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Low Regularity for the Higher Order Nonlinear Dispersive Equation in Sobolev Spaces of Negative Index

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Abstract In this paper, we investigate the initial value problem(IVP henceforth) associated with the higher order nonlinear dispersive equation given in Jones et al. (Int J Math Math Sci 24:371–377, 2000):

$$\begin{cases} \partial_t u + \alpha \partial_x^7 u + \beta \partial_x^5 u + \gamma \partial_x^3 u + \mu \partial_x u + \lambda u \partial_x u = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

with the initial data in the Sobolev space $H^s(\mathbb{R})$. Benefited from the ideas of Huo and Jia (Z Angew Math Phys 59:634–646, 2008), Zhang et al. (Acta Math Sci 37B(2):385–394, 2017) and Zhang and Huang (Math Methods Appl Sci 39(10):2488–2513, 2016) that is, using Fourier restriction norm method, Tao's $[k, Z]$-multiplier method and the contraction mapping principle, we prove that IVP is locally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ with $s \geq -\frac{5}{8}$. Moreover, based on the local well-posedness and conservation law, we establish the global well-posedness for the initial data $u_0 \in H^s(\mathbb{R})$ with $s = 0$.

Keywords Higher order nonlinear dispersive equation · Fourier restriction norm method · Low regularity · Well-posedness

Mathematics Subject Classification 35A07 · 35Q53

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1 Introduction

In this paper, we investigate the IVP associated with the higher order nonlinear dispersive equation as follows:

$$\partial_t u + \alpha\partial_x^7 u + \beta\partial_x^5 u + \gamma\partial_x^3 u + \mu\partial_x u + \lambda u\partial_x u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $\alpha \neq 0, \beta, \gamma, \mu, \lambda$ are real numbers.

Equation (1.1) is called the higher order nonlinear dispersive equation which arises in the study of water waves with surface tension and arises as mathematical models for the weakly nonlinear propagation of long waves, see Ref. [1]. Indeed, Eq. (1.1) models the situation when nonlinearity (i.e $\lambda u\partial_x u$), dispersion (including higher dispersion $\alpha\partial_x^7 u$ and lower dispersion $\beta\partial_x^5 u, \alpha\partial_x^3 u, \gamma\partial_x u$) are taken into account at the same time.

Setting $\alpha = 0$ in (1.1), Eq. (1.1) becomes the Kawahara equation (for short KE)

$$\partial_t u + \beta\partial_x^5 u + \gamma\partial_x^3 u + \mu\partial_x u + \lambda u\partial_x u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{1.3}$$

Hence, Eq. (1.1) can be regarded as a perturbation of KE(1.3)by a higher dispersion term $\alpha\partial_x^7 u$. (1.3) is one of important dispersive equation which was proposed firstly by Kawahara in 1972, see Ref. [2].

When $\beta = 0$, $\gamma \neq 0$ and $\mu \neq 0$, Eq. (1.3) reduces to KdV equation. There is a large literature concerning well-posedness of KdV equation in Sobolev spaces with low regularity. We refer the reader to Ref. [3–8]. In [3], by using the I-method, the authors proved the KdV equation is globally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -\frac{3}{4}$. In [7], by using Cauchy-Schwartz inequality, the authors proved that the KdV equation is globally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -\frac{3}{4}$. In [8], Molinet and Ribaud investigated the local and global Cauchy problem for the generalized KdV equation

$$\partial_t u + \partial_x^3 u + u^k\partial_x u = 0, \quad k \geq 4, \quad k \in N^+$$

with the initial data in homogeneous and non-homogeneous Besov spaces and proved existence and uniqueness of self-similar solutions. In [9], by constructing some special resolution spaces and using dyadic bilinear estimates together with I-method, Guo proved that the KdV equation is globally well-posedness for the initial data $u_0 \in H^s(\mathbb{R})$ with $s = -\frac{3}{4}$. When $\alpha = 0$, $\beta \neq 0$, $\mu \neq 0$, Eq. (1.1) reduces to an usual Kawahara Eq. (1.3) which has been studied by many authors, see [10–15]. In [10], the authors proved that (1.2–1.3) have a local solution for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -1$ and a global solution for $u_0 \in L^2(\mathbb{R})$. In [12], the authors showed that the Cauchy problem (1.2–1.3) is locally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ with $s \geq -\frac{7}{5}$ and globally well-posed for $u_0 \in H^s(\mathbb{R})$ with $s \geq -\frac{1}{2}$. In [13], by using $[K; Z]$-multiplier norm method, the authors studied the local well-posedness for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -\frac{7}{4}$. In [14], the authors established the local well-posedness result of (1.3) (1.2) for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -\frac{7}{4}$.

As for the Cauchy problem (1.2–1.3), Huo [15] obtained the local well-posedness for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -\frac{11}{8}$. Later on, Tao and Cui [16] considered the IVP (1.2–1.3) and proved the local solution for $u_0 \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$, and the global solution for $u_0 \in H^s(R)$ with $s \geq 2$ by using strichartz estimates. Recently, Yan and Li [17] showed that the Cauchy problem (1.2–1.3) is globally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -\frac{63}{58}$ by using I-method as well as L^2 conservation law. Later on, Yan et al. [18] improved the above results. More precisely, the authors proved that the Cauchy problem (1.2–1.3) is globally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > -\frac{3}{22}$ by using the

Fourier restriction norm method, I-method and L^2 conservation law. Quite recently, Chen and Guo [19] improved the results of [17]. More precisely, the authors proved that (1.3) is locally well-posed in $H^{-\frac{7}{4}}$ by using the ideas of $\overline{F}^s$-type space [8]. Next they established that it is globally well-posed in $H^s(\mathbb{R})$ for $s \geq -\frac{7}{4}$ by using the ideas of I-method. Compared to the KdV Eq. (1.3) has less symmetries, such as no invariant scaling transform and not completely integrable. They treated with some new difficulties that are caused by the lack symmetries of this equation.

It is worth mentioning that Zhang et al. [20] studied the initial value problem associated with the generalized Kawahara equation as follows:

$$\begin{cases} \partial_t u + \alpha \partial_x^5 u + \beta \partial_x^3 u + \gamma \partial_x u + \mu \partial_x (u^k) = 0, \quad x \in R, \quad t \geq 0, \\ u(x, 0) = u_0(x) \end{cases}$$

with initial data in the Sobolev space $H^s(R)$. Benefited from ideas of Ref. [21,22], first, we proved that the local well-posedness is established for the initial data $u_0 \in H^s(R)$ with $s \geq -\frac{7}{4}(k = 2)$ and $s \geq -\frac{1}{4}(k = 3)$ respectively. Then,using these results and conservation laws, we also proved that the IVP is globally well-posed for the initial data $u_0 \in H^s(R)$ with $s = 0(k = 2, 3)$. Finally, benefited from ideas of Ref. [21–23], i.e, using complex variables technique and Paley–Wiener theorem, we prove the unique continuation property.

