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Transition Fronts of Combustion Reaction Diffusion Equations in $\mathbb{R}^N$

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Abstract This paper is concerned with combustion transition fronts in $\mathbb{R}^N$ ($N \geq 1$). Firstly, we prove the existence and the uniqueness of the global mean speed which is independent of the shape of the level sets of the fronts. Secondly, we show that the planar fronts can be characterized in the more general class of almost-planar fronts. Thirdly, we show the existence of new types of transitions fronts in $\mathbb{R}^N$ which are not standard traveling fronts. Finally, we prove that all transition fronts are monotone increasing in time, whatever shape their level sets may have.

Keywords Reaction–diffusion equations · Combustion nonlinearity · Transition front · Qualitative properties

Mathematics Subject Classification 35K40 · 35K57 · 35C07 · 35B35 · 35B40

1 Introduction

This paper investigates reaction diffusion equations of the type

$$u_t = \Delta u + f(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

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where $N \in \mathbb{N}$, $u_t = \frac{\partial u}{\partial t}$ and Δ denotes the Laplace operator with respect to the space variables $x \in \mathbb{R}^N$. The nonlinear reaction term $f(u)$ is of the "ignition temperature" type, that is, $f : [0, 1] \to \mathbb{R}$ is a C^1 function such that

$$\exists\, \theta \in (0, 1),\ \ f \equiv 0 \text{ on } [0, \theta] \cup \{1\},\ \ f > 0 \text{ on } (\theta, 1) \text{ and } f'(1) < 0. \tag{2}$$

Such a profile can be derived from the Arrhenius kinetis with a cut-off for low temperatures and from the law of mass action. The real number θ is the ignition temperature, below which no reaction happens.

In any dimension $N \geq 1$, standard planar traveling fronts are solutions of the type

$$u(t, x) = \phi_f\left(x \cdot e - c_f t\right),$$

where e is any given unit vector of $\mathbb{R}^N$, $c_f \in \mathbb{R}$ is the propagation speed and $\phi_f : \mathbb{R} \to [0, 1]$ is the propagation profile, such that

$$\begin{aligned} &\phi_f'' + c_f \phi_f' + f\left(\phi_f\right) = 0 \quad \text{in } \mathbb{R}, \\ &\phi_f(-\infty) = 1 \ \text{ and } \ \phi_f(+\infty) = 0. \end{aligned} \tag{3}$$

The profile ϕ_f is then a heteroclinic connection between the state 0 and the stable state 1. The level sets of such traveling fronts are parallel hyperplanes which are orthogonal to the direction of the propagation e. These fronts are invariant in the moving frame with speed c_f in the direction e. It is well known [1] that such front exists and is unique up to translation. Besides, the speed c_f is positive which has the sign of $\int_0^1 f(s)ds$ [5] and the function ϕ_f is decreasing.

In $\mathbb{R}^N$ with $N \geq 2$, propagating wave fronts contains more types of traveling fronts except planar traveling fronts, such as V-shaped traveling fronts in two-dimensional spaces, pyramidal traveling fronts with non-axisymmetric shape in three-dimensional spaces and conical-shaped axisymmetric traveling fronts in high-dimensional spaces. The profiles of these fronts are still invariant in a moving frame with constant speed. But they have non-planar level sets. For instance, (1) admits the conical-shaped fronts of the type

$$u(t, x) = \phi(|x'|, x_N - ct),$$

where $x' = (x_1, \ldots, x_{N-1})$ and $|x'| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$ whose profiles are invariant and which have non-planar level sets. For the existence, uniqueness, stability and other qualitative properties of these non-planar traveling fronts, we refer to [7,8,12–14,24,25,33–36] and the references therein.

As we introduced above, Eq. (1) admits many types of traveling fronts. However, they have some common properties. For instance, the solutions u converge to the equilibrium states 0 or 1 far away from their moving or stationary level sets, uniformly in time. Their common properties led us to ask whether it is possible to introduce a more general notion of traveling fronts to include all types of waves. Berestycki and Hamel [3,4] give an affirmative answer. They introduce the general notion of transition fronts. Before we describe the definition of transition fronts, we firstly introduce some notions. For any two subsets A and B of $\mathbb{R}^N$ and for $x \in \mathbb{R}^N$, we set

$$d(A, B) = \inf\{|x - y|, (x, y) \in A \times B\} \tag{4}$$

and $d(x, A) = d(\{x\}, A)$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. Let $(\Omega_t^-)_{t\in\mathbb{R}}$ and $(\Omega_t^+)_{t\in\mathbb{R}}$ be two families of open nonempty subsets of $\mathbb{R}^N$, which satisfy

$$\forall\, t \in \mathbb{R}, \quad \begin{cases} \Omega_t^- \bigcap \Omega_t^+ = \emptyset, \\ \partial\Omega_t^- = \partial\Omega_t^+ =: \Gamma_t, \\ \Omega_t^- \bigcup \Gamma_t \bigcup \Omega_t^+ = \mathbb{R}^N, \\ \sup\{d(x, \Gamma_t)\big| x \in \Omega_t^+\} = \sup\{d(x, \Gamma_t)\big| x \in \Omega_t^-\} = +\infty \end{cases} \tag{5}$$

and

$$\begin{aligned} &\inf\{\sup\{d(y, \Gamma_t);\, y \in \Omega_t^+,\, |y - x| \le r\}\big| t \in \mathbb{R}, x \in \Gamma_t\} = +\infty, \\ &\inf\{\sup\{d(y, \Gamma_t);\, y \in \Omega_t^-,\, |y - x| \le r\}\big| t \in \mathbb{R}, x \in \Gamma_t\} = +\infty, \end{aligned} \quad r \to +\infty. \tag{6}$$

Notice that the condition (5) implies that the interface Γ_t is not empty for every $t \in \mathbb{R}$.

Definition 1 (*See* [3,4]) For problem (1), a transition front connecting 0 and 1 is a classical solution $u : \mathbb{R}\times\mathbb{R}^N \to (0, 1)$ for which there exist some sets $(\Omega_t^\pm)_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ satisfying (5) and (6), and, for every $\varepsilon > 0$, there exists $M \ge 0$ such that

$$\begin{aligned} &\forall\, t \in \mathbb{R},\ \forall\, x \in \Omega_t^+, \quad d(x, \Gamma_t) \ge M \Rightarrow u(t, x) \ge 1 - \varepsilon, \\ &\forall\, t \in \mathbb{R},\ \forall\, x \in \Omega_t^-, \quad d(x, \Gamma_t) \ge M \Rightarrow u(t, x) \le \varepsilon. \end{aligned} \tag{7}$$

Furthermore, u is said to have a global mean speed Λ (≥ 0) if

$$\frac{d(\Gamma_t, \Gamma_s)}{|t - s|} \to \Lambda \quad as\ |t - s| \to +\infty. \tag{8}$$

Remark 1 Notice that, for a given transition front u connecting 0 and 1, the sets $(\Omega_t^\pm)_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ are not uniquely determined. In fact, for any sets $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$, if

$$\sup_{t\in\mathbb{R}} \max\left(\sup_{x\in\Gamma_t} d(x, \widetilde{\Gamma}_t),\ \sup_{x\in\widetilde{\Gamma}_t} d(x, \Gamma_t) \right) < +\infty,$$

then the family $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$ with corresponding sets $(\widetilde{\Omega}_t^\pm)_{t\in\mathbb{R}}$ also satisfies (5), (6) and (7). That is, the solution u is also a transition front connecting 0 and 1 with the families $(\widetilde{\Omega}_t^\pm)_{t\in\mathbb{R}}$ and $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$.

Notice furthermore that for any transition front u connecting 0 and 1, the interfaces $(\Gamma_t)_{t\in\mathbb{R}}$ have uniformly bounded local oscillations, that is

$$\forall\, \sigma > 0,\ \ \sup\{d(\Gamma_t, \Gamma_s),\ t, s \in \mathbb{R},\ |t - s| \le \sigma\} < +\infty. \tag{9}$$

In fact, it is shown in Lemma 3 and Remark 3 of [10], in the case of reaction–diffusion equations (1) with nonlinearity f satisfying $f(u) > 0$ for $u \in (1 - \delta, 1)$, where $0 < \delta < 1$. Obviously, the assumptions of nonlinear reaction term f in this paper (see (2)) satisfy the above condition with $\delta = 1 - \theta$.

In [3,4,11], the authors have showed that all the known standard traveling fronts (planar and non-planar traveling fronts) are transition fronts in the sense of Definition 1. In particular, Hamel [11] proved that for Eq. (1) with bistable nonlinearity there exist new types of transition fronts in $\mathbb{R}^N$ which are not invariant in any frame as time runs. This property is different from standard traveling fronts which are invariant in a moving frame with constant speed. It also shows the broadness of Definition 1. In recent years, many papers have been devoted to the investigation of the existence and stability of transition fronts. For bistable transition fronts, we refer to [3,4,10,11]. For Fisher-KPP transition fronts, the readers can see [15,16,21–23,28,31,38]. Transition fronts for equations with combustion nonlinearity, the investigations mainly focus on the case of the heterogeneous equations in $\mathbb{R}$, see [19,20,29,30,32,37,39,40]. In this paper, we prove that even the homogeneous combustion equation (1) in $\mathbb{R}^N$ $(N \geq 1)$ also has many deep properties, such as the existence of new transition fronts and general estimates shared by all transition fronts.

The first main result of this paper proves the existence and uniqueness of the global mean speed for any transition fronts connecting the state 0 and the stable state 1, regardless of the shape of the level sets of the transition fronts.

Theorem 1 *For problem* (1)*, any transition front u connecting* 0 *and* 1 *has a global mean speed Λ. Furthermore, this global mean speed Λ is equal to c_f.*

The second result of this paper gives a characterization of the planar fronts $\phi_f(x \cdot e - c_f t)$ among the more general class of almost-planar transition fronts introduced in [4], and defined as follows.

Definition 2 (*See* [4,11]) A transition front u in the sense of Definition 1 is called almost-planar if, for every $t \in \mathbb{R}$, the set Γ_t can be chosen as the hyperplane

$$\Gamma_t = \left\{ x \in \mathbb{R}^N \middle| x \cdot e_t = \xi_t \right\}$$

for some vector e_t of the unit sphere $\mathbb{S}^{N-1}$ and some real number ξ_t.

From the definition, we can easily see that the level sets of almost-planar fronts are in some sense close to hyperplanes, even if they are not a priori assumed to be planar. The following theorem shows that planar fronts $\phi_f(x \cdot e - c_f t)$ for problem (1) fall within the more general class of almost-planar fronts.

Theorem 2 *For problem* (1)*, any almost-planar transition front u connecting* 0 *and* 1 *is planar, that is, there exist a unit vector e of $\mathbb{R}^N$ and a real number ξ such that*

$$u(t, x) = \phi_f(x \cdot e - c_f t + \xi) \quad \textit{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Thirdly, we show the broadness of transition fronts. In other words, we prove the existence of new types transition fronts of the Eq. (1), which are not invariant as time runs in any moving frame. Recall that the profiles of standard traveling fronts are invariant in a moving frame with constant speed.

Theorem 3 *Let $N \geq 2$. The problem* (1) *admits transition fronts u connecting* 0 *and* 1 *which satisfy the following property: there is no function $\Phi : \mathbb{R}^N \to (0, 1)$ (independent of t) for which there would be some families $(R_t)_{t \in \mathbb{R}}$ and $(x_t)_{t \in \mathbb{R}}$ of rotations and points in $\mathbb{R}^N$ such that $u(t, x) = \Phi(R_t(x - x_t))$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.*

Finally, we establish the time monotonicity of the transition front u.

Theorem 4 *For problem* (1)*, any transition front u connecting* 0 *and* 1 *is monotone increasing in time t. That is, $u_t > 0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.*

In fact, in order to prove Theorem 4, it is sufficient to prove that the transition front u is an invasion of the state 0 by the state 1, in the sense that the sets $(\Omega_t^{\pm})_{t \in \mathbb{R}}$ can be chosen so that

$$\Omega_t^+ \subset \Omega_s^+ \text{ for all } t < s \text{ and } d(\Gamma_t, \Gamma_s) \to +\infty \text{ as } |t - s| \to +\infty,$$

since it is easy to check that the problem (1) and the nonlinearity f satisfy all assumptions of [4, Theorem 1.11]. Similar to [10], it follows from Theorem 1 and Lemma 1 (see Sect. 2) that u is an invasion in the above sense with the families $\big(\widehat{\Omega}_t^{\pm}\big)_{t\in\mathbb{R}}$ and $\big(\widehat{\Gamma}_t\big)_{t\in\mathbb{R}}$, where for some constant $\tau_0 > 0$,

$$\begin{array}{ll}\widehat{\Omega}^{\pm}_{k\tau_0+t} = \Omega^{\pm}_{k\tau_0} & \text{for any } k \in \mathbb{Z} \text{ and } 0 \le t < \tau_0,\\ \widehat{\Gamma}_t := \partial\widehat{\Omega}_t^{+} = \widehat{\Omega}_t^{-} & \text{for any } t \in \mathbb{R}.\end{array}$$

Now we give a brief stated on the methods of our proofs. Firstly, in order to prove the existence and the uniqueness of the global mean speed of the transition fronts connecting 0 and 1, we need introduce two radially symmetric functions and show their dynamical properties, see Lemmas 1 and 2 below. Secondly, using the one-dimensional stability of the planar front and parabolic Liouville type result of Berestycki and Hamel [3, Theorem 3.1], we show that the planar fronts can be characterized by the more general class of almost-planar transition fronts. Thirdly, by mixing three planar fronts moving in three different directions, we show that the new transition fronts exist in dimension $N = 2$. And by trivially extending the two-dimensional solutions in the variables $x_3, \ldots, x_N$, we obtain that the new transition fronts exist in all dimensions $N \ge 3$.

