Delta_{\xi_1,\xi_2}\Big(\widehat{E}+\widehat{D}\Big)\big|_{op(l^1)}\|F_\zeta\|^*_{s,\mathcal{O}} + \big\|\Delta_{\xi_1,\xi_2}R_\zeta\big\|^*_{s,\mathcal{O}}\Big\}.$$

Dividing $\|\Delta_{\xi_1,\xi_2}F_\zeta\|^*_{s,\mathcal{O}}$ by $|\xi_1-\xi_2|$ and taking supreme over $\xi_1\ne\xi_2\in\mathcal{O}$, (note that $\Omega$ does not depending on the parameter $\xi$), we have

$$\begin{aligned}\|F_\zeta\|^L_{s,\mathcal{O}} &\le 2\gamma^{-1}T^\tau_{n,j}\varepsilon_n^{\frac{-1}{120}}\Big\{2T_{n,j}|\overline{\omega}|\,\|F_\zeta\|^*_{s,\mathcal{O}} + \big\|R_\zeta\big\|^L_{s,\mathcal{O}}\Big\}\\ &\le 2\gamma^{-1}T^\tau_{n,j}\varepsilon_n^{\frac{-1}{120}}\Big\{4\gamma^{-1}T^{\tau+1}_{n,j}|\overline{\omega}|\varepsilon_n^{\frac{-1}{120}}\|R_\zeta\|^*_{s,\mathcal{O}} + \big\|R_\zeta\big\|^L_{s,\mathcal{O}}\Big\}\\ &\le 2^4\gamma^{-2}T^{2\tau+1}_{n,j}\varepsilon_n^{\frac{-1}{60}}|\overline{\omega}|\big\|R_\zeta\big\|_{s,\mathcal{O}},\end{aligned}$$

where the second inequality is from (3.27). Then by (3.27) and the inequality above we obtain, note $F_{i,l}=F_\zeta,\ R_{i,l}=R_\zeta$,

$$\|F_{i,l}\|_{s,\mathcal{O}} \le 2^5\gamma^{-2}T^{2\tau+1}_{n,j}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\big\|R_{i,l}\big\|_{s,\mathcal{O}}. \tag{3.28}$$

Thus by the inequality above we have

$$\begin{aligned}\|F^{(1)}_{z_i}\|_{D(s,r),\mathcal{O}} &= \|\sum_{l\in\mathcal{J}}F_{i,l}\overline{z}_l\|_{D(s,r),\mathcal{O}} = \sup_{\|\overline{z}\|_{a,p}\le r}\sum_{l\in\mathcal{J}}\|F_{i,l}\|_{s,\mathcal{O}}|\overline{z}_l|\\ &\le 2^5\gamma^{-2}T^{2\tau+1}_{n,j}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sup_{\|\overline{z}\|_{a,p}\le r}\sum_{l\in\mathcal{J}}\|R_{i,l}\|_{s,\mathcal{O}}|\overline{z}_l|\\ &= 2^5\gamma^{-2}T^{2\tau+1}_{n,j}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\|R^{(1)}_{z_i}\|_{D(s,r),\mathcal{O}},\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{r}\|F_z^{(1)}\|_{a,p,D(s,r),\mathcal{O}} &= \frac{1}{r}\left\{\sum_{i\in\mathcal{J}}\|F_{z_i}^{(1)}\|^2_{D(s,r)\times\mathcal{O}}e^{2ai}i^{2p}\right\}^{\frac{1}{2}} \\ &\le 2^5\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\frac{1}{r}\left\{\sum_{i\in\mathcal{J}}\|R_{z_i}^{(1)}\|^2_{D(s,r),\mathcal{O}}e^{2ai}i^{2p}\right\}^{\frac{1}{2}} \\ &= 2^5\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\frac{1}{r}\|R_z^{(1)}\|_{a,p,D(s,r),\mathcal{O}}.\end{aligned} \tag{3.29}$$

Similarly, we also obtain

$$\frac{1}{r}\|F_{\overline{z}}^{(1)}\|_{a,p,D(s,r),\mathcal{O}} \le 2^5\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\frac{1}{r}\|R_{\overline{z}}^{(1)}\|_{a,p,D(s,r),\mathcal{O}}. \tag{3.30}$$

Moreover, by Cauchy estimate and (3.27) we obtain

$$\begin{aligned}&\frac{1}{r^2}\|F_{\theta_y}^{(1)}\|_{D(s-\sigma,r),\mathcal{O}} \\ &= \frac{1}{r^2}\sup_{\|z\|_{a,p},\|\overline{z}\|_{a,p}\le r}\sum_{i,l\in\mathcal{J},i\ne l}\sum_{k\in\mathbb{Z}^d}\|\widehat{F}_{i,l}^{(1)}(k,\xi)\mathrm{i}k_y\|_{\mathcal{O}}e^{|k|(s-\sigma)}|z_i||\overline{z}_l| \\ &\le \sigma^{-1}\frac{1}{r^2}\sup_{\|z\|_{a,p},\|\overline{z}\|_{a,p}\le r}\sum_{i,l\in\mathcal{J},i\ne l}\sum_{k\in\mathbb{Z}^d}\|\widehat{F}_{i,l}^{(1)}(k,\xi)\|_{\mathcal{O}}e^{|k|s}|z_i||\overline{z}_l| \\ &= \sigma^{-1}\frac{1}{r^2}\sup_{\|z\|_{a,p},\|\overline{z}\|_{a,p}\le r}\sum_{i,l\in\mathcal{J},i\ne l}\|F_{i,l}^{(1)}\|_{s,\mathcal{O}}|z_i||\overline{z}_l| \\ &\le \sigma^{-1}2^5\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\frac{1}{r^2}\sup_{\|z\|_{a,p},\|\overline{z}\|_{a,p}\le r}\sum_{i,l\in\mathcal{J},i\ne l}\|R_{i,l}^{(1)}\|_{s,\mathcal{O}}|z_i||\overline{z}_l| \\ &= 2^5\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\frac{1}{r^2}\|R^{(1)}\|_{D(s,r),\mathcal{O}} \\ &\le 2^5\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\|X_{R^{(1)}}\|_{r,s,r,\mathcal{O}},\ \ y=1,\ldots,d.\end{aligned} \tag{3.31}$$

Thus by (3.29)–(3.31) we obtain

$$\|X_{F^{(1)}}\|_{r,s-\sigma,r,\mathcal{O}} \le 2^6\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\|X_{R^{(1)}}\|_{r,s,r,\mathcal{O}}. \tag{3.32}$$

(II) $\delta=(\cdots,1,\cdots)$, $\alpha=(\cdots,1,\cdots)$, where 1 is the $i$-th ($l$-th) component of the vectors $\delta$ ($\alpha$), and "$\cdots$" stands for zeros, and $\beta=0$, $\eta=0$. Denote

$$R^{(2)}(\theta,z,\rho,\xi)=\sum_{|\delta|=|\alpha|=1}R_{\delta,\alpha}(\theta,\xi)z^\delta\rho^\alpha,$$
$$F^{(2)}(\theta,z,\rho,\xi)=\sum_{|\delta|=|\alpha|=1}F_{\delta,\alpha}(\theta,\xi)z^\delta\rho^\alpha,$$

where $R_{\delta,\alpha}(\theta,\xi)=R_{\delta,\beta,\alpha,\eta}(\theta,\xi)$, $F_{\delta,\alpha}(\theta,\xi)=F_{\delta,\beta,\alpha,\eta}(\theta,\xi)$. By Eq. (3.18) we get

$$\partial_\omega\widetilde{F}_{\delta,\alpha}(\theta,\xi)+\left\{\mathrm{i}\langle\delta,\widetilde{\Omega}+b(\theta,\xi)\rangle-\langle\alpha,\widetilde{\Lambda}-w(\theta,\xi)\rangle\right\}\widetilde{F}_{\delta,\alpha}(\theta,\xi)=\widetilde{R}_{\delta,\alpha}(\theta,\xi). \tag{3.33}$$

We solve the truncated system of (3.33), i.e.,

$$\begin{aligned}&\mathcal{T}_{T_{n,j}}\partial_\omega \widetilde{F}_{\delta,\alpha}+\mathcal{T}_{T_{n,j}}\big\{\big[\mathrm{i}\langle\delta,\widetilde{\Omega}+b(\theta,\xi)\rangle\\&\quad-\langle\alpha,\widetilde{\Lambda}-w(\theta,\xi)\rangle\big]\widetilde{F}_{\delta,\alpha}(\theta,\xi)\big\}=\mathcal{T}_{T_{n,j}}\widetilde{R}_{\delta,\alpha},\ \mathcal{T}_{T_{n,j}}\widetilde{F}_{\delta,\alpha}=\widetilde{F}_{\delta,\alpha},\end{aligned}$$

which is equivalent to, for any $|k|\leq T_{n,j}$ (by comparing the Fourier coefficients)

$$\begin{aligned}&\Big\{\mathrm{i}\langle k,\omega\rangle+\mathrm{i}\langle\delta,\widetilde{\Omega}\rangle-\langle\alpha,\widetilde{\Lambda}\rangle\Big\}\widehat{\widetilde{F}}_{\delta,\alpha}(k,\xi)\\&\quad+\sum_{|k_1|\leq T_{n,j}}\big[\mathrm{i}\langle\delta,\widehat{b}(k-k_1,\xi)\rangle+\langle\alpha,\widehat{w}(k-k_1,\xi)\rangle\big]\widehat{\widetilde{F}}_{\delta,\alpha}(k_1,\xi)=\widehat{\widetilde{R}}_{\delta,\alpha}(k,\xi).\end{aligned}$$

Denote

$$A(k,\alpha,\delta,\xi)=\mathrm{i}\langle k,\xi\overline{\omega}\rangle+\mathrm{i}\langle\delta,\widetilde{\Omega}\rangle-\langle\alpha,\widetilde{\Lambda}\rangle. \tag{3.34}$$