As for the Cauchy problem (1.1)–(1.2), Tao and Cui [24] have established the first results of the well-posedness. More precisely, the authors showed that the Cauchy problem (1.1)–(1.2) is locally well-posed for the initial data $u_0 \in H^s(\mathbb{R})$ with $s \geq \frac{2}{13}$ by using some dispersive estimates and Banach fixed point technique. Related works are present in Ref. [25,26]. Kenig et al. [27] studied the following high-order dispersive equation

$$\partial_t u + \partial_x^{2j+1} u + P\left(u, \partial_x u, \ldots, \partial_x^{2j} u\right) = 0$$

and obtained local well-posedness for initial data $u_0 \in H^s(\mathbb{R}) \cap L^2(|x|^m dx)$, $s, m \in Z^+$, where P is a polynomial without constant or linear terms. Pilod [28] investigated the higher-order nonlinear dispersive equation

$$\partial_t u + \partial_x^{2j+1} u + \sum_{0 \leq j_1 + j_2 \leq 2j} a_{j_1, j_2} \partial_x^{j_1} u \partial_x^{j_2}.$$

The author showed that the associated initial value problem is well-posed in weighted Besov and Sobolev spaces for small initial data and he also proved ill-posedness results when $a_{0,k} \neq 0$ for some $k > j$.

In our contribution, in order to obtain the local well-posedness of (1.1) and (1.2), our novelty is to establish the new bilinear estimates by using Fourier restriction norm method, and is to improve the contribution by Tao and Cui [24]. It is important to point out that the phase function $\phi(\xi)$(see Definition 2.1 in section 2) or their derivatives has one pole and some non-zero singular points. This is different from the phase function of the semigroup of the linear KdV equation and the Kawahara equation, and also makes the problem much more difficulty. Therefore, we need to use Fourier restriction operators

$$P^N f = \int_{|\xi| \geq N} e^{ix\xi} \hat{f}(\xi) d\xi, \quad P_N f = \int_{|\xi| \leq N} e^{ix\xi} \hat{f}(\xi) d\xi, \quad \forall N > 0,$$

to eliminate the singularity of the phase function $\phi(\xi)$. Moreover, the operators will be used to decompose nonlinear term $\partial_x(u^2)$. To deal with the term, we first decompose it as the high-frequency part and the corresponding low-frequency one as follows:

$$\partial_x(u^2) = P^N\{\partial_x(u^2)\} + P_N\{\partial_x(u^2)\}. \tag{1.4}$$

Next, we are going to decompose each term on the right hand side of (1.4) as the sumation of those products which consist of each factor acted on by the Fourier restriction operators P^N or P_N. We shall estimate each resulting term with different methods to overcome the obstacles. To the best of our knowledge, this is the first well-posedness result the IVP (1.1)–(1.2).

The rest of the paper is organized as follows. In Sect. 2, we show some notations and state our main result. In Sect. 3, we show some preliminary results that will play fundamental role in our further analysis. In Sect. 4, using Fourier restriction norm method, we establish the linear estimates. Finally, in Sect. 5, we prove of the main results, including the local well-posedness(LWP) and global well-posedness(GWP) for the IVP (1.1) and (1.2). More precisely, firstly, using the bilinear estimate and the linear estimate, together with contraction mapping principle, we prove the LWP. Secondly, we obtain the GWP, which follows from LWP and the L^2 conservation law by standard method.

2 Some Notations and Main Results

In this section, before proceeding to our analysis, we present some notations which will be used throughout in our paper and state the main result.

Definition 2.1 For $s, b \in \mathbb{R}$, the space $X_{s,b}$ is the complete of the Schwartz function on $\mathbb{R}^2$ with respect to the norm

$$\|u\|_{X_{s,b}} = \|S(t)u\|_{H_x^s H_t^b} = \|\langle\xi\rangle^s \langle\tau - \phi(\xi)\rangle^b \mathcal{F}(u)(\xi,\tau)\|_{L_\xi^2 L_\tau^2},$$

where $\langle\cdot\rangle = 1 + |\cdot|$, the phase function $\phi(\xi) = \alpha\xi^7 - \beta\xi^5 + \gamma\xi^3 - \mu\xi$.

In our arguments, we shall use the trivial embedding

$$\|u\|_{X_{s_1,b_1}} \le \|u\|_{X_{s_2,b_2}}, \quad \text{for} \quad s_1 \le s_2, \quad b_1 \le b_2.$$

We denote $\widehat{u}(\xi,\tau) = \mathcal{F}(u)$ by the Fourier transform of u both variable x and t, and by $\mathcal{F}_{(\cdot)}(u)$ the Fourier transform in the $(\cdot)$ variable.

We use $A \sim B$ using the statement that $A \le C_1 B$ and $B \le C_1 A$ for some constant $C_1 > 0$, use $A \ll B$ to denote the statement that $A \le \frac{1}{C_2} B$ for some large enough constant $C_2 > 0$. We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \le CY$ for some constant $C > 0$.

We introduce some variables for convenience

$$\sigma = \tau - \phi(\xi), \quad \sigma_j = \tau_j - \phi(\xi_j), \quad j = 1, 2.$$

Throughout this paper, we shall denote the following notation $\int_\star d\delta$ as the convolution integral

$$\int_{\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2} \cdot d\tau_1 d\tau_2 d\xi_1 d\xi_2.$$

In what follows, we shall give some useful notations for multilinear expressions in Ref. [29]. Let Z be any Abelian additive group with an invariant measure $d\delta$. For any integer $k \ge 2$, we define $\Gamma_k(Z)$ to be the hyperplane

$$\Gamma_k(Z) = \left\{ (\xi_1, \xi_2, \dots, \xi_k) \in Z^k : \sum_{j=1}^{k} \xi_j = 0 \right\},$$

and define a $[k; Z]$-multiplier to be any function $m : \Gamma_k(Z) \to \mathcal{C}$. If m is a $[k; Z]$-multiplier, then we define $\|m\|_{[k;Z]}$ to be the best constant, such that the following inequality

$$\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_j(\xi_j) \right| \leq \|m\|_{[k;Z]} \prod_{j=1}^{k} \|f_j\|_{L^2(Z)}$$

holds for all the test functions f_j defined on Z. It is obvious that $\|m\|_{[k;Z]}$ determines a norm on m for test functions at least. We are concerned with the good boundedness on the norm. In this paper, we let $Z = \mathbb{R} \times \mathbb{R}$.

We now are position to state the main results.

