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Stability and Spectral Comparison of a Reaction–Diffusion System with Mass Conservation

Evangelos Latos[1] · **Yoshihisa Morita**[2] · **Takashi Suzuki**[3]

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Abstract We study the global-in-time behavior of solutions to a reaction–diffusion system with mass conservation, as proposed in the study of cell polarity, particularly, the second model of the work by Otsuji et al. (PLoS Comput Biol 3:e108, 2007). First, we show the existence of a Lyapunov function and confirm the global-in-time existence of the solution with compact orbit. Then we study the stability and instability of stationary solutions by using the semi-unfolding-minimality property and the spectral comparison. As a result the dynamics near the stationary solutions is qualitatively characterized by a variational function.

Keywords Reaction diffusion system · Mass conservation · Cell polarity · Global-in-time behavior · Lyapunov function · Spectral comparison

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✉ Yoshihisa Morita
morita@rins.ryukoku.ac.jp

Evangelos Latos
evangelos.latos@math.uni-mannheim.de

Takashi Suzuki
suzuki@sigmath.es.osaka-u.ac.jp

1 University of Mannheim, 68131 Mannheim, Germany

2 Department of Applied Mathematics and Informatics, Ryukoku University, Seta Otsu 520-2194, Japan

3 Center for Mathematical Modeling and Data Science, Osaka University, Machikaneyamacho, Toyonakashi 560-8531, Japan

1 Introduction

The purpose of the present paper is to study the mass conserved reaction–diffusion system

$$u_t = D\Delta u + f(u,v), \quad \tau v_t = \Delta v - f(u,v), \quad \text{in } \Omega \times (0,T),$$
$$\frac{\partial}{\partial \nu}(u,v)\Big|_{\partial\Omega} = 0, \quad (u,v)|_{t=0} = (u_0(x), v_0(x)), \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outer unit normal vector, $D, \tau > 0$ are constants, and $(u_0, v_0) = (u_0(x), v_0(x))$ are smooth nonnegative functions. Given sufficiently smooth nonlinearity $f = f(u,v)$, standard theory allows the existence of a unique local-in-time classical solution $(u,v) = (u(\cdot,t), v(\cdot,t))$ to (1). Then mass conservation property for this system writes

$$\frac{d}{dt}\int_\Omega (u + \tau v)\, dx = 0. \tag{2}$$

Several equations in this form are used in the study of cell polarity, e.g. [8,17] and [14]. It is expected that different species inside the cell shall separate according to their diffusion coefficients, i.e. slow and fast diffusions will localize the species near the membrane and in the cytosol, respectively. Although three kinds of molecules are interacting inside the cell in [17], each one of them has two phases, that is, active and inactive phases which are characterized by slow and fast diffusions, respectively. Problem (1), thus focuses on these two phases of a single species, ignoring interactions between the other species.

The system (1) is a closed system in the sense that the active and inactive phases are reversible. Thus, it might be natural to suppose negatively that the system allows the Turing pattern (see [26]), that is, the appearance of spatially inhomogeneous stable stationary states induced by diffusion. In fact, the emergence of Turing patterns is widely observed in open systems including activator-inhibitor systems, where an energy flow in the system is assumed to induce the pattern. In [17], however, the authors presented the following three models in which the Turing type instability certainly takes places:

$$f(u,v) = -\frac{au}{u^2+b} + v,$$
$$f(u,v) = -\alpha_1\left[\frac{u+v}{\left(\alpha_2(u+v)+1\right)^2} - v\right],$$
$$f(u,v) = \alpha_1(u+v)[(\alpha u + v)(u+v) - \alpha_2], \tag{3}$$

where a, b, α, α_1, and α_2 are positive constants. So far, mathematical analysis is done for the first model, noticing the similarity between the Fix–Caginalp model [22] (see [11,13,15] and [16]).

The present paper deals with the second form of the reaction term $f(u,v)$ of (3) and investigates the stability property of stationary solutions We are primarily interested in how mathematically different the second form of $f(u,v)$ is from the first one, though both allow the Turing type instability. In fact, as mentioned above, the first one is transformed into a phase-field type system and this fact helps in those studies, but apparently the second one does not allow such a nice form. We also found a biological model of reaction–diffusion equations with $f(u,v) = \mu[h(u+v)+v]$ in [19], though the diffusion of u is absent by a biological reason. We explain briefly this model in the end of this section.

Here we confirm several classical results. First, this nonlinearity $f = f(u, v)$ satisfies

$$f(0, v),\ -f(u, 0) \geq 0, \quad u, v \geq 0$$

and hence the standard maximum principle applied to the individual equations to u and v guarantees their non-negativity. Next, the solution is global-in-time because of the weak nonlinearity, that is, at most linear growth as $u, v \to \infty$. (see [12]) Thus we obtain the following statement.

Theorem 1 *The classical solution* $(u, v) = (u(\cdot, t), v(\cdot, t))$ *to* (1) *for the second case of* $f(u, v)$ *in* (3) *is global-in-time and satisfies*

$$u(\cdot, t),\ v(\cdot, t) \geq 0 \ \ on\ \overline{\Omega},\ t \geq 0. \tag{4}$$

We remark that the classical solution to (1) with $t > 0$ is ensured by $0 \leq u_0, v_0 \in L^\infty(\Omega)$, therefore we assume the initial values to be smooth.

Now we let

$$h(z) = -\frac{\alpha_1 z}{(\alpha_2 z + 1)^2}, \quad k = \alpha_1, \tag{5}$$

to rewrite the model as

$$\begin{aligned} u_t &= D\Delta u + h(u+v) + kv, \\ \tau v_t &= \Delta v - h(u+v) - kv, \qquad \text{in } \Omega \times (0, T), \\ \frac{\partial}{\partial \nu}(u, v)\Big|_{\partial\Omega} &= 0, \quad (u, v)|_{t=0} = (u_0(x), v_0(x)). \end{aligned} \tag{6}$$

Here we assume $\tau \neq 1$ and furthermore,

$$\xi = \frac{1 - \tau D}{\tau - 1} > 0, \quad \alpha = \frac{1 - D}{\tau - 1} > 0, \tag{7}$$

that is, either $\tau > 1 > \tau D$ or $\tau D > 1 > \tau$. Using

$$\begin{aligned} w &= Du + v, \quad z = u + v, \\ g(z) &= (1 - D)h(z) - kDz, \end{aligned} \tag{8}$$

system (6) transforms into

$$\begin{aligned} z_t &= D\Delta z + (w_t - D\Delta w + kw) + g(z), \\ w_t + \xi z_t &= \alpha \Delta w, \qquad \text{in } \Omega \times (0, T), \\ \frac{\partial}{\partial \nu}(z, w)\Big|_{\partial\Omega} &= 0, \quad (z, w)|_{t=0} = (z_0(x), w_0(x)). \end{aligned} \tag{9}$$

If the second term on the right-hand side of the first equation of system (9) is reduced to kw, we obtain

$$\begin{aligned} & z_t = D\Delta z + kw + g(z), \quad w_t + \xi z_t = \alpha \Delta w \qquad \text{in } \Omega \times (0, T) \\ & \frac{\partial}{\partial \nu}(z, w)\Big|_{\partial\Omega} = 0, \quad (z, w)|_{t=0} = (z_0(x), w_0(x)), \end{aligned} \tag{10}$$

where $z_0 = u_0 + v_0$ and $w_0 = Du_0 + v_0$. It is a generalization of the Fix–Caginalp model [2,5] for $g(z) = z - z^3$. We noticed that the first model of (3) is reduced to (10) (see [16]). Then, as in the Fix–Caginalp model [22], we used a variational structure arising between the Lyapunov function and the stationary state, to clarify the global-in-time dynamics [13] in accordance with a spectral property of the stationary state [15]. Namely, first, the stationary state is described by an elliptic boundary value problem with a nonlocal term where the conservative quantity of (10) is involved as a parameter. Second, this elliptic problem has a variational functional associated with the Lyapunov function to (10). Finally, any non-degenerate minimum of this variational functional is dynamically stable as a stationary solution to (10).

