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Global Classical Solutions, Stability of Constant Equilibria, and Spreading Speeds in Attraction–Repulsion Chemotaxis Systems with Logistic Source on $\mathbb{R}^N$

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Dedicated to the memory of Professor George Sell

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Abstract In this paper, we consider the following chemotaxis systems of parabolic–elliptic–elliptic type on $\mathbb{R}^N$,

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla(u \nabla v_1) + \chi_2 \nabla(u \nabla v_2) + u(a - bu), & x \in \mathbb{R}^N,\ t > 0, \\ 0 = (\Delta - \lambda_1 I)v_1 + \mu_1 u, & x \in \mathbb{R}^N,\ t > 0, \\ 0 = (\Delta - \lambda_2 I)v_2 + \mu_2 u, & x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \mathbb{R}^N, \end{cases}$$

where $\chi_i \geq 0$, $\lambda_i > 0$, $\mu_i > 0$ $(i = 1, 2)$ and $a > 0$, $b > 0$ are constant real numbers, and N is a positive integer. First, under some conditions on the parameters $\chi_i, \mu_i, \lambda_i, a, b$ and N, we prove the global existence and boundedness of classical solutions $(u(x, t; u_0), v_1(x, t; u_0), v_2(x, t; u_0))$ for nonnegative, bounded, and uniformly continuous initial functions $u_0(x)$. Next, we explore the asymptotic stability of the constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$ and prove under some further assumption on the parameters that, for every strictly positive initial $u_0(x)$,

$$\lim_{t\to\infty} \Big[\|u(\cdot, t; u_0) - \frac{a}{b}\|_\infty + \|\lambda_1 v_1(\cdot, t; u_0) - \frac{a}{b}\mu_1\|_\infty + \|\lambda_2 v_2(\cdot, t; u_0) - \frac{a}{b}\mu_2\|_\infty \Big] = 0.$$

Finally, we investigate the spreading properties of the global solutions with compactly supported initial functions. We show that under some conditions on the parameters, there are two positive numbers $0 < c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) \leq c_+^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2)$ such that for every nonnegative initial function $u_0(x)$ with nonempty and compact support, we have

$$\lim_{t\to\infty} \left[\sup_{|x|\leq ct} |u(x, t; u_0) - \frac{a}{b}| + \sup_{|x|\leq ct} |\lambda_1 v_1(x, t; u_0) - \frac{a}{b}\mu_1| + \sup_{|x|\leq ct} |\lambda_2 v_2(x, t; u_0) - \frac{a}{b}\mu_2| \right] = 0$$

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whenever $0 \leq c < c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2)$, and

$$\lim_{t\to\infty}\left[\sup_{|x|\geq ct}|u(x,t;u_0)| + \sup_{|x|\geq ct}|v_1(x,t;u_0)| + \sup_{|x|\geq ct}|v_2(x,t;u_0)|\right] = 0$$

whenever $c > c_+^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2)$. Furthermore we show that

$$\lim_{(\chi_1,\chi_2)\to(0,0)} c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = \lim_{(\chi_1,\chi_2)\to(0,0)} c_+^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = 2\sqrt{a}.$$

Keywords Parabolic–elliptic chemotaxis system · Logistic source · Classical solution · Local existence · Global existence · Asymptotic stability · Spreading speeds

Mathematics Subject Classification 35B35 · 35B40 · 35K57 · 35Q92 · 92C17

1 Introduction and the Statement of the Main Results

Chemotaxis describes the oriented movement of biological cells or organism in response to chemical gradients. The oriented movement of cells has a crucial role in a wide range of biological phenomena. At the beginning of 1970s, Keller and Segel (see [23,24]) introduced systems of partial differential equations of the following form to model the time evolution of both the density $u(x,t)$ of a mobile species and the density $v(x,t)$ of a chemoattractant,

$$\begin{cases} u_t = \nabla \cdot (m(u)\nabla u - \chi(u,v)\nabla v) + f(u,v), & x \in \Omega, \\ \tau v_t = \Delta v + g(u,v), & x \in \Omega, \end{cases} \tag{1.1}$$

complemented with certain boundary condition on $\partial\Omega$ if Ω is bounded, where $\Omega \subset \mathbb{R}^N$ is an open domain; $\tau \geq 0$ is a non-negative constant linked to the speed of diffusion of the chemical; the function $\chi(u,v)$ represents the sensitivity with respect to chemotaxis; and the functions f and g model the growth of the mobile species and the chemoattractant, respectively. In literature, (1.1) is called the Keller–Segel (KS) model or a chemotaxis model.

