



On Centered Co-circular Central Configurations of the n -Body Problem

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Abstract

We study the co-circular central configurations of the n -body problem for which the center of mass and the center of the common circle coincide. In particular, we prove that there are no central configurations of this type with all the masses equal except one. This provides more evidences for the veracity of the conjecture that the regular n -gon with equal masses is the unique co-circular central configuration of the n -body problem whose center of mass is the center of the circle. Our result remains valid if we consider power-law potentials.

Keywords Co-circular central configurations · n -Body problem · Regular n -gon

Mathematics Subject Classification 70F07 · 70F15

1 Introduction

A configuration of the Newtonian n -body problem is *central* if the acceleration vector of each body is a common scalar multiple of its position vector (with respect to the center of mass); i.e. if there exists λ independent of i such that

$$\ddot{\mathbf{q}}_i = -\lambda (\mathbf{q}_i - \mathbf{c}), \quad i = 1, \dots, n,$$

where \mathbf{q}_i is the position of the mass m_i and $\mathbf{c} = \sum_{i=1}^n m_i \mathbf{q}_i / \sum_{i=1}^n m_i$ is the center of mass of the system; or equivalently, if there exists λ such that

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$$\lambda (\mathbf{q}_i - \mathbf{c}) = \sum_{j=1, j \neq i}^n m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{r_{ij}^3}, \quad i = 1, \dots, n. \tag{1}$$

Equations (1) can be written as

$$\nabla U(\mathbf{q}) + \lambda \nabla I(\mathbf{q}) = 0,$$

where

$$U(\mathbf{q}) = \sum_{i < j} \frac{m_i m_j}{r_{ij}}, \quad r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|, \tag{2}$$

is the Newtonian potential and

$$I = \frac{1}{2} \sum_{i=1}^n m_i \|\mathbf{q}_i - \mathbf{c}\|^2,$$

is one half the moment of inertia of the system.

Central configurations play an important role in the study of the n -body problem. They provide the unique solutions of the n -body problem for $n \geq 3$ that are known explicitly, the *homographic solutions* for which the configuration formed by the bodies remains similar to itself for any time; every motion starting or ending in a total collision and every parabolic scape to infinity is asymptotic to a central configuration (see for instance [5,29]); the bifurcations in the topology of the level sets with constant energy and angular momentum are related with central configurations (see for instance [22,30]).

The equations for central configurations (1) are invariant under homotheties and rotations with respect to the center of mass. Thus we consider classes of central configurations module such transformations.

The study of central configurations goes back to Euler and Lagrange in the second part of the eighteenth century. In 1767 Euler [12] found the three collinear central configurations of the 3-body problem and 5 years later Lagrange [19] found the two equilateral triangle central configurations. These five are the only classes of central configurations of the 3-body problem, see [32]. The set of central configurations of the n -body problem with $n > 3$ for an arbitrary given set of the masses is not completely known. Over the years many authors have found several partial results on central configurations of the n -body problem with $n > 3$, see for instance [14,22,23,28,32] for a general background. The difficulty of the problem of finding central configurations of the general n -body problem forces us to consider some simplifications by imposing restrictions, usually on the masses or the geometry of the configuration. The most common simplifications are to take some equal masses or some infinitesimal masses and to impose symmetries or a fixed shape on the configuration, see for instance [1,3,8–10,16,20,27,33] and the reference therein.

One of the main open questions is about the finiteness of the number of central configurations in the plane for fixed positive masses. It was proposed by Wintner [32] and it the sixth problem of the list of eighteen problems of the twenty-first century by Smale [31]. It is also collected as Problem 9 in the list of open problems in celestial mechanics compiled by Albouy, Cabral and Santos [2]. The exact number of central configurations of the n -body for a given set of masses is only known for $n = 3$ (there are have five classes of central configurations, the ones found by Euler and Lagrange) and for the collinear central configurations (Moulton [24] proved that there are $n!/2$ classes of collinear central configurations). A lower bound of the number of classes of planar central configurations can be found in [26]. Recently Hampton and Moeckel [18] and Albouy and Kaloshin [4] proved the finiteness of the number of classes of central configurations for $n = 4$ for any choice of the masses and

Albouy and Kaloshin [4] the finiteness of the number of classes of central configurations for $n = 5$ for almost any choice of the masses.