Here we show similar properties for problem (9). In this model, we still have a Lyapunov function which induces a variational function to formulate a stationary state. Accordingly the orbit of a non-stationary solution is compact (Theorem 2), while any local minimum is dynamically stable (Theorem 3). Furthermore, the Morse index of the stationary solution, defined by the variational function, is equal to the dynamical instability if $\xi\eta_2 > k$, where η_2 denotes the second eigenvalue of $-\Delta$ under the Neumann boundary condition (Theorem 4). As a consequence of Theorem 4 we can see that any stable stationary solution has a monotone profile in one-dimensional space, similar to the first form of (3) even though the Turing instability takes place (see [11]).

Although (9) is derived from (6) for the case of $\tau \neq 1$, system (6) itself has a Lyapunov function even for $\tau = 1$. This fact was noticed by [11] to confirm the existence of global-in-time solutions and the spectral comparison property of stationary solutions. Before the end of this section we shall confirm that the Lyapunov function of $\tau = 1$ used by [11] is regarded as a limit case under suitable scaling.

To begin with, we note that mass conservation (2) takes the form

$$\frac{d}{dt}\int_{\Omega}(\xi z + w)\, dx = 0,$$

in (z, w)-variable of (8). Noticing this property, we set

$$\int_{\Omega}(\xi z + w)\, dx = \int_{\Omega}(\xi z_0 + w_0)\, dx = \lambda. \tag{11}$$

To derive the Lyapunov function of (9), we multiply the first equation of (9) with z_t to obtain

$$\|z_t\|_2^2 + \frac{d}{dt}\int_{\Omega}\left(\frac{D}{2}|\nabla z|^2 - G(z)\right) dx = (w_t - D\Delta w + kw, z_t), \tag{12}$$

where

$$G(z) = \int_0^z g(s)\, ds,$$

and $(\cdot,\cdot)$ denotes the L^2-inner product. Multiplying the second equation of (9) with $w_t - D\Delta w + kw$, next, we obtain

$$\xi(z_t, w_t - D\Delta w + kw) = (-w_t + \alpha\Delta w, w_t - D\Delta w + kw) = -\|w_t\|_2^2 \\ -\alpha D\|\Delta w\|_2^2 - \alpha k\|\nabla w\|_2^2 - \frac{d}{dt}\int_{\Omega}\left(\frac{\alpha + D}{2}|\nabla w|^2 + \frac{k}{2}w^2\right) dx. \tag{13}$$

From (12) and (13) it follows that

$$\xi\|z_t\|_2^2 + \|w_t\|_2^2 + \alpha D\|\Delta w\|_2^2 + \alpha k\|\nabla w\|_2^2 \\ = -\frac{d}{dt}\int_{\Omega}\left(\frac{\alpha + D}{2}|\nabla w|^2 + \frac{k}{2}w^2 + \frac{\xi D}{2}|\nabla z|^2 - \xi G(z)\right) dx.$$

Therefore,

$$L = L(z,w) = \int_\Omega \left(\frac{\alpha + D}{2} |\nabla w|^2 + \frac{k}{2} w^2 + \frac{\xi D}{2} |\nabla z|^2 - \xi G(z) \right) dx, \tag{14}$$

is a Lyapunov function with:

$$\frac{dL}{dt} = -\left\{ \xi \|z_t\|_2^2 + \|w_t\|_2^2 + \alpha D \|\Delta w\|_2^2 + \alpha k \|\nabla w\|_2^2 \right\} \le 0. \tag{15}$$

Now we formulate the stationary state of (9). First,

$$\alpha \Delta w = 0, \quad \left. \frac{\partial w}{\partial \nu} \right|_{\partial \Omega} = 0,$$

holds in the stationary state of (9) and hence $w = w(x)$ is a spatially homogeneous function denoted by $w = \overline{w} \in \mathbf{R}$. Then the total mass conservation (11) implies

$$\lambda = \int_\Omega (\xi z + \overline{w})\, dx, \tag{16}$$

hence

$$\overline{w} = \frac{1}{|\Omega|} \left(\lambda - \xi \int_\Omega z\, dx \right). \tag{17}$$

Plugging (17) into the first equation, we see that the stationary state of (9) is reduced to a single equation concerning $z = z(x)$, that is,

$$-D\Delta z = g(z) + \frac{k}{|\Omega|} \left(\lambda - \xi \int_\Omega z\, dx \right), \quad \left. \frac{\partial z}{\partial \nu} \right|_{\partial \Omega} = 0. \tag{18}$$

This problem is the Euler–Lagrange equation corresponding to the functional

$$J_\lambda(z) = \int_\Omega \left(\frac{D}{2} |\nabla z|^2 - G(z) - \frac{k\lambda}{|\Omega|} z \right) dx + \frac{k\xi}{2|\Omega|} \left(\int_\Omega z\, dx \right)^2 \tag{19}$$

defined for $z \in H^1(\Omega)$.

Our point is to clarify the dynamical stability of the solution to (18), regarded as a stationary solution to (9). First, the Lyapunov function guarantees the global-in-time solution. Let $(u_0, v_0) \in X = C^2(\overline{\Omega})^2$ and E_λ be the set of solutions $z = z(x)$ to (18) for $\lambda \in \mathbf{R}$ defined by

$$\lambda = \int_\Omega (\xi z_0 + w_0) dx = \int_\Omega (u_0 + \tau v_0) dx. \tag{20}$$

Theorem 2 *If (7) holds, the orbit $\mathcal{O} = \{(u(\cdot,t), v(\cdot,t))\}_{t \ge 0} \subset X$ of the solution $(u,v) = (u(\cdot,t), v(\cdot,t))$ to (6) with (5) is compact and hence the ω-limit set defined by*

$$\omega(u_0, v_0) = \{(u_*, v_*) \mid \exists t_k \uparrow +\infty \text{ such that } \|(u(\cdot,t_k), v(\cdot,t_k)) - (u_*, v_*)\|_X = 0\},$$

is nonempty, compact, and connected. Furthermore, any $(u_*, v_*) \in \omega(u_0, v_0)$ admits $z_* \in E_\lambda$ such that

$$u_* = \frac{w_* - z_*}{D - 1}, \quad v_* = \frac{Dz_* - w_*}{D - 1}, \tag{21}$$

for $w_* \in \mathbf{R}$ defined by

$$w_* = \frac{1}{|\Omega|} \left(\lambda - \xi \int_\Omega z_* dx \right). \tag{22}$$

Finally, it holds that

$$\lim_{t\uparrow+\infty}\|w(\cdot,t)-\langle w(t)\rangle\|_{C^2}=0, \tag{23}$$

where

$$\langle w(t)\rangle=\frac{1}{|\Omega|}\int_\Omega w(x,t)dx.$$

As we have seen, any stationary solution (u_*, v_*) to (9) corresponds to a critical point $z_* \in H^1(\Omega)$ of $J_\lambda(z)$ in (19) through (21)–(22). Now we examine its dynamical stability. The first result follows from the *semi-unfolding-minimality* between the Lyapunov function $L(z, w)$ and variational functional $J_\lambda(z)$ (see [21]). Namely, first, it holds that $L(z, \overline{w}) = J_\lambda(z)$ for $\overline{w} \in \mathbf{R}$ defined by (17). This property is called the semi-unfolding. Second, $J_\lambda(z)$ arises as the global minimum of $L(z, w)$ among all w satisfying (11). This property is called the semi-minimality. These structures of the second model are similar to the ones of the first model of (3) studied in [13]. In this paper when a stationary solution of (6) is Lyapunov stable, we call that a stationary solution of (6) is dynamically stable.