Since the works by Keller and Segel, a rich variety of mathematical models for studying chemotaxis has appeared (see [1,6,7,13,17,18,22,33,41–43,46,49–54,57], and the references therein). The reader is referred to [16,19] for some detailed introduction into the mathematics of KS models. In the current paper, we consider chemoattraction-repulsion process in which cells undergo random motion and chemotaxis towards attractant and away from repellent [31]. Moreover, we consider the model with proliferation and death of cells and assume that chemicals diffuse very quickly. These lead to the model of partial differential equations as follows:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla(u\nabla v_1) + \chi_2\nabla(u\nabla v_2) + u(a - bu), & x \in \Omega,\ t > 0, \\ 0 = (\Delta - \lambda_1 I)v_1 + \mu_1 u, & x \in \Omega,\ t > 0, \\ 0 = (\Delta - \lambda_2 I)v_2 + \mu_2 u, & \text{in } x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

complemented with certain boundary condition on $\partial\Omega$ if Ω is bounded.

When Ω is a smooth bounded domain, it is seen that (1.2) complemented with Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v_1}{\partial n} = \frac{\partial v_2}{\partial n} = 0, \tag{1.3}$$

has a unique nonzero constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$. The global existence of classical solutions and the stability of the above equilibrium solution of (1.2)+(1.3) are among central dynamical issues. They have been studied in many papers (see [8,20,21,27,29–31,44,45,55,56] and the references therein). For example, in [55], amount others, the authors proved that

- *If $b > \chi_1\mu_1 - \chi_2\mu_2$, or $N \le 2$, or $\frac{N-2}{N}(\chi_1\mu_1 - \chi_2\mu_2) < b$ and $N \ge 3$, then for every nonnegative initial $u_0 \in C^0(\overline{\Omega})$, (1.2)+(1.3) has a unique global classical solution $(u(\cdot,\cdot), v_1(\cdot,\cdot), v_2(\cdot,\cdot))$ which is uniformly bounded.*
- *If $a = b > 2\chi_1\mu_1$, then for every nonnegative initial $u_0 \in C^0(\overline{\Omega})$, $u_0 \ne 0$, the global classical solution $(u(\cdot,\cdot), v_1(\cdot,\cdot), v_2(\cdot,\cdot))$ of (1.2)+(1.3) satisfies*

$$\lim_{t\to\infty}\Big[\|u(\cdot,t)-1\|_{C^0(\Omega)} + \|v_1(\cdot,t) - \frac{\mu_1}{\lambda_1}\|_{C^0(\Omega)} + \|v_2(\cdot,t) - \frac{\mu_2}{\lambda_2}\|_{C^0(\Omega)}\Big] = 0.$$

While attraction–repulsion chemotaxis systems on bounded domains have been studied in many papers, there is little study of such systems on unbounded domains. The objective of this paper is to study the dynamics of (1.2) with $\Omega = \mathbb{R}^N$, that is,

$$\begin{cases} u_t = \Delta u - \chi_1\nabla(u\nabla v_1) + \chi_2\nabla(u\nabla v_2) + u(a-bu), & x\in\mathbb{R}^N,\ t>0,\\ 0 = (\Delta - \lambda_1 I)v_1 + \mu_1 u, & x\in\mathbb{R}^N,\ t>0,\\ 0 = (\Delta - \lambda_2 I)v_2 + \mu_2 u, & x\in\mathbb{R}^N,\ t>0,\\ u(\cdot,0) = u_0, & x\in\mathbb{R}^N. \end{cases} \tag{1.4}$$

In the case that the chemorepellent is absent, that is, $\chi_2 = 0$, the authors of the current paper studied in [36] the global existence of classical solutions and asymptotic behavior of bounded global classical solutions of (1.4). In the current paper, we investigate the global existence of classical solutions, stability of constant equilibria, and spreading speeds of (1.4) when both chemoattractant and chemorepellent are present. More precisely, we identify the circumstances under which positive classical solutions of (1.4) with nonnegative, bounded, and uniformly continuous initial functions exist globally; investigate the asymptotic stability of the nonzero constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$; and explore the spreading properties of the global solutions with compactly supported initial functions. We pay special attention to the combined effect of the chemoattractant and chemorepellent on the above dynamical issues.

