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A Generalization of Bochner's Theorem and Its Applications in the Study of Impulsive Differential Equations

Liangping Qi[1] · **Rong Yuan**[2]

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Abstract In this paper, we reveal the deep relation between Stepanov and piecewise continuous almost periodic functions and apply it to the study of almost periodic impulsive differential equations. Under the quasi-uniform continuity condition, the equivalence of Stepanov and piecewise continuous almost periodic functions is firstly established, which provides both a generalization of Bochner's theorem and a powerful tool to investigate piecewise continuous almost periodic functions. As applications, the module containment for piecewise continuous almost periodic solutions to linear impulsive differential equations is studied.

Keywords Stepanov and piecewise continuous almost periodic functions · Bochner's theorem · Module containment · Impulsive differential equations

Mathematics Subject Classification 42A75 · 34A37 · 34C27

1 Introduction

Almost periodic functions, which are more often encountered in the study of various phenomena than the rather special periodic ones, are first introduced by H. Bohr and substantially studied by S. Bochner, H. Weyl, A. Besicovitch, J. Favard, J. von Neumann, V.V. Stepanov, N.N. Bogolyubov, and others [35]. The theory of almost periodic functions has many important applications in problems of ordinary differential equations, dynamical systems, stability theory and partial differential equations etc. A vast amount of research

✉ Liangping Qi
lqi@nankai.edu.cn

Rong Yuan
ryuan@bnu.edu.cn

1 Chern Institute of Mathematics, Nankai University, Tianjin, People's Republic of China

2 School of Mathematical Sciences, Beijing Normal University, Beijing, People's Republic of China

has been directed toward studying these phenomena. See [24,31,35,44,60] for surveys and [4,12,13,15,21,25,30,32,36–43,50,52,57] et al. for recent developments.

There are many generalizations of Bohr's almost periodic functions and we refer the readers to [11] for details. One of the generalization is given by Stepanov [56], which successes in removing the continuity restrictions, and characterises the almost periodicity of a locally integrable function f by requiring the set

$$T(f,\epsilon) := \left\{\tau \in \mathbb{R};\ \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(s+\tau) - f(s)|ds < \epsilon\right\}$$

to be relatively dense for all $\epsilon > 0$. Bochner [14] shows that by using a construction, a Stepanov function can be reduced to a Bohr function so that properties of Stepanov functions could be derived from the corresponding ones of Bohr functions. Another important generalization considered in this paper is the class of piecewise continuous almost periodic (p.c.a.p., for short) functions first introduced in [26], which have discontinuities of the first kind only and satisfy the quasi-uniform continuous and almost periodic conditions (Definition 2.13). The class of p.c.a.p. functions characterises an important and complicated kind of oscillations in the study of impulsive differential equations which have a wide scope of applications, not only in mathematics, but also in various fields of science and technology. Many biological phenomena involving thresholds and optimal control models in economics exhibit impulsive effects. See [1,3,5–7,10,27,28,33,34,45–48,53–55,59] et al. and the references therein for the vast amount of research that has been directed toward the study of impulsive differential equations.

By using an ingenious method, Bochner proved an important theorem on the equivalence of Stepanov and Bohr almost periodic functions under the uniform continuity condition (Theorem 2.8), which provides new characterizations for both of the two classes of functions. The remark in [46, p. 400] shows that all bounded p.c.a.p. functions (Definition 2.13) are Stepanov almost periodic, which indicates a possible way to investigate impulsive differential equations. However, the essential condition under which Stepanov and piecewise continuous almost periodic functions are equivalent has not been discovered and the module containment for p.c.a.p. solutions to impulsive differential equations has not been studied. So, the purposes of this paper are to give a generalization of Bochner's theorem for the two classes of functions mentioned above and apply it to the study of impulsive differential equations, which are of great interest. We think that the profound equivalence of Stepanov and piecewise continuous almost periodic functions under the condition of quasi-uniform continuity (Theorem 3.2) provides not only a good understanding of the complicated p.c.a.p. motions, but also a powerful tool to study various properties of p.c.a.p. functions including Fourier analysis and module containment. Our Theorem 3.2 is a generalization of Bochner's important result (Theorem 2.8) in the sense that Bohr almost periodic functions and the uniform continuity condition are extended to p.c.a.p. functions and the quasi-uniform continuity condition, respectively. Moreover, the module containment which serves as one of the few verifiable spectral relation also helps a lot to understand the complexity of p.c.a.p. motions. It is shown that Theorem 3.2 can be applied to investigate the module containment for impulsive differential equations. The idea is to view p.c.a.p. functions as Stepanov almost periodic ones, then vector-valued Bohr almost periodic ones, so that Favard's module containment theorem is applicable.

The module mod(f), defined to be the smallest additive group containing all the frequencies of a Bohr almost periodic function f, reflects the complexity of motions. For instance, f is $2\pi/\omega$-periodic $\Leftrightarrow$ mod$(f) \subset \omega\mathbb{Z}$ and f is constant $\Leftrightarrow$ mod$(f) = \{0\}$. Following the pioneer work of Favard [22,23], many works have been devoted to this topic. For module containment properties, see [16–18,24,35,49,57] et al. for classical results on Bohr almost periodic functions, [2,61,62] et al. for hybrid systems and [51] for almost automorphic functions. In [29], the authors propose an improvement of Favard's theorem concerning the existence of almost periodic solutions to the following differential equations in $\mathbb{R}^d$

$$x' = A(t)x + f(t) \tag{1}$$

whose coefficients A and f are almost periodic too, but this dose not really improve Favard's classical theory as pointed out by Tarallo in [58], which proves that the separation condition on solutions with norm in the sense of Bohr and Stepanov are equivalent. Furthermore, [9] also proves an interesting result that the Stepanov and uniformly almost periodic sequences coincide. However, the situation is different if impulse effect is considered in (1). In fact, Stepanov and piecewise continuous almost periodic functions are naturally related by the quasi-uniform continuity condition, and Stepanov almost periodic functions can indeed help a lot in the study of impulsive differential equations. By a completely new approach, in this paper we shall give a generalization of Bochner's theorem (Theorem 2.8) and apply it to study the following linear impulsive differential equations in $\mathbb{R}^d$

$$\begin{cases} x' = A(t)x + h(t), & t \neq \tau_n, \\ x(\tau_n^+) - x(\tau_n) = B(n)x(\tau_n) + b(n), & n \in \mathbb{Z}, \end{cases} \tag{2}$$

where A and h are respectively Bohr almost periodic matrix-valued and piecewise continuous almost periodic vector-valued functions, B and b are respectively almost periodic matrix-valued and vector-valued sequences, and $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ is a Wexler sequence.

This paper is organized as follows. Section 1 is an introduction and Sect. 2 introduces basic notations and necessary knowledge. In Sect. 3 we give the generalization of Bochner's theorem for Stepanov and piecewise continuous almost periodic functions. In Sect. 4 we establish the module containment theorem for Stepanov almost periodic functions (Theorem 4.7) and reveal the deep relation between the normal sequences in the sense of Bohr and Stepanov (Theorem 4.10). In Sect. 5 we make use of the generalization of Bochner's theorem to study the module of p.c.a.p. solutions to the linear impulsive differential equation (2) (Theorem 5.2).

2 Preliminaries

Let $\mathbb{G} = \mathbb{R}$ or $\mathbb{Z}$, and $(X, |\cdot|)$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$. A two-sided sequence in X is a function $\{u_n\}_{n\in\mathbb{Z}} = \{u(n)\}_{n\in\mathbb{Z}}$ from $\mathbb{Z}$ to X. In the following both notations will be used.

Definition 2.1 [19, p. 45], [35, p. 1], [46, p. 183] A continuous function $f : \mathbb{G} \to X$ is called Bohr almost periodic if given any $\epsilon > 0$, the ϵ-translation set (or ϵ-almost periodic set) of f,

$$T(f, \epsilon) := \{\tau \in \mathbb{G};\ |f(t+\tau) - f(t)| < \epsilon, \forall t \in \mathbb{G}\}$$

is relatively dense, that is, there is a positive number $l = l(\epsilon)$ such that $[a, a+l] \cap T(f, \epsilon) \neq \emptyset$ for all $a \in \mathbb{G}$.

Denote by $AP(\mathbb{G}, X)$ the set of all Bohr almost periodic functions from $\mathbb{G}$ to X. Equipped with the uniform convergence norm $\|f\| = \sup_{t\in\mathbb{G}} |f(t)|$, $AP(\mathbb{G}, X)$ is a Banach space. For each $f \in AP(\mathbb{G}, X)$, the mean value

$$\mathcal{M}\{f\} := \begin{cases} \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(t)dt, & \text{if } \mathbb{G} = \mathbb{R}, \\ \lim_{N\to\infty} \frac{1}{N}\sum_{n=a}^{a+N-1} f(n), & \text{if } \mathbb{G} = \mathbb{Z}, \end{cases}$$

exists uniformly with respect to $a \in \mathbb{G}$. The set

$$\Lambda_f = \{\lambda_k\} := \begin{cases} \left\{\lambda \in \mathbb{R};\ a(f,\lambda) = \mathcal{M}\{fe^{-i\lambda\cdot}\} \neq 0\right\}, & \text{if } \mathbb{G} = \mathbb{R}, \\ \left\{\widetilde{\mu} \in \mathbb{R}/2\pi\mathbb{Z};\ a(f,\widetilde{\mu}) = \mathcal{M}\{fe^{-i\mu\cdot}\} \neq 0, \mu \in \widetilde{\mu}\right\}, & \text{if } \mathbb{G} = \mathbb{Z}, \end{cases}$$

called the spectrum of f, is at most countable. Denoting $a_k = a(f, \lambda_k)$, we associate the Fourier series

$$f(t) \sim \sum_k a_k e^{i\lambda_k t}, \quad t \in \mathbb{G},$$

where $e^{i\widetilde{s}n} := e^{isn}$, $n \in \mathbb{Z}$, for $\mathbb{G} = \mathbb{Z}$. The elements $a_k \in X$ and λ_k are called the Fourier coefficients and exponents of f, respectively. The additive group

$$\mathrm{mod}(f) := \left\{\sum_{k=1}^{n} m_k\lambda_k;\ \lambda_k \in \Lambda_f, m_k \in \mathbb{Z}, n \in \mathbb{Z}_+\right\}$$

is called the module of f.

The following Favard's module containment theorem is a powerful tool to study the module of Bohr almost periodic functions.

Theorem 2.2 [8, p. 34], [24, p. 61], [35, pp. 42–44] *The following statements are equivalent for f and $g \in AP(\mathbb{R}, X)$.*

(i) $\mathrm{mod}(f) \supset \mathrm{mod}(g)$.
(ii) *For every $\epsilon > 0$ there is a $\delta > 0$ so that $T(f,\delta) \subset T(g,\epsilon)$.*
(iii) *$\mathcal{T}_\alpha f$ exists implies $\mathcal{T}_\alpha g$ exists (any sense).*
(iv) *$\mathcal{T}_\alpha f = f$ implies $\mathcal{T}_\alpha g = g$ (any sense).*
(v) *$\mathcal{T}_\alpha f = f$ implies there is $\alpha' \subset \alpha$ so that $\mathcal{T}_{\alpha'} g = g$ (any sense).*

Remark 2.3 The operator $\mathcal{T}_\alpha f = g$ is adopt here to ease the notation for taking limits, which means that $g(t) = \lim_{k\to\infty} f(t+\alpha_k)$, $t \in \mathbb{G}$, $\alpha = \{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{G}$, and is written only when the limit exists. The mode of convergence in Theorem 2.2 includes those pointwise, uniform and uniform on compact intervals and it will be specified at each use of the symbol [24, p. 3]. The mode of convergence in the sense of Stepanov is proven in Theorem 4.10. The symbol $\beta \subset \alpha$ means that $\beta = \{\beta_k\}_{k=1}^{\infty}$ is a subsequence of $\alpha = \{\alpha_k\}_{k=1}^{\infty}$.

Remark 2.4 Except for the Banach space X, the module containment theorem above is the same to that in [24, p. 61]. One can check that the proof of Theorem 4.5 in [24, p. 61] indeed remains true when X is a Banach space. In addition, results in [8, p. 34] and [35, pp. 42–44] are stated for Bohr almost periodic functions taking values in Banach spaces and metric spaces, respectively.

Set

$$\mathrm{span}(E) = \{n_1\beta_1 + n_2\beta_2 + \cdots + n_k\beta_k;\ k \in \mathbb{Z}_+, \beta_j \in E, n_j \in \mathbb{Z}, 1 \leq j \leq k\}$$

for $E \subset \mathbb{R}$ or $E \subset \mathbb{R}/2\pi\mathbb{Z}$. The following generalization of Favard's module containment theorem, which connects the continuous and discrete, is crucial in the study of the module of almost periodic solutions to both hybrid and impulsive systems. See [62] for more applications.