In this paper we deal with another of the important open questions related with central configurations, collected also in [2] as Problem 12 and proposed first by Chenciner in 2004 [7]. The question is

Is there any central configuration of the n -body problem, different from the n -gon with equal masses, with all the masses lying on a common circle with origin at its center of mass?

This question arose while Chenciner studied a particular kind of solutions of the n -body problem, called *choreographies*, in which all the bodies move on a common path. The n -gon with equal masses is the simplest choreography of the n -body problem, for more examples of choreographies see for instance [6, 13, 25].

It is easy to show that the open question proposed by Chenciner has positive answer for $n = 3$. Hampton in [15] proved that the question has positive answer for $n = 4$. For $n = 5$ the problem has been studied in [21]. Many other authors have been interested in proving the positive answer of the question for $n > 5$. For the planar n -vortex problem, Cors et al. [11] proved that the regular n -gon with equal-strength circulations is the only co-circular central configuration whose center of vorticity is located at the center of the circle containing the vortices. But only few partial results have been found for the Newtonian n -body problem. Next we give the most important ones. In what follows the central configurations with n masses lying on a common circle having the center of mass at the center of the circle will be called, as in [17], *centered co-circular central configurations*.

Cors et al. in [11] proved the following result for the family of power-law potentials of the form

$$U_\alpha(\mathbf{q}) = \sum_{i < j} \frac{m_i m_j}{r_{ij}^\alpha}. \quad (3)$$

Notice that $\alpha = 1$ corresponds to the Newtonian potential (2).

Theorem 1 *For any $\alpha > 0$ and given a set of positive masses, for each ordering of the bodies on the unit circle, there exist a unique centered co-circular central configuration.*

As a corollary of Theorem 1 (see again [11]), if all the masses are equal then for any $\alpha > 0$ the regular n -gon is the only centered co-circular central configuration.

Hampton in [17] proved that the regular n -gon with equal masses is isolated among the set of centered co-circular central configurations of the Newtonian n -body problem (i.e. $\alpha = 1$); in other words, small perturbations of the masses and of the positions of the regular n -gon with equal masses does not provide centered co-circular central configurations. He also proved that there are no centered co-circular central configurations of the $n + 1$ -body problem formed by n equal masses on a regular n -gon plus one infinitesimal mass.

In this paper we give more evidences than the question proposed by Chenciner has positive answer for all n . In particular we go a step further than Hampton and we prove that there are no centered co-circular central configurations with $n - 1$ equal masses and an arbitrary non equal mass, not necessarily small. We extend the result to general power-law potentials, and we get the following result.

Theorem 2 *For any $\alpha > 0$, there are no centered co-circular central configurations of the n -body problem with power law potentials (3) having all the masses equal except one.*

The paper is structured as follows. In Sect. 2 we give the equations of the centered co-circular central configurations. In Sect. 3 we give the proof of Theorem 2.

2 Equations for the Centered Co-circular Central Configurations

In this paper we will use the equations derived in [11], which are summarized in the following result.

