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Ground State Solutions of Discrete Asymptotically Linear Schrödinger Equations with Bounded and Non-periodic Potentials

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Abstract
We study the existence of ground state solutions for a class of discrete nonlinear Schrödinger equations with a sign-changing potential V that converges at infinity and a nonlinear term being asymptotically linear at infinity. The resulting problem engages two major difficulties: one is that the associated functional is strongly indefinite and the other is that, due to the convergency of V at infinity, the classical methods such as periodic translation technique and compact inclusion method cannot be employed directly to deal with the lack of compactness of the Cerami sequence. New techniques are developed in this work to overcome these two major difficulties. This enables us to establish the existence of a ground state solution and derive a necessary and sufficient condition for a special case. To the best of our knowledge, this is the first attempt in the literature on the existence of a ground state solution for the strongly indefinite problem under no periodicity condition on the bounded potential and the nonlinear term being asymptotically linear at infinity. Moreover, our conditions can also be used to significantly improve the well-known results of the corresponding continuous nonlinear Schrödinger equation.

Keywords Discrete nonlinear Schrödinger equations · Gap solitons · Ground state solutions · Saturable nonlinearity · Linking theorem · Variational methods

Mathematics Subject Classification 39A12 · 35Q55

1 Introduction

Discrete nonlinear Schrödinger (DNLS) equations are very important nonlinear lattice models in the nonlinear science, ranging from condensed matter physics to biology [5,7–9,11,17]. For DNLS equations, one central problem is the existence of gap solitons [2,10,12,19,24,26]. Gap solitons in the DNLS equations are solitary standing waves with temporal frequencies in

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gaps of continuous spectrum which decay to zero at infinity. The main tools in establishing the existence of gap solitons include the principle of anticontinuity [2,19], centre manifold reduction [10] and variational methods [24]. Gap solitons observed in the optical pulse propagation in saturable nonlinear media [7,32,35] can be modelled by the DNLS equations with a sign-changing potential in the linear term. The DNLS equations also lead to the discrete nonlinear Laplacian equations with a potential containing a negative part [3,7,24,32]. The DNLS equations with sign-changing potentials are not well studied.

In this paper, we consider the following DNLS equation

$$-\Delta u(m)+V(m)u(m)=f(u(m)),\quad m\in\mathbb{Z} \tag{1.1}$$

and establish the existence of a nontrivial solution of (1.1) satisfying the boundary condition

$$\lim_{|m|\to\infty} u(m)=0. \tag{1.2}$$

In (1.1), the operator Δ is the discrete Laplacian defined as

$$\Delta u(m):=u(m+1)-2u(m)+u(m-1),\quad m\in\mathbb{Z}.$$

The potential V satisfies the following assumptions:

- (V_1) $0<\lim_{|m|\to\infty} V(m)=V_\infty<\infty$.
- (V_2) $V(m)\le V_\infty - C_0e^{-r_0|m|}$ for $m\in\mathbb{Z}$, where $0<C_0$ and $0<r_0<\cosh^{-1}(V_\infty/2+1)$, where $\cosh^{-1}(x)=\ln(x+\sqrt{x^2-1})$, $x\in[1,\infty)$ is the inverse function of the hyperbolic cosine $\cosh(x):=(e^x+e^{-x})/2$ for $x\in[0,\infty)$.
- (V_3) $\inf\sigma(L)<0$, where $\sigma(L)$ is the spectrum of $L:=-\Delta+V$ in l^2. Here for $1\le p$, l^p is defined by

$$l^p\equiv\left\{u=\{u(m)\}_{m\in\mathbb{Z}}: u(m)\in\mathbb{R},\ \|u\|_{l^p}=\left(\sum_{m\in\mathbb{Z}}|u(m)|^p\right)^{\frac{1}{p}}<\infty\right\},$$

  which exhibits the following property

$$l^q\subset l^p,\ \|u\|_{l^p}\le\|u\|_{l^q},\ 1\le q\le p\le\infty.$$

The nonlinear term $f\in C(\mathbb{R},\mathbb{R})$ is assumed to satisfy the following conditions:

- (f_1) f is locally Lipschitz, i.e., for every $x\in\mathbb{R}$ there exist a neighborhood U_x of x and a constant L_x such that, for all $u,\ v\in U_x$, one has

$$|f(u)-f(v)|\le L_x|u-v|.$$

- (f_2) $\lim_{u\to0}(f(u)/u)=0$ and $0<\lim_{|u|\to\infty}(f(u)/u)=a<\infty$.
- (f_3) The function $u\mapsto f(u)/|u|$ is strictly increasing in $u\in\mathbb{R}\backslash\{0\}$.
- (f_4) If $F(u):=\int_0^u f(s)ds$ and $\widetilde{F}(u):=\frac{1}{2}f(u)u-F(u)$, then

$$\lim_{|u|\to\infty}\widetilde{F}(u)=\infty.$$

It is seen from (V_1) that V is bounded and $\sigma_{ess}(L)=[V_\infty, V_\infty+4]$ [34], where $\sigma_{ess}(L)$ is the essential spectrum of L. Hypothesis (V_3) implies that

$$\sigma_-:=\sup\{\sigma(L)\cap(-\infty,0)\}<0<\sigma_+:=\inf\{\sigma(L)\cap(0,\infty)\}. \tag{1.3}$$

Thus the potential V is also sign-changing, non-periodic and approaches a limit V_∞ at infinity. Moreover, the assumptions on f show that f is asymptotically linear at infinity.

Typical functions satisfying hypotheses $(f_1) - (f_4)$ include $f(u) = au^3/(1+u^2)$ and $f(u) = au(1-e^{-u^2})$, for $u \in \mathbb{R}$ and $a > 0$. These functions are also widely used in the literature. For instance, $f(u) = au^3/(1+u^2)$ was used in [32,35] to study the optical pulse propagation in 1D equidistant nonlinear waveguide arrays.

We point out that mainly using the variational method [23,37,38], the existence of nontrivial solutions of (1.1) has been studied under different assumptions on the potential V and the nonlinearity f. For example, Pankov [24] obtained the existence of a nontrivial solution of the nonautonomous problem with $f(u(m)) = f(m, u(m))$ in (1.1) and 0 belonging to a spectral gap of $-\Delta + V$, and both V and f are periodic in m with f satisfying the Ambrosseti–Rabinowitz (AR) condition. Later Zhou and Yu [41] improved the classical AR superlinear condition to a general superlinear one. The existence of a nontrivial solution of (1.1) with a constant potential V and an asymptotically linear term f, was given by Pankov and Rothos [25] using the Nehari manifold approach and the mountain pass argument. Further, by using the mountain pass lemma of [29] in combination with periodic approximations, Zhou and Yu [40] studied the existence of nontrivial solutions of (1.1) under the assumption that V and f are both periodic, $u \mapsto f(m, u)/|u|$ is strictly increasing for $u \in \mathbb{R}\backslash\{0\}$, and f is asymptotically linear at infinity. Recently, Chen et al. [3] considered the nonautonomous problem of (1.1) with the potential V being periodic and f being asymptotically linear at infinity. When either 0 is a spectral endpoint of $-\Delta + V$, or it is in a finite spectral gap of $-\Delta + V$, the authors obtained the existence of nontrivial solitons by using a generalized weak linking theorem introduced by Schechter and Zou [30]. In the above mentioned work, the periodicity assumptions on V and f play an essential role since the periodicity ensures that (1.1) is invariant under periodic translation. This property is used to overcome the lack of compactness of a Palais–Smale or Cerami sequence, due to the fact that (1.1) is defined in $\mathbb{Z}$.

Existence of nontrivial solutions of (1.1) with an unbounded potential V (i.e., $\lim_{|m|\to\infty} V(m) = \infty$), has also been studied in the literature. In contrast to the periodic case, (1.1) with an unbounded potential V is no longer translating-invariant. The unbounded potential V ensures a compact inclusion from a weighted subspace of l^2 into l^p $(p \geq 2)$, which allows us to be able to handle the lack of compactness of a Palais–Smale or Cerami sequence. Zhang and Pankov [36] investigated the existence of nontrivial solutions of the nonautonomous problem with the unbounded potential V and $f(m, u) = \gamma_m |u|^{p-2}u$ in (1.1). The method used the minimization method on the Nehari manifold and the compact embedding technique. By using the fountain theorem of Zou [42] and the compact inclusion, Zhou and Ma [39] obtained infinitely many high-energy solutions for the nonautonomous problem with an unbounded potential V and a suplinear nonlinearity f at infinity. Chen and Schechter [4] studied the unbounded potential problem of (1.1) with superlinear nonlinearity f at infinity. By using the weak linking theorem of Schechter and Zou [30], they obtained the existence of ground state solutions. By using the critical point theory, Pankov and Zhang [27] proved the existence and multiplicity results for nontrivial solutions for the nonautonomous problem with an unbounded potential V and a saturable nonlinearity f at infinity. Recently, Lin and Zhou [16] obtained infinitely many high-energy solutions for the nonautonomous problem provided that V is unbounded and f is of mixed nonlinearity at infinity, by using the fountain theorem of Zou [42]. Other related results for (1.1) can be found in [6,12,14,15,18,20,26,33].

Notice that the existence of a nontrivial weak solution for the corresponding continuous version of (1.1) with indefinite and non-periodic linear part V and asymptotically linear f was obtained by Maia et al. [21,22]. In [21,22], the authors proved the existence of a nontrivial weak solution in $H^1(\mathbb{R}^N)$, $N \geq 3$ by employing spectral theory arguments, the geometry of the linking theorem, and the interaction between translated solutions of the problem at infinity. In their celebrated works, f is a C^3 function with the following crucial restriction on f [1,21,22]: There exist $C_2 > 0$ and $1 < p_1 \leq p_2$ such that $p_1,\ p_2 < (N+2)/(N-2)$ and $|f^{(k)}(s)| \leq C_2(|s|^{p_1-k}+|s|^{p_2-k})$ for $k \in \{0,\ 1,\ 2,\ 3\}$ and $s \in \mathbb{R}$, which plays a crucial role in their proof. In this paper, we only need f to be locally Lipschitz. Obviously, if $f \in C^1(\mathbb{R}, \mathbb{R})$, then it is locally Lipschitz. Thus, it is possible to use our condition (f_1) to improve those in the existing results obtained in [1,21,22] for the continuous nonlinear Schrödinger equations.

Discrete Schrödinger operators of the form $-\Delta + V$ appear in a wide range of fields, such as the description of random walks, the propagation of waves in crystals, and the theory of nonlinear integrable lattices (see [7,34] and references therein). It can be seen from [1,21,22] that the exponential decay estimate on eigenfunctions corresponding to eigenvalues below the essential spectrum for the continuous Schrödinger operators has been thoroughly established very early. The estimate plays a significant role in establishing the compactness of Cerami sequences. However, an analogous result for $-\Delta + V$ has not been established yet. By using the Combes–Thomas method, Smith [31] proved the existence of exponential decay of eigenfunctions corresponding to eigenvalues below the essential spectrum for $-\Delta + V$, and proposed a conjecture on finding upper bounds of exponential decay rate on these eigenfunctions. Based on Phragmen–Lindelöf principle of pseudodifference equations, Rabinovich and Roch in [28] obtained an upper bound for $-\Delta + V$ depending on a bounded and slowly oscillating potential V. Without the assumption of V being slowly oscillating and by using elementary method, the upper bound of [28] has been reformulated with some separate interest to fit in with our setting in this paper (Lemma 2.1).

In this work, one difficulty in problem (1.1) is that the associated functional J (defined in Sect. 2) is strongly indefinite. To tackle this difficulty, we adapt the classical linking theorem with a Cerami sequence introduced by Li and Wang [13]. It is convenient to decompose the functional space l^2 into a direct sum of two subspaces H^+ and H^-, one of them being finite dimensional. It is possible to prove that the limiting problem

$$-\Delta u(m) + V_\infty u(m) = f(u(m)), \quad m \in \mathbb{Z}, \tag{1.4}$$

admits a ground state solution u_0 in l^2. After projecting u_0 on the subspace H^+, the linking set M is constructed, under which we do not need to use the minimization method on the generalized Nehari manifold. Although it is rather intricate to estimate the interactions of the translations of u_0, we are able to find the linking geometry (Lemma 3.6). This allows us to find a Cerami sequence at level c that is given by the linking minmax structure.