Theorem 2.1 (Local well-posedness) *Assume that $\alpha\beta < 0$ and $\gamma > 0$. Let $s \geq -\frac{5}{8}$ and $u_0 \in H^s(\mathbb{R})$. Then there exist a real number $b > \frac{1}{2}$, which is close enough to $\frac{1}{2}$ and a constant $T > 0$ such that the Cauchy problem (1.1) and (1.2) admits a unique local solution $u(x, t) \in C([0, T]; H^s) \cap X_{s,b}$. Moreover, given $t \in [0, T]$, the map $u_0 \to u(t)$ is Lipschitz continuous from H^s to $C([0, T]; H^s)$.*

Theorem 2.2 (Global well-posedness) *For $s = 0$, the solution obtained in Theorem 2.1 can be extended to a global one.*

3 Preliminary Estimates

In this section, we shall deduce several estimates. To facilitate further on our analysis, we introduce some notations as follows:

$$\begin{aligned}
a &= \max\left\{1, \left(\left|\frac{5\beta}{7\alpha}\right|\right)^{\frac{1}{2}}\right\},\\
\mathcal{F}F_\rho(\xi, \tau) &= \frac{F(\xi, \tau)}{(1 + |\tau - \phi(\xi)|)^\rho},\\
D_x^{-s} &= \mathcal{F}_x^{-1}|\xi|^{-s}\mathcal{F}_x,\\
\|f\|_{L_x^p L_t^q} &= \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x, t)|^q dt\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}},\\
\|f\|_{L_t^q L_x^p} &= \left(\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x, t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}},\\
\|f\|_{L_t^\infty H_x^s} &= \| \|f\|_{H_x^s} \|_{L_t^\infty}.
\end{aligned}$$

Next, we shall show preliminary estimates.

Lemma 3.1 [24,30] *Assume that $\alpha \in \mathbb{R}$ and $0 \leq \alpha \leq \frac{5}{2}$. Then for $\forall\delta > 0$, there exists a constant $C > 0$, such that $\sup|D^\alpha G(x, t)| \leq C|t|^{-\frac{1}{7}(\alpha+1)}$, where $x \in \mathbb{R}$, $0 < |t| \leq \delta$.*

Lemma 3.2 *The group $\{S(t)\}_{-\infty}^{\infty}$ satisfies*

$$\|S(t)\varphi\|_{L_x^{12} L_t^{12}} \lesssim \|\varphi\|_{L^2}. \tag{3.1}$$

Proof More generally, we are going to prove

$$\|S(t)\varphi\|_{L_x^p L_t^q} \lesssim \|\varphi\|_{L^2}, \tag{3.2}$$

where $p \geq 2, q \geq 2, \frac{2}{p} = \frac{1}{5}\left(1 - \frac{2}{q}\right)$. When $p = q = 12$, (3.2) is corresponding to (3.1).

In fact, by duality, we need to bound

$$\left\| \int S(t) f(t, \cdot) dt \right\|_{L_x^2} \tag{3.3}$$

for $f \in L_t^{p'} L_x^{q'}$, $\|f\|_{L_t^{p'} L_x^{q'}} = 1$, where $\frac{1}{p'} + \frac{1}{p} = 1$ and $\frac{1}{q'} + \frac{1}{q} = 1$.

Squaring (3.3) and utilizing $\|f\|_{L_t^{p'} L_x^{q'}} = 1$, it follows from the unitary property of the linear group that

$$\int \int \langle S(s-t) f(t, \cdot), f(s, \cdot) \rangle ds dt \lesssim \left\| \int S(t) f(t, \cdot) dt \right\|_{L_s^p L_x^2}, \tag{3.4}$$

where $\langle \cdot, \cdot \rangle$ is defined as the inner product in L_x^2.

On the other hand, applying interpolation technique (see Ref. [20–22,31]) and Strichartz esimate (see Ref. [24,30]), we obtain

$$\|S(t)\varphi\|_{L_x^p} \lesssim |t|^{-\frac{1}{7}\left(1-\frac{2}{p}\right)} \|\varphi\||_{L_x^{p'}}, \quad \text{where} \quad \frac{1}{p'} + \frac{1}{p} = 1. \tag{3.5}$$

Substituting (3.5) in (3.4), we get the bound

$$\| \int |s-t|^{-\frac{1}{7}(1-\frac{2}{p})} \|f(t)\|_{L_x^{q'}} dt \|_{L_s^p} \lesssim \|f\|_{L_t^{p'} L_x^{q'}} \lesssim 1,$$

by using the Hardy–Littlewood–Sobolev inequality [32]. This proves (3.2). □

Remark 3.1 Indeed, note that $S(t) f(x) = (G(\cdot, t) * f)(x)$, $S(0) f = f$, and phase function $\phi(\xi) = \alpha\xi^7 - \beta\xi^5 + \gamma\xi^3 - \mu\xi$, then we have $u(x, t) = S(t)u_0(x)$ and

$$\begin{aligned} D_x^\alpha G(x, t) &= c \int_{\mathbb{R}} |\xi|^\alpha e^{i(x,t)\cdot(\xi, \phi(\xi))} d\xi \\ &= c \int_{\mathbb{R}} |\xi|^\alpha e^{i(x\xi + t\phi(\xi))} d\xi. \end{aligned}$$

Using Young inequality and Strichartz estimate in Ref. [30],as well as taking $\alpha = 0$ in Lemma 3.1, we have

$$\|S(t)\varphi\|_{L_x^\infty} \leq C|t|^{-\frac{1}{7}} \|\varphi\||_{L_x^1}.$$

On the other hand, it is easy to see that

$$\|S(t)\varphi\|_{L_x^2} \leq C\|\varphi\||_{L_x^2}.$$

Therefore, (3.5) follows by interpolation the above inequality.

Lemma 3.3 *The group* $\{S(t)\}_{-\infty}^{\infty}$ *satisfies*

$$\left\| D_x^3 P^{2a} S(t)\varphi \right\|_{L_x^\infty L_t^2} \lesssim \|\varphi\|_{L_x^2}, \tag{3.6}$$

$$\left\| D_x^{-\frac{1}{4}} P^{2a} S(t)\varphi \right\|_{L_x^4 L_t^\infty} \lesssim \|\varphi\|_{L_x^2}, \tag{3.7}$$

$$\left\| D_x^{\frac{5}{6}} P^{2a} S(t)\varphi \right\|_{L_x^6 L_t^6} \lesssim \|\varphi\|_{L_x^2}. \tag{3.8}$$

Proof First, we prove (3.6). It is easy to see that

$$\begin{aligned} \phi(\xi) &= \alpha\xi^7 - \beta\xi^5 + \gamma\xi^3 - \mu\xi, \\ \phi'(\xi) &= 7\alpha\xi^6 - 5\beta\xi^4 + 3\gamma\xi^2 - \mu, \\ \phi''(\xi) &= 42\alpha\xi^5 - 20\beta\xi^3 + 6\gamma\xi, \end{aligned}$$

where $\alpha\beta < 0$, $\gamma > 0$.