Theorem 3 *Given $0 \le (u_0, v_0) \in X$, let $z_* \in H^1(\Omega)$ be a local minimum of $J_\lambda(z)$ in (19) for λ defined by (20). Then (u_*, v_*) derived from (21)–(22) is a dynamically stable stationary state of (6).*

Finally we pay attention to the linearized stability. We write (9) as

$$\begin{aligned}&(1+D\xi/\alpha)z_t-\xi\alpha w_t=D\Delta z+g(z)+kw,\\ &w_t+\xi z_t=\alpha\Delta w, \qquad\qquad \text{in } \Omega\times(0,T)\\ &\left.\frac{\partial}{\partial\nu}(z,w)\right|_{\partial\Omega}=0,\end{aligned} \tag{24}$$

recalling $1 - D/\alpha = \xi/\alpha$. Then the linearlized equation of (24) around (z_*, w_*) is given as

$$\frac{\partial}{\partial t}M\begin{pmatrix}Z\\W\end{pmatrix}+\mathcal{A}_1\begin{pmatrix}Z\\W\end{pmatrix}=0,\quad \left.\frac{\partial}{\partial\nu}\begin{pmatrix}Z\\W\end{pmatrix}\right|_{\partial\Omega}=0$$

where

$$M=\begin{pmatrix}1+D\xi/\alpha & -\xi/\alpha\\ \xi & 1\end{pmatrix},\qquad \mathcal{A}_1=\begin{pmatrix}-D\Delta-g'(z_*) & -k\\ 0 & -\alpha\Delta\end{pmatrix}.$$

Therefore, an index of the instability for (z_*, w_*) to (9), or equivalently, that of (u_*, v_*) to (6), is indicated by the number of eigenvalues with negative real parts of the operator $\mathcal{A} = M^{-1}\mathcal{A}_1$. This operator is actually realized in $L^2(\Omega; \mathbf{C})^2$, the Hilbert space composed of square integrable complex-valued functions on Ω, with the domain

$$D(\mathcal{A})=\left\{\begin{pmatrix}Z\\W\end{pmatrix}\in H^2(\Omega;\mathbf{C})^2 \;\middle|\; \int_\Omega(W+\xi Z)\,dx=0,\ \left.\frac{\partial}{\partial\nu}\begin{pmatrix}Z\\W\end{pmatrix}\right|_{\partial\Omega}=0\right\}.$$

On the other hand, the element z_* is also a stationary state of

$$z_t=-\delta J_\lambda(z), \tag{25}$$

that is,

$$z_t = D\Delta z + g(z) + \frac{k}{|\Omega|}\left(\lambda - \xi \int_\Omega z\,dx\right), \quad \left.\frac{\partial z}{\partial \nu}\right|_{\partial\Omega} = 0. \tag{26}$$

The index of the instability of the solution for z_* to (26), on the other hand, is indicated by the number of negative eigenvalues of $\mathcal{L}$, which is the self-adjoint operator in $L^2(\Omega)$ defined by

$$\mathcal{L}\varphi = -\left(D\Delta\varphi + g'(z_*)\varphi - \frac{k\xi}{|\Omega|}\int_\Omega \varphi\,dx\right), \tag{27}$$

with the domain

$$D(\mathcal{L}) = \left\{\varphi \in H^2(\Omega) \mid \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0\right\}.$$

The following theorem assures that these two Morse indices coincide, provided that

$$\xi\eta_2 > k, \tag{28}$$

where η_2 is the second eigenvalue of $-\Delta$ with the Neumann boundary condition.

Theorem 4 *Any eigenvalue $\sigma \in \mathbf{C}$ of $\mathcal{A}$ in $Re\,\sigma < \alpha k/2\xi$ is real, and has the same algebraic and geometric multiplicities under an additional condition $\sigma < \alpha\eta_2$. In addition, if the condition (28) is satisfied, the numbers of negative and zero eigenvalues of $\mathcal{A}$ and $\mathcal{L}$ coincide.*

We note that the assumption (28) is a technical condition to ensure that the assertion of the above theorem (see the fifth step of the proof in Sect. 4) holds. Hence, we might relax the condition by improving the argument in the proof, which would be a future work.

Theorem 4 is regarded as a spectral comparison property first observed by [1]. It has been examined for the first model of (3) by [15] and for the second model with $\tau = 1$ by [11]. Here we use a similar argument as in [3] for the proof. By virtue of this theorem we can see that there is a restriction on the profile of stable stationary solutions. For instance, in a cylindrical domain [23] tells that every stable solution z_*, given by the critical point of J_λ, must be monotone in the axial direction. Therefore, such a property to the solution (u_*, v_*) to (9) is inherited from z_* through (21) (see [11] for a similar application).

We give a remark on the case $\tau = 1$. Some of the above results are similar to those in ([11]) for the case $\tau = 1$, therefore we make clear the connection by taking the limit $\tau \to 1$. We confirm that the Lyapunov function $L(u, v)$ and stationary state valid to $\tau \neq 1$, that is, (14) and (18), respectively, are reduced to those for $\tau = 1$ used in [11], under suitable scaling. In the following, we assume $D \neq 1$, because $\tau = D = 1$ is the trivial case of (1).

First, given $\tau \neq 1$, we define $\hat{L}(z, w; \tau)$ by

$$L(z, w) = \xi\hat{L}(z, w; \tau), \quad \xi = \xi(\tau) = \frac{1 - \tau D}{\tau - 1}.$$

Since

$$\lim_{\tau\to 1}\frac{\alpha + D}{\xi} = \lim_{\tau\to 1}\left\{\frac{1 - D}{1 - \tau D} + \frac{D(\tau - 1)}{1 - \tau D}\right\} = 1,$$
$$\lim_{\tau\to 1}\frac{k}{\xi} = 0,$$

it follows that

$$\hat{L}(z, w) \equiv \lim_{\tau\to 1}\hat{L}(z, w; \tau) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 + \frac{D}{2}|\nabla z|^2 - G(z)\right)dx,$$

which is the Lyapunov function used in [11].

Next, to derive the limit problem of (18) we take

$$\hat{\lambda} = \lambda/\xi = \int_\Omega (z + w/\xi)dx.$$

By taking $\tau \to 1$, it holds that

$$\hat{\lambda} = \int_\Omega z. \tag{29}$$

On the other hand, since $\lambda = \xi\hat{\lambda}$ we write (18) as

$$-D\Delta z = g(z) + \frac{k\xi}{|\Omega|}\left(\hat{\lambda} - \int_\Omega z\,dx\right), \quad \left.\frac{\partial z}{\partial \nu}\right|_{\partial\Omega} = 0.$$

Because of $\xi \to \infty$ as $\tau \to 1$, we need to assume $\int_\Omega z\,dx \to \hat{\lambda}$ so that the limit problem makes sense. Therefore, we can require the limit problem as $\tau \to 1$ to be

$$-D\Delta z = g(z) + \mu, \quad \left.\frac{\partial z}{\partial \nu}\right|_{\partial\Omega} = 0 \tag{30}$$

with some $\mu \in \mathbf{R}$

$$\mu = -\frac{1}{|\Omega|}\int_\Omega g,$$

and $\int_\Omega z\,dx = \hat{\lambda}$. Hence we end up with

$$-D\Delta z = g(z) - \frac{1}{|\Omega|}\int_\Omega g(z), \quad \left.\frac{\partial z}{\partial \nu}\right|_{\partial\Omega} = 0. \tag{31}$$

The stationary state of (6) with $\tau = 1$ is now formulated by (29)–(31), using $z = u + v$. This is the Euler–Lagrange equation of the functional

$$\hat{J}_{\hat{\lambda}}(z) = \int_\Omega \left(\frac{D}{2}|\nabla z|^2 - G(z)\right)dx,$$

defined for

$$H = \left\{z \in H^1(\Omega) \mid \int_\Omega z\,dx = \hat{\lambda}\right\}.$$

As mentioned in the explanation of the model equations, there is a biological model which has a similar form in the kinetics. In [19], the authors propose the next system:

$$\begin{aligned} u_t &= -\mu[\Gamma(u+v)u - (1-\Gamma(u+v))v], \\ v_t &= D_v\Delta v + \mu[\Gamma(u+v)u - (1-\Gamma(u+v))v], \end{aligned}$$

where u and v stand for the density of a proliferating population and a migrating population in tumour cells, respectively. $\Gamma(u+v)$ is the probability that an immotile cells becomes motile. They assume that the proliferating population does not migrate, hence no diffusion in the u equation. For an explicit function

$$\Gamma(z) := \frac{1}{2}(1 + \tanh(\alpha[z^* - z])),$$

they show a Turing type instability and wave patterns by using numerical methods, where α and z^* are positive parameters. Since we write

$$\Gamma(u+v)u - (1-\Gamma(u+v))v = \Gamma(u+v)(u+v) - v,$$

the right hand side of the u equation can be written as $g(z) + \mu v$, where $g(z) = -\mu\Gamma(z)z$. Our final remark is that with a slightly little modification our results can be extended to the case when $h(z)$ is replaced by $g(z)$.