Another difficulty is the lack of compactness of the Cerami sequence. As a result, neither the periodic translation technique nor the compact inclusion method can be adapted. To overcome this difficulty, we give exponential decay bounds on eigenfunctions corresponding to eigenvalues below the essential spectrum of $-\Delta + V$ (Lemma 2.1). Moreover, an upper bound of the exponential decay rate that depends only on V_∞ is also explicitly given. This bound allows a priori estimate on exponential decay for nontrivial solutions of (1.1) and (1.4). These new estimates on exponential decay rate are crucial to our proof. Assuming that $u \mapsto f(u)/|u|$ is strictly increasing for $u \in \mathbb{R}\backslash\{0\}$, we can successfully compare the energy level c of the Cerami sequence with the ground state level c_∞ of the limiting problem (Lemma 3.7). Thus, the concentration-compactness method can be used.

To the best of our knowledge, this is the first attempt to obtain the existence of a ground state solution of (1.1) with a sign-changing and bounded potential V that does not need to be periodic, and a nonlinearity f which is asymptotically linear at infinity. Moreover, we also derive a necessary and sufficient condition on the existence of ground state solutions for a special case.

Our main results are stated as follows.

Theorem 1.1 *Assume that* $(V_1)-(V_3)$ *hold and* $f \in C(\mathbb{R}, \mathbb{R})$ *satisfies* $(f_1)-(f_4)$. *If* $V_\infty < a$, *then* (1.1) *has a ground state solution* u *in* l^2. *Moreover, the solution decays exponentially at infinity, that is, for any* $\tau \in (0, \cosh^{-1}(V_\infty/2+1))$, *there exists a constant* C, *depending only on* τ *and* V_∞, *such that*

$$|u(m)| \le C\|u\|_{l^\infty} e^{-\tau|m|}, \quad m \in \mathbb{Z}. \tag{1.5}$$

We remark that although for (1.1) with periodic potentials, the existence of nontrivial solutions which decay exponentially has been extensively studied in [3,18,24,25,33,40,41], no explicit bounds on the exponential decay rate have been provided. Nevertheless, our estimate (1.5) explicitly gives an exponential decay bound on a nontrivial solution of (1.1). Indeed, (1.5) is a byproduct of Lemma 2.1 (given in Sect. 2). Lemma 2.1 can also provide us more precise information on the upper bound of exponential decay rate of a nontrivial solution of (1.1) with periodic potentials.

Theorem 1.1 only presents a sufficient condition on the existence of a ground state solution of (1.1) in l^2. We mention that if some of these conditions fail, then (1.1) has no nontrivial solution in l^2.

Proposition 1.2 *Assume that* (V_1) *and* (V_3) *hold and* $f \in C(\mathbb{R}, \mathbb{R})$ *satisfies* (f_2) *and* (f_3). *If* $0 \notin \sigma(L)$ *and* $a \le \min\{\sigma_+, -\sigma_-\}$, *then* (1.1) *has no nontrivial solution in* l^2.

Combining Theorem 1.1 and Proposition 1.2, we obtain a necessary and sufficient condition on the existence of ground state solutions of (1.1).

Theorem 1.3 *Under conditions* $(V_1)-(V_3)$ *and* $(f_1)-(f_4)$, *if* $0 \notin \sigma(L)$, $\sigma_+ = V_\infty$ *and*

$$\min\{V_\infty - a,\ a + \sigma_-\} < 0, \tag{1.6}$$

then (1.1) has at least one ground state solution in l^2 *if and only if* $V_\infty < a$.

We remark that $\sigma_+ = V_\infty$ is possible for a class of sign-changing potentials. In fact, if $\sigma_+ < V_\infty$, then $\lambda_m^+ := \sup\{\sigma(L) \cap (0, V_\infty)\}$ exists and $\lambda_m^+ < V_\infty$ provided that V_∞ is not a cluster point of isolated eigenvalues of L. Define a new potential V_1 as $V - \lambda_m^+ - \varepsilon$ for a small enough $\varepsilon > 0$. Then $\sigma_+ = V_\infty$ is satisfied by rewriting L with V_1 in the place of V. We should mention that there is no published result focusing on a necessary and sufficient condition of (1.1) with a sign-changing potential V going to a limit V_∞ at infinity and an asymptotically linear term f at infinity. Notice that, if $V_\infty < a$, then (1.6) is satisfied automatically. Thus it follows from Theorem 1.1 that the ground state solution obtained in Theorem 1.3 also shares the exponential decay estimate (1.5).

We organize the rest of the paper as follows. In Sect. 2 we present some preliminaries including the variational setting associated with (1.1) and some auxiliary lemmas. In Sect. 3, we first prove that every Cerami sequence of the corresponding functional J of (1.1) is bounded, and then we show that J satisfies the linking geometry. We present the proofs of our main results in Sect. 4.

2 Preliminaries

In this section, we build the variational setting associated with (1.1) and present some auxiliary lemmas which are crucial to the proofs of our main results.

2.1 Variational Setting

Let $E := l^2$. The energy functional $J : E \to \mathbb{R}$ associated with (1.1) is given by

$$J(u) = \frac{1}{2}((-\Delta + V)u, u)_E - \sum_{m\in\mathbb{Z}} F(u(m)), \quad u \in E,$$

where $(\cdot,\cdot)_E$ is the inner product in l^2. The corresponding norm in E is denoted by $\|\cdot\|_E$. Then $J \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\langle J'(u), v\rangle = ((-\Delta + V)u, v)_E - \sum_{m\in\mathbb{Z}} f(u(m))v(m), \quad u, v \in E. \tag{2.1}$$

Thus, (1.1) is the corresponding Euler-Lagrange equation for J. To find nontrivial solutions of (1.1), we only need to look for nonzero critical points of J in E.

It is known from conditions (V_1) and (V_3) that the eigenvalue problem

$$-\Delta u(m) + V(m)u(m) = \lambda u(m), \quad m \in \mathbb{Z} \tag{2.2}$$

has a sequence of eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{k_*} < 0$. Denote by φ_i the eigenfunction corresponding to λ_i for $i \in \{1, 2, \ldots, k_*\}$ in E. Setting $E^- := \text{span}\{\varphi_i : i = 1, 2, \ldots, k_*\}$, we know that $\dim E^- < \infty$ since the essential spectrum of $-\Delta+V$ equals $[V_\infty, V_\infty+4]$ [34]. Denote $E^0 := \ker(-\Delta+V)$. If $0 \notin \sigma(-\Delta+V)$, then $E^0 = \{0\}$, if not, then 0 is an eigenvalue of finite multiplicity. Thus, E^0 is finite dimensional. Denote by $\{e_i : i = 1, 2, \ldots, k_{**}\}$ the basis of E^0, and if $E^0 = \{0\}$, then one has $e_i = 0, i = 1, 2, \ldots, k_{**}$ for convenience. Setting $E^+ := (E^- \oplus E^0)^\perp$, we know that $E = E^+ \oplus E^- \oplus E^0$ and $\dim(E^- \oplus E^0) < \infty$. We call E^+ and E^- the positive and negative spectral subspaces of $-\Delta + V$ in E, respectively. Then, we have

$$((-\Delta + V)u, u)_E \ge \sigma_+ \|u\|_E^2, \quad u \in E^+, \tag{2.3}$$

and

$$-((-\Delta + V)u, u)_E \ge -\sigma_- \|u\|_E^2, \quad u \in E^-, \tag{2.4}$$

where σ_+ and σ_- are given by (1.3). For any $u, v \in E = E^+ \oplus E^- \oplus E^0$, $u = u^+ + u^- + u^0$ and $v = v^+ + v^- + v^0$, we define an equivalent inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on E by

$$(u, v) = ((-\Delta + V)u^+, v^+)_E - ((-\Delta + V)u^-, v^-)_E + (u^0, v^0)_E \text{ and } \|u\| = (u, v)^{\frac{1}{2}}, \tag{2.5}$$

respectively. Clearly, the decomposition $E = E^+ \oplus E^- \oplus E^0$ is also orthogonal with respect to both inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_E$. Therefore, J can be written as

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{m\in\mathbb{Z}} F(u(m))$$

for each $u = u^+ + u^- + u^0 \in E$. We also have

$$\begin{aligned}\langle J'(u), v\rangle &= (u^+ - u^-, v) - \sum_{m\in\mathbb{Z}} f(u(m))v(m)\\ &= ((-\Delta + V)u^+, v^+)_E - ((-\Delta + V)u^-, v^-)_E\\ &\quad - \sum_{m\in\mathbb{Z}} f(u(m))v(m)\end{aligned} \tag{2.6}$$

for $u = u^+ + u^- + u^0 \in E$ and $v = v^+ + v^- + v^0 \in E$.

Definition 2.1 Assume that the set $\Lambda = \{u : u \in E\backslash\{0\},\ J'(u) = 0\}$ of all nontrivial critical points of J is nonempty. A solution $u_0 \in E$ of (1.1) is called a ground state solution if its energy level $J(u_0) > 0$ satisfies

$$J(u_0) = \inf\{J(u) : u \in \Lambda\}.$$

For the limiting problem (1.4) with the boundary condition (1.2), the energy functional associated with (1.4) is given by

$$J_\infty(u) = \frac{1}{2}((-\Delta + V_\infty)u, u)_E - \sum_{m\in\mathbb{Z}} F(u(m)), \quad u \in E.$$

We have

$$\langle J'_\infty(u), v\rangle = ((-\Delta + V_\infty)u, v)_E - \sum_{m\in\mathbb{Z}} f(u(m))v(m), \quad u, v \in E.$$

Let $u_0 \in E$ be a ground state solution of (1.4), that is, $J_\infty(u_0) = c_\infty > 0$ with

$$c_\infty := \inf\{J_\infty(u) : u \in E\backslash\{0\},\ J'_\infty(u) = 0\}. \tag{2.7}$$

The existence of u_0 has been proved in [3] as $V_\infty < a$.

2.2 Some Auxiliary Lemmas

To prove the main results, we need some auxiliary lemmas.

Lemma 2.1 *Let $V \in l^\infty$, and $u \in \ker(-\Delta + V - \lambda)$ in E for some $\lambda < \alpha := \liminf_{|m|\to\infty} V(m)$. Assume $\beta \in [\lambda, \alpha)$. Then for any $\mu \in (0, \cosh^{-1}(1 + (\alpha - \beta)/2))$, there exists a constant C, depending only on μ and β, such that*

$$|u(m)| \le C\|u\|_{l^\infty} e^{-\mu|m|}, \quad m \in \mathbb{Z}. \tag{2.8}$$

Proof A simple calculation yields

$$\Delta e^{-\mu|m|} = (e^\mu + e^{-\mu} - 2)e^{-\mu|m|}, \quad m \in \mathbb{Z}\backslash\{0\}.$$

Since $\mu \in (0, \cosh^{-1}(1 + (\alpha - \beta)/2))$, we have

$$e^\mu + e^{-\mu} - 2 < \alpha - \beta.$$

By the definition of α, there exists an integer $N = N(\mu, \beta) > 0$ such that

$$V(m) > \beta + e^\mu + e^{-\mu} - 2, \quad |m| \ge N,$$

and thus, for all $\lambda \le \beta$, we further have

$$V(m) > \lambda,\ \lambda - V(m) + e^\mu + e^{-\mu} - 2 < 0,\ |m| \ge N.$$

Denote $C = e^{\mu N}$. For any $u \in \ker(-\Delta + V - \lambda)\backslash\{0\}$ with $\lambda \le \beta$, we define a sequence $w = \{w(m)\}$ as

$$w(m) = u(m) - C\|u\|_{l^\infty} e^{-\mu|m|}, \quad m \in \mathbb{Z}.$$

Then $w \in l^2$ and $w^+ \in l^2$, where $w^+ = \{w^+(m)\}$ is defined by $w^+(m) = \max\{w(m), 0\}$ for $m \in \mathbb{Z}$. The definition of C ensures that $w(m) \le 0$ for all $|m| \le N$. Therefore, $w^+(m) \equiv 0$ for $|m| \le N$. Let

$$A = \{m : w^+(m) > 0,\ m \in \mathbb{Z}\} \quad \text{and} \quad B = \{m \in \mathbb{Z} : |m| \le N\}.$$

Clearly, $A \subset D(N) \equiv \mathbb{Z}\backslash B$. Denote the forward difference operator ∇ by $\nabla u(m) = u(m+1) - u(m)$. We claim that $A = \emptyset$. Otherwise,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |\nabla w^+(m)|^2 &= \sum_{m\in D(N)} (\nabla w(m)) \cdot \nabla w^+(m)\\ &= -\sum_{m\in D(N)} (\Delta w(m+1)) \cdot w^+(m+1)\\ &= \sum_{m\in A} (-\Delta w(m)) \cdot w(m)\\ &\leq \sum_{m\in A} (\lambda - V(m) + e^{\mu} + e^{-\mu} - 2) C\|u\|_{l^\infty} e^{-\mu|m|} w(m),\end{aligned}$$

since $\lambda - V(m) \leq 0$ and $u(m) > C\|u\|_{l^\infty} e^{-\mu|m|}$ for $m \in A$. However, it is impossible as $u(m) > 0$ for $m \in A \subset D(N)$ and N is chosen such that

$$\lambda - V(m) + e^{\mu} + e^{-\mu} - 2 < 0, \quad m \in D(N).$$

This proves the claim. It follows from the claim that $w(m) \leq 0$ for all $m \in \mathbb{Z}$, that is, $u(m) \leq C\|u\|_{l^\infty} e^{-\mu|m|}$ for all $m \in \mathbb{Z}$. Replacing u by $-u$ finishes the proof. □

Next we show that every solution of (1.1) in l^2 decays exponentially at infinity.