If $|\xi| \geq 2a$, then the phase function $\phi(\xi)$ is invertible, and we have

$$\begin{aligned} P^{2a} S(t)\varphi &= \int_{|\xi|\geq 2a} e^{ix\xi} e^{-it\phi(\xi)} \varphi(\xi) d\xi \\ &= \int_{|\phi^{-1}|\geq a} e^{ix\phi^{-1}} e^{-it\phi(\xi)} \varphi(\phi^{-1}) \frac{1}{\phi'} d\phi \\ &= \mathcal{F}_t \left(e^{ix\phi^{-1}} \chi_{|\phi^{-1}|\geq 2a} \varphi(\phi^{-1}) \frac{1}{\phi'} \right). \end{aligned}$$

In what follows, we shall use the change of variable $\xi = \phi^{-1}$. By Plancherel equality,we have

$$\begin{aligned} \|P^{2a} S(t)\varphi\|_{L_t^2}^2 &= \left\| \mathcal{F}_t \left(e^{ix\phi^{-1}} \chi_{|\phi^{-1}|\geq 2a} \varphi(\phi^{-1}) \frac{1}{\phi'} \right) \right\|_{L_\phi^2}^2 \\ &= \int_{|\phi^{-1}|\geq 2a} |\widehat{\varphi}(\phi^{-1})|^2 \frac{1}{|\phi'(\xi)|^2} \phi'(\xi) d\xi = \int_{|\xi|\geq 2a} |\varphi(\xi)|^2 \frac{1}{|\phi'(\xi)|^2} |\phi'(\xi)| d\xi \\ &\leq \int_{|\xi|\geq 2a} |\varphi(\xi)|^2 \frac{1}{|\phi'(\xi)|} d\xi = \int_{|\xi|\geq 2a} |\varphi(\xi)|^2 \frac{1}{|7\alpha\xi^6||1 - \frac{5\beta}{7\alpha\xi^2} + \frac{3\gamma}{7\alpha\xi^4} - \frac{\mu}{7\alpha\xi^6}|} d\xi \\ &\lesssim \int_{|\xi|\geq 2a} |\varphi(\xi)|^2 \frac{1}{|\xi|^6} d\xi \\ &\lesssim \|\varphi\|_{\dot{H}^{-3}}^2. \end{aligned}$$

This implies the estimate (3.6).

Next, we turn to the proof of (3.7). If $|\xi| \geq 2a$, then we can also obtain $|\phi''(\xi)| \gtrsim |\xi|^5$. In fact, with the help of Theorem 2.5 in Ref. [33], we conclude that

$$\begin{aligned}
\|P^{2a}S(t)\varphi\|^2_{L^4_xL^\infty_t} &\leq \int_{|\xi|\geq 2a} |\mathcal{F}P^{2a}\varphi(\xi)|^2 \left|\frac{\phi'(\xi)}{\phi''(\xi)}\right|^{\frac{1}{2}} d\xi \\
&\leq C\int_{|\xi|\geq 2a} |\mathcal{F}P^{2a}\varphi(\xi)|^2 \left(\frac{|7\alpha\xi^6||1-\frac{5\beta}{7\alpha\xi^2}+\frac{3\gamma}{7\alpha\xi^4}-\frac{\mu}{7\alpha\xi^6}|}{|42\alpha\xi^5||1-\frac{20\beta}{42\alpha\xi^2}+\frac{6\gamma}{42\alpha\xi^4}|}\right)^{\frac{1}{2}} d\xi \\
&\lesssim \int_{|\xi|\geq 2a} |\mathcal{F}P^{2a}\varphi(\xi)|^2 |\xi|^{\frac{1}{2}} d\xi \\
&\lesssim \|P^{2a}\varphi\|^2_{H^{\frac{1}{4}}}.
\end{aligned}$$

which implies the estimate (3.7). Finally, (3.8) follows by interpolation between (3.6) and (3.7). □

Lemma 3.4 *If* $\rho > \frac{1}{2}$, *for any fixed* $N > 0$, *then*

$$\|P_N F_\rho\|_{L^2_xL^\infty_t} \leq C\|f\|_{L^2_\xi L^2_\tau}.$$

Proof The proof is similar with that of Lemma 2.2 in Ref. [33], we omit the details here. □

Lemma 3.5 *Suppose* $\rho > \frac{1}{2}\frac{6(q-2)}{5q}$. *Then, for* $2 \leq q \leq 12$, *we have*

$$\|F_\rho\|_{L^q_xL^q_t} \leq C\|f\|_{L^2_\xi L^2_\tau}. \tag{3.9}$$

Proof Using the change of the variable $\tau = \lambda + \phi(\xi)$, we have

$$\begin{aligned}
F_\rho(x,t) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(x\xi+t\tau)}\frac{f(\xi,\tau)}{(1+|\tau-\phi(\xi)|)^\rho}d\xi d\tau \\
&= \int_{-\infty}^{+\infty} e^{it\lambda}\left(\int_{-\infty}^{+\infty} e^{i(x\xi+t\phi(\xi))}f(\xi,\lambda+\phi(\xi))d\xi\right)\frac{d\lambda}{(1+|\lambda|)^\rho}.
\end{aligned}$$

Using (3.1) and Minkowski's integral inequality and taking into account $\rho > \frac{1}{2}$, we get

$$\begin{aligned}
\|F_\rho\|_{L^{12}_xL^{12}_t} &\lesssim \int_{-\infty}^{+\infty} \|f(\xi,\lambda+\phi(\xi))\|_{L^2_\xi}\frac{d\lambda}{(1+|\lambda|)^\rho} \\
&\lesssim \|f\|_{L^2_\xi L^2_\tau}.
\end{aligned} \tag{3.10}$$

It is easy to see that

$$\|F_0\|_{L^2_xL^2_t} \lesssim \|f\|_{L^2_\xi L^2_\tau}. \tag{3.11}$$

Then, (3.9) follows by interpolation between (3.10) and (3.11). □

Lemma 3.6 *If* $\rho > \frac{3}{8}$, *then*

$$\left\|D_x^{\frac{5}{8}}P^{2a}F_\rho\right\|_{L^4_xL^4_t} \lesssim \|f\|_{L^2_\xi L^2_\tau}. \tag{3.12}$$

Proof Taking into account $\rho > \frac{1}{2}$, and using (3.8) and Minkowski's integral inequality, we have

$$\begin{aligned}\left\| D_x^{\frac{5}{8}} P^{2a} F_\rho \right\|_{L_x^6 L_t^6} &\lesssim \int_{-\infty}^{+\infty} \|f(\xi, \lambda + \phi(\xi))\|_{L_\xi^2} \frac{d\lambda}{(1+|\lambda|)^\rho} \\ &\lesssim \|f\|_{L_\xi^2 L_\tau^2}. \end{aligned} \tag{3.13}$$

