

Global Hopf Bifurcation for Differential-Algebraic Equations with State-Dependent Delay

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Abstract We develop a global Hopf bifurcation theory for differential equations with a state-dependent delay governed by an algebraic equation, using the S^1 -equivariant degree. We apply the global Hopf bifurcation theory to a model of genetic regulatory dynamics with threshold type state-dependent delay vanishing at the stationary state, for a description of the global continuation of the periodic oscillations.

Keywords State-dependent delay · Hopf bifurcation · Differential-algebraic equations · S^1 -equivariant degree · Regulatory dynamics

1 Introduction

Consider the following system of differential-algebraic equations (DAEs) with state-dependent delay,

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - \tau(t)), \sigma), \\ \tau(t) = g(x(t), x(t - \tau(t)), \sigma), \end{cases} \quad (1.1)$$

where we assume that

- (S1) The map $f: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \mapsto f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ is C^2 (twice continuously differentiable).
- (S2) The map $g: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\gamma_1, \gamma_2, \sigma) \mapsto g(\gamma_1, \gamma_2, \sigma) \in \mathbb{R}$ is C^2 .
- (S3) $(\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2})f(\theta_1, \theta_2, \sigma)|_{\sigma=\sigma_0, \theta_1=\theta_2=x_{\sigma_0}}$ is nonsingular, where $\sigma_0 \in \mathbb{R}$, and where $(x_{\sigma_0}, \tau_{\sigma_0})$ (or, for simplicity, $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$) is a stationary state of (1.1). That is,

$$f(x_{\sigma_0}, x_{\sigma_0}, \sigma_0) = 0, \quad g(x_{\sigma_0}, x_{\sigma_0}, \sigma_0) = \tau_{\sigma_0}.$$

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(S3) implies that there exists $\epsilon_0 > 0$ and a C^1 -smooth curve $(\sigma_0 - \epsilon_0, \sigma_0 + \epsilon_0) \ni \sigma \mapsto (x_\sigma, \tau_\sigma) \in \mathbb{R}^{N+1}$ such that (x_σ, τ_σ) is the unique stationary state of (1.1) in a small neighborhood of $(x_{\sigma_0}, \tau_{\sigma_0})$ for σ close to σ_0 . In the following, we write $\partial_i f = \frac{\partial}{\partial \theta_i} f$ for $i = 1, 2$, and similarly we define $\partial_i g$ for $i = 1, 2$.

The state-dependent delay of system (1.1) arises in several applications. To mention a few, in the model of turning processes [7], the delay τ is the time duration for one around of cutting, in the echo control model [18], the state-dependent delay is the echo traveling time between the object’s positions when the sound is emitted and received. See [4] for a review. To model diffusion processes in genetic regulatory dynamics with time delay, we considered in [5] the following system:

$$\begin{cases} \frac{dx(t)}{dt} = -\mu_m x(t) + f_0(y(t - \tau(t))), \\ \frac{dy(t)}{dt} = -\mu_p y(t) + g_0(x(t - \tau(t))), \\ \tau(t) = \epsilon_0 + c(x(t) - x(t - \tau(t))), \end{cases} \tag{1.2}$$

where $f_0, g_0 : \mathbb{R} \rightarrow \mathbb{R}$ are three times continuously differentiable functions; μ_m, μ_p, c and ϵ_0 are positive constants. The time delay $\tau(t) = \epsilon_0 + c(x(t) - x(t - \tau(t)))$ models the homogenization time of the substances produced in the regulatory processes. Since the equation for τ can be written as

$$\int_{t-\tau(t)}^t \frac{1 - c\dot{x}(s)}{\epsilon_0} ds = 1, \tag{1.3}$$

we call τ a threshold type state-dependent delay and we have shown in [5] that using the time transformation $t \mapsto \int_0^t (1 - c\dot{x}(s)) ds$ inspired by [15, 16], system (1.2) can be transformed into a system with constant delay and distributed delay under certain conditions. In such a case, the theory we developed in [1] is applicable to system (1.2) for a local and global Hopf bifurcation theory. However, if $\epsilon_0 = 0$ in (1.2), then the integral equation for τ becomes

$$\int_{t-\tau(t)}^t (1 - c\dot{x}(s)) ds = 0, \tag{1.4}$$

which cannot be employed to remove the state-dependent delay using the time transformation $t \mapsto \int_0^t (1 - c\dot{x}(s)) ds$. Thus the global Hopf bifurcation theory developed in [1] is no longer applicable. We remark that if we obtain a differential equation of τ from $\tau(t) = \epsilon_0 + c(x(t) - x(t - \tau))$ by taking derivatives on both sides, the resulting system will have a foliation of equilibria and at least one zero eigenvalue. Thus the global Hopf bifurcation theory developed in [6] is not applicable either. With these facts, we are motivated to develop a global Hopf bifurcation theory for system (1.1) and apply it to an extended three dimensional Goodwin’s model with state-dependent delay where the delay vanishes at the stationary state. [See system (5.1) at Sect. 5 for a brief description of the model.]

To start the discussion, we denote by $C(\mathbb{R}; \mathbb{R}^N)$ the normed space of bounded continuous functions from \mathbb{R} to \mathbb{R}^N equipped with the usual supremum norm $\|x\| = \sup_{t \in \mathbb{R}} |x(t)|$ for $x \in C(\mathbb{R}; \mathbb{R}^N)$, where $|\cdot|$ denotes the Euclidean norm. We also denote by $C^1(\mathbb{R}; \mathbb{R}^N)$ the normed space of continuously differentiable bounded functions with bounded derivatives from \mathbb{R} to \mathbb{R}^N equipped with the usual C^1 norm

$$\|x\|_{C^1} = \max\{\sup_{t \in \mathbb{R}} |x(t)|, \sup_{t \in \mathbb{R}} |\dot{x}(t)|\}$$

for $x \in C^1(\mathbb{R}; \mathbb{R}^N)$.

We wish to drop the algebraic equation in (1.1) for the application of S^1 -equivariant degree. We have

Lemma 1 Assume that (S2) holds. The following statements are true:

- (i) For every $(x, \sigma) \in C(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}$, where x is periodic, there exists a function $\tau : \mathbb{R} \rightarrow \mathbb{R}$ such that $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$.
- (ii) Let $\sigma \in \mathbb{R}$ be fixed and x_0 be a constant function in $C(\mathbb{R}; \mathbb{R}^N)$. There exists an open ball $O(\epsilon) \subset C^1(\mathbb{R}; \mathbb{R}^N)$ centered at x_0 with radius $\epsilon > 0$ such that for every periodic $x \in O(\epsilon)$ with period $p > 0$, there exists a unique periodic $\tau \in C^1(\mathbb{R}; \mathbb{R})$ with period p such that $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$, $t \in \mathbb{R}$. Moreover, τ depends on x continuously under the supremum norm.

Proof Fix an arbitrary $t \in \mathbb{R}$ and let $a = \tau(t)$. Consider the graphs of $h = a$ and $h = g(x(t), x(t - a), \sigma)$ in the h - a plane. The graphs must have an intersection since $x \in C(\mathbb{R}; \mathbb{R}^N)$ is periodic and $h = g(x(t), x(t - a), \sigma)$ is continuous and bounded with respect to a . Since t is arbitrary, there exists a function $\tau : \mathbb{R} \rightarrow \mathbb{R}$ with $a = \tau(t)$ such that $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$.

Define $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $F(a, t) = a - g(x(t), x(t - a), \sigma)$ where $x \in C^1(\mathbb{R}; \mathbb{R}^N)$ is periodic. By (S2) F is continuously differentiable in $(a, t) \in \mathbb{R}^2$. Moreover, $a = \tau(t)$ is such that $F(a, t) = 0$. Note that we have

$$\frac{\partial F}{\partial a} = 1 + \partial_2 g(x(t), x(t - a), \sigma)\dot{x}(t - a).$$

Since by (S2) $\partial_2 g(\gamma_1, \gamma_2, \sigma)$ is continuous, there exists an open ball $O(\epsilon) \subset C^1(\mathbb{R}; \mathbb{R}^N)$ near x_0 with $\epsilon > 0$, such that for every $x \in O(\epsilon)$ with x periodic, we have $\frac{\partial F}{\partial a} \neq 0$. Indeed, we can choose $\epsilon > 0$ small enough so that $\frac{\partial F}{\partial a}$ assumes values in a small neighborhood of 1 in \mathbb{R} . By the Implicit Function Theorem, the solution $a = \tau(t)$ of $a = g(x(t), x(t - a), \sigma)$ for a is continuously differentiable with respect to t . Moreover, by taking derivatives on both sides of $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$ we know that $\dot{\tau}$ is bounded in \mathbb{R} . That is, $\tau \in C^1(\mathbb{R}; \mathbb{R})$.

Next we show that we can choose $\epsilon > 0$ small enough so that τ is unique. Suppose not. Then for every $\epsilon > 0$, there exists $x \in O(\epsilon)$ with period p such that τ is not unique. That is, there exists $\tau_0 \neq \tau$ such that $\tau_0(t) = g(x(t), x(t - \tau_0(t)), \sigma)$. Let $\epsilon_1 > 0$ and $L > 0$ be such that

$$L = \sup_{x, y \in O(\epsilon_1)} \sup_{t \in \mathbb{R}} |\partial_2 g(x(t), y(t), \sigma)|.$$

Then by the Integral Mean Value Theorem, for $x \in O(\epsilon) \subset O(\epsilon_1)$, we have

$$\begin{aligned} 0 &< \sup_{t \in \mathbb{R}} |\tau(t) - \tau_0(t)| \\ &= \sup_{t \in \mathbb{R}} |g(x(t), x(t - \tau(t)), \sigma) - g(x(t), x(t - \tau_0(t)), \sigma)| \\ &= \sup_{t \in \mathbb{R}} \left| \int_0^1 \partial_2 g(x(t), x(t) + s(x(t - \tau(t)) - x(t - \tau_0(t))), \sigma) ds \right. \\ &\quad \left. \times (x(t - \tau(t)) - x(t - \tau_0(t))) \right| \\ &= \sup_{t \in \mathbb{R}} \left| \int_0^1 \partial_2 g(x(t), x(t) + s(x(t - \tau(t)) - x(t - \tau_0(t))), \sigma) ds \right| \\ &\quad \times \sup_{t \in \mathbb{R}} \left| \int_0^1 \dot{x}(t + \theta(\tau(t) - \tau_0(t))) d\theta(\tau(t) - \tau_0(t)) \right| \\ &\leq L \epsilon \sup_{t \in \mathbb{R}} |\tau(t) - \tau_0(t)|, \end{aligned}$$

which leads to $L\epsilon \geq 1$ for every $\epsilon \in (0, \epsilon_1)$. This is a contradiction. Therefore, τ is uniquely determined by x with $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$, $t \in \mathbb{R}$.

Next, we show that τ is p -periodic. Indeed, we have for every $t \in \mathbb{R}$,

$$\tau(t + p) = g(x(t + p), x(t + p - \tau(t + p)), \sigma) = g(x(t), x(t - \tau(t + p)), \sigma),$$

which combined with $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$ and the uniqueness of the solution for τ leads to $\tau(t) = \tau(t + p)$, $t \in \mathbb{R}$.

Lastly, we show that τ depends on x continuously. Suppose that x and \bar{x} both lie in $O(\epsilon) \subset C^1(\mathbb{R}; \mathbb{R}^N)$ and that τ and $\bar{\tau}$ are the corresponding solutions of $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$, $t \in \mathbb{R}$ for τ . Let $L_1 > 0$ and $L_2 > 0$ be the supremum of $|\partial_1 g|$ and $|\partial_2 g|$ when x is in $O(\epsilon)$, respectively. We have

$$\begin{aligned} &|\tau(t) - \bar{\tau}(t)| \\ &= |g(x(t), x(t - \tau(t)), \sigma) - g(\bar{x}(t), \bar{x}(t - \bar{\tau}(t)), \sigma)| \\ &= |g(x(t), x(t - \tau(t)), \sigma) - g(\bar{x}, x(t - \tau(t)), \sigma) \\ &\quad + g(\bar{x}, x(t - \tau(t)), \sigma) - g(\bar{x}(t), \bar{x}(t - \bar{\tau}(t)), \sigma)| \\ &\leq L_1 \|x - \bar{x}\| + L_2 |x(t - \tau(t)) - x(t - \bar{\tau}(t)) + x(t - \bar{\tau}(t)) - \bar{x}(t - \bar{\tau}(t))| \\ &\leq L_1 \|x - \bar{x}\| + L_2(\epsilon |\tau(t) - \bar{\tau}(t)| + \|x - \bar{x}\|). \end{aligned}$$

Take $\epsilon > 0$ so that $0 < L_2\epsilon < 1$. We have $|\tau(t) - \bar{\tau}(t)| < \frac{(L_1+L_2)\|x-\bar{x}\|}{1-L_2\epsilon}$ and

$$\|\tau - \bar{\tau}\| < \frac{(L_1 + L_2)\|x - \bar{x}\|}{1 - L_2\epsilon},$$

which shows the continuous dependency of τ on x under the supremum norm. □

By Lemma 1, we notice that if $x \in C(\mathbb{R}; \mathbb{R}^N)$ is periodic, the function τ satisfying $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$ is not necessarily continuous and neither is $f(x(t), x(t - \tau(t)), \sigma)$, while continuity is crucial for applying topological degree theory for a Hopf bifurcation. However, if $x \in C^1(\mathbb{R}; \mathbb{R}^N)$ is periodic and is in a small neighborhood of a constant function, τ is continuously differentiable. The complexity is caused by the implicitly given τ in the algebraic equation of system (1.1). If we replace the delayed term $x(t - \tau(t))$ with $x(t - \tau_\sigma)$ in the algebraic equation where τ_σ is the stationary state of τ defined after (S3), we obtain the following system with state-dependent delay,

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - \tau), \sigma), \\ \tau(t) = g(x(t), x(t - \tau_\sigma), \sigma), \end{cases} \tag{1.5}$$

where τ is continuous if x is continuous. We notice that system (1.5) shares the same set of stationary states of system (1.1) and it has interest on its own right since it also represents a class of differential-algebraic equations with state-dependent delay. Due to the similarities between systems (1.5) and (1.1), we are interested in developing global Hopf bifurcation theories for both systems, while for systems (1.5) we use the state space of $C(\mathbb{R}; \mathbb{R}^N)$ and for system (1.1) we use $C^1(\mathbb{R}; \mathbb{R}^N)$. Moreover, we show that if system (1.5) undergoes Hopf bifurcation at $(x_{\sigma_0}, \tau_{\sigma_0})$, then system (1.1) also undergoes Hopf bifurcation at the same bifurcation point. Namely, we show that systems (1.5) and (1.1) share the same set of Hopf bifurcation points.

We organize the remaining part of the paper as following: Using the framework for a Hopf bifurcation theory established in [6], we develop a local Hopf bifurcation theory for system (1.5) in Sect. 2 and for system (1.1) in Sect. 3. We develop global Hopf bifurcation

theories for both systems (1.1) and (1.5) in Sect. 4. In Sect. 5 we apply the developed local and global Hopf bifurcation theories to the prototype system (1.2) with $\epsilon_0 = 0$. We conclude the discussion in Sect. 6.

2 Local Hopf Bifurcation for System (1.5)

We begin with definitions of notations. We denote by $V = C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ the space of 2π -periodic continuous functions from \mathbb{R} to \mathbb{R}^N equipped with the supremum norm. We denote by $C^1_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ the Banach space of 2π -periodic and continuously differentiable functions equipped with the C^1 norm.

Note that if $x \in C(\mathbb{R}; \mathbb{R}^N)$ is p -periodic, then $\tau(t) = g(x(t), x(t - \tau_\sigma), \sigma)$, $t \in \mathbb{R}$ is continuous and p -periodic. We call x a solution if (x, τ) satisfies system (1.5). For a stationary state x_{σ_0} of system (1.5) with the parameter σ_0 , we say that (x_{σ_0}, σ_0) is a Hopf bifurcation point of system (1.5), if there exist a sequence $\{(x_k, \sigma_k, T_k)\}_{k=1}^{+\infty} \subseteq C(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}^2$ and $T_0 > 0$ such that

$$\lim_{k \rightarrow +\infty} \|(x_k, \sigma_k, T_k) - (x_{\sigma_0}, \sigma_0, T_0)\|_{C(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}^2} = 0,$$

and (x_k, σ_k) is a nonconstant T_k -periodic solution of system (1.5).

Due to the nature of the same approach of using the S^1 -equivariant degree, the presentation of the remaining part of this section is similar to that of [6], even though the systems in question are different. We study Hopf bifurcation of (1.5) through the system obtained through the formal linearization [2]. Namely, we freeze the state-dependent delay in system (1.5) at its stationary state and linearize the resulting differential equation of x with constant delay at the stationary state. For $\sigma \in (\sigma_0 - \epsilon_0, \sigma_0 + \epsilon_0)$, the following system is called the formal linearization of system (1.5) at the stationary point x_σ :

$$\dot{x}(t) = \partial_1 f(\sigma) (x(t) - x_\sigma) + \partial_2 f(\sigma) (x(t - \tau_\sigma) - x_\sigma), \tag{2.1}$$

where

$$\partial_1 f(\sigma) := \partial_1 f(x_\sigma, \tau_\sigma, \sigma), \quad \partial_2 f(\sigma) := \partial_2 f(x_\sigma, \tau_\sigma, \sigma), \quad \tau_\sigma = g(x_\sigma, x_\sigma, \sigma).$$

Letting $x(t) = e^{\omega t} \cdot C + x_\sigma$ with $C \in \mathbb{R}^N$, we obtain the following characteristic equation of the linear system corresponding to the inhomogeneous linear system (2.1),

$$\det \Delta_{(x_\sigma, \sigma)}(\omega) = 0, \tag{2.2}$$

where $\Delta_{(x_\sigma, \sigma)}(\omega)$ is an $N \times N$ complex matrix defined by

$$\Delta_{(x_\sigma, \sigma)}(\omega) = \omega I - \partial_1 f(\sigma) - \partial_2 f(\sigma) e^{-\omega \tau_\sigma}. \tag{2.3}$$

A solution ω to the characteristic equation (2.2) is called a characteristic value of the stationary state (x_σ, σ) . If zero is not a characteristic value of (x_{σ_0}, σ_0) , (x_{σ_0}, σ_0) is said to be a nonsingular stationary state. We say that (x_{σ_0}, σ_0) is a *center* if the set of nonzero purely imaginary characteristic values of (x_{σ_0}, σ_0) is nonempty and discrete. (x_{σ_0}, σ_0) is called an *isolated center* if it is the only center in some neighborhood of (x_{σ_0}, σ_0) in $\mathbb{R}^N \times \mathbb{R}$.