Proposition 2.2 *Under assumptions of Theorem 1.1, any nontrivial solution $u \in l^2$ of (1.1) decays exponentially at infinity, that is, for any $\tau \in (0, \cosh^{-1}(V_\infty/2 + 1))$, there exists a constant C, depending only on τ and V_∞, such that (1.5) holds.*

Proof Define $\widetilde{V}(m) := V(m) - U(m)$, where $U(m) = f(u(m))/u(m)$ if $u(m) \neq 0$ and $U(m) = 0$ if $u(m) = 0$. Since $f(u) = o(u)$ as $u \to 0$ and $\lim_{|m|\to\infty} u(m) = 0$, it follows that $\lim_{|m|\to\infty} U(m) = 0$. Then $\lim_{|m|\to\infty} \widetilde{V}(m) = V_\infty$. By using Lemma 2.1 with $\widetilde{V}(m)$ for $V(m)$, it is easy to obtain the desired estimate (1.5). □

Lemma 2.3 *Under assumption (f_1), for any $C_1 > 0$, there exists a constant $C_2 > 0$ such that*

$$|F(u+v) - F(u) - F(v)| \leq C_2|u||v|$$

for all $u, v \in \mathbb{R}$ with $|u|, |v| \leq C_1$.

Proof It is well known that f is locally Lipschitz if and only if it is Lipschitz on every bounded and closed subset of $\mathbb{R}$. Thus, for any $C_1 > 0$, there is a constant $C_2 > 0$ such that

$$|f(u) - f(v)| \leq C_2|u - v|$$

for all $u, v \in \mathbb{R}$ with $|u|, |v| \leq C_1$. If $u = 0$ or $v = 0$, there is nothing to show. Thus assume $0 < |u|, |v| \leq C_1$. Then we have

$$\begin{aligned}&|F(u+v) - F(u) - F(v)|\\ &= \left|\int_0^1 [f(su+v) - f(su)]u ds\right|\\ &\leq |u| \int_0^1 |f(su+v) - f(su)| ds\\ &\leq C_2|u||v|,\end{aligned}$$

which completes the proof. □

Lemma 2.4 *If $\mu_2 > \mu_1 \geq 0$, there exists $C > 0$ such that, for all $m_1,\ m_2 \in \mathbb{Z}$, one has*

$$\sum_{m\in\mathbb{Z}} e^{-\mu_1|m-m_1|}e^{-\mu_2|m-m_2|} \leq Ce^{-\mu_1|m_1-m_2|}.$$

Proof It follows from

$$\begin{aligned}&\mu_1|m_1-m_2| + (\mu_2-\mu_1)|m-m_2|\\ &\quad\leq \mu_1(|m-m_1|+|m-m_2|) + (\mu_2-\mu_1)|m-m_2|\\ &\quad= \mu_1|m-m_1| + \mu_2|m-m_2|,\end{aligned}$$

that

$$\sum_{m\in\mathbb{Z}} e^{-\mu_1|m-m_1|}e^{-\mu_2|m-m_2|} \leq \sum_{m\in\mathbb{Z}} e^{-\mu_1|m_1-m_2|}e^{-(\mu_2-\mu_1)|m-m_2|} = Ce^{-\mu_1|m_1-m_2|}.$$

Thus the desired result follows. □

3 Linking Geometry with a Bounded Cerami Sequence

3.1 Boundedness of a Cerami Sequence

Given a Banach space $(E, \|\cdot\|)$, we say that a functional $J \in C^1(E,\mathbb{R})$ satisfies the Cerami condition if every sequence $\{u_k\} \subset E$ with $|J(u_k)| < M$, for some constant $M > 0$, and $\|J'(u_k)\|_{E^*}(1+\|u_k\|) \to 0$ has a subsequence $u_{k_n} \to u$ in E.

Lemma 3.1 *Let $\{u_k\} \subset E$ be a sequence such that $J(u_k) \to c > 0$ and $\|J'(u_k)\|_{E^*}(1 + \|u_k\|) \to 0$, as $k \to \infty$. Then, $\{u_k\}$ has a bounded subsequence.*

Proof Assumptions (f_1) and (f_2) imply that, given $\varepsilon > 0$ and $2 \leq p$, there exists C_ε such that

$$|f(u)| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1} \quad \text{and} \quad |F(u)| \leq \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{3.1}$$

for all $u \in \mathbb{R}$.

Let $B(c, r)$ be the open ball in a Hilbert space with radius r and center c. If $\{w_k\}$ is a bounded sequence in E, then it satisfies one of the following cases:

(i) Nonvanishing: there exist constants $r,\ \eta > 0$ and a sequence $\{n_k\} \subset \mathbb{Z}$ such that $\limsup_{k\to\infty} \sum_{m\in B(n_k,r)} |w_k(m)|^2 > \eta$.
(ii) Vanishing: for all $r > 0$, $\limsup_{k\to\infty} \sup_{n\in\mathbb{Z}} \sum_{m\in B(n,r)} |w_k(m)|^2 = 0$.

By way of contradiction, we assume that $\|u_k\| \to \infty$. Setting $v_k = u_k/\|u_k\|$ yields $\|v_k\| = 1$. The sequence $\{v_k\}$ is bounded. We finish the proof with contradictory arguments to show that neither (i) or (ii) is satisfied by $\{v_k\}$ as follows.

Claim 3.2 *Nonvanishing of the sequence $\{v_k\}$ is impossible.*

Proof First assume that (i) holds for the sequence $\{v_k\}$. By equivalence of the norms, there exist constants $c_1,\ c_2 > 0$ such that

$$\|u\| \leq c_1\|u\|_E \leq c_2\|u\|, \quad \text{for} \quad u \in E. \tag{3.2}$$

Denote

$$l_0^s = \{u \in l^s : \{n \in \mathbb{Z} : |u(n)| > 0\} \text{ is a finite set},\ 2 \leq s\}.$$

Then l_0^2 is dense in l^2. Moreover, for every $\phi \in l_0^2$, there exists $m_0 \in \mathbb{N}$ such that $\phi(m) = 0$ for all $|m| > m_0$. Let $\{n_k\} \subset \mathbb{Z}$ be the sequence given by (i). Denote $\phi_k = \{\phi_k(m)\}$ by $\phi_k(m) = \phi(m - n_k)$ for any $\phi = \{\phi(m)\} \in l_0^2$. Noting that the sequence $\{u_k\}$ is a Cerami sequence, we have from (3.2) that

$$\begin{aligned}|\langle J'(u_k), \phi_k\rangle| &\le \|J'(u_k)\|_{E^*}\|\phi_k\| \le c_1\|J'(u_k)\|_{E^*}\|\phi_k\|_E \\ &= c_1\|J'(u_k)\|_{E^*}\|\phi\|_E \to 0.\end{aligned}$$

Since $\|u_k\| \to \infty$, the cardinality of the set $A_k = \{m \in \mathbb{Z} : |u(m)| > 0\}$ is positive. Let $o_\nu(1)$ be a quantity that approaches zero as ν goes to infinity. Denote $f_\infty(u) = f(u) - au$. Then,

$$\begin{aligned}o_k(1) &= \frac{1}{\|u_k\|}\langle J'(u_k), \phi_k\rangle = (v_k^+ - v_k^-, \phi_k) - \sum_{m\in\mathbb{Z}} \frac{f(u_k(m))}{\|u_k\|}\phi_k(m) \\ &= (v_k^+ - v_k^-, \phi_k) - \sum_{m\in\mathbb{Z}} av_k(m)\phi_k(m) - \sum_{m\in\mathbb{Z}} \frac{f_\infty(u_k(m))}{\|u_k\|}\phi_k(m) \\ &= (v_k^+ - v_k^-, \phi_k) - \sum_{m\in\mathbb{Z}} av_k(m)\phi_k(m) - \sum_{m\in A_k} \frac{f_\infty(u_k(m))}{u_k(m)}v_k(m)\phi_k(m). \quad (3.3)\end{aligned}$$

Define $\widetilde{v}_k(m) = v_k(m + n_k)$ and $\widetilde{u}_k(m) = u_k(m + n_k)$. Note that $\{\widetilde{v}_k\}$ is bounded in E. In fact, it follows from (3.2) that

$$\|\widetilde{v}_k\| \le c_1\|\widetilde{v}_k\|_E = c_1\|v_k\|_E \le c_2\|v_k\| = c_2.$$

Thus, passing to a subsequence if necessary, we have

$$\widetilde{v}_k \rightharpoonup \widetilde{v} \text{ in } E, \quad \text{and} \quad \widetilde{v}_k \to \widetilde{v} \text{ in } l_0^2. \tag{3.4}$$

Let $\Omega = \{m \in \mathbb{Z} : |\phi(m)| > 0\}$. By (f_2) and (f_3), $|f(\cdot)|/|\cdot|$ is a bounded function in $\mathbb{R}\backslash\{0\}$ with $|f(\cdot)|/|\cdot| \le a$. From (3.4), there exists $g \in l^1$ such that $|\widetilde{v}_k(m)| \le g(m)$ in Ω. Thus, we obtain

$$\left|\frac{f_\infty(\widetilde{u}_k(m))}{\widetilde{u}_k(m)}\widetilde{v}_k(m)\phi(m)\right| \le 2ag(m)\phi(m). \tag{3.5}$$

We have that $\widetilde{v} \ne 0$. In fact, it follows from (i) and (3.4) that

$$\sum_{m\in B(0,r)} |\widetilde{v}(m)|^2 = \limsup_{k\to\infty} \sum_{m\in B(0,r)} |\widetilde{v}_k(m)|^2 = \limsup_{k\to\infty} \sum_{m\in B(n_k,r)} |v_k(m)|^2 > \eta > 0.$$

By (f_2), we have that $f_\infty(u)/u \to 0$ if $|u| \to \infty$. From (3.5) and the Lebesgue Dominated Convergence Theorem, it holds that

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} \frac{f_\infty(u_k(m))}{u_k(m)}v_k(m)\phi_k(m) \\ &\quad= \sum_{m\in\mathbb{Z}} \frac{f_\infty(\widetilde{u}_k(m))}{\widetilde{u}_k(m)}\widetilde{v}_k(m)\phi(m) \\ &\quad= \sum_{m\in\Omega} \frac{f_\infty(\widetilde{u}_k(m))}{\widetilde{u}_k(m)}\widetilde{v}_k(m)\phi(m) \\ &\quad= \sum_{m\in\Omega} \frac{f_\infty(\widetilde{v}_k(m)\|\widetilde{u}_k\|)}{\widetilde{v}_k(m)\|\widetilde{u}_k\|}\widetilde{v}_k(m)\phi(m) \to 0. \quad (3.6)\end{aligned}$$