Note that, due to biological interpretations, only nonnegative initial functions will be of interest. We call $(u(x,t), v_1(x,t), v_2(x,t))$ a *classical solution* of (1.4) on $[0,T)$ if $u, v_1, v_2 \in C(\mathbb{R}^N\times[0,T))\cap C^{2,1}(\mathbb{R}^N\times(0,T))$ and satisfies (1.4) for $(x,t)\in\mathbb{R}^N\times(0,T)$ in the classical sense. A classical solution $(u(x,t), v_1(x,t), v_2(x,t))$ of (1.4) on $[0,T)$ is called *nonnegative* if $u(x,t)\ge 0$, $v_1(x,t)\ge 0$ and $v_2(x,t)\ge 0$ for all $(x,t)\in\mathbb{R}^N\times[0,T)$. A *global classical solution* of (1.4) is a classical solution on $[0,\infty)$.

Let

$$C^b_{\rm unif}(\mathbb{R}^N) = \{u\in C(\mathbb{R}^N)\,|\, u(x) \text{ is uniformly continuous in } x\in\mathbb{R}^N \text{ and } \sup_{x\in\mathbb{R}^N}|u(x)|<\infty\} \tag{1.5}$$

equipped with the norm $\|u\|_\infty = \sup_{x\in\mathbb{R}^N}|u(x)|$. We have the following result on the global existence of classical solutions of (1.4) for initial functions belonging to $C^b_{\rm unif}(\mathbb{R}^N)$.

Theorem A *Suppose that*

$$\chi_1 = a = b = 0 \tag{1.6}$$

or

$$b > \chi_1\mu_1 - \chi_2\mu_2 + M, \tag{1.7}$$

where

$$M := \min\Big\{\frac{1}{\lambda_2}\big((\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big),\\ \frac{1}{\lambda_1}\big((\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_2\mu_2(\lambda_1-\lambda_2)_+\big)\Big\}. \tag{1.8}$$

Then for every nonnegative initial function $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$, (1.4) has a unique nonnegative global classical solution $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$ with $u(\cdot,0;u_0) = u_0$. Furthermore, it holds that

$$\|u(\cdot,t;u_0)\|_\infty \le \begin{cases} \|u_0\|_\infty & \textit{if } (1.6) \textit{ holds} \\ \max\{\|u_0\|_\infty, \frac{a}{b+\chi_2\mu_2-\chi_1\mu_1-M}\} & \textit{if } (1.7) \textit{ holds}. \end{cases} \tag{1.9}$$

Remark 1.1 $M \le \chi_2\mu_2$. (1.6) and (1.7) provide explicit conditions for the global existence of classical solutions. The following special and important conditions follow from (1.7).

- (i) If $b > \chi_1\mu_1$, (1.4) always has global bounded classical solution for any initial $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ with $u_0 \ge 0$.
- (ii) If $\lambda_1 \le \lambda_2$ and $\chi_2\mu_2\lambda_2 \ge \chi_1\mu_1\lambda_1$, we have that $M = \chi_2\mu_2 - \frac{\lambda_1}{\lambda_2}\chi_1\mu_1$. In this case, it follows from Theorem A that for every nonnegative bounded and uniformly continuous initial data u_0, (1.4) has a unique bounded global classical solution $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$, whenever $b > \chi_1\mu_1(1-\frac{\lambda_1}{\lambda_2})$. Thus, in the absence of chemoattractant, i.e $\chi_1 = 0$, for every nonnegative bounded and uniformly continuous initial data u_0, (1.4) has a unique bounded global classical solution $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$, whenever $b > 0$.
- (iii) If $\lambda_1 \le \lambda_2$ and $\chi_2\mu_2\lambda_2 \le \chi_1\mu_1\lambda_1$, we have that $M = 0$. In this case, it follows from Theorem A that for every nonnegative bounded and uniformly continuous initial data u_0, (1.4) has a unique bounded global classical solution $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$, whenever $b > \chi_1\mu_1 - \chi_2\mu_2$.
- (iv) We note that if $\lambda_1 \ge \lambda_2$ and $\chi_2\mu_2\lambda_2 \ge \chi_1\mu_1\lambda_1$, then $M = \chi_2\mu_2 - \chi_1\mu_1$. Thus, if $\lambda_1 \ge \lambda_2$ and $\chi_2\mu_2\lambda_2 \ge \chi_1\mu_1\lambda_1$, it follows from Theorem A that for every $b > 0$ and for every nonnegative bounded and uniformly continuous initial data u_0, (1.4) has a unique bounded global classical solution $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$.
- (v) If $\lambda_1 \ge \lambda_2$ and $\chi_2\mu_2\lambda_2 \le \chi_1\mu_1\lambda_1$, we have that $M = \frac{(\lambda_1-\lambda_2)\chi_2\mu_2}{\lambda_1}$. In this case, it follows from Theorem A that for every nonnegative bounded and uniformly continuous initial data u_0, (1.4) has a unique bounded global classical solution $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$, whenever $b > \chi_1\mu_1 - \frac{\lambda_2}{\lambda_1}\chi_2\mu_2$.