Thus

$$\sup_{\xi\in\mathcal{O}}|A^{-1}(k,\alpha,\delta,\xi)|=\sup_{\xi\in\mathcal{O}}\frac{1}{\big|\mathrm{i}\langle k,\xi\overline{\omega}\rangle+\mathrm{i}\langle\delta,\widetilde{\Omega}\rangle-\langle\alpha,\widetilde{\Lambda}\rangle\big|}\leq\frac{1}{|\langle\alpha,\widetilde{\Lambda}\rangle|}<1,$$

where the last inequality is from (3.20). Then with the similar discussions in case I) we obtain

$$\big\|X_{F^{(2)}}\big\|_{r,s-\sigma,r,\mathcal{O}}\leq 2^6T_{n,j}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\big\|X_{R^{(2)}}\big\|_{r,s,r,\mathcal{O}}.$$

(III) $\alpha=(\cdots,1,\cdots),\ \eta=(\cdots,1,\cdots)$, where 1 is the $i$-th ($l$-th) component of the vectors $\alpha\ (\eta),\ i\neq l$, and "$\cdots$" stands for zeros, and $\delta=0,\beta=0$. Denote $\zeta=\alpha-\eta$ and

$$\begin{aligned}R^{(3)}(\theta,\rho,\overline{\rho},\xi)&=\sum_{\zeta=\alpha-\eta}R_\zeta(\theta,\xi)(\theta,\xi)\rho^\alpha\overline{\rho}^\eta,\\F^{(3)}(\theta,\rho,\overline{\rho},\xi)&=\sum_{\zeta=\alpha-\eta}F_\zeta(\theta,\xi)(\theta,\xi)\rho^\alpha\overline{\rho}^\eta,\end{aligned}$$

where $R_\zeta(\theta,\xi)=R_{\delta,\beta,\alpha,\eta}(\theta,\xi),\ F_\zeta(\theta,\xi)=F_{\delta,\beta,\alpha,\eta}(\theta,\xi)$. By Eq. (3.18) we obtain

$$\partial_\omega\widetilde{F}_\zeta(\theta,\xi)-\langle\zeta,\widetilde{\Lambda}-w(\theta,\xi)\rangle\widetilde{F}_\zeta(\theta,\xi)=\widetilde{R}_\zeta(\theta,\xi). \tag{3.35}$$

Note that, by (3.20),

$$\sup_{\xi\in\mathcal{O}}\frac{1}{\big|\mathrm{i}\langle k,\xi\overline{\omega}\rangle-\langle\zeta,\widetilde{\Lambda}\rangle\big|}\leq\frac{1}{|\langle\zeta,\widetilde{\Lambda}\rangle|}<1.$$

Then with the similar discussions in case II) we know that the homological equation (3.35) possesses an approximate solution $F^{(3)}$ satisfying

$$\big\|X_{F^{(3)}}\big\|_{r,s-\sigma,r,\mathcal{O}}\leq 2^6T_{n,j}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\big\|X_{R^{(3)}}\big\|_{r,s,r,\mathcal{O}}.$$

With the similar calculations in the cases I)–III) above we can also get the estimate about the rest terms of the function $F$. Thus, we obtain

$$\big\|X_F\big\|_{r,s-\sigma,r,\mathcal{O}}\leq 2^6\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\big\|X_R\big\|_{r,s,r,\mathcal{O}}.$$

Obviously, the error term $R^{(er)}$ has the formula gave by (3.21). Moreover,

$$\begin{aligned}\|R^{(er)}_{\delta,\beta,\alpha,\eta}\|_{s-\sigma,\mathcal{O}} &\le e^{-T\sigma}\epsilon_n^{\frac{1}{120}}\Big\{\|R_{\delta,\beta,\alpha,\eta}\|_{s,\mathcal{O}}+2\epsilon_n\|F_{\delta,\beta,\alpha,\eta}\|_{s,\mathcal{O}}\Big\}\\ &\le 2\widetilde{\varepsilon}_{n,j}\epsilon_n^{\frac{1}{120}}\|F_{\delta,\beta,\alpha,\eta}\|_{s,\mathcal{O}}\\ &< 2^6\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{40}}\widetilde{\varepsilon}_{n,j}\|R_{\delta,\beta,\alpha,\eta}\|_{s,\mathcal{O}},\end{aligned}$$

where the last inequality is by (3.28) [note that the estimate in (3.28) is the biggest bounds for the coefficient function $F_{\delta,\beta,\alpha,\eta}$]. Then with the same calculations to get (3.32) we obtain

$$\big\|X_{R^{(er)}}\big\|_{r,s-2\sigma,r,\mathcal{O}} \le 2^7\gamma^{-2}T_{n,j}^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{40}}\widetilde{\varepsilon}_{n,j}\sigma^{-1}\big\|X_R\big\|_{r,s,r,\mathcal{O}}.$$

In the case $n=0$, note that $B_0(\theta,\xi)=0$, $W_0(\theta,\xi)=0$, we will not make the change $\widetilde{R}_{\delta,\beta,\alpha,\eta}(\theta,\xi)=e^{-\mathrm{i}\langle\delta-\beta,\mathcal{B}(\theta,\xi)\rangle-\langle\alpha-\eta,\mathcal{W}(\theta,\xi)\rangle}R_{\delta,\beta,\alpha,\eta}(\theta,\xi)$ and $\widetilde{F}_{\delta,\beta,\alpha,\eta}(\theta,\xi)=e^{-\mathrm{i}\langle\delta-\beta,\mathcal{B}(\theta,\xi)\rangle-\langle\alpha-\eta,\mathcal{W}(\theta,\xi)\rangle}F_{\delta,\beta,\alpha,\eta}(\theta,\xi)$ and deal with the equation between $F_{\delta,\beta,\alpha,\eta}(\theta,\xi)$ and $R_{\delta,\beta,\alpha,\eta}(\theta,\xi)$ directly. In this case

$$\begin{aligned}R^{(er)}_{\delta,\beta,\alpha,\eta}(\theta,\xi)=\mathcal{R}_{T_{n,j}}\Big\{R_{\delta,\beta,\alpha,\eta}(\theta,\xi)-\big[\mathrm{i}\langle\delta-\beta,b(\theta,\xi)\rangle\\ +\langle\alpha-\eta,w(\theta,\xi)\rangle\big]F_{\delta,\beta,\alpha,\eta}(\theta,\xi)\Big\},\end{aligned}$$

and the estimates about $F$ and $R^{(er)}$ also hold. □

***Remark 3.1*** For the function $A(k,\alpha,\delta,\xi)$ defined by (3.34), with the discussion in case II) we know that $\|A^{-1}(k,\alpha,\delta)\|^*_{\mathcal{O}}$ is bounded for all $k$, $\alpha$ and $\delta$. However, $\|A^{-1}(k,\alpha,\delta)\|^L_{\mathcal{O}}$ may not be bounded. See the calculations below.

$$\begin{aligned}\|A^{-1}(k,\alpha,\delta)\|^L_{\mathcal{O}} &= \sup_{\xi_1,\xi_2\in\mathcal{O},\xi_1\neq\xi_2}\left|\frac{1}{A(k,\alpha,\delta,\xi_1)}-\frac{1}{A(k,\alpha,\delta,\xi_2)}\right||\xi_1-\xi_2|^{-1}\\ &=\sup_{\xi_1,\xi_2}\frac{\big|(\xi_1-\xi_2)\mathrm{i}\langle k,\overline{\omega}\rangle+\mathrm{i}\langle\delta,\Delta_{\xi_1,\xi_2}\big[B_n(\theta)\big]_\theta\rangle+\langle\alpha,\Delta_{\xi_1,\xi_2}\big[W_n(\theta)\big]_\theta\rangle\big||\xi_1-\xi_2|^{-1}}{\big|\mathrm{i}\langle k,\xi_1\overline{\omega}\rangle+\mathrm{i}\langle\delta,\widetilde{\Omega}(\xi_1)\rangle-\langle\alpha,\widetilde{\Lambda}(\xi_1)\rangle\big|\big|\mathrm{i}\langle k,\xi_2\overline{\omega}\rangle+\mathrm{i}\langle\delta,\widetilde{\Omega}(\xi_2)\rangle-\langle\alpha,\widetilde{\Lambda}(\xi_2)\rangle\big|}\\ &\ge\sup_{\xi\in\mathcal{O}}\frac{\big|\langle k,\overline{\omega}\rangle\big|}{\big|\mathrm{i}\langle k,\xi\overline{\omega}\rangle+\mathrm{i}\langle\delta,\widetilde{\Omega}(\xi)\rangle-\langle\alpha,\widetilde{\Lambda}(\xi)\rangle\big|^2}\\ &\quad-\sup_{\xi\in\mathcal{O}}\frac{2\big(\|B_n\|_{s,\mathcal{O}}+\|W_n\|_{s,\mathcal{O}}\big)}{\big|\mathrm{i}\langle k,\xi\overline{\omega}\rangle+\mathrm{i}\langle\delta,\widetilde{\Omega}\rangle-\langle\alpha,\widetilde{\Lambda}\rangle\big|^2}\\ &>\sup_{\xi\in\mathcal{O}}\frac{\big|\langle k,\overline{\omega}\rangle\big|}{\big(\langle k,\xi\overline{\omega}\rangle+\langle\delta,\widetilde{\Omega}\rangle\big)^2+\big|\langle\alpha,\widetilde{\Lambda}\rangle\big|^2}-\frac{4\epsilon_0}{\|\widetilde{\Lambda}\|^2_{\mathcal{O}}}\\ &>\sup_{\xi\in\mathcal{O}}\frac{\big|\langle k,\overline{\omega}\rangle\big|}{\big(\langle k,\xi\overline{\omega}\rangle+\langle\delta,\widetilde{\Omega}\rangle\big)^2+\big|\langle\alpha,\widetilde{\Lambda}\rangle\big|^2}-4\epsilon_0.\end{aligned}\tag{3.36}$$