Theorem 2.5 [62] *Assume that $\eta > 0$ is fixed and $f, g \in AP(\mathbb{R}, X)$, then the following statements are equivalent.*

(i) $\mathrm{mod}(g) \subset \mathrm{span}\left(\mathrm{mod}(f) \cup \{\frac{2\pi}{\eta}\}\right)$.
(ii) *For any sequence $\alpha' \subset \eta\mathbb{Z}$, $\mathcal{T}_{\alpha'} f = f$ implies the existence of a subsequence $\alpha \subset \alpha'$ with $\mathcal{T}_\alpha g = g$ (any sense).*

Theorem 2.6 [62] *The following two statements are equivalent for u and $v \in AP(\mathbb{Z}, X)$.*

(i) $\mathrm{mod}(u) \supset \mathrm{mod}(v)$.
(ii) *For any sequence $\alpha' \subset \mathbb{Z}$, $\mathcal{T}_{\alpha'} f = f$ implies the existence of a subsequence $\alpha \subset \alpha'$ with $\mathcal{T}_\alpha g = g$ (any sense).*

Next we introduce the definition and basic properties of Stepanov almost periodic functions.

Definition 2.7 [8, p. 77], [19, p. 173] A function $f \in L^p_{loc}(\mathbb{R}, X)$, $p \geq 1$, is called S^p-almost periodic if for each $\epsilon > 0$, the ϵ-translation set (or ϵ-almost periodic set) of f,

$$T(f, \epsilon) := \left\{ \tau \in \mathbb{R};\ \sup_{t\in\mathbb{R}} \left[\int_0^1 |f(t+\tau+s) - f(t+s)|^p ds \right]^{\frac{1}{p}} < \epsilon \right\}$$

is relatively dense.

Denote by $S^p(\mathbb{R}, X)$ the Banach space of all Stepanov almost periodic functions (of order p) with the norm

$$\|f\|_{S^p} = \sup_{t\in\mathbb{R}} \left[\int_t^{t+1} |f(s)|^p ds \right]^{\frac{1}{p}}.$$

If $p = 1$, we write $S(\mathbb{R}, X)$ instead of $S^1(\mathbb{R}, X)$ for simplicity.

Clearly, Bohr almost periodic functions are Stepanov almost periodic. Their relationship was established by Bochner as the following important result which gives new characterizations for both of the two classes of functions.

Theorem 2.8 (Bochner [8, p. 78], [19, p. 174], [35, p. 34]) *If $f \in S^p(\mathbb{R}, X)$ is uniformly continuous, then $f \in AP(\mathbb{R}, X)$.*

A Fourier series can be constructed for a Stepanov almost periodic function.

Theorem 2.9 [8, p. 79], [35, p. 35] *Let $f \in S^p(\mathbb{R}, X)$. The formal Fourier expansion then holds*

$$f(t) \sim \sum_k a_k e^{i\lambda_k t}, \tag{3}$$

where

$$a_k = \lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} f(t) e^{-i\lambda_k t} dt$$

uniformly with respect to $a \in \mathbb{R}$.

Remark 2.10 The expression (3) does not imply any convergence. However, from Bochner's construction and the approximation theorem for Bohr almost periodic functions it follows that for every $\epsilon > 0$ there is a trigonometric polynomial

$$P_\epsilon(t) = \sum_{k=1}^{N_\epsilon} b_{k,\epsilon} e^{i\lambda_k t}, \quad b_{k,\epsilon} \in X,$$

such that $\|P_\epsilon - f\|_{S^p} < \epsilon$.

We begin with characterizing the discontinuities before giving the definition of p.c.a.p. functions. A sequence $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ will be called admissible if $\lim_{j\to\pm\infty} \tau_j = \pm\infty$ and $\tau_j < \tau_{j+1}$ for all $j \in \mathbb{Z}$. Let $\tau_j^k = \tau_{j+k} - \tau_j$ for $j, k \in \mathbb{Z}$.

Definition 2.11 [46, p. 195] An admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ is called a Wexler sequence if it satisfies the separation condition $\inf_{j\in\mathbb{Z}} \tau_j^1 > 0$, and the family of derived sequences

$$\{\{\tau_j^k\}\} := \{\{\tau_j^k\}_{j\in\mathbb{Z}}\}_{k\in\mathbb{Z}}$$

is equi-potentially (or uniformly, see e.g. [1,5]) almost periodic (e.p.a.p., for short), that is, for each $\epsilon > 0$ the common ϵ-translation set of all the sequences $\{\{\tau_j^k\}\}$,

$$T(\{\{\tau_j^k\}\}, \epsilon) = \left\{p \in \mathbb{Z};\, |\tau_{j+p}^k - \tau_j^k| < \epsilon \text{ for all } j, k \in \mathbb{Z}\right\}$$

is relatively dense.

Lemma 2.12 [46, p. 377] *Suppose that* $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ *is an admissible sequence and the derived family* $\{\{\tau_j^k\}\}$ *is e.p.a.p. Then there exist unique* $\xi \in \mathbb{R}$ *and* $\zeta \in AP(\mathbb{Z}, \mathbb{R})$ *such that*

$$\tau_n = \xi n + \zeta(n), \quad n \in \mathbb{Z}.$$

This lemma illustrates the condition imposed on the sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ containing the discontinuities of a p.c.a.p. function. Since $\{\tau_j\}_{j\in\mathbb{Z}}$ is admissible, $\xi > 0$ and $\xi + \zeta(n+1) - \zeta(n) > 0$ for all $n \in \mathbb{Z}$.

Let $PC(\mathbb{R}, X)$ be the set of all piecewise continuous functions $h : \mathbb{R} \to X$ which have discontinuities of the first kind (both $h(t+0)$ and $h(t-0)$ exist) only at the points of a subset of an admissible sequence $\{\tau_j = \tau_j(h)\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ and are continuous from the left at $\{\tau_j\}_{j\in\mathbb{Z}}$, i.e. $\lim_{t\to\tau_j-0} h(t) = h(\tau_j)$ for all $j \in \mathbb{Z}$.

Note that different functions in $PC(\mathbb{R}, X)$ do not necessarily have the same points of discontinuities. Since the empty set is a subset of every admissible sequence, $PC(\mathbb{R}, X)$ contains all continuous functions.

Definition 2.13 [46, p. 201] A function $h \in PC(\mathbb{R}, X)$ is called piecewise continuous almost periodic (p.c.a.p.) if the following conditions hold:

(i) There is an admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ which contains possible discontinuities of h and has an e.p.a.p. family of derived sequences $\{\{\tau_j^k\}\}$.
(ii) For each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|h(s) - h(t)| < \epsilon$ whenever s, $t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$.
(iii) For each $\epsilon > 0$, the ϵ-translation set (or ϵ-almost periodic set) of h,

$$T(h, \epsilon) := \{\tau \in \mathbb{R};\, |h(t+\tau) - h(t)| < \epsilon \text{ for all } t \in \mathbb{R} \text{ such that } |t - \tau_j| > \epsilon,\ j \in \mathbb{Z}\}$$

is relatively dense.

Let $PCAP(\mathbb{R}, X)$ be the set of all p.c.a.p. functions.

3 A Generalization of Bochner's Theorem

In this section, we prove the deep equivalent relation between Stepanov and piecewise continuous almost periodic functions under the quasi-uniform continuity condition. Consequently, it will be natural to derive properties of p.c.a.p. functions and solutions to impulsive differential equations from the corresponding ones of Stepanov almost periodic functions, for instance, properties of Fourier series and module containment.

Definition 3.1 A function $h \in PC(\mathbb{R}, X)$ which has discontinuities at the points of a subset of an admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$, is said to be quasi-uniformly continuous on $\mathbb{R}$, if for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|h(s)-h(t)| < \epsilon$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s-t| < \delta$.

Let

$$\begin{aligned} PUCW(\mathbb{R}, X) = \{&h \in PC(\mathbb{R}, X);\ h \text{ is quasi-uniformly continuous on } \mathbb{R} \text{ and} \\ &\text{has discontinuities at the points of a subset of a Wexler} \\ &\text{sequence}\}, \\ PCAPW(\mathbb{R}, X) = \{&h \in PCAP(\mathbb{R}, X);\ h \text{ has discontinuities at the points of} \\ &\text{a subset of a Wexler sequence}\}. \end{aligned}$$

We shall prove the following generalization of Bochner's theorem.

Theorem 3.2 $S^p(\mathbb{R}, X) \cap PUCW(\mathbb{R}, X) = PCAPW(\mathbb{R}, X)$ *for any* $p \geq 1$.

To show this, we need three lemmas. Define for every $h \in PC(\mathbb{R}, X)$ a continuous function

$$h_\sigma(t) = \frac{1}{\sigma}\int_t^{t+\sigma} h(\nu)d\nu, \quad t \in \mathbb{R},$$

where $\sigma > 0$, and a quantity

$$\|h\| = \sup_{t\in\mathbb{R}} |h(t)| = \sup_{j\in\mathbb{Z}} \sup_{\tau_j < t \leq \tau_{j+1}} |h(t)|,$$

which may be infinite.

Lemma 3.3 *Suppose that* $h \in PCAP(\mathbb{R}, X)$ *has discontinuities at the points of a subset of a Wexler sequence* $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$*, then* h *is bounded on* $\mathbb{R}$.

Proof Let $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$. By (ii) of Definition 2.13, for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$, $0 < \delta < \theta/3$ such that $|h(s) - h(t)| < \epsilon$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s-t| < \delta$. Let an inclusion length for $T(h, \delta)$ be l and $M := \sup_{t\in[0,l]} |h(t)|$. We shall consider three cases of the points in $\mathbb{R}$.

Case 1 $|t - \tau_j| > \delta$ for all $j \in \mathbb{Z}$. From (iii) of Definition 2.13, there exists an $r \in [-t, -t+l] \cap T(h, \delta)$. Hence $t + r \in [0, l]$ and

$$\begin{aligned} |h(t)| &\leq |h(t+r) - h(t)| + |h(t+r)| \\ &< \delta + M. \end{aligned}$$

Case 2 $\tau_k - \delta \leq t \leq \tau_k$ for some $k \in \mathbb{Z}$. Since $\delta < \theta/3$, there exists an s such that

$$\begin{aligned} &\tau_{k-1} + \delta < \tau_k - 2\delta < s < \tau_k - \delta, \\ &0 < t - s < 2\delta. \end{aligned}$$

Let $r \in [-s, -s+l] \cap T(h, \delta)$. By using the inequalities in Case 1 for $h(s)$ it follows that

$$\begin{aligned}|h(t)| &\le \left|h(t) - h\left(t - \frac{t-s}{2}\right)\right| + \left|h\left(t - \frac{t-s}{2}\right) - h(s)\right| + |h(s)| \\ &< 2\epsilon + \delta + M.\end{aligned}$$

Case 3 $\tau_k < t \le \tau_k + \delta$ for some $k \in \mathbb{Z}$. Since $\delta < \theta/3$ there exists an s such that

$$\tau_k + \delta < s < t + \delta \le \tau_k + 2\delta < \tau_{k+1} - \delta.$$

Let $r \in [-s, -s+l] \cap T(h, \delta)$. A direct calculation shows that

$$\begin{aligned}|h(t)| &\le |h(t) - h(s)| + |h(s)| \\ &< \epsilon + \delta + M.\end{aligned}$$

□

Remark 3.4 Note that p.c.a.p. functions are not necessarily bounded on $\mathbb{R}$. See the supplement written by S. I. Trofimchuk in [46, p. 399] for an example of an unbounded p.c.a.p. function which has discontinuities at the points of a sequence with finite limit points. Since Theorem 75 on the boundedness of p.c.a.p. functions in [46, p. 203] does not assume the separation condition $\inf_{j\in\mathbb{Z}} \tau_j^1 > 0$, its proof is not sufficient. We provide a correct one.

Lemma 3.5 *Suppose that* $h \in PUCW(\mathbb{R}, X)$ *has discontinuities at the points of a subset of a Wexler sequence* $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$, *and* $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$. *Then given any* $\epsilon > 0$, *there exists a* δ, $0 < \delta < \min\{\epsilon, \theta/2\}$ *such that* $|h_\sigma(t) - h(t)| < \epsilon$ *for all* $\sigma \in \mathbb{R}$, $0 < \sigma < \delta$ *and* $t \in \mathbb{R}$, $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$.