It follows from Remark 1.1 (iii)&(v), that when $\chi_2 = 0$, we recover as a special case Theorem 1.5 in [36] for the case $b > \chi_1$ and $\mu_1 = 1$. When (1.7) does not hold, we leave it open whether for any nonnegative initial function $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ global solution to (1.4) exists.

Theorem A is fundamental. Assume the conditions in Theorem A. Then (1.4) generates a dynamical system on the infinite dimensional space $X^+ = \{u \in C^b_{\text{unif}}(\mathbb{R}^N) \,|\, u \ge 0\}$. Methods and theorems for general infinite dimensional dynamical systems in literature (e.g. [14,38]) may then be utilized for the further study of many important dynamical aspects, including the long time behavior of bounded solutions, stability of certain special solutions, existence of global attractor, etc. In the following, we explore the stability of the nonzero constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$.

We first study the stability of $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$ with respect to strictly positive initial functions. From now on, we shall always suppose that $a > 0$, unless otherwise specified. We prove

Theorem B *Suppose that*

$$b > \chi_1\mu_1 - \chi_2\mu_2 + K, \tag{1.10}$$

where

$$K := \min\Big\{\frac{1}{\lambda_2}\Big(|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_1\mu_1|\lambda_1-\lambda_2|\Big), \\ \frac{1}{\lambda_1}\Big(|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_2\mu_2|\lambda_1-\lambda_2|\Big)\Big\}. \tag{1.11}$$

Then for every initial function $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $\inf_{x\in\mathbb{R}^N} u_0(x) > 0$, (1.4) has a unique bounded global classical solution $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$ with $u(\cdot,0;u_0) = u_0$. Furthermore we have that

$$\lim_{t\to\infty}\|u(\cdot,t;u_0) - \frac{a}{b}\|_\infty = 0 \tag{1.12}$$

and

$$\lim_{t\to\infty}\|\lambda_i v_i(\cdot,t;u_0) - \frac{a}{b}\mu_i\|_\infty = 0,\ \forall\, i = 1,2. \tag{1.13}$$

Remark 1.2 (1) (1.10) provides explicit conditions for the global stability of the constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$ with respect to strictly positive initial functions. We point out the following special and important equivalent conditions of (1.10).

(i) If $\lambda_1 \le \lambda_2$, and $\chi_2\mu_2\lambda_2 \ge \chi_1\mu_1\lambda_1$, then (1.10) holds if and only if $b > 2\chi_1\mu_1 - 2\frac{\lambda_1}{\lambda_2}\chi_1\mu_1$.

(ii) If $\lambda_1 \le \lambda_2$, and $\chi_2\mu_2\lambda_2 \le \chi_1\mu_1\lambda_1$, then (1.10) holds if and only if $b > 2\chi_1\mu_1 - 2\chi_2\mu_2$.

(iii) If $\lambda_1 \ge \lambda_2$, and $\chi_2\mu_2\lambda_2 \ge \chi_1\mu_1\lambda_1$, then (1.10) holds if and only if $b > 0$.

(iv) If $\lambda_1 \ge \lambda_2$, and $\chi_1\mu_1\lambda_1 \ge \chi_2\mu_2\lambda_2$, then (1.10) holds if and only if $b > 2\chi_1\mu_1 - 2\frac{\lambda_2}{\lambda_1}\chi_2\mu_2$.