Then, (3.12) follows by interpolation between (3.11) and (3.13). □

Lemma 3.7 (1) *Let* $\rho > \frac{1}{2}\theta$, $\theta \in [0, 1]$, *then*

$$\left\| D_x^{3\theta} P^{2a} F_\rho \right\|_{L_x^{\frac{2}{1-\theta}} L_t^2} \lesssim \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.14}$$

(2) *Let* $\rho > \frac{1}{2}$, *then*

$$\left\| D_x^{-\frac{1}{4}} P^{2a} F_\rho \right\|_{L_x^4 L_t^\infty} \lesssim \|f\|_{L_\xi^2 L_\tau^2}. \tag{3.15}$$

Proof From the arguments in arriving at Lemma 3.5 and (3.6), we obtain

$$\left\| D_x^3 P^{2a} F_\rho \right\|_{L_x^\infty L_t^2} \lesssim \|\varphi\|_{L_x^2}. \tag{3.16}$$

It is clear that (3.14) follows by interpolation between (3.16) and (3.11). Meanwhile, (3.15) follows by interpolation between (3.7) and (3.11). □

Lemma 3.8 [14,15] *Assume that* f, f_1 *and* f_2 *belong to Schwartz space on* $\mathbb{R}^2$. *Then, we have*

$$\int_\star \overline{\widehat{f}}(\xi, \tau) \widehat{f_1}(\xi_1, \tau_1) \widehat{f_2}(\xi_2, \tau_2) d\delta = \int_{R^2} \overline{f} f_1 f_2(x, t) dx dt, \tag{3.17}$$

Lemma 3.9 [34] *If* m *and* M *are* $[k; Z]$ *multipliers and satisfy* $|m(\xi)| \leq |M(\xi)|$ *for all* $\xi \in \Gamma_k(Z)$, *then* $\|m\|_{[k;Z]} \leq \|M\|_{[k;Z]}$.

Lemma 3.10 *If* $|\xi| \geq 2a$, *then we have*

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi_1||\xi_2||\xi|^5,$$

where $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$; $\sigma = \tau - \phi(\xi)$, $\sigma_1 = \tau_1 - \phi(\xi_1)$, $\sigma_2 = \tau_2 - \phi(\xi_2)$; $\phi(\xi) = \phi(\xi) = \alpha\xi^7 - \beta\xi^5 + \gamma\xi^3 - \mu\xi$.

Proof It is easy to see that

$$\sigma_1 + \sigma_2 - \sigma = 7\alpha\xi_1\xi_2\xi\left(\xi^2 - \xi_1\xi + \xi_1^2\right)^2 - 5\beta\xi_1\xi_2\xi\left(\xi^2 - \xi_1\xi + \xi_1^2\right) + 3\gamma\xi_1\xi_2\xi.$$

Observing that $\xi^2 - \xi_1\xi + \xi_1^2 \geq \frac{3}{4}\xi^2$ and $\alpha\beta < 0$, $\gamma > 0$, we get $|\sigma_1 + \sigma_2 - \sigma| \gtrsim |\xi_1||\xi_2||\xi|^5$ which implies Lemma 3.10. □

4 Bilinear Estimates

In this section, using Fourier restriction norm method, we shall state a bilinear estimate which will be the main tool in the proof of local existence for the IVP(1.1) and (1.2).

Lemma 4.1 *Let b be close enough to $\frac{1}{2}$ satisfying $b > \frac{1}{2}$. For $\frac{1}{2} < b'$ and $s \geq -\frac{5}{8}$, we have*

$$\|\partial_x(u_1u_2)\|_{X_{s,b-1}} \lesssim \|u_1\|_{X_{s,b'}}\|u_2\|_{X_{s,b'}}. \tag{4.1}$$

Proof By duality and Plancherel identity, it suffices to show that

$$\begin{aligned}\Gamma &= \int_\star \langle\xi\rangle^s|\xi|\frac{\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\mathcal{F}u_1(\xi_1,\tau_1)\mathcal{F}u_2(\xi_2,\tau_2)d\delta \\ &= \int_\star \frac{\langle\xi\rangle^s|\xi|}{\langle\sigma\rangle^{1-b}\prod_{j=1}^2\langle\xi_j\rangle^s\langle\sigma_j\rangle^{b'}}\overline{f}(\xi,\tau)f_1(\xi_1,\tau_1)f_2(\xi_2,\tau_2)d\delta \\ &\leq \left\|\frac{\langle\xi\rangle^s|\xi|}{\langle\sigma\rangle^{1-b}\prod_{j=1}^2\langle\xi_j\rangle^s\langle\sigma_j\rangle^{b'}}\right\|_{[3;\mathbb{R}\times\mathbb{R}]}\|\overline{f}\|_{L^2_\xi L^2_\tau}\prod_{j=1}^2\|f_j\|_{L^2_\xi L^2_\tau},\end{aligned}$$

for $\overline{f} \in L^2(\mathbb{R}^2)$, $\overline{f} \geq 0$, where $f_j = \langle\xi_j\rangle^s\langle\sigma_j\rangle^{b'}\widehat{u}_j$, $j = 1, 2$, $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$. □

For convenience to our further analysis, we introduce some notations as follows

$$\begin{aligned}\mathcal{F}F^j_\rho(\xi,\tau) &= \frac{f_j(\xi,\tau)}{(1+|\tau-\phi(\xi)|)^\rho}, \quad j = 1, 2, \\ K(\xi,\xi_1,\xi_2) &= \frac{\langle\xi\rangle^s|\xi|}{\prod_{j=1}^2\langle\xi_j\rangle^s}.\end{aligned}$$

In order to bound the integral Γ, we shall split the domain of the integral into several pieces. Here, we consider the most interesting case $s \leq 0$. Otherwise, it is easy for us to see that $K(\xi,\xi_1,\xi_2) \lesssim 1$. Let $r = -s$. By symmetry, it suffices to estimate the integral Γ in the domain $|\xi_1| \leq |\xi_2|$.

Situation I. Assume that $|\xi| \leq 4a$.

Case 1. If $|\xi_1| \leq 2a$, then we have $|\xi_2| \leq |\xi - \xi_1| \leq 6a$ and $K_1(\xi,\xi_1,\xi_2) \lesssim 1$. Consequently, the integral Γ restricted to this domain is bounded by

$$\begin{aligned}\Gamma &= \int_\star \frac{|\xi|\chi_{|\xi|\leq 4a}\overline{f}(\xi,\tau)}{\langle\xi\rangle^r\langle\sigma\rangle^{1-b}}\frac{\langle\xi_1\rangle^r\chi_{|\xi_1|\leq 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\langle\xi_2\rangle^r\chi_{|\xi_2|\leq 6a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta \\ &\lesssim \int_\star \frac{\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\frac{f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta \\ &\lesssim \int \overline{F}_{1-b}\cdot F^1_{b'}\cdot F^2_{b'}(x,t)dxdt \\ &\lesssim \|F_{1-b}\|_{L^2_xL^2_t}\|F^1_{b'}\|_{L^4_xL^4_t}\|F^2_{b'}\|_{L^4_xL^4_t} \\ &\lesssim \|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau},\end{aligned}$$

which follows by Lemma 3.5 (with $q = 4$) and Lemma 3.7.