Proof Since h is quasi-uniformly continuous on $\mathbb{R}$, for each ϵ, $0 < \epsilon < \theta/2$, there exists a $\delta = \delta(\epsilon)$, $0 < \delta < \epsilon$, such that $|h(s) - h(t)| < \epsilon$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$. Let $\sigma \in (0, \delta)$ and $t \in \mathbb{R}$ with $\tau_k + \epsilon < t < \tau_{k+1} - \epsilon$ for some $k \in \mathbb{Z}$. Therefore,

$$\tau_k + \epsilon + \sigma < t + \sigma < \tau_{k+1} - \epsilon + \sigma < \tau_{k+1}$$

and

$$\begin{aligned}|h_\sigma(t) - h(t)| &= \left|\frac{1}{\sigma}\int_t^{t+\sigma} [h(\nu) - h(t)] d\nu\right| \\ &\le \frac{1}{\sigma}\int_t^{t+\sigma} |h(\nu) - h(t)| d\nu < \epsilon.\end{aligned}$$

For any $\epsilon' > 0$ choose an ϵ so small that $0 < \epsilon < \min\{\epsilon', \theta/2\}$. It follows that there is a δ, $0 < \delta < \epsilon$, with $|h_\sigma(t) - h(t)| < \epsilon$ for all $\sigma \in \mathbb{R}$, $0 < \sigma < \delta$ and $t \in \mathbb{R}$, $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$. Consequently, if $t \in \mathbb{R}$, $|t - \tau_j| > \epsilon'$, $j \in \mathbb{Z}$, then $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$ and $|h_\sigma(t) - h(t)| < \epsilon < \epsilon'$. □

Lemma 3.6 *Suppose that* $h \in PUCW(\mathbb{R}, X)$ *has discontinuities at the points of a subset of a Wexler sequence* $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$. *If for each* $\epsilon > 0$ *there exists an* $f_\epsilon \in AP(\mathbb{R}, X)$ *such that* $|f_\epsilon(t) - h(t)| < \epsilon$ *for all* $t \in \mathbb{R}$, $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$, *then* $h \in PCAPW(\mathbb{R}, X)$.

Proof It suffices to prove that h satisfies (iii) of Definition 2.13. Let $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$ and ϵ be a number with $0 < \epsilon < \theta/6$. We shall show that $T(h, 3\epsilon)$ is relatively dense. Consider the following inequalities in $(r, q) \in \mathbb{R} \times \mathbb{Z}$,

$$|f_\epsilon(t + r) - f_\epsilon(t)| < \epsilon, \quad t \in \mathbb{R}, \tag{4}$$

$$|\tau_j^q - r| < \epsilon, \quad j \in \mathbb{Z}. \tag{5}$$

Lemma 29 in [46, p. 198] implies that the following two sets

$$\Gamma = \{r \in \mathbb{R};\ \text{there exists } q \in \mathbb{Z} \text{ such that } (r, q) \text{ satisfies(4)and(5)}\},$$
$$Q = \{q \in \mathbb{Z};\ \text{there exists } r \in \mathbb{R} \text{ such that } (r, q) \text{ satisfies(4)and(5)}\},$$

are relatively dense. Let $(r, q) \in \Gamma \times Q$ satisfy (4) and (5). If $\tau_k + 3\epsilon < t < \tau_{k+1} - 3\epsilon$ for some $k \in \mathbb{Z}$, from (5) it follows that

$$\tau_{k+q} - \tau_k - \epsilon < r < \tau_{k+1+q} - \tau_{k+1} + \epsilon,$$
$$\tau_{k+q} + 2\epsilon < t + r < \tau_{k+1+q} - 2\epsilon.$$

Therefore, $|t - \tau_j| > 3\epsilon > \epsilon$ and $|t + r - \tau_j| > 2\epsilon > \epsilon$ for all $j \in \mathbb{Z}$. A direct calculation shows that

$$|h(t + r) - h(t)| \le |h(t + r) - f_\epsilon(t + r)| + |f_\epsilon(t + r) - f_\epsilon(t)| + |f_\epsilon(t) - h(t)| < 3\epsilon.$$

Consequently, $\Gamma \subset T(h, 3\epsilon)$ and $h \in PCAPW(\mathbb{R}, X)$. □

Proof of Theorem 3.2 Let $h \in PCAPW(\mathbb{R}, X)$ have discontinuities at the points of a subset of a Wexler sequence $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$. It is obvious that $h \in PUCW(\mathbb{R}, X)$. Next we show that $h \in S^p(\mathbb{R}, X)$. Let $L > \sup_{j\in\mathbb{Z}} \tau_j^1$, $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$ and $m \in \mathbb{Z}_+$ satisfy $m\theta > 1$. A direct calculation shows that

$$\tau_{n+m} - \tau_n = \sum_{j=0}^{m-1} (\tau_{n+j+1} - \tau_{n+j}) = \sum_{j=0}^{m-1} \tau_{n+j}^1 \ge m\theta > 1$$

for all $n \in \mathbb{Z}$. For every $t \in \mathbb{R}$ there exists a unique $k \in \mathbb{Z}$ with $\tau_k < t \le \tau_{k+1}$. Consequently, $t + 1 \le \tau_{k+m+1}$ and

$$\begin{aligned}\int_t^{t+1} |h(s + r) - h(s)|^p ds &\le \int_{\tau_k}^{\tau_{k+1+m}} |h(s + r) - h(s)|^p ds \\ = \sum_{j=k}^{k+m} \Bigg[\int_{\tau_j}^{\tau_j+\epsilon} |h(s + r) - h(s)|^p ds &+ \int_{\tau_j+\epsilon}^{\tau_{j+1}-\epsilon} |h(s + r) - h(s)|^p ds \\ + \int_{\tau_{j+1}-\epsilon}^{\tau_{j+1}} |h(s + r) - h(s)|^p ds\Bigg] &\le \sum_{j=k}^{k+m} [2\epsilon(2\|h\|)^p + (\tau_j^1 - 2\epsilon)\epsilon^p] \\ < (m + 1)[2^{p+1}\|h\|^p + (L - 2\epsilon)\epsilon^{p-1}]\epsilon\end{aligned}$$

for all $r \in T(h, \epsilon)$, where $\|h\| < \infty$ by Lemma 3.3 and $0 < \epsilon < \theta/2$. Hence $h \in S^p(\mathbb{R}, X)$.

For the reverse containment, assume that $h \in S^p(\mathbb{R}, X) \cap PUCW(\mathbb{R}, X)$ has discontinuities at the points of a subset of a Wexler sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ with $\inf_{j\in\mathbb{Z}} \tau_j^1 = \theta$. Hölder's inequality yields

$$\int_t^{t+1} |h(s+\tau) - h(s)|ds \le \left[\int_t^{t+1} |h(s+\tau) - h(s)|^p ds\right]^{\frac{1}{p}}, \quad p \ge 1.$$

Hence $h \in S(\mathbb{R}, X)$. From Lemma 3.5 it follows that for every $\epsilon > 0$ there exists a δ, $0 < \delta < \min\{\theta/2, 1\}$ such that $|h_\sigma(t) - h(t)| < \epsilon$ for all $\sigma \in \mathbb{R}$, $0 < \sigma < \delta$ and $t \in \mathbb{R}$, $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$. Moreover, $h \in S(\mathbb{R}, X)$ implies that $h_\sigma \in AP(\mathbb{R}, X)$ for $0 < \sigma < \delta$ [11, p. 80]. Therefore, $h \in PCAPW(\mathbb{R}, X)$ by Lemma 3.6. □

Remark 3.7 Theorem 3.2 provides a powerful tool by using Stepanov functions to study various properties including Fourier analysis, module containment and almost periodicity of the primitives of p.c.a.p. functions, etc.

The following theorem shows that a p.c.a.p. function with possible discontinuities at the points of a Wexler sequence is Bohr almost periodic if and only if it is continuous on $\mathbb{R}$. Note that this result is not obvious from the definitions of piecewise continuous and Bohr almost periodic functions.

Theorem 3.8 $PCAPW(\mathbb{R}, X) \cap C(\mathbb{R}, X) = AP(\mathbb{R}, X)$.

Proof It is easy to check that a Bohr almost periodic function is a p.c.a.p. one which has discontinuities at the points of the empty subset of any Wexler sequence.

Next we prove the converse inclusion. Let $h \in PCAPW(\mathbb{R}, X) \cap C(\mathbb{R}, X)$. Theorem 3.2 yields $h \in S(\mathbb{R}, X)$. By Theorem 2.8, it suffices to show that h is uniformly continuous on $\mathbb{R}$. Let $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ be the Wexler sequence in Definition 2.13 for h and $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$. Hence for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$, $0 < \delta < \theta$ such that $|h(s) - h(t)| < \epsilon$ whenever s, $t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$. Letting $s \to \tau_j + 0$ we arrive at

$$|h(\tau_j^+) - h(t)| \le \epsilon, \quad \tau_j < t < \tau_j + \delta, j \in \mathbb{Z}.$$

On the other hand, if $|s - t| < \delta$ and

$$\tau_{k-1} < s \le \tau_k < t < \tau_{k+1}$$

for some $k \in \mathbb{Z}$, a direct calculation shows that

$$\begin{aligned}|h(s) - h(t)| &\le |h(s) - h(\tau_j^+)| + |h(\tau_j^+) - h(t)| \\ &< 2\epsilon.\end{aligned}$$

This proves the uniform continuity of h on $\mathbb{R}$. □

4 Module Containment for Stepanov Almost Periodic Functions

From Theorem 3.2 it is reasonable to investigate the module containment for p.c.a.p. solutions to differential equations with impulses at fixed times by using Stepanov almost periodic functions. We shall prove relevant properties of this class of functions.

4.1 A Space Isometrically Isomorphic to $S^p(\mathbb{R}, X)$

Bochner showed that by using a construction, a Stepanov almost periodic function can be reduced to a Bohr one which is vector valued. Consequently, the theory of Stepanov almost periodic functions can be included in that of Bohr almost periodic functions. In this subsection we construct a space which is isometrically isomorphic to $S^p(\mathbb{R}, X)$ on the basis of Bochner's method and useful in consequent sections.

For any p, $1 \le p < \infty$, consider function spaces $L^p_{loc}(\mathbb{R}, X)$, $Y = L^p([0,1], X)$ and $C(\mathbb{R}, Y)$. For every $f \in L^p_{loc}(\mathbb{R}, X)$, put

$$\tilde{f}(t)(s) = f(t+s), \quad s \in [0,1]\ a.e., t \in \mathbb{R}. \tag{6}$$

It is obvious that the function $\tilde{f}(t) : [0,1] \to X$ belongs to Y for all $t \in \mathbb{R}$. Hence $\tilde{f}$ is a function from $\mathbb{R}$ to Y. For any t and $\tau \in \mathbb{R}$, a direct calculation shows that

$$\begin{aligned}\|\tilde{f}(t) - \tilde{f}(\tau)\|_Y^p &= \int_0^1 |\tilde{f}(t)(s) - \tilde{f}(\tau)(s)|^p ds \\ &= \int_0^1 |f(t+s) - f(\tau+s)|^p ds \\ &= \int_\tau^{\tau+1} |f(t-\tau+s) - f(s)|^p ds,\end{aligned}$$

which implies $\tilde{f} \in C(\mathbb{R}, Y)$.

Furthermore, from (6) it follows that

$$\tilde{f}(t)(s) = \tilde{f}(\tau)(t-\tau+s) \tag{7}$$

for $s \in [0,1] \cap [\tau - t, \tau - t + 1]\ a.e.$, which is a translation invariant property of $\tilde{f}$ in some sense and turns out to be the essential condition to construct the space isometrically isomorphic to $S^p(\mathbb{R}, X)$. Let

$$\widetilde{C}(\mathbb{R}, Y) = \{\tilde{f} \in C(\mathbb{R}, Y);\ \tilde{f} \text{ satisfies}(7)\} \tag{8}$$

and define a linear map by

$$\Phi : L^p_{loc}(\mathbb{R}, X) \to \widetilde{C}(\mathbb{R}, Y), \quad f \mapsto \tilde{f}. \tag{9}$$

Lemma 4.1 $\Phi : L^p_{loc}(\mathbb{R}, X) \to \widetilde{C}(\mathbb{R}, Y)$ *is a linear isomorphism.*

Proof It is obvious that the linear map Φ is injective. Next we prove that Φ is surjective. For every $g \in \widetilde{C}(\mathbb{R}, Y)$ define

$$f(n+s) = g(n)(s), \quad s \in (0,1]\ a.e., n \in \mathbb{Z}.$$

Because $g(n) \in Y$ for all $n \in \mathbb{Z}$, $f \in L^p_{loc}(\mathbb{R}, X)$. Given $t \in \mathbb{R}$ and $s \in [0,1]$, let $n = n(t,s)$ be the unique integer such that $n < t + s \le n+1$. Then

$$f(t+s) = g(n)(t+s-n) = g(t)(s)$$

for $s \in [0,1]\ a.e.$ by the definition of f and (7). Therefore, $\Phi(f) = g$. □

Let

$$M^p(\mathbb{R}, X) = \left\{ f \in L^p_{loc}(\mathbb{R}, X);\ \sup_{t\in\mathbb{R}} \int_0^1 |f(t+s)|^p ds < \infty \right\} \tag{10}$$

be the Banach space [20, p. 39] of functions bounded in the mean (of order p) equipped with the norm

$$\|f\|_{M^p} = \sup_{t\in\mathbb{R}} \left[\int_0^1 |f(t+s)|^p ds\right]^{\frac{1}{p}} \tag{11}$$

and

$$\widetilde{BC}(\mathbb{R}, Y) = \widetilde{C}(\mathbb{R}, Y) \cap BC(\mathbb{R}, Y)$$

be a subspace of bounded and continuous functions equipped with the uniform convergence norm $\|\cdot\|$.