If (x_{σ_0}, σ_0) is an isolated center of (2.1), then there exist $\beta_0 > 0$ and $\delta \in (0, \epsilon_0)$ such that

$$\det \Delta_{(x_{\sigma_0}, \sigma_0)}(i\beta_0) = 0,$$

and

$$\det \Delta_{(x_\sigma, \sigma)}(i\beta) \neq 0, \tag{2.4}$$

for any $\sigma \in (\sigma_0 - \delta, \sigma_0 + \delta)$ and any $\beta \in (0, +\infty) \setminus \{\beta_0\}$. Hence, we can choose constants $\alpha_0 = \alpha_0(\sigma_0, \beta_0) > 0$ and $\varepsilon = \varepsilon(\sigma_0, \beta_0) > 0$ such that $\det \Delta_{(x_{\sigma_0}, \sigma_0)}(\cdot)$ has no zeros in $\partial\Omega$ where $\Omega := (0, \alpha_0) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \subset \mathbb{R}^2 \cong \mathbb{C}$. We note that $\det \Delta_{(x_\sigma, \sigma)}(\omega)$ is analytic in ω and is continuous in σ . If $\delta > 0$ is small enough, then there is no zero of $\det \Delta_{(x_{\sigma_0 \pm \delta}, \sigma_0 \pm \delta)}(\omega)$ in $\partial\Omega$. So we can define the number

$$\gamma_\pm(x_{\sigma_0}, \sigma_0, \beta_0) = \text{deg}_B(\det \Delta_{(x_{\sigma_0 \pm \delta}, \sigma_0 \pm \delta)}(\cdot), \Omega),$$

and the crossing number of $(x_{\sigma_0}, \sigma_0, \beta_0)$ as

$$\gamma(x_{\sigma_0}, \sigma_0, \beta_0) = \gamma_- - \gamma_+, \tag{2.5}$$

where deg_B is the Brouwer degree in finite-dimensional spaces. See, e.g., [8] for details.

To formulate the Hopf bifurcation problem as a fixed point problem in $C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$, we normalize the period of a $2\pi/\beta$ -periodic solution x of (1.5) and the associated $\tau \in C(\mathbb{R}; \mathbb{R})$ by setting $(x(t), \tau(t)) = (y(\beta t), z(\beta t))$ and obtain

$$\begin{pmatrix} \dot{y}(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma) \\ g(y(t), y(t - \beta z(t)), \sigma) \end{pmatrix}, \tag{2.6}$$

where $(y_\sigma, z_\sigma) = (x_\sigma, \tau_\sigma)$.

Define $N_0 : V \times \mathbb{R} \times (0, +\infty) \ni (y, \sigma, \beta) \mapsto N_0(y, \sigma, \beta) \in V$ by

$$N_0(y, \sigma, \beta)(t) = f(y(t), y(t - \beta z(t)), \sigma), \tag{2.7}$$

where z is chosen to satisfy the second line of (2.6) in light of Lemma 1.

Then the differential equation part of system (2.6) is rewritten as

$$\dot{y}(t) = \frac{1}{\beta} N_0(y, \sigma, \beta)(t). \tag{2.8}$$

Correspondingly, (2.1) is transformed into

$$\dot{y}(t) = \frac{1}{\beta} \tilde{N}_0(y, \sigma, \beta)(t), \tag{2.9}$$

where $\tilde{N}_0 : V \times \mathbb{R} \times (0, +\infty) \ni (y, \sigma, \beta) \mapsto \tilde{N}_0(y, \sigma, \beta) \in V$ is defined by

$$\tilde{N}_0(y, \sigma, \beta)(t) = \partial_1 f(\sigma)(y(t) - y_\sigma) + \partial_2 f(\sigma)(y(t - \beta z_\sigma) - y_\sigma).$$

We note that y is 2π -periodic if and only if x is $(2\pi/\beta)$ -periodic.

Let $L_0 : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow V$ be defined by $L_0 y(t) = \dot{y}(t)$, $t \in \mathbb{R}$ and $K : V \rightarrow \mathbb{R}^N$ be defined by

$$K(y) = \frac{1}{2\pi} \int_0^{2\pi} y(t) dt. \tag{2.10}$$

Define the map $\tilde{\mathcal{F}} : V \times \mathbb{R} \times (0, +\infty) \rightarrow V$ by

$$\tilde{\mathcal{F}}(y, \sigma, \beta) := y - (L_0 + K)^{-1} \left[\frac{1}{\beta} \tilde{N}_0(y, \sigma, \beta) + K(y) \right]. \tag{2.11}$$

We call the set defined by

$$B(y_0, \sigma_0, \beta_0; r, \rho) = \{(y, \sigma, \beta) : \|y - y_\sigma\| < r, |(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho\},$$

a special neighborhood of $\tilde{\mathcal{F}}$, if it satisfies

- (i) $\tilde{\mathcal{F}}(y, \sigma, \beta) \neq 0$ for every $(y, \sigma, \beta) \in \overline{B(y_0, \sigma_0, \beta_0; r, \rho)}$ with $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and $\|y - y_\sigma\| \neq 0$;
- (ii) (y_0, σ_0, β_0) is the only isolated center of (2.9) in $\overline{B(y_0, \sigma_0, \beta_0; r, \rho)}$.

We cite some technical Lemmas from [6] with necessary notational adaptations, before we state and prove the local Hopf bifurcation theorem for system (1.5).

Lemma 2 ([6]) *Let $L_0 : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow V$ be defined by $L_0 y(t) = \dot{y}(t)$, $t \in \mathbb{R}$ and let $K : V \rightarrow \mathbb{R}^N$ be defined at (2.10). Then $L_0 + K$ has a compact inverse $(L_0 + K)^{-1} : V \rightarrow V$.*

Lemma 3 ([6]) *For any $\sigma \in \mathbb{R}$ and $\beta > 0$, the map $N_0(\cdot, \sigma, \beta) : V \rightarrow V$ defined by (2.7) is continuous.*

Lemma 4 ([6]) *If system (2.1) has a nonconstant periodic solution with period $T > 0$, then there exists an integer $m \geq 1$, $m \in \mathbb{N}$ such that $\pm im 2\pi/T$ are characteristic values of the stationary state $(x_\sigma, \tau_\sigma, \sigma)$.*

For the purpose of establishing the S^1 -degree on some special neighborhood near the stationary state, we have

Lemma 5 *Assume (S1)–(S3) hold. Let L_0 and K be as in Lemma 2 and $\tilde{N}_0 : V \times \mathbb{R} \times (0, +\infty) \rightarrow V$ be as in (2.9). Let $\tilde{\mathcal{F}} : V \times \mathbb{R} \times (0, +\infty) \rightarrow V$ be defined at (2.11). If $B(y_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood of $\tilde{\mathcal{F}}$ with $0 < \rho < \beta_0$, then there exists $r' \in (0, r]$ such that the neighborhood*

$$B(y_0, \sigma_0, \beta_0; r', \rho) = \{(u, \sigma, \beta) : \|y - y_\sigma\| < r', |(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho\}$$

satisfies

$$\dot{y}(t) \cong \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma)$$

for $(y, \sigma, \beta) \in \overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$ with $y \neq y_\sigma$ and $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$.

Proof We prove by contradiction. Suppose the statement is not true, then for any $0 < r' \leq r$, there exists (y, σ, β) such that $0 < \|y - y_\sigma\| < r'$, $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and

$$\dot{y}(t) = \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma) \text{ for } t \in \mathbb{R}. \tag{2.12}$$

Then there exists a sequence of nonconstant periodic solutions $\{(y_k, \sigma_k, \beta_k)\}_{k=1}^\infty$ of (2.12) such that

$$\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\| = 0, |(\sigma_k, \beta_k) - (\sigma_0, \beta_0)| = \rho, \tag{2.13}$$

and

$$\dot{y}_k(t) = \frac{1}{\beta_k} f(y_k(t), y_k(t - \beta_k z_k(t)), \sigma_k) \text{ for } t \in \mathbb{R}, \tag{2.14}$$

where z_k is determined by y_k by the second line of system (2.6).

Note that $0 < \rho < \beta_0$ implies that $\beta_k \geq \beta_0 - \rho > 0$ for every $k \in \mathbb{N}$. Also, since the sequence $\{\sigma_k, \beta_k\}_{k=1}^\infty$ belongs to a bounded neighborhood of (σ_0, β_0) in \mathbb{R}^2 , there exists a convergent subsequence, still denoted by $\{(\sigma_k, \beta_k)\}_{k=1}^\infty$ for notational simplicity, that converges to (σ^*, β^*) so that $|(\sigma^*, \beta^*) - (\sigma_0, \beta_0)| = \rho$ and $\beta^* > 0$. Then we have

$$\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\| = 0, \quad \lim_{k \rightarrow +\infty} |(\sigma_k, \beta_k) - (\sigma^*, \beta^*)| = 0, \tag{2.15}$$

and

$$\dot{y}_k(t) = \frac{1}{\beta_k} f(y_k(t), y_k(t - \beta_k z_k(t)), \sigma_k) \text{ for } t \in \mathbb{R}. \tag{2.16}$$

In the following we show that the system

$$\dot{v}(t) = \frac{1}{\beta^*} \partial_1 f(\sigma^*)v(t) + \frac{1}{\beta^*} \partial_2 f(\sigma^*)v(t - \beta^* z_{\sigma^*}), \tag{2.17}$$

has a nonconstant periodic solution which contradicts the assumption that $(y_{\sigma_0}, \sigma_0, \beta_0)$ is the only center of (2.9) in $\overline{B(y_0, \sigma_0, \beta_0; r, \rho)}$.

By (S1), $f: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \rightarrow f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ is C^2 in (θ_1, θ_2) . It follows from the Integral Mean Value Theorem that

$$\begin{aligned} \dot{y}_k(t) &= \frac{1}{\beta_k} \int_0^1 \partial_1 f_k(\sigma_k, s)(t) ds (y_k(t) - y_{\sigma_k}) \\ &\quad + \frac{1}{\beta_k} \int_0^1 \partial_2 f_k(\sigma_k, s)(t) ds (y_k(t - \beta_k z_k(t)) - y_{\sigma_k}), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} \partial_1 f_k(\sigma_k, s)(t) &:= \partial_1 f(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), y_{\sigma_k} + s(y_k(t - z_k(t)) - y_{\sigma_k}), \sigma_k), \\ \partial_2 f_k(\sigma_k, s)(t) &:= \partial_2 f(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), y_{\sigma_k} + s(y_k(t - z_k(t)) - y_{\sigma_k}), \sigma_k). \end{aligned}$$

Put

$$v_k(t) = \frac{y_k(t) - y_{\sigma_k}}{\|y_k - y_{\sigma_k}\|}. \tag{2.19}$$

Then we have

$$v_k(t - \beta_k z_k(t)) = \frac{y_k(t - \beta_k z_k(t)) - y_{\sigma_k}}{\|y_k - y_{\sigma_k}\|}. \tag{2.20}$$

By (2.18) and (2.20) we have

$$\dot{v}_k(t) = \frac{1}{\beta_k} \int_0^1 \partial_1 f_k(\sigma_k, s)(t) ds v_k(t) + \frac{1}{\beta_k} \int_0^1 \partial_2 f_k(\sigma_k, s)(t) ds v_k(t - \beta_k z_k(t)). \tag{2.21}$$

We claim that there exists a convergent subsequence of $\{v_k\}_{k=1}^{+\infty}$. Indeed, by (2.13) and system (2.6), we know that $\{z_k, \beta_k\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R}; \mathbb{R}) \times \mathbb{R}$ and hence $\lim_{t \rightarrow +\infty} [t - \beta_k z_k(t)] = +\infty$. Then by (2.19) and (2.20), we have

$$\|v_k\| = 1, \quad \|v_k(\cdot - \beta_k z_k(\cdot))\| = 1.$$

Recall that $\partial_i f(\sigma^*)$ and $\partial_i g(\sigma^*)$, $i = 1, 2$, are defined in (2.1). By (2.15), we know that $(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), y_{\sigma_k} + s(y_k(t - z_k(t)) - y_{\sigma_k}), \sigma_k)$ converges to the stationary state $(y_{\sigma^*}, y_{\sigma^*}, \sigma^*)$ in $C(\mathbb{R}; \mathbb{R}^{2N}) \times \mathbb{R}$ uniformly for all $s \in [0, 1]$. By (S1) we know that $f(\theta_1, \theta_2, \sigma)$ is C^2 in $(\theta_1, \theta_2, \sigma)$ and $\partial_1 f(\theta_1, \theta_2, \sigma)$ is C^1 in σ . Also, by (2.13), the sequence

$\{u_k, \beta_k, \sigma_k\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Then there exists a constant $\tilde{L}_1 > 0$ so that

$$|\partial_1 f_k(\sigma_k, s)(t) - \partial_1 f(\sigma^*)| \leq \tilde{L}_1 |(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), y_{\sigma_k} + s(y_k(t - z_k(t)) - y_{\sigma_k}), \sigma_k) - (y_{\sigma^*}, y_{\sigma^*}, \sigma^*)|,$$

for all $t \in \mathbb{R}, k \in \mathbb{N}$ and $s \in [0, 1]$. Therefore, we have $\lim_{k \rightarrow +\infty} \|\partial_1 f_k(\sigma_k, s) - \partial_1 f(\sigma^*)\| = 0$ uniformly for $s \in [0, 1]$. By the same argument we obtain that

$$\lim_{k \rightarrow +\infty} \|\partial_1 f_k(\sigma_k, s) - \partial_1 f(\sigma^*)\| = 0, \quad \lim_{k \rightarrow +\infty} \|\partial_2 f_k(\sigma_k, s) - \partial_2 f(\sigma^*)\| = 0, \quad (2.22)$$

uniformly for $s \in [0, 1]$. From (2.22) we know that $\|\partial_1 f_k(\sigma_k, s)\|$ and $\|\partial_2 f_k(\sigma_k, s)\|$ are both uniformly bounded for all $k \in \mathbb{N}$ and $s \in [0, 1]$. Then it follows from (2.21) that there exists a constant $\tilde{L}_2 > 0$ such that $\|\dot{v}_k\| < \tilde{L}_2$ for any $k \in \mathbb{N}$. By the Arzela–Ascoli Theorem, there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{+\infty}$ of $\{v_k\}_{k=1}^{+\infty}$. That is, there exists $v^* \in \{v \in V : \|v\| = 1\}$ such that

$$\lim_{j \rightarrow +\infty} \|v_{k_j} - v^*\| = 0. \quad (2.23)$$

By the Integral Mean Value Theorem, we have

$$\begin{aligned} &|v_{k_j}(t - \beta_{k_j} z_{k_j}(t)) - v_{k_j}(t - \beta^* z_{\sigma^*})| \\ &= \left| \int_0^1 \dot{v}_{k_j}(t - \theta(\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*})) d\theta(\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*}) \right| \\ &\leq \|\dot{v}_{k_j}\| \cdot |\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*}| \\ &\leq \tilde{L}_2 (\beta_{k_j} |z_{k_j}(t) - z_{\sigma^*}| + |\beta_{k_j} - \beta^*| |z_{\sigma^*}|). \end{aligned} \quad (2.24)$$

By (2.15) and (2.24) we have

$$\lim_{j \rightarrow +\infty} \|v_{k_j}(\cdot - \beta_{k_j} z_{k_j}(\cdot)) - v_{k_j}(\cdot - \beta^* z_{\sigma^*})\| = 0. \quad (2.25)$$

Therefore, it follows from (2.23) and (2.25) that

$$\lim_{j \rightarrow +\infty} \|v_{k_j}(\cdot - \beta_{k_j} z_{k_j}(\cdot)) - v^*(\cdot - \beta^* z_{\sigma^*})\| = 0. \quad (2.26)$$

It follows from (2.15), (2.22), (2.23) and (2.26) that the right hand side of (2.21) converges uniformly to the right hand side of (2.17). Therefore, v^* is differentiable and we have

$$\lim_{k \rightarrow +\infty} |\dot{v}_k(t) - \dot{v}^*(t)| = 0,$$

and

$$\dot{v}^*(t) = \frac{1}{\beta^*} \partial_1 f(\sigma^*) v^*(t) + \frac{1}{\beta^*} \partial_2 f(\sigma^*) v^*(t - \beta^* z_{\sigma^*}). \quad (2.27)$$

Since by (S3) the matrix $\partial_1 f(\sigma^*) + \partial_2 f(\sigma^*)$, is nonsingular, $v = 0$ is the only constant solution of (2.27). Also, we have $v^* \in \{v \in V : \|v\| = 1\}$, $\|v^*\| \neq 0$. Therefore, $(v^*(t), \sigma^*, \beta^*)$ is a nonconstant periodic solution of the linear equation (2.27). Then by Lemma 4 $(y_{\sigma^*}, \sigma^*, \beta^*)$ is also a center of (2.9) in $\overline{B(y_0, \sigma_0, \beta_0; r, \rho)}$. This contradicts the assumption that $B(y_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood of (2.6). This completes the proof. \square

To apply the homotopy argument of S^1 -degree, we show the following

Lemma 6 Assume (S1)–(S3) hold. Let $L_0, K, \tilde{N}_0, \tilde{\mathcal{F}}$ be as in Lemma 5 and $N_0 : V \times \mathbb{R} \times (0, +\infty) \rightarrow V$ be as in (2.6). Define the map $\mathcal{F} : V \times \mathbb{R} \times (0, +\infty) \rightarrow V$ by

$$\mathcal{F}(y, \sigma, \beta) := y - (L_0 + K)^{-1} \left[\frac{1}{\beta} N_0(y, \sigma, \beta) + K(y) \right].$$

If $\mathcal{U} = B(y_0, \sigma_0, \beta_0; r, \rho) \subseteq V \times \mathbb{R} \times (0, +\infty)$ is a special neighborhood of $\tilde{\mathcal{F}}$ with $0 < \rho < \beta_0$, then there exists $r' \in (0, r]$ such that $\mathcal{F}_\theta = (\mathcal{F}, \theta)$ and $\tilde{\mathcal{F}}_\theta = (\tilde{\mathcal{F}}, \theta)$ are homotopic on $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$, where θ is a completing function (or Ize’s function) defined on $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$ which satisfies

- (i) $\theta(y_\sigma, \sigma, \beta) = -|(\sigma, \beta) - (\sigma_0, \beta_0)|$ if $(y_\sigma, \sigma, \beta) \in \tilde{\mathcal{U}}$;
- (ii) $\theta(y, \sigma, \beta) = r'$ if $\|y - y_\sigma\| = r'$.