Here we would like to point out that the main results of this paper (Theorems 1, 2, 3 and 4) are similar to those established for Eq. (1) with *bistable* nonlinearity by Hamel [11] and Guo and Hamel [10], where the reaction term $f : [0, 1] \to \mathbb{R}$ is a C^1 function such that

$$f(0) = f(1) = 0,\ \ f'(0) < 0 \text{ and } f'(1) < 0.$$

But in this paper we treat the combustion case, in particular, the reaction term f satisfies $f(u) = 0$ for any $u \in [0, \theta]$ with some $\theta \in (0, 1)$, which is essentially different from the assumption $f'(0) < 0$ in the bistable case. Since the signs of $f'(0)$ and $f'(1)$ play important roles in the estimates of speeds and constructing the super-sub solutions, some new difficulties occur in the combustion reaction diffusion equations. To overcome these difficulties, we need some new techniques and establish some new estimates. See Lemmas 1, 2, 4 and Proposition 1 below for more details.

The rest of this paper is organized as follows. Section 2 proves the existence and the uniqueness of the global mean speed among all transition fronts. That is, we give the proof of Theorem 1. In Sect. 3, we prove Theorem 2. That is, we give a characterization of the planar fronts among the more general class of almost-planar transition fronts. In Sect. 4, we construct new types transition fronts. That is, we are devoted to the proof of Theorem 3.

2 The Global Mean Speed

In this section, we prove that any transition front of the Eq. (1) has a global mean speed and this speed is unique. We first introduce auxiliary notations for some radially symmetric functions and we show some of their dynamical properties. The following two key properties, Lemmas 1 and 2 below, will provide a sharp lower bound and a upper bound for the speed of the interfaces Γ_t of any transition front connecting 0 and 1 for the problem (1), respectively.

In the following, let $\theta < \beta < 1$. For any $R > 0$, let v_R^f denote the solution of the Cauchy problem

$$\big(v_R^f\big)_t = \Delta v_R^f + f\big(v_R^f\big), \quad t > 0, \quad x \in \mathbb{R}^N, \tag{10}$$

with initial value

$$v_R^f(0,x)=\begin{cases}\beta, & |x|<R,\\ 0, & |x|\geq R.\end{cases} \tag{11}$$

Lemma 1 *There is $R>0$ such that the following holds: for any $\varepsilon\in(0,c_f]$, there is $T_\varepsilon>0$ such that*

$$v_R^f(t,x)\geq\beta \quad for\ all\ t\geq T_\varepsilon\ and\ |x|\leq(c_f-\varepsilon)t. \tag{12}$$

In fact,

$$v_R^f(t,\cdot)\to 1 \quad uniformly\ in\ \left\{x\in\mathbb{R}^N\middle||x|\leq(c_f-\varepsilon)t\right\}\ as\ t\to+\infty. \tag{13}$$

Proof Let g be any given $C^1([0,1])$ function which satisfies

$$g(0)=g(\theta)=g(1)=0,\ g'(0)<0,\ g'(1)<0,\ g'(\theta)>0,$$
$$g<0 \text{ on } (0,\theta),\ 0<g\leq f \text{ on } (\theta,1),\ \int_0^1 g(s)ds>0$$

and

$$0<c_f-c_g\leq\frac{\varepsilon}{2}, \tag{14}$$

where c_g is the wave speed of the planar front ϕ_g which satisfies (3) with the nonlinearity g. In fact, g is of the bistable type. Such fronts exist, see [2,9,17]. It is easy to see that $f\geq g$ on [0, 1]. Then the comparison principle implies that

$$1\geq v_R^f(t,x)\geq v_R^g(t,x),\quad \forall\,(t,x)\in[0,+\infty)\times\mathbb{R}^N. \tag{15}$$

For the solution v_R^g of the equation (10)-(11) with replacing f by g, it follows from Lemma 4.1 of [11] that we have

$$v_R^g(t,\cdot)\to 1 \quad \text{uniformly in } \left\{x\in\mathbb{R}^N\middle||x|\leq\left(c_g-\frac{\varepsilon}{2}\right)t\right\} \text{ as } t\to+\infty.$$

Inequalities (14) and (15), together with the above formula, yield that (13) holds. This completes the proof. □

Lemma 2 *For any $\varepsilon>0$, there exist some positive real numbers α_ε, T_ε and R_ε such that for all $R\geq R_\varepsilon$, the solution w_R of the following Cauchy problem*

$$(w_R)_t=\Delta w_R+f(w_R),\quad t>0,\quad x\in\mathbb{R}^N,$$

with initial value

$$w_R(0,x)=\begin{cases}\alpha_\varepsilon, & |x|<R,\\ 1, & |x|\geq R\end{cases}$$

satisfies

$$w_R(t,x)\leq 3\alpha_\varepsilon \quad for\ all\ T_\varepsilon\leq t\leq\frac{R}{c_f+\varepsilon}\ and\ |x|\leq R-(c_f+\varepsilon)t. \tag{16}$$

Proof Let δ be chosen so that

$$0<\delta<\frac{\theta}{2} \text{ and } f'\leq\frac{f'(1)}{2} \text{ on } [1-\delta,1]. \tag{17}$$

Since $\phi''_f(s) \sim \nu e^{-c_f s}$ as $s \to +\infty$ with $\nu > 0$, one can choose $C > 0$ such that

$$\phi_f \geq 1 - \delta \text{ on } (-\infty, -C],\ \phi_f \leq \delta \text{ on } [C, +\infty) \text{ and } \phi''_f \geq 0 \text{ on } [C, +\infty). \tag{18}$$

Since ϕ'_f is negative and continuous on $\mathbb{R}$, there is $\kappa > 0$ such that

$$-\phi'_f \geq \kappa > 0 \quad \text{on } [-C, C]. \tag{19}$$

Set $L = \max\limits_{u \in [0,1]} |f'(u)|$. For every $\varepsilon > 0$, let

$$0 < \alpha_\varepsilon < \min\left(\frac{\theta}{4}, \frac{\kappa\varepsilon}{8L}\right).$$

Choose $D_\varepsilon > 0$ such that

$$\phi_f \geq 1 - 2\alpha_\varepsilon \text{ on } (-\infty, -D_\varepsilon] \text{ and } \phi_f \leq \alpha_\varepsilon \text{ on } [D_\varepsilon, +\infty). \tag{20}$$

Let $\rho_{\alpha_\varepsilon}$ be the solution of the following ordinary differential equation

$$\rho'_{\alpha_\varepsilon}(t) = f(\rho_{\alpha_\varepsilon}),$$
$$\rho_{\alpha_\varepsilon}(0) = \alpha_\varepsilon.$$

Since $\alpha_\varepsilon \in (0, \theta)$, f is Lipschitz-continuous and $f \equiv 0$ on $[0, \theta]$, then $\rho_{\alpha_\varepsilon}(t) \equiv \alpha_\varepsilon$ by the existence and uniqueness of solution of the ordinary differential equation. It follows from the maximum principle and (2) that for any $R > 0$,

$$0 \leq \rho_{\alpha_\varepsilon}(t) \leq w_R(t, x) \leq 1 \quad \text{for all } t \geq 0, \quad x \in \mathbb{R}^N.$$

Then the following inequality holds

$$(w_R - \rho_{\alpha_\varepsilon})_t \leq \Delta(w_R - \rho_{\alpha_\varepsilon}) + L(w_R - \rho_{\alpha_\varepsilon}).$$

Thus for the above equation, the assumptions of the initial value yield

$$0 \leq w_R(t, x) - \rho_{\alpha_\varepsilon}(t) \leq \frac{e^{Lt}}{(4\pi t)^{\frac{N}{2}}} \int\limits_{|y| \geq R} e^{-\frac{|x-y|^2}{4t}} dy \quad \text{for all} \quad t > 0 \text{ and } x \in \mathbb{R}^N.$$

Therefore, if $0 < B \leq R$ and $|x| \leq R - B$, one infers that

$$0 \leq w_R(t, x) - \rho_{\alpha_\varepsilon}(t) \leq \frac{e^{Lt}}{(4\pi)^{\frac{N}{2}}} \int\limits_{|z| \geq \frac{B}{\sqrt{t}}} e^{-|z|^2} dz.$$

Thus, take a real number $T > 0$ and there exists $B > 0$ such that for all $R \geq B$ and $|x| \leq R - B$,

$$w_R(T, x) - \rho_{\alpha_\varepsilon}(T) \leq \alpha_\varepsilon,$$

whence

$$w_R(T, x) \leq \rho_{\alpha_\varepsilon}(T) + \alpha_\varepsilon = 2\alpha_\varepsilon \quad \text{for all } R \geq B \text{ and } |x| \leq R - B. \tag{21}$$

It is elementary to check that for every $\varepsilon > 0$, there is a C^2 function $h_\varepsilon : [0, +\infty) \to \mathbb{R}$ satisfying the following properties:

$$\begin{aligned} &0 \le h'_\varepsilon \le 1 \quad \text{on } [0, +\infty), \\ &h'_\varepsilon = 0 \quad \text{on a neighborhood of } 0, \\ &h_\varepsilon(r) = r \quad \text{on } [H_\varepsilon, +\infty) \text{ for some } H_\varepsilon > 0, \\ &\frac{(N-1)h'_\varepsilon(r)}{r} + h''_\varepsilon(r) \le \frac{\varepsilon}{4} \quad \text{on } [0, +\infty). \end{aligned} \tag{22}$$

Notice in particular that

$$r \le h_\varepsilon(r) \le r + h_\varepsilon(0) \quad \text{for all } r \ge 0. \tag{23}$$

We choose $T_\varepsilon > T > 0$ such that

$$\frac{\varepsilon t}{2} \ge h_\varepsilon(0) + B + 2D_\varepsilon \quad \text{for all } t \ge T_\varepsilon, \tag{24}$$

and $R_\varepsilon > 0$ such that

$$R_\varepsilon \ge \max(B, (c_f + \varepsilon)T_\varepsilon) \quad \text{and} \quad \frac{\varepsilon R_\varepsilon}{2(c_f + \varepsilon)} \ge B + D_\varepsilon + C + H_\varepsilon. \tag{25}$$

In the sequel, R is arbitrary real number such that

$$R \ge R_\varepsilon. \tag{26}$$

For all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, we set

$$\overline{W}(t, x) = \min\left(\phi_f(\bar{\xi}(t, x)) + 2\alpha_\varepsilon, 1\right),$$

where

$$\bar{\xi}(t, x) = -h_\varepsilon(|x|) - \left(c_f + \frac{\varepsilon}{2}\right)(t - T) + R - B - D_\varepsilon.$$

In the set $\Sigma = \left[T, \frac{R}{c_f+\varepsilon}\right] \times \mathbb{R}^N$, let us then check that $\overline{W}$ is a supersolution for the problem satisfied by w_R.

Since $f(1) = 0$, it is sufficient to check that

$$\mathscr{L}(t, x) = \overline{W}_t(t, x) - \Delta\overline{W}(t, x) - f(\overline{W}(t, x)) \ge 0 \text{ for all } (t, x) \in \Sigma \text{ such that } \overline{W}(t, x) < 1.$$

Since ϕ_f is of class C^2 and h vanishes in the neighborhood of 0, then $\overline{W}(t, x) = \phi_f(\bar{\xi}(t, x)) + 2\alpha_\varepsilon$ is of class C^2 in the set where $\overline{W}(t, x) < 1$.