Lemma 4.2 $\Phi : (M^p(\mathbb{R}, X), \|\cdot\|_{M^p}) \to (\widetilde{BC}(\mathbb{R}, Y), \|\cdot\|)$ *is an isometric isomorphism.*

Proof Lemma 4.1 and the equalities

$$\begin{aligned}
\|\Phi(f)\| &= \sup_{t\in\mathbb{R}} \|\tilde{f}(t)\|_Y \\
&= \sup_{t\in\mathbb{R}} \left[\int_0^1 |\tilde{f}(t)(s)|^p ds\right]^{\frac{1}{p}} \\
&= \sup_{t\in\mathbb{R}} \left[\int_0^1 |f(t+s)|^p ds\right]^{\frac{1}{p}} \\
&= \|f\|_{M^p}
\end{aligned}$$

imply that $\Phi : M^p(\mathbb{R}, X) \to \widetilde{BC}(\mathbb{R}, Y)$ is both an isomorphism and an isometry. □

Corollary 4.3 *The space* $(\widetilde{BC}(\mathbb{R}, Y), \|\cdot\|)$ *is complete.*

Let

$$\widetilde{AP}(\mathbb{R}, Y) = \widetilde{C}(\mathbb{R}, Y) \cap AP(\mathbb{R}, Y).$$

Lemma 4.2 and the fact that $f \in S^p(\mathbb{R}, X) \Leftrightarrow \tilde{f} \in AP(\mathbb{R}, Y)$ [8, p. 78] together yield the following

Lemma 4.4 $\Phi : (S^p(\mathbb{R}, X), \|\cdot\|_{M^p}) \to (\widetilde{AP}(\mathbb{R}, Y), \|\cdot\|)$ *is an isometric isomorphism.*

Corollary 4.5 *The space* $(\widetilde{AP}(\mathbb{R}, Y), \|\cdot\|)$ *is complete.*

4.2 Module Containment

By Theorem 2.9 we define the module of $f \in S^p(\mathbb{R}, X)$, denoted by $\mathrm{mod}(f)$, to be the additive group

$$\mathrm{mod}(f) := \left\{\sum_{k=1}^{n} m_k\lambda_k;\ \lambda_k \in \Lambda_f, m_k \in \mathbb{Z}, n \in \mathbb{Z}_+\right\}.$$

Furthermore, from the proof of Theorem 2.9 in [8, p. 79] it follows that if

$$\tilde{f}(t) \sim \sum_k \widetilde{a_k} e^{i\lambda_k t},$$

then

$$\widetilde{a_k}(s) = a_k e^{i\lambda_k s}, \quad s \in [0, 1] \text{ a.e.}$$

and

$$f(t) \sim \sum_k a_k e^{i\lambda_k t}.$$

Hence

$$\operatorname{mod}(f) = \operatorname{mod}(\tilde{f}). \tag{12}$$

We first show the equivalence between pointwise and uniform convergence for Bohr almost periodic functions.

Lemma 4.6 *Suppose that f, $f^* \in AP(\mathbb{G}, X)$ and $\alpha \subset \mathbb{G}$ is a sequence with $\mathcal{T}_\alpha f = f^*$ pointwise for all $t \in \mathbb{G}$, then already $\mathcal{T}_\alpha f = f^*$ uniformly for all $t \in \mathbb{G}$.*

Proof Assume the contrary that $\{f(t + \alpha_k)\}_{k=1}^{\infty}$ dose not converge uniformly to $f^*(t)$. Therefore, there exist $\epsilon_0 > 0$ and a subsequence $\alpha' \subset \alpha$ such that

$$\sup_{t \in \mathbb{G}} |f(t + \alpha'_k) - f^*(t)| > \epsilon_0 \tag{13}$$

for all $k \in \mathbb{Z}_+$. Since $f \in AP(\mathbb{G}, X)$, there are $f_* \in AP(\mathbb{R}, X)$ and a subsequence $\alpha'' \subset \alpha'$ satisfying

$$\lim_{k \to \infty} \sup_{t \in \mathbb{G}} |f(t + \alpha''_k) - f_*(t)| = 0,$$

by Bochner's criterion. Because $\mathcal{T}_\alpha f = f^*$ pointwise for all $t \in \mathbb{G}$, $f^* = f_*$. This contradicts (13). □

We formulate Favard's module containment theorem for Stepanov almost periodic functions as follows.

Theorem 4.7 *The following statements are equivalent for f and $g \in S^p(\mathbb{R}, X)$.*

(i) $\operatorname{mod}(f) \supset \operatorname{mod}(g)$.
(ii) *For every $\epsilon > 0$ there is a $\delta > 0$ so that $T(f, \delta) \subset T(g, \epsilon)$.*
(iii) *For any sequence $\alpha \subset \mathbb{R}$ satisfying*

$$\lim_{k \to \infty} \left[\int_t^{t+1} |f(\alpha_k + s) - f^*(s)|^p ds \right]^{\frac{1}{p}} = 0$$

for all $t \in \mathbb{R}$ (in any sense, e.g. pointwise, uniform with respect to t etc.) and some $f^* \in S^p(\mathbb{R}, X)$, there exists $g^* \in S^p(\mathbb{R}, X)$ such that

$$\lim_{k \to \infty} \left[\int_t^{t+1} |g(\alpha_k + s) - g^*(s)|^p ds \right]^{\frac{1}{p}} = 0$$

for all $t \in \mathbb{R}$ (any sense).
(iv) *For any sequence $\alpha \subset \mathbb{R}$ satisfying*

$$\lim_{k \to \infty} \left[\int_t^{t+1} |f(\alpha_k + s) - f(s)|^p ds \right]^{\frac{1}{p}} = 0$$

for all $t \in \mathbb{R}$ *(any sense), there results*

$$\lim_{k\to\infty}\left[\int_t^{t+1}|g(\alpha_k+s)-g(s)|^p ds\right]^{\frac{1}{p}}=0$$

for all $t \in \mathbb{R}$ *(any sense).*

(v) *For any sequence* $\alpha \subset \mathbb{R}$ *satisfying*

$$\lim_{k\to\infty}\left[\int_t^{t+1}|f(\alpha_k+s)-f(s)|^p ds\right]^{\frac{1}{p}}=0$$

for all $t \in \mathbb{R}$ *(any sense), there exists a subsequence* $\alpha' \subset \alpha$ *such that*

$$\lim_{k\to\infty}\left[\int_t^{t+1}|g(\alpha'_k+s)-g(s)|^p ds\right]^{\frac{1}{p}}=0$$

for all $t \in \mathbb{R}$ *(any sense).*

Proof If $\tilde{f}=\Phi(f)$ and $\tilde{g}=\Phi(g)$, where Φ is given by (9), then Theorem 4.4 implies that $\tilde{f},\ \tilde{g} \in \widetilde{AP}(\mathbb{R},Y)$ and Theorem 2.2 is applicable. By (12), (i) is equivalent to

$$\operatorname{mod}(\tilde{f}) \supset \operatorname{mod}(\tilde{g}). \tag{14}$$

The following proof is divided into five steps.

1. (14) $\Rightarrow$ (ii). For every $\epsilon>0$ find an ϵ', $0<\epsilon'<\epsilon$. A direct calculation shows that

$$\begin{aligned}\sup_{t\in\mathbb{R}}&\left[\int_0^1|g(t+\tau+s)-g(t+s)|^p ds\right]^{\frac{1}{p}}\\ &=\sup_{t\in\mathbb{R}}\left[\int_0^1|\tilde{g}(t+\tau)(s)-\tilde{g}(t)(s)|^p ds\right]^{\frac{1}{p}}\\ &=\sup_{t\in\mathbb{R}}\|\tilde{g}(t+\tau)-\tilde{g}(t)\|_Y,\end{aligned} \tag{15}$$

which implies $T(\tilde{g},\epsilon')\subset T(g,\epsilon)$. By Theorem 2.2 there is a $\delta>0$ so that $T(\tilde{f},\delta)\subset T(\tilde{g},\epsilon')$. Since (15) also yields $T(f,\delta)\subset T(\tilde{f},\delta)$, one arrives at $T(f,\delta)\subset T(g,\epsilon)$.

2. (ii) $\Rightarrow$ (14). For each $\epsilon>0$, (15) implies $T(g,\epsilon)\subset T(\tilde{g},\epsilon)$. From (ii) it follows that there is a $\delta'>0$ such that $T(f,\delta')\subset T(g,\epsilon)$. By (15) again, $T(\tilde{f},\delta)\subset T(f,\delta')$ for all δ with $0<\delta<\delta'$. Hence $T(\tilde{f},\delta)\subset T(\tilde{g},\epsilon)$ and Theorem 2.2 implies (14).

3. (14) $\Rightarrow$ (iii). Let $f^*\in S^p(\mathbb{R},X)$ and $\alpha\subset\mathbb{R}$ be a sequence such that

$$\lim_{k\to\infty}\left[\int_t^{t+1}|f(\alpha_k+s)-f^*(s)|^p ds\right]^{\frac{1}{p}}=0$$

for all $t\in\mathbb{R}$ (any sense). A straightforward computation shows that

$$\begin{aligned}&\left[\int_t^{t+1}|f(\alpha_k+s)-f^*(s)|^p ds\right]^{\frac{1}{p}}\\ &\quad=\left[\int_0^1|f(t+\alpha_k+s)-f^*(t+s)|^p ds\right]^{\frac{1}{p}}\end{aligned}$$

$$= \left[\int_0^1 |\tilde{f}(t+\alpha_k)(s) - \widetilde{f^*}(t)(s)|^p ds \right]^{\frac{1}{p}}$$
$$= \|\tilde{f}(t+\alpha_k) - \widetilde{f^*}(t)\|_Y$$

for all $k \in \mathbb{Z}_+$. So

$$\lim_{k\to\infty} \|\tilde{f}(t+\alpha_k) - \widetilde{f^*}(t)\|_Y = 0$$

for all $t \in \mathbb{R}$ (any sense). By Theorem 2.2, Lemma 4.6, Corollary 4.5 and Theorem 4.4 there exists a $\widetilde{g^*} \in \widetilde{AP}(\mathbb{R}, Y)$ such that $g^* = \Phi^{-1}(\widetilde{g^*}) \in S^p(\mathbb{R}, X)$ and

$$\lim_{k\to\infty} \|\tilde{g}(t+\alpha_k) - \widetilde{g^*}(t)\|_Y = 0$$

for all $t \in \mathbb{R}$ (any sense). Therefore,

$$\lim_{k\to\infty} \left[\int_t^{t+1} |g(\alpha_k+s) - g^*(s)|^p ds \right]^{\frac{1}{p}} = \lim_{k\to\infty} \|\tilde{g}(t+\alpha_k) - \widetilde{g^*}(t)\|_Y = 0$$

for all $t \in \mathbb{R}$ (any sense).

4. (iii) $\Rightarrow$ (14). Let $f^* \in S^p(\mathbb{R}, X)$ and $\alpha \subset \mathbb{R}$ be a sequence such that

$$\lim_{k\to\infty} \left[\int_t^{t+1} |f(\alpha_k+s) - f^*(s)|^p ds \right]^{\frac{1}{p}} = 0$$

for all $t \in \mathbb{R}$ (any sense). Then

$$\lim_{k\to\infty} \|\tilde{f}(t+\alpha_k) - \widetilde{f^*}(t)\|_Y = 0$$

for all $t \in \mathbb{R}$ (any sense). From (iii) it follows that there is a $g^* \in S^p(\mathbb{R}, X)$ with

$$\lim_{k\to\infty} \|\tilde{g}(t+\alpha_k) - \widetilde{g^*}(t)\|_Y = \lim_{k\to\infty} \left[\int_t^{t+1} |g(\alpha_k+s) - g^*(s)|^p ds \right]^{\frac{1}{p}} = 0$$

for all $t \in \mathbb{R}$ (any sense). Hence (14) follows from Theorem 2.2.

5. (14) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). The proof is similar to that in steps 3 and 4. So we omit it. □

4.3 Normal Sequences

In this subsection we prove the deep relation that for any $f \in AP(\mathbb{R}, X)$ and any sequence $\alpha \subset \mathbb{R}$,

$$\mathcal{T}_\alpha f = f^* \text{ (any sense) } \Leftrightarrow \mathcal{T}_\alpha \tilde{f} = \widetilde{f^*} \text{ (any sense)},$$

by which the mode of convergence in Theorem 2.2 could be that in $S^p(\mathbb{R}, X)$. This equivalent relation is useful in the study of the module containment for impulsive differential equations.