Proof Since $\mathcal{U} = B(y_0, \sigma_0, \beta_0; r, \rho) \subseteq V \times \mathbb{R} \times (0, +\infty)$ is a special neighborhood of $\tilde{\mathcal{F}}$ with $0 < \rho < \beta_0$, then by Lemma 5, both $\mathcal{F}_\theta = (\mathcal{F}, \theta)$ and $\tilde{\mathcal{F}}_\theta = (\tilde{\mathcal{F}}, \theta)$ are \mathcal{U} -admissible. That is, the S^1 degrees of \mathcal{F}_θ and $\tilde{\mathcal{F}}_\theta$ are well-defined on \mathcal{U} . Note that we introduce the completing function θ in order to distinguish nontrivial solutions from trivial ones. See [8] for details.

Suppose, for contradiction, that the conclusion is not true. Then for any $r' \in (0, r]$, $\mathcal{F}_\theta = (\mathcal{F}, \theta)$ and $\tilde{\mathcal{F}}_\theta = (\tilde{\mathcal{F}}, \theta)$ are not homotopic on $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$. That is, any homotopy map between \mathcal{F}_θ and $\tilde{\mathcal{F}}_\theta$ has a zero on the boundary of $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$. In particular, the linear homotopy $h(\cdot, \alpha) := \alpha \mathcal{F}_\theta + (1 - \alpha) \tilde{\mathcal{F}}_\theta = (\alpha \mathcal{F} + (1 - \alpha) \tilde{\mathcal{F}}, \theta)$ has a zero on the boundary of $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$, where $\alpha \in [0, 1]$.

Note that $\theta(y, \sigma, \beta) > 0$ if $\|y - y_\sigma\| = r'$. Then, there exist (y, σ, β) and $\alpha \in [0, 1]$ such that $\|y - y_\sigma\| < r', |(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and

$$H(y, \sigma, \beta, \alpha) := \alpha \mathcal{F} + (1 - \alpha) \tilde{\mathcal{F}} = 0. \tag{2.28}$$

Since $r' > 0$ is arbitrary in the interval $(0, r]$, there exists a nonconstant sequence $\{(y_k, \sigma_k, \beta_k, \alpha_k)\}_{k=1}^\infty$ of solutions of (2.28) such that

$$\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\| = 0, |(\sigma_k, \beta_k) - (\sigma_0, \beta_0)| = \rho, 0 \leq \alpha_k \leq 1, \tag{2.29}$$

and

$$H(y_k, \sigma_k, \beta_k, \alpha_k) = 0, \text{ for all } k \in \mathbb{N}. \tag{2.30}$$

Note that $0 < \rho < \beta_0$ implies that $\beta_k \geq \beta_0 - \rho > 0$ for every $k \in \mathbb{N}$. From (2.29) we know that $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^\infty$ belongs to a compact subset of \mathbb{R}^3 . Therefore, there exist a convergent subsequence, denoted for notational simplicity by $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^\infty$ without loss of generality, and $(\sigma^*, \beta^*, \alpha^*) \in \mathbb{R}^3$ such that $\beta^* \geq \beta_0 - \rho > 0, \alpha^* \in [0, 1]$ and

$$\lim_{k \rightarrow +\infty} |(\sigma_k, \beta_k, \alpha_k) - (\sigma^*, \beta^*, \alpha^*)| = 0. \tag{2.31}$$

Similarly as in the proof of Lemma 5, we show that the system

$$\dot{v}(t) = \frac{1}{\beta^*} \partial_1 f(\sigma^*)v(t) + \frac{1}{\beta^*} \partial_2 f(\sigma^*)v(t - \beta^* z_{\sigma^*}) \tag{2.32}$$

with $\partial_i f(\sigma^*), \partial_i g(\sigma^*), i = 1, 2$, defined at (2.1), has a nonconstant periodic solution which contradicts the assumption that $B(u_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood which contains an isolated center of (2.9).

By (2.30), we know that the subsequence $\{(y_k, \sigma_k, \beta_k, \alpha_k)\}_{k=1}^\infty$ satisfies

$$H(y_k, \sigma_k, \beta_k, \alpha_k) = 0. \tag{2.33}$$

By (S1), $f: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \rightarrow f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ is C^2 in (θ_1, θ_2) . Then it follows from the Integral Mean Value Theorem and from (2.33) that

$$\begin{aligned} \dot{u}_k(t) &= \frac{\alpha_k}{\beta_k} \int_0^1 \partial_1 f_k(\sigma_k, s)(t) ds (y_k(t) - y_{\sigma_k}) \\ &+ \frac{\alpha_k}{\beta_k} \int_0^1 \partial_2 f_k(\sigma_k, s)(t) ds (y_k(t - \beta_k z_k(t)) - y_{\sigma_k}) \\ &+ \frac{1 - \alpha_k}{\beta_k} \int_0^1 \partial_1 f_k(\sigma_k, s)(t) ds (y_k(t) - y_{\sigma_k}) \\ &+ \frac{1 - \alpha_k}{\beta_k} \int_0^1 \partial_2 f_k(\sigma_k, s)(t) ds (y_k(t - \beta_k z_{\sigma_k}) - y_{\sigma_k}), \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} \partial_1 f_k(\sigma_k, s)(t) &:= \partial_1 f(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), y_{\sigma_k} + s(y_k(t - \beta_k z_k(t)) - y_{\sigma_k}), \sigma_k), \\ \partial_2 f_k(\sigma_k, s)(t) &:= \partial_2 f(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), y_{\sigma_k} + s(y_k(t - \beta_k z_k(t)) - y_{\sigma_k}), \sigma_k). \end{aligned}$$

Put

$$v_k(t) = \frac{y_k(t) - y_{\sigma_k}}{\|y_k - y_{\sigma_k}\|}. \tag{2.35}$$

Then we have

$$v_k(t - \beta_k z_k(t)) = \frac{y_k(t - \beta_k z_k(t)) - y_{\sigma_k}}{\|y_k - y_{\sigma_k}\|}. \tag{2.36}$$

By (2.34) and (2.36), we have

$$\begin{aligned} \dot{v}_k(t) &= \frac{\alpha_k}{\beta_k} \int_0^1 \partial_1 f_k(\sigma_k, s)(t) ds v_k(t) \\ &+ \frac{\alpha_k}{\beta_k} \int_0^1 \partial_2 f_k(\sigma_k, s)(t) ds v_k(t - \beta_k z_{\sigma_k}) \\ &+ \frac{1 - \alpha_k}{\beta_k} \int_0^1 \partial_1 f_k(\sigma_k, s)(t) ds v_k(t) \\ &+ \frac{1 - \alpha_k}{\beta_k} \int_0^1 \partial_2 f_k(\sigma_k, s)(t) ds v_k(t - \beta_k z_{\sigma_k}). \end{aligned} \tag{2.37}$$

We show that there exists a convergent subsequence of $\{v_k\}_{k=1}^{+\infty}$. Indeed, by (2.29) we know that $\{z_k, \beta_k\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R}; \mathbb{R}) \times \mathbb{R}$. Therefore we have

$$\lim_{t \rightarrow +\infty} t - \beta_k z_k(t) = +\infty. \tag{2.38}$$

By (2.35), (2.36) and (2.38), we have $\|v_k\| = 1, \|v_k(\cdot - \beta_k z_k)\| = 1$. By (S1) and (2.31) and by a similar argument yielding (2.22), we know that

$$\lim_{k \rightarrow +\infty} \|\partial_1 f_k(\sigma_k, s) - \partial_1 f(\sigma^*)\| = 0, \quad \lim_{k \rightarrow +\infty} \|\partial_2 f_k(\sigma_k, s) - \partial_2 f(\sigma^*)\| = 0, \tag{2.39}$$

uniformly for $s \in [0, 1]$. We know from (2.39) that $\|\partial_1 f_k(\sigma_k, s)\|, \|\partial_2 f_k(\sigma_k, s)\|$, are both uniformly bounded for every $k \in \mathbb{N}$ and $s \in [0, 1]$. It follows from (2.37) that there exists $\tilde{L}_3 > 0$ such that $\|\dot{v}_k\| < \tilde{L}_3$ for every $k \in \mathbb{N}$. By the Arzela–Ascoli Theorem, there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{+\infty}$ of $\{v_k\}_{k=1}^{+\infty}$. That is, there exists $v^* \in \{v \in V : \|v\| = 1\}$ such that

$$\lim_{j \rightarrow +\infty} \|v_{k_j} - v^*\| = 0. \tag{2.40}$$

By the Integral Mean Value Theorem, we obtain for all $t \in \mathbb{R}$,

$$\begin{aligned} & |v_{k_j}(t - \beta_{k_j} z_{k_j}(t)) - v_{k_j}(t - \beta^* z_{\sigma^*})| \\ &= \left| \int_0^1 \dot{v}_{k_j}(t - \beta^* z_{\sigma^*} - \theta(\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*})) d\theta(\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*}) \right| \\ &\leq \|\dot{v}_{k_j}\| \cdot |\beta_{k_j} z_{k_j}(t) - \beta^* z_{\sigma^*}| \\ &\leq \tilde{L}_3(\beta_{k_j} |z_{k_j}(t) - z_{\sigma^*}| + |\beta_{k_j} - \beta^*| z_{\sigma^*}). \end{aligned} \tag{2.41}$$

Then by (2.31) and (2.41) we have

$$\lim_{j \rightarrow +\infty} \|v_{k_j}(\cdot - \beta_{k_j} z_{k_j}(\cdot)) - v_{k_j}(\cdot - \beta^* z_{\sigma^*})\| = 0. \tag{2.42}$$

From (2.40) and (2.42) we have

$$\lim_{j \rightarrow +\infty} \|v_{k_j}(\cdot - \beta_{k_j} z_{k_j}(\cdot)) - v^*(\cdot - \beta^* z_{\sigma^*})\| = 0. \tag{2.43}$$

It follows from (2.31), (2.39), (2.40) and (2.43) that the right hand side of (2.37) converges uniformly to the right hand side of (2.32). Therefore,

$$\lim_{j \rightarrow +\infty} |\dot{v}_{k_j}(t) - \dot{v}^*(t)| = 0, \tag{2.44}$$

and

$$\dot{v}^*(t) = \frac{1}{\beta^*} \partial_1 f(\sigma^*) v^*(t) + \frac{1}{\beta^*} \partial_2 f(\sigma^*) v^*(t - \beta^* \tau_{\sigma^*}). \tag{2.45}$$

Noticing that $v^* \in \{v : \|v\| = 1\}$, we have $\|v^*\| \neq 0$. Since the matrix $\partial_1 f(\sigma^*) + \partial_2 f(\sigma^*)$ is nonsingular, v^* is a nonconstant periodic solution of (2.45). Then by Lemma 4 ($y_{\sigma^*}, \sigma^*, \beta^*$) is also a center of (2.9) in $B(y_0, \sigma_0, \beta_0; r, \rho)$. This contradicts the assumption that $B(y_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood of (2.9) which contains only one center (y_0, σ_0, β_0) . This completes the proof. \square

Now we are in the position to prove a local Hopf bifurcation theorem for system (1.5).

Theorem 1 *Assume (S1)–(S3) hold. Let (x_{σ_0}, σ_0) be an isolated center of system (2.1). If the crossing number defined by (2.5) satisfies*

$$\gamma(x_{\sigma_0}, \sigma_0, \beta_0) \neq 0,$$

then there exists a bifurcation of nonconstant periodic solutions of (1.5) near (x_{σ_0}, σ_0) . More precisely, there exists a sequence $\{(x_n, \sigma_n, \beta_n)\}$ such that $\sigma_n \rightarrow \sigma_0, \beta_n \rightarrow \beta_0$ as $n \rightarrow \infty$, and $\lim_{n \rightarrow \infty} \|x_n - x_{\sigma_0}\| = 0$, where

$$(x_n, \sigma_n) \in C(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}$$

is a nonconstant $2\pi/\beta_n$ -periodic solution of system (1.5).

Proof Let (x, τ) be a solution of system (1.5) with x being $2\pi/\beta$ -periodic and $\beta > 0$. Let $(x(t), \tau(t)) = (y(\beta t), z(\beta t))$. Then system (1.5) is transformed to

$$\begin{cases} \dot{y}(t) = \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma), \\ z(t) = g(y(t), y(t - \beta z_\sigma), \sigma). \end{cases} \tag{2.46}$$

Let $V = C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$. For any $\xi = e^{i\nu} \in S^1, u \in V, (\xi u)(t) := u(t + \nu)$. Recall that δ and ε are defined before (2.5). Let $\mathcal{D}(\sigma_0, \beta_0) = (\sigma_0 - \delta, \sigma_0 + \delta) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$ and define the maps

$$\begin{aligned} L_0 y(t) &:= \dot{y}(t), \quad y \in C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N), \\ N_0(y, \sigma, \beta)(t) &:= f(y(t), y(t - \beta z(t)), \sigma), \quad y \in V, \\ \tilde{N}_0(y, \sigma, \beta)(t) &:= \partial_1 f(\sigma)(y(t) - y_\sigma) + \partial_2 f(\sigma)(y(t - \beta z_\sigma) - y_\sigma), \quad y \in V, \end{aligned}$$

where $(\sigma, \beta) \in \mathcal{D}(\sigma_0, \beta_0)$ and $t \in \mathbb{R}$, and (y_σ, z_σ) is the stationary state of the system at σ such that $y_{\sigma_0} = x_{\sigma_0}$. The space V is a Banach representation of the group $G = S^1$.

Define the operator $K : V \rightarrow \mathbb{R}^N$ by

$$K(y) := \frac{1}{2\pi} \int_0^{2\pi} y(t) dt, \quad y \in V.$$

By Lemma 2, the operator $L_0 + K : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow V$ has a compact inverse $(L_0 + K)^{-1} : V \rightarrow V$. Then, finding a $2\pi/\beta$ -periodic solution for the system (1.5) is equivalent to finding a solution of the following fixed point problem:

$$y = (L_0 + K)^{-1} \left[\frac{1}{\beta} N_0(y, \sigma, \beta) + K(y) \right], \tag{2.47}$$

where $(y, \sigma, \beta) \in V \times \mathbb{R} \times (0, +\infty)$.

Define the following maps $\mathcal{F} : V \times \mathbb{R} \times (0, +\infty) \rightarrow V$ and $\tilde{\mathcal{F}} : V \times \mathbb{R} \times (0, +\infty) \rightarrow V$ by

$$\begin{aligned} \mathcal{F}(y, \sigma, \beta) &:= y - (L_0 + K)^{-1} \left[\frac{1}{\beta} N_0(y, \sigma, \beta) + K(y) \right], \\ \tilde{\mathcal{F}}(y, \sigma, \beta) &:= y - (L_0 + K)^{-1} \left[\frac{1}{\beta} \tilde{N}_0(y, \sigma, \beta) + K(y) \right]. \end{aligned}$$

Finding a $2\pi/\beta$ -periodic solution of system (1.5) is equivalent to finding the solution of the problem

$$\mathcal{F}(y, \sigma, \beta) = 0, \quad (y, \sigma, \beta) \in V \times \mathbb{R} \times (0, +\infty).$$

By results in [6], it is sufficient to verify the following conditions:

- (A1) V has an S^1 -isotypical decomposition $V = \overline{\bigoplus_{k=0}^\infty V_k}$ and for each integer $k = 0, 1, 2, \dots$, the subspace V_k is of finite dimension.
- (A2) There exists a compact resolvent K of L_0 such that for every fixed parameter $(\sigma, \beta) \in \mathbb{R}^2, (L_0 + K)^{-1} \circ [N_0(\cdot, \sigma, \beta) + K] : V \rightarrow V$ is a condensing map.
- (A3) There exists a 2-dimensional submanifold $M \subset V_0 \times \mathbb{R}^2$ such that i) $M \subset \mathcal{F}^{-1}(0)$; ii) if $(y_0, \sigma_0, \beta_0) \in M$, then there exists an open neighborhood $U_{(\sigma_0, \beta_0)}$ of (σ, β) in \mathbb{R}^2 , an open neighborhood U_{y_0} of U_0 in V_0 , and a C^1 -map $\eta : U_{(\sigma_0, \beta_0)} \rightarrow U_{y_0}$ such that $M \cap (U_{y_0} \times U_{(\sigma_0, \beta_0)}) = \{(\eta(\sigma, \beta), (\sigma, \beta)) : (\sigma, \beta) \in U_{(\sigma_0, \beta_0)}\}$.
- (A4) $M \subset \tilde{\mathcal{F}}^{-1}(0)$ and for every fixed parameter $(\sigma, \beta) \in \mathbb{R}^2, (L_0 + K)^{-1} \circ [\tilde{N}_0(\cdot, \sigma, \beta) + K] : V \rightarrow V$ is a condensing map.