In this paragraph, let (t, x) be any point in Σ such that $\overline{W}(t, x) < 1$. Since $\phi''_f + c_f\phi'_f + f(\phi_f) = 0$ in $\mathbb{R}$, then by (22) and $\phi'_f \le 0$, there holds

$$\begin{aligned} \mathscr{L}(t, x) =& f(\phi_f(\bar{\xi}(t, x))) - f(\overline{W}(t, x)) + (1 - (h'_\varepsilon(|x|))^2)\phi''_f(\bar{\xi}(t, x)) \\ &- \left(\frac{\varepsilon}{2} - \frac{(N-1)h'_\varepsilon(|x|)}{|x|} - h''_\varepsilon(|x|)\right)\phi'_f(\bar{\xi}(t, x)) \\ \ge& f(\phi_f(\bar{\xi}(t, x))) - f(\overline{W}(t, x)) - \frac{\varepsilon}{4}\phi'_f(\bar{\xi}(t, x)) + (1 - (h'_\varepsilon(|x|))^2)\phi''_f(\bar{\xi}(t, x)). \end{aligned}$$

Firstly, if $\bar{\xi}(t, x) \le -C$, then (18) and the definition of $\overline{W}$ yield $1 - \delta \le \phi_f(\bar{\xi}(t, x)) \le \overline{W}(t, x) < 1$. Whence by (17), one gets

$$f(\phi_f(\bar{\xi}(t, x))) - f(\overline{W}(t, x)) \ge -f'(1)\alpha_\varepsilon.$$

In addition, it follows from (25) and (26) that the inequalitues $\bar{\xi}(t,x) \le -C$ and $T \le t \le \frac{R}{c_f+\varepsilon}$ yield

$$h_\varepsilon(|x|) \ge -\left(c_f + \frac{\varepsilon}{2}\right)(t-T) + R - B - D_\varepsilon + C \ge \frac{\varepsilon R}{2(c_f+\varepsilon)} - B - D_\varepsilon + C \ge H_\varepsilon.$$

From the properties (22), the inequality $h_\varepsilon(|x|) \ge H_\varepsilon$ implies that $h'_\varepsilon(|x|) = 1$. Therefore, if $\bar{\xi}(t,x) \le -C$, then $\phi'_f \le 0$ implies

$$\mathscr{L}(t,x) \ge -f'(1)\alpha_\varepsilon - \frac{\varepsilon}{4}\phi'_f(\bar{\xi}(t,x)) \ge 0.$$

Secondly, if $\bar{\xi}(t,x) \ge C$, then by (18), $\phi_f(\bar{\xi}(t,x)) \le \delta$. Thus,

$$0 < \phi_f(\bar{\xi}(t,x)) \le \overline{W}(t,x) \le \delta + 2\alpha_\varepsilon < \theta.$$

Since $f = 0$ on $[0,\theta]$, $\phi''_f \ge 0$ on $[C,+\infty)$ from (18), $0 \le h'_\varepsilon(|x|) \le 1$ on $[0,+\infty)$ and $\phi'_f \le 0$ on $\mathbb{R}$, one gets that, if $\bar{\xi}(t,x) \ge C$, then

$$\mathscr{L}(t,x) \ge -\frac{\varepsilon}{4}\phi'_f(\bar{\xi}(t,x)) + (1-(h'_\varepsilon(|x|))^2)\phi''_f(\bar{\xi}(t,x)) \ge 0.$$

Lastly, if $-C \le \bar{\xi}(t,x) \le C$, then

$$f(\phi_f(\bar{\xi}(t,x))) - f(\overline{W}(t,x)) \ge -2L\alpha_\varepsilon,$$

recall that $L = \max_{u\in[0,1]} |f'(u)|$. It follows from (24) and (26) that $\bar{\xi}(t,x) \le C$ and $T \le t \le \frac{R}{c_f+\varepsilon}$ imply

$$h_\varepsilon(|x|) \ge -\left(c_f + \frac{\varepsilon}{2}\right)(t-T) + R - B - D_\varepsilon - C \ge \frac{\varepsilon R}{2(c_f+\varepsilon)} - B - D_\varepsilon - C \ge H_\varepsilon.$$

Thus by (22), $h'_\varepsilon(|x|) = 1$. Consequently, it follows from the definition of α_ε and (19) that

$$\mathscr{L}(t,x) \ge -2L\alpha_\varepsilon + \frac{\kappa\varepsilon}{4} \ge 0.$$

On the other hand, at the time T, it follows from (21), (24), (26) and the definition of $\overline{W}$ that

$$w_R(T,x) \le 2\alpha_\varepsilon \le \overline{W}(T,x) \quad \text{for all } |x| \le R - B.$$

If $|x| \ge R - B$, then $h_\varepsilon(|x|) \ge |x| \ge R - B$ from (23), whence $\bar{\xi}(T,x) \le -D_\varepsilon$ and

$$\overline{W}(T,x) = \min\left(\phi_f(\bar{\xi}(T,x)) + 2\alpha_\varepsilon, 1\right) \ge \min\left((1-2\alpha_\varepsilon) + 2\alpha_\varepsilon, 1\right) = 1 \ge w_R(T,x)$$

from (20) and the fact that $w_R \le 1$ on $(0,+\infty) \times \mathbb{R}^N$. Thus

$$w_R(T,x) \le \overline{W}(T,x) \quad \text{for all } x \in \mathbb{R}^N.$$

As a conclusion, the maximum principle implies that, for all $T \le t \le \frac{R}{c_f+\varepsilon}$ and $x \in \mathbb{R}^N$,

$$w_R(t,x) \le \overline{W}(t,x) \le \phi_f(\bar{\xi}(t,x)) + 2\alpha_\varepsilon.$$

For all $T_\varepsilon \le t \le \frac{R}{c_f+\varepsilon}$ and $|x| \le R - (c_f+\varepsilon)t$, there hold

$$h_\varepsilon(|x|) \le |x| + h_\varepsilon(0) \le R - (c_f+\varepsilon)t + h_\varepsilon(0)$$

and

$$\begin{aligned}\bar{\xi}(t,x) \geq & -R + (c_f+\varepsilon)t - h_\varepsilon(0) - \left(c_f + \frac{\varepsilon}{2}\right)(t-T) + R - B - D_\varepsilon \\ \geq & \frac{\varepsilon t}{2} - h_\varepsilon(0) - B - D_\varepsilon \\ \geq & D_\varepsilon\end{aligned}$$

from (24). Thus, (20) yields $\phi_f(\bar{\xi}(t,x)) \leq \alpha_\varepsilon$. Whence, if $T_\varepsilon \leq t \leq \frac{R}{c_f+\varepsilon}$ and $|x| \leq R - (c_f+\varepsilon)t$, then

$$w_R(t,x) \leq \phi_f(\bar{\xi}(t,x)) + 2\alpha_\varepsilon \leq \alpha_\varepsilon + 2\alpha_\varepsilon = 3\alpha_\varepsilon.$$

This completes the proof. □

Proof of Theorem 1 Let u be any transition front of problem (1) which connects the equilibrium points 0 and 1. For any $\varepsilon \in (0, c_f]$, let α_ε be defined as in Lemma 2 and $\theta < \beta < 1$. It follows from Definition 1 that there is $M \geq 0$ such that

$$\begin{aligned}&\forall\, t \in \mathbb{R},\ \forall\, x \in \overline{\Omega_t^+},\quad d(x,\Gamma_t) \geq M \Rightarrow \beta \leq u(t,x) < 1,\\ &\forall\, t \in \mathbb{R},\ \forall\, x \in \overline{\Omega_t^-},\quad d(x,\Gamma_t) \geq M \Rightarrow 0 < u(t,x) \leq \alpha_\varepsilon.\end{aligned} \tag{27}$$

Let $R > 0$ be as in Lemma 1. Without loss of generality, one can assume that $R \geq M$ (since the functions v_R^f are nondecreasing with respect to the parameter $R > 0$). By (6), there exists a real number $r > 0$ such that

$$\forall\, t \in \mathbb{R},\quad \forall\, x \in \Gamma_t,\quad \exists\, y^\pm \in \Omega_t^\pm,\quad |x - y^\pm| \leq r \quad \text{and} \quad d(y^\pm, \Gamma_t) \geq 2R. \tag{28}$$

Our goal is to prove

$$\frac{d(\Gamma_t,\Gamma_s)}{|t-s|} \to c_f \quad \text{as } |t-s| \to +\infty.$$

For this purpose, we divide our proof into two steps. In the first step, we prove inequality for the lim inf. At the second step, we show inequality for the lim sup.

Step 1. the lower estimate We show that

$$\liminf_{|t-s|\to+\infty} \frac{d(\Gamma_t,\Gamma_s)}{|t-s|} \geq c_f. \tag{29}$$

We assume that (29) does not hold, then one has

$$\liminf_{|t-s|\to+\infty} \frac{d(\Gamma_t,\Gamma_s)}{|t-s|} < c_f - 2\varepsilon \tag{30}$$

for some $\varepsilon > 0$ small enough. Thus, there exist two sequences $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ such that $|t_k - s_k| \to +\infty$ as $k \to +\infty$ and

$$d(\Gamma_{t_k}, \Gamma_{s_k}) < (c_f - 2\varepsilon)|t_k - s_k| \quad \text{for } k \text{ large enough}.$$

Without loss of generality, we assume that $t_k < s_k$ for all $k \in \mathbb{N}$. The definition of the distance $d(\Gamma_{t_k}, \Gamma_{s_k})$ implies that there exist two sequences $(x_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$x_k \in \Gamma_{t_k},\ z_k \in \Gamma_{s_k} \text{ and } |x_k - z_k| < (c_f - 2\varepsilon)(s_k - t_k) \quad \text{for } k \text{ large enough}.$$

First of all, by (28), there exists a sequence $(y_k^+)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that

$$y_k^+ \in \Omega_{t_k}^+,\ |x_k - y_k^+| \leq r \text{ and } d(y_k^+, \Gamma_{t_k}) \geq 2R \text{ for all } k \in \mathbb{N}.$$

Thus, for every $k \in \mathbb{N}$ and $y \in B(y_k^+, R)$, one has $y \in \Omega_{t_k}^+$ and $d(y, \Gamma_{t_k}) \geq R \geq M$, whence $u(t_k, y) \geq \beta$ from (27). By (11), one has

$$u(t_k, x) \geq v_R^f(0, x - y_k^+), \quad x \in \mathbb{R}^N.$$

Thus the maximum principle yields

$$u(t, x) \geq v_R^f(t - t_k, x - y_k^+) \quad \text{for all } t > t_k \text{ and } x \in \mathbb{R}^N.$$

Let T_ε be defined as in Lemma 1, thus Lemma 1 yields that for every $k \in \mathbb{N}$,

$$u(t, x) \geq \beta \quad \text{for all } t \geq t_k + T_\varepsilon \text{ and } |x - y_k^+| \leq (c_f - \varepsilon)(t - t_k). \tag{31}$$

Next, it follows from (28) that there exists a sequence $(y_k^-)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that

$$y_k^- \in \Omega_{s_k}^-, \ |z_k - y_k^-| \leq r \ \text{ and } \ d(y_k^-, \Gamma_{s_k}) \geq 2R \geq M \ \text{ for all } k \in \mathbb{N}.$$

Property (27) implies that

$$u(s_k, y_k^-) \leq \alpha_\varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{32}$$

Finally, notice that for all $k \in \mathbb{N}$,

$$|y_k^- - y_k^+| \leq |y_k^- - z_k| + |z_k - x_k| + |x_k - y_k^+| \leq r + (c_f - 2\varepsilon)(s_k - t_k) + r$$

Thus, it follows from $s_k - t_k \to +\infty$ as $k \to +\infty$ that $s_k \geq t_k + T_\varepsilon$ for k large enough and

$$|y_k^- - y_k^+| \leq (c_f - \varepsilon)(s_k - t_k) \quad \text{for } k \text{ large enough.}$$

Choose $t = s_k$ and $x = y_k^-$ in (31) for k large enough. Thus,

$$u(s_k, y_k^-) \geq \beta \quad \text{for } k \text{ large enough.}$$

But $\alpha_\varepsilon < \beta$ contradicting (32). Therefore, the assumption (30) cannot hold. That is,

$$\liminf_{|t-s|\to+\infty} \frac{d(\Gamma_t, \Gamma_s)}{|t - s|} \geq c_f.$$

Step 2: the upper estimate We show that

$$\limsup_{|t-s|\to+\infty} \frac{d(\Gamma_t, \Gamma_s)}{|t - s|} \leq c_f. \tag{33}$$

Let us assume by contradiction that

$$\limsup_{|t-s|\to+\infty} \frac{d(\Gamma_t, \Gamma_s)}{|t - s|} > c_f + 3\varepsilon \tag{34}$$

for some $\varepsilon > 0$ small enough. Then there exist two sequences $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ of real numbers such that $|t_k - s_k| \to +\infty$ as $k \to +\infty$ and

$$d(\Gamma_{t_k}, \Gamma_{s_k}) > (c_f + 3\varepsilon)|t_k - s_k| \quad \text{for } k \text{ large enough.}$$

Without loss of generality, one can assume that $t_k < s_k$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, pick a point z_k on Γ_{s_k}. It follows from (28) that there are two sequences $(y_k^\pm)_{k\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that

$$y_k^\pm \in \Omega_{s_k}^\pm, \ |z_k - y_k^\pm| \leq r \text{ and } d(y_k^\pm, \Gamma_{s_k}) \geq M \quad \text{for all } k \in \mathbb{N}.$$

Thus, by (27), one has

$$0 < u(s_k, y_k^-) \leq \alpha_\varepsilon < 3\alpha_\varepsilon < \theta < \beta \leq u(s_k, y_k^+) < 1 \quad \text{for all } k \in \mathbb{N}. \tag{35}$$

It follows from $d(z_k, \Gamma_{t_k}) > (c_f + 3\varepsilon)(s_k - t_k) > 0$ for k large enough that there holds

$$\text{either } B(z_k, (c_f + 3\varepsilon)(s_k - t_k)) \subset \Omega^+_{t_k} \text{ or } B(z_k, (c_f + 3\varepsilon)(s_k - t_k)) \subset \Omega^-_{t_k}.$$

We claim that $B(z_k, (c_f + 3\varepsilon)(s_k - t_k)) \subset \Omega^-_{t_k}$ for k large enough. If not, up to extraction of a subsequence,

$$B(z_k, (c_f + 3\varepsilon)(s_k - t_k)) \subset \Omega^+_{t_k} \quad \text{for all } k \text{ large enough.} \tag{36}$$