Definition 4.8 [35, p. 42] Let $f \in AP(\mathbb{R}, X)$. A sequence $\alpha \subset \mathbb{R}$ is called f-normal if $\mathcal{T}_\alpha f$ exists uniformly with respect to $t \in \mathbb{R}$. In particular, α is called f-increasing if $\mathcal{T}_\alpha f = f$ uniformly.

The main tool in this subsection is as follows. It is clear that the module reflects the mode of convergence of almost periodic functions.

Theorem 4.9 [35, pp. 42–43] *Let $f \in AP(\mathbb{R}, X)$, then for a sequence $\alpha \subset \mathbb{R}$ to be f-normal it is necessary and sufficient that there exists a unique function $\theta(\lambda)$ with*

$$\lim_{k\to\infty} e^{i\lambda\alpha_k} = \theta(\lambda), \quad \lambda \in \mathrm{mod}(f). \tag{16}$$

In particular, α is f-increasing if and only if $\theta(\lambda) \equiv 1$.

The main results in this subsection is the following

Theorem 4.10 *Suppose that $f, f^* \in AP(\mathbb{R}, X)$ and $\alpha \subset \mathbb{R}$ is a sequence, then $\mathcal{T}_\alpha f = f^*$ (any sense) if and only if $\mathcal{T}_\alpha \tilde{f} = \widetilde{f^*}$ (any sense).*

Proof By Lemma 4.6, it is sufficient to consider only the uniform convergence in $\mathcal{T}_\alpha f = f^*$ and $\mathcal{T}_\alpha \tilde{f} = \widetilde{f^*}$.

Suppose that $\mathcal{T}_\alpha \tilde{f} = g$ uniformly for all $t \in \mathbb{R}$. Hence α is $\tilde{f}$-normal and (16) holds by Theorem 4.9. Since $\mathrm{mod}(f) = \mathrm{mod}(\tilde{f})$, α is f-normal by Theorem 4.9 again. Assume that $\mathcal{T}_\alpha f = f^*$ uniformly for all $t \in \mathbb{R}$. It is easy to check that

$$\begin{aligned}\lim_{k\to\infty}\sup_{t\in\mathbb{R}} \|\tilde{f}(t+\alpha_k) - \widetilde{f^*}(t)\|_Y &= \lim_{k\to\infty}\sup_{t\in\mathbb{R}} \left[\int_t^{t+1} |f(t+\alpha_k) - f^*(t+s)|^p ds\right]^{\frac{1}{p}} \\ &= 0.\end{aligned} \tag{17}$$

From the uniqueness of the limit $\widetilde{f^*} \in AP(\mathbb{R}, Y)$ it follows that $g = \widetilde{f^*}$.

At last, $\mathcal{T}_\alpha f = f^*$ implies $\mathcal{T}_\alpha \tilde{f} = \widetilde{f^*}$ by (17). □

5 Module Containment for Linear Impulsive Differential Equations

In this section, we make use of Theorem 3.2 to study impulsive differential equations. Consider the linear differential equation with impulses at fixed times

$$\begin{cases} x' = A(t)x + h(t), & t \neq \tau_n, \\ x(\tau_n^+) - x(\tau_n) = B(n)x(\tau_n) + b(n), & n \in \mathbb{Z}, \end{cases} \tag{18}$$

which satisfies the following conditions:

(H1) $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ is a Wexler sequence such that

$$\tau_n = \xi n + \zeta(n), \quad n \in \mathbb{Z},$$

where $\xi > 0$, $\zeta \in AP(\mathbb{Z}, \mathbb{R})$ and $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$.

(H2) $A \in AP(\mathbb{R}, \mathbb{R}^{d\times d})$, $h \in PCAP(\mathbb{R}, \mathbb{R}^d)$ has discontinuities at the points of a subset of $\{\tau_j\}_{j\in\mathbb{Z}}$, $B \in AP(\mathbb{Z}, \mathbb{R}^{d\times d})$, $b \in AP(\mathbb{Z}, \mathbb{R}^d)$, where $d \in \mathbb{Z}_+$. $\det[I + B(n)] \neq 0$ for all $n \in \mathbb{Z}$.

(H3) $\rho_1^* + D\ln\rho_2^* < 0$, where

$$D = \lim_{T\to\infty} \frac{i(t, t+T)}{T}, \;\; \rho_1^* = \sup_{t\in\mathbb{R}} \rho_1(t), \;\; \rho_2^* = \left[\sup_{j\in\mathbb{Z}} \rho_2(j)\right]^{1/2},$$

$i(t, t+T)$ is the number of the terms of $\{\tau_j\}_{j\in\mathbb{Z}} \cap [t, t+T]$, $\rho_1(t)$ and $\rho_2(j)$ are respectively the largest eigenvalues of the matrices

$$\frac{1}{2}[A(t) + A^T(t)], \quad [I + B(j)]^T \cdot [I + B(j)].$$

Denote by $U(t,s)$ and $W(t,s)$ respectively the Cauchy matrices of the linear system

$$x' = A(t)x$$

and the homogeneous impulsive one

$$\begin{cases} x' = A(t)x, & t \neq \tau_n, \\ x(\tau_n^+) - x(\tau_n) = B(n)x(\tau_n), & n \in \mathbb{Z}. \end{cases}$$

Set

$$|Q|_M = \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{|Qx|}{|x|}$$

for any matrix $Q \in \mathbb{R}^{d\times d}$ and

$$\|A\| = \sup_{t\in\mathbb{R}} |A(t)|_M, \quad \|B\| = \sup_{n\in\mathbb{Z}} |B(n)|_M,$$

respectively. By [46, p. 212], (H3) yields

$$|W(t,s)|_M \leq C_1 e^{-C_2(t-s)}, \quad t \geq s, \tag{19}$$

for some positive constants C_1 and C_2. [46, p. 215] proves the following result

Theorem 5.1 *Suppose that* (18) *satisfies (H1)–(H3), then* (18) *admits a unique p.c.a.p. solution ϕ, which is asymptotically stable and given by*

$$\phi(t) = \int_{\infty}^{t} W(t,s)h(s)ds + \sum_{\tau_j < t} W(t,\tau_j)b(j). \tag{20}$$

Our goal is to characterize the module of p.c.a.p. solutions to linear impulsive differential equation. Denote by $E^{(r)}$ the representative set

$$E^{(r)} = \{\beta \in [0, 2\pi); (\beta + 2\pi\mathbb{Z}) \in E\}$$

of a set $E \subset \mathbb{R}/2\pi\mathbb{Z}$. The main result in this section is formulated as follows.

Theorem 5.2 *Suppose that* (18) *satisfies (H1)–(H3), then* (18) *admits a unique p.c.a.p. solution ϕ, which is asymptotically stable and given by*

$$\phi(t) = \int_{\infty}^{t} W(t,s)h(s)ds + \sum_{\tau_j < t} W(t,\tau_j^+)b(j). \tag{21}$$

Furthermore, the solution ϕ satisfies

$$\operatorname{mod}(\phi) \subset \operatorname{span}\left(\operatorname{mod}(A,h) \cup \left[\frac{1}{\xi} \cdot \left\{[\operatorname{mod}(B,b,\zeta)]^{(r)} \cup \{2\pi\}\right\}\right]\right). \tag{22}$$

Remark 5.3 Note that (20) and (21) are different at the second sum. The correct one is (21) by [10,34]. For completeness, we shall give a detailed proof of Theorem 5.2.

The main tools to study the module containment for p.c.a.p. solutions are Theorems 2.5, 3.2, 4.10 and Lemma 4.4. To make use of these tools in impulsive differential equations we introduce the following results.

Lemma 5.4 [46, pp. 207–208] *The following statements are true.*

(i) *If I is the identity matrix in $\mathbb{R}^{d\times d}$, then*

$$|U(t,s)-I|_M < e^{\|A\|\cdot|t-s|}-1, \quad |U(t,s)|_M < e^{\|A\|\cdot|t-s|}$$

for all $s, t \in \mathbb{R}$.

(ii) *Let $L>0$ be fixed, then for any $\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that*

$$|U(t',s')-U(t,s)|_M < \epsilon$$

whenever $|s'-s|<\delta$, $|t'-t|<\delta$ and $|s-t|\le L$.

(iii) *If $r \in T\big(A, \epsilon/(Le^{L\|A\|})\big)$, where $L>0$, then*

$$|U(t+r,s+r)-U(t,s)|_M < \epsilon$$

whenever $|s-t|<L$.

Lemma 5.5 [46, p. 210] *Suppose that (19) holds for some positive constants C_1 and C_2, then the diagonal of the matrix $W(t,s)$ is almost periodic, i.e. for any $\epsilon>0$, $t\ge s$, $|t-\tau_j|>\epsilon$, $|s-\tau_j|>\epsilon$, $j\in\mathbb{Z}$, there exists a relatively dense set of almost periods, Γ, such that*

$$|W(t+r,s+r)-W(t,s)|_M < \epsilon C_3 e^{-C_2(t-s)/2} \tag{23}$$

for all $r\in\Gamma$, where C_3 is a positive constant.

Remark 5.6 It is not convenient to use Lemma 5.5 in our applications since the set Γ is not clear. However, from the proof of Lemma 5.5 it follows that if (19) is true and $(r,q)\in\mathbb{R}\times\mathbb{Z}$ satisfies

$$|A(t+r)-A(t)|_M < \epsilon, \quad t\in\mathbb{R}, \tag{24}$$

$$|B(n+q)-B(n)|_M < \epsilon, \quad n\in\mathbb{Z}, \tag{25}$$

$$|\tau_j^q - r| < \epsilon, \quad j\in\mathbb{Z}, \tag{26}$$

then (23) holds for all $t\ge s$ with $|t-\tau_j|>\epsilon$, $|s-\tau_j|>\epsilon$, $j\in\mathbb{Z}$. This remark will be used in the proof of Lemma 5.7 and Theorem 5.2.

A result similar to Lemma 5.5 is true if the Wexler sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ is taken into consideration.

Lemma 5.7 *Suppose that $(r,q)\in\mathbb{R}\times\mathbb{Z}$ satisfies (24)–(26), where $0<\epsilon<\theta/3$, $\theta=\inf_{j\in\mathbb{Z}}\tau_j^1$. Then (19) implies*

$$\begin{aligned}|W(t+r,\tau_{k+q})-W(t,\tau_k)|_M &< 2C_1e^{-C_2(t+r-\tau_{k+q})}(e^{3\epsilon\|A\|}-1)\\ &\quad+\epsilon C_3e^{-C_2(t+r-\tau_{k+q})/2}\end{aligned} \tag{27}$$

and

$$\begin{aligned}|W(t+r,\tau_{k+q}^+)-W(t,\tau_k^+)|_M &< 2C_1e^{-C_2(t+r-\tau_{k+q}-\epsilon)}(e^{3\epsilon\|A\|}-1)\\ &\quad+\epsilon C_3e^{-C_2(t+r-\tau_{k+q}-3\epsilon)/2}\end{aligned} \tag{28}$$

for all $t\in\mathbb{R}$, $|t-\tau_j|>\epsilon$, $j\in\mathbb{Z}$ and $\tau_k<t$, where C_1, C_2 and C_3 are the constants in Lemma 5.5.

Proof Assume that $t \in \mathbb{R}$ satisfies $\tau_m + \epsilon < t < \tau_{m+1} - \epsilon$ for some $m \in \mathbb{Z}$. A direct calculation shows that

$$\tau_{m+q} - \tau_m - \epsilon < r < \tau_{m+1+q} - \tau_{m+1} + \epsilon,$$
$$\tau_{m+q} < t + r < \tau_{m+1+q}$$

by (26).