Case 2. If $2a \leq |\xi_1| \leq |\xi_2|$, then, for $r = -s < \frac{5}{8}$ we conclude that $K_(\xi,\xi_1,\xi_2) \leq C|\xi_1|^{\frac{5}{8}}|\xi_2|^{\frac{5}{8}}$. Therefore, by Lemmas 3.6 and 3.7, the integral Γ restricted to this domain is bounded by

$$
\begin{aligned}
\Gamma &= \int_\star \frac{|\xi|\chi_{|\xi|\le 4a}\overline{f}(\xi,\tau)}{\langle\xi\rangle^r\langle\sigma\rangle^{1-b}} \frac{\langle\xi_1\rangle^r\chi_{|\xi_1|\ge 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}} \frac{\langle\xi_2\rangle^r\chi_{|\xi_2|\ge 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta \\
&\le C\int_\star \frac{\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{|\xi_1|^{\frac{5}{8}}\chi_{|\xi_1|\ge 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}} \frac{|\xi_2|^{\frac{5}{8}}\chi_{|\xi_2|\ge 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta \\
&= C\int \overline{F}_{1-b}\cdot D_x^{\frac{5}{8}}P^{2a}F_{b'}^1\cdot D_x^{\frac{5}{8}}P^{2a}F_{b'}^2(x,t)dxdt \\
&\lesssim \|F_{1-b}\|_{L_x^2L_t^2}\left\|D_x^{\frac{5}{8}}P^{2a}F_{b'}^1\right\|_{L_x^4L_t^4}\left\|D_x^{\frac{5}{8}}P^{2a}F_{b'}^2\right\|_{L_x^4L_t^4} \\
&\lesssim \|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2}.
\end{aligned}
$$

Situation II. Assume that $|\xi|\le 4a$.

Case 1. If $|\xi_1|\le 2a$, then we have $|\xi_2|\ge 2a$, $|\xi|\sim|\xi_2|$ and $K_1(\xi,\xi_1,\xi_2)\le C|\xi_2|$. Consequently, the integral Γ restricted to this domain is bounded by

$$
\begin{aligned}
\Gamma &= \int_\star \frac{|\xi|\chi_{|\xi|\ge 4a}\overline{f}(\xi,\tau)}{\langle\xi\rangle^r\langle\sigma\rangle^{1-b}} \frac{\langle\xi_1\rangle^r\chi_{|\xi_1|\le 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}} \frac{\langle\xi_2\rangle^r\chi_{|\xi_2|\ge 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta \\
&\le C\int_\star \frac{\chi_{|\xi|\ge 4a}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}} \frac{\chi_{|\xi_1|\le 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}} \frac{|\xi_2|\chi_{|\xi_2|\ge 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta \\
&\lesssim \int P^{4a}\overline{F}_{1-b}\cdot P_{2a}F_{b'}^1\cdot D_xP^{2a}F_{b'}^2(x,t)dxdt \\
&\lesssim \|F_{1-b}\|_{L_x^2L_t^2}\|P_{2a}F_{b'}^1\|_{L_x^2L_t^\infty}\|D_xP^{2a}F_{b'}^2\|_{L_x^\infty L_t^2} \\
&\lesssim \|f\|_{L_\xi^2L_\tau^2}\|f_1\|_{L_\xi^2L_\tau^2}\|f_2\|_{L_\xi^2L_\tau^2},
\end{aligned}
$$

which follows by Lemma 3.5 (with $q=2$), Lemma 3.7 ((3.14) with $\theta=1$) and 3.8.

Case 2. If $2a\le|\xi_1|\le|\xi_2|$, then, it follows from Lemma 3.10 that, if $|\xi|\ge 2a$, $|\xi_1|\ge 2a$ and $|\xi_2|\ge 2a$, then we have

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\gtrsim|\xi_1||\xi_2||\xi|^5.$$

This implies that one of the following cases always occurs:

Case (a) $|\sigma|\gtrsim|\xi_1||\xi_2||\xi|^5$;
Case (b) $|\sigma_1|\gtrsim|\xi_1||\xi_2||\xi|^5$;
Case (c) $|\sigma_2|\gtrsim|\xi_1||\xi_2||\xi|^5$.

Next, we are going to consider the three Cases (**a**–**c**) respectively.

Case 2.1. If **Case (a)** holds, for $b-s-1\le\frac{5}{8}$ and $s\le 4-5b$, then we have

$$K(\xi,\xi_1,\xi_2)\le C|\xi_1|^{\frac{5}{8}}|\xi_2|^{\frac{5}{8}}.$$

Therefore, by Lemmas 3.6 and 3.8, the integral Γ restricted to this domain is bounded by

$$\begin{aligned}\Gamma &= \int_\star \frac{|\xi|\chi_{|\xi|\geq 4a}\overline{f}(\xi,\tau)}{\langle\xi\rangle^r\langle|\xi_1||\xi_2||\xi|^5\rangle^{1-b}}\frac{\langle\xi_1\rangle^r\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{\langle\xi_2\rangle^r\chi_{|\xi_2|\geq 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&= \int_\star |\xi|^{s-4+5b}\chi_{|\xi|\geq 4a}\overline{f}(\xi,\tau)\frac{|\xi_1|^{b-s-1}\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{|\xi_2|^{b-s-1}\chi_{|\xi_2|\geq 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C|\xi|^{s-4+5b}\chi_{|\xi|\geq 4a}\overline{f}(\xi,\tau)\frac{|\xi_1|^{\frac{5}{8}}\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)}{\langle\sigma_1\rangle^{b'}}\frac{|\xi_2|^{\frac{5}{8}}\chi_{|\xi_2|\geq 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&= C\int \overline{F}_0\cdot D_x^{\frac{5}{8}}P^{2a}F^1_{b'}\cdot D_x^{\frac{5}{8}}P^{2a}F^2_{b'}(x,t)dxdt\\
&\lesssim \|F_0\|_{L^2_xL^2_t}\|D_x^{\frac{5}{8}}P^{2a}F^1_{b'}\|_{L^4_xL^4_t}\|D_x^{\frac{5}{8}}P^{2a}F^2_{b'}\|_{L^4_xL^4_t}\\
&\lesssim \|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.\end{aligned}$$