Since $u_k^0 \in E^0$, it follows from (3.3), (3.4) and (3.6) that

$$\begin{aligned}
o_k(1) &= \frac{1}{\|u_k\|}\langle J'(u_k), \phi_k\rangle \\
&= (v_k^+ - v_k^-, \phi_k) - \sum_{m\in\mathbb{Z}} a v_k(m)\phi_k(m) - \sum_{m\in\mathbb{Z}} \frac{f_\infty(u_k(m))}{u_k(m)} v_k(m)\phi_k(m) \\
&= \sum_{m\in\mathbb{Z}} [\nabla v_k^+(m)\cdot\nabla\phi(m-n_k) + V(m)v_k^+(m)\phi(m-n_k)] \\
&\quad + \sum_{m\in\mathbb{Z}} [\nabla v_k^-(m)\cdot\nabla\phi(m-n_k) + V(m)v_k^-(m)\phi(m-n_k)] \\
&\quad - \sum_{m\in\mathbb{Z}} a v_k(m)\phi_k(m) - o_k(1) \\
&= \sum_{m\in\mathbb{Z}} [\nabla\widetilde{v}_k^+(m)\cdot\nabla\phi(m) + V(m+n_k)\widetilde{v}_k^+(m)\phi(m)] \\
&\quad + \sum_{m\in\mathbb{Z}} [\nabla\widetilde{v}_k^-(m)\cdot\nabla\phi(m) + V(m+n_k)\widetilde{v}_k^-(m)\phi(m)] \\
&\quad + \sum_{m\in\mathbb{Z}} [\nabla\widetilde{v}_k^0(m)\cdot\nabla\phi(m) + V(m+n_k)\widetilde{v}_k^0(m)\phi(m)] \\
&\quad - \sum_{m\in\mathbb{Z}} a\widetilde{v}_k(m)\phi(m).
\end{aligned} \tag{3.7}$$

We distinguish two cases to finish the proof. *Case 1.* $|n_k| \to \infty$. In this case, it follows from (V_1) that $V(m+n_k)$ converges to V_∞ in $\mathbb{Z}$ as $k\to\infty$. From (3.7), we have

$$\begin{aligned}
o_k(1) = &\sum_{m\in\Omega} [\nabla\widetilde{v}_k^+(m)\cdot\nabla\phi(m) + (V_\infty + o_k(1))\widetilde{v}_k^+(m)\phi(m)] \\
&+ \sum_{m\in\Omega} [\nabla\widetilde{v}_k^-(m)\cdot\nabla\phi(m) + (V_\infty + o_k(1))\widetilde{v}_k^-(m)\phi(m)] \\
&+ \sum_{m\in\Omega} [\nabla\widetilde{v}_k^0(m)\cdot\nabla\phi(m) + (V_\infty + o_k(1))\widetilde{v}_k^0(m)\phi(m)] \\
&- \sum_{m\in\Omega} a\widetilde{v}_k(m)\phi(m).
\end{aligned} \tag{3.8}$$

Taking $k\to\infty$ in (3.8) and noticing that (3.4) holds, then for any $\phi \in l_0^2$, we get

$$((-\Delta + V_\infty)\widetilde{v}, \phi)_E = (a\widetilde{v}, \phi)_E,$$

that is, $\widetilde{v} \neq 0$ is a solution of the problem $-\Delta\widetilde{v}(m) + V_\infty\widetilde{v}(m) = a\widetilde{v}(m)$ in $\mathbb{Z}$. This contradicts to the fact that there is no eigenfunction of $-\Delta$ in l^2 [34] since $V_\infty < a$.

Case 2. $\{n_k\}$ is a bounded sequence. From (3.2), we have

$$\|\widetilde{u}_k\| \geq \frac{c_1}{c_2}\|\widetilde{u}_k\|_E = \frac{c_1}{c_2}\|u_k\|_E \geq \frac{1}{c_2}\|u_k\|,$$

which goes to infinity as $k\to\infty$. We see from (3.4) that

$$0 \neq |\widetilde{v}(m_0)| = \lim_{k\to\infty} |\widetilde{v}_k(m_0)| = \lim_{k\to\infty} \frac{|\widetilde{u}_k(m_0)|}{\|\widetilde{u}_k\|}$$

with some $m_0 \in B(0, r)$. As $\|\widetilde{u}_k\| \to \infty$, we obtain $|\widetilde{u}_k(m_0)| \to \infty$. Thus, combining (f_4) with Fatou's Lemma gives

$$\begin{aligned}
&\liminf_{k\to\infty} \sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f(u_k(m))u_k(m) - F(u_k(m))\right] \\
&\geq \sum_{m\in\mathbb{Z}} \liminf_{k\to\infty} \left[\frac{1}{2} f(\widetilde{u}_k(m))\widetilde{u}_k(m) - F(\widetilde{u}_k(m))\right] \\
&\geq \liminf_{k\to\infty} \left[\frac{1}{2} f(\widetilde{u}_k(m_0))\widetilde{u}_k(m_0) - F(\widetilde{u}_k(m_0))\right] \\
&= \infty.
\end{aligned}$$

This is impossible since it contradicts with

$$\sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f(u_k(m))u_k(m) - F(u_k(m))\right] = J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle = c + o_k(1).$$

In summary, (i) is impossible for the sequence $\{v_k\}$. □

Claim 3.3 *Vanishing of the sequence* $\{v_k\}$ *is impossible.*

Proof Now we assume that (ii) is true for the sequence $\{v_k\}$. Since the sequence $\{v_k\}$ is a Cerami sequence, we have $\langle J'(u_k), u_k^+\rangle \to 0$ and $\langle J'(u_k), u_k^-\rangle \to 0$. Thus,

$$\begin{aligned}
o_k(1) &= \left\langle J'(u_k), \frac{u_k^+}{\|u_k\|^2}\right\rangle = \frac{1}{\|u_k\|}\langle J'(u_k), v_k^+\rangle \\
&= \|v_k^+\|^2 - \sum_{m\in\mathbb{Z}} \frac{f(u_k(m))}{u_k(m)} v_k(m)v_k^+(m)
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
o_k(1) &= \left\langle J'(u_k), \frac{u_k^-}{\|u_k\|^2}\right\rangle = \frac{1}{\|u_k\|}\langle J'(u_k), v_k^-\rangle \\
&= -\|v_k^-\|^2 - \sum_{m\in\mathbb{Z}} \frac{f(u_k(m))}{u_k(m)} v_k(m)v_k^-(m).
\end{aligned} \tag{3.10}$$

Subtracting (3.10) from (3.9) gives

$$\begin{aligned}
o_k(1) &= \|v_k^+\|^2 + \|v_k^-\|^2 - \sum_{m\in\mathbb{Z}} \frac{f(u_k(m))}{u_k(m)} v_k(m)(v_k^+(m) - v_k^-(m)) \\
&= \|v_k\|^2 - \|v_k^0\|^2 - \sum_{m\in\mathbb{Z}} \frac{f(u_k(m))}{u_k(m)} v_k(m)(v_k^+(m) - v_k^-(m)) \\
&= 1 - \|v_k^0\|^2 - \sum_{m\in\mathbb{Z}} \frac{f(u_k(m))}{u_k(m)} v_k(m)(v_k^+(m) - v_k^-(m)).
\end{aligned}$$

Since $\{v_k\}$ vanishes, we have $v_k^0 \rightharpoonup 0$ in l^2 as $k \to \infty$. It follows from $\dim E^0 < \infty$ that $\|v_k^0\| \to 0$ as $k \to \infty$. Then

$$\sum_{m\in\mathbb{Z}} \frac{f(u_k(m))}{u_k(m)} v_k(m)(v_k^+(m) - v_k^-(m)) \to 1. \tag{3.11}$$

By equivalence of the norms, there exists a constant $\rho_0 > 0$ such that

$$\|u\|^2 \geq \rho_0 \|u\|_E^2, \quad u \in E. \tag{3.12}$$

It follows from (f_2) that, given $0 < \varepsilon < \frac{1}{2}\rho_0$, there exists $\delta > 0$ such that

$$\frac{|f(u)|}{|u|} \leq \varepsilon \quad \text{for} \quad 0 < |u| \leq \delta.$$

For each $k \in \mathbb{N}$, consider the set $B_k = \{m \in \mathbb{Z} : |u_k(m)| < \delta\}$. By (3.12) and Hölder's inequality,

$$\begin{aligned}
&\sum_{m \in B_k} \frac{f(u_k(m))}{u_k(m)} v_k(m)(v_k^+(m) - v_k^-(m)) \\
&\leq \varepsilon \sum_{m \in B_k} |v_k(m)||v_k^+(m) - v_k^-(m)| \\
&\leq \varepsilon (\|v_k\|_E \|v_k^+\|_E + \|v_k\|_E \|v_k^-\|_E) \\
&\leq 2\varepsilon \|v_k\|_E^2 \leq \frac{2\varepsilon}{\rho_0} \|v_k\|^2 = \frac{2\varepsilon}{\rho_0} < 1.
\end{aligned}$$

It also follows from (3.11) that

$$\liminf_{k \to \infty} \sum_{m \in \mathbb{Z} \setminus B_k} \frac{f(u_k(m))}{u_k(m)} v_k(m)(v_k^+(m) - v_k^-(m)) > 0. \tag{3.13}$$

Denote $|\mathbb{Z} \setminus B_k|$ the cardinality of $\mathbb{Z} \setminus B_k$. We claim that

$$\limsup_{k \to \infty} |\mathbb{Z} \setminus B_k| = \infty. \tag{3.14}$$

Otherwise,

$$\limsup_{k \to \infty} |\mathbb{Z} \setminus B_k| < \infty.$$

Then, since the vanishing of $\{v_k\}$ implies $v_k(m) \to 0$ in $\mathbb{Z}$ as $k \to \infty$, taking into account the above inequality and the boundedness of $f(s)/s$ for $s \in \mathbb{R} \setminus \{0\}$, we have

$$\lim_{k \to \infty} \sum_{m \in \mathbb{Z} \setminus B_k} \frac{f(u_k(m))}{u_k(m)} v_k(m)(v_k^+(m) - v_k^-(m)) \to 0.$$

This contradicts with (3.13) and hence (3.14) holds. Condition (f_3) shows that there exists R with $R > \delta > 0$ such that if $|u| > R$ then $\frac{1}{2} f(u)u - F(u) > 1$. For each $k \in \mathbb{N}$, let $\Omega_k = \{m \in \mathbb{Z} : |u_k(m)| > R\}$. Then

$$c + o_k(1) \geq \sum_{m \in \Omega_k} \left[\frac{1}{2} f(u_k(m)) u_k(m) - F(u_k(m)) \right] > |\Omega_k|,$$

which implies that the sequence $\{|\Omega_k|\}$ is bounded. Let $\widetilde{\Omega}_k = \{m \in \mathbb{Z} : \delta \leq |u_k(m)| \leq R\}$. Since $\widetilde{\Omega}_k = (\mathbb{Z} \setminus B_k) \setminus \Omega_k$, we have $|\mathbb{Z} \setminus B_k| = |\Omega_k| + |\widetilde{\Omega}_k|$. It follows from (3.14) and the boundedness of $\{|\Omega_k|\}$ that

$$|\widetilde{\Omega}_k| \to \infty. \tag{3.15}$$

We see from (f_3) that $\delta_0 = \inf_{u\in[\delta,R]}(\frac{1}{2}f(u)u - F(u)) > 0$. Hence, from (3.15), we have

$$\sum_{m\in\mathbb{Z}}\left[\frac{1}{2}f(u_k(m))u_k(m) - F(u_k(m))\right]$$
$$\geq \sum_{m\in\widetilde{\Omega}_k}\left[\frac{1}{2}f(u_k(m))u_k(m) - F(u_k(m))\right] \geq \delta_0|\widetilde{\Omega}_k| \to \infty.$$

This contradicts with

$$\sum_{m\in\mathbb{Z}}\left[\frac{1}{2}f(u_k(m))u_k(m) - F(u_k(m))\right] = J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle = c + o_k(1).$$

Thus we have proved (ii) is impossible for the sequence $\{v_k\}$. □

To sum up, we have proved that $\{u_k\}$ has a bounded subsequence. □

3.2 Linking Geometry

Now we show that the functional J satisfies the geometry of the linking theorem with a Cerami sequence [13].