Theorem 3 *Let*

$$V_\alpha = \sum_{i < j}^n \frac{m_i m_j}{r_{ij}^\alpha} \quad \text{with} \quad r_{ij} = \sqrt{2 - 2 \cos(\theta_j - \theta_i)} = 2 \sin\left(\frac{|\theta_j - \theta_i|}{2}\right),$$

and let $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ with $\mathbf{q}_i = (\cos \theta_i, \sin \theta_i)$ and $\theta_i \in [0, 2\pi)$ for $i = 1, \dots, n$ be a centered co-circular central configuration with masses m_1, \dots, m_n . Then there exists $\lambda \in \mathbb{R}$ such that the angles θ_i and the masses m_i satisfy the $2n$ equations

$$\frac{1}{\alpha m_i} \cdot \frac{\partial V_\alpha}{\partial \theta_i} = - \sum_{j \neq i}^n \frac{\delta_{ij} m_j \cos\left(\frac{\theta_j - \theta_i}{2}\right)}{r_{ij}^{\alpha+1}} = 0, \quad \text{for } i = 1, \dots, n, \tag{4}$$

$$\sum_{j \neq i}^n \frac{m_j}{r_{ij}^\alpha} = \frac{2\lambda}{\alpha}, \quad \text{for } i = 1, \dots, n, \tag{5}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } \theta_i > \theta_j, \\ -1 & \text{if } \theta_i < \theta_j. \end{cases}$$

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ and $\mathbf{m} = (m_1, \dots, m_n)$. We consider the function

$$F(\boldsymbol{\theta}, \mathbf{m}, \lambda) = V_\alpha - \frac{2\lambda}{\alpha} \sum_{i=1}^n m_i = \sum_{i < j}^n \frac{m_i m_j}{r_{ij}^\alpha} - \frac{2\lambda}{\alpha} \sum_{i=1}^n m_i.$$

It is easy to check that Eqs. (4) and (5) are equivalent, respectively, to the vectorial equations

$$\nabla_{\boldsymbol{\theta}} F = 0 \quad \text{and} \quad \nabla_{\mathbf{m}} F = 0. \tag{6}$$

Therefore the centered co-circular central configurations are solutions of (6).

Our aim is to analyze the solutions with all the masses equal except one. Without loss of generality we can take the unit of mass such that all the equal masses be equal to 1. Due to the invariance of central configurations under rotations, we can assume that the different mass is m_1 and that $\theta_1 = 0$. It is not restrictive to assume that the bodies are arranged so that

$$0 = \theta_1 < \theta_2 < \dots < \theta_n < 2\pi,$$

this corresponds to ordering the masses (m_1, \dots, m_n) counterclockwise around the circle.

We define $\tilde{\Omega} = \{\tilde{\boldsymbol{\theta}} \in [0, 2\pi)^{n-1} : 0 < \theta_2 < \dots < \theta_n < 2\pi\}$ and $\tilde{F} : \tilde{\Omega} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ where

$$\tilde{F}(\theta_2, \dots, \theta_n, m_1, \lambda) = F(0, \theta_2, \dots, \theta_n, m_1, 1, \dots, 1, \lambda)$$

and we consider the system of equations

$$\nabla_{m_1} \tilde{F} = 0 \quad \text{and} \quad \nabla_{\theta_i} \tilde{F} = 0 \quad \text{for } i = 2, \dots, n. \tag{7}$$

We observe that a solution of the centered co-circular central configurations, that is, of $\nabla_{\boldsymbol{\theta}} F = 0$ and $\nabla_{\mathbf{m}} F = 0$ with $\theta_1 = 0$ and $m_j = 1$ for $j = 2, \dots, n$ if it exists, must in particular satisfy (7).

3 Proof of Theorem 2

We start giving some preliminary results that will be useful in the proof of our main result.

Lemma 1 *The solutions of system (6) satisfy $F(\theta, \mathbf{m}, \lambda/2) = 0$.*

Proof By simple algebra it can be shown that

$$F(\theta, \mathbf{m}, \lambda/2) = \frac{1}{2} \mathbf{m} \cdot \nabla_{\mathbf{m}} F(\theta, \mathbf{m}, \lambda),$$

so the solutions of $\nabla_{\mathbf{m}} F = 0$ are zeroes of $F(\theta, \mathbf{m}, \lambda/2)$. □