- (A5) There exist $r > 0$ and $\rho > 0$ so that $B(y_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood of $\tilde{\mathcal{F}}$ and there exists $r' \in (0, r]$ such that $\mathcal{F}(y, \sigma, \beta) \neq 0$ for $(y, \sigma, \beta) \in \overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$ with $y \neq \eta(\sigma, \beta)$ and $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$.
- (A6) $D_u\tilde{\mathcal{F}}(y_0, \sigma_0, \beta_0) : V_0 \rightarrow V_0$ is an isomorphism.

By (S1) we know that the linear operator \tilde{N}_0 is continuous. By Lemma 3, we know that $N_0(\cdot, \sigma, \beta) : V \rightarrow V$ is continuous. Moreover, by Lemma 2 the operator $(L_0 + K)^{-1} : V \rightarrow V$ is compact and hence $(L_0 + K)^{-1} \circ (\frac{1}{\beta}N_0(\cdot, \alpha, \beta) + K) : V \rightarrow V$ and $(L_0 + K)^{-1} \circ (\frac{1}{\beta}\tilde{N}_0(\cdot, \alpha, \beta) + K) : V \rightarrow V$ are completely continuous and hence are condensing maps. That is, (A2) and (A4) are satisfied.

Since $(x_{\sigma_0}, \sigma_0) = (y_{\sigma_0}, \sigma_0)$ is an isolated center of system (2.1) with a purely imaginary characteristic value $i\beta_0, \beta_0 > 0, (y_{\sigma_0}, \sigma_0, \beta_0) \in V \times \mathbb{R} \times (0, +\infty)$ is an isolated V -singular point of $\tilde{\mathcal{F}}$. That is, $(y_{\sigma_0}, \sigma_0, \beta_0)$ is the only point in V such that the derivative $D_y\mathcal{F}(y_{\sigma_0}, \sigma_0, \beta_0)$ is not an automorphism of V . We define the following two-dimensional submanifold $M \subset V_0 \times \mathbb{R} \times (0, +\infty)$ by

$$M := \{(y_\sigma, \sigma, \beta) : \sigma \in (\sigma_0 - \delta, \sigma_0 + \delta), \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\},$$

such that the point $(y_{\sigma_0}, \sigma_0, \beta_0)$ is the only V -singular point of $\tilde{\mathcal{F}}$ in M . M is the set of trivial solutions to the system (2.1) and satisfies the assumption (A3).

Since $(y_{\sigma_0}, \sigma_0, \beta_0) \in V \times \mathbb{R} \times (0, +\infty)$ is an isolated V -singular point of $\tilde{\mathcal{F}}$, for $\rho > 0$ sufficiently small, the linear operator $D_u\tilde{\mathcal{F}}(y_\sigma, \sigma, \beta) : V \rightarrow V$ with $|(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho$, is not an automorphism only if $(\sigma, \beta) = (\sigma_0, \beta_0)$. Then, by the Implicit Function Theorem, there exists $r > 0$ such that for every $(y, \sigma, \beta) \in V \times \mathbb{R} \times (0, +\infty)$ with $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and $0 < \|y - y_\sigma\| \leq r$, we have $\tilde{\mathcal{F}}(y, \sigma, \beta) \neq 0$. Then the set $B(x_0, \sigma_0, \beta_0; r, \rho)$ defined by

$$\{(y, \sigma, \beta) \in V \times \mathbb{R} \times (0, +\infty); |(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho, \|y - y_\sigma\| < r\},$$

is a special neighborhood for $\tilde{\mathcal{F}}$.

By Lemma 5, there exists a special neighborhood $\mathcal{U} = \overline{B(y_{\sigma_0}, \sigma_0, \beta_0; r', \rho)}$ such that \mathcal{F} and $\tilde{\mathcal{F}}$ are nonzero for $(y, \sigma, \beta) \in \overline{B(y_{\sigma_0}, \sigma_0, \beta_0; r', \rho)}$ with $y \neq y_\sigma$ and $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$. That is, (A5) is satisfied.

Let θ be a completing function on \mathcal{U} . It follows from Lemma 6 that (\mathcal{F}, θ) is homotopic to $(\tilde{\mathcal{F}}, \theta)$ on \mathcal{U} .

It is known that V has the following isotypical direct sum decomposition

$$V = \overline{\bigoplus_{k=0}^{\infty} V_k},$$

where V_0 is the space of all constant mappings from \mathbb{R} into \mathbb{R}^N , and V_k with $k > 0, k \in \mathbb{N}$ is the vector space of all mappings of the form

$$x \cos k \cdot + y \sin k \cdot : \mathbb{R} \ni t \rightarrow x \cos kt + y \sin kt \in \mathbb{R}^N,$$

where $x, y \in \mathbb{R}^N$. Then $V_k, k > 0, k \in \mathbb{N}$, are finite dimensional. Then, (A1) is satisfied.

For $(\sigma, \beta) \in \mathcal{D}(\sigma_0, \beta_0)$, we denote by $\Psi(\sigma, \beta)$ the map $D_y\tilde{\mathcal{F}}(y(\sigma), \sigma, \beta) : V \rightarrow V$. Then we have $\Psi(\sigma, \beta)(V_k) \subset V_k$ for all $k = 0, 1, 2, \dots$. Therefore, we can define $\Psi_k : \mathcal{D}(\sigma_0, \beta_0) \rightarrow L(V_k, V_k)$ by

$$\Psi_k(\sigma, \beta) := \Psi(\sigma, \beta)|_{V_k}.$$

We note that $V_k, k \geq 1, k \in \mathbb{N}$, can be endowed with the natural complex structure $J : V_k \rightarrow V_k$ defined by

$$J(x \cos k \cdot + y \sin k \cdot) = -(x \sin k \cdot + y \cos k \cdot), \quad x, y \in \mathbb{R}^N.$$

By extending the linearity of J to the vector space spanned over the field of complex numbers by $e^{ik \cdot} \cdot \epsilon_j : \mathbb{R} \ni t \rightarrow e^{ikt} \cdot \epsilon_j \in \mathbb{C}^N, j = 1, 2, \dots, N$, we know that

$$\{e^{ik \cdot} \cdot \epsilon_j, J(e^{ik \cdot} \cdot \epsilon_j)\}_{j=1}^N = \{e^{ik \cdot} \cdot \epsilon_j, ie^{ik \cdot} \cdot \epsilon_j\}_{j=1}^N$$

is a basis of V_k , where $\{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$ denotes the standard basis of \mathbb{R}^N . Then we identify V_k with the vector space over the complex numbers spanned by $e^{ik \cdot} \cdot \epsilon_j, j = 1, 2, \dots, N$.

Then we have for $v_k \in V_k, k \in \mathbb{Z}, k \geq 1$,

$$\begin{aligned} \Psi_k(\sigma, \beta)v_k &= v_k - (L_0 + K)^{-1} \left(\frac{1}{\beta} D_u \tilde{N}_0(u(\sigma), \sigma, \beta) + K \right) v_k \\ &= v_k - \frac{1}{\beta} (L_0 + K)^{-1} (\partial_1 f(\sigma)v_k + \partial_2 f(\sigma)(v_k)_{\beta z_\sigma}), \end{aligned}$$

where $(v_k)_{\beta z_\sigma} = v_k(\cdot - \beta z_\sigma)$. Then we have, for $e^{ik \cdot} \epsilon_j \in V_k$,

$$\begin{aligned} &\Psi_k(\sigma, \beta)(e^{ik \cdot} \epsilon_j) \\ &= \frac{1}{ik\beta} \left(ik\beta \text{Id} - \partial_1 f(\sigma) - \partial_2 f(\sigma)e^{-ik\beta z_\sigma} \right) \cdot (e^{ik \cdot} \epsilon_j) \\ &= \frac{1}{ik\beta} \Delta_{(u(\sigma), \sigma)}(ik\beta) \cdot (e^{ik \cdot} \epsilon_j), \end{aligned}$$

where the last equality follows from (2.3). Therefore, the matrix representation $[\Psi_k]$ of $\Psi_k(\sigma, \beta)$ with respect to the ordered \mathbb{C} -basis $\{e^{ik \cdot} \epsilon_j\}_{j=1}^N$ is given by

$$\frac{1}{ik\beta} \Delta_{(y_\sigma, \sigma)}(ik\beta).$$

Next we show that there exists some $k \in \mathbb{Z}, k \geq 1$, such that $\mu_k(y_{\sigma_0}, \sigma_0, \beta_0) := \deg_B(\det_{\mathbb{C}}[\Psi_k]) \neq 0$.

Define $\Psi_H : \mathcal{D}(\sigma_0, \beta_0) \rightarrow \mathbb{R}^2 \simeq \mathbb{C}$ by

$$\Psi_H(\sigma, \beta) = \det \Delta_{(y_\sigma, \sigma)}(i\beta).$$

The number $\mu_1(y_{\sigma_0}, \sigma_0, \beta_0)$ can be written as follows (see Theorem 7.1.5 of [8]):

$$\mu_1(u(\sigma_0), \sigma_0, \beta_0) = \epsilon \cdot \deg(\Psi_H, \mathcal{D}(\sigma_0, \beta_0)),$$

where $\epsilon = \text{sign det } \Psi_0(\sigma, \beta)$ for $(\sigma, \beta) \in \mathcal{D}(\sigma_0, \beta_0)$. For a constant map $v_0 \in V_0$,

$$\Psi_0(\sigma, \beta)v_0 = -\frac{1}{\beta}(\partial_1 f(\sigma) + \partial_2 f(\sigma))v_0.$$

Then, by (S3), we have $\epsilon \neq 0$ and therefore (A6) is satisfied.

Note that $\alpha_0, \beta_0, \delta$ and ε are chosen at (2.5). Define the function $H : [\sigma_0 - \delta, \sigma_0 + \delta] \times \overline{\Omega} \rightarrow \mathbb{R}^2 \simeq \mathbb{C}$ by

$$H(\sigma, \alpha, \beta) := \det \Delta_{(y_\sigma, \sigma)}(\alpha + i\beta),$$

where $\Omega = (0, \alpha_0) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$, $\alpha_0 = \alpha_0(\sigma_0, \beta_0) > 0$. By the same argument for (2.4) and (2.5), we know that H satisfies all the conditions of Lemma 2.1 of [6] (or Lemma 7.2.1 of [8]) by the choice of $\alpha_0, \beta_0, \varepsilon$ and δ . So we have

$$\text{deg}(\Psi_H, \mathcal{D}(\sigma_0, \beta_0)) = \gamma(y_{\sigma_0}, \sigma_0, \beta_0) \neq 0.$$

Thus, $\mu_1(y_{\sigma_0}, \sigma_0, \beta_0) \neq 0$ which, by Theorem 2.4 of [6], implies that $(y_{\sigma_0}, \sigma_0, \beta_0)$ is a bifurcation point of the system (2.46). Consequently, there exists a sequence of non-constant periodic solutions (x_n, σ_n, β_n) such that $\sigma_n \rightarrow \sigma_0, \beta_n \rightarrow \beta_0$ as $n \rightarrow \infty$, and x_n is a $2\pi/\beta_n$ -periodic solution of (1.5) such that the associated pair (x_n, τ_n) with $\tau_n(t) = g(x_n(t), x_n(t - \tau_n(t)), \sigma_n)$ satisfies (1.5) with $\lim_{n \rightarrow +\infty} \|(x_n, \tau_n) - (x_{\sigma_0}, \tau_{\sigma_0})\| = 0$. \square

3 Local Hopf Bifurcation for System (1.1)

Now we consider the local Hopf bifurcation problem of system (1.1). By Lemma 1, we know that if $x \in C^1(\mathbb{R}; \mathbb{R}^N)$ is p -periodic and is in a small neighborhood $O(\epsilon)$ of x_σ , there exists a unique p -periodic $\tau \in C^1(\mathbb{R}; \mathbb{R})$ such that $\tau(t) = g(x(t), x(t - \tau(t)), \sigma)$, $t \in \mathbb{R}$. We call x a solution if (x, τ) satisfies system (1.1).

For a stationary state x_{σ_0} of system (1.1) with the parameter σ_0 , we say that (x_{σ_0}, σ_0) is a Hopf bifurcation point of system (1.1), if there exist a sequence $\{(x_k, \sigma_k, T_k)\}_{k=1}^{+\infty} \subseteq C^1(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}^2$ and $T_0 > 0$ such that

$$\lim_{k \rightarrow +\infty} \|(x_k, \sigma_k, T_k) - (x_{\sigma_0}, \sigma_0, T_0)\|_{C^1(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}^2} = 0,$$

and (x_k, σ_k) is a nonconstant T_k -periodic solution of system (1.1).

We freeze the state-dependent delay in system (1.1) at its stationary state and linearize the resulting differential equation of x with constant delay at the stationary state. For $\sigma \in (\sigma_0 - \epsilon_0, \sigma_0 + \epsilon_0)$, the following formal linearization of system (1.1) at the stationary point x_σ :

$$\dot{x}(t) = \partial_1 f(\sigma) (x(t) - x_\sigma) + \partial_2 f(\sigma) (x(t - \tau_\sigma) - x_\sigma), \tag{3.1}$$

where

$$\partial_1 f(\sigma) := \partial_1 f(x_\sigma, \tau_\sigma, \sigma), \quad \partial_2 f(\sigma) := \partial_2 f(x_\sigma, \tau_\sigma, \sigma), \quad \tau_\sigma = g(x_\sigma, x_\sigma, \sigma).$$

Notice that the system (3.1) is the same as system (2.1) and hence they share the same characteristic equations.

Let (x_{σ_0}, σ_0) be an isolated center of (2.1) and let $O(\epsilon^*) \subset C^1(\mathbb{R}; \mathbb{R}^N)$ be a neighborhood of x_{σ_0} . In the following we confine the discussion with $x \in O(\epsilon^*) \subset C^1(\mathbb{R}; \mathbb{R}^N)$, where by Lemma 1, $\epsilon^* > 0$ is chosen so that every p -periodic $x \in O(\epsilon^*) \subset C^1(\mathbb{R}; \mathbb{R}^N)$ determines a unique continuously differentiable p -periodic τ .

Now we formulate the Hopf bifurcation problem as a fixed point problem in $C^1(\mathbb{R}; \mathbb{R}^N)$. We normalize the period of the $2\pi/\beta$ -periodic solution $x \in O(\epsilon^*)$ of (1.5) and the associated $\tau \in C^1(\mathbb{R}; \mathbb{R})$ by setting $(x(t), \tau(t)) = (y(\beta t), z(\beta t))$. We obtain

$$\begin{pmatrix} \dot{y}(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma) \\ g(y(t), y(t - \beta z(t)), \sigma) \end{pmatrix}. \tag{3.2}$$

Let $W = O(\epsilon^*) \cap C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$. Define $N_1 : W \ni (y, \sigma, \beta) \times \mathbb{R}^2 \rightarrow N_1(y, \sigma, \beta) \in C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ by

$$N_1(y, \sigma, \beta)(t) = f(y(t), y(t - \beta z(t)), \sigma), \tag{3.3}$$

where z is chosen according to y in light of Lemma 1 so that (y, z) is a solution of the second line of (3.2). Then the equation for \dot{y} in system (3.2) is rewritten as

$$\dot{y}(t) = \frac{1}{\beta} N_1(y, \sigma, \beta)(t). \tag{3.4}$$

Correspondingly, (3.1) is transformed into

$$\dot{y}(t) = \frac{1}{\beta} \tilde{N}_1(y, \sigma, \beta)(t), \tag{3.5}$$

where $\tilde{N}_1 : W \ni (y, \sigma, \beta) \times \mathbb{R}^2 \rightarrow \tilde{N}_1(y, \sigma, \beta) \in C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ is defined by

$$\tilde{N}_1(y, \sigma, \beta)(t) = \partial_1 f(\sigma)(y(t) - y_\sigma) + \partial_2 f(\sigma)(y(t - \beta z_\sigma) - y_\sigma).$$

with $(y_\sigma, z_\sigma) = (x_\sigma, \tau_\sigma)$. We note that y is 2π -periodic if and only if x is $(2\pi/\beta)$ -periodic.

Let $L_0 : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ be defined by $L_0 y(t) = \dot{y}(t), t \in \mathbb{R}$ and $K : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow \mathbb{R}^N$ be defined by

$$K(y) = \frac{1}{2\pi} \int_0^{2\pi} y(t) dt. \tag{3.6}$$

Define the map $\tilde{\mathcal{F}} : W \times \mathbb{R}^2 \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ by

$$\tilde{\mathcal{F}}(y, \sigma, \beta) := y - (L_0 + K)^{-1} \left[\frac{1}{\beta} \tilde{N}_0(y, \sigma, \beta) + K(y) \right]. \tag{3.7}$$

We suppose that the set defined by

$$B(y_0, \sigma_0, \beta_0; r, \rho) = \{(y, \sigma, \beta) : \|y - y_\sigma\|_{C^1} < r, |(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho\},$$

is a special neighborhood of $\tilde{\mathcal{F}}$ which satisfies

- (i) $\tilde{\mathcal{F}}(y, \sigma, \beta) \neq 0$ for every $(y, \sigma, \beta) \in \overline{B(y_0, \sigma_0, \beta_0; r, \rho)}$ with $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and $\|y - y_\sigma\|_{C^1} \neq 0$;
- (ii) (y_0, σ_0, β_0) is the only isolated center in $\overline{B(y_0, \sigma_0, \beta_0; r, \rho)}$.

Before we state and prove the local Hopf bifurcation theorem for system (1.1), we need the following technical Lemmas.