Lemma 3.4 (Linking Theorem with a Cerami sequence [13]) *Let $H = H^+ \oplus H^-$ be a Banach space with* $\dim H^- < \infty$. *Let $R > \rho > 0$, and let $u \in H^+$ be a fixed element such that $\|u\| = \rho$. Define*

$$M := \{w = tu + v^- : \|w\| \leq R,\ t \geq 0,\ v^- \in H^-\},$$
$$M_0 := \{w = tu + v^- : v^- \in H^-,\ \|w\| = R,\ t \geq 0 \text{ or } \|w\| \leq R,\ t = 0\},$$
$$N_\rho := \{w \in H^+ : \|w\| = \rho\}.$$

Let $J \in C^1(H, \mathbb{R})$ be such that

$$b := \inf_{N_\rho} J > a := \max_{M_0} J.$$

Then, $c \geq b$, and there exists a Cerami sequence at level c for the functional J with

$$c := \inf_{\gamma\in\Gamma}\max_{w\in M} J(\gamma(w)),\ \ \Gamma := \{\gamma \in C(M, H) : \gamma|_{M_0} = \mathrm{Id}\}.$$

To simplify the notation, given $w \in E$ and $n \in \mathbb{Z}$, we respectively let $w^+(\cdot - n)$, $w^-(\cdot - n)$ and $w^0(\cdot - n)$ be the projections in E^+, E^- and E^0 of the translation $w(\cdot - n)$.

Remark 3.5 If $u,\ v \in l^2$, then

$$\sum_{m\in\mathbb{Z}} u(m-n)v(m) \to 0 \quad \text{as} \quad |n| \to \infty.$$

Let $u_0 \in E$ be a ground state solution of the limiting Eq. (1.4) such that $J_\infty(u_0) = c_\infty > 0$ where c_∞ is given by (2.7). For $R > 0$ and $n \in \mathbb{Z}$, consider

$$M := \{w = tu_0^+(\cdot - n) + v^- + v^0 : \|w\| \leq R,\ t \geq 0,\ v^- + v^0 \in E^- \oplus E^0\}$$

and

$$M_0 := \big\{w = tu_0^+(\cdot - n) + v^- + v^0 : v^- + v^0 \in E^- \oplus E^0,$$
$$\|w\| = R,\ t \geq 0 \text{ or } \|w\| \leq R,\ t = 0\big\}.$$

Lemma 3.6 *There exist* $R > 0$ *and* $n \in \mathbb{Z}$, *with* R *and* $|n|$ *sufficiently large, such that* $J|_{M_0} \le 0$.

Proof The subset M_0 can be written as a disjoint union of M_1 and M_2 where

$$M_1 := \{w = tu_0^+(\cdot - n) + v^- + v^0 : v^- + v^0 \in E^- \oplus E^0,\ \|w\| \le R,\ t = 0\}$$

and

$$M_2 := \{w = tu_0^+(\cdot - n) + v^- + v^0 : v^- + v^0 \in E^- \oplus E^0,\ \|w\| = R,\ t > 0\}.$$

As $M_1 \subset E^- \oplus E^0$, we have $J(w) \le 0$ for any $w \in M_1$. Let $R > 0$ and $w \in M_2$ with $\|w\| = R$. Writing

$$w = \|w\| w/\|w\| = \|w\| u_w = \|w\|(\lambda_w u_0^+(\cdot - n) + v_w^- + v_w^0),$$

we have

$$\begin{aligned} J(w) &= J(\|w\|u_w) \\ &= \frac{1}{2}\|w\|^2 \lambda_w^2 \|u_0^+(\cdot - n)\|^2 - \frac{1}{2}\|w\|^2\|v_w^-\|^2 - \sum_{m\in\mathbb{Z}} F(\|w\|u_w(m)) \\ &= \frac{1}{2}\|w\|^2 \left\{ \lambda_w^2 \|u_0^+(\cdot - n)\|^2 - \|v_w^-\|^2 - 2\sum_{m\in\mathbb{Z}} \frac{F(Ru_w(m))}{|Ru_w(m)|^2}|u_w(m)|^2 \right\}. \end{aligned}$$

To simplify the notation, we write λ, u, v^- and v^0 instead of λ_w, u_w, v_w^- and v_w^0, respectively. By (f_2) and (f_3), we have $\lim_{|s|\to\infty}(F(s)/s^2) = a/2$ and $|F(s)/s^2| < a/2$ for all $s \ne 0$, which ensure

$$\frac{|F(Ru(m))|}{|Ru(m)|^2}|u(m)|^2 \le \frac{a}{2}|u(m)|^2.$$

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{R\to\infty} \sum_{m\in\mathbb{Z}} \left(\frac{a}{2} - \frac{F(Ru(m))}{|Ru(m)|^2} \right) |u(m)|^2 = 0 \tag{3.16}$$

for all $u \in E$ with $\|u\| = 1$. Since M_2 is contained in a finite-dimensional subspace of E, for $w = \|w\|u \in M_2$ with $\|u\| = 1$, we claim that the limit in (3.16) is uniform in u. Let ∂B_1 be the boundary of $B(0, 1)$ in a finite-dimensional space generated by the terms $u_0^+(\cdot - n),\ \varphi_1, \ldots, \varphi_{k_*},\ e_1, \ldots, e_{k_{**}}$. It is sufficient to prove that (3.16) holds uniformly for $u \in \partial B_1$. In fact, for each $R = j \in \mathbb{N}$, consider $J_j : \partial B_1 \to \mathbb{R}$ with

$$J_j(u) = \sum_{m\in\mathbb{Z}} \left[\frac{a}{2} - F(ju(m))/|ju(m)|^2 \right] |u(m)|^2.$$

From the continuity of the function F, we see that J_j is a continuous functional for each fixed j. By equivalence of the norms, (f_2) shows that there exists a constant $C > 0$ such that

$$0 \le J_j(u) = \sum_{m\in\mathbb{Z}} \left[\frac{a}{2} - F(ju(m))/|ju(m)|^2 \right] |u(m)|^2 \le a\|u\|_E^2 \le C$$

for all $u \in \partial B_1$. Since J_j is continuous in the compact set ∂B_1, for each fixed j, J_j reaches its maximum at some $u_j \in \partial B_1$. Let $\{u_j\}$ be the sequence of these maxima. Since $\|u_j\|$ equals 1 for each j in the finite-dimensional space spanned by $u_0^+(\cdot - n)$, $\varphi_1, \ldots, \varphi_{k_*}$, $e_1, \ldots, e_{k_{**}}$, there exists $\overline{u} \in \partial B_1$ such that, passing to a subsequence if needed,

$$\|u_j - \overline{u}\| \to 0 \qquad \text{as } j \to \infty. \tag{3.17}$$

For all $u \in \partial B_1$ and for each j, we have $0 \le J_j(u) \le J_j(u_j)$, that is,

$$\begin{aligned} 0 &\le \sum_{m\in\mathbb{Z}} \left[\frac{a}{2} - F(ju(m))/|ju(m)|^2\right] |u(m)|^2 \\ &\le \sum_{m\in\mathbb{Z}} \left[\frac{a}{2} - F(ju_j(m))/|ju_j(m)|^2\right] |u_j(m)|^2. \end{aligned} \tag{3.18}$$

Note that $u_j(m) \to \overline{u}(m)$ in $\mathbb{Z}$ as $j \to \infty$ for some $\overline{u}(m)$. If $\overline{u}(m) \ne 0$, then $|j\overline{u}(m)| \to \infty$ as $j \to \infty$. Thus, (f_2) implies that

$$\left[\frac{a}{2} - F(ju_j(m))/|ju_j(m)|^2\right] |u_j(m)|^2 \to 0 \tag{3.19}$$

as $j \to \infty$. If $\overline{u}(m) = 0$, we also have (3.19). It follows from (3.17) that there exists $h \in l^1$ such that, passing to a subsequence if necessary,

$$0 \le \left[\frac{a}{2} - F(ju_j(m))/|ju_j(m)|^2\right] |u_j(m)|^2 \le a|u_j(m)|^2 \le ah(m). \tag{3.20}$$

Finally, by using (3.19), (3.20) and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{j\to\infty} \sum_{m\in\mathbb{Z}} \left[\frac{a}{2} - F(ju_j(m))/|ju_j(m)|^2\right] |u_j(m)|^2 = 0.$$

Thus, taking $j \to \infty$ in (3.18) produces

$$\lim_{j\to\infty} \sum_{m\in\mathbb{Z}} \left[\frac{a}{2} - F(ju(m))/|ju(m)|^2\right] |u(m)|^2 = 0$$

uniformly for $u \in \partial B_1$. This proves the claim.

Recalling that $E = E^+ \oplus E^- \oplus E^0$ is orthogonal with respect to $(\cdot,\cdot)_E$, we have

$$\begin{aligned} J(w) &= \frac{1}{2}\|w\|^2 \left\{\lambda^2 \|u_0^+(\cdot - n)\|^2 - \|v^-\|^2 - a\sum_{m\in\mathbb{Z}} |u(m)|^2 + o_R(1)\right\} \\ &= \frac{1}{2}\|w\|^2 \left\{\lambda^2 \|u_0^+(\cdot - n)\|^2 - \|v^-\|^2 - a\lambda^2 \sum_{m\in\mathbb{Z}} |u_0^+(m-n)|^2 \right. \\ &\quad \left. -a\sum_{m\in\mathbb{Z}} |v^-(m)|^2 - a\sum_{m\in\mathbb{Z}} |v^0(m)|^2 + o_R(1)\right\} \\ &\le \frac{1}{2}\|w\|^2 \left\{\lambda^2 \left[\|u_0^+(\cdot - n)\|^2 - a\sum_{m\in\mathbb{Z}} |u_0^+(m-n)|^2\right] + o_R(1)\right\}. \end{aligned} \tag{3.21}$$

Define another norm in E by

$$\|u\|_{V_\infty} := ((-\Delta + V_\infty)u, u)_E^{\frac{1}{2}}, \quad u \in E.$$

It is easy to check that the three norms, $\|\cdot\|_E$, $\|\cdot\|$ and $\|\cdot\|_{V_\infty}$, are equivalent. It follows from (V_1) and (V_2) that

$$\begin{aligned}\|u_0^+(\cdot - n)\|^2 &= ((-\Delta + V)u_0^+(\cdot - n), u_0^+(\cdot - n))_E \\ &\le ((-\Delta + V_\infty)u_0^+(\cdot - n), u_0^+(\cdot - n))_E \\ &= \|u_0^+(\cdot - n)\|_{V_\infty}^2 \le \|u_0(\cdot - n)\|_{V_\infty}^2. \end{aligned} \tag{3.22}$$

Since J_∞ is translation invariant, it is true that u_0 and $u_0(\cdot - n)$ are critical points of J_∞. Thus, $\langle J'_\infty(u_0(\cdot - n)), u_0(\cdot - n)\rangle = 0$, that is,

$$\|u_0(\cdot - n)\|_{V_\infty}^2 = \sum_{m\in\mathbb{Z}} f(u_0(m-n))u_0(m-n). \tag{3.23}$$

In terms of (3.22) and (3.23), we have

$$\|u_0^+(\cdot - n)\|^2 \le \sum_{m\in\mathbb{Z}} f(u_0(m-n))u_0(m-n). \tag{3.24}$$

Subtracting (3.24) into (3.21) gives us

$$\begin{aligned} J(w) \le \frac{1}{2}\|w\|^2 &\left\{\lambda^2 \left[\sum_{m\in\mathbb{Z}} f(u_0(m-n))u_0(m-n) \right.\right. \\ &\left.\left. -a\sum_{m\in\mathbb{Z}} |u_0^+(m-n)|^2\right] + o_R(1)\right\} \\ = \frac{1}{2}\|w\|^2 &\left\{\lambda^2 \left[\sum_{m\in\mathbb{Z}} f(u_0(m-n))u_0(m-n) - a\sum_{m\in\mathbb{Z}} |u_0(m-n)|^2 \right.\right. \\ &\left.\left. + a\sum_{m\in\mathbb{Z}}[|u_0(m-n)|^2 - |u_0^+(m-n)|^2]\right] + o_R(1)\right\} \\ = \frac{1}{2}\|w\|^2 &\left\{\lambda^2 \left[\sum_{k\in\mathbb{Z}} f(u_0(k))u_0(k) - a\sum_{k\in\mathbb{Z}} |u_0(k)|^2 \right.\right. \\ &\left.\left. + a\sum_{m\in\mathbb{Z}}[|u_0(m-n)|^2 - |u_0^+(m-n)|^2]\right] + o_R(1)\right\}. \end{aligned} \tag{3.25}$$