Note that for the fixed $k$ with $\big|\langle k,\overline{\omega}\rangle\big|>1$, there is $\delta$ such that

$$\big|\langle k,\xi\overline{\omega}\rangle+\langle\delta,\widetilde{\Omega}\rangle\big|<1,$$

which implies

$$\big(\langle k,\xi\overline{\omega}\rangle+\langle\delta,\widetilde{\Omega}\rangle\big)^2+\big|\langle\alpha,\widetilde{\Lambda}\rangle\big|^2<2\big\|\langle\alpha,\widetilde{\Lambda}\rangle\big\|^2_{\mathcal{O}}.$$

Thus, by the inequality above and (3.36) we obtain

$$\|A^{-1}(k,\alpha,\delta)\|_{\mathcal{O}}^{L} > \frac{\big|\langle k,\overline{\omega}\rangle\big|}{2\big\|\langle\alpha,\widetilde{\Lambda}\rangle\big\|_{\mathcal{O}}^{2}} - 4\epsilon_0 > \frac{\big|\langle k,\overline{\omega}\rangle\big|}{8} - 4\epsilon_0.$$

Note that there are $k \in \mathbb{Z}^d$ such that $\big|\langle k,\overline{\omega}\rangle\big| \to \infty$, as $|k|$ goes to $\infty$, which means, together with the inequality above, for these $(k,\delta,\alpha)$,

$$\|A^{-1}(k,\alpha,\delta)\|_{\mathcal{O}}^{L} \to \infty.$$

That is $\|A^{-1}(k,\alpha,\delta)\|_{\mathcal{O}}^{L}$ is unbounded.

By the discussions above we know that there are $(k,\delta,\alpha)$ such that $\|A^{-1}(k,\alpha,\delta)\|_{\mathcal{O}}$ is unbounded. This is the reason we also make the change in (3.23) to kill the function $W(\theta,\xi)$ and solve the truncated equations for the homological equation (3.18) along the hyperbolic directions.

### 3.3 Iteration Lemma

Beside the parameters defined in (3.1), we will also define the sequences, for $0<\gamma<1,\ 0<r_0<1$, $(\gamma_n)_{n\ge 0}$, $(r_n)_{n\ge 0}$, and $(D_n)_{n\ge 0}$ in the following manner:

$$r_0 = r,\ \gamma_0=\gamma,\ r_{n+1}=\varepsilon_{n+1}^{\frac{4}{3}}r_n,\ \gamma_n=\gamma_0\varsigma_n,\ D_n=D(s_n,r_n),\ n\ge 1.$$

**Lemma 3.3** (Iteration Lemma) *Suppose that the real analytic Hamiltonian system $H_n = H_{n-1}\circ\Phi_n = N_n + P_n$ defined on $D_n\times\mathcal{O}_n$, where*

$$N_n = E_n(\theta,\xi) + \langle\omega, I\rangle + \langle\Omega + B_n(\theta,\xi)z,\bar z\rangle - \langle\Lambda - W_n(\theta,\xi)\rho,\overline{\rho}\rangle,$$
$$P_n = P_n(\theta, z,\bar z,\rho,\overline{\rho},\xi),$$

*and*

$$\mathcal{O}_n = \left\{\xi\in\mathcal{O}_{n-1} : \big|\langle k,\omega\rangle + \langle\zeta,\widetilde{\Omega}_n\rangle\big| \ge \frac{\gamma_n}{\langle k\rangle^{\tau}},\ \forall 0<|\zeta|\le 2,\ k\in\mathbb{Z}^d\right\}$$

*where $\Omega$ and $\Lambda$ are the ones defined in (2.2), $B_n$ is the one defined in (3.7) and satisfies the estimates (3.8) and the function $W_n(\theta,\xi) = (W_n^l(\theta,\xi) : l\in\mathbb{Z}_1)^T$ have the same decomposition in (3.7) and satisfy the same estimate in (3.8) with $\widetilde{\Omega}_n = \Omega + [B_n]_\theta$. Moreover,*

$$\|X_{E_n - E_{n-1}}\|_{r_n,s_n,r_n,\mathcal{O}_n} \le 2\varepsilon_{n-1}, \tag{3.37}$$
$$\|X_{P_n}\|_{r_n,s_n,r_n,\mathcal{O}_n} \le \varepsilon_n. \tag{3.38}$$

*Then there is a subset $\mathcal{O}_{n+1}\subset\mathcal{O}_n$ with*

$$\mathcal{O}_{n+1} = \left\{\xi\in\mathcal{O}_n : \big|\langle k,\omega\rangle + \langle\zeta,\widetilde{\Omega}_{n+1}\rangle\big| \ge \frac{\gamma_{n+1}}{\langle k\rangle^{\tau}},\ \forall 0<|\zeta|\le 2, k\in\mathbb{Z}^d\right\} \tag{3.39}$$

*and a real analytic symplectic change of variables*

$$\Phi_{n+1} : D_{n+1}\times\mathcal{O}_{n+1} \to D_n\times\mathcal{O}_n$$

*such that $H_{n+1} = H_n\circ\Phi_{n+1}$ has the analogous form of $H_n$ and satisfies the conditions (3.37) and (3.38) and $B_{n+1}$ is the one with the formula in (3.7) and satisfies the estimates (3.8) with $(n+1)$ in place of $n$, the function $W_{n+1}$ has the same decomposition and satisfies* *the same estimate with the ones of the function $B_{n+1}$. Furthermore, we have the following estimates*

$$\|\Phi_{n+1} - id\|_{r_{n+1},s_{n+1},r_{n+1},\mathcal{O}_n} \leq 2\varepsilon_n^{\frac{1}{2}}, \tag{3.40}$$

$$\|D\Phi_{n+1} - Id\|_{r_{n+1},s_{n+1},r_{n+1},\mathcal{O}_n} \leq 2\varepsilon_n^{\frac{1}{2}}. \tag{3.41}$$

### 3.4 Proof of Lemma 3.3

In order to prove Lemma 3.3 we let

$$\widetilde{\varepsilon}_0 = \varepsilon_n,\ \widetilde{r}_0 = r_n,\ \widetilde{s}_0 = s_n(1-\eta_n),\ \gamma = \gamma_n,\ \mathcal{O} = \mathcal{O}_n$$

and define $\widetilde{\delta}_j = \widetilde{\varepsilon}_{n,j}^{\frac{1}{3}},\ \widetilde{r}_{j+1} = \widetilde{\delta}_j\widetilde{r}_j,\ \widetilde{s}_{j+1} = \widetilde{s}_j - 5\sigma_{n,j},\ j \geq 0.$

Consider the real analytic Hamiltonian system $H_n = N_n + P_n$ defined on $D_n \times \mathcal{O}_n$, we rewrite it as

$$\begin{aligned}\widetilde{H}_0 = \widetilde{N}_0 + \widetilde{P}_0 = E(\theta,\xi) + \langle\omega, I\rangle + \langle(\Omega + B(\theta,\xi))z, \bar{z}\rangle \\ - \langle(\Lambda - W(\theta,\xi))\rho, \overline{\rho}\rangle + \widetilde{P}_0(\theta, z, \bar{z}, \rho, \overline{\rho}, \xi)\end{aligned} \tag{3.42}$$

defined on $\widetilde{D}_0 \times \mathcal{O}$, where $\widetilde{D}_j = D(\widetilde{s}_j, \widetilde{r}_j)$, $\mathcal{O} = \mathcal{O}_n$, $E = E_n$, $B(\theta,\xi) = B_n(\theta,\xi)$, $W(\theta,\xi) = W_n(\theta,\xi)$ and $\widetilde{P}_0 = P_n$. Obviously,

$$\|X_{\widetilde{P}_0}\|_{\widetilde{r}_0,\widetilde{s}_0,\widetilde{r}_0,\mathcal{O}} \leq \widetilde{\varepsilon}_{n,0}.$$

Denote $\widetilde{H}_j = \widetilde{N}_j + \widetilde{P}_j$, where

$$\widetilde{N}_j = E + \sum_{l=0}^{j} \widetilde{E}_{n+1,l}(\theta,\xi) + \langle\omega, I\rangle + \left\langle \left[\Omega + \left(B + \sum_{l=0}^{j} b_{n+1,l}\right)(\theta,\xi)\right] z, \bar{z}\right\rangle$$

$$-\left\langle \left[\Lambda - \left(W + \sum_{l=0}^{j} w_{n+1,l}\right)(\theta,\xi)\right] \rho, \overline{\rho}\right\rangle, \tag{3.43}$$

$$\|X_{\widetilde{P}_j}\|_{\widetilde{r}_j,\widetilde{s}_j,\widetilde{r}_j,\mathcal{O}} < \widetilde{\varepsilon}_{n,j} \tag{3.44}$$

with

$$\widetilde{E}_{n+1,0}(\theta,\xi) = 0,\ b_{n+1,l}(\theta,\xi) = (b^i_{n+1,l}(\theta,\xi),\ i \in \mathcal{J})^T,\ b_{n+1,0}(\theta,\xi) = 0,$$

and

$$w_{n+1,l}(\theta,\xi) = (w^i_{n+1,l}(\theta,\xi),\ i \in \mathbb{Z}_1)^T,\ \ w_{n+1,0}(\theta,\xi) = 0.$$

Moreover,

$$b_{n+1,l}(\theta,\xi) = \sum_{|k|\leq T_{n,l-1}} \widehat{b}_{n+1,l}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle},\ \|b_{n+1,l}\|_{\widetilde{s}_{l-1},\mathcal{O}} \leq \widetilde{\varepsilon}_{n,l-1}, \tag{3.45}$$

$$w_{n+1,l}(\theta,\xi) = \sum_{|k|\leq T_{n,l-1}} \widehat{w}_{n+1,l}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle},\ \|w_{n+1,l}\|_{\widetilde{s}_{l-1},\mathcal{O}} \leq \widetilde{\varepsilon}_{n,l-1}, \tag{3.46}$$

and

$$\|X_{\widetilde{E}_{n+1,l}}\|_{\widetilde{r}_{l-1},\widetilde{s}_{l-1},\widetilde{r}_{l-1},\mathcal{O}} < \widetilde{\varepsilon}_{n,l-1}. \tag{3.47}$$