(2) By (i)-(iv), if $b > 2\chi_1\mu_1$, then (1.10) holds. Hence the hypothesis (1.10) is weaker than the known result on bounded domain.

(3) If $\chi_2 = 0$, then (ii) and (iv) extend [36, Theorem 1.7].

(4) By (i) and (iii), if $\chi_1 = 0$, then the constant solution $\frac{a}{b}$ is stable with respect to strictly positive perturbation whenever $b > 0$.

(5) It is interesting to know whether hypothesis (1.7) is enough to have the stability of the constant steady solution $(\frac{a}{b}, \frac{a\mu_1}{b\lambda_1}, \frac{a\mu_2}{b\lambda_2})$ with respect to strictly positive perturbation. We plan to study this question in our future work.

Next, we study the attraction of $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$ with respect to global classical solutions of (1.4) with compactly supported initial functions, or equivalently, the spreading properties of global classical solutions of (1.4) with compactly supported initial functions. For $x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$, let $|x| = \big(\sum_{i=1}^N x_i^2\big)^{\frac{1}{2}}$. We obtain the following main results.

Theorem C *Suppose that (1.7) holds and define*

$$D := \min\Big\{\frac{|\chi_2\mu_2 - \chi_1\mu_1|}{2\sqrt{\lambda_2}} + \frac{\chi_1\mu_1|\sqrt{\lambda_1} - \sqrt{\lambda_2}|}{2\sqrt{\lambda_1\lambda_2}}, \frac{|\chi_1\mu_1 - \chi_2\mu_2|}{2\sqrt{\lambda_1}} + \frac{\chi_2\mu_2|\sqrt{\lambda_2} - \sqrt{\lambda_1}|}{2\sqrt{\lambda_1\lambda_2}}\Big\}. \tag{1.14}$$

Then for every $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ *with* $u_0 \geq 0$ *and* $supp(u_0)$ *being compact and non-empty, we have that*

$$\lim_{t\to\infty}\left[\sup_{|x|\geq ct}|u(x,t;u_0)| + \sup_{|x|\geq ct}|v_1(x,t;u_0)| + \sup_{|x|\geq ct}|v_2(x,t;u_0)|\right] = 0 \tag{1.15}$$

for every $c > c_+^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)$*, where*

$$c_+^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) = 2\sqrt{a} + \frac{\sqrt{a}(D\sqrt{Na}+\chi_2\mu_2)}{b+\chi_2\mu_2-\chi_1\mu_1-M}, \tag{1.16}$$

and M *is given by* (1.8).

Remark 1.3 (i) If $\lambda_1 \leq \lambda_2$ and $\chi_2\mu_2\lambda_2 \geq \chi_1\mu_1\lambda_1$, then

$$c_+^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) = 2\sqrt{a} + \frac{\sqrt{a}(D\sqrt{Na}+\chi_2\mu_2)}{b-\left(1-\frac{\lambda_1}{\lambda_2}\right)\chi_1\mu_1}.$$

(ii) If $\lambda_1 \leq \lambda_2$ and $\chi_2\mu_2\lambda_2 \leq \chi_1\mu_1\lambda_1$, then

$$c_+^*(\chi_1,\mu_1,\lambda_1,\chi_1,\mu_2,\lambda_2) = 2\sqrt{a} + \frac{\sqrt{a}(D\sqrt{Na}+\chi_2\mu_2)}{b+\chi_2\mu_2-\chi_1\mu_1}.$$

(iii) If $\lambda_1 \geq \lambda_2$ and $\chi_2\mu_2\lambda_2 \geq \chi_1\mu_1\lambda_1$ then

$$c_+^*(\chi_1,\mu_1,\lambda_1,\chi_1,\mu_2,\lambda_2) = 2\sqrt{a} + \frac{\sqrt{a}(D\sqrt{Na}+\chi_2\mu_2)}{b}.$$

(iv) If $\lambda_1 \geq \lambda_2$ and $\chi_2\mu_2\lambda_2 \leq \chi_1\mu_1\lambda_1$, then

$$c_+^*(\chi_1,\mu_1,\lambda_1,\chi_1,\mu_2,\lambda_2) = 2\sqrt{a} + \frac{\sqrt{a}(D\sqrt{Na}+\chi_2\mu_2)}{b-\frac{1}{\lambda_1}(\chi_1\mu_1\lambda_1-\chi_2\mu_2\lambda_2)}.$$