In what follows, we will estimate

$$\sum_{k\in\mathbb{Z}} f(u_0(k))u_0(k) - a\sum_{k\in\mathbb{Z}} |u_0(k)|^2 \tag{3.26}$$

and

$$\sum_{m\in\mathbb{Z}} \left[|u_0(m-n)|^2 - |u_0^+(m-n)|^2\right]. \tag{3.27}$$

Since $u_0 \neq 0$ is bounded, the function $f(u_0(\cdot))/u_0(\cdot)$ assumes its maximum at some $m_0 \in \mathbb{Z}$. Thus, since $|f(s)/s| < a$ for all $s \in \mathbb{R}\backslash\{0\}$, we have

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} f(u_0(k))u_0(k) - a\sum_{k\in\mathbb{Z}} |u_0(k)|^2 \\ &= \sum_{k\in\mathbb{Z}} \left(\frac{f(u_0(k))}{u_0(k)} - a\right)|u_0(k)|^2 \\ &\le \left(\frac{f(u_0(m_0))}{u_0(m_0)} - a\right)\|u_0\|_E^2 < -\gamma, \end{aligned}$$

where $\gamma = \frac{1}{2}(a - f(u_0(m_0))/u_0(m_0))\|u_0\|_E^2 > 0$. This means that there exists $\gamma > 0$ such that

$$\sum_{k\in\mathbb{Z}} f(u_0(k))u_0(k) - a\sum_{k\in\mathbb{Z}} |u_0(k)|^2 < -\gamma. \tag{3.28}$$

For (3.27), as $u_0^+(\cdot - n)$, $u_0^-(\cdot - n)$ and $u_0^0(\cdot - n)$ are orthogonal with respect to $(\cdot,\cdot)_E$, we get

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \left[|u_0(m-n)|^2 - |u_0^+(m-n)|^2\right] \\
&\quad= \sum_{m\in\mathbb{Z}} \left[|u_0^+(m-n) + u_0^-(m-n) + u_0^0(m-n)|^2 - |u_0^+(m-n)|^2\right] \\
&\quad= \sum_{m\in\mathbb{Z}} \left[|u_0^+(m-n)|^2 + |u_0^-(m-n)|^2 + |u_0^0(m-n)|^2 - |u_0^+(m-n)|^2\right] \\
&\quad= \sum_{m\in\mathbb{Z}} \left[|u_0^-(m-n)|^2 + |u_0^0(m-n)|^2\right].
\end{aligned}$$

We claim that $\sum_{m\in\mathbb{Z}} \left[|u_0^-(m-n)|^2 + |u_0^0(m-n)|^2\right] \to 0$ as $|n| \to \infty$. In fact, since $\{\varphi_1, \ldots, \varphi_{k_*}\}$ and $\{e_1, \ldots, e_{k_{**}}\}$ are respectively the bases for the subspaces E^- and E^0, (V_1), (V_2) and Remark 3.5 indicate that, given $\varepsilon > 0$, for each $i \in \{1, \ldots, k_*\}$, there exists $N_i > 0$ such that if $|n| \geq N_i$, then

$$\begin{aligned}
&(u_0(\cdot - n), \varphi_i) \\
&\quad= -\sum_{m\in\mathbb{Z}} [\nabla u_0^-(m-n) \cdot \nabla\varphi_i(m)] - \sum_{m\in\mathbb{Z}} V(m)u_0^-(m-n)\varphi_i(m) < \varepsilon,
\end{aligned}$$

and for each $j \in \{1, \ldots, k_{**}\}$, there exists $K_j > 0$ such that if $|n| \geq K_j$, then

$$(u_0(\cdot - n), e_j) = \sum_{m\in\mathbb{Z}} u_0^0(m-n)e_j(m) < \varepsilon.$$

Taking $N^* = \max\{N_1, \ldots, N_{k_*}, K_1, \ldots, K_{k_{**}}\}$ gives us that, for $i \in \{1, \ldots, k_*\}$ and $j \in \{1, \ldots, k_{**}\}$,

$$(u_0(\cdot - n), \varphi_i) < \varepsilon \text{ and } (u_0(\cdot - n), e_j) < \varepsilon, \text{ if } |n| \geq N^*. \tag{3.29}$$

Since $u_0^-(\cdot - n) + u_0^0(\cdot - n) \in E^- \oplus E^0$ is a linear combination of $\varphi_1, \ldots, \varphi_{k_*}, e_1, \ldots, e_{k_{**}}$, that is,

$$u_0^-(\cdot - n) + u_0^0(\cdot - n) = \sum_{i=1}^{k_*} a_i(n)\varphi_i + \sum_{j=1}^{k_{**}} b_j(n)e_j,$$

it follows from (3.29) that there exists $N^* > 0$ such that, if $|n| \geq N^*$, then

$$\begin{aligned}
&\|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|^2 \\
&\quad= (u_0(\cdot - n), u_0^-(\cdot - n) + u_0^0(\cdot - n)) \\
&\quad= \left(u_0(\cdot - n), \sum_{i=1}^{k_*} a_i(n)\varphi_i + \sum_{j=1}^{k_{**}} b_j(n)e_j\right) \\
&\quad< \varepsilon(k_* + k_{**}) \max\{|a_1(n)|, \ldots, |a_{k_*}(n)|, |b_1(n)|, \ldots, |b_{k_{**}}(n)|\}.
\end{aligned} \tag{3.30}$$

In the following, we show that there exists a constant $C > 0$ that does not depend on n, such that

$$\max\{|a_1(n)|, \dots, |a_{k_*}(n)|, |b_1(n)|, \dots, |b_{k_{**}}(n)|\} < C \quad \text{for } n \in \mathbb{Z}. \tag{3.31}$$

Indeed, as $\dim(E^- \oplus E^0) < \infty$, by equivalence of the norms in a finite-dimensional space, there exists $D > 0$, which does not depend on n, such that

$$\left\| \sum_{i=1}^{k_*} a_i(n)\varphi_i + \sum_{j=1}^{k_{**}} b_j(n)e_j \right\|_{V_\infty}^2 \geq D(\max\{|a_1(n)|, \dots, |a_{k_*}(n)|, |b_1(n)|, \dots, |b_{k_{**}}(n)|\})^2.$$

Thus, we obtain

$$\|u_0\|_{V_\infty}^2 \geq \|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|_{V_\infty}^2 = \left\| \sum_{i=1}^{k_*} a_i(n)\varphi_i + \sum_{j=1}^{k_{**}} b_j(n)e_j \right\|_{V_\infty}^2 \geq D(\max\{|a_1(n)|, \dots, |a_{k_*}(n)|, |b_1(n)|, \dots, |b_{k_{**}}(n)|\})^2. \tag{3.32}$$

This implies (3.31) by taking $C = \|u_0\|_{V_\infty}^2/\sqrt{D} > 0$. Substituting (3.31) into (3.30) yields $\|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|^2 < \varepsilon(k_* + k_{**})C$ for $|n| \geq N^*$. By equivalence of $\|\cdot\|$ and $\|\cdot\|_{V_\infty}$ in E, we have that $\|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|_{V_\infty} \to 0$ as $|n| \to \infty$. Thus,

$$\sum_{m\in\mathbb{Z}} \left[|u_0^-(m-n)|^2 + |u_0^0(m-n)|^2\right] \leq \frac{1}{V_\infty}\|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|_{V_\infty}^2 \to 0 \text{ as } |n| \to \infty. \tag{3.33}$$

Substituting (3.28) and (3.33) into (3.25), we see that

$$J(w) \leq \frac{1}{2}\|w\|^2 \left\{\lambda^2[-\gamma + o_{|n|}(1)] + o_R(1)\right\} \tag{3.34}$$

for $|n|$ and R sufficiently large.

Now, we are in a position to finish the proof of the lemma. Indeed, assume by contradiction that there exists $w_j = \|w_j\|u_j = \|w_j\|(\lambda_j u_0^+(\cdot - n) + v_j^- + v_j^0)$ such that

$$0 \leq \frac{J(w_j)}{\|w_j\|^2} = \frac{1}{2}(\lambda_j^2\|u_0^+(\cdot - n)\|^2 - \|v_j^-\|^2) - \sum_{m\in\mathbb{Z}} \frac{F(w_j(m))}{|w_j(m)|^2}|u_j(m)|^2 \tag{3.35}$$

for all j and $\|w_j\| \to \infty$ as $j \to \infty$. Since $\|\lambda_j u_0^+(\cdot - n) + v_j^- + v_j^0\|^2 = 1$, it follows that $\lambda_j^2\|u_0^+(\cdot - n)\|^2 + \|v_j^-\|^2 + \|v_j^0\|^2 = 1$. Thus, noting that F is a nonnegative function according to (f_3), we have $\|v_j^-\|^2 \leq \lambda_j^2\|u_0^+(\cdot - n)\|^2 = 1 - \|v_j^-\|^2 - \|v_j^0\|^2$ and therefore

$$(1 - \|v_j^0\|^2)/2 \leq \lambda_j^2\|u_0^+(\cdot - n)\|^2 \leq 1. \tag{3.36}$$

Passing to a subsequence if necessary, we may assume that $u_j \rightharpoonup u = \lambda_0^2 u_0^+(\cdot - n) + v^- + v^0$ in E. We claim that there exists a constant $b_0 > 0$ such that $\|v_j^0\| < b_0 < 1$ for j sufficiently large. If not, we may assume that $\|v_j^0\| \to 1$ as $j \to \infty$. Then

$$\lambda_j^2\|u_0^+(\cdot - n)\|^2 - \|v_j^-\|^2 \leq \lambda_j^2\|u_0^+(\cdot - n)\|^2 + \|v_j^-\|^2 = 1 - \|v_j^0\|^2 \to 0$$

as $j \to \infty$. It follows from $\dim E^0 < \infty$ that $v_j^0 \to v^0 \neq 0$ as $j \to \infty$, which implies $u = \lambda_0^2 u_0^+(\cdot - n) + v^- + v^0 \neq 0$. Thus, there exists $m_* \in \mathbb{Z}$ such that $u_j(m_*) \to u(m_*) \neq 0$ and $|w_j(m_*)| = \|w_j\||u_j(m_*)| \to \infty$ as $j \to \infty$. Then by (f_2) and (f_3), we have

$$\frac{F(w_j(m_*))}{|w_j(m_*)|^2}|u_j(m_*)|^2 \to \frac{a}{2}|u(m_*)|^2 > 0$$

as $j \to \infty$. This contradicts with (3.35), since F is a nonnegative function and $\lambda_j^2\|u_0^+(\cdot - n)\|^2 - \|v_j^-\|^2 \to 0$ as $j \to \infty$. Thus $\|v_j^0\| < b_0 < 1$ for j sufficiently large. By equivalence of the norms and translation invariance of $\|\cdot\|_{V_\infty}$, there exists $C > 0$, which does not depend on n, such that $2\|u_0^+(\cdot - n)\|^2 \leq C\|u_0\|_{V_\infty}^2$. It follows from (3.36) that

$$0 < k_0 := \frac{1 - b_0^2}{C\|u_0\|_{V_\infty}^2} \leq \lambda_j^2$$

for j sufficiently large. Considering (3.34), if $\lambda^2 \geq k_0$, we take $n \in \mathbb{Z}$ with $|n|$ sufficiently large such that $-\gamma + o_{|n|}(1) < -\gamma/2$. Thus, (3.34) becomes

$$J(w) \leq \frac{1}{2}\|w\|^2\left[-\frac{\lambda^2\gamma}{2} + o_R(1)\right].$$

Since $-\lambda^2 \leq -k_0$ and since R does not depend on n according to the uniform convergence in u in (3.16), taking R sufficiently large such that $-k_0\gamma/2 + o_R(1) < 0$, we obtain

$$J(w) \leq \frac{1}{2}\|w\|^2\left[-\frac{\lambda^2\gamma}{2} + o_R(1)\right] \leq \frac{1}{2}\|w\|^2\left[-\frac{k_0\gamma}{2} + o_R(1)\right] < 0.$$

Letting $\lambda = \lambda_j$ and $w = w_j$ in the above inequality leads to a contradiction to (3.35). Thus the proof of the lemma is complete. □

Lemma 3.7 *For c_∞ given in (2.7) and c given in Lemma 3.4, one has $c < c_\infty$.*

Proof Note that the set M defined in Lemma 3.4 is bounded and closed, and is contained in the finite-dimensional space $E^- \oplus E^0 \oplus \mathbb{R}u_0^+(\cdot - n)$. Thus, M is a compact set. Since J is a continuous functional, for all $n \in \mathbb{Z}$, there exists $w_n = v_n^- + v_n^0 + t_n u_0^+(\cdot - n) \in M$ with

$$\max_{w \in M} J(w) = J(v_n^- + v_n^0 + t_n u_0^+(\cdot - n)).$$

We claim that there are $A_1,\ A_2 \in \mathbb{R}$, independent of n, such that $0 < A_1 \leq t_n \leq A_2$ for $|n|$ sufficiently large.