Suppose that for $j=0,\ldots,\nu-1$, there exists real analytic $F_{j+1}$ defined on $\widetilde{D}_{j+1}\times\mathcal{O}$ such that one gets the real analytic Hamiltonian systems $\widetilde{H}_{j+1}$ defined in $\widetilde{D}_{j+1}\times\mathcal{O}$ :

$$\widetilde{H}_{j+1}=\widetilde{H}_j\circ X^1_{F_{j+1}}=\widetilde{N}_{j+1}+\widetilde{P}_{j+1},$$

where $\widetilde{N}_{j+1}$ is the one in (3.43) and satisfies, together with $\widetilde{P}_{j+1}$, (3.44)–(3.47) with $(j+1)$ in place of $j$ in the domain $\widetilde{D}_{j+1}\times\mathcal{O}$. Moreover, the real analytic symplectic map $X^1_{F_{j+1}}$ satisfies

$$\|X^1_{F_{j+1}}-id\|_{\widetilde{r}_{j+1},\widetilde{s}_{j+1},\widetilde{r}_{j+1},\mathcal{O}}<\widetilde{\varepsilon}^{\frac{1}{2}}_{n,j}, \tag{3.48}$$

$$\|DX^1_{F_{j+1}}-Id\|_{\widetilde{r}_{j+1},\widetilde{s}_{j+1},\widetilde{r}_{j+1},\mathcal{O}}<\widetilde{\varepsilon}^{\frac{1}{2}}_{n,j}. \tag{3.49}$$

Then, one wants to finds $F_{\nu+1}$ defined in $\widetilde{D}_{\nu+1}\times\mathcal{O}$ such that $\widetilde{H}_{\nu+1}=\widetilde{H}_\nu\circ X^1_{F_{\nu+1}}=\widetilde{N}_{\nu+1}+\widetilde{P}_{\nu+1}$ with $\widetilde{N}_{\nu+1}$ being the one in (3.43) and satisfies, together with $\widetilde{P}_{j+1}$, satisfies (3.44)–(3.47) with $(\nu+1)$ in place of $j$, and $F_{\nu+1}$ satisfies (3.48) and (3.49) with $\nu$ in place of $j$, on the domain $\widetilde{D}_{\nu+1}\times\mathcal{O}$.

In the following we will construct such function $F_{\nu+1}$. For $j=\nu$, by using the Taylor–Fourier expansion, we separate $\widetilde{P}_\nu$

$$\widetilde{P}_\nu(\theta,z,\bar{z},\rho,\overline{\rho},\xi)=\sum_{\delta,\beta,\alpha,\eta,k}\widehat{\widetilde{P}}_{\nu,\delta,\beta,\alpha,\eta}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle}z^\delta\bar{z}^\beta\rho^\alpha\overline{\rho}^\eta$$

into three parts:

$$\widetilde{P}_\nu=P_\nu^{(el)}+P_\nu^{(nf)}+P_\nu^{(pe)},$$

where

$$\begin{aligned}
P_\nu^{(el)}(\theta,z,\bar{z},\rho,\overline{\rho},\xi)&\sum_{\substack{0<|\delta+\beta|+|\alpha+\eta|\le 2\\ \delta\ne\beta,\alpha\ne\eta,k}}\widehat{\widetilde{P}}_{\nu,\delta,\beta,\alpha,\eta}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle}z^\delta\bar{z}^\beta\rho^\alpha\overline{\rho}^\eta,\\
P_\nu^{(nf)}(\theta,z,\bar{z},\rho,\overline{\rho},\xi)&=\sum_{k\in\mathbb{Z}^d}\widehat{\widetilde{P}}_{\nu,0,0,0,0}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle}\\
&\quad+\sum_{\substack{0<|\delta+\beta|+|\alpha+\eta|\le 2\\ \delta=\beta,\alpha=\eta,|k|\le T_{n,\nu}}}\widehat{\widetilde{P}}_{\nu,\delta,\beta,\alpha,\eta}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle}z^\delta\bar{z}^\beta\rho^\alpha\overline{\rho}^\eta\\
&=\widetilde{E}_{n+1,\nu+1}(\theta,\xi)+\langle b_{n+1,\nu+1}(\theta,\xi)z,\bar{z}\rangle+\langle w_{n+1,\nu+1}(\theta,\xi)\rho,\overline{\rho}\rangle,\\
P_\nu^{(pe)}(\theta,z,\bar{z},\rho,\overline{\rho},\xi)&=\sum_{\substack{0<|\delta+\beta|+|\alpha+\eta|\le 2\\ \delta=\beta,\alpha=\eta,|k|>T_{n,\nu}}}\widehat{\widetilde{P}}_{\nu,\delta,\beta,\alpha,\eta}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle}z^\delta\bar{z}^\beta\rho^\alpha\overline{\rho}^\eta\\
&\quad+\sum_{|\delta+\beta|+|\alpha+\eta|>2,k}\widehat{\widetilde{P}}_{\nu,\delta,\beta,\alpha,\eta}(k,\xi)e^{\mathrm{i}\langle k,\theta\rangle}z^\delta\bar{z}^\beta\rho^\alpha\overline{\rho}^\eta\\
&=:P_\nu^{(pe1)}+P_\nu^{(pe2)}.
\end{aligned} \tag{3.50}$$

Shorten the notations $\widetilde{\delta}_\nu,\widetilde{r}_\nu,\widetilde{s}_\nu,\sigma_{n,\nu},\ T_{n,\nu}$ as $\widetilde{\delta},\widetilde{r},\widetilde{s},\sigma$ and $T$, respectively. Obviously,

$$\|X_{P_\nu^{(el)}}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}},\ \|X_{P_\nu^{(nf)}}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}}\le\|X_{\widetilde{P}_\nu}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}}, \tag{3.51}$$

and

$$\|X_{P_\nu^{(pe1)}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}}\le e^{-T\sigma}\widetilde{\delta}^{-1}\|X_{\widetilde{P}_\nu}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}}<\widetilde{\delta}^{-1}\|X_{\widetilde{P}_\nu}\|^2_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}}.$$

For $P_\nu^{(pe2)}$ we have

$$\|X_{P_\nu^{(pe2)}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s},4\widetilde{\delta}\widetilde{r},\mathcal{O}} \le \widetilde{\delta}\|X_{\widetilde{P}_\nu}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}} = \|X_{\widetilde{P}_\nu}\|^{\frac{4}{3}}_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}}.$$

The discussions above yield

$$\|X_{P_\nu^{(pe)}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}} < 2\|X_{\widetilde{P}_\nu}\|^{\frac{4}{3}}_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}}. \tag{3.52}$$

We rewrite $\widetilde{H}_\nu$ as $\widetilde{H}_\nu = \widetilde{N}_{\nu+1} + P_\nu^{(el)} + P_\nu^{(pe)}$, where

$$\begin{aligned}\widetilde{N}_{\nu+1} &= \widetilde{N}_\nu + P_\nu^{(nf)}\\ &=: E(\theta,\xi) + \sum_{l=0}^{\nu+1}\widetilde{E}_{n+1,l}(\theta,\xi) + \langle\omega, I\rangle + \left\langle\left[\Omega + \left(B + \sum_{i=0}^{\nu+1} b_{n+1,i}\right)(\theta,\xi)\right]z, \bar{z}\right\rangle\\ &\quad - \left\langle\left[\Lambda - \left(W + \sum_{i=0}^{\nu+1} w_{n+1,i}\right)(\theta,\xi)\right]\rho, \overline{\rho}\right\rangle.\end{aligned}$$

Note that

$$\|b_{n+1,\nu+1}\|_{\widetilde{s},\mathcal{O}},\ \|w_{n+1,\nu+1}\|_{\widetilde{s},\mathcal{O}} \le \|X_{P_\nu^{(nf)}}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}} \le \|X_{\widetilde{P}_\nu}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}} < \widetilde{\epsilon}_{n,\nu},$$

and

$$\|X_{E_{n+1,\nu+1}}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}} \le \|X_{\widetilde{P}_\nu}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}} \le \widetilde{\epsilon}_{n,\nu}.$$

Thus, $\widetilde{N}_{\nu+1}$ owns the formula in (3.43) and satisfies (3.45)–(3.47) with $j = \nu + 1$.