Since $s_k - t_k \to +\infty$ as $k \to +\infty$, then for k large enough,

$$B(z_k, R) \subset \Omega^+_{t_k} \text{ and } d(y, \Gamma_{t_k}) \geq M \quad \text{for all } y \in B(z_k, R),$$

recall that $R > 0$ is defined as in Lemma 1. Thus for k large enough,

$$u(t_k, y) \geq \beta \quad \text{for all } y \in B(z_k, R),$$

from (27). It follows from (11) that one has

$$u(t_k, x) \geq v^f_R(0, x - z_k), \quad \text{for all } x \in \mathbb{R}^N.$$

By the maximum principle, one gets

$$u(t, x) \geq v^f_R(t - t_k, x - z_k) \quad \text{for all } t > t_k \text{ and } x \in \mathbb{R}^N. \tag{37}$$

Let $T_{\varepsilon'} > 0$ be defined as in Lemma 1 with $\varepsilon' = \frac{c_f}{2}$. The inequality (37) and Lemma 1 yield that, for k large enough,

$$u(t, x) \geq \beta \text{ for all } t \geq t_k + T_{\varepsilon'} \text{ and } |x - z_k| \leq (c_f - \varepsilon')(t - t_k) = \frac{c_f}{2}(t - t_k).$$

Since $c_f > 0$ and $s_k - t_k \to +\infty$ as $k \to +\infty$, then for k large enough,

$$s_k \geq t_k + T_{\varepsilon'} \text{ and } |y^-_k - z_k| \leq r \leq \frac{c_f}{2}(s_k - t_k).$$

Thus, the previous inequality implies $u(s_k, y^-_k) \geq \beta$ for k large enough. This is in contradiction with (35). Whence for k large enough,

$$B(z_k, (c_f + 3\varepsilon)(s_k - t_k)) \subset \Omega^-_{t_k}.$$

Since $s_k - t_k \to +\infty$ as $k \to +\infty$, then for k large enough,

$$B(z_k, (c_f + 2\varepsilon)(s_k - t_k)) \subset \Omega^-_{t_k} \text{ and } d(y, \Gamma_{t_k}) \geq M \text{ for all } y \in B(z_k, (c_f + 2\varepsilon)(s_k - t_k)).$$

Thus by (27), one has

$$u(t_k, y) \leq \alpha_\varepsilon \quad \text{for all } y \in B(z_k, (c_f + 2\varepsilon)(s_k - t_k)).$$

It follows from the definition of $w_{(c_f+2\varepsilon)(s_k-t_k)}$ (as in Lemma 1) that for k large enough,

$$u(t_k, x) \leq w_{(c_f+2\varepsilon)(s_k-t_k)}(0, x - z_k) \quad \text{for all } x \in \mathbb{R}^N.$$

Thus the maximum principle implies

$$u(t, x) \leq w_{(c_f+2\varepsilon)(s_k-t_k)}(t - t_k, x - z_k) \quad \text{for all } t > t_k \text{ and } x \in \mathbb{R}^N.$$

Since $s_k - t_k \to +\infty$ as $k \to +\infty$, then for k large enough, one has $(c_f + 2\varepsilon)(s_k - t_k) \geq R_\varepsilon$ and

$$T_\varepsilon \leq s_k - t_k \leq \frac{(c_f + 2\varepsilon)(s_k - t_k)}{c_f + \varepsilon},$$
$$|y_k^+ - z_k| \leq r \leq \varepsilon(s_k - t_k) = (c_f + 2\varepsilon)(s_k - t_k) - (c_f + \varepsilon)(s_k - t_k),$$

where $T_\varepsilon > 0$ and $R_\varepsilon \geq (c_f + \varepsilon)T_\varepsilon > 0$ are given by Lemma 2 so that (16) is valid for all $R \geq R_\varepsilon$. Choose $R = (c_f + 2\varepsilon)(s_k - t_k)$, $t = s_k - t_k$ and $x = y_k^+ - z_k$ in (16) for k large enough, one can obtain

$$u(s_k, y_k^+) \leq w_{(c_f+2\varepsilon)(s_k-t_k)}(s_k - t_k, y_k^+ - z_k) \leq 3\alpha_\varepsilon.$$

This is in contradiction with (35). Whence the conclusion (33) follows.

Combining with the *Step 1* and *Step 2*, the proof of Theorem 1 is thereby complete.

3 Almost-Planar Fronts

In this section, we characterize the planar fronts $\phi_f(x \cdot e - c_f t)$ for Eq. (1) among the more general class of almost-planar fronts. The proof of Theorem 2 mainly uses the one-dimensional stability of the planar front ϕ_f [18] and the parabolic Liouville type result of Berestycki and Hamel [3, Theorem 3.1]. Before the proof, we first give some auxiliary lemmas.

Lemma 3 *Let $u : \mathbb{R} \times \mathbb{R}^N \to [0, 1]$ be a solution of* (1) *for which there are a real number $t_0 \in \mathbb{R}$ and a unit vector $e \in \mathbb{S}^{N-1}$ such that*

$$\sup_{x\in\mathbb{R}^N, x\cdot e\geq A} u(t_0, x) \to 0 \quad \left(resp. \inf_{x\in\mathbb{R}^N, x\cdot e\leq -A} u(t_0, x) \to 1 \right) \quad as\ A \to +\infty. \tag{38}$$

Then property (38) *holds at every time $t_1 > t_0$ with the same vector e.*

Proof Since for the case $\inf_{x\in\mathbb{R}^N, x\cdot e\leq -A} u(t_0, x) \to 1$ as $A \to +\infty$, the proof of Lemma 3 is similar to [11, Lemma 3.1], we only give the proof for the case $\sup_{x\in\mathbb{R}^N, x\cdot e\geq A} u(t_0, x) \to 0$ as $A \to +\infty$.

For any $\delta \in (0, 1)$, let v^δ be the solution of the following one-dimensional Cauchy problem

$$v_t^\delta = v_{yy}^\delta + f(v^\delta), \quad t > 0,\ y \in \mathbb{R},$$
$$v^\delta(0, y) = \begin{cases} 1, & y \leq 0, \\ \delta, & y > 0. \end{cases}$$

Let $\rho^\delta : \mathbb{R} \to (0, 1)$ be the solution of the following ordinary differential equation

$$\left(\rho^\delta\right)'(t) = f(\rho^\delta(t)), \quad t > 0,$$
$$\rho^\delta(0) = \delta.$$

Then by the maximum principle, one has

$$0 \leq \rho^\delta(t) \leq v^\delta(t, y) \leq 1, \quad t \geq 0, \quad y \in \mathbb{R}.$$

Thus

$$0 \le v^\delta(t,x) - \rho^\delta(t) \le \frac{e^{Lt}}{(4\pi t)^{\frac{N}{2}}} \int_{y\le 0} e^{-\frac{|x-y|^2}{4t}} dy,$$

where $L = \max_{u\in[0,1]} |f'(u)|$. Then the maximum principle and standard parabolic estimates imply that for each $t > 0$, $v^\delta(t,\cdot)$ is decreasing in $\mathbb{R}$, $v^\delta(t,-\infty) = 1$ and $v^\delta(t,+\infty) = \rho^\delta(t)$.

Assume that $\sup_{x\in\mathbb{R}^N, x\cdot e\ge A} u(t_0,x) \to 0$ as $A \to +\infty$. Let $\varepsilon \in (0,\theta)$ be arbitrary. Then there exists a constant M such that

$$u(t_0,x) \le v^\varepsilon(0, x\cdot e - M), \quad x \in \mathbb{R}^N.$$

Thus it follows from the maximum principle that

$$u(t_1,x) \le v^\varepsilon(t_1 - t_0, x\cdot e - M), \quad t_1 > t_0,\ x \in \mathbb{R}^N,$$

and whence

$$\limsup_{A\to+\infty} \left(\sup_{x\in\mathbb{R}^N, x\cdot e\ge A} u(t_1,x) \right) \le v^\varepsilon(t_1 - t_0, +\infty) = \rho^\varepsilon(t_1 - t_0), \quad t_1 > t_0.$$

Since $\varepsilon \in (0,\theta)$, f is C^1 on $[0,1]$ and $f = 0$ on $[0,\theta]$, then the existence and uniqueness of solution of the ordinary differential equation yield that $\rho^\varepsilon(t) \equiv \varepsilon$ for all $t \ge 0$. Therefore, one has

$$0 \le \limsup_{A\to+\infty} \left(\sup_{x\in\mathbb{R}^N, x\cdot e\ge A} u(t_1,x) \right) \le \rho^\varepsilon(t_1 - t_0) = \varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

This completes the proof. □

The following corollary can be obtained immediately from Lemma 3.

Corollary 1 *Let $u : \mathbb{R} \times \mathbb{R}^N \to [0,1]$ be a solution of (1) such that, for every time $t \in \mathbb{R}$, there is a unit vector $e_t \in \mathbb{S}^{N-1}$ such that*

$$\inf_{x\in\mathbb{R}^N, x\cdot e_t\le -A} u(t,x) \to 1 \quad \textit{and} \quad \sup_{x\in\mathbb{R}^N, x\cdot e_t\ge A} u(t,x) \to 0 \quad \textit{as } A \to +\infty. \tag{39}$$

Then $e_t = e$ is independent of time t.

Let u be an almost-planar transition front connecting 0 and 1, in the sense of Definition 2, for problem (1). That is, there exist some families $(e_t)_{t\in\mathbb{R}}$ in $\mathbb{S}^{N-1}$ and $(\xi_t)_{t\in\mathbb{R}}$ in $\mathbb{R}$ such that

$$\Gamma_t = \left\{ x \in \mathbb{R}^N \middle| x\cdot e_t = \xi_t \right\}$$

for every $t \in \mathbb{R}$. Up to changing e_t into $-e_t$, (5) and Definition 1 yields that (39) holds for every $t \in \mathbb{R}$. It follows from Corollary 1 that $e_t = e$ is a constant vector, whence

$$\Omega_t^+ = \left\{ x \in \mathbb{R}^N \middle| x\cdot e < \xi_t \right\} \quad \text{and} \quad \Omega_t^- = \left\{ x \in \mathbb{R}^N \middle| x\cdot e > \xi_t \right\} \tag{40}$$

for all $t \in \mathbb{R}$.

In Sect. 2, we have already proved that any transition front connecting equilibrium points 0 and 1 has a global mean speed c_f. Here, for almost planar fronts, one has that

$$\frac{|\xi_t - \xi_s|}{|t-s|} \to c_f \quad \text{as } |t-s| \to +\infty.$$

Then for any $\gamma \in (0,1)$, there exists a constant $K > 0$ large enough such that

$$\gamma c_f |t-s| \le |\xi_t - \xi_s| \le 2c_f|t-s| \quad \text{for } |t-s| \ge K. \tag{41}$$

For $n \in \mathbb{Z}$, we define $\widetilde{\xi}_t$ such that

$$\widetilde{\xi}_t = \begin{cases} \xi_t, & t = nK, \\ \xi_{nK} + \frac{\xi_{(n+1)K} - \xi_{nK}}{K}(t-nK), & nK \le t \le (n+1)K. \end{cases}$$

It follows from (9) that one has

$$\forall\, \sigma > 0, \qquad \sup_{(t,s)\in\mathbb{R}^2, |t-s|\le\sigma} |\xi_t - \xi_s| < +\infty.$$

Thus, one gets

$$\sup_{t\in\mathbb{R}} |\widetilde{\xi}_t - \xi_t| < +\infty. \tag{42}$$

Moreover, one has

$$\gamma c_f \le \frac{\xi_{(n+1)K} - \xi_{nK}}{K} \le 2c_f.$$

Now we mollify the function $\widetilde{\xi}_t$ to make it smooth. Define $\eta \in C^\infty(\mathbb{R})$ by

$$\eta(z) = \begin{cases} C\exp\left(\frac{1}{|z|^2-1}\right) & \text{if } |z| < 1, \\ 0 & \text{if } |z| \ge 1, \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}} \eta dz = 1$. For each $\epsilon > 0$, set $\eta_\epsilon(z) = \frac{1}{\epsilon}\eta\left(\frac{z}{\epsilon}\right)$. Let

$$\xi_t^\epsilon = \widetilde{\xi}_t * \eta_\epsilon = \int_{-\epsilon}^{\epsilon} \eta_\epsilon(z)\widetilde{\xi}_{t-z} dz, \quad t \in \mathbb{R}$$

such that

$$\sup_{t\in\mathbb{R}} |\xi_t^\epsilon - \widetilde{\xi}_t| \le 1 \quad \text{and} \quad \gamma c_f \le \frac{d\xi_t^\epsilon}{dt} \le 2c_f \quad \text{for } t \in \mathbb{R}. \tag{43}$$

Whence, (42) and (43) yield

$$\sup_{t\in\mathbb{R}} |\xi_t^\epsilon - \xi_t| \le +\infty,$$

and hence, $u(t,x)$ is still an almost-planar front with sets

$$\widetilde{\Gamma}_t = \left\{x \in \mathbb{R}^N \middle| x\cdot e = \xi_t^\epsilon\right\}$$

and

$$\widetilde{\Omega}_t^+ = \left\{x \in \mathbb{R}^N \middle| x\cdot e < \xi_t^\epsilon\right\}, \quad \widetilde{\Omega}_t^- = \left\{x \in \mathbb{R}^N \middle| x\cdot e > \xi_t^\epsilon\right\}.$$

from Remark 1. Let α and β be two given real numbers such that

$$0 < \alpha < \theta < \beta < 1, \tag{44}$$

where we recall that θ is defined in (2). By the Definition 1, there is $M > 0$ such that

$$\forall\, (t,x) \in \mathbb{R}\times\mathbb{R}^N, \quad \begin{cases} x\cdot e - \xi_t^\epsilon \le -M \Rightarrow \beta \le u(t,x) < 1, \\ x\cdot e - \xi_t^\epsilon \ge M \Rightarrow 0 < u(t,x) \le \alpha. \end{cases} \tag{45}$$

Lemma 4 *For any* $\gamma \in (0,1)$,

$$u(t,x) \le \theta e^{-\gamma c_f (x\cdot e - \xi_t^\epsilon - M)} \tag{46}$$

in $\Sigma = \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^N \middle| x\cdot e - \xi_t^\epsilon \ge M\right\}$.