We first prove (27). Let ϵ' be a number such that $\epsilon < \epsilon' < 2\epsilon$. For any $\tau_k < t$, it is easy to see that $k \leq m$ and

$$\tau_{k-1} + \epsilon < \tau_k - \epsilon' < \tau_k - \epsilon,$$
$$\tau_{k-1+q} < \tau_{k+q} - \epsilon - \epsilon' < \tau_k - \epsilon' + r < \tau_{k+q} + \epsilon - \epsilon' < \tau_{k+q},$$

where the last inequality is obtained by using (26) again. Therefore, from Remark 5.6, Lemma 5.4 and the inequalities above it follows that

$$\begin{aligned}
&|W(t+r, \tau_{k+q}) - W(t, \tau_k)|_M \\
&\quad \leq |W(t+r, \tau_{k+q}) - W(t+r, \tau_k - \epsilon' + r)|_M + |W(t+r, \tau_k - \epsilon' + r) \\
&\qquad - W(t, \tau_k - \epsilon')|_M + |W(t, \tau_k - \epsilon') - W(t, \tau_k)|_M \\
&\quad < |W(t+r, \tau_{k+q})[I - U(\tau_{k+q}, \tau_k - \epsilon' + r)]|_M + \epsilon C_3 e^{-C_2(t-\tau_k+\epsilon')/2} \\
&\qquad + |W(t, \tau_k - \epsilon')[I - U(\tau_k - \epsilon', \tau_k)]|_M \\
&\quad < C_1 e^{-C_2(t+r-\tau_{k+q})} \left[e^{\|A\|(\tau_{k+q}-\tau_k-r+\epsilon')} - 1\right] + \epsilon C_3 e^{-C_2(t-\tau_k+\epsilon')/2} \\
&\qquad + C_1 e^{-C_2(t-\tau_k+\epsilon')} \left(e^{\|A\|\epsilon'} - 1\right) \\
&\quad < C_1 e^{-C_2(t+r-\tau_{k+q})} \left(e^{3\epsilon\|A\|} - 1\right) + \epsilon C_3 e^{-C_2(t+r-\tau_{k+q})/2} \\
&\qquad + C_1 e^{-C_2(t+r-\tau_{k+q})} \left(e^{2\epsilon\|A\|} - 1\right) \\
&\quad < 2C_1 e^{-C_2(t+r-\tau_{k+q})} \left(e^{3\epsilon\|A\|} - 1\right) + \epsilon C_3 e^{-C_2(t+r-\tau_{k+q})/2}.
\end{aligned}$$

Next we prove (28). For any $\tau_k < t$, it is easy to see that $k \leq m$ and $\tau_k < t - \epsilon$. Let ϵ' be a number such that $\epsilon < \epsilon' < \min\{2\epsilon, t - \tau_k\}$. A straightforward computation shows that

$$\tau_k + \epsilon < \tau_k + \epsilon' < \tau_k + 2\epsilon < \tau_{k+1} - \epsilon,$$
$$\tau_{k+q} < \tau_{k+q} - \epsilon + \epsilon' < \tau_k + \epsilon' + r < \tau_{k+q} + \epsilon + \epsilon' < \tau_{k+q} + 3\epsilon < \tau_{k+q+1},$$

where the last inequality is obtained by using (26). Therefore, from Remark 5.6, Lemma 5.4 and the inequalities above it follows that

$$\begin{aligned}
&|W(t+r, \tau_{k+q}^+) - W(t, \tau_k^+)|_M \\
&\quad \leq |W(t+r, \tau_{k+q}^+) - W(t+r, \tau_k + \epsilon' + r)|_M + |W(t+r, \tau_k + \epsilon' + r) \\
&\qquad - W(t, \tau_k + \epsilon')|_M + |W(t, \tau_k + \epsilon') - W(t, \tau_k^+)|_M \\
&\quad < |W(t+r, \tau_{k+q}^+)[I - U(\tau_{k+q}, \tau_k + \epsilon' + r)]|_M + \epsilon C_3 e^{-C_2(t-\tau_k-\epsilon')/2} \\
&\qquad + |W(t, \tau_k^+)[U(\tau_k, \tau_k + \epsilon') - I]|_M \\
&\quad < C_1 e^{-C_2(t+r-\tau_{k+q})} \left[e^{\|A\|(\tau_k+\epsilon'+r-\tau_{k+q})} - 1\right] + \epsilon C_3 e^{-C_2(t-\tau_k-\epsilon')/2} \\
&\qquad + C_1 e^{-C_2(t-\tau_k)} \left(e^{\|A\|\epsilon'} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
&< C_1 e^{-C_2(t+r-\tau_{k+q})}\left(e^{3\epsilon\|A\|}-1\right)+\epsilon C_3 e^{-C_2(t+r-\tau_{k+q}-3\epsilon)/2} \\
&\quad + C_1 e^{-C_2(t+r-\tau_{k+q}-\epsilon)}\left(e^{2\epsilon\|A\|}-1\right) \\
&< 2C_1 e^{-C_2(t+r-\tau_{k+q}-\epsilon)}\left(e^{3\epsilon\|A\|}-1\right)+\epsilon C_3 e^{-C_2(t+r-\tau_{k+q}-3\epsilon)/2}.
\end{aligned}
$$

□

The relation between the spectra of an almost periodic sequence and the function defined by filling in the gaps linearly reads as follows.

Lemma 5.8 *Suppose that* $f \in AP(\mathbb{R}, X)$*,* $u \in AP(\mathbb{Z}, X)$ *and*

$$
f(t) = (n+1-t)\cdot u(n) + (t-n)\cdot u(n+1), \quad n < t \le n+1, n \in \mathbb{Z},
$$

then

$$
\begin{aligned}
\Lambda_u &= \Lambda_f/2\pi\mathbb{Z}, \quad \operatorname{mod}(u) = \operatorname{mod}(f)/2\pi\mathbb{Z}, \\
\Lambda_f &= \left\{(\Lambda_u - \{0\})^{(r)} + 2k\pi\right\}_{k\in\mathbb{Z}} \cup \left(\Lambda_f \cap \{0\}\right), \\
\operatorname{mod}(f) &= \operatorname{span}\left(\left\{(\Lambda_u - \{0\})^{(r)} + 2k\pi\right\}_{k\in\mathbb{Z}}\right).
\end{aligned}
$$

Proof The first two equations are consequences of Theorem 2.3 in [62], from the proof of which we know that $a(f,0) = a(u,0)$ and

$$
a(f,\lambda) = \frac{2(1-\cos\lambda)}{\lambda^2}\cdot a(u,\widetilde{\lambda}), \quad \lambda \in \mathbb{R}, \lambda \ne 0.
$$

It is easy to see that $a(f,\lambda) = 0 \Leftrightarrow a(u,\widetilde{\lambda})$ for all $\lambda \in \mathbb{R}, \lambda \ne 0, \widetilde{\lambda} \ne 0$, and $a(f,\lambda) \equiv 0$ for all $\lambda \in \mathbb{R}, \lambda \ne 0, \widetilde{\lambda} = 0$. Therefore,

$$
\begin{aligned}
&\lambda \in \Lambda_f \Leftrightarrow \widetilde{\lambda} \in \Lambda_u, \quad \lambda \in \mathbb{R}, \lambda \ne 0, \widetilde{\lambda} \ne 0, \\
&0 \in \Lambda_f \Leftrightarrow 0 \in \Lambda_u, \\
&(2\pi\mathbb{Z} - \{0\}) \cap \Lambda_f = \emptyset,
\end{aligned}
$$

which imply the last two equalities. □

We are in the position proving Theorem 5.2.

Proof of Theorem 5.2 Let $L > \sup_{j\in\mathbb{Z}} \tau_j^1$. The proof is divided into five steps.

1. We prove that for any $C > 0$,

$$
\sum_{\tau_j < t} e^{-C(t-\tau_j)} < \frac{1}{1-e^{-C\theta}}, \tag{29}
$$

which will be used later. If $\tau_m < t \le \tau_{m+1}$ for some $m \in \mathbb{Z}$, from

$$
t - \tau_j > \tau_m - \tau_j \ge (m-j)\theta, \quad j \le m
$$

it follows that

$$
e^{-C(t-\tau_j)} < e^{-C(m-j)\theta}, \quad j \le m
$$

and

$$\sum_{\tau_j<t} e^{-C(t-\tau_j)} = \sum_{j=-\infty}^{m} e^{-C(t-\tau_j)} < \sum_{j=-\infty}^{m} e^{-C(m-j)\theta}$$
$$= \sum_{j=0}^{\infty} e^{-Cj\theta} = \frac{1}{1-e^{-C\theta}}.$$

2. We prove that the function ϕ given by (21) is a bounded solution to (18). By (29), (19) and Lemma 3.3,

$$|\phi(t)| \le \int_{-\infty}^{t} C_1 e^{-C_2(t-s)} \|h\| ds + \sum_{\tau_j<t} C_1 e^{-C_2(t-\tau_j)} \|b\|$$
$$< C_1 \left(\frac{\|h\|}{C_2} + \frac{\|b\|}{1-e^{-C_2\theta}} \right),$$

which implies the boundedness of ϕ on $\mathbb{R}$. Moreover, by (2.14) in [10],

$$\frac{d}{dt}\phi(t) = \int_{-\infty}^{t} \frac{d}{dt} W(t,s)h(s)ds + h(t) + \sum_{\tau_j<t} \frac{d}{dt} W(t,\tau_j^+)b(j)$$
$$= \int_{-\infty}^{t} A(t)W(t,s)h(s)ds + \sum_{\tau_j<t} A(t)W(t,\tau_j^+)b(j) + h(t)$$
$$= A(t)\phi(t) + h(t)$$

for $t \neq \tau_n$, and

$$\phi(\tau_n^+) - \phi(\tau_n) = \int_{-\infty}^{\tau_n} [W(\tau_n^+, s) - W(\tau_n, s)]h(s)ds + \sum_{\tau_j \le \tau_n} W(\tau_n^+, \tau_j^+)b(j)$$
$$- \sum_{\tau_j<\tau_n} W(\tau_n, \tau_j^+)b(j)$$
$$= \int_{-\infty}^{\tau_n} B(n)W(\tau_n, s)h(s)ds + \sum_{\tau_j<\tau_n} B(n)W(\tau_n, \tau_j^+)b(j)$$
$$+ W(\tau_n^+, \tau_n^+)b(n) = B(n)\phi(\tau_n) + b(n)$$

for $n \in \mathbb{Z}$. So ϕ is a solution to (18) on $\mathbb{R}$.

3. We prove that ϕ is p.c.a.p. by Definition 2.13.

It is obvious that ϕ has discontinuities at the points of a subset of $\{\tau_j\}_{j\in\mathbb{Z}}$.

If $t, \tau \in (\tau_m, \tau_{m+1}]$ for some $m \in \mathbb{Z}$ and $t \ge \tau$, then

$$\sum_{\tau_j<t} W(t,\tau_j^+)b(j) = \sum_{\tau_j<\tau} W(t,\tau_j^+)b(j).$$

From (i) of Lemma 5.4, (19), the boundedness of ϕ and h, and the equality

$$W(t,s) = U(t,\tau)W(\tau,s), \quad s \in \mathbb{R}$$

it follows that

$$\begin{aligned}
\phi(t) &= \int_{-\infty}^{\tau} W(t,s)h(s)ds + \int_{\tau}^{t} W(t,s)h(s)ds + \sum_{\tau_j<\tau} W(t,\tau_j^+)b(j) \\
&= \int_{-\infty}^{\tau} U(t,\tau)W(\tau,s)h(s)ds + \sum_{\tau_j<\tau} U(t,\tau)W(\tau,\tau_j^+)b(j) \\
&\quad + \int_{\tau}^{t} W(t,s)h(s)ds \\
&= U(t,\tau)\phi(\tau) + \int_{\tau}^{t} W(t,s)h(s)ds
\end{aligned}$$

and

$$\begin{aligned}
|\phi(t)-\phi(\tau)| &\le |U(t,\tau)-I|_M \cdot \|\phi\| + \int_{\tau}^{t} |W(t,s)|_M \cdot \|h\| ds \\
&< [e^{\|A\|(t-\tau)} - 1] \cdot \|\phi\| + \int_{\tau}^{t} C_1 e^{-C_2(t-s)} \|h\| ds \\
&\le [e^{\|A\|(t-\tau)} - 1] \cdot \|\phi\| + C_1 \|h\|(t-\tau).
\end{aligned}$$

Therefore, ϕ satisfies (ii) of Definition 2.13.