The change of variables we need is the time-1-map of the flow $X^t_{F_{\nu+1}}\big|_{t=1}$. Using Taylor formula to expand $\widetilde{H}_\nu \circ X^t_{F_{\nu+1}}\big|_{t=1}$, we obtain

$$\begin{aligned}\widetilde{H}_\nu \circ X^1_{F_{\nu+1}} &= \widetilde{N}_{\nu+1} \circ X^1_{F_{\nu+1}} + P_\nu^{(el)} \circ X^t_{F_{\nu+1}} + P_\nu^{(pe)} \circ X^1_{F_{\nu+1}}\\ &= \widetilde{N}_{\nu+1} + P_\nu^{(el)} + \int_0^1 \left\{P_\nu^{(el)}, F_{\nu+1}\right\} \circ X^t_{F_{\nu+1}} dt + P_\nu^{(pe)} \circ X^1_{F_{\nu+1}}\\ &\quad + \left\{\widetilde{N}_{\nu+1}, F_{\nu+1}\right\} + \int_0^1 (1-t)\left\{\left\{\widetilde{N}_{\nu+1}, F_{\nu+1}\right\}, F_{\nu+1}\right\} \circ X^t_{F_{\nu+1}} dt.\end{aligned}$$

We want to find $F_{\nu+1}$ such that

$$\{F_{\nu+1}, N_{\nu+1}\} = P_\nu^{(el)}.$$

From Proposition 3.1 we know that the homological equation above have a real analytic approximate solution $F_{\nu+1}$ and satisfies

$$\left\|X_{F_{\nu+1}}\right\|_{\widetilde{r},\widetilde{s}-\sigma,\widetilde{r},\mathcal{O}} \le 2^6\gamma^{-2}T^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-1}\|X_{P_\nu^{(el)}}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}}. \tag{3.53}$$

Moreover, the error term $P_\nu^{(er)}$ of the homological equation above satisfies

$$\begin{aligned}\|X_{P_\nu^{(er)}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-2\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}} &\le \widetilde{\delta}^{-1}\|X_{P_\nu^{(er)}}\|_{\widetilde{r},\widetilde{s}-2\sigma,\widetilde{r},\mathcal{O}}\\ &\le 2^7\widetilde{\delta}^{-1}\gamma^{-2}T^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{40}}\sigma^{-1}\widetilde{\epsilon}_{n,\nu}^2\\ &\le 2^7\gamma^{-2}T^{2\tau+2}|\overline{\omega}|\varepsilon_n^{\frac{-1}{40}}\widetilde{\epsilon}_{n,\nu}^{\frac{5}{3}}\\ &\le 2^7\gamma^{-2}\varepsilon_n^{\frac{-1}{40}}|\overline{\omega}|\varepsilon_n^{\frac{-1}{40}}\widetilde{\epsilon}_{n,\nu}^{\frac{5}{3}}\\ &= 2^7\gamma_0^{-2}(n+2)^2(\nu+2)^2|\overline{\omega}|\varepsilon_n^{\frac{-1}{40}}\varepsilon_n^{\frac{-1}{40}}\widetilde{\varepsilon}_{n,\nu}^{\frac{5}{3}} \le \widetilde{\varepsilon}_{n,\nu}^{\frac{3}{2}},\end{aligned} \tag{3.54}$$

where the third inequality is by $\sigma^{-1} = T\big(\ln\widetilde{\varepsilon}_{n,\nu}^{-1}\big)^{-1} < T$, the last but one inequality is from the following [the third inequality below is from (3.5)]

$$\begin{aligned}T^{2\tau+2} < T^{2\tau+3} \le \widetilde{T}_{n,\mathcal{N}_n-1}^{2\tau+3} < \exp\left\{\frac{2(2\tau+3)}{3}K_{n-1}^{\frac{1}{2}}\right\}\\ \le \exp\left\{\frac{1}{40}(40(2\tau+1))K_{n-1}^{\frac{1}{2}}\right\} = \varepsilon_n^{\frac{-1}{40}}.\end{aligned}$$

By (3.53) and Cauchy estimate and with the similar calculations above we have

$$\|DX_{F_{\nu+1}}\|_{\widetilde{r},\widetilde{s}-2\sigma,\widetilde{r},\mathcal{O}} \le 2^6\gamma^{-2}T^{2\tau+1}|\overline{\omega}|\varepsilon_n^{\frac{-1}{60}}\sigma^{-2}\widetilde{\varepsilon}_{n,\nu} \le \widetilde{\varepsilon}_{n,\nu}^{\frac{6}{7}}. \tag{3.55}$$

From (3.53) and (3.55) we also obtain

$$\|X_{F_{\nu+1}}\|_{\widetilde{r},\widetilde{s}-\sigma,\widetilde{r},\mathcal{O}} \le \widetilde{\varepsilon}_{n,\nu}^{\frac{6}{7}}. \tag{3.56}$$

Then the flow $X_{F_{\nu+1}}^t$ of the vector field $X_{F_{\nu+1}}$ exists on $D(\widetilde{s}-3\sigma,\frac{\widetilde{r}}{2})$ for $0\le t\le 1$ and takes this domain into $D(\widetilde{s}-2\sigma,\widetilde{r})$. Similarly, it takes $D(\widetilde{s}-4\sigma,\frac{\widetilde{r}}{4})$ into $D(\widetilde{s}-3\sigma,\frac{\widetilde{r}}{2})$. Thus by Gronwall's inequality and the inequalities (3.55) and (3.56) we obtain

$$\begin{aligned}&\|X_{F_{\nu+1}}^t - id\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-5\sigma,\widetilde{\delta}\widetilde{r},\mathcal{O}} \le c\widetilde{\delta}^{-1}\|X_{F_{\nu+1}}\|_{\widetilde{r},\widetilde{s}-2\sigma,\widetilde{r},\mathcal{O}} < \widetilde{\varepsilon}_{n,\nu}^{\frac{1}{2}},\ 0\le t\le 1,\\ &\|DX_{F_{\nu+1}}^t - Id\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-5\sigma,\widetilde{\delta}\widetilde{r},\mathcal{O}} \le c\widetilde{\delta}^{-1}\|DX_{F_{\nu+1}}\|_{\widetilde{r},\widetilde{s}-2\sigma,\widetilde{r},\mathcal{O}} < \widetilde{\varepsilon}_{n,\nu}^{\frac{1}{2}},\ 0\le t\le 1.\end{aligned}$$

Also in the same way to obtain (20.7) in [15], for any vector field $Y$ we obtain

$$\|(X_{F_{\nu+1}}^1)^*Y\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-5\sigma,\widetilde{\delta}\widetilde{r},\mathcal{O}} \le c\|Y\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-3\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}}. \tag{3.57}$$

By the definition of $X_{F_{\nu+1}}^1$ and from (3.54) and (3.57) we know that

$$\begin{aligned}\widetilde{H}_\nu\circ X_{F_{\nu+1}}^1 &= \widetilde{N}_{\nu+1} + \widetilde{P}_{\nu+1}\\ &= \widetilde{N}_{\nu+1} + \int_0^1 t\big\{P_\nu^{(el)}, F_{\nu+1}\big\}\circ X_{F_{\nu+1}}^t dt + P_\nu^{(pe)}\circ X_{F_{\nu+1}}^1\\ &\quad + P_\nu^{(er)} + \int_0^1 (1-t)\big\{P_\nu^{(er)}, F_{\nu+1}\big\}\circ X_{F_{\nu+1}}^t dt\end{aligned} \tag{3.58}$$

is well defined on $D(\widetilde{s}_{\nu+1},\widetilde{r}_{\nu+1})\times\mathcal{O}$. Moreover,

$$\begin{aligned}X_{\widetilde{P}_{\nu+1}} &= \int_0^1 (X_{F_{\nu+1}}^t)^*[X_{tP_\nu^{(el)}}, X_{F_{\nu+1}}]dt + (X_{F_{\nu+1}}^1)^*X_{P_\nu^{(pe)}}\\ &\quad + X_{P_\nu^{(er)}} + \int_0^1 (X_{F_{\nu+1}}^t)^*[X_{(1-t)P_\nu^{(er)}}, X_{F_{\nu+1}}]dt,\end{aligned}$$

where $[X_{tP_\nu^{(el)}}, X_{F_{\nu+1}}]$ is the commutator of the two vector fields $X_{tP_\nu^{(el)}}, X_{F_{\nu+1}}$. In view of (3.51), (3.55), (3.56) and Cauchy estimate, we get

$$\begin{aligned}
&\|[X_{tP_\nu^{(el)}}, X_{F_{\nu+1}}]\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-3\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}} \\
&\leq \|DX_{tP_\nu^{(el)}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-3\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}}\|X_{F_{\nu+1}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-3\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}} \\
&\quad + \|DX_{F_{\nu+1}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-3\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}}\|X_{tP_\nu^{(el)}}\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-3\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}} \\
&\leq 2\widetilde{\delta}^{-1}\sigma^{-1}\|X_{F_{\nu+1}}\|_{\widetilde{r},\widetilde{s}-\sigma,\widetilde{r},\mathcal{O}}\|X_{P_\nu^{(el)}}\|_{\widetilde{r},\widetilde{s},\widetilde{r},\mathcal{O}} \\
&\leq 2\widetilde{\delta}^{-1}\sigma^{-1}\widetilde{\varepsilon}_{n,\nu}^{\frac{13}{7}}.
\end{aligned} \tag{3.59}$$

Then, by (3.57) and (3.59), we obtain

$$\begin{aligned}
&\|(X^t_{F_{\nu+1}})^*[X_{tP_\nu^{(el)}}, X_{F_{\nu+1}}]\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-5\sigma,\widetilde{\delta}\widetilde{r},\mathcal{O}} \\
&\leq c\|[X_{P^{(el)}}, X_{F_{\nu+1}}]\|_{\widetilde{\delta}\widetilde{r},\widetilde{s}-3\sigma,4\widetilde{\delta}\widetilde{r},\mathcal{O}} \\
&< c2\widetilde{\delta}^{-1}\sigma^{-1}\widetilde{\varepsilon}_{n,\nu}^{\frac{13}{7}} < \frac{\widetilde{\varepsilon}_{n,\nu}^{\frac{5}{4}}}{4} = \frac{\widetilde{\varepsilon}_{n,\nu+1}}{4}.
\end{aligned}$$

With (3.52) and (3.54), we can also obtain the same bound for the rest three terms of $X_{\widetilde{P}_{\nu+1}}$, we omit the details. Then we arrive at the estimate

$$\|X_{\widetilde{P}_{\nu+1}}\|_{\widetilde{r}_{\nu+1},\widetilde{s}_{\nu+1},\widetilde{r}_{\nu+1},\mathcal{O}} < \widetilde{\varepsilon}_{n,\nu+1}.$$