Proof Let $\bar{u}(t,x) = \theta e^{-\gamma c_f (x\cdot e - \xi_t^\epsilon - M)}$. On $\partial\Sigma = \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^N \middle| x\cdot e - \xi_t^\epsilon = M\right\}$, it follows from (44) and (45) that

$$\bar{u}(t,x) = \theta > \alpha \ge u(t,x).$$

Define

$$\varepsilon_* = \inf\left\{\varepsilon > 0 \middle| u - \varepsilon \le \bar{u} \text{ in } \Sigma\right\}.$$

Since u is bounded, ε_* is a well-defined real number and $\varepsilon_* \ge 0$. Furthermore, one has

$$w := \bar{u} - (u - \varepsilon_*) \ge 0 \quad \text{in } \Sigma.$$

In particular,

$$w > \varepsilon_* \quad \text{on } \partial\Sigma. \tag{47}$$

One only has to prove that $\varepsilon_* = 0$.

Assume by contradiction that $\varepsilon_* > 0$. Then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence of points $(t_n, x_n)_{n\in\mathbb{N}}$ in Σ such that

$$\varepsilon_n \to \varepsilon_* \text{ as } n \to +\infty \text{ and } \bar{u}(t_n,x_n) < u(t_n,x_n) - \varepsilon_n \text{ for all } n \in \mathbb{N}.$$

We claim that the sequence $(x_n\cdot e - \xi_{t_n}^\epsilon)_{n\in\mathbb{N}}$ is bounded. Assume not, up to extraction of some sequence, one has

$$x_n\cdot e - \xi_{t_n}^\epsilon \to +\infty, \text{ and then } u(t_n,x_n) \to 0 \quad \text{as } n \to +\infty.$$

But

$$u(t_n,x_n) > \bar{u}(t_n,x_n) + \varepsilon_n \ge \varepsilon_n \to \varepsilon_* > 0 \quad \text{as } n \to +\infty.$$

This gives a contradiction. Thus, the sequence $(x_n\cdot e - \xi_{t_n}^\epsilon)_{n\in\mathbb{N}}$ is bounded.

It follows from (9) that for any $\sigma > 0$, there holds

$$\sup\{d(\widetilde{\Gamma}_t, \widetilde{\Gamma}_s), t, s \in \mathbb{R}, |t-s| \le \sigma\} < +\infty.$$

Since $(\widetilde{\Gamma}_t)_{t\in\mathbb{R}}$ are all parallel hyperplanes, it then follows that for any fix $\tau > 0$, there exists a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ such that

$$\tilde{x}_n \in \widetilde{\Gamma}_{t_n-\tau} \text{ for all } n \in \mathbb{N} \text{ and } \sup\{d(x_n,\tilde{x}_n)\} < +\infty.$$

By (6), there exist $r > 0$ and a sequence $(y_n)_{n\in\mathbb{N}}$ such that

$$d(\tilde{x}_n, y_n) \le r \quad \text{and} \quad y_n\cdot e - \xi_{t_n-\tau}^\epsilon \ge M \text{ for all } n \in \mathbb{N}.$$

Then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ such that

$$z_n \in \overline{\widetilde{\Omega}_{t_n-\tau}^-} \quad \text{and} \quad M = z_n\cdot e - \xi_{t_n-\tau}^\epsilon = y_n\cdot e - \xi_{t_n-\tau}^\epsilon - d(y_n,z_n) \text{ for all } n \in \mathbb{N}. \tag{48}$$

Since $d(y_n,z_n) \le y_n\cdot e - \xi_{t_n-\tau}^\epsilon \le d(\tilde{x}_n,y_n) \le r$ and since the sequence $(d(\tilde{x}_n,x_n))_{n\in\mathbb{N}}$ is bounded, then the sequence $(d(x_n,z_n))_{n\in\mathbb{N}}$ is bounded.

Choose $\rho > 0$ so that

$$\rho\|(\bar{u}-u)_t\|_{L^\infty(\mathbb{R}\times\Sigma)} + 2\rho\|\nabla_x(\bar{u}-u)\|_{L^\infty(\mathbb{R}\times\Sigma)} < \varepsilon_*, \tag{49}$$

which is possible since $\bar{u}$ and u have bounded derivatives. Choose $K \in \mathbb{N} \setminus \{0\}$ so that

$$K\rho \geq \max\left(\tau, \sup\{d(x_n, z_n)\big| n \in \mathbb{N}\}\right). \tag{50}$$

For each $n \in \mathbb{N}$, then there exists a sequence of points $(X_{n,0}, X_{n,1}, \ldots, X_{n,K})$ in Σ such that

$$X_{n,0} = x_n, \ X_{n,K} = z_n \text{ and } d(X_{n,i}, X_{n,i+1}) \leq \rho \text{ for each } 0 \leq i \leq K-1.$$

For each $n \in \mathbb{N}$ and $0 \leq i \leq K-1$, set

$$E_{n,i} = \left[t_n - \frac{i+1}{K}\tau, t_n - \frac{i}{K}\tau\right] \times \overline{B(X_{n,i}, 2\rho)}.$$

Since $w(t_n, x_n) \to 0$ as $n \to +\infty$, (49) and (50) yield that $w < \varepsilon_*$ in $E_{n,0}$ for large n. It follows from (47) and the connectivity of $E_{n,0}$ that $E_{n,0} \subset \Sigma$ for large n.

By the definition of Σ and $\bar{u}$, one has $0 \leq \bar{u} \leq \theta$ in Σ. Then from (2) and (43), one has

$$\begin{aligned}\bar{u}_t - \Delta\bar{u} - f(\bar{u}) &= \gamma c_f \bar{u}\frac{d\xi_t^\epsilon}{dt} - (\gamma c_f)^2\bar{u} \\ &= \bar{u}\gamma c_f\left[\frac{d\xi_t^\epsilon}{dt} - \gamma c_f\right] \\ &\geq 0\end{aligned}$$

in Σ. On the other hand, $u - \varepsilon_* < u \leq \alpha < \theta$ in Σ. Assumption (2) implies that $u - \varepsilon_*$ is a subsolution of (1) in Σ. Since f is of class C^1, the function w satisfies inequations of the type

$$w_t \geq \Delta w + b(t,x)w \quad \text{in } E_{n,0}$$

for n large enough, where the sequence $(\|b\|_{L^\infty(E_{n,0})})_{n\in\mathbb{N}}$ is bounded. Since $w(t_n, X_{n,0}) = w(t_n, x_n) \to 0$ as $n \to +\infty$, it follows from the linear parabolic estimates that

$$w\left(t_n - \frac{\tau}{K}, X_{n,1}\right) \to 0 \quad \text{as } n \to +\infty.$$

An immediate induction yields $w\left(t_n - \frac{i\tau}{K}, X_{n,i}\right) \to 0$ as $n \to +\infty$ for each $i = 1, \ldots, K$. In particular, for $i = K$,

$$w\left(t_n - \tau, z_n\right) \to 0 \quad \text{as } n \to +\infty.$$

But $z_n \in \overline{\widetilde{\Omega}^-_{t_n-\tau}}$ and $z_n \cdot e - \xi^\epsilon_{t_n-\tau} = M$ for all $n \in \mathbb{N}$. As a consequence, for all $n \in \mathbb{N}$, $w\left(t_n - \tau, z_n\right) > \varepsilon_*$ from (47).

One has reached a contradiction, which means that $\varepsilon_* = 0$. Thus,

$$u(t,x) \leq \theta e^{-\gamma c_f(x\cdot e - \xi_t^\epsilon - M)}$$

for all $(t,x) \in \Sigma$. This completes the proof. □

Proof of Theorem 2 For any fixed $\gamma \in (0,1)$, let $\underline{v}_\beta$ and $\bar{v}_\alpha$ be the solution of the one-dimensional Cauchy problem

$$v_t = v_{yy} + f(v), \quad t > 0, \ y \in \mathbb{R} \tag{51}$$

with initial condition

$$\underline{v}_\beta(0,y) \in C(\mathbb{R}, [0,\beta]) \quad \text{and} \quad \underline{v}_\beta(0,y) = \begin{cases} \beta & \text{if } y \leq -1, \\ 0 & \text{if } y \geq 0, \end{cases} \tag{52}$$

and $\bar{v}_\alpha(0, y) \in C(\mathbb{R}, [0, 1])$,

$$\bar{v}_\alpha(0, y) \geq \alpha \text{ on } [-1, 1] \quad \text{and} \quad \bar{v}_\alpha(0, y) = \begin{cases} 1 & \text{if } y \leq 0, \\ \theta e^{-\gamma c_f y} & \text{if } y \geq 1, \end{cases} \tag{53}$$

respectively. It follows from (45) and Lemma 4 that for every $t_0 \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

$$\underline{v}_\beta(0, x \cdot e - \xi^\epsilon_{t_0} + M) \leq u(t_0, x) \leq \bar{v}_\alpha(0, x \cdot e - \xi^\epsilon_{t_0} - M).$$

Thus,

$$\underline{v}_\beta(t - t_0, x \cdot e - \xi^\epsilon_{t_0} + M) \leq u(t, x) \leq \bar{v}_\alpha(t - t_0, x \cdot e - \xi^\epsilon_{t_0} - M). \tag{54}$$

for all $t > t_0$ and $x \in \mathbb{R}^N$, from the maximum principle. By [18], there exist two constants $\underline{\omega} > 0$ and $\bar{\omega} > 0$ such that

$$\left| \frac{\underline{v}_\beta(s, y) - \phi_f(y - c_f s + \underline{\xi})}{\phi_f^\gamma(y - c_f s + \underline{\xi})} \right| \leq \underline{A} e^{-\underline{\omega} t}, \quad s \geq 0, \quad y \in \mathbb{R},$$

and

$$\left| \frac{\bar{v}_\alpha(s, y) - \phi_f(y - c_f s + \bar{\xi})}{\phi_f^\gamma(y - c_f s + \bar{\xi})} \right| \leq \bar{A} e^{-\bar{\omega} t}, \quad s \geq 0, \quad y \in \mathbb{R},$$

for some $\underline{A} > 0$, $\bar{A} > 0$, $\underline{\xi} \in \mathbb{R}$ and $\bar{\xi} \in \mathbb{R}$. In particular, since $\phi_f(-\infty) = 1$ and $\phi_f(+\infty) = 0$, there exist $T > 0$ and $B > 0$ such that, for all $s \geq T$,

$$\underline{v}_\beta(s, y) > \alpha \quad \text{if } y \leq c_f s - B,$$
$$\bar{v}_\alpha(s, y) < \beta \quad \text{if } y \geq c_f s + B.$$

It follows from (54) that for all $t_0 < t_0 + T \leq t$,

$$\begin{aligned} u(t, x) > \alpha \quad &\text{if } x \cdot e - \xi^\epsilon_{t_0} + M \leq c_f(t - t_0) - B, \\ u(t, x) < \beta \quad &\text{if } x \cdot e - \xi^\epsilon_{t_0} - M \geq c_f(t - t_0) + B. \end{aligned} \tag{55}$$

By (45) and (55), for all $t_0 < t_0 + T \leq t$, we have

$$\begin{aligned} \xi^\epsilon_{t_0} - M + c_f(t - t_0) - B < \xi^\epsilon_t + M, \\ \xi^\epsilon_{t_0} + M + c_f(t - t_0) + B > \xi^\epsilon_t - M. \end{aligned} \tag{56}$$

By fixing $t = 0$, one gets that $\limsup_{t_0 \to -\infty} |\xi^\epsilon_{t_0} - c_f t_0| \leq |\xi^\epsilon_0| + 2M + B$. For any arbitrary $t \in \mathbb{R}$, letting $t_0 \to -\infty$ in (56) then leads to

$$|\xi^\epsilon_t - c_f t| \leq |\xi_0| + 4M + 2B.$$

Thus, by Definition 1 and (40), our solution $u : \mathbb{R} \times \mathbb{R}^N \to (0, 1)$ of (1) satisfies

$$\inf_{(t,x) \in \mathbb{R} \times \mathbb{R}^N, x \cdot e - c_f t \leq -A} u(t, x) \to 1 \textit{ and } \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^N, x \cdot e - c_f t \geq A} u(t, x) \to 0 \quad \textit{as } A \to +\infty.$$

It follows from Theorem 3.1 of [3] and the uniqueness of the planar fronts that there exists $\xi \in \mathbb{R}$ such that $u(t, x) = \phi_f(x \cdot e - c_f t + \xi)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. This completes the proof of Theorem 2.