Lemma 7 *Let $L_0 : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ be defined by $L_0 y(t) = \dot{y}(t), t \in \mathbb{R}$ and let $K : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow \mathbb{R}^N$ be defined at (3.6). Then the inverse $(L_0 + K)^{-1} : C_{2\pi}(\mathbb{R}; \mathbb{R}^N) \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ exists and is continuous.*

Proof By the proof of Lemma 3.1 in [6], $L_0 + K : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ is one-to-one and onto. Moreover, $(L_0 + K)^{-1} : C_{2\pi}(\mathbb{R}; \mathbb{R}^N) \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ is continuous. □

Lemma 8 *For any $\sigma \in \mathbb{R}$ and $\beta > 0$, the map $N_1(\cdot, \sigma, \beta) : W \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ defined by (3.3) is continuous.*

Proof Let $\{y_n\}_{n=1}^\infty \subset W$ be a convergent sequence with limit $y_0 \in W$. By Lemma 1, $\{y_n\}_{n=1}^\infty \subset W$ uniquely determines a sequence $\{z_n\}_{n=1}^\infty \subset C_{2\pi}^1(\mathbb{R}; \mathbb{R})$ satisfying

$$z_n(t) = g(y_n(t), y_n(t - \beta z_n(t)), \sigma), t \in \mathbb{R}.$$

Moreover, there exists $z_0 \in C_{2\pi}^1(\mathbb{R}; \mathbb{R})$ such that $z_0(t) = g(y_0(t), y_0(t - \beta z_0(t)), \sigma), t \in \mathbb{R}$ and by Lemma 1 we have,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi]} |z_n(t) - z_0(t)| = 0. \tag{3.8}$$

By taking derivatives on both sides of $z_n(t) = g(y_n(t), y(t - \beta z_n(t)), \sigma)$ we obtain that

$$\dot{z}_n(t) = \frac{\partial_1 g(y_n(t), y_n(t - \beta z_n(t)), \sigma) \dot{y}_n(t)}{1 + \beta \partial_2 g(y_n(t), y_n(t - \beta z_n(t)), \sigma) \dot{y}_n(t - \beta z_n(t))}. \tag{3.9}$$

Notice that, we have

$$\dot{z}_0(t) = \frac{\partial_1 g(y_0(t), y_0(t - \beta z_0(t)), \sigma) \dot{y}_0(t)}{1 + \beta \partial_2 g(y_0(t), y_0(t - \beta z_0(t)), \sigma) \dot{y}_0(t - \beta z_0(t))}. \tag{3.10}$$

Using the Triangle Inequality and the Integral Mean Value Theorem, and noticing from (S2) that g is C^2 , we can show that

$$\begin{aligned} & \sup_{t \in [0, 2\pi]} \left| \frac{\partial_1 g(y_n(t), y_n(t - \beta z_n(t)), \sigma) \dot{y}_n(t)}{1 + \beta \partial_2 g(y_n(t), y_n(t - \beta z_n(t)), \sigma) \dot{y}_n(t - \beta z_n(t))} \right. \\ & \quad \left. - \frac{\partial_1 g(y_0(t), y_0(t - \beta z_0(t)), \sigma) \dot{y}_0(t)}{1 + \beta \partial_2 g(y_0(t), y_0(t - \beta z_0(t)), \sigma) \dot{y}_0(t - \beta z_0(t))} \right| \\ & \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.11}$$

Therefore, by (3.9) and (3.10) we have $\lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi]} |\dot{z}_n(t) - \dot{z}_0(t)| = 0$ which combined with (3.8) leads to $\lim_{n \rightarrow \infty} \|z_n - z_0\|_{C^1} = 0$.

Next we show that $N_1 : W \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ defined by $N_1(y, \sigma, \beta)(t) = f(y(t), y(t - \beta z(t)), \sigma)$ is continuous. That is,

$$\lim_{n \rightarrow \infty} \|N_1(y_n(t), y_n(t - \beta z_n(t)), \sigma) - N_1(y_0(t), y_0(t - \beta z_0(t)), \sigma)\|_{C^1} = 0. \tag{3.12}$$

By the proof of Lemma 3.2 in [6], we know that the restriction $N_1|_{C_{2\pi}(\mathbb{R}; \mathbb{R}^N)}$ is a continuous map from $C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ to $C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$. Therefore, we have

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi]} |f(y_n(t), y_n(t - \beta z_n(t)), \sigma) - f(y_0(t), y_0(t - \beta z_0(t)), \sigma)| = 0. \tag{3.13}$$

Moreover, since $\lim_{n \rightarrow \infty} \|y_n - y_0\|_{C^1} = 0$ and $\lim_{n \rightarrow \infty} \|z_n - z_0\|_{C^1} = 0$. We can use the Triangle Inequality and the Integral Mean Value Theorem to obtain that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi]} \left| \frac{d}{dt} N_1(y_n, y_n(t - \tau_n(t)), \sigma) - \frac{d}{dt} N_1(y_0, y_0(t - \tau_0(t)), \sigma) \right| \\ & = \lim_{n \rightarrow \infty} \sup_{t \in [0, 2\pi]} |\partial_1 f(y_n(t), y_n(t - \beta z_n(t)), \sigma) \dot{y}_n(t) + \partial_2 f(y_n(t), y_n(t - \beta z_n(t)), \sigma) \\ & \quad \times \dot{y}_n(t - \beta z_n(t))(1 - \beta \dot{z}_n(t)) - \partial_1 f(y_0(t), y_0(t - \beta z_0(t)), \sigma) \dot{y}_0(t)| \end{aligned}$$

$$\begin{aligned}
 & -\partial_2 f(y_0(t), y_0(t - \beta z_0(t)), \sigma) \\
 & \times \dot{y}_0(t - \beta z_0(t))(1 - \beta \dot{z}_0(t)), \\
 & = 0.
 \end{aligned}
 \tag{3.14}$$

By (3.13) and (3.14), $N_1 : W \rightarrow C^1_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ is continuous. □

To establish the S^1 -degree on some special neighborhood near the stationary state, we have

Lemma 9 *Assume (S1)–(S3) hold. Let L_0 and K be as in Lemma 7 and $\tilde{N}_1 : W \times \mathbb{R} \times (0, +\infty) \rightarrow C^1_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ be as in (3.5). Let $\tilde{\mathcal{F}} : W \times \mathbb{R} \times (0, +\infty) \rightarrow C^1_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ be defined at (2.11). If $B(y_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood of $\tilde{\mathcal{F}}$ with $0 < \rho < \beta_0$, then there exists $r' \in (0, r]$ such that the neighborhood*

$$B(y_0, \sigma_0, \beta_0; r', \rho) = \{(u, \sigma, \beta) : \|y - y_\sigma\|_{C^1} < r', |(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho\}$$

satisfies

$$\dot{y}(t) \neq \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma)$$

for $(y, \sigma, \beta) \in \overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$ with $y \neq y_\sigma$ and $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$.

Proof We prove by contradiction. Suppose the statement is not true, then for any $0 < r' \leq r$, there exists (y, σ, β) such that $0 < \|y - y_\sigma\|_{C^1} < r'$, $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and

$$\dot{y}(t) = \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma) \text{ for } t \in \mathbb{R}.
 \tag{3.15}$$

Then there exists a sequence of nonconstant periodic solutions $\{(y_k, \sigma_k, \beta_k)\}_{k=1}^\infty$ of (3.15) such that

$$\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\|_{C^1} = 0, \quad |(\sigma_k, \beta_k) - (\sigma_0, \beta_0)| = \rho,
 \tag{3.16}$$

and

$$\dot{y}_k(t) = \frac{1}{\beta_k} f(y_k(t), y_k(t - \beta_k z_k(t)), \sigma_k) \text{ for } t \in \mathbb{R},
 \tag{3.17}$$

where z_k is chosen according to y_k in light of Lemma 1 so that (y_k, z_k) is a solution of system (3.2).

Note that $0 < \rho < \beta_0$ implies that $\beta_k \geq \beta_0 - \rho > 0$ for every $k \in \mathbb{N}$. Also, since the sequence $\{\sigma_k, \beta_k\}_{k=1}^\infty$ belongs to a bounded neighborhood of (σ_0, β_0) in \mathbb{R}^2 , there exists a convergent subsequence, still denoted by $\{(\sigma_k, \beta_k)\}_{k=1}^\infty$ for notational simplicity, that converges to (σ^*, β^*) so that $|(\sigma^*, \beta^*) - (\sigma_0, \beta_0)| = \rho$ and $\beta^* > 0$. Then we have

$$\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\|_{C^1} = 0, \quad \lim_{k \rightarrow +\infty} |(\sigma_k, \beta_k) - (\sigma^*, \beta^*)| = 0,
 \tag{3.18}$$

and

$$\dot{y}_k(t) = \frac{1}{\beta_k} f(y_k(t), y_k(t - \beta_k z_k(t)), \sigma_k) \text{ for } t \in \mathbb{R}.
 \tag{3.19}$$

We need to show that the system

$$\dot{v}(t) = \frac{1}{\beta^*} \partial_1 f(\sigma^*)v(t) + \frac{1}{\beta^*} \partial_2 f(\sigma^*)v(t - \beta^* z_{\sigma^*}),
 \tag{3.20}$$

has a nonconstant periodic solution which contradicts the assumption that $(y_{\sigma_0}, \sigma_0, \beta_0)$ is the only center of (3.5) in $\overline{B(y_0, \sigma_0, \beta_0; r, \rho)}$. But (3.18) implies that

$$\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\| = 0, \quad \lim_{k \rightarrow +\infty} |(\sigma_k, \beta_k) - (\sigma^*, \beta^*)| = 0, \tag{3.21}$$

Then by the same argument in the proof of Lemma 5, (3.20) has a nonconstant periodic solution. This completes the proof. \square

To apply the homotopy argument of S^1 -degree, we show the following

Lemma 10 *Assume (S1)–(S3) hold. Let $L_0, K, \tilde{N}_1, \tilde{\mathcal{F}}$ be as in Lemma 9 and $N_1 : W \times \mathbb{R}^2 \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ be as in (3.2). Define the map $\mathcal{F} : W \times \mathbb{R} \times (0, +\infty) \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ by*

$$\mathcal{F}(y, \sigma, \beta) := y - (L_0 + K)^{-1} \left[\frac{1}{\beta} N_1(y, \sigma, \beta) + K(y) \right].$$

If $\mathcal{U} = B(y_0, \sigma_0, \beta_0; r, \rho) \subseteq W \times \mathbb{R} \times (0, +\infty)$ is a special neighborhood of $\tilde{\mathcal{F}}$ with $0 < \rho < \beta_0$, then there exists $r' \in (0, r]$ such that $\mathcal{F}_\theta = (\mathcal{F}, \theta)$ and $\tilde{\mathcal{F}}_\theta = (\tilde{\mathcal{F}}, \theta)$ are homotopic on $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$, where θ is a completing function (or Ize’s function) defined on $B(y_0, \sigma_0, \beta_0; r', \rho)$ which satisfies

- (i) $\theta(y_\sigma, \sigma, \beta) = -|(\sigma, \beta) - (\sigma_0, \beta_0)|$ if $(y_\sigma, \sigma, \beta) \in \tilde{\mathcal{U}}$;
- (ii) $\theta(y, \sigma, \beta) = r'$ if $\|y - y_\sigma\|_{C^1} = r'$.

Proof Since $\mathcal{U} = B(y_0, \sigma_0, \beta_0; r, \rho) \subseteq W \times \mathbb{R}^2$ is a special neighborhood of $\tilde{\mathcal{F}}$ with $0 < \rho < \beta_0$, then by Lemma 9, both $\mathcal{F}_\theta = (\mathcal{F}, \theta)$ and $\tilde{\mathcal{F}}_\theta = (\tilde{\mathcal{F}}, \theta)$ are \mathcal{U} -admissible. For contradiction, suppose that the conclusion is not true. Then for any $r' \in (0, r]$, $\mathcal{F}_\theta = (\mathcal{F}, \theta)$ and $\tilde{\mathcal{F}}_\theta = (\tilde{\mathcal{F}}, \theta)$ are not homotopic on $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$. That is, any homotopy map between \mathcal{F}_θ and $\tilde{\mathcal{F}}_\theta$ has a zero on the boundary of $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$. In particular, the linear homotopy $h(\cdot, \alpha) := \alpha \mathcal{F}_\theta + (1 - \alpha) \tilde{\mathcal{F}}_\theta = (\alpha \mathcal{F} + (1 - \alpha) \tilde{\mathcal{F}}, \theta)$ has a zero on the boundary of $\overline{B(y_0, \sigma_0, \beta_0; r', \rho)}$, where $\alpha \in [0, 1]$.

Note that $\theta(y, \sigma, \beta) > 0$ if $\|y - y_\sigma\|_{C^1} = r'$. Then, there exist (y, σ, β) and $\alpha \in [0, 1]$ such that $\|y - y_\sigma\|_{C^1} < r', |(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and

$$H(y, \sigma, \beta, \alpha) := \alpha \mathcal{F} + (1 - \alpha) \tilde{\mathcal{F}} = 0. \tag{3.22}$$

Since $r' > 0$ is arbitrary in the interval $(0, r]$, there exists a nonconstant sequence $\{(y_k, \sigma_k, \beta_k, \alpha_k)\}_{k=1}^\infty$ of solutions of (3.22) such that

$$\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\|_{C^1} = 0, \quad |(\sigma_k, \beta_k) - (\sigma_0, \beta_0)| = \rho, \quad 0 \leq \alpha_k \leq 1, \tag{3.23}$$

and

$$H(y_k, \sigma_k, \beta_k, \alpha_k) = 0, \quad \text{for all } k \in \mathbb{N}. \tag{3.24}$$

Note that $0 < \rho < \beta_0$ implies that $\beta_k \geq \beta_0 - \rho > 0$ for every $k \in \mathbb{N}$. From (3.23) we know that $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^\infty$ belongs to a compact subset of \mathbb{R}^3 . Therefore, there exist a convergent subsequence, denoted for notational simplicity by $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^\infty$ without loss of generality, and $(\sigma^*, \beta^*, \alpha^*) \in \mathbb{R}^3$ such that $\beta^* \geq \beta_0 - \rho > 0, \alpha^* \in [0, 1]$ and

$$\lim_{k \rightarrow +\infty} |(\sigma_k, \beta_k, \alpha_k) - (\sigma^*, \beta^*, \alpha^*)| = 0. \tag{3.25}$$

By the same token for the proof of Lemma 5, we show that the system

$$\dot{v}(t) = \frac{1}{\beta^*} \partial_1 f(\sigma^*)v(t) + \frac{1}{\beta^*} \partial_2 f(\sigma^*)v(t - \beta^* z_{\sigma^*}) \tag{3.26}$$

with $\partial_i f(\sigma^*)$, $\partial_i g(\sigma^*)$, $i = 1, 2$, defined at (2.1), has a nonconstant periodic solution which contradicts the assumption that $B(u_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood which contains an isolated center of (3.5). Since (3.23) implies $\lim_{k \rightarrow +\infty} \|y_k - y_{\sigma_k}\| = 0$, by the same argument in the proof of Lemma 10 we know that system (3.26) has a nonconstant periodic solution. This is a contradiction. \square

Now we are in the position to prove the local Hopf bifurcation theorem for system (1.1).

Theorem 2 *Assume (S1)–(S3) hold. Let (x_{σ_0}, σ_0) be an isolated center of system (2.1). If the crossing number defined by (2.5) satisfies*

$$\gamma(x_{\sigma_0}, \sigma_0, \beta_0) \neq 0,$$

then there exists a bifurcation of nonconstant periodic solutions of (1.1) near (x_{σ_0}, σ_0) . More precisely, there exists a sequence $\{(x_n, \sigma_n, \beta_n)\}$ such that $\sigma_n \rightarrow \sigma_0$, $\beta_n \rightarrow \beta_0$ as $n \rightarrow \infty$, and $\lim_{n \rightarrow \infty} \|x_n - x_{\sigma_0}\|_{C^1} = 0$, where

$$(x_n, \sigma_n) \in C^1(\mathbb{R}; \mathbb{R}^N) \times \mathbb{R}$$

is a nonconstant $2\pi/\beta_n$ -periodic solution of system (1.1).

Proof Let (x, τ) be a solution of system (1.1) with x being $2\pi/\beta$ -periodic and $\beta > 0$. Let $(x(t), \tau(t)) = (y(\beta t), z(\beta t))$. Then system (1.1) is transformed to

$$\begin{cases} \dot{y}(t) = \frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma), \\ z(t) = g(y(t), y(t - \beta z(t)), \sigma). \end{cases} \tag{3.27}$$

Let $W = O(\epsilon^*) \cap C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$. For any $\xi = e^{i\nu} \in S^1$, $u \in W$, $(\xi u)(t) := u(t + \nu)$. By results in [6], it is sufficient to verify the following conditions (A1)–(A6) listed in the proof of Theorem 1.

Recall that δ and ε are defined before (2.5). Let $\mathcal{D}(\sigma_0, \beta_0) = (\sigma_0 - \delta, \sigma_0 + \delta) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$ and define the maps

$$\begin{aligned} L_0 y(t) &:= \dot{y}(t), \quad y \in C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N), \\ N_1(y, \sigma, \beta)(t) &:= f(y(t), y(t - \beta z(t)), \sigma), \quad y \in W, \\ \tilde{N}_1(y, \sigma, \beta)(t) &:= \partial_1 f(\sigma)(y(t) - y_\sigma) + \partial_2 f(\sigma)(y(t - \beta z_\sigma) - y_\sigma), \quad y \in W, \end{aligned}$$

where $(\sigma, \beta) \in \mathcal{D}(\sigma_0, \beta_0)$ and $t \in \mathbb{R}$, and (y_σ, z_σ) is the stationary state of the system at σ such that $y_{\sigma_0} = x_{\sigma_0}$. The space $C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ is a Banach representation of the group $G = S^1$.

Define the operator $K : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow \mathbb{R}^N$ by

$$K(y) := \frac{1}{2\pi} \int_0^{2\pi} y(t) dt, \quad y \in C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N).$$

By Lemma 2, the operator $L_0 + K : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \rightarrow C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ has a compact inverse $(L_0 + K)^{-1}$. Then, finding a $2\pi/\beta$ -periodic solution for the system (1.1) is equivalent to finding a solution of the following fixed point problem:

$$y = (L_0 + K)^{-1} \left[\frac{1}{\beta} N_1(y, \sigma, \beta) + K(y) \right], \tag{3.28}$$

where $(y, \sigma, \beta) \in W \times \mathbb{R} \times (0, +\infty)$.