This paper is composed of four sections. Theorems 2, 3, and 4 are proven in Sects. 2, 3, and 4, respectively. The standard L^p norm on Ω is denoted by $\| \ \|_p$, $1 \le p \le \infty$.

2 Proof of Theorem 2

In this section we will prove several a priori estimates. Henceforth, C_i, $i = 1, 2, \dots, 19$ denote positive constants independent of t.

The first observation is the inequality

$$\|u(\cdot,t)\|_1 + \|v(\cdot,t)\|_1 \le C_1, \tag{32}$$

which follows from (11) and $\xi > 0$. Now we show the following lemma.

Lemma 2.1 *It holds that*

$$\|v(\cdot,t)\|_\infty \le C_2. \tag{33}$$

Proof Since

$$0 \ge h(z) \ge -C_3, \quad z \ge 0, \tag{34}$$

we have

$$\tau v_t \le \Delta v + C_3 - kv, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0(x) \ge 0.$$

Hence it holds that $v(x,t) \le \overline{v}(t)$, where $\overline{v} = \overline{v}(t)$ is the solution to

$$\frac{d\overline{v}}{dt} = \tau^{-1}C_3 - \tau^{-1}k\overline{v}, \quad \overline{v}(0) = \|v_0\|_\infty,$$

that is,

$$\overline{v}(t) = e^{-\tau^{-1}kt}\|v_0\|_\infty + \frac{C_3}{k}(1 - e^{-\tau^{-1}kt}).$$

Then we obtain

$$\|\overline{v}(\cdot,t)\|_\infty \le C_4,$$

and hence (33). □

Lemma 2.2 *We have*

$$\|z(\cdot,t)\|_{H^1}^2 + \|w(\cdot,t)\|_{H^1}^2 + \int_0^t (\|z_t(\cdot,t')\|_2^2 + \|\nabla w(\cdot,t')\|_2^2)\,dt' \le C_5. \tag{35}$$

Proof First, (15) implies

$$\begin{aligned} L(z(\cdot,t), w(\cdot,t)) &+ \int_0^t (\xi\|z_t(\cdot,t')\|_2^2 + \|w_t(\cdot,t')\|_2^2)dt' \\ &+ \int_0^t (\alpha D\|\Delta w(\cdot,t')\|_2^2 + k\alpha\|\nabla w(\cdot,t')\|_2^2)dt' = L(z_0, w_0). \end{aligned}$$

By (8) and (34), we have

$$g(z) \le (1+D)C_3, \quad z \ge 0. \tag{36}$$

In (14):

$$L = L(z,w) = \int_\Omega \left(\frac{\alpha+D}{2}|\nabla w|^2 + \frac{k}{2}w^2 + \frac{\xi D}{2}|\nabla z|^2 - \xi G(z) \right) dx,$$

it holds that

$$G(z) \le (1+D)C_3 z, \quad z \ge 0.$$

Then (35) follows from (32), which is derived from the conservation of the total mass, and Wirtinger's inequality:

$$\mu_2 \|z - \langle z \rangle\|_2^2 \le \|\nabla z\|_2^2, \quad z \in H^1(\Omega),$$

with $\mu_2 > 0$ defined to be the second eigenvalue of $-\Delta$ under the Neumann boundary condition where, $\langle z \rangle = \frac{1}{|\Omega|}\int_\Omega z\, dx$. □

Lemma 2.3 *It holds that*

$$\|w(\cdot,t)\|_\infty \le C_6. \tag{37}$$

Proof Taking $\mu > 0$, we write the second equation of (9) as

$$w_t = (\alpha\Delta - \mu)w + \mu w - \xi z_t, \quad \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = 0, \quad w|_{t=0} = w_0(x).$$

Then it follows that

$$w(\cdot,t) = e^{t(\alpha\Delta-\mu)}w_0 + \int_0^t e^{(t-t')(\alpha\Delta-\mu)}[\mu w(\cdot,t') - \xi z_t(\cdot,t')]\, dt'. \tag{38}$$

To estimate the second term on the right-hand side of (38), we use the semigroup estimate (see [20])

$$\|e^{t\Delta}\phi\|_r \le C_7(q,r)\max\left\{1, t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\right\}\|\phi\|_q, \quad 1 \le q \le r \le \infty, \tag{39}$$

recalling that N is the space dimension.

First, we apply this to $q = 2$ and $r = \infty$ for $N = 1$ and $1 \le r < \frac{2N}{(N-2)_+}$ for $N \ge 2$. Then it follows that

$$\frac{N}{2}\left(\frac{1}{2} - \frac{1}{r}\right) < \frac{1}{2},$$

and hence

$$\|w(\cdot,t)\|_r \le C_8\|w_0\|_r + C_8\int_0^t (t-t')^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)} e^{-\mu(t-t')}(\|w(\cdot,t')\|_2 + \|z_t(\cdot,t')\|_2)\, dt' \le C_9, \tag{40}$$

from (38). Here we used (35) for the second inequality to derive.

If $N \ge 2$, we also have

$$\|z(\cdot,t)\|_r \le C_{10}, \quad 1 \le r < \frac{2N}{(N-2)_+}, \tag{41}$$

derived from (35), which implies

$$\|u(\cdot,t)\|_q \leq C_{11}, \quad 1 \leq q < \frac{2N}{(N-2)_+}, \tag{42}$$

by (33). Using (34), now we have

$$u_t \leq D\Delta u + kv, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0.$$

Then it holds that

$$0 \leq u(\cdot,t) \leq \overline{u}(\cdot,t), \tag{43}$$

for

$$\overline{u}(\cdot,t) = e^{(D\Delta-\mu)t}u_0 + \int_0^t e^{(D\Delta-\mu)(t-t')}[\mu u(\cdot,t') + kv(\cdot,t')]\,dt',$$

where the semigroup estimate (39) is applicable. From (35), (42), and (43) it thus follows that

$$\|\overline{u}(\cdot,t)\|_r \leq C_{12},$$

for $1 \leq r \leq \infty$ satisfying

$$\frac{N}{2}\left(\frac{1}{q} - \frac{1}{r}\right) < 1.$$

Thus we obtain

$$\|u(\cdot,t)\|_\infty \leq C_{13}, \tag{44}$$

for $N \leq 5$, while (42) is improved as

$$\|u(\cdot,t)\|_q \leq C_{14}, \quad 1 \leq q < \frac{2N}{(N-6)_+},$$

for $N \geq 6$. Continuing this procedure, we reach (44) for any N and then (37) follows from (35). □

Proof of Theorem 2 By Lemmas 2.1 and 2.3 we have

$$\|(u(\cdot,t), v(\cdot,t))\|_\infty \leq C_{15}.$$

This implies $T = +\infty$ and the relative compactness of the orbit

$$\mathcal{O} = \{(u(\cdot,t), v(\cdot,t))\}_{t\geq 0} \subset C^2(\overline{\Omega})^2.$$

From the general theory (see [6,7]) the ω-limit set $\omega(u_0, v_0)$ is contained in the set of equilibria, that is, $L(z,w)$ is constant on $\omega(u_0,v_0)$ by LaSalle's principle.

Given $(u_*, v_*) \in \omega(u_0, v_0)$, let $(\tilde{u}, \tilde{v}) = (\tilde{u}(\cdot,t), \tilde{v}(\cdot,t))$ be the solution to (6) for $(u_0, v_0) = (u_*, v_*)$ and

$$\tilde{w} = D\tilde{u} + \tilde{v}, \quad \tilde{z} = \tilde{u} + \tilde{v}.$$

From the above property we have

$$\frac{d}{dt}L(\tilde{z}(\cdot,t), \tilde{w}(\cdot,t)) = 0, \quad t \geq 0,$$

and then it follows that

$$\tilde{z}_t = 0, \quad \tilde{w}_t = 0, \quad \nabla \tilde{w} = 0,$$

from (15). Hence we have

$$D\Delta z_* + kw_* + g(z_*) = 0, \quad \left.\frac{\partial z_*}{\partial \nu}\right|_{\partial\Omega} = 0,$$

and $w_* \in \mathbf{R}$. This w_* is determined by the total mass

$$\lambda = \int_\Omega (\xi z_* + w_*) dx,$$

for λ in (20). Then (22) follows, and $z = z_*$ is a solution to (18).