Proof On the one hand, since $w_n = v_n^- + v_n^0 + t_n u_0^+(\cdot - n) \in M$, and since the number $R > 0$ given by Lemma 3.6 does not depend on n, we have that

$$\begin{aligned} R^2 \geq \|w_n\|^2 &= \|v_n^- + v_n^0\|^2 + t_n^2\|u_0^+(\cdot - n)\|^2 \\ &\geq t_n^2(\|u_0(\cdot - n)\|^2 - \|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|^2). \end{aligned}$$

As shown in (3.33), we can take $|n|$ large enough such that

$$\|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|^2 \leq \frac{C}{2}\|u_0\|_{V_\infty}^2,$$

where $C > 0$ does not depend on n and satisfies $\|u_0(\cdot - n)\|^2 \geq C\|u_0\|^2_{V_\infty}$. Thus,

$$R^2 \geq t_n^2(\|u_0(\cdot - n)\|^2 - \|u_0^-(\cdot - n) + u_0^0(\cdot - n)\|^2) \geq \frac{t_n^2 C}{2}\|u_0\|^2_{V_\infty}.$$

In other words, we have

$$t_n^2 \leq A_2^2 := 2R^2/(C\|u_0\|^2_{V_\infty}).$$

On the other hand, by (3.1) with $2 < p$, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, if $u \in E^+$ with $\|u\| = \rho > 0$, then

$$J(u) = \frac{1}{2}\|u\|^2 - \sum_{m\in\mathbb{Z}} F(u(m)) \geq \frac{1}{2}\rho^2 - \varepsilon\|u\|^2_E - C_\varepsilon\|u\|^p_{l^p}. \tag{3.37}$$

By equivalence of the norms, there are $C_1,\ C_2 > 0$ such that

$$J(u) \geq \frac{1}{2}\rho^2 - \varepsilon C_1\|u\|^2 - C_2\|u\|^p = \left(\frac{1}{2} - \varepsilon C_1\right)\rho^2 - C_2\rho^p.$$

Let $\varepsilon > 0$ satisfy $D_\varepsilon := 1/2 - \varepsilon C_1 > 0$. Take $\rho > 0$ sufficiently small such that $\rho_0 := D_\varepsilon\rho^2 - C_2\rho^p > 0$, that is, $0 < \rho < (D_\varepsilon/C_2)^{1/(p-2)}$. Then $J(u) \geq \rho_0 > 0$ for all $u \in E^+$ with $\|u\| = \rho$.

In fact, note that ρ_0 does not depend on n. Thus, by taking $t_0 > 0$, which does not depend on n, sufficiently small such that $\|t_0 u_0^+(\cdot - n)\| \leq \rho < R$, we can prove that $I(t_0 u_0^+(\cdot - n)) \geq \rho_0 > 0$. Therefore,

$$J(v_n^- + v_n^0 + t_n u_0^+(\cdot - n)) = \max_{w\in M} J(w) \geq I(t_0 u_0^+(\cdot - n)) \geq \rho_0,$$

that is,

$$\begin{aligned}&\frac{t_n^2}{2}\|u_0^+(\cdot - n)\|^2 - \frac{1}{2}\|v_n^-\|^2 - \sum_{m\in\mathbb{Z}} F(v_n^-(m) + v_n^0(m) + t_n u_0^+(m - n))\\ &= J(v_n^- + v_n^0 + t_n u_0^+(\cdot - n)) \geq \rho_0.\end{aligned}$$

By the nonnegativity of F, we have

$$\frac{t_n^2}{2}\|u_0^+(\cdot - n)\|^2 \geq \rho_0,$$

which indicates that

$$t_n^2 \geq A_1^2 := \frac{2\rho_0}{C\|u_0\|^2_{V_\infty}},$$

where $C > 0$ does not depend on n and satisfies $\|u_0^+(\cdot - n)\|^2 \leq C\|u_0\|^2_{V_\infty}$. This proves the claim. □

Now, for simplicity, we denote $u_{0,n}(\cdot) := u_0(\cdot - n)$, and denote C a positive constant, which may not necessarily be the same in every situation. Since F is nonnegative, we see from definitions of J and J_∞ that

$$\begin{aligned}&J(v_n^- + v_n^0 + t_n u_{0,n}^+)\\ &= -\sum_{m\in\mathbb{Z}} F(v_n^-(m) + v_n^0(m) + t_n u_{0,n}^+(m)) + \frac{t_n^2}{2}\|u_{0,n}^+\|^2 - \frac{1}{2}\|v_n^-\|^2\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{m\in\mathbb{Z}}[F(t_n u_{0,n}(m)) - F(v_n^-(m) + v_n^0(m) + t_n u_{0,n}^+(m))]\\
&\quad + \frac{t_n^2}{2}\|u_{0,n}\|^2 - \sum_{m\in\mathbb{Z}} F(t_n u_{0,n}(m)) - \frac{t_n^2}{2}\|u_{0,n}^- + u_{0,n}^0\|^2\\
&\leq \sum_{m\in\mathbb{Z}}[F(v_n^-(m) + v_n^0(m) - t_n(u_{0,n}^-(m) + u_{0,n}^0(m)))\\
&\quad + F(t_n u_{0,n}(m)) - F(v_n^-(m) + v_n^0(m) + t_n u_{0,n}^+(m))]\\
&\quad + J_\infty(t_n u_{0,n}) + \frac{t_n^2}{2}\sum_{m\in\mathbb{Z}}(V(m) - V_\infty)|u_{0,n}(m)|^2. \qquad (3.38)
\end{aligned}$$

We firstly estimate the first term in the last inequality of (3.38). Taking $w_n^* = v_n^- + v_n^0 - t_n(u_{0,n}^- + u_{0,n}^0)$, we want to estimate

$$J_n = \left|\sum_{m\in\mathbb{Z}}[F(w_n^*(m)) + F(t_n u_{0,n}(m)) - F(w_n^*(m) + t_n u_{0,n}(m))]\right|.$$

Since $w_n^* \in M$, $\|w_n^*\|^2 \leq R^2$, and hence we may repeat the estimates in (3.32) with w_n^* replacing $u_{0,n}^- + u_{0,n}^0$, and use the claim just proved to show that there is a constant $C > 0$, not depending on n, such that

$$\begin{aligned}
|w_n^*(m)| &\leq C\sum_{i=1}^{k_*}|\varphi_i(m)| + C\sum_{j=1}^{k_{**}}|e_j(m)|\\
&\leq D := C\sum_{i=1}^{k_*}\sup_{m\in\mathbb{Z}}|\varphi_i(m)| + C\sum_{j=1}^{k_{**}}\sup_{m\in\mathbb{Z}}|e_j(m)|, \qquad (3.39)
\end{aligned}$$

for all $m \in \mathbb{Z}$. Without loss of generality, we may take D with $|u_{0,n}(m)| \leq D$ for all $m \in \mathbb{Z}$ since $u_{0,n} \in l^\infty$. Thus, in terms of (f_1) and Lemma 2.3, we conclude that there exists a constant $C > 0$ such that

$$J_n \leq C t_n \sum_{m\in\mathbb{Z}}|w_n^*(m)||u_{0,n}(m)|. \qquad (3.40)$$

Since u_0 in l^2 is a nontrivial solution of (1.4), given by Lemma 2.1, we have that

$$|u_0(m)| \leq Ce^{-\mu_1|m|}, \quad m \in \mathbb{Z},$$

for some $\mu_1 \in (r_0, \cosh^{-1}(V_\infty/2 + 1))$, where r_0 is given by (V_2). Now, with $\beta = \lambda_i < 0 < V_\infty$ in Lemma 2.1, any eigenfunction φ_i, $i = 1, \ldots, k_*$, satisfies

$$|\varphi_i(m)| \leq Ce^{-\mu_2|m|}, \quad m \in \mathbb{Z},$$

for some $\mu_2 \in (\mu_1, \cosh^{-1}(V_\infty/2+1))$. Similarly, it is ture that e_j, $j = 1, \ldots, k_{**}$, satisfies

$$|e_j(m)| \leq Ce^{-\mu_2|m|}, \quad m \in \mathbb{Z}.$$

Thus, from the first inequality in (3.39), one has for $|n|$ sufficiently large that

$$|w_n^*(m)| \leq Ce^{-\mu_2|m|}, \quad m \in \mathbb{Z}.$$

It follows from Lemma 2.4 that

$$\sum_{m\in\mathbb{Z}} |w_n^*(m)||u_{0,n}(m)| \le C \sum_{m\in\mathbb{Z}} e^{-\mu_2|m|}e^{-\mu_1|m-n|} \le Ce^{-\mu_1|n|}. \tag{3.41}$$

The above, together with (3.40), yields

$$J_n \le Ce^{-\mu_1|n|}, \tag{3.42}$$

where $C > 0$ does not depend on n, since t_n is uniformly bounded by the claim.

Now, we estimate the term

$$\frac{t_n^2}{2}\sum_{m\in\mathbb{Z}}(V(m) - V_\infty)|u_{0,n}(m)|^2.$$

It follows from (V_2) that

$$\begin{aligned}\frac{t_n^2}{2}\sum_{m\in\mathbb{Z}}(V(m) - V_\infty)|u_{0,n}(m)|^2 &= \frac{t_n^2}{2}\sum_{m\in\mathbb{Z}}(V(m+n) - V_\infty)|u_0(m)|^2\\ &\le -\frac{t_n^2}{2}\sum_{m\in\mathbb{Z}} C_0e^{-r_0|m+n|}|u_0(m)|^2\\ &\le -\frac{t_n^2}{2}C_0\sum_{m\in\mathbb{Z}} e^{-r_0|n|}e^{-r_0|m|}|u_0(m)|^2\\ &= -Ce^{-r_0|n|},\end{aligned}$$

for $|n|$ sufficiently large, where $C > 0$ does not depend on n.

Thus, (3.38) combined with (3.42) gives

$$J(v_n^- + v_n^0 + t_n u_{0,n}^+) \le J_\infty(t_n u_{0,n}) - Ce^{-r_0|n|} + Ce^{-\mu_1|n|}.$$

Since $0 < r_0 < \mu_1$, we see from the above inequality that

$$J(v_n^- + v_n^0 + t_n u_{0,n}^+) < \max_{t\ge 0} J_\infty(tu_0),$$

for $|n|$ sufficiently large. Since u_0 is ground state solution of (1.4), it is seen from (f_3) that $\max_{t\ge0} J_\infty(tu_0)$ is attained exactly at $t = 1$. In fact, assume that $\max_{t\ge0} J_\infty(tu_0)$ is attained at some $t = t_0$. Obviously, $t_0 \ne 0$ and

$$\left.\frac{\partial J_\infty(tu_0)}{\partial t}\right|_{t=t_0} = t_0((-\Delta + V_\infty)u_0, u_0) - \sum_{m\in\mathbb{Z}} f(t_0u_0(m))u_0(m) = 0.$$

Since u_0 is a nontrivial critical point of J_∞, we have

$$((-\Delta + V_\infty)u_0, u_0) - \sum_{m\in\mathbb{Z}} f(u_0(m))u_0(m) = 0.$$

Denote $\Omega_0 = \{m \in \mathbb{Z} : |u_0(m)| > 0\}$. Then combining the two equations above gives us that

$$\sum_{m\in\Omega_0}\left[\frac{f(u_0(m))}{u_0(m)} - \frac{f(t_0u_0(m))}{t_0u_0(m)}\right]|u_0(m)|^2 = 0.$$

Thus, it follows from (f_3) that $t_0 = 1$. Since u_0 is a ground state solution for (1.4), it follows from the definition of $c > 0$ that

$$c \leq \max_{w \in M} J(w) = J(v_n^- + v_n^0 + t_n u_{0,n}^+) < c_\infty,$$

and the lemma is proved. □

4 Proofs of Main Results

In this section, we present the proofs of our main results of this paper.