Consider the inequalities (24)–(26) and the following ones

$$|h(t+r)-h(t)| < \epsilon, \quad |t-\tau_j| > \epsilon, \, j \in \mathbb{Z}, \tag{30}$$

$$|b(n+q)-b(n)| < \epsilon, \quad n \in \mathbb{Z}. \tag{31}$$

Using the method of common almost periods as in Lemma 35 in [46, p. 208], the following two sets

$$\begin{aligned}
\Gamma &= \{r \in \mathbb{R};\ \text{there exists } q \in \mathbb{Z} \text{ such that } (r,q) \text{ satisfies}(24)-(26)\text{and}(30),(31)\}, \\
Q &= \{q \in \mathbb{Z};\ \text{there exists } r \in \mathbb{R} \text{ such that } (r,q) \text{ satisfies}(24)-(26)\text{and}(30),(31)\}
\end{aligned}$$

are relatively dense. Let $(r,q) \in \Gamma \times Q$ satisfy (24)–(26) and (30), (31), then (23) and (28) hold by Remark 5.6 and Lemma 5.7, respectively. If $\tau_m + \epsilon < t < \tau_{m+1} - \epsilon$ for some $m \in \mathbb{Z}$, from (26) it follows that

$$\begin{aligned}
&\tau_{m+q} - \tau_m - \epsilon < r < \tau_{m+1+q} - \tau_{m+1} + \epsilon, \\
&\tau_{m+q} < t + r < \tau_{m+q+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi(t+r) &= \int_{-\infty}^{t+r} W(t+r,s)h(s)ds + \sum_{\tau_j<t+r} W(t+r,\tau_j^+)b(j) \\
&= \int_{-\infty}^{t} W(t+r,s+r)h(s+r)ds + \sum_{\tau_j<t} W(t+r,\tau_{j+q}^+)b(j+q)
\end{aligned}$$

and

$$\begin{aligned}
|\phi(t+r)-\phi(t)| &\le \int_{-\infty}^{t} |W(t+r,s+r)h(s+r) - W(t,s)h(s)|ds \\
&\quad + \sum_{\tau_j<t} |W(t+r,\tau_{j+q}^+)b(j+q) - W(t,\tau_j^+)b(j)|.
\end{aligned}$$

For convenience we denote

$$E(r,t,s) = |W(t+r,s+r)h(s+r) - W(t,s)h(s)|$$

for all $r, t, s \in \mathbb{R}$ with $t \geq s$. (19) yields

$$E(r,t,s) \leq 2C_1\|h\|e^{-C_2(t-s)}.$$

A straightforward computation shows that

$$\begin{aligned}
\int_{-\infty}^{t} E(r,t,s)ds &= \int_{\tau_m}^{t} E(r,t,s)ds + \sum_{j=-\infty}^{m} \int_{\tau_{j-1}}^{\tau_j} E(r,t,s)ds \\
&= \int_{\tau_m+\epsilon}^{t} E(r,t,s)ds + \int_{\tau_m}^{\tau_m+\epsilon} E(r,t,s)ds \\
&\quad + \sum_{j=-\infty}^{m} \Bigg[\int_{\tau_{j-1}}^{\tau_{j-1}+\epsilon} E(r,t,s)ds + \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} E(r,t,s)ds \\
&\quad + \int_{\tau_j-\epsilon}^{\tau_j} E(r,t,s)ds \Bigg] \\
&\leq \int_{\tau_m+\epsilon}^{t} E(r,t,s)ds + 2\epsilon C_1\|h\|e^{-C_2(t-\tau_m-\epsilon)} \\
&\quad + \sum_{j=-\infty}^{m} \Big\{ 2\epsilon C_1\|h\|\big[e^{-C_2(t-\tau_{j-1}-\epsilon)} + e^{-C_2(t-\tau_j)}\big] \\
&\quad + \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} E(r,t,s)ds \Big\} \\
&= \int_{\tau_m+\epsilon}^{t} E(r,t,s)ds + 2\epsilon C_1\|h\|e^{-C_2(t-\tau_m-\epsilon)} \\
&\quad + \sum_{j=-\infty}^{m} \Big[2\epsilon C_1\|h\|\big(1+e^{C_2\epsilon}\big)e^{-C_2(t-\tau_j)} + \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} E(r,t,s)ds \Big] \\
&< \int_{\tau_m+\epsilon}^{t} E(r,t,s)ds + 2\epsilon C_1\|h\|e^{-C_2(t-\tau_m-\epsilon)} \\
&\quad + 2\epsilon C_1\|h\|\frac{1+e^{C_2\epsilon}}{1-e^{-C_2\theta}} + \sum_{j=-\infty}^{m} \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} E(r,t,s)ds,
\end{aligned}$$

where (29) is used to obtain the last inequality. By Remark 5.6 and (19),

$$\begin{aligned}
\int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} E(r,t,s)ds &\leq \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} \big\{ |[W(t+r,s+r) - W(t,s)]h(s+r)| \\
&\quad + |W(t,s)[h(s+r) - h(s)]| \big\} ds \\
&\leq (\tau_j - \tau_{j-1} - 2\epsilon)\big[\epsilon C_3\|h\|e^{-C_2(t-\tau_j+\epsilon)/2} \\
&\quad + \epsilon C_1 e^{-C_2(t-\tau_j+\epsilon)}\big] \\
&< \epsilon L\big[C_3\|h\|e^{-C_2(t-\tau_j+\epsilon)/2} + C_1 e^{-C_2(t-\tau_j+\epsilon)}\big]
\end{aligned}$$

for $j \le m$ and similarly,

$$\int_{\tau_m+\epsilon}^{t} E(r,t,s)ds \le (t-\tau_m-\epsilon)(\epsilon C_3\|h\| + \epsilon C_1) < \epsilon L(C_3\|h\| + C_1).$$

Consequently,

$$\begin{aligned}\int_{-\infty}^{t} E(r,t,s)ds &< \epsilon L(C_3\|h\| + C_1) + 2\epsilon C_1\|h\|\frac{1+e^{C_2\epsilon}}{1-e^{-C_2\theta}} \\ &\quad + \sum_{j=-\infty}^{m} \epsilon L\big[C_3\|h\|e^{-C_2(t-\tau_j+\epsilon)/2} + C_1 e^{-C_2(t-\tau_j+\epsilon)}\big] \\ &< \epsilon L(C_3\|h\| + C_1) + 2\epsilon C_1\|h\|\frac{1+e^{C_2\epsilon}}{1-e^{-C_2\theta}} \\ &\quad + \epsilon L\frac{C_3\|h\|e^{-C_2\epsilon/2}}{1-e^{-C_2\theta/2}}\cdot\frac{C_1 e^{-C_2\epsilon}}{1-e^{-C_2\theta}} =: F_1(\epsilon).\end{aligned}$$

On the other hand, from (19) and (28) it follows that

$$\begin{aligned}&\Big|\sum_{\tau_j<t}\big[W(t+r,\tau_{j+q}^{+})b(j+q) - W(t,\tau_j^{+})b(j)\big]\Big| \\ &\le \sum_{j=-\infty}^{m}\big\{\big|\big[W(t+r,\tau_{j+q}^{+}) - W(t,\tau_j^{+})\big]b(j+q)\big| \\ &\quad + \big|W(t,\tau_j^{+})[b(j+q)-b(j)]\big|\big\} \\ &< \sum_{j=-\infty}^{m}\big\{\big[2C_1 e^{-C_2(t+r-\tau_{j+q}-\epsilon)}\big(e^{3\epsilon\|A\|}-1\big) + \epsilon C_3 e^{-C_2(t+r-\tau_{j+q}-3\epsilon)/2}\big]\cdot\|b\| \\ &\quad + \epsilon C_1 e^{-C_2(t-\tau_j)}\big\} \\ &< \sum_{j=-\infty}^{m}\big[2C_1 e^{-C_2(t-\tau_j-2\epsilon)}\big(e^{3\epsilon\|A\|}-1\big) + \epsilon C_3 e^{-C_2(t-\tau_j-4\epsilon)/2}\big]\cdot\|b\| + \frac{\epsilon C_1}{1-e^{-C_2\theta}} \\ &< \Big[2C_1 e^{2C_2\epsilon}\cdot\frac{e^{3\epsilon\|A\|}-1}{1-e^{-C_2\theta}} + \frac{\epsilon C_3 e^{2C_2\epsilon}}{1-e^{-C_2\theta/2}}\Big]\cdot\|b\| + \frac{\epsilon C_1}{1-e^{-C_2\theta}} =: F_2(\epsilon).\end{aligned}$$

Therefore,

$$|\phi(t+r) - \phi(t)| < F_1(\epsilon) + F_2(\epsilon)$$

for all $t\in\mathbb{R}$, $|t-\tau_j| > \epsilon$, $j\in\mathbb{Z}$. Hence $T(\phi, F_1(\epsilon)+F_2(\epsilon))$ contains the relatively dense set Γ. Summing up, ϕ is p.c.a.p.

4. We make use of Theorems 2.5, 3.2, 4.4 and 4.10 to prove the module containment. This step is divided into four substeps. Let $0<\epsilon<\theta/3$ and $L > \sup_{j\in\mathbb{Z}}\tau_j^1$ be an integer.

4.1. We construct suitable functions and sequences. By filling in the gaps linearly we define Bohr almost periodic functions

$$\begin{aligned}\bar{B}(t) &= (n+1-t)B(n) + (t-n)B(n+1), \quad n<t\le n+1, n\in\mathbb{Z}, \\ \bar{b}(t) &= (n+1-t)b(n) + (t-n)b(n+1), \quad n<t\le n+1, n\in\mathbb{Z}, \\ \bar{\zeta}(t) &= (n+1-t)\zeta(n) + (t-n)\zeta(n+1), \quad n<t\le n+1, n\in\mathbb{Z}.\end{aligned}$$

Note that all of A, $\bar{B}(\cdot/\xi)$, $\bar{b}(\cdot/\xi)$, $\bar{\zeta}(\cdot/\xi)$ and h are also Stepanov almost periodic functions. Let $\alpha' \subset \xi\mathbb{Z}$ be a sequence such that

$$\int_t^{t+1} |A(s+\alpha_k') - A(s)|_M ds < \frac{1}{k}, \quad \int_t^{t+1} \Big|\bar{B}\Big(\frac{s+\alpha_k'}{\xi}\Big) - \bar{B}\Big(\frac{s}{\xi}\Big)\Big|_M ds < \frac{1}{k},$$
$$\int_t^{t+1} \Big|\bar{b}\Big(\frac{s+\alpha_k'}{\xi}\Big) - \bar{b}\Big(\frac{s}{\xi}\Big)\Big| ds < \frac{1}{k}, \quad \int_t^{t+1} \Big|\bar{\zeta}\Big(\frac{s+\alpha_k'}{\xi}\Big) - \bar{\zeta}\Big(\frac{s}{\xi}\Big)\Big| ds < \frac{1}{k},$$
$$\int_t^{t+1} |h(s+\alpha_k') - h(s)| ds < \frac{1}{k}, \quad \forall t \in \mathbb{R},$$

for all $k \in \mathbb{Z}_+$. Since all of A, $\bar{B}(\cdot/\xi)$, $\bar{b}(\cdot/\xi)$, $\bar{\zeta}(\cdot/\xi)$ are Bohr almost periodic, Theorem 4.10 implies that

$$\|A(\cdot+\alpha_k') - A(\cdot)\| \to 0, \quad \Big\|\bar{B}\Big(\frac{\cdot+\alpha_k'}{\xi}\Big) - \bar{B}\Big(\frac{\cdot}{\xi}\Big)\Big\| \to 0,$$
$$\Big\|\bar{b}\Big(\frac{\cdot+\alpha_k'}{\xi}\Big) - \bar{b}\Big(\frac{\cdot}{\xi}\Big)\Big\| \to 0, \quad \Big\|\bar{\zeta}\Big(\frac{\cdot+\alpha_k'}{\xi}\Big) - \bar{\zeta}\Big(\frac{\cdot}{\xi}\Big)\Big\| \to 0,$$

as $k \to \infty$. Consequently, there exists a subsequence $\alpha \subset \alpha'$ such that

$$|A(t+\alpha_k) - A(t)| < \frac{1}{k}, \quad \Big|\bar{B}\Big(t+\frac{\alpha_k}{\xi}\Big) - \bar{B}(t)\Big| < \frac{1}{k},$$
$$\Big|\bar{b}\Big(t+\frac{\alpha_k}{\xi}\Big) - \bar{b}(t)\Big| < \frac{1}{k}, \quad \Big|\bar{\zeta}\Big(t+\frac{\alpha_k}{\xi}\Big) - \bar{\zeta}(t)\Big| < \frac{1}{k},$$
$$\int_t^{t+1} |h(s+\alpha_k) - h(s)| ds < \frac{1}{k}, \quad \forall t \in \mathbb{R},$$

for all $k \in \mathbb{Z}_+$. Moreover, it is easy to check that

$$\begin{aligned} |\tau_j^{\alpha_k/\xi} - \alpha_k| &= |\tau_{j+\alpha_k/\xi} - \tau_j - \alpha_k| \\ &= \Big|\zeta\Big(j+\frac{\alpha_k}{\xi}\Big) - \zeta(j)\Big| < \frac{1}{k} \end{aligned} \tag{32}$$

for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$.

4.2. We prove that $|\phi(t+\alpha_k) - \phi(t)|$ is sufficiently small uniformly for all $t \in \mathbb{R}$, $|t-\tau_j| > \epsilon$, $j \in \mathbb{Z}$ and large k.