4 Existence of Non-standard Transition Fronts

In this section, we prove Theorem 3. That is, we prove the existence of new kinds of transition fronts, which are not invariant in any moving frame. We first consider the case $N = 2$ and construct two-dimensional transition fronts satisfying the conclusion of Theorem 3. The conclusion in $\mathbb{R}^N$ with $N > 2$ will be then obtained immediately by trivially extending the constructed two-dimensional fronts in variables $x_3, \ldots, x_N$. Now we first give some preliminaries.

For the standard planar traveling fronts ϕ_f, it is well known that there exist some positive constants λ_1, C_0, C_1 and C_2 such that

$$\phi_f(s) \le C_0 e^{-c_f s}, \quad s \ge 0, \tag{57}$$

$$1 - \phi_f(s) \le C_1 e^{\lambda_1 s}, \quad s \le 0, \tag{58}$$

$$|\phi_f'(s)| \le C_2 e^{-\lambda_1 |s|}, \quad s \in \mathbb{R}. \tag{59}$$

Fix an angle α such that $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$. Consider the quasilinear parabolic equation

$$W_t = \frac{W_{xx}}{1 + W_x^2} + c_f \sqrt{1 + W_x^2}, \quad x \in \mathbb{R}, \quad t > 0. \tag{60}$$

It follows from Propositions 1.1 and 2.5 of [27] that for any $c > c_f$, there exists a unique solution $\varphi(x; c)$ of (60) with asymptotic lines $y = |x| \cot \alpha$ satisfying

$$c = \frac{\varphi_{xx}}{1 + \varphi_x^2} + c_f \sqrt{1 + \varphi_x^2}, \quad x \in \mathbb{R}.$$

Lemma 5 (Brazhnik [6], Ninomiya and Taniguchi [24,26,27]) *There exist positive constants γ_1, k_i $(i = 1, 2, 3)$ and $\omega_\pm$ such that*

$$\begin{aligned}
&\max\left\{|\varphi''(x)|,\ |\varphi'''(x)|\right\} \le k_1 \operatorname{sech}(\gamma_1 x), \\
&k_2 \operatorname{sech}(\gamma_1 x) \le \frac{c}{\sqrt{1 + \varphi'(x)^2}} - c_f \le k_3 \operatorname{sech}(\gamma_1 x), \\
&|x| \cot \alpha \le \varphi(x), \\
&\omega_- \le \tilde{\omega}(x) \le \omega_+
\end{aligned}$$

for any $x \in \mathbb{R}$, where

$$\tilde{\omega}(x) = \frac{c(\varphi(x) - |x| \cot \alpha)}{c - c_f \sqrt{1 + \varphi'(x)^2}}.$$

By Lemma 5, it is easy to obtain that

$$1 > \frac{1}{\sqrt{1 + \varphi'(x)^2}} > \frac{c_f}{c} \quad \text{for all } x \in \mathbb{R}$$

and that there exists a constant $a > 0$ such that

$$|x| \cot \alpha \le \varphi(x) \le |x| \cot \alpha + a \quad \text{for all } x \in \mathbb{R}.$$

It follows from [12,36] that there exists a unique V-shaped traveling front $\phi(x_1, x_2 - ct)$ (Fig. 1) of the problem (1) in $\mathbb{R}^2$ satisfying the following properties: $0 < \phi < 1$ in $\mathbb{R}^2$, ϕ is of class $C^2(\mathbb{R}^2)$, $c = \frac{c_f}{\sin \alpha}$ and

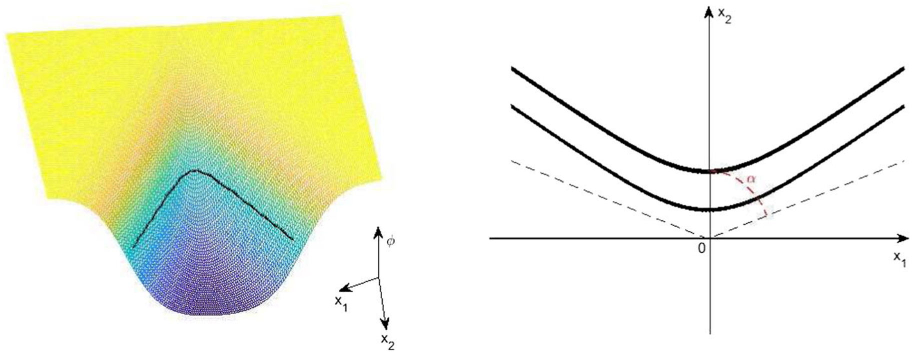

Fig. 1 The profiles (left figure) and the contour lines (right figure) of the V-shaped traveling front

$$\liminf_{A\to+\infty}\left(\inf_{x_2\le|x_1|\cot\alpha-A}\phi(x_1,x_2)\right)=1,$$
$$\liminf_{A\to+\infty}\left(\inf_{x_2\ge|x_1|\cot\alpha+A}\phi(x_1,x_2)\right)=0. \tag{61}$$

Furthermore, for any $\beta_1\in(0,1)$, there exist two positive constants $\varepsilon_0^+(\beta_1)$ and $\alpha_0^+(\beta_1,\varepsilon)$ so that, for $0<\varepsilon<\varepsilon_0^+(\beta_1)$ and $0<\vartheta<\alpha_0^+(\beta_1,\varepsilon)$,

$$\begin{aligned}&\phi_f\left(x_2\sin\alpha-|x_1|\cos\alpha\right)\\&\quad<\phi(x_1,x_2)\\&\quad<\phi_f\left(\frac{x_2-\varphi(\vartheta x_1)/\vartheta}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}\right)+\varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2-\varphi(\vartheta x_1)/\vartheta)\right).\end{aligned}\tag{62}$$

Fix $\beta_1\in(0,1)$, $0<\varepsilon<\varepsilon_0^+(\beta_1)$ and $0<\vartheta<\alpha_0^+(\beta_1,\varepsilon)$. Now we show that ϕ is asymptotically planar along the directions $(\pm\sin\alpha,\cos\alpha)$. This property plays an important role in the proof of Theorem 3.

Proposition 1 *There exist two positive constants ρ_1 and ω_1 such that*

$$0\le\phi(x_1,x_2)-\phi_f(x_2\sin\alpha-|x_1|\cos\alpha)\le\rho_1 e^{-\omega_1\sqrt{x_1^2+x_2^2}}\quad\text{for all }(x_1,x_2)\in\mathbb{R}^2. \tag{63}$$

Proof Let

$$\begin{aligned}\rho_1=\max\Bigg\{&C_1,2\left(C_2\max_{s\in(-\infty,0]}\left|se^{\lambda_1\frac{c_f}{c}s}\right|\frac{k_3}{c}+\max_{s\in\mathbb{R}}\left|\phi_f'(s)\right|\frac{\omega_+k_3}{c}+\varepsilon\right),\\&2\left(\frac{\omega_+k_3}{c}+\beta_1\frac{c_f}{c}a\varepsilon\right)C_2e^{\lambda_1 a}+2C_0^{\beta_1}\Bigg\}.\end{aligned}$$

Choose $\mu\in(0,1)$ such that

$$\mu\frac{\lambda_1c_f\cot\alpha}{c}<\frac{1}{2}\gamma_1\vartheta\quad\text{and}\quad\mu\frac{\beta_1c_f^2\cot\alpha}{c}<\frac{1}{2}\gamma_1\vartheta.$$

Fix a real number ω_1 such that

$$0<\omega_1<\min\left\{\frac{\lambda_1c_f\cot\alpha}{c},\frac{1}{2}\gamma_1\vartheta,\frac{\gamma_1\vartheta}{2\cot\alpha},\frac{\mu c_f\lambda_1}{c},\frac{\mu\beta_1c_f^2}{c}\right\}.$$

Now we divide our proof into three cases.

Case 1 when $x_2 \le 0$, by (58), one has

$$\begin{aligned} 0 &< \phi(x_1, x_2) - \phi_f(x_2 \sin\alpha - |x_1| \cos\alpha) \\ &< 1 - \phi_f(x_2 \sin\alpha - |x_1| \cos\alpha) \\ &\le C_1 e^{\lambda_1 (x_2 \sin\alpha - |x_1| \cos\alpha)} \\ &\le \rho_1 e^{-\omega_1 \sqrt{x_1^2 + x_2^2}}. \end{aligned}$$

Case 2 when $x_2 > 0$ and $x_2 \sin\alpha - |x_1| \cos\alpha < 0$, one has

$$x_1^2 > \frac{x_2^2}{\cot^2\alpha} \quad \text{and} \quad \frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} - \frac{c_f}{c}(x_2 - |x_1| \cot\alpha) < 0$$

from $\frac{1}{\sqrt{1+|\varphi'(x_1)|^2}} > \frac{c_f}{c}$ and $|x_1| \cot\alpha \le \varphi(x_1)$ for any $x_1 \in \mathbb{R}$. Thus it follows from (59), Lemma 5 and $c = \frac{c_f}{\sin\alpha}$ that

$$\begin{aligned} 0 &< \phi(x_1, x_2) - \phi_f(x_2 \sin\alpha - |x_1| \cos\alpha) \\ &< \phi_f\left(\frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}}\right) + \varepsilon \operatorname{sech}(\gamma_1 \vartheta x_1) \phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2 - \varphi(\vartheta x_1)/\vartheta)\right) \\ &\quad - \phi_f\left(\frac{c_f}{c}(x_2 - |x_1| \cot\alpha)\right) \\ &< \int_0^1 \phi_f'\left(\frac{c_f}{c}(x_2 - |x_1| \cot\alpha) + \left(\frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} - \frac{c_f}{c}(x_2 - |x_1| \cot\alpha)\right) \upsilon\right) d\upsilon \\ &\quad \times \left(\frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} - \frac{c_f}{c}(x_2 - |x_1| \cot\alpha)\right) + \varepsilon \operatorname{sech}(\gamma_1 \vartheta x_1) \\ &= \int_0^1 \phi_f'\left(\frac{c_f}{c}(x_2 - |x_1| \cot\alpha) + \left(\frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} - \frac{c_f}{c}(x_2 - |x_1| \cot\alpha)\right) \upsilon\right) d\upsilon \\ &\quad \times \left(\frac{1}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} - \frac{c_f}{c}\right)(x_2 - |x_1| \cot\alpha) \\ &\quad - \int_0^1 \phi_f'\left(\frac{c_f}{c}(x_2 - |x_1| \cot\alpha) + \left(\frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} - \frac{c_f}{c}(x_2 - |x_1| \cot\alpha)\right) \upsilon\right) d\upsilon \\ &\quad \times \frac{\varphi(\vartheta x_1)/\vartheta - |x_1| \cot\alpha}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} + \varepsilon \operatorname{sech}(\gamma_1 \vartheta x_1) \\ &\le - C_2 e^{\lambda_1 \frac{c_f}{c}(x_2 - |x_1| \cot\alpha)} \left(\frac{1}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} - \frac{c_f}{c}\right)(x_2 - |x_1| \cot\alpha) \\ &\quad + \max_{s \in \mathbb{R}} \left|\phi_f'(s)\right| \frac{\varphi(\vartheta x_1)/\vartheta - |x_1| \cot\alpha}{\sqrt{1 + |\varphi'(\vartheta x_1)|^2}} + \varepsilon \operatorname{sech}(\gamma_1 \vartheta x_1) \end{aligned}$$

$$\le \left(C_2 \max_{s\in(-\infty,0]} \left|se^{\lambda_1 \frac{c_f}{c}s}\right| \frac{k_3}{c} + \max_{s\in\mathbb{R}} \left|\phi_f'(s)\right| \frac{\omega_+ k_3}{c} + \varepsilon\right) \operatorname{sech}(\gamma_1\vartheta x_1)$$
$$< 2\left(C_2 \max_{s\in(-\infty,0]} \left|se^{\lambda_1 \frac{c_f}{c}s}\right| \frac{k_3}{c} + \max_{s\in\mathbb{R}} \left|\phi_f'(s)\right| \frac{\omega_+ k_3}{c} + \varepsilon\right) e^{-\gamma_1\vartheta|x_1|}$$
$$\le \rho_1 e^{-\omega_1\sqrt{x_1^2+x_2^2}}.$$