Since each $(u_*, v_*) \in \omega(u_0, v_0)$ satisfies $w_* = Du_* + v_* \in \mathbf{R}$, it holds that

$$\lim_{t\uparrow+\infty} \|\nabla w(\cdot, t)\|_{C^1} = 0.$$

Then we obtain (23). □

3 Proof of Theorem 3

We have derived (25) by reducing the second equation of (9) to the stationary state. This process is valid even in the variational level, that is, between the functionals $L(z, w)$ and $J_\lambda(z)$. In Lemma 3.1 below, we shall show the semi-unfolding-minimality property, observed in several models in non-equilibrium thermodynamics [9,10,22–25] and [18] (see also [21]).

For the moment we regard $L(z, w)$ and $J_\lambda(z)$ as smooth functionals of $(z, w) \in H^1(\Omega) \times H^1(\Omega)$ and $z \in H^1(\Omega)$, defined by (14) and (19), respectively.

Lemma 3.1 *Given $\lambda \in \mathbf{R}$, let $(z, w) \in H^1(\Omega) \times H^1(\Omega)$ satisfy*

$$\int_\Omega (\xi z + w) dx = \lambda,$$

and define $\overline{w} \in \mathbf{R}$ by (17). Then it holds that

$$L(z, w) \geq L(z, \overline{w}) = \xi J_\lambda(z) + \frac{\lambda^2 k}{2|\Omega|}. \tag{45}$$

Proof We have

$$\overline{w} = \frac{1}{|\Omega|} \int_\Omega w \, dx,$$

and hence

$$\int_\Omega w^2 \, dx \geq \int_\Omega \overline{w}^2 \, dx,$$

by Jensen's inequality. Then $L(z, w) \geq L(z, \overline{w})$ follows.

The second identity of (45) is now derived as

$$L(z, \overline{w}) = \frac{1}{2} \int_\Omega (k\overline{w}^2 + \xi D|\nabla z|^2 - 2\xi G(z)) dx = \xi J_\lambda(z) + \frac{\lambda^2 k}{2|\Omega|}.$$

□

The following lemma holds because $h = h(z)$ is real analytic in $z \geq 0$. The proof is similar to Lemma 7 of [13] and is omitted.

Lemma 3.2 *Let $z_* = z_*(x)$ be a local minimizer of functional $J_\lambda(z)$, $z \in H^1(\Omega)$, defined by (19). Since $h = h(z)$ is a real-analytic function of $z \in \mathbf{R}$, there exists an $\varepsilon_0 > 0$ such that any $\varepsilon \in (0, \varepsilon_0/4]$ admits a $\delta_0 > 0$ so that*

$$\|z - z_*\|_{H^1} < \varepsilon_0, \ J_\lambda(z) - J_\lambda(z_*) < \delta_0 \quad \Rightarrow \quad \|z - z_*\|_{H^1} < \varepsilon. \tag{46}$$

We are ready to give the following proof using semi-duality.

Proof of Theorem 3 First, the solution $(z, w) = (z(\cdot, t), w(\cdot, t))$ lies on the affine space $\{(z, w) \mid \int_\Omega \xi z + w \, dx = \lambda\}$. Let (z_0, w_0) be the initial value and let $0 \leq z_* \in H^1(\Omega)$ be a local minimum of $J_\lambda(z)$, $z \in H^1(\Omega)$, for λ defined by (16). Given $\varepsilon > 0$, we shall show the existence of $\delta > 0$ such that

$$\|z_0 - z_*\|_{H^1} + \|w_0 - \overline{w}\|_{H^1} < \delta, \tag{47}$$

implies

$$\|z(\cdot, t) - z_*\|_{H^1} + \|w(\cdot, t) - \overline{w}\|_{H^1} < C_{16}\varepsilon, \quad t \geq 0, \tag{48}$$

for $\overline{w} \in \mathbf{R}$ defined by (17). This property will imply the stability of $(z_*, \overline{w})$ concerning (9) in $X = C^2(\overline{\Omega})^2$, because the orbit

$$\mathcal{O} = \{(u(\cdot, t), v(\cdot, t))\}_{t \geq 0},$$

is relatively compact in X.

First, we take $\varepsilon_0 > 0$ be as in Lemma 3.2. Then the total mass conservation in the form of (11) implies

$$\xi J_\lambda(z(\cdot, t)) - \xi J_\lambda(z_*) \leq L(z_0, w_0) - L(z_*, \overline{w}), \quad t \geq 0,$$

by (15). Given $\varepsilon \in (0, \varepsilon_0/4]$, next, we take δ_0 as in Lemma 3.2. Then we determine $\delta > 0$ such that (47) implies

$$\|z_0 - z_*\|_{H^1} < \varepsilon_0/2, \quad L(z_0, w_0) - L(z_*, \overline{w}) < \xi \delta_0. \tag{49}$$

From the second inequality of (49) we have

$$J_\lambda(z(\cdot, t)) - J_\lambda(z_*) < \delta_0, \quad t \geq 0. \tag{50}$$

Now we show

$$\|z(\cdot, t) - z_*\|_{H^1} < \varepsilon_0/2, \quad t \geq 0. \tag{51}$$

In fact, if this is not the case we have $t_0 > 0$ such that

$$\|z(\cdot, t_0) - z_*\|_{H^1} = \varepsilon_0/2 < \varepsilon_0, \tag{52}$$

because of the first inequality of (49) and the continuity of $t \mapsto z(\cdot, t) \in H^1(\Omega)$. Then Lemma 3.2, based on (50) and (52), implies

$$\|z(\cdot, t_0) - z_*\|_{H^1} < \varepsilon \leq \varepsilon_0/4,$$

a contradiction. Having (50) and (51), we obtain

$$\|z(\cdot, t) - z_*\|_{H^1} < \varepsilon, \quad t \geq 0. \tag{53}$$

Since

$$\langle w(t)\rangle = \frac{1}{|\Omega|}\int_\Omega w(\cdot,t)\,dx = \frac{1}{\tau|\Omega|}\left(\lambda - \xi\int_\Omega z\,dx\right),$$

it holds that

$$\begin{aligned}|\langle w(t)\rangle - \overline{w}| &\le \frac{\xi}{\tau|\Omega|}\|z_* - z(\cdot,t)\|_1\\ &\le \frac{\xi}{\tau|\Omega|^{1/2}}\|z_* - z(\cdot,t)\|_2 < \frac{\xi\varepsilon}{\tau|\Omega|^{1/2}}.\end{aligned}$$

Then (48) follows from (23) and

$$\|\overline{w} - w_*\|_{H^1} \le \frac{\xi}{|\Omega|^{1/2}}\|z - z_*\|_2.$$

□

4 Proof of Theorem 4

The eigenvalue problem of $\mathcal{A}$ in $L^2(\Omega : \mathbf{C})^2$ takes the form

$$\mathcal{A}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \sigma\begin{pmatrix}\phi\\ \psi\end{pmatrix},\quad \begin{pmatrix}\phi\\ \psi\end{pmatrix}\in D(\mathcal{A})\setminus\{0\}$$

which means $(\phi,\psi)\neq(0,0)$ and by

$$M = \begin{pmatrix}1 + D\xi/\alpha & -\xi/\alpha\\ \xi & 1\end{pmatrix},\quad \mathcal{A}_1 = \begin{pmatrix}-D\Delta - g'(z_*) & -k\\ 0 & -\alpha\Delta\end{pmatrix}.$$

it is written as

$$\mathcal{A}_1\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \sigma M\begin{pmatrix}\phi\\ \psi\end{pmatrix},\quad \begin{pmatrix}\phi\\ \psi\end{pmatrix}\in D(\mathcal{A}_1)\setminus\{0\},$$

namely,

$$\begin{aligned}&-(D\Delta\phi + g'(z_*)\phi + k\psi) = \sigma\{(1 + D\xi/\alpha)\phi - (\xi/\alpha)\psi\},\\ &-\alpha\Delta\psi = \sigma(\psi + \xi\phi),\quad \text{in } \Omega,\\ &\frac{\partial}{\partial\nu}(\phi,\psi)\Big|_{\partial\Omega} = 0,\quad \int_\Omega(\psi + \xi\phi)dx = 0.\end{aligned}\tag{54}$$

Henceforth, $(\,\cdot\,,\,\cdot\,)$ and $\|\cdot\|$ indicate the inner product and norm in $L^2(\Omega;\mathbf{C})^2$, respectively.