By (S1) we know that the linear operator \tilde{N}_1 is continuous. By Lemma 8, we know that $N_1(\cdot, \sigma, \beta) : W \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ is continuous. Moreover, by Lemma 7 the operator

$(L_0 + K)^{-1} : C_{2\pi}(\mathbb{R}; \mathbb{R}^N) \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ is continuous. Noticing that the embedding $j : C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) \hookrightarrow C_{2\pi}(\mathbb{R}; \mathbb{R}^N)$ is compact, we obtain that $(L_0 + K)^{-1} \circ (\frac{1}{\beta} N_1(\cdot, \alpha, \beta) + K) : W \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ and $(L_0 + K)^{-1} \circ (\frac{1}{\beta} \tilde{N}_1(\cdot, \alpha, \beta) + K) : W \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ are completely continuous and hence are condensing maps. That is, (A2) and (A4) are satisfied.

Define the following maps $\mathcal{F} : W \times \mathbb{R} \times (0, +\infty) \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ and $\tilde{\mathcal{F}} : W \times \mathbb{R} \times (0, +\infty) \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ by

$$\begin{aligned} \mathcal{F}(y, \sigma, \beta) &:= y - (L_0 + K)^{-1} \left[\frac{1}{\beta} N_1(y, \sigma, \beta) + K(y) \right], \\ \tilde{\mathcal{F}}(y, \sigma, \beta) &:= y - (L_0 + K)^{-1} \left[\frac{1}{\beta} \tilde{N}_1(y, \sigma, \beta) + K(y) \right], \end{aligned}$$

which are equivariant condensing fields. Finding a $2\pi/\beta$ -periodic solution of system (1.1) is equivalent to finding the solution of the problem

$$\mathcal{F}(y, \sigma, \beta) = 0, \quad (y, \sigma, \beta) \in W \times \mathbb{R} \times (0, +\infty).$$

Since $(x_{\sigma_0}, \sigma_0) = (y_{\sigma_0}, \sigma_0)$ is an isolated center of system (2.1) with a purely imaginary characteristic value $i\beta_0$, $\beta_0 > 0$, $(y_{\sigma_0}, \sigma_0, \beta_0) \in W \times \mathbb{R} \times (0, +\infty)$ is an isolated singular point of $\tilde{\mathcal{F}}$. That is, $(y_{\sigma_0}, \sigma_0, \beta_0)$ is the only point in W such that the derivative $D_y \mathcal{F}(y_{\sigma_0}, \sigma_0, \beta_0)$ is not an automorphism of $C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$. One can define the following two-dimensional submanifold $M \subset V_0 \times \mathbb{R} \times (0, +\infty)$ by

$$M := \{(y_\sigma, \sigma, \beta) : \sigma \in (\sigma_0 - \delta, \sigma_0 + \delta), \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\},$$

such that the point $(y_{\sigma_0}, \sigma_0, \beta_0)$ is the only singular point of $\tilde{\mathcal{F}}$ in M . M is the set of trivial solutions to the system (2.1) and satisfies the assumption (A3).

Since $(y_{\sigma_0}, \sigma_0, \beta_0) \in W \times \mathbb{R} \times (0, +\infty)$ is an isolated singular point of $\tilde{\mathcal{F}}$, for $\rho > 0$ sufficiently small, the linear operator $D_u \tilde{\mathcal{F}}(y_\sigma, \sigma, \beta) : W \rightarrow C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ with $|(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho$, is not an automorphism only if $(\sigma, \beta) = (\sigma_0, \beta_0)$. Then, by the Implicit Function Theorem, there exists $r > 0$ such that for every $(y, \sigma, \beta) \in W \times \mathbb{R} \times (0, +\infty)$ with $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and $0 < \|y - y_\sigma\| \leq r$, we have $\tilde{\mathcal{F}}(y, \sigma, \beta) \neq 0$. Then the set $B(x_0, \sigma_0, \beta_0; r, \rho)$ defined by

$$\{(y, \sigma, \beta) \in W \times \mathbb{R} \times (0, +\infty); |(\sigma, \beta) - (\sigma_0, \beta_0)| < \rho, \|y - y_\sigma\|_{C^1} < r\},$$

is a special neighborhood for $\tilde{\mathcal{F}}$.

By Lemma 9, there exists a special neighborhood $\mathcal{U} = B(y_{\sigma_0}, \sigma_0, \beta_0; r', \rho)$ such that \mathcal{F} and $\tilde{\mathcal{F}}$ are nonzero for $(y, \sigma, \beta) \in \overline{B(y_{\sigma_0}, \sigma_0, \beta_0; r', \rho)}$ with $y \neq y_\sigma$ and $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$. That is, (A5) is satisfied.

Let θ be a completing function on \mathcal{U} . It follows from Lemma 10 that (\mathcal{F}, θ) is homotopic to $(\tilde{\mathcal{F}}, \theta)$ on \mathcal{U} . It is known that $C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N)$ has the following isotypical direct sum decomposition

$$C_{2\pi}^1(\mathbb{R}; \mathbb{R}^N) = \overline{\bigoplus_{k=0}^{\infty} V_k},$$

where V_0 is the space of all constant mappings from \mathbb{R} into \mathbb{R}^N , and V_k with $k > 0, k \in \mathbb{N}$ is the vector space of all mappings of the form

$$x \cos k \cdot + y \sin k \cdot : \mathbb{R} \ni t \rightarrow x \cos kt + y \sin kt \in \mathbb{R}^N,$$

where $x, y \in \mathbb{R}^N$. Then $V_k, k > 0, k \in \mathbb{N}$, are finite dimensional. Then, (A1) is satisfied.

The verification of (A6) and the computation of the crossing number $\gamma(y_{\sigma_0}, \sigma_0, \beta_0) \neq 0$ is the same as that in the proof of Theorem 1. We omit the details here. Then by Theorem 2.4 of [6], $(y_{\sigma_0}, \sigma_0, \beta_0)$ is a bifurcation point of the system (3.27). Consequently, there exists a sequence of non-constant periodic solutions (x_n, σ_n, β_n) such that $\sigma_n \rightarrow \sigma_0, \beta_n \rightarrow \beta_0$ as $n \rightarrow \infty$, and x_n is a $2\pi/\beta_n$ -periodic solution of (1.1) such that x_n satisfies (1.1) with $\lim_{n \rightarrow +\infty} \|x_n - x_{\sigma_0}\|_{C^1} = 0$. □

4 Global Bifurcation of DAEs with State-Dependent Delays

In this section we use a Rabinowitz-type global Hopf bifurcation theorem (Theorem 2.5 developed in [6]) to describe the maximal continuation of bifurcated periodic solutions with large amplitudes when the bifurcation parameter σ is far away from the bifurcation value. Note that systems (1.1) and (1.5) share the same differential equation for x and differ only in the algebraic equation for the state-dependent delay τ . Moreover, by Theorems 1 and 2, both systems share the same set of Hopf bifurcation points. In the following, we state results in terms of system (1.1), which are also applicable to system (1.5).

Lemma 11 (Vidossich [17]) *Let X be a Banach space, $v : \mathbb{R} \rightarrow X$ be a \mathfrak{p} -periodic function with the following properties:*

- (i) $v \in L^1_{loc}(\mathbb{R}, X)$;
- (ii) *there exists $U \in L^1([0, \frac{\mathfrak{p}}{2}]; \mathbb{R}_+)$ such that $|v(t) - v(s)| \leq U(t - s)$ for almost every (in the sense of the Lebesgue measure) $s, t \in \mathbb{R}$ such that $s \leq t, t - s \leq \frac{\mathfrak{p}}{2}$;*
- (iii) $\int_0^{\mathfrak{p}} v(t) dt = 0$.

Then

$$\mathfrak{p} \|v\|_{L^\infty} \leq 2 \int_0^{\frac{\mathfrak{p}}{2}} U(t) dt.$$

We make the following assumption on f :

(S4) There exists constant $L_f > 0$ such that

$$|f(\theta_1, \theta_2, \sigma) - f(\bar{\theta}_1, \bar{\theta}_2, \sigma)| \leq L_f(|\theta_1 - \bar{\theta}_1| + |\theta_2 - \bar{\theta}_2|),$$

for every $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2, \sigma \in \mathbb{R}$.

Lemma 12 *Suppose that system (1.1) satisfies the assumption (S4) and x is a nonconstant periodic solution. The following statements are true.*

- (i) *If $\|\tau\|_{L^\infty} < \frac{1}{2L_f}$, then the minimal period \mathfrak{p} of x satisfies*

$$\mathfrak{p} \geq \frac{2}{1 - 2L_f \|\tau\|_{L^\infty}}.$$

- (ii) *If τ is continuously differentiable in \mathbb{R} , then the minimal period \mathfrak{p} of x satisfies*

$$\mathfrak{p} \geq \frac{4}{L_f(2 + \|\dot{\tau}\|_{L^\infty})}.$$

(iii) Suppose there exists a constant $L_g > 0$ such that

$$|g(\theta_1, \theta_2, \sigma) - g(\bar{\theta}_1, \bar{\theta}_2, \sigma)| \leq L_g(|\theta_1 - \bar{\theta}_1| + |\theta_2 - \bar{\theta}_2|),$$

for every $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2, \sigma \in \mathbb{R}$. If $\|\dot{x}\|_{L^\infty} < \frac{1}{L_g}$, then the minimal period \mathfrak{p} of x satisfies

$$\mathfrak{p} \geq \frac{2(1 - L_g \|\dot{x}\|_{L^\infty})}{L_f}.$$

Proof Assume that x is a nonconstant periodic solution with minimal period \mathfrak{p} . Let $v(t) = \dot{x}(t)$. Then we have $\int_0^{\mathfrak{p}} v(t)dt = 0$. For $s \leq t$, by (S4) and the Integral Mean Value Theorem, we have

$$\begin{aligned} |v(t) - v(s)| &\leq |\dot{x}(t) - \dot{x}(s)| \\ &\leq L_f(|x(t) - x(s)| + |x(t - \tau(t)) - x(s - \tau(s))|) \\ &\leq L_f \|\dot{x}\|_{L^\infty}(t - s) + L_f \|\dot{x}\|_{L^\infty}(t - s + |\tau(t) - \tau(s)|) \\ &\leq (2L_f \|\dot{x}\|_{L^\infty} + L_f \|\dot{x}\|_{L^\infty} \cdot \|\dot{\tau}\|_{L^\infty})(t - s). \end{aligned} \tag{4.1}$$

(i) By (4.1) we have

$$\begin{aligned} |v(t) - v(s)| &\leq L_f \|\dot{x}\|_{L^\infty}(t - s) + L_f \|\dot{x}\|_{L^\infty}(t - s + |\tau(t) - \tau(s)|) \\ &\leq 2L_f \|\dot{x}\|_{L^\infty}(t - s) + 2L_f \|\tau\|_{L^\infty} \cdot \|\dot{x}\|_{L^\infty}. \end{aligned}$$

Let

$$U(t) = 2L_f \|\dot{x}\|_{L^\infty}t + 2L_f \|\tau\|_{L^\infty} \cdot \|\dot{x}\|_{L^\infty}.$$

Then, by Lemma 11, we obtain

$$\mathfrak{p} \|\dot{x}\|_{L^\infty} \leq 2 \int_0^{\frac{\mathfrak{p}}{2}} U(t)dt = \frac{\mathfrak{p}^2}{4} \cdot 2L_f \|\dot{x}\|_{L^\infty} + \mathfrak{p} \cdot 2L_f \|\tau\|_{L^\infty} \cdot \|\dot{x}\|_{L^\infty}.$$

If $\|\tau\|_{L^\infty} < \frac{1}{2L_f}$, then

$$\mathfrak{p} \geq \frac{2}{(1 - 2L_f \|\tau\|_{L^\infty})}.$$

(ii) If τ is continuously differentiable in \mathbb{R} , then we have $\|\dot{\tau}\|_{L^\infty} < \infty$. Moreover, by (4.1) we have

$$\begin{aligned} |v(t) - v(s)| &\leq L_f \|\dot{x}\|_{L^\infty}(t - s) + L_f \|\dot{x}\|_{L^\infty}(t - s + |\tau(t) - \tau(s)|) \\ &\leq (2 + \|\dot{\tau}\|_{L^\infty})L_f \cdot \|\dot{x}\|_{L^\infty}(t - s). \end{aligned}$$

Let

$$U(t) = (2 + \|\dot{\tau}\|_{L^\infty})L_f \cdot \|\dot{x}\|_{L^\infty}t.$$

Then, by Lemma 11, we obtain

$$\mathfrak{p} \|\dot{x}\|_{L^\infty} \leq 2 \int_0^{\frac{\mathfrak{p}}{2}} U(t)dt = \frac{\mathfrak{p}^2}{4} \cdot (2 + \|\dot{\tau}\|_{L^\infty})L_f \cdot \|\dot{x}\|_{L^\infty}.$$

Therefore,

$$\mathfrak{p} \geq \frac{4}{L_f(2 + \|\dot{\tau}\|_{L^\infty})}.$$

(iii) If g is Lipschitz continuous, then we have

$$\begin{aligned}
 |\tau(t) - \tau(s)| &\leq L_g|x(t) - x(s)| + L_g|x(t - \tau(t)) - x(s - \tau(s))| \\
 &\leq L_g\|\dot{x}\|_{L^\infty}(t - s) + L_g\|\dot{x}\|_{L^\infty}(t - s + |\tau(t) - \tau(s)|).
 \end{aligned}$$

If $\|\dot{x}\|_{L^\infty} < \frac{1}{L_g}$, then we have

$$|\tau(t) - \tau(s)| \leq \frac{2L_g\|\dot{x}\|_{L^\infty}(t - s)}{1 - L_g\|\dot{x}\|_{L^\infty}}. \tag{4.2}$$

By (4.1) and (4.2)

$$\begin{aligned}
 |v(t) - v(s)| &\leq L_f\|\dot{x}\|_{L^\infty}(t - s) + L_f\|\dot{x}\|_{L^\infty}(t - s + |\tau(t) - \tau(s)|) \\
 &\leq 2L_f \cdot \|\dot{x}\|_{L^\infty}(t - s) + \frac{2L_fL_g\|\dot{x}\|_{L^\infty}^2(t - s)}{1 - L_g\|\dot{x}\|_{L^\infty}} \\
 &= \frac{2L_f\|\dot{x}\|_{L^\infty}}{1 - L_g\|\dot{x}\|_{L^\infty}}(t - s).
 \end{aligned}$$

Let

$$U(t) = \frac{2L_f\|\dot{x}\|_{L^\infty}}{1 - L_g\|\dot{x}\|_{L^\infty}}t.$$

We obtain

$$\rho\|\dot{x}\|_{L^\infty} \leq 2 \int_0^{\frac{\rho}{2}} U(t)dt = \frac{\rho^2}{4} \cdot \frac{2L_f\|\dot{x}\|_{L^\infty}}{1 - L_g\|\dot{x}\|_{L^\infty}},$$

and

$$\rho \geq \frac{2(1 - L_g\|\dot{x}\|_{L^\infty})}{L_f}.$$

□

To describe the minimal periods of the periodic solutions near the bifurcation point, we need the following result which was first established in [9] for ordinary differential equations and was extended to other types of delay differential equations in [6, 19].

Lemma 13 *Suppose that system (1.1) satisfies (S1–S4). Assume further that there exists a sequence of real numbers $\{\sigma_k\}_{k=1}^\infty$ such that:*

- (i) *For each k , system (1.5) with $\sigma = \sigma_k$ has a nonconstant periodic solution $x_k \in C(\mathbb{R}; \mathbb{R}^{N+1})$ with the minimal period $T_k > 0$, and one of the conditions (i), (ii) and (iii) at Lemma 12 is satisfied by (x_k, τ_k) ;*
- (ii) *$\lim_{k \rightarrow \infty} \sigma_k = \sigma_0 \in \mathbb{R}$, $\lim_{k \rightarrow \infty} T_k = T_0 < \infty$, and $\lim_{k \rightarrow \infty} \|x_k - x_0\| = 0$, where $x_0 : \mathbb{R} \rightarrow \mathbb{R}^N$ is a constant map with the value x_0 .*

Then x_0 is a stationary state of (1.1) and there exists $m \geq 1$, $m \in \mathbb{N}$ such that $\pm im 2\pi/T_0$ are the roots of the characteristic equation (2.2) with $\sigma = \sigma_0$.

Proof By Lemma 12 and the uniform convergence of $\{(x_k, \sigma_k, T_k)\}_{k=1}^\infty$ we conclude that there exists $T^* > 0$ such that $T_k \geq T^*$ and therefore $T_0 \geq T^*$. We can show that (x_0, σ_0) is a stationary state of (1.5), and that the following linear system

$$\dot{v}(t) = \partial_1 f(\sigma_0)v(t) + \partial_2 f(\sigma_0)v(t - \tau_0) \tag{4.3}$$

has a nonconstant periodic solution, the proofs of which are just simplified versions of the proof for Lemma 4.3 in [6] without the equations for τ_k . Hence we omit the details here. Then by Lemma 4, there exists $m \geq 1, m \in \mathbb{N}$, such that $\pm im 2\pi/T_0$ are characteristic values of (2.2). This completes the proof. \square

Now we can describe the relation between $2\pi/\beta_k$ and the minimal period of u_k in Theorem 1.

Theorem 3 *Assume (S1–S4) hold and every point in the sequence $\{(x_k, \tau_k)\}_{k=1}^\infty$ at Theorem 1 satisfies one of the conditions among (i), (ii) and (iii) at Lemma 12, then every limit point of the minimal period of x_k as $k \rightarrow +\infty$ is contained in the set*

$$\left\{ \frac{2\pi}{(n\beta_0)} : \pm im n\beta_0 \text{ are characteristic values of } (x_0, \sigma_0), m, n \geq 1, m, n \in \mathbb{N} \right\}.$$

Moreover, if $\pm im n\beta_0$ are not characteristic values of (x_0, σ_0) for any integers $m, n \in \mathbb{N}$ such that $mn > 1$, then $2\pi/\beta_k$ is the minimal period of $u_k(t)$ and $2\pi/\beta_k \rightarrow 2\pi/\beta_0$ as $k \rightarrow \infty$.