If $\tau_m + \epsilon < t < \tau_{m+1} - \epsilon$ for some $m \in \mathbb{Z}$, from (32) it follows that

$$\tau_{m+\alpha_k/\xi} - \tau_m - \frac{1}{k} < \alpha_k < \tau_{m+1+\alpha_k/\xi} - \tau_{m+1} + \frac{1}{k},$$
$$\tau_{m+\alpha_k/\xi} - \frac{1}{k} + \epsilon < t + \alpha_k < \tau_{m+1+\alpha_k/\xi} + \frac{1}{k} - \epsilon.$$

Hence

$$\tau_{m+\alpha_k/\xi} < t + \alpha_k < \tau_{m+1+\alpha_k/\xi}$$

for all $k > 1/\epsilon$. Therefore,

$$\begin{aligned}\phi(t+\alpha_k) &= \int_{-\infty}^{t+\alpha_k} W(t+\alpha_k, s)h(s)ds + \sum_{\tau_j < t+\alpha_k} W(t+\alpha_k, \tau_j^+)b(j) \\ &= \int_{-\infty}^{t} W(t+\alpha_k, s+\alpha_k)h(s+\alpha_k)ds \\ &\quad + \sum_{\tau_j < t} W(t+\alpha_k, \tau_{j+\alpha_k/\xi}^+)b\Big(j+\frac{\alpha_k}{\xi}\Big).\end{aligned}$$

For convenience we denote

$$G_k(t,s) = W(t+\alpha_k, s+\alpha_k)h(s+\alpha_k) - W(t,s)h(s),$$

where $t, s \in \mathbb{R}$, $t \geq s$, and $k \in \mathbb{Z}_+$. (19) implies that

$$|G_k(t,s)| \leq 2C_1\|h\|e^{-C_2(t-s)}.$$

A straightforward computation shows that,

$$\begin{aligned}\Big|\int_{-\infty}^{t} G_k(t,s)ds\Big| &\leq \int_{\tau_m}^{t} |G_k(t,s)|ds + \sum_{j=-\infty}^{m} \int_{\tau_{j-1}}^{\tau_j} |G_k(t,s)|ds \\ &= \int_{\tau_m}^{\tau_m+\epsilon} |G_k(t,s)|ds + \int_{\tau_m+\epsilon}^{t} |G_k(t,s)|ds \\ &\quad + \sum_{j=-\infty}^{m} \Big[\int_{\tau_{j-1}}^{\tau_{j-1}+\epsilon} |G_k(t,s)|ds + \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} |G_k(t,s)|ds \\ &\quad + \int_{\tau_j-\epsilon}^{\tau_j} |G_k(t,s)|ds\Big] \\ &\leq 2\epsilon C_1\|h\|e^{-C_2(t-\tau_m-\epsilon)} + \int_{\tau_m+\epsilon}^{t} |G_k(t,s)|ds \\ &\quad + \sum_{j=-\infty}^{m} \Big\{2\epsilon C_1\|h\|\big[e^{-C_2(t-\tau_{j-1}-\epsilon)} + e^{-C_2(t-\tau_j)}\big] \\ &\quad + \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} |G_k(t,s)|ds\Big\} \\ &= 2\epsilon C_1\|h\|e^{-C_2(t-\tau_m-\epsilon)} + \int_{\tau_m+\epsilon}^{t} |G_k(t,s)|ds \\ &\quad + \sum_{j=-\infty}^{m} \Big[2\epsilon C_1\|h\|\big(1+e^{C_2\epsilon}\big)e^{-C_2(t-\tau_j)} + \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} |G_k(t,s)|ds\Big] \\ &< 2\epsilon C_1\|h\|e^{-C_2(t-\tau_m-\epsilon)} + \int_{\tau_m+\epsilon}^{t} |G_k(t,s)|ds \\ &\quad + 2\epsilon C_1\|h\| \cdot \frac{1+e^{C_2\epsilon}}{1-e^{-C_2\theta}} + \sum_{j=-\infty}^{m} \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} |G_k(t,s)|ds\end{aligned}$$

for all $k > 1/\epsilon$, where (29) is used to obtain the last inequality. If $k > 1/\epsilon$, by Remark 5.6 and (19),

$$\begin{aligned}\int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} |G_k(t,s)|ds &\le \int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon} \Big\{|[W(t+\alpha_k, s+\alpha_k) - W(t,s)]h(s+\alpha_k)| \\ &\quad + |W(t,s)[h(s+\alpha_k) - h(s)]|\Big\}ds \\ &\le (\tau_j - \tau_{j-1} - 2\epsilon)\epsilon C_3\|h\|e^{-C_2(t-\tau_j+\epsilon)/2} \\ &\quad + C_1 e^{-C_2(t-\tau_j+\epsilon)} \int_{\tau_{j-1}}^{\tau_{j-1}+L} |h(s+\alpha_k) - h(s)|ds \\ &< (L-2\epsilon)\epsilon C_3\|h\|e^{-C_2(t-\tau_j+\epsilon)/2} + \frac{LC_1}{k}e^{-C_2(t-\tau_j+\epsilon)} \\ &< (L-2\epsilon)\epsilon C_3\|h\|e^{-C_2(t-\tau_j+\epsilon)/2} + \epsilon LC_1 e^{-C_2(t-\tau_j+\epsilon)}\end{aligned}$$

for $j \le m$ and similarly,

$$\begin{aligned}\int_{\tau_m+\epsilon}^{t} |G_k(t,s)|ds &\le (t-\tau_m-\epsilon)\epsilon C_3\|h\| + \frac{LC_1}{k} \\ &\le (L-\epsilon)\epsilon C_3\|h\| + \epsilon LC_1.\end{aligned}$$

Consequently,

$$\begin{aligned}\Big|\int_{-\infty}^{t} G_k(t,s)ds\Big| &< 2\epsilon C_1\|h\|e^{-C_2(t-\tau_m-\epsilon)} + (L-\epsilon)\epsilon C_3\|h\| + \epsilon LC_1 \\ &\quad + 2\epsilon C_1\|h\| \cdot \frac{1+e^{C_2\epsilon}}{1-e^{-C_2\theta}} + (L-2\epsilon)\epsilon C_3\|h\| \cdot \frac{e^{-C_2\epsilon/2}}{1-e^{-C_2\theta/2}} \\ &\quad + \frac{\epsilon LC_1 e^{-C_2\epsilon}}{1-e^{-C_2\theta}} =: F_3(\epsilon)\end{aligned}$$

by the above inequalities and (29). On the other hand, from Lemma 5.7 and (19) it follows that for any $k > 1/\epsilon$,

$$\begin{aligned}&\Big|\sum_{\tau_j<t}\Big[W(t+\alpha_k, \tau^+_{j+\alpha_k/\xi})b\Big(j+\frac{\alpha_k}{\xi}\Big) - W(t,\tau_j^+)b(j)\Big]\Big| \\ &\le \sum_{j=-\infty}^{m}\Big\{\Big|[W(t+\alpha_k,\tau^+_{j+\alpha_k/\xi}) - W(t,\tau_j^+)]b\Big(j+\frac{\alpha_k}{\xi}\Big)\Big| \\ &\quad + \Big|W(t,\tau_j^+)\Big[b\Big(j+\frac{\alpha_k}{\xi}\Big) - b(j)\Big]\Big|\Big\} \\ &< \sum_{j=-\infty}^{m}\Big\{\Big[2C_1 e^{-C_2(t+\alpha_k-\tau_{j+\alpha_k/\xi}-\epsilon)}\Big(e^{3\epsilon\|A\|}-1\Big) \\ &\quad + \epsilon C_3 e^{-C_2(t+\alpha_k-\tau_{j+\alpha_k/\xi}-3\epsilon)/2}\Big]\cdot\|b\| + \frac{C_1}{k}e^{-C_2(t-\tau_j)}\Big\} \\ &< \sum_{j=-\infty}^{m}\Big\{\Big[2C_1 e^{-C_2(t-\tau_j-1/k-\epsilon)}\Big(e^{3\epsilon\|A\|}-1\Big) + \epsilon C_3 e^{-C_2(t-\tau_j-1/k-3\epsilon)/2}\Big]\cdot\|b\| \\ &\quad + \frac{C_1}{k}e^{-C_2(t-\tau_j)}\Big\}\end{aligned}$$

and by (29),

$$\Big|\sum_{\tau_j<t}\Big[W(t+\alpha_k,\tau^+_{j+\alpha_k/\xi})b\Big(j+\frac{\alpha_k}{\xi}\Big)-W(t,\tau^+_j)b(j)\Big]\Big|$$
$$<\Big[2C_1e^{C_2(1/k+\epsilon)}\cdot\frac{e^{3\epsilon\|A\|}-1}{1-e^{-C_2\theta}}+\frac{\epsilon C_3e^{C_2(1/k+3\epsilon)/2}}{1-e^{-C_2\theta/2}}\Big]\cdot\|b\|+\frac{C_1}{k(1-e^{-C_2\theta})}$$
$$<\Big[2C_1e^{2\epsilon C_2}\cdot\frac{e^{3\epsilon\|A\|}-1}{1-e^{-C_2\theta}}+\frac{\epsilon C_3e^{2\epsilon C_2}}{1-e^{-C_2\theta/2}}\Big]\cdot\|b\|+\frac{C_1\epsilon}{1-e^{-C_2\theta}}=:F_4(\epsilon).$$

Therefore,

$$|\phi(t+\alpha_k)-\phi(t)|<F_3(\epsilon)+F_4(\epsilon) \tag{33}$$

for all $t\in\mathbb{R}$, $|t-\tau_j|>\epsilon$, $j\in\mathbb{Z}$ and $k>1/\epsilon$.

4.3. We prove that

$$\lim_{k\to\infty}\sup_{t\in\mathbb{R}}\int_t^{t+1}|\phi(s+\alpha_k)-\phi(s)|ds=0. \tag{34}$$

Let $N\in\mathbb{Z}_+$, $N\theta>1$. It is obvious that

$$\tau_{j+N}-\tau_j\geq N\theta>1$$

for all $j\in\mathbb{Z}$. If $\tau_m+\epsilon<t<\tau_{m+1}-\epsilon$ for some $m\in\mathbb{Z}$, then

$$t+1<\tau_{m+1}+1<\tau_{m+1+N}.$$

From (33) it follows that

$$\int_t^{t+1}|\phi(s+\alpha_k)-\phi(s)|ds\leq\int_{\tau_m}^{\tau_{m+1+N}}|\phi(s+\alpha_k)-\phi(s)|ds$$
$$=\sum_{j=m+1}^{m+1+N}\Big[\int_{\tau_{j-1}}^{\tau_{j-1}+\epsilon}|\phi(s+\alpha_k)-\phi(s)|ds+\int_{\tau_{j-1}+\epsilon}^{\tau_j-\epsilon}|\phi(s+\alpha_k)-\phi(s)|ds$$
$$+\int_{\tau_j-\epsilon}^{\tau_j}|\phi(s+\alpha_k)-\phi(s)|ds\Big]$$
$$<(N+1)\{4\epsilon\|\phi\|+(L-2\epsilon)[F_1(\epsilon)+F_2(\epsilon)]\}$$

for all $k>1/\epsilon$. Since ϵ is arbitrarily small, (34) holds.

4.4. We prove the module containment. (34) implies that α is $\widetilde{\phi}$-increasing. By (12), Theorem 2.5 and Lemma 5.8,

$$\mathrm{mod}(\phi)=\mathrm{mod}(\widetilde{\phi})\subset\mathrm{span}\Big(\mathrm{mod}\Big(\widetilde{A},\widetilde{\bar{B}}\Big(\frac{\cdot}{\xi}\Big),\widetilde{\bar{b}}\Big(\frac{\cdot}{\xi}\Big),\widetilde{\bar{\zeta}}\Big(\frac{\cdot}{\xi}\Big),\tilde{h}\Big)\cup\Big\{\frac{2\pi}{\xi}\Big\}\Big)$$
$$=\mathrm{span}\Big(\mathrm{mod}\Big(A,\bar{B}\Big(\frac{\cdot}{\xi}\Big),\bar{b}\Big(\frac{\cdot}{\xi}\Big),\bar{\zeta}\Big(\frac{\cdot}{\xi}\Big),h\Big)\cup\Big\{\frac{2\pi}{\xi}\Big\}\Big)$$
$$=\mathrm{span}\Big(\mathrm{mod}(A,h)\cup\Big[\frac{1}{\xi}\cdot\mathrm{mod}(\bar{B},\bar{b},\bar{\zeta})\Big]\cup\Big\{\frac{2\pi}{\xi}\Big\}\Big)$$
$$=\mathrm{span}\Big(\mathrm{mod}(A,h)\cup\Big\{\frac{1}{\xi}\cdot\bigcup_{k\in\mathbb{Z}}[(\Lambda_B\cup\Lambda_b\cup\Lambda_\zeta-\{0\})^{(r)}+2k\pi]\Big\}$$

$$\cup \left\{ \frac{2\pi}{\xi} \right\} \Big)$$

$$= \text{span}\Big(\text{mod}(A, h) \cup \Big[\frac{1}{\xi} \cdot \big\{ [\text{mod}(B, b, \zeta)]^{(r)} \cup \{2\pi\} \big\} \Big] \Big).$$

5. We prove that ϕ is asymptotically stable. By (2.18) in [10], any solution x to (18) can be represented as

$$x(t) = W(t, t_0^+)x(t_0^+) + \int_{t_0}^{t} W(t, s)h(s)ds + \sum_{t_0<\tau_j<t} W(t, \tau_j^+)b(j), \quad t > t_0.$$

If φ and ψ are two distinct solutions to (18), then by (19),

$$\begin{aligned} |\varphi(t) - \psi(t)| &\le |W(t, t_0^+)[\varphi(t_0^+) - \psi(t_0^+)]| \\ &\le C_1 e^{-C_2(t-t_0)} |\varphi(t_0^+) - \psi(t_0^+)|, \quad t > t_0. \end{aligned}$$

Thus (18) admits a unique almost periodic solution ϕ and ϕ is asymptotically stable. □

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