Case 2.2. If **Case (b)** holds, for $2r-2b'\leq\frac{5}{8}$ and $1-r\leq 5b'$, $(r=-s)$ then we have

$$K(\xi,\xi_1,\xi_2)\leq C|\xi_2|^{\frac{5}{8}}.$$

Therefore, by Lemmas 3.6, 3.7 and 3.8, the integral Γ restricted to this domain is bounded by

$$\begin{aligned}\Gamma &= \int_\star \frac{|\xi|\chi_{|\xi|\geq 4a}\overline{f}(\xi,\tau)}{\langle\xi\rangle^r\langle\sigma\rangle^{1-b}}\frac{\langle\xi_1\rangle^r\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)}{\langle|\xi_1||\xi_2||\xi|^5\rangle^{b'}}\frac{\langle\xi_2\rangle^r\chi_{|\xi_2|\geq 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&= \int_\star \frac{|\xi|^{1-r-5b'}\chi_{|\xi|\geq 4a}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}|\xi_1|^{r-b'}\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)\frac{|\xi_2|^{r-b'}\chi_{|\xi_2|\geq 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int_\star \frac{\chi_{|\xi|\geq 4a}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)\frac{|\xi_2|^{2r-2b'}\chi_{|\xi_2|\geq 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&\leq C\int_\star \frac{\chi_{|\xi|\geq 4a}\overline{f}(\xi,\tau)}{\langle\sigma\rangle^{1-b}}\chi_{|\xi_1|\geq 2a}f_1(\xi_1,\tau_1)\frac{|\xi_2|^{\frac{5}{8}}\chi_{|\xi_2|\geq 2a}f_2(\xi_2,\tau_2)}{\langle\sigma_2\rangle^{b'}}d\delta\\
&= C\int P^{4a}\overline{F}_{1-b}\cdot P^{2a}F^1_0\cdot D_x^{\frac{5}{8}}P^{2a}F^2_{b'}(x,t)dxdt\\
&\lesssim \|P^{4a}\overline{F}_{1-b}\|_{L^4_xL^4_t}\|P^{2a}F^1_0\|_{L^2_xL^2_t}\|D_x^{\frac{5}{8}}P^{2a}F^2_{b'}\|_{L^4_xL^4_t}\\
&\lesssim \|f\|_{L^2_\xi L^2_\tau}\|f_1\|_{L^2_\xi L^2_\tau}\|f_2\|_{L^2_\xi L^2_\tau}.\end{aligned}$$

Case 2.3. If **Case (c)** holds, the argument is similar to **Case 2.2**.
This completes the proof of Lemma 4.1.

5 Proof of the Main Result

In this section, in order to prove Theorems 2.1 and 2.2, we first establish the linear estimates as follows.

Let $\psi\in C_0^\infty(\mathbb{R})$ with $\psi=1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $supp\psi\subset[-1,1]$. We denote $\psi_\delta(\cdot)=\psi(\delta^{-1})(\cdot)$ for some non-zero $\delta\in\mathbb{R}$.

Lemma 5.1 (See [6,7]) *If* $s \in \mathbb{R}$ *and* $\frac{1}{2} < b < 1$, *then, for* $\varphi \in H^s$, *we have*

$$\|\psi(t)S(t)\varphi\|_{X_{s,b}} \leq C\|\varphi\|_{H^s}.$$

Lemma 5.2 (See [6,7]) *If* $s \in \mathbb{R}$, $\frac{1}{2} < b < b' < 1$ *and* $0 < \delta \leq 1$, *then we have*

$$\|\psi_\delta(t)S(t)\varphi\|_{X_{s,b}} \leq C\delta^{\frac{1}{2}-b}\|\varphi\|_{H^s},$$
$$\left\|\psi_\delta(t)\int_0^t S(t-t')F(t')dt'\right\|_{X_{s,b}} \leq C\delta^{\frac{1}{2}-b}\|F\|_{X_{s,b-1}},$$
$$\|\psi_\delta(t)F\|_{X_{s,b-1}} \leq C\delta^{b'-b}\|F\|_{X_{s,b'-1}}.$$

Now, we turn to the proof of Theorems 2.1, 2.2. More precisely, using the bilinear estimate (Lemma 4.1) and the linear estimate (Lemmas 5.1, 5.2) together with contraction mapping principle, we shall prove the local well-posedness for the IVP (1.1) and (1.2).

Proof of Theorem 2.1 For $u_0 \in H^s (s \geq -\frac{5}{8})$, we define the operator

$$\Phi(u) = \psi_1(t)S(t)u_0 + \lambda\psi_1(t)\int_0^t S(t-t')\psi_\delta(t')[uu_x](t')dt'$$

and the ball

$$\mathbb{B} = \{u \in X_{s,b} : \|u\|_{X_{s,b}} \leq 2C\|u_0\|_{H^s}\}.$$

Next, we are going to prove Φ is a contraction mapping on the ball $\mathbb{B}$. For this purpose, we first prove

$$\Phi(\mathbb{B}) \subset \mathbb{B}.$$

□

By Lemmas 4.1 and 5.1, 5.2 for $\frac{1}{2} < b < b' < 1$, we have

$$\begin{aligned}\|\Phi(u)\|_{X_{s,b}} &\leq \|\psi_1(t)S(t)u_0\|_{X_{s,b}} + \|\lambda\psi_1(t)\int_0^t S(t-t')\psi_\delta(t')[uu_x](t')dt'\|_{X_{s,b}}\\ &\leq C\|u_0\|_{H^s} + C\|\psi_\delta(t)[uu_x]\|_{X_{s,b}}\\ &\leq C\|u_0\|_{H^s} + C\delta^{b-b'}\|uu_x\|_{X_{s,b'-1}}\\ &\leq C\|u_0\|_{H^s} + C\delta^{b-b'}\|u\|^2_{X_{s,b}}.\end{aligned}$$

Therefore, if we fix δ such that $C\delta^{b-b'}\|u_0\|_{H^s} < \frac{1}{2}$, then we obtain $\Phi(\mathbb{B}) \subset \mathbb{B}$.
On the other hand, for $u, v \in \mathbb{B}$, we have

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{X_{s,b}} &\leq C\delta^{b-b'}(\|u\|_{X_{s,b}} + \|v\|_{X_{s,b}})\|u-v\|_{X_{s,b}}\\ &\leq \frac{1}{2}\|u-v\|_{X_{s,b}}.\end{aligned}$$

Consequently, Φ is a contraction mapping on the ball $\mathbb{B}$. There exists a unique fixed point which solves the IVP (1.1) and (1.2) for $T < \frac{1}{2}\delta$.

Proof of Theorem 2.2 First, we establish the L^2 conservation law and we obtain the global well-posedness of the solution which follows from local well-posedness and the L^2 conservation law by standard method. In order to prove the global well-posedness for the initial data $u_0 \in H^s(\mathbb{R})$ with $s = 0$, we establish the L^2 conservation law as follows. □

Lemma 5.3 *Let $u_0 \in L^2$, and $u \in C([0,T], L^2)$ be a solution of IVP* (1.1) *and* (1.2)*. Then we have $\|u\|_{L^2} = \|u_0\|_{L^2}$.*

Proof Multiplying (1.1) by u and integrating the resulting equation over $\mathbb{R}$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2} = 0,$$

that is, $\|u\|_{L^2} = \|u_0\|_{L^2}$.