For the proof of Theorem 4, we have to compare nonpositive eigenvalues of $\mathcal{A}$ and $\mathcal{L}$, which is defined in (27) as,

$$\mathcal{L}\varphi = -(D\Delta\varphi + g'(z_*)\varphi - k\xi\langle\varphi\rangle)$$

To carry out it, we will take the following steps:

1. Prove that every eigenvalue of $\mathcal{A}$ in $\{\lambda\in\mathbf{C} \mid \mathrm{Re}\lambda\le 0\}$ is real.
2. Show the coincidence of the algebraic multiplicity and geometric one of each nonpositive eigenvalue.
3. Write the equations of (54) as $\mathcal{L}\phi = \sigma\mathcal{M}(-\sigma/\alpha)\phi$ by an appropriate nonlocal operator $\mathcal{M}(s)$, which is bounded for each $s\ge 0$ provided (28) holds.

4. Consider the weighted eigenvalue problem for $\mathcal{L}\phi = \mu\mathcal{M}(s)\phi$ and show that the number of negative eigenvalues of this problem is equal to that of $\mathcal{L}$, that is, $\mathcal{L}\phi = \mu^*\phi$.
5. For each negative eigenvalue $\mu = \mu(s)$ of the weighted problem prove the monotonicity of $\mu(s)/s$ together with the continuity in $s \in (0, \infty)$ and conclude the existence of s enjoying $s = -\mu(s)/\alpha$.

We first show the next lemma.

Lemma 4.1 *Any eigenvalue $\sigma \in \mathbf{C}$ of $\mathcal{A}$ satisfying*

$$Re\,\sigma < \alpha k/2\xi \tag{55}$$

is real.

Proof We may assume $\sigma \neq 0$. Letting

$$\sigma_1 = \text{Re}\,\sigma, \quad \sigma_2 = \text{Im}\,\sigma, \quad J_1 = \text{Re}\,(\phi, \psi), \quad J_2 = \text{Im}\,(\phi, \psi),$$

we have

$$\begin{aligned} D\|\nabla\phi\|^2 - \int_\Omega g'(z_*)|\phi|^2 dx - k(J_1 - \imath J_2) \\ = (\sigma_1 + \imath\sigma_2)\{(1 + D\xi/\alpha)\|\phi\|^2 - \xi/\alpha(J_1 - \imath J_2)\}, \\ \alpha\|\nabla\psi\|^2 = (\sigma_1 + \imath\sigma_2)\|\psi\|^2 + (\sigma_1 + \imath\sigma_2)\xi(J_1 + \imath J_2), \end{aligned}$$

by (54). Then it follows that

$$\begin{aligned} kJ_2 &= \sigma_2(1 + D\xi/\alpha)\|\phi\|^2 - (\xi/\alpha)(\sigma_2 J_1 - \sigma_1 J_2), \\ \alpha\|\nabla\psi\|^2 &= \sigma_1\|\psi\|^2 + \xi(\sigma_1 J_1 - \sigma_2 J_2), \\ 0 &= \sigma_2\|\psi\|^2 + \xi(\sigma_2 J_1 + \sigma_1 J_2). \end{aligned} \tag{56}$$

The last two equalities of (56) imply

$$\alpha\sigma_2\|\nabla\psi\|^2 = -(\sigma_1^2 + \sigma_2^2)\xi J_2, \tag{57}$$

while from the first and the third equalities we have

$$(\sigma_2/\alpha)\|\psi\|^2 + \sigma_2(1 + D\xi/\alpha)\|\phi\|^2 = (k - 2\xi\sigma_1/\alpha)J_2. \tag{58}$$

Equalities (57)–(58) are reduced to

$$(\sigma_2/\alpha)\|\psi\|^2 + \sigma_2(1 + D\xi/\alpha)\|\phi\|^2 = -\alpha\sigma_2\frac{k - 2\xi\sigma_1/\alpha}{(\sigma_1^2 + \sigma_2^2)\xi}\|\nabla\psi\|^2.$$

Thus $\sigma_1 < \alpha k/2\xi$ implies $\sigma_2 = 0$. □

Henceforth, we define $-\Delta_N$ by $-\Delta_N\phi = -\Delta\phi$, $\phi \in D(-\Delta_N)$, and

$$D(-\Delta_N) = \left\{\phi \in H^2(\Omega) \cap L_0^2(\Omega) \,\middle|\, \frac{\partial\phi}{\partial\nu}\bigg|_{\partial\Omega} = 0\right\},$$

$$L_0^2(\Omega) = \{\phi \in L^2(\Omega) \mid \int_\Omega \phi\, dx = 0\}.$$

Since

$$\int_\Omega (-\Delta\phi)dx = 0, \quad \phi \in D(A),$$

the operator $-\Delta_N$ is a self-adjoint operator in $L^2_0(\Omega)$. We put also

$$Q\phi = \phi - \langle \phi \rangle, \quad \langle \phi \rangle = \frac{1}{|\Omega|}\int_\Omega \phi\, dx,$$

for $\phi \in L^2(\Omega)$.

The proof of the following lemma is similar to that of Lemma 3.3 of [3], although more careful computation is needed.

Lemma 4.2 *In addition to the condition for the spectrum σ in Lemma 4.1 assume $\sigma < \alpha\eta_2$. Then the algebraic and geometric multiplicities of the eigenvalue σ of $\mathcal{A}$ in (55) coincide.*

Proof Let

$$(\mathcal{A} - \sigma I)\begin{pmatrix}\phi_0 \\ \psi_0\end{pmatrix} = 0, \quad \begin{pmatrix}\phi_0 \\ \psi_0\end{pmatrix} \in D(\mathcal{A}) \setminus \{0\}.$$

To prove

$$\text{Ker}\,(\mathcal{A} - \sigma I) = \text{Ker}\,(\mathcal{A} - \sigma I)^m, \quad m \geq 2, \tag{59}$$

it suffices to show the nonexistence of the solution to

$$(\mathcal{A} - \sigma I)\begin{pmatrix}\phi \\ \psi\end{pmatrix} = \begin{pmatrix}\phi_0 \\ \psi_0\end{pmatrix}, \quad \begin{pmatrix}\phi \\ \psi\end{pmatrix} \in D(\mathcal{A}). \tag{60}$$

First, Eq. (60) yields

$$\mathcal{A}_1\begin{pmatrix}\phi_0 \\ \psi_0\end{pmatrix} = \sigma M\begin{pmatrix}\phi_0 \\ \psi_0\end{pmatrix}, \quad \int_\Omega (\xi\phi_0 + \psi_0)dx = 0, \tag{61}$$

and hence

$$-\alpha\Delta\psi_0 = \sigma(\xi\phi_0 + \psi_0) = \sigma(\xi Q\phi_0 + Q\psi_0),$$

from the second component. Applying Q to both sides, we obtain

$$Q\psi_0 = \sigma\xi/\alpha(-\Delta_N - \sigma/\alpha)^{-1}Q\phi_0, \quad \langle\psi\rangle = -\xi\langle\phi\rangle. \tag{62}$$

Then the first component of (61) implies

$$\begin{aligned}&-D\Delta\phi_0 - g'(z_*)\phi_0 + \{k\xi - \sigma(1 + \xi(D+\xi)/\alpha)\}\langle\phi_0\rangle \\ &\quad = \sigma\{(1 + D\xi/\alpha) + \xi/\alpha(k - \sigma\xi/\alpha)(-\Delta_N - \sigma/\alpha)^{-1}\}Q\phi_0.\end{aligned} \tag{63}$$

Similarly, (60) implies

$$(\mathcal{A}_1 - \sigma M)\begin{pmatrix}\phi \\ \psi\end{pmatrix} = M\begin{pmatrix}\phi_0 \\ \psi_0\end{pmatrix},$$

and hence

$$\begin{aligned}&-D\Delta\phi - g'(z_*)\phi - \sigma(1 + D\xi/\alpha)\phi - (k - \sigma\xi/\alpha))\psi \\ &\quad = (1 + D\xi/\alpha)\phi_0 - (\xi/\alpha)\psi_0, \\ &-\alpha\Delta\psi - \sigma\psi - \sigma\xi\phi = \xi\phi_0 + \psi_0.\end{aligned} \tag{64}$$