4.1 Proof of Theorem 1.1

Recall that a functional I is said to be weakly sequentially lower semi-continuous if for any $u_j \rightharpoonup u$ in E, one has $I(u) \leq \liminf_{j\to\infty} I(u_j)$, and I' is said to be weakly sequentially continuous if $\lim_{j\to\infty} < I'(u_j), v >=< I'(u), v >$ for each $v \in E$. By a standard argument, one checks easily the following.

Lemma 4.1 *Under assumptions of Theorem 1.1, the functional*

$$I(u) := \sum_{m\in\mathbb{Z}} F(u(m))$$

in E is non-negative, weakly sequentially lower semi-continuous, and I' is weakly sequentially continuous.

Remark 4.2 Since $\Psi : E \to \mathbb{R}$ with $\Psi(u) = \|u\|^2$ is a functional of class C^1 and Ψ' is weakly sequentially continuous, by the above lemma and by equivalence of the norms in E, we obtain that both J' and J'_∞ are weakly sequentially continuous.

According to Sect. 3, we deduce that there exists a bounded sequence $\{u_k\} \subset E$ satisfying

$$J(u_k) \to c > 0 \text{ and } \|J'(u_k)\|_{E^*}(1 + \|u_k\|) \to 0 \text{ as } k \to \infty.$$

Thus, there exists a constant $C > 0$ such that $\|u_k\| \leq C$. Passing to a subsequence if necessary, we have $u_k \rightharpoonup u$ in E. Next we show that $u \neq 0$.

Arguing by contradiction, suppose that $u = 0$, i.e., $u_k \rightharpoonup 0$ in E, and so $u_k \to 0$ in l_0^s, $2 \leq s$ and $u_k(m) \to 0$ in $\mathbb{Z}$. By (V_1), (V_2) and (f_2), it is easy to show that

$$\lim_{k\to\infty} \sum_{m\in\mathbb{Z}} (V(m) - V_\infty) |u_k(m)|^2 = 0, \ \lim_{k\to\infty} \sum_{m\in\mathbb{Z}} F(u_k(m)) = 0, \ u \in E$$

and

$$\lim_{k\to\infty} \sum_{m\in\mathbb{Z}} (V(m) - V_\infty) u_k(m)v(m) = 0, \ \lim_{k\to\infty} \sum_{m\in\mathbb{Z}} f(u_k(m))v(m) = 0, \ u, v \in E.$$

Note that

$$J_\infty(u) = J(u) - \frac{1}{2} \sum_{m\in\mathbb{Z}} (V(m) - V_\infty) |u(m)|^2, \ u \in E$$

and

$$\langle J'_\infty(u), v\rangle = \langle J'(u), v\rangle - \sum_{m\in\mathbb{Z}} (V(m) - V_\infty)\, u(m)v(m), \;\; u,\, v \in E.$$

Thus, we have

$$J_\infty(u_k) \to c \text{ and } \|J'_\infty(u_k)\|_{E^*}(1 + \|u_k\|_{V_\infty}) \to 0, \;\; \text{as } k \to \infty.$$

We claim that there exist $\delta > 0$ and $m_k \in \mathbb{Z}$ such that $|u_k(m_k)| \geq \delta$. If not, then $u_k \to 0$ in l^∞ as $k \to \infty$. Since $\dim E^0 < \infty$, we have $u_k^0 \to 0$ in E, that is, $\|u_k^0\| \to 0$, as $k \to \infty$. For $p > 2$, we have

$$\|u_k\|_{l^p}^p \leq \|u_k\|_{l^\infty}^{p-2} \|u_k\|_E^2.$$

It follows from boundedness of $\{u_k\}$ and equivalence of the norms in E that $u_k \to 0$ in l^p for all $p > 2$. In terms of (3.1) and (3.2), we have

$$\begin{aligned}
&\left|\sum_{m\in\mathbb{Z}} f(u_k(m))(u_k^+(m) - u_k^-(m))\right| \\
&\leq \varepsilon \sum_{m\in\mathbb{Z}} |u_k(m)||u_k^+(m) - u_k^-(m)| + C_\varepsilon \sum_{m\in\mathbb{Z}} |u_k(m)|^{p-1}|u_k^+(m) - u_k^-(m)| \\
&\leq \varepsilon \|u_k\|_E(\|u_k^+\|_E + \|u_k^-\|_E) + C_\varepsilon \|u_k\|_{l^p}^{p-1}(\|u_k^+\|_{l^p} + \|u_k^-\|_{l^p}) \\
&\leq 2\varepsilon \|u_k\|_E^2 + C_\varepsilon \|u_k\|_{l^p}^{p-1}(\|u_k^+\|_{l^p} + \|u_k^-\|_{l^p}) \\
&\leq \frac{2\varepsilon c_2^2}{c_1^2}\|u_k\|^2 + C_\varepsilon \|u_k\|_{l^p}^{p-1}(\|u_k^+\|_{l^p} + \|u_k^-\|_{l^p}).
\end{aligned}$$

Take a small enough ε with $1 - 2\varepsilon c_2^2/c_1^2 > 0$, which together with $J'(u_k) \to 0$, $\|u_k^0\| \to 0$ and

$$\|u_k\|^2 = \left\langle J'(u_k), u_k^+ - u_k^-\right\rangle - \|u_k^0\|^2 + \sum_{m\in\mathbb{Z}} f(u_k(m))(u_k^+ - u_k^-(m)),$$

indicates that $u_k \to 0$ in E as $k \to \infty$. This further implies that $c = 0$, a contradiction.

As J_∞ and J'_∞ are invariant under translation, writing $v_k = \{v_k(m)\}$ with $v_k(m) = u_k(m + m_k)$, we have $\|v_k\|_{V_\infty} = \|u_k\|_{V_\infty}$, $J_\infty(v_k) = J_\infty(u_k)$, $J'_\infty(v_k) = J'_\infty(u_k)$, and $|v_k(0)| \geq \delta$ for each k. Thus $\{v_k\}$ is also a bounded Cerami sequence at level c, that is,

$$J_\infty(v_k) \to c \text{ and } \|J'_\infty(v_k)\|_{E^*}(1 + \|v_k\|_{V_\infty}) \to 0, \;\; \text{as } k \to \infty. \tag{4.1}$$

Passing to a subsequence if necessary, we have $v_k \rightharpoonup v$ in E, and $v_k \to v$ in l_0^s, $2 \leq s$ and $v_k(m) \to v(m)$ in $\mathbb{Z}$ as $k \to \infty$. Obviously, $v \neq 0$ and $J'_\infty(v) = 0$ according to Remark 4.2. This shows that $J_\infty(v) \geq c_\infty$ since c_∞ is the least energy level. On the other hand, by using (4.1), we have

$$\begin{aligned}
c_\infty > c &= \lim_{k\to\infty}\left[J_\infty(v_k) - \frac{1}{2}\langle J'_\infty(v_k), v_k\rangle\right] \\
&= \lim_{k\to\infty}\sum_{m\in\mathbb{Z}}\left[\frac{1}{2}f(v_k(m))v_k(m) - F(v_k(m))\right] \\
&\geq \sum_{m\in\mathbb{Z}}\lim_{k\to\infty}\left[\frac{1}{2}f(v_k(m))v_k(m) - F(v_k(m))\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f(v(m))v(m) - F(v(m))\right] \\
&= J_\infty(v) - \frac{1}{2}\langle J'_\infty(v), v\rangle = J_\infty(v) \geq c_\infty.
\end{aligned}
$$

This contradiction implies that $u \neq 0$. By a standard argument, we can verify that $J'(u) = 0$. This shows that $u \in E$ is a nontrivial solution of (1.1).

Now we try to find a ground state solution of (1.1). In fact, let

$$
\Lambda = \{u : u \in E\backslash\{0\},\ J'(u) = 0\}
$$

be the set of all nontrivial critical points of J and

$$
c_* = \inf\{J(u) : u \in \Lambda\}.
$$

From (f_2) and (f_3), we have

$$
\frac{1}{2} f(s)s - F(s) > \frac{1}{2} f(s)s - \frac{f(s)}{s}\int_0^s t\,dt = 0, \quad s \neq 0. \tag{4.2}
$$

Therefore, for any $u \in \Lambda$, we have

$$
J(u) = J(u) - \frac{1}{2}\langle J'(u), u\rangle = \sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f(u(m))u(m) - F(u(m))\right] > 0.
$$

Thus $0 \leq c_* \leq J(u)$. Suppose that there exists $\{u_k\} \subset \Lambda$ such that $J(u_k) \to c_*$ as $k \to \infty$. Then $\{u_k\}$ is a Cerami sequence at level c_*. By Lemma 3.1, $\{u_k\}$ is bounded in E. Up to a subsequence if necessary, we have $u_k \rightharpoonup u_*$ in E. Repeating the previous procedure in obtaining the compactness of the Cerami sequence of J, we can prove that u_* is a nontrivial critical point of J. Therefore, by (4.2) and Fatou's lemma, we have

$$
\begin{aligned}
c_* &= \lim_{k\to\infty} \left[J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle\right] \\
&= \lim_{k\to\infty} \sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f(u_k(m))u_k(m) - F(u_k(m))\right] \\
&\geq \sum_{m\in\mathbb{Z}} \lim_{k\to\infty} \left[\frac{1}{2} f(u_k(m))u_k(m) - F(u_k(m))\right] \\
&= \sum_{m\in\mathbb{Z}} \left[\frac{1}{2} f(u_*(m))u_*(m) - F(u_*(m))\right] \\
&= J(u_*) - \frac{1}{2}\langle J'(u_*), u_*\rangle = J(u_*) \geq c_*.
\end{aligned}
$$

Hence $J(u_*) = c_* > 0$. In other words, $u_* \in E$ is a ground state solution of (1.1). Estimate (1.5) follows from Proposition 2.2. This completes the proof.

4.2 Proofs of Proposition 1.2 and Theorem 1.3

We first prove Proposition 1.2.

Proof By way of contradiction, we assume that (1.1) has a nontrivial solution $u = \{u(m)\}$ in l^2. Then u is a nonzero critical point of J, that is, $\langle J'(u), u\rangle = 0$. We see from (2.6), (f_2) and (f_3) that

$$\begin{aligned}
&((-\Delta + V)u^+, u^+)_E - ((-\Delta + V)u^-, u^-)_E \\
&= \sum_{m\in\mathbb{Z}} f(u(m))(u^+(m) - u^-(m)) \\
&\le \sum_{m\in\mathbb{Z}} \left|\frac{f(u(m))}{u(m)}\right| |(u^+(m) + u^-(m))(u^+(m) - u^-(m))| \\
&< a \sum_{m\in\mathbb{Z}} ||u^+(m)|^2 - |u^-(m)|^2| \le a \sum_{m\in\mathbb{Z}} (|u^+(m)|^2 + |u^-(m)|^2) \\
&= a\|u^+\|_E^2 + a\|u^-\|_E^2 \le \sigma_+\|u^+\|_E^2 - \sigma_-\|u^-\|_E^2.
\end{aligned}$$

This is impossible since it follows from (2.3) and (2.4) that

$$((-\Delta + V)u^+, u^+)_E - ((-\Delta + V)u^-, u^-)_E \ge \sigma_+\|u^+\|_E^2 - \sigma_-\|u^-\|_E^2.$$

Thus the proof of Proposition 1.2 is complete. □

Finally we prove Theorem 1.3.

Proof Note that $\sigma_+ = V_\infty$. The proof of Theorem 1.3 consists of the following two steps. (i) Assume that $V_\infty < a$. Then (1.6) is satisfied automatically in this case. Thus it follows from Theorem 1.1 that (1.1) has a ground state solution u in l^2. (ii) Assume that (1.1) has a ground state solution u in l^2. Then we need to show that $a > V_\infty$. By way of contradiction, we assume that $a \le V_\infty$. It follows from (1.6) that $a + \sigma_- < 0$, which further includes that $a \le \min\{\sigma_+, -\sigma_-\}$. Thus, by virtue of Proposition 1.2, we know that (1.1) has no nontrivial solution in l^2. This contradicts with our assumptions. The proof is complete. □

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