Once we reach the $\mathcal{N}_n$-th step, we terminate the above iteration. Denote

$$H_{n+1} := \widetilde{H}_{\mathcal{N}_n} = H_n \circ \Phi_{n+1} = \widetilde{N}_{\mathcal{N}_n} + \widetilde{P}_{\mathcal{N}_n} = N_{n+1} + P_{n+1},$$

defined on $D(s_{n+1}, r_{n+1}) \times \mathcal{O}_{n+1}$ and

$$\Phi_{n+1} = X^1_{F_1} \circ X^1_{F_2} \circ \cdots \circ X^1_{F_{\mathcal{N}_n}}$$

where

$$\begin{aligned}
N_{n+1} &= E_{n+1} + \langle\omega, I\rangle + \langle[\Omega + B_{n+1}(\theta,\xi)]z, \bar{z}\rangle - \langle[\Lambda - W_{n+1}(\theta,\xi)]\rho, \overline{\rho}\rangle, \\
P_{n+1} &= \widetilde{P}_{\mathcal{N}_n}, \ s_{n+1} = \widetilde{s}_{\mathcal{N}_n}, \ r_{n+1} = \widetilde{r}_{\mathcal{N}_n},
\end{aligned}$$

and $\mathcal{O}_{n+1}$ is the one defined by (3.39) with

$$E_{n+1} = E_n + \sum_{i=0}^{\mathcal{N}_n} E_{n+1,i}, \ B_{n+1} = B_n + \sum_{i=0}^{\mathcal{N}_n} b_{n+1,i}, \ W_{n+1} = W_n + \sum_{i=0}^{\mathcal{N}_n} w_{n+1,i}.$$

Recall that

$$\widetilde{r}_{j+1} = \widetilde{\varepsilon}_{n,j}^{\frac{1}{3}}\widetilde{r}_j, \quad \widetilde{s}_{j+1} = \widetilde{s}_j - 5\sigma_j.$$

Since

$$\Pi_{j=0}^{\mathcal{N}_n-1}\widetilde{\varepsilon}_{n,j}^{\frac{1}{3}} = \Pi_{j=0}^{\mathcal{N}_n-1}\widetilde{\varepsilon}_{n,0}^{\frac{1}{3}\left(\frac{5}{4}\right)^j} = \widetilde{\varepsilon}_{n,0}^{\frac{4}{3}[\left(\frac{5}{4}\right)^{\mathcal{N}_n}-1]} > \varepsilon_{n+1}^{\frac{4}{3}},$$

where the last inequality is from $\widetilde{\varepsilon}_{n,\mathcal{N}_n-1} > \varepsilon_{n+1}$ and $\widetilde{\varepsilon}_{n,0}^{\frac{-4}{3}} = \varepsilon_n^{\frac{-4}{3}} \gg 1$, which implies

$$r_{n+1} = \widetilde{r}_{\mathcal{N}_n} = r_n\Pi_{j=0}^{\mathcal{N}_n-1}\widetilde{\varepsilon}_j^{\frac{1}{3}} \geq r_n\varepsilon_{n+1}^{\frac{4}{3}}.$$

Moreover,

$$s_{n+1} = \widetilde{s}_{\mathcal{N}_n} = \widetilde{s}_0 - 5\sum_{j=0}^{\mathcal{N}_n-1}\sigma_{n,j} = \widetilde{s}_0 - \widetilde{s}_0\varsigma_n\sum_{j=0}^{\mathcal{N}_n-1}\varsigma_j \geq \widetilde{s}_0 - \varsigma_n\widetilde{s}_0 = s_n(1-\varsigma_n)^2.$$

As for the estimates about the change of variables $\Phi_{n+1}$, with the standard calculations in the KAM iteration and the inequalities (3.48) and (3.49) we know that $\Phi_{n+1}$ satisfies (3.40) and (3.41). We omit the details.

The above estimates imply that $H_{n+1}$ is well defined on $D(r_{n+1}, s_{n+1}) \times \mathcal{O}_{n+1}$. Moreover, $B_{n+1}$ owns the formula in (3.7) and satisfies the estimates in (3.8), the functions $E_{n+1}$ and $P_{n+1}$ satisfy the estimates in (3.37) and (3.38), respectively, with $n$ in place of $(n+1)$. The vector $W_{n+1} = (W^l_{n+1} : l \in \mathbb{Z}_1)^T$ has the same decomposition and the satisfies the same estimates as the ones of $B_{n+1}$. □

## 4 Proof of Main Results

### 4.1 Convergence and the Proof of Theorem 2.1

Consider the Hamiltonian system $H$(2.1) defined on $D(s,r) \times \mathcal{O}$ where $\omega = \xi\overline{\omega}$, $\Omega$ and $\Lambda$ are the ones defined in (2.2) and

$$\|X_P\|_{r,s,r,\mathcal{O}_*} \leq \varepsilon,$$

with

$$\mathcal{O}_* = \left\{\xi \in [1,\ 2] : |\langle k,\omega\rangle + \langle\zeta,\Omega\rangle| \geq \frac{\gamma}{\langle k\rangle^\tau},\ 0 < |\zeta| \leq 2, k \in \mathbb{Z}^d\right\}.$$

Set $s_0 = s,\ r_0 = r, \gamma_0 = \gamma, \mathcal{O}_0 = \mathcal{O}_*$, and assume that $\varepsilon_0 = \varepsilon \leq \varepsilon_*$ with $\ln \varepsilon_*^{\frac{-1}{40(2\tau+1)}} \geq 3\exp\left\{(3^{-1}e^{-4}s_0\varsigma_2)^{\frac{-1}{a}}\right\}$. Obviously, $E_0(\theta,\xi) = 0$, $B_0(\theta,\xi) = 0$ and $W_0(\theta,\xi) = 0$, and it is easy to check that system (2.1) satisfies all hypotheses of Lemma 3.3 with $n = 0$. Note that

$$\begin{aligned} s_\infty &= s_0\Pi_{n=0}^\infty(1-\varsigma_n)^2 \geq s_0\Pi_{n=0}^\infty(1-2\varsigma_n) = s_0\Pi_{n=0}^\infty[1-2(n+2)^{-2}] \\ &= s_0\exp\left\{\sum_{n=0}^\infty \ln[1-2(n+2)^{-2}]\right\} \geq s_0\exp\left\{\sum_{n=0}^\infty -4(n+2)^{-2}\right\} \\ &= s_0e^{-2} := s_*. \end{aligned}$$

Moreover,

$$r_\infty = r_0\Pi_{n=0}^\infty\varepsilon_{n+1}^{\frac{4}{3}} = 0 := r_*,$$

then

$$D(s_0,r_0) \supset D(s_1,r_1) \supset \cdots \supset D(s_\infty,r_\infty) \supset D(s_*,r_*).$$

Let $\Phi^n = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_n$. Then

$$H_n = H \circ \Phi^n = N_n + P_n$$

is the one in Lemma 3.3. Denote $\mathcal{O}_\gamma = \cap_{j=0}^\infty \mathcal{O}_j$. Note that $\Omega$ does not depend on the parameter $\xi$, then by the definitions of $\mathcal{O}_j,\ j \geq 0$, we know that the calculations to estimate the measures of set $\mathcal{O}_j,\ \ j \geq 0$, are the same with the ones in the proof of Lemma 5.2, we omit the details. By Lemma 5.2 we have

$$meas\mathcal{O}_\gamma \geq 1 - c\gamma.$$

From Lemma 3.3 we know that $H_n, N_n, P_n, \Phi^n$ and $D\Phi^n$ converge uniformly on $D(s_*, r_*) \times \mathcal{O}_\gamma$. Let the limits be $H_*, N_*, P_*, \Phi$ and $D\Phi$ respectively. Moreover, by (3.40) and (3.41) we know that

$$\begin{aligned} &\|\Phi - id\|_{r_*,s_*,r_*,\mathcal{O}_\gamma} \leq 4\varepsilon_0^{\frac{1}{2}}, \\ &\|D\Phi - Id\|_{r_*,s_*,r_*,\mathcal{O}_\gamma} \leq 4\varepsilon_0^{\frac{1}{2}}. \end{aligned} \tag{4.1}$$

Then

$$\begin{aligned} N_* &= E_*(\theta, \xi) + \langle \omega, I \rangle + \langle (\Omega + B_*(\theta, \xi))z, \overline{z} \rangle - \langle (\Lambda - W_*(\theta, \xi))\rho, \overline{\rho} \rangle, \\ P_* &= \sum_{|\delta+\beta|+|\alpha+\eta|\geq 3} P^*_{\delta,\beta,\alpha,\eta}(\theta, \xi) z^\delta \overline{z}^\beta \rho^\alpha \overline{\rho}^\eta. \end{aligned}$$

Moreover, from Sect. 3.4 we know that two inequalities in (2.3) hold.