Therefore, for Cauchy problem (1.1)–(1.2) with the initial data $u_0 \in L^2$, global well-posedness of the solution follows from local well-posedness and the L^2 conservation law by standard method. □

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References

1. Jones, K.L., He, X.G., Chen, Y.K.: Existence of periodic traveling wave solution to the forced generalized nearly concentric Korteweg-de Vries equation. Int. J. Math. Math. Sci. **24**, 371–377 (2000)
2. Kawahara, T.: Oscillatory solitary waves in dispersive media. J. Phys. Soc. Jpn. **33**, 260–264 (1972)
3. Colliander, J., Keel, M., Stalfilanti, G., Takaoka, H., Tao, T.: Global wel-posedness for the KdV in Sobolev spaces of negative indices. EJDE **26**, 1–7 (2001)
4. Colliander, J., Keel, M., Stalfilanti, G., Takaoka, H., Tao, T.: Sharp global wel-posedness for the KdV and modified KdV on R and T. J. Am. Math. Soc. **16**, 705–749 (2003)
5. Kenig, C.E., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Commun. Pure Appl. Math. **46**, 527–620 (1993)
6. Kenig, C.E., Ponce, G., Vega, L.: The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices. Duke Math. J. **71**, 1–21 (1993)
7. Kenig, C.E., Ponce, G., Vega, L.: A bilinear estimate with application to the KdV equation. J. Am. Math. Soc. **9**, 573–603 (1996)
8. Molinet, L., Ribaud, F.: On the Cauchy problem for the generalized Korteweg-de Vries equation. Commun. Partial Differ. Equ. **28**, 2065–2091 (2003)
9. Guo, Z.H.: Global well-posedness of the Korteweg-de Vries equation in $H^{-\frac{3}{4}}(R)$. J. Math. Pure Appl. **9**, 583–597 (2009)
10. Cui, S.B., Deng, D.G., Tao, S.P.: Global existence of solutions for the Cauchy problem of the Kawahara equation with L^2 initial data. Acta Math. Sin. Engl. Ser. **22**, 1457–1466 (2006)
11. Cui, S.B., Tao, S.P.: Strichartz estimates for dispersive equations and sovability of Cauchy problems of the Kawahara equation. J. Math. Anal. Appl. **304**, 683–702 (2005)
12. Wang, H., Cui, S.B., Deng, D.G.: Global existence of solutions of the Kawahara equation in Sobolev spaces of negative indices. Acta Math. Sin. Engl. Ser. **23**, 1435–1446 (2007)
13. Chen, W.G., Li, J.F., Miao, C.X., Wu, J.H.: Low regularity solutions of two fifth-order KdV type equations. arXiv:0710.2704v3 [math.AP] 7 April (2009)
14. Jia, Y.L., Huo, Z.H.: Well-posedness of the fifth-order shallow water equations. J. Differ. Equ. **246**, 2448–2467 (2009)
15. Huo, Z.H.: The Cauchy problem for the fifth-order shallow water equations. Acta Math. Sin. Engl. Ser. **21**, 441–454 (2005)
16. Tao, S.P., Cui, S.B.: Local and global existence of solutions to initial value probles of modified nonlinear Kawahara equations. Acta Math. Sin. Engl. Ser. **21**, 1035–1044 (2005)
17. Yan, W., Li, Y.S.: The Cauchy problem for Kawahara equation in Sobolev spaces with low regularity. Math. Methods Appl. Sci. **33**, 1647–1660 (2010)

18. Yan, W., Li, Y.S., Yang, X.Y.: The Cauchy problem for the modified Kawahara equation in Sobolev spaces with low regularity. Math. Comput. Model. **54**, 1251–1261 (2011)
19. Chen, W.G., Guo, Z.H.: Global well-posedness and I-method for the fifth order Korteweg-de Vries equation. J. Anal. Math. **114**, 121–156 (2011)
20. Zhang, Z.Y., Liu, Z.H., Sun, M.B., Li, S.H.: Well-posedness and unique continuation property for the solutions to the generalized Kawahara equation below the energy space. Appl. Anal. (2017). https://doi.org/10.1080/00036811.2017.1385064
21. Zhang, Z.Y., Huang, J.H., Sun, M.B.: Almost conservation laws and global rough solutions of the defocusing nonlinear wave equation on R^2. Acta Math. Sci. **37B**(2), 385–394 (2017)
22. Zhang, Z.Y., Huang, J.H.: Well-posedness and unique continuation property for the generalized Ostrovsky equation with low regularity. Math. Methods Appl. Sci. **39**(10), 2488–2513 (2016)
23. Zhang, Z.Y., Liu, Z.H., Huang, J.H., Sun, M.B.: On the unique continuation property for the modified Kawahara equation. Adv. Math. (China) **45**(1), 80–88 (2016)
24. Tao, S.P., Cui, S.B.: The local and global exsitence of solution of the Cauchy problem for a seven-order nonlinear equation. Acta Math. Sin. **25A**(4), 451–460 (2005)
25. Tao, S.P.: Initial Value Problems for Some Classes of Higher Order Nonlinear Dispersive Equations. A Doctoral Dissertation, Lanzhou University (2001)
26. Tao, S.P., Cui, S.B.: Local and global existence of solutions to initial value problems of nonlinear Kaup–Kupershmidt equations. Acta Math. Sin. Engl. Ser. **21**(4), 1–14 (2005)
27. Kenig, C.E., Ponce, G., Vega, L.: Higher-order nonlinear dispersive equations. Proc. Am. Math. Soc. **122**, 157–166 (1994)
28. Pilod, D.: On the Cauchy problem for higher-order nonlinear dispersive equations. J. Differ. Equ. **245**, 2055–2077 (2008)
29. Tao, T.: Multi-linear weighted convolution of L^2 functions and applications to nonlinear dispersive equations. Am. J. Math. **123**, 839–908 (2000)
30. Ben-Artzi, M., Saut, J.C.: Uniform decay esimates for a class of oscillatory integrals and applications. Differ. Integral Equ. **12**, 137–145 (1999)
31. Tian, L.X., Gui, G.L., Liu, Y.: On the Cauchy problem for the generalized shallow water wave equation. J. Differ. Equ. **245**, 1838–1852 (2008)
32. Wang, B.X., Huo, Z.H., Chao, C.C., Guo, Z.H.: Harmonic Analysis Method for Nonlinear Evolution Equations. World Scientific Press, Singapore (2011)
33. Kenig, C.E., Ponce, G., Vega, L.: Oscillatory integrals and regularity of dispersive equations. Iniana Univ. Math. J. **40**, 33–69 (1991)
34. Huo, Z.H., Jia, Y.L.: Low regularity solution for a nonlocal perturbation of KdV equation. Z. Angew. Math. Phys. **59**, 634–646 (2008)