From the second equation of (64) it follows that

$$Q\psi = \frac{\sigma\xi}{\alpha}B^{-1}Q\phi + \frac{1}{\alpha}B^{-1}(\xi Q\phi_0 + Q\psi_0),$$

by putting $B = -\Delta_N - \sigma/\alpha$. We note that B has an inverse by the assumption. Plug this into the first equation of (64). Since $\xi\langle\phi\rangle + \langle\psi\rangle = 0$ and $\xi\langle\phi_0\rangle + \langle\psi_0\rangle = 0$, we obtain

$$\tilde{\mathcal{L}}(\phi) = W, \tag{65}$$

where

$$\begin{aligned}\tilde{\mathcal{L}}(\phi) = &-D\Delta\phi - g'(z_*)\phi + \{k\xi - \sigma(1+\xi(D+\xi)/\alpha)\}\langle\phi\rangle \\ &-\sigma\{(1+D\xi/\alpha) + (\xi/\alpha)(k-\sigma(\xi/\alpha))B^{-1}\}Q\phi,\end{aligned} \tag{66}$$

and

$$\begin{aligned}W = &(1+\xi(D+\xi)/\alpha)\langle\phi_0\rangle + (1+D\xi/\alpha)Q\phi_0 - (\xi/\alpha)Q\psi_0 \\ &+\frac{\xi}{\alpha}(k-\sigma\xi/\alpha)B^{-1}Q\phi_0 + \frac{1}{\alpha}(k-\sigma\xi/\alpha)B^{-1}Q\psi_0.\end{aligned}$$

The operator $\tilde{\mathcal{L}}$ in (66) is realized as a self-adjoint operator in $L^2(\Omega)$ with the domain $D(\tilde{\mathcal{L}}) = \{\phi \in H^2(\Omega) \mid \frac{\partial\phi}{\partial\nu}\Big|_{\partial\Omega} = 0\}$. It holds that $\tilde{\mathcal{L}}(\phi_0) = 0$ by (63). Hence (65) implies

$$(W, \phi_0) = 0. \tag{67}$$

Here we have

$$\begin{aligned}(W,\phi_0) = &(1+\xi(D+\xi)/\alpha)\|\langle\phi_0\rangle\|^2 + (1+D\xi/\alpha)\|Q\phi_0\|^2 \\ &+\frac{\xi}{\alpha}(k-\sigma\xi/\alpha)(B^{-1}Q\phi_0, Q\phi_0) - \frac{\xi}{\alpha}(Q\psi_0, Q\phi_0) \\ &+\frac{1}{\alpha}(k-\sigma\xi/\alpha)(B^{-1}Q\psi_0, Q\phi_0).\end{aligned}$$

Due to (62), the sum of the last three terms on the right-hand side of the above equality is equal to

$$\begin{aligned}&\frac{\xi}{\alpha}(k-\sigma\xi/\alpha)(B^{-1}Q\phi_0, Q\phi_0) - \frac{\xi}{\alpha}\cdot\frac{\sigma\xi}{\alpha}(B^{-1}Q\phi_0, Q\phi_0) \\ &\quad+\frac{1}{\alpha}(k-\sigma\xi/\alpha)((\sigma\xi/\alpha)B^{-1}Q\phi_0, B^{-1}Q\phi_0) \\ &= \frac{\xi}{\alpha}\left\{(k-2\sigma\xi/\alpha)\|B^{-1/2}Q\phi_0\|^2 + \frac{\sigma}{\alpha}(k-\sigma\xi/\alpha)\|B^{-1}Q\phi\|^2\right\}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}(W,\phi_0) = &(1+\xi(D+\xi)/\alpha)\|\langle\phi_0\rangle\|^2 + (1+D\xi/\alpha)\|Q\phi_0\|^2 \\ &+\frac{\xi}{\alpha}\left\{(k-2\sigma\xi/\alpha)\|B^{-1/2}Q\phi_0\|^2 + \frac{\sigma}{\alpha}(k-\sigma\xi/\alpha)\|B^{-1}Q\phi_0\|^2\right\}.\end{aligned} \tag{68}$$

By the assumption (55) in Lemma 4.1 we have

$$k - 2\sigma\xi/\alpha > 0,$$

which implies that the right-hand side of the equality (68) is positive. Hence, $(W, \phi_0) > 0$, which is a contradiction. If $\sigma < 0$, utilizing

$$\begin{aligned}&\|B^{-1}Q\phi_0\|^2 \le \frac{1}{\eta_2 - \sigma/\alpha}\|B^{-1/2}Q\phi_0\|^2, \\ &k - 2\sigma\xi/\alpha + \frac{\sigma}{\alpha}\left(\frac{k-\sigma\xi/\alpha}{\eta_2-\sigma/\alpha}\right) > (k-\sigma\xi/\alpha)\left(1+\frac{\sigma/\alpha}{\eta_2-\sigma/\alpha}\right) > 0,\end{aligned}$$

we can assert $(W, \phi_0) > 0$, which is a contradiction. □

To go to the third step of the proof of Theorem 4, we write the second equation of (54) as

$$\alpha(-\Delta_N - (\sigma/\alpha))Q\psi = \sigma\langle\psi\rangle + \sigma\xi\phi = \sigma\xi Q\phi,$$

that is,

$$Q\psi = \sigma(\xi/\alpha)(-\Delta_N - (\sigma/\alpha))^{-1}Q\phi. \tag{69}$$

Next, the first equation of (54) writes

$$\begin{aligned} &-D\Delta\phi - g'(z_*)\phi - k(\langle\psi\rangle + Q\psi) \\ &\quad = \sigma\left[(1 + D\xi/\alpha)(\langle\phi\rangle + Q\phi) - (\xi/\alpha)(\langle\psi\rangle + Q\psi)\right] \\ &\quad = \sigma\left[(1 + \xi(D+\xi)/\alpha)\langle\phi\rangle + (1 + D\xi/\alpha)Q\phi - (\xi/\alpha)Q\psi\right]. \end{aligned}$$

Therefore, it holds that

$$\begin{aligned} -D\Delta\phi - g'(z_*)\phi + k\xi\langle\phi\rangle &= \sigma\left[(1 + \xi(D+\xi)/\alpha)\langle\phi\rangle + (1 + D\xi/\alpha)Q\phi \right.\\ &\quad \left. -\sigma(\xi/\alpha)^2(-\Delta_N - \sigma/\alpha)^{-1}Q\phi + (k\xi/\alpha)(-\Delta_N - \sigma/\alpha)^{-1}Q\phi\right] \\ &= \sigma[(1 + \xi(D+\xi)/\alpha)\langle\phi\rangle \\ &\quad + \{(1 + D\xi/\alpha) + (\xi/\alpha)(k - \sigma(\xi/\alpha))(-\Delta_N - \sigma/\alpha)^{-1}\}Q\phi]. \end{aligned} \tag{70}$$

By Lemmas 4.1 and 4.2, any eigenvalue σ of $\mathcal{A}$ satisfying (55) is real, with equal algebraic and geometric multiplicities. Then it holds that

$$\frac{\sigma}{\alpha} < \frac{k}{2\xi} < \frac{k}{\xi} < \eta_2,$$

by (28) and the assumptions. Here we put

$$\begin{aligned} \mathcal{M}(s) &= (1 + \xi(D+\xi)/\alpha)(1 - Q) \\ &\qquad + \{(1 + D\xi/\alpha) + (\xi/\alpha)(k + s\xi)(-\Delta_N + s)^{-1}\}Q, \end{aligned}$$

for each $s > -k/\xi (> -\eta_2)$. From (27) and (70), the complex number σ satisfying (55) is an eigenvalue of $\mathcal{A}$ if and only if it is real, $\sigma/\alpha < \eta_2$, and

$$\mathcal{L}\phi = \sigma\mathcal{M}(-\sigma/\alpha)\phi, \quad \phi \in D(\mathcal{L}) \setminus \{0\}. \tag{71}$$

Associated with this problem we consider the eigenvalue problem

$$\mathcal{L}\phi = \mu\mathcal{M}(s)\phi, \quad \phi \in D(\mathcal{L}) \setminus \{0\}. \tag{72}$$

For $s > -k/\xi$, $\mathcal{M}(s)$ is self-adjoint and bounded positive operator. Thus problem (72) admits an infinite number of eigenvalues, which are all real, denoted by

$$\mu_1(s) \le \mu_2(s) \le \cdots \le \mu_j(s) \le \cdots \to +\infty,$$

according to their multiplicities. For fixed $s > -k/\xi$, let $\Sigma(s) = \{\mu_j(s)\}_{j=1}^{\infty}$. We note that the problem (71) implies $\sigma \in \Sigma(-\sigma/\alpha)$.