Case 3 when $x_2 > 0$ and $x_2 \sin\alpha - |x_1|\cos\alpha \ge 0$, one has $x_2 > |x_1|\cot\alpha$. It follows from (57), (59), Lemma 5 and $c = \frac{c_f}{\sin\alpha}$ that

$$\begin{aligned}
0 &< \phi(x_1,x_2) - \phi_f(x_2\sin\alpha - |x_1|\cos\alpha)\\
&< \phi_f\left(\frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}\right) + \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2 - \varphi(\vartheta x_1)/\vartheta)\right)\\
&\quad - \phi_f\left(\frac{c_f}{c}(x_2 - |x_1|\cot\alpha)\right)\\
&\le \phi_f\left(\frac{x_2 - \varphi(\vartheta x_1)/\vartheta}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}\right) - \phi_f\left(\frac{x_2 - |x_1|\cot\alpha}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}\right)\\
&\quad + \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2 - \varphi(\vartheta x_1)/\vartheta)\right)\\
&= \int_0^1 -\phi_f'\left(\frac{x_2 - |x_1|\cot\alpha - (\varphi(\vartheta x_1)/\vartheta - |x_1|\cot\alpha)\upsilon}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}\right) d\upsilon\\
&\quad \times \frac{\varphi(\vartheta x_1)/\vartheta - |x_1|\cot\alpha}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}} + \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2 - \varphi(\vartheta x_1)/\vartheta)\right)\\
&\quad - \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2 - |x_1|\cot\alpha)\right)\\
&\quad + \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2 - |x_1|\cot\alpha)\right)\\
&= \int_0^1 -\phi_f'\left(\frac{x_2 - |x_1|\cot\alpha - (\varphi(\vartheta x_1)/\vartheta - |x_1|\cot\alpha)\upsilon}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}\right) d\upsilon\\
&\quad \times \frac{\varphi(\vartheta x_1)/\vartheta - |x_1|\cot\alpha}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}\\
&\quad + \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\int_0^1 -\beta_1\phi_f'\left(\frac{c_f}{c}\left(x_2 - |x_1|\cot\alpha - (\varphi(\vartheta x_1)/\vartheta - |x_1|\cot\alpha)\upsilon\right)\right)\\
&\quad \times \phi_f^{\beta_1-1}\left(\frac{c_f}{c}\left(x_2 - |x_1|\cot\alpha - (\varphi(\vartheta x_1)/\vartheta - |x_1|\cot\alpha)\upsilon\right)\right) d\upsilon\\
&\quad \times \frac{c_f}{c}\left(\varphi(\vartheta x_1)/\vartheta - |x_1|\cot\alpha\right) + \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\phi_f^{\beta_1}\left(\frac{c_f}{c}(x_2 - |x_1|\cot\alpha)\right)\\
&\le C_2 e^{-\lambda_1\frac{x_2 - |x_1|\cot\alpha}{\sqrt{1+|\varphi'(\vartheta x_1)|^2}}} e^{\lambda_1 a}\frac{\omega_+ k_3}{c}\operatorname{sech}(\gamma_1\vartheta x_1)\\
&\quad + \beta_1 C_2 e^{-\lambda_1\frac{c_f}{c}(x_2 - |x_1|\cot\alpha)} e^{\lambda_1 a}\frac{c_f}{c} a\varepsilon\operatorname{sech}(\gamma_1\vartheta x_1)\\
&\quad + \varepsilon\operatorname{sech}(\gamma_1\vartheta x_1) C_0^{\beta_1} e^{-\beta_1\frac{c_f^2}{c}(x_2 - |x_1|\cot\alpha)}\\
&\le 2\left(\frac{\omega_+ k_3}{c} + \beta_1\frac{c_f}{c}a\varepsilon\right) C_2 e^{\lambda_1 a} e^{-\frac{c_f}{c}\lambda_1(x_2 - |x_1|\cot\alpha)} e^{-\gamma_1\vartheta|x_1|}
\end{aligned}$$

$$
\begin{aligned}
&+2C_0^{\beta_1} e^{-\frac{\beta_1 c_f^2}{c}(x_2-|x_1|\cot\alpha)} e^{-\gamma_1\vartheta|x_1|}\\
&\le 2\left(\frac{\omega_+ k_3}{c}+\beta_1\frac{c_f}{c}a\varepsilon\right)C_2 e^{\lambda_1 a} e^{-\mu\frac{c_f}{c}\lambda_1(x_2-|x_1|\cot\alpha)} e^{-\gamma_1\vartheta|x_1|}\\
&\quad +2C_0^{\beta_1} e^{-\mu\frac{\beta_1 c_f^2}{c}(x_2-|x_1|\cot\alpha)} e^{-\gamma_1\vartheta|x_1|}\\
&\le 2\left(\frac{\omega_+ k_3}{c}+\beta_1\frac{c_f}{c}a\varepsilon\right)C_2 e^{\lambda_1 a} e^{-\mu\frac{c_f}{c}\lambda_1 x_2-\frac{1}{2}\gamma_1\vartheta|x_1|}+2C_0^{\beta_1} e^{-\mu\frac{\beta_1 c_f^2}{c}x_2-\frac{1}{2}\gamma_1\vartheta|x_1|}\\
&<\left[2\left(\frac{\omega_+ k_3}{c}+\beta_1\frac{c_f}{c}a\varepsilon\right)C_2 e^{\lambda_1 a}+2C_0^{\beta_1}\right]e^{-\omega_1(|x_1|+|x_2|)}\\
&\le \rho_1 e^{-\omega_1\sqrt{x_1^2+x_2^2}}.
\end{aligned}
$$

Combining the above three cases, the proof of Proposition 1 is thereby complete. □

It follows from Proposition 1 and the Schauder interior estimates that there exist two positive constants ρ_2 and ω_2 such that

$$
\left|\nabla\phi(x_1,x_2)-\nabla(\phi_f(x_2\sin\alpha-x_1\cos\alpha))\right|\le\rho_2 e^{-\omega_2\sqrt{x_1^2+x_2^2}}\quad\text{for all}\quad x_1\ge 0, x_2\in\mathbb{R}.
$$

Whence

$$
\begin{aligned}
&\left|\phi_{x_1}(x_1,x_2)+\phi_f'(x_2\sin\alpha-x_1\cos\alpha)\cos\alpha\right|\le\rho_2 e^{-\omega_2\sqrt{x_1^2+x_2^2}},\\
&\left|\phi_{x_2}(x_1,x_2)-\phi_f'(x_2\sin\alpha-x_1\cos\alpha)\sin\alpha\right|\le\rho_2 e^{-\omega_2\sqrt{x_1^2+x_2^2}}
\end{aligned}
\tag{64}
$$

for all $x_1\ge 0,\ x_2\in\mathbb{R}$. Since the standard planar traveling fronts $\phi_f(s)$ converges exponentially fast to 0 and 1 as $s\to\pm\infty$, Proposition 1 yields that the V-shaped traveling front ϕ also converges exponentially fast to 0 and 1 as $x_2-|x_1|\cot\alpha\to\pm\infty$. By the Schauder interior estimates, there exist two positive constants ρ_3 and ω_3 such that

$$
|\nabla\phi(x_1,x_2)|\le\rho_3 e^{-\omega_3|x_2-|x_1|\cot\alpha|}\quad\text{for all }(x_1,x_2)\in\mathbb{R}^2. \tag{65}
$$

It follows from Corollary 3.3 and Lemma 3.4 of [36] that

$$
\forall A\ge 0,\qquad \sup_{-A\le x_2-|x_1|\cot\alpha\le A}\phi_{x_2}(x_1,x_2)<0 \tag{66}
$$

and that ϕ is decreasing in any direction $(\cos\hat{\alpha},\sin\hat{\alpha})$ such that $\pi/2-\alpha<\hat{\alpha}<\pi/2+\alpha$, see also [12]. In particular, the function ϕ is nonincreasing along the directions $(\pm\sin\alpha,\cos\alpha)$.

Define

$$
\psi(x_1,x_2)=\phi(x_1\sin\alpha-x_2\cos\alpha,x_1\cos\alpha+x_2\sin\alpha)\quad\text{for all }(x_1,x_2)\in\mathbb{R}^2, \tag{67}
$$

which rotates the function ϕ with angle $\alpha-\frac{\pi}{2}$ clockwise. Then the function ψ (Fig. 2) is decreasing in any direction $(\cos\hat{\beta},\sin\hat{\beta})$ with $0<\hat{\beta}<2\alpha$. In particular, ψ is nonincreasing in the horizontal direction $(1,0)$ and it converges to the planar front $\phi_f(x_2)$ along this direction. Set

$$
\begin{aligned}
\underline{v}(t,x_1,x_2)&=\psi(x_1-ct\cos\alpha,x_2-ct\sin\alpha)\\
&=\phi(x_1\sin\alpha-x_2\cos\alpha,x_1\cos\alpha+x_2\sin\alpha-ct).
\end{aligned}
\tag{68}
$$

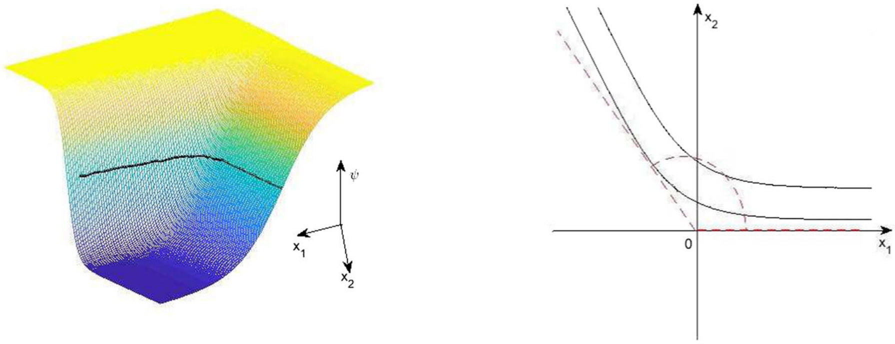

Fig. 2 The profiles (left figure) and the contour lines (right figure) of function ψ

Since $\phi(x_1, x_2 - ct)$ solves the Eq. (1) in $\mathbb{R}^2$, then the $C^2(\mathbb{R} \times \mathbb{R}^2)$ function $\underline{v}$ also satisfies (1) in $\mathbb{R}^2$. Moreover, the definition of $\underline{v}$ yields $\underline{v}_t(t, x_1, x_2) > 0$ and $\underline{v}_{x_1}(t, x_1, x_2) \leq 0$ in $\mathbb{R} \times \mathbb{R}^2$.

Now we consider the following Neumann boundary value problem in half-space $H = \{(x_1, x_2) \in \mathbb{R}^2, x_1 < 0\}$

$$\begin{aligned} v_t &= \Delta v + f(v), (t, x_1, x_2) \in \mathbb{R} \times H, \\ v_{x_1} &= 0, (t, x_1, x_2) = (t, 0, x_2) \in \mathbb{R} \times \partial H. \end{aligned} \tag{69}$$

It is easy to see that the function $\underline{v}$ is a subsolution of (69).

In the following lemma, we construct a supersolution which looks like the function $\underline{v}$ for very negative times, up to some exponentially small terms.

Lemma 6 *There exist some constants $\sigma > 0$, $\delta > 0$ and $T < 0$ such that the function*

$$\bar{v}(t, x_1, x_2) = \min\left\{\underline{v}\left(t + \sigma e^{\delta t}, x_1, x_2\right) + \delta e^{\delta(x_1+t)}, 1\right\} \tag{70}$$

is a supersolution of (69) *for $t \leq T$ and $(x_1, x_2) \in \overline{H}$.*

Proof Let

$$\omega_4 = \frac{\min(\omega_2 c_f \cos\alpha, \omega_3 c)}{2} > 0,$$

where ω_2 and ω_3 are given in (64) and (65). Choose δ such that

$$0 < \delta < \min(1, \omega_4, \theta/2) \text{ and } f' \leq 0 \text{ on } [1-\delta, 1]. \tag{71}$$

It follows from (61) that there exists a real number $A > 0$ such that

$$\begin{aligned} \phi(x_1, x_2) &\geq 1 - \delta \quad \text{for all } x_2 \leq |x_1| \cot\alpha - A, \\ \phi(x_1, x_2) &\leq \delta \quad \text{for all } x_2 \geq |x_1| \cot\alpha + A. \end{aligned} \tag{72}$$

Equation (66) implies that there exists a constant $\kappa > 0$ such that

$$\sup_{-A \leq x_2 - |x_1|\cot\alpha \leq A} \phi_{x_2}(x_1, x_2) = -\kappa < 0. \tag{73}$$

Choose $\sigma > 0$ such that

$$\sigma c \kappa \geq L = \max_{u \in [0,1]} |f'(u)|. \tag{74}$$

Set

$$\rho_4 = (\sin\alpha + \cos\alpha)\max(\rho_2, \rho_3 e^{\omega_3 c\sigma}) > 0.$$

Let $T < 0$ be such that

$$T \leq -2\sigma < 0 \ \text{ and } \ \delta^2 e^{\delta t} \geq \rho_4 e^{\omega_4 t} \quad \text{for all } t \leq T.$$

Similar to the Lemma 5.1 of Hamel [11] combining with Proposition 1, we can prove that $\bar{v}_{x_1} \geq 0$ on $(-\infty, T] \times \partial H$ in the region where $\bar{v} < 1$.