Proof Let T_k denote the minimal period of $x_k(t)$. Then there exists a positive integer n_k such that $2\pi/\beta_k = n_k T_k$. Since $T_k \leq 2\pi/\beta_k \rightarrow 2\pi/\beta_0$ as $k \rightarrow \infty$, there exists a subsequence $\{T_{k_j}\}_{j=1}^\infty$ and T_0 such that $T_0 = \lim_{j \rightarrow \infty} T_{k_j}$. Since $2\pi/\beta_{k_j} \rightarrow 2\pi/\beta_0, T_{k_j} \rightarrow T_0$ as $j \rightarrow \infty, n_{k_j}$ is identical to a constant n for k large enough. Therefore, $2\pi/\beta_0 = nT_0$. Thus $T_{k_j} \rightarrow 2\pi/(n\beta_0)$ as $j \rightarrow \infty$. By Lemma 13, $\pm im 2\pi/T_0 = \pm im n\beta_0$ are characteristic values of (x_0, σ_0) for some $m \geq 1, m \in \mathbb{N}$.

Moreover, if $\pm im n\beta_0$ are not characteristic values of (u_0, σ_0) for any integers $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $mn > 1$, then $m = n = 1$. Therefore, for k large enough $n_{k_j} = 1$ and $2\pi/\beta_k = T_k$ is the minimal period of $x_k(t)$ and $2\pi/\beta_k \rightarrow 2\pi/\beta_0$ as $k \rightarrow \infty$. This completes the proof. \square

The following lemma shows that we can locate all the possible Hopf bifurcation points of system (1.1) with state-dependent delay at the centers of its corresponding formal linearization. Since the proof is similar to that for Lemma 4.5 in [6], we omit the details here.

Lemma 14 *Assume (S1–S3) hold. If (x_0, σ_0) is a Hopf bifurcation point of system (1.1), then it is a center of (2.1).*

Now we are in the position to consider the global Hopf bifurcation problem of system (1.1). Letting $(x(t), \tau(t)) = (y(\frac{2\pi}{p}t), z(\frac{2\pi}{p}t))$, we can reformulate the problem as a problem of finding 2π -period solutions to the following equation:

$$\dot{y}(t) = \frac{p}{2\pi} N_0(y(t), \sigma, 2\pi/p), \tag{4.4}$$

where the z satisfies the algebraic equation $z(t) = g(y(t), y(t - \frac{p}{2\pi}z(t)), \sigma)$. Accordingly, the formal linearization (2.1) becomes

$$\dot{x}(t) = \frac{p}{2\pi} \tilde{N}_0(x(t), \sigma, 2\pi/p). \tag{4.5}$$

Using the same notations as in the proof of Theorem 1, we can define

$$\mathcal{N}_0(x, \sigma, p) = N_0(x, \sigma, 2\pi/p), \mathcal{N}_0(x, \sigma, p) = \tilde{N}_0(x, \sigma, 2\pi/p).$$

Then the following system

$$L_0x = \frac{p}{2\pi} \mathcal{N}_0(x, \sigma, p), \quad p > 0, \tag{4.6}$$

is equivalent to (4.4) and

$$L_0x = \frac{p}{2\pi} \tilde{\mathcal{N}}_0(x, \sigma, p), \quad p > 0, \tag{4.7}$$

is equivalent to (4.5). Let \mathcal{S} denote the closure of the set of all nontrivial periodic solutions of system (4.6) in the space $V \times \mathbb{R} \times \mathbb{R}_+$, where \mathbb{R}_+ is the set of all nonnegative reals. It follows from Lemma 12 that the constant solution $(x_0, \sigma_0, 0)$ does not belong to this set if the sequence $\{(x_k, \tau_k)\}_{k=1}^\infty$ in Theorem 1 satisfies one of the conditions among i), ii) and iii) at Lemma 12. Consequently, we can assume that problem (4.6) is well posed on the whole space $V \times \mathbb{R}^2$, in the sense that if \mathcal{S} exists in $V \times \mathbb{R}^2$, then it must be contained in $V \times \mathbb{R} \times \mathbb{R}_+$.

Then by the global Hopf bifurcation theorem 2.5 developed in [6] and with similar arguments leading to Theorem 4.6 in [6], we obtain the following global Hopf bifurcation theorem for system (1.1) with state-dependent delay.

Theorem 4 *Suppose that system (1.1) satisfies (S1–S4) and (S3) holds at every center of (4.7). Assume that all the centers of (4.7) are isolated and every periodic solution x of system (1.1) satisfies one of the conditions among (i), (ii) and (iii) at Lemma 12. Let \mathcal{M} be the set of trivial periodic solutions of (4.6) and assume that \mathcal{M} is complete. If $(x_0, \sigma_0, p_0) \in \mathcal{M}$ is a bifurcation point, then either the connected component $C(x_0, \sigma_0, p_0)$ of (x_0, σ_0, p_0) in \mathcal{S} is unbounded, or*

$$C(x_0, \sigma_0, p_0) \cap \mathcal{M} = \{(x_0, \sigma_0, p_0), (x_1, \sigma_1, p_1), \dots, (x_q, \sigma_q, p_q)\},$$

where $p_i \in \mathbb{R}_+, (x_i, \sigma_i, p_i) \in M, i = 0, 1, 2, \dots, q$. Moreover, in the latter case, we have

$$\sum_{i=0}^q \epsilon_i \gamma(x_i, \sigma_i, 2\pi/p_i) = 0,$$

where $\gamma(x_i, \sigma_i, 2\pi/p_i)$ is the crossing number of (x_i, σ_i, p_i) defined by (2.5) and

$$\epsilon_i = \text{sgn det}(\partial_1 f(\sigma_i) + \partial_2 f(\sigma_i)).$$

5 Global Hopf Bifurcation of a Model of Regulatory Dynamics

We consider the following extended Goodwin’s model for regulatory dynamics:

$$\begin{cases} \frac{dx(t)}{dt} = -\mu_m x(t) + \frac{\alpha_m}{1 + \left(\frac{z(t-\tau)}{\tilde{z}}\right)^h}, \\ \frac{dy(t)}{dt} = -\mu_p y(t) + \alpha_p x(t - \tau), \\ \frac{dz(t)}{dt} = -\mu_e z(t) + \alpha_e y(t - \tau), \\ \tau(t) = c(x(t) - x(t - \tau)), \end{cases} \tag{5.1}$$

where x is the concentration of mRNA, y is the concentration of the related protein; z is the concentration of an active enzyme which controls the level of the metabolite functioning as repressor at the DNA level; μ_m, μ_p and μ_e are nonnegative degradation rates; α_m, α_p and α_e are positive coefficients for the inhibition/activation terms; c and \tilde{z} are positive constants; h is an even positive integer. The Goodwin’s model [3] without delay ($\tau = 0$) has been

extensively studied in system biology modeling various regulatory dynamics. Note that if we freeze the delay τ at the stationary state in system (5.1), it becomes the classic Goodwin’s model without delay.

We are interested in the onset and termination of each Hopf bifurcation branch of periodic solutions which are described as one of the alternatives given in Theorem 4. To be specific, we need to obtain the boundedness or unboundedness of the connected component of the pairs of nonconstant periodic solution and parameter in the product space of the state and the parameter space. In the following, we first analyze the local Hopf bifurcation of system (5.1) and then consider the boundedness of periodic solutions of system (5.1) for a global Hopf bifurcation in light of Theorem 4.

5.1 Local Hopf Bifurcation

Note that h is an even positive integer. Every stationary point (x, y, z) of System (5.1) satisfies that

$$\begin{cases} -\mu_m x + \frac{\alpha_m}{1 + \left(\frac{z}{z_0}\right)^h} = 0, \\ -\mu_p y + \alpha_p x = 0, \\ -\mu_e z + \alpha_e y = 0, \end{cases} \tag{5.2}$$

and $(x, y, z) = \left(x_0, \frac{\alpha_p}{\mu_p} x_0, \frac{\alpha_e \alpha_p}{\mu_e \mu_p} x_0\right)$, where by Descartes’ rule of signs we know that $x = x_0$ is the unique solution of

$$\mu_m \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p z}\right)^h x^{h+1} + \mu_m x - \alpha_m = 0.$$

Freezing the delay of system (5.1) at $\tau = 0$ and linearizing the resulting nonlinear system at the stationary state $(x, y, z) = \left(x_0, \frac{\alpha_p}{\mu_p} x_0, \frac{\alpha_e \alpha_p}{\mu_e \mu_p} x_0\right)$ leads to the characteristic polynomial

$$\begin{aligned} \det \left(\lambda I - \begin{bmatrix} -\mu_m & 0 & -\frac{h\alpha_m z^{h-1}}{\tilde{z}^h \left(1 + \left(\frac{z}{z_0}\right)^h\right)^2} \\ \alpha_p & -\mu_p & 0 \\ 0 & \alpha_e & -\mu_e \end{bmatrix} \right) \\ = (\lambda + \mu_m)(\lambda + \mu_p)(\lambda + \mu_e) + \frac{h\alpha_m z^{h-1}}{\tilde{z}^h \left(1 + \left(\frac{z}{z_0}\right)^h\right)^2}, \end{aligned} \tag{5.3}$$

which has a unique negative root and a pair of imaginary roots. In the following, we discuss the existence of purely imaginary eigenvalues as the parameter α_m varies. We have

Lemma 15 *Let (x, y, z) be a stationary state of system (5.1). Then the following equation of (x, α_m)*

$$\begin{cases} (\mu_m + \mu_p)(\mu_e + \mu_p)(\mu_e + \mu_m) = \frac{h\alpha_m^3}{\tilde{z}^h \mu_m^2} \cdot \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p}\right)^{h-1} x^{h-3}, \\ \mu_m x = \frac{\alpha_m}{1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p z} x\right)^h}, \end{cases} \tag{5.4}$$

has a unique solution for $(x, \alpha_m) = (x^*, \alpha_m^*)$.

Proof Noticing that by the second equation of (5.4), $\frac{\alpha_m}{x} = \mu_m \left(1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \bar{z}} x \right)^h \right)$, we rewrite the first equation of (5.4) into

$$x^h \left(1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \bar{z}} x \right)^h \right)^3 = \frac{(\mu_m + \mu_p)(\mu_e + \mu_p)(\mu_e + \mu_m)}{\frac{h\mu_m}{\bar{z}^h} \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \bar{z}} \right)^{h-1}},$$

which has a unique positive solution for x^h and hence for x with $x = x^*$ for some $x^* > 0$. Then $\alpha_m = \alpha_m^*$ with $\alpha_m^* = x^* \mu_m \left(1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \bar{z}} x^* \right)^h \right)$. The solution of (5.4) is $(x, \alpha_m) = (x^*, \alpha_m^*)$. □

Lemma 16 *Let α_m^* be as in Lemma 15 and $\lambda = u \pm iv$ be the imaginary roots of the characteristic polynomial at (5.3). Then u and v are continuously differentiable with respect to α_m and $u = 0$ if and only if $\alpha_m = \alpha_m^*$. Moreover,*

$$\frac{du}{d\alpha_m} \Big|_{\alpha_m = \alpha_m^*} > 0.$$

Proof Let $(x, y, z) = \left(x_0, \frac{\alpha_p}{\mu_p} x_0, \frac{\alpha_e \alpha_p}{\mu_e \mu_p} x_0 \right)$ be a stationary state of System (5.1) and let

$$F(\lambda, \alpha_m) = (\lambda + \mu_m)(\lambda + \mu_p)(\lambda + \mu_e) + \frac{h\alpha_m z^{h-1}}{\bar{z}^h \left(1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \bar{z}} \right)^h \right)^2}.$$

Noticing that $z = \frac{\alpha_e \alpha_p}{\mu_e \mu_p} x_0$ and

$$\frac{dx_0}{d\alpha_m} = \frac{1}{\mu_m + \mu_m(h + 1) \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \bar{z}} \right)^h x_0^h},$$

we know that F is continuously differentiable with respect to (λ, α_m) . Let (λ, α_m) be such that $F(\lambda, \alpha_m) = 0$. Then we have

$$\begin{aligned} \frac{dF}{d\lambda} &= (\lambda + \mu_m)(\lambda + \mu_p)(\lambda + \mu_e) \left(\frac{1}{\lambda + \mu_m} + \frac{1}{\lambda + \mu_p} + \frac{1}{\lambda + \mu_e} \right) \\ &= -\frac{h\alpha_m z^{h-1}}{\bar{z}^h \left(1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \bar{z}} \right)^h \right)^2} \left(\frac{1}{\lambda + \mu_m} + \frac{1}{\lambda + \mu_p} + \frac{1}{\lambda + \mu_e} \right). \end{aligned}$$

Next we show that $\frac{dF}{d\lambda} \neq 0$ at every solution of $F(\lambda, \alpha_m) = 0$. Otherwise, F has a repeated root and the root satisfies

$$\frac{1}{\lambda + \mu_m} + \frac{1}{\lambda + \mu_p} + \frac{1}{\lambda + \mu_e} = 0$$

which leads to two distinct negative roots:

$$\lambda = \frac{-(\mu_m + \mu_p + \mu_e) \pm \sqrt{(\mu_m + \mu_e)^2 + \mu_p^2 - \mu_p(\mu_m + \mu_e)}}{3}.$$

This is a contradiction. Then by the Implicit Function Theorem, λ is continuously differentiable with respect to α_m .

Next we bring $\lambda = u + iv$ into the characteristic polynomial at (5.3) we have

$$\begin{cases} ((u + \mu_m)(u + \mu_p) - v^2)(u + \mu_e) - (\mu_m + \mu_p + 2u)v^2 + c_0 = 0 \\ [(u + \mu_m)(u + \mu_p) - v^2 + (u + \mu_e)(\mu_m + \mu_p + 2u)]v = 0, \end{cases} \tag{5.5}$$

where $c_0 = \frac{h\alpha_m z^{h-1}}{\tilde{z}^h \left(1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \tilde{z}}\right)^h\right)^2} = \frac{h\alpha_m^3}{\tilde{z}^h \mu_m^2} \cdot \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p}\right)^{h-1} x^{h-3}$. If $u = 0$, then (5.5) leads to

$$\begin{cases} (\mu_m + \mu_p)(\mu_e + \mu_p)(\mu_e + \mu_m) = c_0, \\ \mu_m \mu_p + \mu_e(\mu_m + \mu_p) = v^2. \end{cases} \tag{5.6}$$

where x satisfies $\mu_m x = \frac{\alpha_m}{1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \tilde{z}} x\right)^h}$. By Lemma 5.4, we have $\alpha_m = \alpha_m^*$. By the uniqueness of α_m^* , $u = 0$ if and only if $\alpha_m = \alpha_m^*$.

To compute $\frac{du}{d\alpha_m}$ at $\alpha_m = \alpha_m^*$, we take derivatives with respect to α_m on both sides of the equations at (5.5) and then letting $u = 0$, we obtain

$$\begin{cases} [(\mu_e \mu_p + \mu_e \mu_m + \mu_m \mu_p) - 3v^2]u' - 2v(\mu_m + \mu_p)v' + c'_0 = 0, \\ [2(\mu_m + \mu_p + \mu_e)v]u' + [(\mu_e \mu_p + \mu_e \mu_m + \mu_m \mu_p) - 3v^2]v' = 0. \end{cases}$$

Then we have

$$\frac{du}{d\alpha_m} \Big|_{\alpha_m = \alpha_m^*} = \frac{-c'_0((\mu_e \mu_p + \mu_e \mu_m + \mu_m \mu_p) - 3v^2)}{[(\mu_e \mu_p + \mu_e \mu_m + \mu_m \mu_p) - 3v^2]^2 + 4v^2(\mu_m + \mu_p)(\mu_m + \mu_p + \mu_e)}.$$

By the second equation of (5.6), we have

$$\frac{du}{d\alpha_m} \Big|_{\alpha_m = \alpha_m^*} = \frac{2c'_0(\mu_e \mu_p + \mu_e \mu_m + \mu_m \mu_p)}{[(\mu_e \mu_p + \mu_e \mu_m + \mu_m \mu_p) - 3v^2]^2 + 4v^2(\mu_m + \mu_p)(\mu_m + \mu_p + \mu_e)}.$$

Noticing that

$$\begin{aligned} c_0 &= \frac{h\alpha_m^3}{\tilde{z}^h \mu_m^2} \cdot \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p}\right)^{h-1} x^{h-3} \\ &= \frac{h}{\tilde{z}^h \mu_m^2} \cdot \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p}\right)^{h-1} x^h \left(\frac{\alpha_m}{x}\right)^3 \\ &= \frac{h\mu_m}{\tilde{z}^h} \cdot \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p}\right)^{h-1} x^h \left(1 + \left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \tilde{z}} x\right)^h\right)^3 \end{aligned}$$

can be regarded as a fourth order polynomial of x^h with positive coefficients, and that $\frac{dx_0}{d\alpha_m} =$

$$\frac{1}{\mu_m + \mu_m(h+1)\left(\frac{\alpha_e \alpha_p}{\mu_e \mu_p \tilde{z}}\right)^h x_0^h} > 0, \text{ we have}$$

$$\frac{dc_0}{d\alpha_m} \Big|_{\alpha_m = \alpha_m^*} > 0,$$

hence $\frac{du}{d\alpha_m} \Big|_{\alpha_m = \alpha_m^*} > 0$. □

Notice that $\frac{du}{d\alpha_m} \Big|_{\alpha_m = \alpha_m^*} > 0$ implies the crossing number at the stationary point $(x(\alpha_m^*), y(\alpha_m^*), z(\alpha_m^*))$ satisfies:

$$\gamma(x(\alpha_m^*), y(\alpha_m^*), z(\alpha_m^*), \alpha_m^*, v(\alpha_m^*)) \neq 0.$$

Moreover, we can check that conditions (S1–S3) for Theorem 1 are satisfied. Then we have the following local Hopf bifurcation theorem for system (5.1).