Lemma 2 *Assume that $\tilde{\theta} = (\theta_2, \dots, \theta_n)$ for $i = 2, \dots, n$ is a solution of $\partial \tilde{F} / \partial \theta_j = 0$ for $j = 2, \dots, n$. Then $\theta_j = \theta_j(m_1)$ is an analytic function of the variable m_1 for $j = 2, \dots, n$. Moreover, we have*

$$\begin{pmatrix} \frac{\partial^2 \tilde{F}}{\partial \theta_2 \partial m_1} \\ \vdots \\ \frac{\partial^2 \tilde{F}}{\partial \theta_n \partial m_1} \end{pmatrix} = -A \begin{pmatrix} \frac{\partial \theta_2}{\partial m_1} \\ \vdots \\ \frac{\partial \theta_n}{\partial m_1} \end{pmatrix} \tag{8}$$

where $A = (a_{ij})_{2 \leq i, j \leq n}$ with

$$a_{ii} = \frac{\partial^2 F}{\partial \theta_i^2}, \quad a_{ij} = a_{ji} = \frac{\partial^2 F}{\partial \theta_j \partial \theta_i}. \tag{9}$$

Proof The theorem follows readily from the implicit function theorem together with the fact that the Hessian matrix $A = (a_{ij})_{2 \leq i, j \leq n}$ given in (9) coincides with the Hessian matrix of V_α with respect to the variables θ_j for $j = 2, \dots, n$ with $\theta_1 = 0$ and so $\det A \neq 0$ (it was proved by Cors et al. in [11] that if $\theta_1 = 0$, then $\mathbf{u}^T A \mathbf{u} > 0$ for all $\mathbf{u} \neq 0$). □

Note that the solutions $\theta_j = \theta_j(m_1)$ for $j = 2, \dots, n$ given by Lemma 2 do not depend on λ and that we can rewrite (8) as

$$\frac{\partial^2 \tilde{F}}{\partial \theta_j \partial m_1} = - \sum_{i=2}^n \frac{\partial^2 \tilde{F}}{\partial \theta_j \partial \theta_i} \frac{\partial \theta_i}{\partial m_1}, \quad j = 2, \dots, n. \tag{10}$$

Proof of Theorem 2 Assume that $\tilde{\theta} = (\theta_2(m_1), \dots, \theta_n(m_1))$ is the unique solution of $\nabla_{\theta_i} \tilde{F} = 0$ for $i = 2, \dots, n$ given by Lemma 2. From equation $\nabla_{m_1} \tilde{F} = 0$ we get

$$\frac{\partial \tilde{F}}{\partial m_1} = \sum_{j=2}^n \frac{1}{r_{1j}^\alpha} - \frac{2\lambda}{\alpha} = 0.$$

Isolating λ we have

$$\lambda = \lambda(m_1) = \frac{\alpha}{2} \sum_{j=2}^n \frac{1}{r_{1j}^\alpha}.$$

So, given m_1 there exist unique $\theta_j(m_1)$ and $\lambda(m_1)$ for $j = 2, \dots, n$ which are a solution of $\nabla_{\theta_i} \tilde{F} = 0$ and $\nabla_{m_1} \tilde{F} = 0$.

Note that if $\theta_j(m_1)$ is constant for $j = 2, \dots, n$ then $\lambda(m_1)$ is also constant. Substituting this solution into equation $\partial F/\partial m_2 = 0$ we find a unique solution m_1 . So system (6) has at most one solution and since the n -gon with equal masses is a solution of system (6), the possible solution that we have found is in fact a solution of (6) and it is the n -gon.