Since $f(1) = 0$, it is sufficient to show that

$$\bar{v}_t \geq \Delta\bar{v} + f(\bar{v})$$

on the region $(t, x_1, x_2) \in (-\infty, T] \times \overline{H}$ such that $\bar{v} < 1$. Since $\underline{v}$ satisfies (1) in $\mathbb{R}^2$ and $\delta < 1$, thus

$$\begin{aligned}\overline{\mathscr{L}}(t, x_1, x_2) :=& \bar{v}_t(t, x_1, x_2) - \Delta\bar{v}(t, x_1, x_2) - f(\bar{v}(t, x_1, x_2))\\ =& \underline{v}_t(t + \sigma e^{\delta t}, x_1, x_2) + \sigma\delta\underline{v}_t(t + \sigma e^{\delta t}, x_1, x_2)e^{\delta t} + \delta^2 e^{\delta(x_1+t)}\\ &- \Delta\underline{v}(t + \sigma e^{\delta t}, x_1, x_2) - \delta^3 e^{\delta(x_1+t)} - f(\bar{v}(t, x_1, x_2))\\ \geq& f\left(\underline{v}(t + \sigma e^{\delta t}, x_1, x_2)\right) - f(\bar{v}(t, x_1, x_2)) + \sigma\delta\underline{v}_t\left(t + \sigma e^{\delta t}, x_1, x_2\right)e^{\delta t}\end{aligned} \tag{75}$$

For simplicity, by (68), we can set

$$\underline{v}\left(t + \sigma e^{\delta t}, x_1, x_2\right) = \phi(\xi_1(x_1, x_2), \xi_2(t, x_1, x_2)),$$

where

$$\xi_1(x_1, x_2) = x_1\sin\alpha - x_2\cos\alpha \ \text{ and } \ \xi_2(t, x_1, x_2) = x_1\cos\alpha + x_2\sin\alpha - ct - c\sigma e^{\delta t}.$$

Firstly, if $\xi_2(t, x_1, x_2) \leq |\xi_1(x_1, x_2)|\cot\alpha - A$, then (72) implies that

$$1 > \bar{v}(t, x_1, x_2) > \underline{v}\left(t + \sigma e^{\delta t}, x_1, x_2\right) = \phi(\xi_1(x_1, x_2), \xi_2(t, x_1, x_2)) \geq 1 - \delta.$$

It follows from (71), (75) and $\underline{v}_t > 0$ that one has

$$\begin{aligned}\overline{\mathscr{L}}(t, x_1, x_2) \geq& f\left(\underline{v}(t + \sigma e^{\delta t}, x_1, x_2)\right) - f(\bar{v}(t, x_1, x_2)) + \sigma\delta\underline{v}_t\left(t + \sigma e^{\delta t}, x_1, x_2\right)e^{\delta t}\\ \geq& 0.\end{aligned} \tag{76}$$

Secondly, if $\xi_2(t, x_1, x_2) \geq |\xi_1(x_1, x_2)|\cot\alpha + A$, then it follows from (71), (72), $x_1 \leq 0$ and $t \leq T < 0$ that

$$0 < \underline{v}\left(t + \sigma e^{\delta t}, x_1, x_2\right) < \bar{v}(t, x_1, x_2) = \underline{v}\left(t + \sigma e^{\delta t}, x_1, x_2\right) + \delta e^{\delta(x_1+t)} \leq 2\delta < \theta.$$

Since $f = 0$ on $[0, \theta]$ and $\underline{v}_t > 0$, then

$$\begin{aligned}\overline{\mathscr{L}}(t, x_1, x_2) \geq& f\left(\underline{v}(t + \sigma e^{\delta t}, x_1, x_2)\right) - f(\bar{v}(t, x_1, x_2)) + \sigma\delta\underline{v}_t\left(t + \sigma e^{\delta t}, x_1, x_2\right)e^{\delta t}\\ \geq& 0.\end{aligned} \tag{77}$$

Lastly, if $-A \leq \xi_2(t, x_1, x_2) - |\xi_1(x_1, x_2)|\cot\alpha \leq A$, then

$$\begin{aligned}& f\left(\underline{v}(t + \sigma e^{\delta t}, x_1, x_2)\right) - f(\bar{v}(t, x_1, x_2))\\ &= f\left(\underline{v}(t + \sigma e^{\delta t}, x_1, x_2)\right) - f\left(\underline{v}(t + \sigma e^{\delta t}, x_1, x_2) + \delta e^{\delta(x_1+t)}\right)\\ &\geq -L\delta e^{\delta(x_1+t)}\end{aligned}$$

and

$$\underline{v}_t(t+\sigma e^{\delta t}, x_1, x_2) = -c\phi_{x_2}(\xi_1(x_1,x_2), \xi_2(t,x_1,x_2)) \geq c\kappa.$$

It follows from (74) and $x_1 \leq 0$ that

$$\overline{\mathscr{L}}(t,x_1,x_2) \geq -L\delta e^{\delta(x_1+t)} + \sigma\delta c\kappa e^{\delta t} \geq \delta(\sigma c\kappa - L)e^{\delta t} \geq 0. \tag{78}$$

Combining with (76), (77) and (78), one has $\overline{\mathscr{L}}(t,x_1,x_2) \geq 0$ for all $(t,x_1,x_2) \in (-\infty, T] \times \overline{H}$ such that $\bar{v}(t,x_1,x_2) < 1$. This completes the proof. □

Proof of Theorem 3 It follows from the positivity of $\underline{v}_t$ and the definition of $\bar{v}$ that $\underline{v}(t,x_1,x_2) < \bar{v}(t,x_1,x_2)$ in $\mathbb{R}\times\overline{H}$. For any $n \in \mathbb{N}$ such that $n > |T|$, let v^n be the solution of the Cauchy problem associated to (69) for times $t > -n$, with initial condition

$$v^n(-n,x_1,x_2) = \underline{v}(-n,x_1,x_2) \quad \text{for all } (x_1,x_2) \in H.$$

Since $(\underline{v}, \bar{v})$ is a couple of sub-supersolution of the problem (69), the maximum principle implies that

$$0 < \underline{v}(t,x_1,x_2) \leq v^n(t,x_1,x_2) \leq \bar{v}(t,x_1,x_2) \leq 1$$

for all $-n < t \leq T$ and $(x_1,x_2) \in \overline{H}$ and that

$$0 < \underline{v}(t,x_1,x_2) \leq v^n(t,x_1,x_2) \leq 1 \ \text{ for all } (t,x_1,x_2) \in (-n,+\infty)\times\overline{H}. \tag{79}$$

In particular, for every $(t,x_1,x_2) \in \mathbb{R}\times\overline{H}$, the sequence $(v^n(t,x_1,x_2))_{n>\max(|T|,|t|)}$ is nondecreasing. Furthermore, since $\underline{v}_t > 0$, (79) and the maximum principle yield that v^n is increasing with respect to time t in $\overline{H}$.

It follows from monotone convergence and standard parabolic estimates up to the boundary that the functions v^n converge to a solution v of (69) as $n \to +\infty$ in $C^{1,2}_{loc}(\mathbb{R}\times\overline{H})$. Furthermore, one has

$$0 < \underline{v}(t,x_1,x_2) \leq v(t,x_1,x_2) \leq \bar{v}(t,x_1,x_2) \leq 1 \quad \text{for all } t \leq T \text{ and } (x_1,x_2) \in \overline{H}, \tag{80}$$

and

$$0 < \underline{v} \leq v \leq 1, \quad v_t \geq 0 \quad \text{in } \mathbb{R}\times\overline{H}.$$

In particular, since for each fixed $(x_1,x_2) \in \overline{H}$, the function $\bar{v}(t,x_1,x_2) \to 0 < 1$ as $t \to -\infty$, then it follows from (80) and the strong maximum principle that $0 < v < 1$ in $\mathbb{R}\times\overline{H}$.

Now we construct a solution u of (1) in $\mathbb{R}^2$. Define u in $\mathbb{R}\times\mathbb{R}^2$ as

$$u(t,x_1,x_2) = \begin{cases} v(t,x_1,x_2) & t\in\mathbb{R}, \quad x_1 \leq 0, \quad x_2 \in \mathbb{R}, \\ v(t,-x_1,x_2) & t\in\mathbb{R}, \quad x_1 > 0, \quad x_2 \in \mathbb{R}. \end{cases}$$

Since v satisfies (69) in the half-plane H with Neumann boundary conditions, then u is a classical time-global solution of (1) in the whole plane $\mathbb{R}^2$. Furthermore, $0 < u < 1$ in $\mathbb{R}\times\mathbb{R}^2$,

$$\underline{v}(t,-|x_1|,x_2) \leq u(t,x_1,x_2) \quad \text{for all } (t,x_1,x_2) \in \mathbb{R}\times\mathbb{R}^2$$

and

$$\underline{v}(t,-|x_1|,x_2) \leq u(t,x_1,x_2) \leq \bar{v}(t,-|x_1|,x_2) \quad \text{for all } t \leq T \text{ and } (x_1,x_2) \in \mathbb{R}^2.$$

Therefore, by the definition of $\underline{v}$ and (62), one has

$$\max(\phi_f(-|x_1|\sin(2\alpha)-x_2\cos(2\alpha)-c_ft),\phi_f(x_2-c_ft))\le u(t,x_1,x_2)$$

for all $(t,x_1,x_2)\in\mathbb{R}\times\mathbb{R}^2$. And it follows from the definition $\bar{v}$ and Proposition 1 that

$$\begin{aligned}&u(t,x_1,x_2)\\&\quad\le\max(\phi_f(-|x_1|\sin(2\alpha)-x_2\cos(2\alpha)-c_ft-c_f\sigma e^{\delta t}),\phi_f(x_2-c_ft-c_f\sigma e^{\delta t}))\\&\qquad+\rho_1 e^{-\omega_1\sqrt{(|x_1|\sin\alpha+x_2\cos\alpha)^2+(|x_1|\cos\alpha-x_2\sin\alpha+ct+c\sigma e^{\delta t})^2}}+\delta e^{\delta(t-|x_1|)}\end{aligned}$$

for all $t\le T$ and $(x_1,x_2)\in\mathbb{R}^2$.

For $t\le 0$, let

$$\begin{aligned}P_t^l&=(ct\cos\alpha,ct\sin\alpha),\quad L_t^l=P_t^l+\mathbb{R}_+(\cos(2\alpha),\sin(2\alpha)),\\P_t^r&=(-ct\cos\alpha,ct\sin\alpha),\quad L_t^r=P_t^r+\mathbb{R}_+(-\cos(2\alpha),\sin(2\alpha))\end{aligned}$$

and

$$\Gamma_t=L_t^l\cup[P_t^l,P_t^r]\cup L_t^r\quad\text{for all } t\le 0,\tag{81}$$

where the superscript l (resp. r) stands for left (resp. right). Define

$$\Gamma_t=\left\{(x_1,x_2)\in\mathbb{R}^2\middle|x_2=|\tan(2\alpha)||x_1|+\frac{c_ft}{|\cos(2\alpha)|}\right\}\quad\text{for all } t>0.\tag{82}$$

Thus, for every $t\in\mathbb{R}$, Γ_t can be written as a graph $\Gamma_t=\{(x_1,x_2)\in\mathbb{R}^2;x_2=\hat{\varphi}_t(x_1)\}$, where $\hat{\varphi}_t:\mathbb{R}\to\mathbb{R}$ is a Lipschitz-continuous function. For all $t\in\mathbb{R}$, define

$$\Omega_t^+=\{(x_1,x_2)\in\mathbb{R}^2\big|x_2<\hat{\varphi}_t(x_1)\}\text{ and }\Omega_t^-=\{(x_1,x_2)\in\mathbb{R}^2\big|x_2>\hat{\varphi}_t(x_1)\}.\tag{83}$$

Obviously, the sets $(\Omega_t^\pm)_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$ satisfy the general properties (5) and (6).

Similar to the proof of Lemma 5.2 of [11], the function u is a transition front connecting 0 and 1 for problem (1) in $\mathbb{R}^2$ with the sets $(\Omega_t^\pm)_{t\in\mathbb{R}}$ and $(\Gamma_t)_{t\in\mathbb{R}}$.

Now we prove that the solution u is not invariant as time runs with any moving frame. That is, it satisfies the conclusion of Theorem 3. Assume by contradiction that there exist a function $\Phi:\mathbb{R}^2\to(0,1)$ and some families $(R_t)_{t\in\mathbb{R}}$ and $(X_t)_{t\in\mathbb{R}}=(x_{1,t},x_{2,t})_{t\in\mathbb{R}}$ of rotations and points in $\mathbb{R}^2$ such that

$$u(t,x_1,x_2)=\Phi(R_t(x_1-x_{1,t},x_2-x_{2,t}))\quad\text{for all }(t,x_1,x_2)\in\mathbb{R}\times\mathbb{R}^2.$$

Then there is $M\ge 0$ such that

$$R_t(\Gamma_t-X_t)\subset\{(x_1,x_2)\in\mathbb{R}^2\big|d((x_1,x_2),R_s(\Gamma_t-X_s))\le M\}\text{ for all }(t,s)\in\mathbb{R}^2,$$

which is contradicted with the definitions of the sets Γ_t defined as (81) and (82). Whence, Theorem 3 holds in $\mathbb{R}^2$.

Now, we extend the transition front u trivially in $\mathbb{R}^N$ ($N\ge 3$). Let

$$\widetilde{u}(t,x_1,\dots,x_N)=u(t,x_1,x_2)\quad\text{for all }(t,x_1,\dots,x_N)\in\mathbb{R}\times\mathbb{R}^N.$$

Obviously, the function $\widetilde{u}$ is a transition front connecting 0 and 1 for problem (1) in $\mathbb{R}^N$ with the sets

$$\widetilde{\Omega}_t^\pm=\{(x_1,\dots,x_N)\in\mathbb{R}^N\big|(x_1,x_2)\in\Omega_t^\pm\}\quad\text{for all } t\in\mathbb{R}$$

and satisfies the desired conclusion. This completes the proof of Theorem 3.

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