Theorem 5 *Let α_m^* be as in Lemma 15. Then system (5.1) undergoes Hopf bifurcation near the stationary point $(x(\alpha_m^*), y(\alpha_m^*), z(\alpha_m^*))$ as α_m varies near α_m^* .*

5.2 Global Hopf Bifurcation

In this section, we develop a global Hopf bifurcation theory for system (5.1). By Lemma 5.4 and Theorem 5, we know that $(x(\alpha_m^*), y(\alpha_m^*), z(\alpha_m^*))$ is the only Hopf bifurcation point and is an isolated center. To apply the global Hopf bifurcation Theorem 4, it remains to check condition (S4) and one of the conditions among (i), (ii) and (iii) at Lemma 12. We first consider the boundedness of periodic solutions.

Theorem 6 *Let (x, y, z) be a periodic solution of system (5.1). Then (x, y, z) satisfies for every $t \in \mathbb{R}$,*

$$0 < x(t) \leq \frac{\alpha_m}{\mu_m}, \quad 0 < y(t) \leq \frac{\alpha_p \alpha_m}{\mu_p \mu_m}, \quad 0 < z(t) \leq \frac{\alpha_e \alpha_p \alpha_m}{\mu_e \mu_p \mu_m}.$$

Proof Note that $h > 0$ is an even integer. We have $\dot{x}(t) \leq -\mu_m x(t) + \alpha_m$, which by Gronwall’s inequality leads to

$$x(t) \leq e^{-\mu_m t} x(0) + \frac{\alpha_m}{\mu_m} (1 - e^{-\mu_m t}). \tag{5.7}$$

Since x is periodic, there exists $p > 0$ such that $x(t) = x(t + p)$ for every $t \in \mathbb{R}$ and for every $n \in \mathbb{N}$, we have $x(t) = x(t + np)$. Then for every $t \in \mathbb{R}$ we have $x(t) = x(t + np) \leq e^{-\mu_m(t+np)} x(0) + \frac{\alpha_m}{\mu_m} (1 - e^{-\mu_m(t+np)}) \rightarrow \frac{\alpha_m}{\mu_m}$ as $n \rightarrow \infty$. Therefore, we have $x(t) \leq \frac{\alpha_m}{\mu_m}$ for every $t \in \mathbb{R}$.

By the same token, with $x(t - \tau) \leq \frac{\alpha_m}{\mu_m}$, we obtain from the second equation of system (5.1) that $y(t) \leq \frac{\alpha_p \alpha_m}{\mu_p \mu_m}$, $t \in \mathbb{R}$, and subsequently from the third equation of system (5.1) that $z(t) \leq \frac{\alpha_e \alpha_p \alpha_m}{\mu_e \mu_p \mu_m}$ for every $t \in \mathbb{R}$.

To obtain lower bounds of x, y and z , let $\bar{x} = -x, \bar{y} = -y$ and $\bar{z} = -z$. Then system (5.1) becomes

$$\begin{cases} \frac{d\bar{x}(t)}{dt} = -\mu_m \bar{x}(t) - \frac{\alpha_m}{1 + \left(\frac{\bar{z}(t-\tau)}{\bar{z}}\right)^h}, \\ \frac{d\bar{y}(t)}{dt} = -\mu_p \bar{y}(t) + \alpha_p \bar{x}(t - \tau), \\ \frac{d\bar{z}(t)}{dt} = -\mu_e \bar{z}(t) + \alpha_e \bar{y}(t - \tau), \\ \tau(t) = c(\bar{x}(t - \tau) - \bar{x}(t)). \end{cases} \tag{5.8}$$

We have $\dot{\bar{x}}(t) < -\mu_m \bar{x}(t)$, which leads to

$$\bar{x}(t) < e^{-\mu_m t} \bar{x}(0). \tag{5.9}$$

Note that \bar{x} is also p -periodic. For every $t \in \mathbb{R}$ we have

$$\bar{x}(t) = \bar{x}(t + np) < e^{-\mu_m(t+np)} \bar{x}(0) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, we have $\bar{x}(t) \leq 0$ for every $t \in \mathbb{R}$. By the same token, with $\bar{x}(t - \tau) \leq 0$, we obtain from the second equation of system (5.8) that $\bar{y}(t) \leq 0, t \in \mathbb{R}$, and subsequently from the third equation of system (5.8) that $\bar{z}(t) \leq 0$ for every $t \in \mathbb{R}$. Then by the definition of $(\bar{x}, \bar{y}, \bar{z})$, we obtain that for every $t \in \mathbb{R}, x(t) \geq 0, y(t) \geq 0, z(t) \geq 0$.

If there exists $t_0 \in \mathbb{R}$ such that $x(t_0) = 0$, then by the first equation of system (5.1) we have $\dot{x}(t_0) > 0$. By the continuity of \dot{x} , there exists $\delta > 0$ such that x is strictly increasing in $(t_0 - \delta, t_0 + \delta)$. so that $x(t) < 0$ for $t \in (t_0 - \delta, t_0)$. This is a contradiction. By the same token we have $y(t) > 0$ and $z(t) > 0$ for every $t \in \mathbb{R}$. \square

Lemma 17 Let $f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$ be defined by

$$f_0(\theta_1, \theta_2) = - \begin{pmatrix} \mu_m \\ \mu_p \\ \mu_e \end{pmatrix} \cdot \theta_1 + \begin{pmatrix} \frac{\alpha_m}{1 + \left(\frac{z_2}{z}\right)^h} \\ \alpha_p x_2 \\ \alpha_e y_2 \end{pmatrix}$$

where $\theta_1 = (x_1, y_1, z_1)$ and $\theta_2 = (x_2, y_2, z_2)$. Then f_0 is Lipschitz continuous with a Lipschitz constant

$$L_f = \max \left\{ \mu_m, \mu_p, \mu_e, \alpha_p, \alpha_e, \frac{\alpha_m h_0}{z} \right\},$$

where $h_0 = \frac{h \left(1 - \frac{2}{h+1}\right)^{\frac{h-1}{h}}}{\left(1 + \frac{h-1}{h+1}\right)^2}$.

Proof We use the Mean Value theorem for integrals to obtain a Lipschitz constant. Let $\tilde{\theta}_1 = (\tilde{x}_1, \tilde{y}_1, \tilde{z}_1)$ and $\tilde{\theta}_2 = (\tilde{x}_2, \tilde{y}_2, \tilde{z}_2)$. Then we have

$$\begin{aligned} \left| f_0(\theta_1, \theta_2) - f_0(\tilde{\theta}_1, \tilde{\theta}_2) \right| &\leq \max \{ \mu_m, \mu_p, \mu_e \} |\theta_1 - \tilde{\theta}_1| \\ &+ \max \left\{ \alpha_p, \alpha_e, \sup_{z_2} \left| \frac{d}{dz_2} \frac{\alpha_m}{1 + \left(\frac{z_2}{z}\right)^h} \right| \right\} |\theta_2 - \tilde{\theta}_2|. \end{aligned} \tag{5.10}$$

We have

$$\frac{d}{dz_2} \frac{\alpha_m}{1 + \left(\frac{z_2}{z}\right)^h} = \frac{\alpha_m h}{z} \frac{\left(\frac{z_2}{z}\right)^{h-1}}{\left(1 + \left(\frac{z_2}{z}\right)^h\right)^2}.$$

Noticing that the map $\mathbb{R} \ni t \rightarrow \frac{t^{h-1}}{(1+t^h)^2}$ vanishes at $t = 0$ and $t = \infty$ and that

$$\frac{d}{dt} \frac{t^{h-1}}{(1+t^h)^2} = 0,$$

if and only if $t = \pm \left(1 - \frac{2}{h+1}\right)^{\frac{1}{h}}$, we obtain that $\sup_{z_2} \left| \frac{d}{dz_2} \frac{\alpha_m}{1 + \left(\frac{z_2}{z}\right)^h} \right| = \frac{\alpha_m h_0}{z}$ with

$$h_0 = \frac{h \left(1 - \frac{2}{h+1}\right)^{\frac{h-1}{h}}}{\left(1 + \frac{h-1}{h+1}\right)^2},$$

and the supremum is achieved at $\frac{z_2}{z} = \left(1 - \frac{2}{h+1}\right)^{\frac{1}{h}}$. Then by (5.10) f_0 is Lipschitz continuous with a Lipschitz constant $L_f = \max \left\{ \mu_m, \mu_p, \mu_e, \alpha_p, \alpha_e, \frac{\alpha_m h_0}{z} \right\}$. \square

To apply the global Hopf bifurcation theorem, we also use Lemma 12 to show the closure of all nontrivial periodic solutions bifurcating from the stationary point $(x(\alpha_m^*), y(\alpha_m^*), z(\alpha_m^*))$ will not include constant solution with zero period.

Lemma 18 *Let (x, y, z) be a periodic solution of system (5.1). If $\alpha_m < \frac{1}{c}$, then $\tau : \mathbb{R} \rightarrow \mathbb{R}$ given by $\tau(t) = c(x(t) - x(t - \tau(t)))$ exists and is continuously differentiable.*

Proof The existence and continuity of τ follows from Lemma 1. Let $f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f_1(\tau, t) = \tau - c(x(t) - x(t - \tau)).$$

Then f_1 is continuously differentiable with respect to (τ, t) . Moreover, by (5.11) we have

$$\frac{\partial f_1(\tau, t)}{\partial \tau} = 1 - c\dot{x}(t - \tau).$$

By the first equation of system (5.1) and by Lemma 6 we have for every $t \in \mathbb{R}$, $\dot{x}(t) < \alpha_m$ and

$$\dot{x}(t) \geq -\mu_m \frac{\alpha_m}{\mu_m} + \frac{\alpha_m}{1 + \left(\frac{\alpha_e \alpha_p \alpha_m}{\mu_e \mu_p z}\right)^h} > -\alpha_m. \tag{5.11}$$

Then we have $|\dot{x}| < \alpha_m$ and by (5.11) we have

$$\frac{\partial f_1(\tau, t)}{\partial \tau} = 1 - c\dot{x}(t - \tau) > 0.$$

By the Implicit Function Theorem, τ is continuously differentiable at $t \in \mathbb{R}$. □

It follows from Lemma 18 and (ii) of Lemma 12 that if $\alpha_m < \frac{1}{c}$, then the period p of every nonconstant periodic solution satisfies $p \geq \frac{4}{L_f(2+\|\bar{\tau}\|_{L^\infty})} > 0$.

Now we are in the position to state the global Hopf bifurcation theorem.

Theorem 7 *Let α_m^* be as in Lemma 15 and $p^* = \frac{2\pi}{v^*}$ where $v^* > 0$ is the imaginary part of eigenvalue of the formal linearization of system (5.1) at $\alpha_m = \alpha_m^*$. Suppose that $\alpha_m^* < \frac{1}{c}$. There exists a connected component \mathcal{C} of the closure of all nonconstant periodic solution of system (5.1) bifurcating from $(\alpha_m^*, p^*, x(\alpha_m^*), y(\alpha_m^*), z(\alpha_m^*)) \in \mathbb{R}^2 \times C(\mathbb{R}; \mathbb{R}^3)$, which satisfies that*

- (i) *either the projection of \mathcal{C} onto the parameter space of the period p is unbounded.*
- (ii) *or the projection of \mathcal{C} onto the parameter space of α_m does not cross $\alpha = 0$ but is not contained in any compact subset of the interval $(0, \frac{1}{c})$.*

Proof We first show that if $\alpha_m = 0$, system (5.1) has no nonconstant periodic solutions. Otherwise, let (x, y, z) be a nonconstant periodic solution with $\alpha_m = 0$. Then from system (5.1) $\dot{x} = -\mu_m x$ implies that $x = 0$ and subsequently $y = z = 0$. This is a contradiction. □

In the following we consider α_m in $(0, \frac{1}{c})$ and introduce the following change of variables:

$$\alpha_m = q(\alpha) = \frac{2}{c\pi} \left(\arctan \alpha - \frac{\pi}{2} \right) + \frac{1}{c}, \tag{5.12}$$

where q is an increasing function of α with $\lim_{\alpha \rightarrow -\infty} q(\alpha) = 0$ and $\lim_{\alpha \rightarrow +\infty} q(\alpha) = \frac{1}{c}$. Then system (5.1) is rewritten as

$$\begin{cases} \frac{dx(t)}{dt} = -\mu_m x(t) + \frac{q(\alpha)}{1 + \left(\frac{z(t-\tau)}{z}\right)^h}, \\ \frac{dy(t)}{dt} = -\mu_p y(t) + \alpha_p x(t - \tau), \\ \frac{dz(t)}{dt} = -\mu_e z(t) + \alpha_e y(t - \tau), \\ \tau(t) = c(x(t) - x(t - \tau)), \end{cases} \tag{5.13}$$

with $\alpha \in \mathbb{R}$ and $\alpha^* = q^{-1}(\alpha_m^*)$ the critical value of α for a unique Hopf bifurcation point. By Theorem 5 There exists a connected component C_0 of the closure of all nonconstant periodic solution of system (5.13) bifurcating from the stationary point $(\alpha^*, p^*, x(\alpha^*), y(\alpha^*), z(\alpha^*)) \in \mathbb{R}^2 \times C(\mathbb{R}; \mathbb{R}^3)$.

By Lemma 17, condition (S4) is satisfied by system (5.13). By Lemma 18, the function τ defined by $\tau(t) = c(x(t) - x(t - \tau(t)))$ for a nonconstant periodic solution (x, y, z) of system (5.13) is continuously differentiable. Hence by Lemma 12, the period p of every nonconstant periodic solution (x, y, z) of system (5.13) is positive. Notice that $(\alpha^*, p^*, x(\alpha^*), y(\alpha^*), z(\alpha^*))$ is the only bifurcation point of system (5.13), by Theorem 4, the connected component C_0 is unbounded in $\mathbb{R}^2 \times C(\mathbb{R}; \mathbb{R}^3)$.

Notice that by Theorem 6, the projection of C_0 onto the space of $(x, y, z) \in C(\mathbb{R}; \mathbb{R}^3)$ is bounded. The unboundedness of C_0 is either because of the unbounded projection onto the parameter space of the period p , or the projection of C onto the parameter space of α .

Notice that q induces a homeomorphism $(q, id) : \mathbb{R}^2 \times C(\mathbb{R}; \mathbb{R}^3) \rightarrow \mathbb{R}^2 \times C(\mathbb{R}; \mathbb{R}^3)$ defined by

$$(q, id)(\alpha, h) = (q(\alpha), h).$$

The image $\mathcal{C} = (q, id)(C_0)$ of C_0 under (q, id) is a connected component of the closure of all nonconstant periodic solution of system (5.1) bifurcating from the bifurcation point

$$(\alpha_m^*, p^*, x(\alpha_m^*), y(\alpha_m^*), z(\alpha_m^*)) \in \mathbb{R}^2 \times C(\mathbb{R}; \mathbb{R}^3),$$

which satisfies that either the projection of \mathcal{C} onto the parameter space of the period p is unbounded, or the projection of \mathcal{C} onto the parameter space of α_m does not cross the hyperplane $\alpha_m = 0$ but is not contained in any compact subset of the interval $(0, \frac{1}{c})$. \square

6 Concluding Remarks

Motivated by the extended Goodwin’s model with a state-dependent delay governed by an algebraic equation, we developed global Hopf bifurcation theories for differential-algebraic equations with state-dependent delay, using the S^1 -equivariant degree. This is based on the framework described in [6] where the technique of formal linearization is employed to obtain auxiliary linear systems at the stationary states which indicate local and global Hopf bifurcation using a homotopy argument. We remark that the work discussed two types of state-dependent delays: the one in system (1.5) depends on the state variable explicitly and the other in system (1.1) depends on the state variable implicitly. We also remark that the local and global Hopf bifurcation theory we developed for system (1.5) is also applicable for systems with the delay given by $\tau(t) = h(x_t)$ where h is a function of $x_t(s) = x(t + s)$, $-r \leq s \leq 0$, $r > 0$, provided that τ is continuous.

The local and global Hopf bifurcation theories are applied to the extended Goodwin's model which describes intracellular processes in the genetic regulatory dynamics. We obtained two alternatives for the connected component \mathcal{C} of periodic solutions in the Fuller space $\mathbb{R}^2 \times C(\mathbb{R}; \mathbb{R}^3)$. Namely, the projection of \mathcal{C} onto the parameter space of the period \mathbf{p} is unbounded, or the projection onto the parameter space of α_m is not contained in any compact subset of the interval $(0, \frac{1}{c})$. We remark that in the previous case, there exists a sequence of periodic solutions with periods going to ∞ . From (4.6), system (5.1) can be represented as

$$\frac{2\pi}{\mathbf{p}} \frac{dx}{dt} = \mathcal{N}_0(x, \alpha_m, \mathbf{p}), \quad \mathbf{p} > 0,$$

where x is normalized to be 2π -periodic. Notice from the definition of \mathcal{N}_0 at (2.7) that \mathbf{p} appears only in the time domain of \mathcal{N}_0 . Note also that the periodic solutions are uniformly bounded with $\alpha_m \in (0, \frac{1}{c})$. Then with $\mathbf{p} \rightarrow \infty$, this alternative implies the possibility that the system has a sequence of nonconstant periodic solutions with the limiting profile satisfying the algebraic equation $\mathcal{N}_0(x, \alpha_m, \mathbf{p}) = 0$. See ([10–12]) for a discussion of limiting profiles for differential equations with state-dependent delays.

If the projection of \mathcal{C} onto the parameter space of the period \mathbf{p} is bounded, we have the latter alternative that the projection of \mathcal{C} onto the parameter space of the period α_m is not contained in any compact subset of the interval $(0, \frac{1}{c})$. Since \mathcal{C} will not cross the hyperplane $\alpha_m = 0$, and will not blow up at $\alpha_m = \frac{1}{c}$ with the boundedness of the solutions and periods, \mathcal{C} must cross the hyperplane $\alpha_m = \frac{1}{c}$ leaving the solutions at $\alpha_m \geq \frac{1}{c}$ out of the scope of the discussion.

We also remark that the state-dependent delay in system (1.5) may be negative or positive and is not a priori advanced or retarded type delay differential equations. It remains open to investigate this type of systems in general settings for a qualitative theory including existence and uniqueness of solutions. For systems with mixed type constant delays, see, among many others [13, 14].

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