Assume now that not all the functions $\theta_j(m_1)$ are constant and we compute $\partial\lambda/\partial m_1$. Clearly

$$\frac{\partial\lambda}{\partial m_1} = \frac{\alpha}{2} \sum_{j=2}^n \frac{\partial}{\partial\theta_j} \left(\frac{1}{r_{1j}^\alpha} \right) \frac{\partial\theta_j}{\partial m_1}.$$

Since

$$\frac{\partial^2 \tilde{F}}{\partial m_1 \partial \theta_j} = \frac{\partial}{\partial \theta_j} \left(\frac{1}{r_{1j}^\alpha} \right),$$

using (10) and (8) we have that

$$\begin{aligned} \frac{\partial\lambda}{\partial m_1} &= \frac{\alpha}{2} \sum_{j=2}^n \frac{\partial^2 \tilde{F}}{\partial m_1 \partial \theta_j} \frac{\partial\theta_j}{\partial m_1} \\ &= -\frac{\alpha}{2} \sum_{j=2}^n \sum_{i=2}^n \frac{\partial^2 \tilde{F}}{\partial \theta_j \partial \theta_i} \frac{\partial\theta_j}{\partial m_1} \frac{\partial\theta_i}{\partial m_1} \\ &= -\frac{\alpha}{2} \left(\frac{\partial\theta_2}{\partial m_1} \dots \frac{\partial\theta_n}{\partial m_1} \right) A \begin{pmatrix} \frac{\partial\theta_2}{\partial m_1} \\ \vdots \\ \frac{\partial\theta_n}{\partial m_1} \end{pmatrix} < 0 \end{aligned} \tag{11}$$

because the matrix A is positive definite.

Since any centered co-circular central configuration is a zero of $F(\theta, \mathbf{m}, \lambda/2)$ (see Lemma 1), the unique solution $\theta_j(m_1)$ and $\lambda(m_1)$ for $j = 2, \dots, n$ of $\nabla_{\theta_i} \tilde{F} = 0$ and $\nabla_{m_1} \tilde{F} = 0$ defined above should be a zero of the function

$$\bar{F}(m_1) = V_\alpha - \frac{\lambda}{\alpha} \sum_{i=1}^n m_i = m_1 \sum_{j=2}^n \frac{1}{r_{1j}^\alpha} + \sum_{2 < j} \frac{1}{r_{ij}^\alpha} - \frac{\lambda}{\alpha} (n - 1 + m_1).$$

Note that $\bar{F}(m_1)$ is a function in the variable m_1 . We want to show that

$$\frac{d\bar{F}(m_1)}{dm_1} > 0.$$

This will clearly imply that $\bar{F}(m_1)$ at most has one solution. As above, since the n -gon with equal masses is a solution of system (6), the possible solution that we have found is in fact a solution of (6) and it is the n -gon, so the proof will follow.

Clearly,

$$\begin{aligned}
 \frac{d\bar{F}}{dm_1} &= \sum_{j=2}^n \frac{\partial V_\alpha}{\partial \theta_j} \frac{\partial \theta_j}{\partial m_1} + \frac{\partial V_\alpha}{\partial m_1} + \frac{\partial \bar{F}}{\partial \lambda} \frac{\partial \lambda}{\partial m_1} - \frac{\lambda}{\alpha} \\
 &= \frac{\partial V_\alpha}{\partial m_1} + \frac{\partial \bar{F}}{\partial \lambda} \frac{\partial \lambda}{\partial m_1} - \frac{\lambda}{\alpha} \\
 &= \sum_{j=2}^n \frac{1}{r_{1j}^\alpha} - \frac{\lambda}{\alpha} + \frac{\partial \bar{F}}{\partial \lambda} \frac{\partial \lambda}{\partial m_1} \\
 &= \frac{2\lambda}{\alpha} - \frac{\lambda}{\alpha} - \frac{1}{\alpha} (n-1+m_1) \frac{\partial \lambda}{\partial m_1} \\
 &= \frac{\lambda}{\alpha} - \frac{1}{\alpha} (n-1+m_1) \frac{\partial \lambda}{\partial m_1} > 0
 \end{aligned}$$

in view of (11). Recall that, from (4), $\partial V_\alpha / \partial \theta_j$ evaluated at the solution $\theta_j(m_1)$ and $\lambda(m_1)$ for $j = 2, \dots, n$ is zero. This concludes the proof. \square

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