### 4.2 Proof of Theorem 1.1

Denote $u_t = v$, then the system (1.6) becomes the following system, take the $\sqrt{m}$ as the example,

$$\begin{cases} u_t = v, \\ v_t = 2mu - u_{xxxx} + g(\varepsilon, \omega t, x, u), \end{cases} \tag{4.2}$$

where

$$g(\varepsilon, \omega t, x, u) = \varepsilon u^3 + 3\varepsilon^{\frac{1}{2}}\sqrt{m}u^2 + \varepsilon^{\frac{1}{2}} f(\omega t, x, \sqrt{m} + \varepsilon^{\frac{1}{2}} u).$$

The Hamiltonian of the above system is

$$H = \frac{1}{2}\langle v, v \rangle + \frac{1}{2}\langle Au, u \rangle + \int_0^\pi G(\varepsilon, \omega t, x, u)dx, \tag{4.3}$$

with the symplectic form $du \wedge dv$ on the space $H^2([0, \pi]) \times L^2([0, \pi])$, where

$$A = -2m + \partial^4_{xxxx}, \ \ \partial_u G(\varepsilon, \omega t, x, u) = -g(\varepsilon, \omega t, x, u).$$

We assume that the function $u$ is even in $x \in [0,\ \pi]$, which implies that we restrict the function $u$ in the space spanned by $\{\psi_j(x) := \sqrt{2\pi^{-1}} \cos jx\}_{j\geq 0}$. Note that $\psi_j(x),\ \ j \geq 0$, is the eigenfunction of the operator $A^{\frac{1}{2}}$ belonging to the eigenvalue $\sqrt{2m - j^4},\ \ j \geq 0$. Thus we make the assumption

$$\begin{aligned} u(t, x) &= \sum_{j\in\mathcal{J}} \frac{1}{\sqrt{\lambda_j}} q_j(t)\psi_j(x) + \sum_{j\in\mathbb{Z}_1} \frac{1}{\sqrt{\lambda_j}} p_j(t)\psi_j(x), \\ v(t, x) &= \sum_{j\in\mathcal{J}} \sqrt{\lambda_j}\widetilde{q}_j(t)\psi_j(x) + \sum_{j\in\mathbb{Z}_1} \sqrt{\lambda_j}\widetilde{p}_j(t)\psi_j(x), \end{aligned} \tag{4.4}$$

where

$$\lambda_j = \begin{cases} \sqrt{2m - j^4}, & j \in \mathbb{Z}_1, \\ \sqrt{j^4 - 2m}, & j \in \mathcal{J}. \end{cases}$$

Then the Hamiltonian defined in (4.3) is changed into

$$H = \frac{1}{2}\sum_{j\in\mathcal{J}} \lambda_j (q_j^2 + \widetilde{q}_j^2) + \frac{1}{2}\sum_{j\in\mathbb{Z}_1} \lambda_j (\widetilde{p}_j^2 - p_j^2) + R(\omega t, q, p) \tag{4.5}$$

with symplectic form $dq \wedge d\widetilde{q} + dp \wedge d\widetilde{p}$ on the space $\ell_{a,p} \times \ell_{a,p} \times \widetilde{\ell}_{a,p} \times \widetilde{\ell}_{a,p}$, and

$$R = \int_0^{\pi} G(\omega t, x, \sum_{j\in\mathcal{J}} \frac{1}{\sqrt{\lambda_j}} q_j(t)\psi_j(x) + \sum_{j\in\mathbb{Z}_1} \frac{1}{\sqrt{\lambda_j}} p_j(t)\psi_j(x))dx.$$

Moreover, we make the change of variables

$$z_j = \frac{1}{\sqrt{2}}(q_j - \mathrm{i}\widetilde{q}_j), \quad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j + \mathrm{i}\widetilde{q}_j), \ j \in \mathcal{J},$$
$$\rho_j = \frac{1}{\sqrt{2}}(p_j - \widetilde{p}_j), \quad \overline{\rho}_j = \frac{1}{\sqrt{2}}(p_j + \widetilde{p}_j), \ j \in \mathbb{Z}_1.$$

Note that $\overline{\rho}$ is not the complex conjugate of $\rho$. Then (4.5) is changed into

$$H = \langle \omega, I\rangle + \sum_{j\in\mathcal{J}} \lambda_j |z_j|^2 - \sum_{j\in\mathbb{Z}_1} \lambda_j \rho_j \overline{\rho}_j + P(\theta, z, \bar{z}, \rho, \overline{\rho}) \tag{4.6}$$

where $\theta = \omega t$, the added variable $I \in \mathbb{C}^d$ is canonically conjugate to $\theta \in \mathbb{T}_c^d$ with symplectic form $d\theta \wedge dI + \mathrm{i}dz \wedge d\bar{z} + d\rho \wedge d\overline{\rho}$ on the space $\mathbb{C}^d \times \mathbb{C}^d \times \ell_{a,p} \times \ell_{a,p} \times \widetilde{\ell}_{a,p} \times \widetilde{\ell}_{a,p}$. Moreover,

$$P(\theta, z, \bar{z}, \rho, \overline{\rho}) = R\left(\theta, \frac{z + \bar{z}}{\sqrt{2}}, \frac{\rho + \overline{\rho}}{\sqrt{2}}\right).$$

The motion equation of the Hamiltonian function defined by (4.6) is

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{z}_j = \mathrm{i}\{\lambda_j z_j + \partial_{\bar{z}_j} P(\theta, z, \bar{z}, \rho, \overline{\rho})\}, \ j \in \mathcal{J}, \\ \dot{\bar{z}}_j = -\mathrm{i}\{\lambda_j \bar{z}_j + \partial_{z_j} P(\theta, z, \bar{z}, \rho, \overline{\rho})\}, \ j \in \mathcal{J}, \\ \dot{\rho}_j = -\lambda_j \rho_j + \partial_{\overline{\rho}_j} P(\theta, z, \bar{z}, \rho, \overline{\rho}), \quad j \in \mathbb{Z}_1, \\ \dot{\overline{\rho}}_j = -\{-\lambda_j \overline{\rho}_j + \partial_{\rho_j} P(\theta, z, \bar{z}, \rho, \overline{\rho})\}, \quad j \in \mathbb{Z}_1. \end{cases} \tag{4.7}$$

Denote

$$\Omega = \mathrm{diag}(\Omega_j = \lambda_j, \quad j \in \mathcal{J}), \quad \Lambda = \mathrm{diag}(\Lambda_j = \lambda_j, \quad j \in \mathbb{Z}_1).$$

Consider the linear operator of (4.7), which we denote as $\mathcal{A}$, i.e., $\mathcal{A} = \mathrm{diag}(\mathrm{i}\Omega, \ -\mathrm{i}\Omega, \ -\Lambda, \ \Lambda)$. The spectrum of $\mathcal{A}$ has the decomposition

$$Spec(\mathcal{A}) = \sigma_s \cup \sigma_c \cup \sigma_u,$$

where

$$\sigma_s = \left\{-\Lambda_j,\ j \in \mathbb{Z}_1\right\}, \quad \sigma_c = \left\{\pm \mathrm{i}\Omega_j,\ j \in \mathcal{J}\right\}, \quad \sigma_u = \left\{\Lambda_j,\ j \in \mathbb{Z}_1\right\}.$$

We call the spectrum belong to $\sigma_s \cup \sigma_u$ and $\sigma_c$ as the hyperbolic spectrum and the center spectrum, respectively. Obviously, the hyperbolic spectrum is finite and the following Lemma 5.1 implies that the hyperbolic spectrum is well separated from the center spectrum.

Set $r = 1,\ 0 < s < 1$, and

$$\mathcal{O} = \left\{\xi \in [1,\ 2] : \left|\langle k, \omega\rangle + \langle l, \Omega(\xi)\rangle\right| \geq \frac{\gamma}{\langle k\rangle^{\tau}},\ 0 < |l| \leq 2,\ \forall k \in \mathbb{Z}^d\right\}.$$

By **(H)** we know that the function $P$ in (4.6) satisfies

$$\|X_P\|_{r,s,r,\mathcal{O}} \leq \varepsilon^{\frac{1}{2}}.$$

We set $r_0 = r = 1,\ s_0 = s < 1,\ \mathcal{O}_0 = \mathcal{O}$ and $\varepsilon_0 := \varepsilon^{\frac{1}{2}} \leq \varepsilon_*$ with $\varepsilon_*$ being the one in Theorem 2.1. Then by the inequality above we know that the Hamiltonian (4.6) satisfies all the hypotheses in Theorem 2.1. Thus by Theorem 2.1 we know that the sympletic change of variables $\Phi$ defined in $D(s_*, r_*) \times \mathcal{O}_\gamma$ casts Hamiltonian system $H$ defined (4.6) into $H_*$. The motion equation of $H_*$ is

$$\begin{cases} \dot{\theta} = \omega, \\ \dot{I} = -\partial_\theta H_*, \\ \dot{z} = \mathrm{i}(\Omega + B_*)z + \mathrm{i}\partial_{\bar{z}} P_*, \\ \dot{\bar{z}} = -\mathrm{i}(\Omega + B_*)\bar{z} - \mathrm{i}\partial_z P_*, \\ \dot{\rho} = (-\Lambda + W_*)\rho + \partial_{\overline{\rho}} P_*, \\ \dot{\overline{\rho}} = -(-\Lambda + W_*)\overline{\rho} - \partial_\rho P_*. \end{cases} \tag{4.8}$$

Equation (4.8) possesses invariant tori

$$\theta = \omega t,\ I_* = I_*(\theta, \xi), \quad z = \bar{z} = 0, \quad \rho = \overline{\rho} = 0.$$

Let $(\theta(t), I(\theta, \xi),\ z(t), \bar{z}(t), \rho(t), \overline{\rho}(t)) = \Phi(\theta_*(0) + \omega t, I_*(\theta, \xi), 0, 0, 0, 0)$. Then (omit the added variable $I$)

$$\begin{aligned}&(\theta(t), z(t), \bar{z}(t), \rho(t), \overline{\rho}(t)) \\ &\quad = (\theta_*(0) + \omega t, X(\theta_*(0) + \omega t), \overline{X}(\theta_*(0) + \omega t), Y(\theta_*(0) + \omega t), \overline{Y}(\theta_*(0) + \omega t))\end{aligned}$$

is a solution of (4.2), where

$$\begin{aligned} X(\theta_*(0) + \omega t) &= (X_j(\theta_*(0) + \omega t) \in \mathbb{C},\ j \in \mathcal{J}) \in \ell_{a,p}, \\ \overline{X}(\theta_*(0) + \omega t) &= (\overline{X}_j(\theta_*(0) + \omega t) \in \mathbb{C},\ j \in \mathcal{J}) \in \ell_{a,p}, \end{aligned}$$

$\overline{X}(\theta_*(0) + \omega t)$ is the complex conjugate of $X(\theta_*(0) + \omega t)$ and $\|X\|_{a,p} \leq 4\varepsilon^{\frac{1}{4}}$. Moreover,

$$\begin{aligned} Y(\theta_*(0) + \omega t) &= (Y_j(\theta_*(0) + \omega t) \in \mathbb{R},\ j \in \mathbb{Z}_1) \in \widetilde{\ell}_{a,p}, \\ \overline{Y}(\theta_*(0) + \omega t) &= (\widetilde{Y}_j(\theta_*(0) + \omega t) \in \mathbb{R},\ j \in \mathbb{Z}_1) \in \widetilde{\ell}_{a,p}, \end{aligned}$$

$\overline{Y}(\theta_*(0)+\omega t)$ is conjugate (not the complex conjugate) to $Y(\theta_*(0)+\omega t)$ and $\|Y\|_{a,p} \leq 4\varepsilon^{\frac{1}{4}}$, $\|\overline{Y}\|_{a,p} \leq 4\varepsilon^{\frac{1}{4}}$. Then

$$u(x,t) = \sum_{j\in\mathbb{Z}_1} \frac{1}{\sqrt{2\lambda_j}}(Y_j + \overline{Y}_j)(\theta_*(0)+\omega t)\psi_j(x) + \sum_{j\in\mathcal{J}} \frac{1}{\sqrt{2\lambda_j}}(X_j + \overline{X}_j)(\theta_*(0)+\omega t)\psi_j(x).$$

Thus the solution of (1.4) we obtain is $y = \sqrt{m} + \varepsilon^{\frac{1}{2}}u(x,t)$.