We next go to the fourth step. We use the weighted L^2 norm $\|\cdot\|_s$ defined by

$$\|u\|_s^2 = (u, u)_s, \quad (u, v)_s = (\mathcal{M}(s)u, v).$$

Then the min-max principle is available to define the spectrum $\Sigma(s)$ through the Rayleigh quotient (see, e.g. [4])

$$R(\phi, s) = \frac{D\|\nabla\phi\|^2 - (g'(z_*)\phi, \phi) + k\xi\|\langle\phi\rangle\|^2}{\|\phi\|_s^2}.$$

Thus, it holds that

$$\begin{aligned}\mu_j(s) &= \inf\{\sup_{\phi\in X_j\setminus\{0\}} R(\phi,s)\mid X_j\subset H^1(\Omega),\ \operatorname{codim} X_j=j-1\}\\ &= \inf\{R(\phi,s)\mid \phi\in H^1(\Omega),\ (\phi,\phi_\ell(s))=0,\ 1\le\ell\le j-1\},\end{aligned}\tag{73}$$

where $\phi_\ell(s)$ denotes an eigenfunction of (72) corresponding to $\mu_\ell(s)$ such that $\|\phi_\ell(s)\|_s=1$. We note that the eigenvalues are arranged in an increasing order with respect to counting the multiplicity and a corresponding eigenfunction is uniquely determined up to multiplication of the nonzero constant.

We compare the spectrum $\Sigma(s)$ with that of the operator $\mathcal{L}$. Let the eigenvalues of $-\Delta_N$ be $\{\eta_i\}_{i=2}^\infty$,

$$0<\eta_2\le\eta_3\le\cdots\le\eta_i\le\cdots\to+\infty,$$

and $\{\Phi_i\}_{i=2}^\infty$ be its L^2 ortho-normal eigenfunctions. Then we have

$$\|\phi\|^2=\langle\phi\rangle^2+\sum_{i=2}^\infty|(\phi,\Phi_i)|^2,\tag{74}$$

$$\begin{aligned}\|\phi\|_s^2 &= (1+\xi(D+\xi)/\alpha)\langle\phi\rangle^2\\ &\quad+\sum_{i=2}^\infty\{(1+D\xi/\alpha)+(\xi/\alpha)(k+s\xi)(\eta_i+s)^{-1}\}|(\phi,\Phi_i)|^2.\end{aligned}\tag{75}$$

By (28) we have

$$0\le\frac{k+s\xi}{\eta_i+s}\le C_{17},\quad s\ge -k/\xi,\ i=2,3,\ldots.$$

Then it holds that

$$C_{18}^{-1}R(\phi)\le R(\phi,s)\le C_{18}R(\phi),\quad s\ge -k/\xi,$$

where

$$R(\phi)=\frac{D\|\nabla\phi\|^2-(g'(z_*)\phi,\phi)+k\xi\|\langle\phi\rangle\|^2}{\|\phi\|^2}.$$

Hence the number of non-positive elements of $\{\mu_j(s)\}_{j=1}^\infty$ is equal to that of the non-positive eigenvalues of $\mathcal{L}$. More precisely, we have

$$C_{18}^{-1}\mu_j^*\le\mu_j(s)\le C_{18}\mu_j^*,\quad s\in[-k/\xi,+\infty),\tag{76}$$

for each j, where μ_j^* denote the j-th eigenvalue of $\mathcal{L}$.

We now go to the final step. From (71), the real number σ in $\sigma<\alpha k/2\xi$ is an eigenvalue of $\mathcal{A}$ if and only if

$$\mu_j(-\sigma/\alpha)=\sigma,\tag{77}$$

for some $j\ge 1$. In particular, the number of zero eigenvalues of $\mathcal{A}$ is equal to that of zero elements of $\{\mu_j(0)\}_{j=1}^\infty$. Namely, this number is equal to m^*. Rewriting (77) with $s=-\sigma/\alpha$, on the other hand, we see that the number of negative eigenvalues of $\mathcal{A}$ is equal to that of $s>0$ such that

$$\frac{\mu_j(s)}{s}=-\alpha,\tag{78}$$

for some $j=1,\ldots,m$.

We prove the monotonicity of $\mu_j(s)/s$ in s. Here we have

$$\frac{\partial}{\partial s}\frac{R(\phi,s)}{s} = -\frac{R(\phi,s)}{s^2}\cdot\frac{1}{\|\phi\|_s^2}\cdot\frac{\partial}{\partial s}(s\|\phi\|_s^2), \tag{79}$$

and

$$\frac{\partial}{\partial s}(s\|\phi\|_s^2) = (1+\xi(D+\xi)/\alpha)\langle\phi\rangle^2 + \sum_{i=2}^{\infty}\{(1+D\xi/\alpha)+(\xi/\alpha)c_i(s)\}|(\phi,\Phi_i)|^2,$$

with

$$0 \le c_i(s) = \frac{\eta_i(k+s\xi)}{(\eta_i+s)^2} + \frac{\xi s}{\eta_i+s} \le C_{19}, \quad s>0.$$

Hence it follows that

$$C_{19}^{-1} \le \frac{1}{\|\phi\|_s^2}\frac{\partial}{\partial s}(s\|\phi\|_s^2) \le C_{19}. \tag{80}$$

From (79) and (80) we have $c_0>0$ independent of $s>0$ and $\phi\in H^1(\Omega)\setminus\{0\}$ such that

$$\begin{aligned}\frac{R(\phi,s')}{s'} &\ge \frac{R(\phi,s)}{s} - c_0\frac{R(\phi,s)}{s^2}(s'-s)+o(s'-s)\\ &= \left(1-\frac{c_0}{s}(s'-s)\right)\frac{R(\phi,s)}{s}+o(s'-s),\end{aligned} \tag{81}$$

as $s'\downarrow s$ uniformly in s and ϕ.

By (73) and (81) it holds that

$$\begin{aligned}\frac{\mu_j(s')}{s'} &\ge \left(1-\frac{c_0}{s}(s'-s)\right)\frac{\mu_j(s)}{s}+o(s'-s)\\ &= \frac{\mu_j(s)}{s} - \frac{c_0\mu_j(s)}{s^2}(s'-s)+o(s'-s),\end{aligned}$$

as $s'\downarrow s>0$. In particular, the mapping

$$s\in(0,+\infty) \quad\mapsto\quad \frac{\mu_j(s)}{s}<0,$$

is strictly increasing if $\mu_j(s)<0$, that is,

$$\frac{\mu_j(s')}{s'} > \frac{\mu_j(s)}{s}, \quad s'>s>0, \tag{82}$$

for $j=1,\ldots,m$.

To confirm the continuity of

$$s\in(0,+\infty)\mapsto\mu_j(s), \tag{83}$$

we use its monotonicity (non-increasing) derived from

$$\frac{d}{ds}\|\phi\|_s^2 \ge 0, \quad \phi\in L^2(\Omega). \tag{84}$$

Indeed, invoking (75) and noticing that

$$\frac{d}{ds}\left(\frac{k+s\xi}{\eta_\ell+s}\right) = \frac{\xi\eta_\ell-k}{(\eta_\ell+s)^2}>0,$$

by the condition (28), we obtain (84). Then (82) and (84) imply

$$\mu_j(s_1) \le \mu_j(s_2) \le \frac{s_2}{s_1}\mu_j(s_1), \quad 0 < s_2 \le s_1,$$

and hence the continuity of (83).

Since (76) implies

$$\lim_{s\downarrow 0} \frac{\mu_j(s)}{s} = -\infty, \quad \lim_{s\uparrow +\infty} \frac{\mu_j(s)}{s} = 0,$$

each $j = 1, \ldots, m$ admits a unique $s = s_j > 0$ such that (78) holds. Thus, the number of negative eigenvalues of $\mathcal{A}$ is equal to m. The proof of Theorem 4 is complete. □

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