## 5 Appendix

**Lemma 5.1** *Assume* $2m > 1$, $(2m)^{\frac{1}{4}} - [(2m)^{\frac{1}{4}}] \in [\frac{1}{100}, \frac{1}{2}]$, *then the following conclusions hold:*

(i) $|\pm\Omega_j| \geq \frac{j^{3/2}}{2}$, $|\Omega_j - \Omega_i| \geq |j^2 - i^2|$, $i, j \in \mathcal{J}$, $i \neq j$. (ii) $\frac{(2m)^{\frac{1}{4}}}{10} < |\Lambda_j| \leq \sqrt{2m}$, $|\Lambda_j - \Lambda_i| > \frac{|j^2 - i^2|}{2m}$, $i, j \in \mathbb{Z}_1$, $i \neq j$.

***Proof*** Consider $\Omega_j = \lambda_j = \sqrt{j^4 - 2m}$, $j \in \mathcal{J}$. It follows that

$$|\Omega_j|^2 = j^4 - 2m = (j^2 + \sqrt{2m})(j + (2m)^{\frac{1}{4}})(j - (2m)^{\frac{1}{4}}) > j^3 \left\{ j - \left( \left[ (2m)^{\frac{1}{4}} \right] + \frac{1}{2} \right) \right\} > \frac{1}{2} j^3 > \frac{j^3}{4}$$

as $j \geq 1 + [(2m)^{\frac{1}{4}}]$. Thus, we have $|\Omega_j| > \frac{j^{3/2}}{2}$. Moreover,

$$|\Omega_j - \Omega_i| = \frac{(i^2+j^2)(i^2-j^2)}{\sqrt{j^4-2m}\sqrt{i^4-2m}} > |j^2 - i^2|.$$

This proves the conclusion (i).

For $\Lambda_j = \lambda_j = \sqrt{2m - j^4}$, $j \in \mathbb{Z}_1$, in view of $j \leq [(2m)^{\frac{1}{4}}]$ and $(2m)^{\frac{1}{4}} - [(2m)^{\frac{1}{4}}] \in [\frac{1}{100}, \frac{1}{2}]$, one has

$$\begin{aligned} |\Lambda_j|^2 = 2m - j^4 &= \left(\sqrt{2m} + j^2\right)\left((2m)^{\frac{1}{4}} + j\right)\left((2m)^{\frac{1}{4}} - j\right) \\ &\geq \left(\sqrt{2m} + j^2\right)\left((2m)^{\frac{1}{4}} + j\right)\left(\left[(2m)^{\frac{1}{4}}\right] + \frac{1}{100} - j\right) \\ &\geq \frac{\sqrt{2m}}{100}. \end{aligned}$$

Thus, $|\Lambda_j| \geq \frac{(2m)^{\frac{1}{4}}}{10}$, and, obviously, $|\Lambda_j| \leq \sqrt{2m}$. Moreover,

$$|\Lambda_j - \Lambda_i| = \frac{(i^2+j^2)(i^2-j^2)}{\sqrt{2m-j^4}\sqrt{2m-i^4}} > \frac{|j^2-i^2|}{2m}.$$

This proves the conclusion (ii). □

**Lemma 5.2** *Assume the frequency vector* $\omega = \xi\overline{\omega}$ *with* $\overline{\omega}$ *being the one defined in* (1.2). *For* $B(\xi) = (B_j(\xi), \quad j \in \mathcal{J})$ *with* $\|B(\xi)\|_{\mathcal{O}} \le c\varepsilon$, *denote* $\widetilde{\Omega}(\xi) = \Omega + B(\xi)$, *where* $\Omega = (\Omega_j, \ j \in \mathcal{J})$ *is the one defined in* (2.2). *Define the set*

$$\mathcal{O}_* = \left\{\xi \in \mathcal{O} : \left|\langle k, \omega\rangle + \langle l, \widetilde{\Omega}(\xi)\rangle\right| \ge \frac{\gamma}{\langle k\rangle^\tau},\ 0 < |l| \le 2,\ \forall k \in \mathbb{Z}^d\right\}.$$

*Then for* $0 < \gamma \ll 1$ *and* $\tau > d+2$, *we have* $\text{meas}\mathcal{O}_* \ge 1 - c\gamma$.

***Proof*** By Lemma 5.1 we know that that

$$\left|\langle k, \omega\rangle + \langle l, \widetilde{\Omega}(\xi)\rangle\right| = \left|\langle l, \widetilde{\Omega}(\xi)\rangle\right| > \left|\langle l, \Omega\rangle\right| - 2\|B\|_{\mathcal{O}} > 1,\ k = 0.$$

In the following we assume $k \ne 0$. Denote

$$\mathcal{O}_{**} = \left\{\xi \in \mathcal{O} : \left|\langle k, \overline{\omega}\rangle + \frac{\langle l, \widetilde{\Omega}(\xi)\rangle}{\xi}\right| \ge \frac{\gamma}{\langle k\rangle^\tau},\ 0 < |l| \le 2,\ \forall k \in \mathbb{Z}^d\right\},$$

$$\mathcal{R}(\gamma) = \bigcup_{k\in\mathbb{Z}^d, 0<|l|\le 2} \mathcal{R}_{k,l}(\gamma)$$

with

$$\mathcal{R}_{k,l}(\gamma) = \left\{\xi \in \mathcal{O} : \left|g_{k,l}\right| < \frac{\gamma}{\langle k\rangle^\tau}\right\}$$

and

$$g_{k,l} := \langle k, \overline{\omega}\rangle + \frac{\langle l, \widetilde{\Omega}(\xi)\rangle}{\xi}.$$

Obviously, $\mathcal{O}_{**} \subset \mathcal{O}_*$ and $\mathcal{O}_{**} = \mathcal{O} \setminus \mathcal{R}(\gamma)$.
*Case one*: $|\langle l, \Omega\rangle| \ge 5|k||\overline{\omega}|$. Then

$$\begin{aligned}
\left|\langle k, \overline{\omega}\rangle + \frac{\langle l, \widetilde{\Omega}(\xi)\rangle}{\xi}\right| &\ge \left|\frac{\langle l, \widetilde{\Omega}(\xi)\rangle}{\xi}\right| - |k||\overline{\omega}| \\
&\ge 2^{-1}|\langle l, \widetilde{\Omega}(\xi)\rangle| - |k||\overline{\omega}| \\
&\ge 2^{-1}|\langle l, \Omega\rangle| - |\langle l, B(\xi)\rangle| - |k||\overline{\omega}| \\
&\ge 2^{-1}|\langle l, \Omega\rangle| - 2|k||\overline{\omega}| \\
&\ge (10)^{-1}|k||\overline{\omega}|,
\end{aligned}$$

that is $\mathcal{R}_{k,l}(\gamma) = \emptyset$.
*Case two*: $|\langle l, \Omega\rangle| < 5|k||\overline{\omega}|$. For $\xi \in \mathcal{O}$, we have

$$\begin{aligned}
\left|\frac{d}{d\xi} g_{k,l}\right| &= \left|\frac{\langle l, \widetilde{\Omega}\rangle}{\xi^2} - \frac{\partial_\xi \langle l, \widetilde{\Omega}\rangle}{\xi}\right| \\
&= \left|\frac{\langle l, \Omega\rangle + \langle l, B\rangle}{\xi^2} - \frac{\partial_\lambda \langle l, B\rangle}{\xi}\right| \\
&\ge \frac{|\langle l, \Omega\rangle|}{4} - 2\|\langle l, B\rangle\|_{\mathcal{O}} \ge \frac{|\langle l, \Omega\rangle|}{8}.
\end{aligned}$$

Therefore,

$$\text{meas}\mathcal{R}_{k,l}(\gamma) \le c\frac{\gamma}{\langle k\rangle^\tau}.$$

Then,

$$\begin{aligned}
&\operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^d} \bigcup_{|\langle l,\Omega\rangle|<5|k||\overline{\omega}|,|l|\leq 2} \mathcal{R}_{k,l}(\gamma)\\
&\leq \sum_{0\neq k\in\mathbb{Z}^d} \sum_{|\langle l,\Omega\rangle|<5|k||\overline{\omega}|,|l|\leq 2} c\frac{\gamma}{|k|^\tau}\\
&\leq 125|\overline{\omega}|^2\gamma \sum_{0\neq k\in\mathbb{Z}^d} \frac{1}{|k|^{\tau-2}} \leq C\gamma, \quad (\tau > d+2),
\end{aligned}$$

where $C$ is a constant depending on $\overline{\omega}$ and $\tau$. This implies that meas$\mathcal{R} \leq C\gamma$, which means meas$\mathcal{O}_{**} \geq 1 - C\gamma$. Thus,

$$\operatorname{meas}\mathcal{O}_* \geq 1 - C\gamma.$$

□

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