(v) Note that $\chi_2 = 0$ implies that $D = \frac{\chi_1\mu_1}{2\sqrt{\lambda_1}}$ and $M = 0$. Hence if $\chi_2 = 0$, it follows from Theorem C that $c_+^*(\chi_1,\mu_1,\lambda_1,0,\mu_2,\lambda_2) = 2\sqrt{a} + \frac{a\chi_1\mu_1\sqrt{N}}{2(b-\chi_1\mu_1)\sqrt{\lambda_1}}$. Thus, in the case $\chi_2 = 0$, and $\mu_1 = \lambda_1 = 1$, we obtain a better estimate for $c_+^*(\chi_1,\mu_1,\lambda_1,0,\mu_2,\lambda_2)$ compare to the one giving by [37, Remark 1.2(iii)].

Theorem D *Suppose that* (1.10) *holds and*

$$4a(1-L) - \frac{Na^2D^2}{(b+\chi_2\mu_2-\chi_1\mu_1-M)^2} > 0, \tag{1.17}$$

where M *is given by* (1.8) *and*

$$L := \min\left\{\frac{(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-}{\lambda_2(b+\chi_2\mu_2-\chi_1\mu_1-M)},\right.$$
$$\left.\frac{(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_2\mu_2(\lambda_1-\lambda_2)_-}{\lambda_1(b+\chi_2\mu_2-\chi_1\mu_1-M)}\right\}. \tag{1.18}$$

Then for every $u_0 \in C^b_{\text{unif}}(\mathbb{R}^N)$ *with* $u_0 \geq 0$ *and* $supp(u_0)$ *being non-empty, we have that*

$$\lim_{t\to\infty}\left[\sup_{|x|\leq ct}|u(x,t;u_0)-\frac{a}{b}| + \sup_{|x|\leq ct}|\lambda_1 v_1(x,t;u_0)-\frac{a}{b}\mu_1| + \sup_{|x|\leq ct}|\lambda_2 v_2(x,t;u_0)-\frac{a}{b}\mu_2|\right] = 0 \tag{1.19}$$

for every $0 \le c < c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2)$*, where*

$$c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = 2\sqrt{a(1-L)} - \frac{aD\sqrt{N}}{b + \chi_2\mu_2 - \chi_1\mu_1 - M}.$$

Remark 1.4 (i) If $\lambda_1 \le \lambda_2$ and $\chi_2\mu_2\lambda_2 \ge \chi_1\mu_1\lambda_1$, then $L = \frac{\chi_1\mu_1(1-\frac{\lambda_1}{\lambda_2})}{b-\chi_1\mu_1(1-\frac{\lambda_1}{\lambda_2})}$ and

$$c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = 2\sqrt{\frac{a\left(b - 2\chi_1\mu_1\left(1-\frac{\lambda_1}{\lambda_2}\right)\right)}{b - \chi_1\mu_1\left(1-\frac{\lambda_1}{\lambda_2}\right)}} - \frac{aD\sqrt{N}}{b - \chi_1\mu_1\left(1-\frac{\lambda_1}{\lambda_2}\right)}.$$

(ii) If $\lambda_1 \le \lambda_2$ and $\chi_2\mu_2\lambda_2 \le \chi_1\mu_1\lambda_1$, then $L = \frac{\chi_1\mu_1-\chi_2\mu_2}{b+\chi_2\mu_2-\chi_1\mu_1}$ and

$$c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = 2\sqrt{\frac{a(b - 2(\chi_1\mu_1 - \chi_2\mu_2))}{b + \chi_2\mu_2 - \chi_1\mu_1}} - \frac{aD\sqrt{N}}{b + \chi_2\mu_2 - \chi_1\mu_1}.$$

(iii) If $\lambda_1 \ge \lambda_2$ and $\chi_2\mu_2\lambda_2 \ge \chi_1\mu_1\lambda_1$, then $L = 0$ and

$$c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = 2\sqrt{a} - \frac{aD\sqrt{N}}{b}.$$

(iv) If $\lambda_1 \ge \lambda_2$ and $\chi_2\mu_2\lambda_2 \le \chi_1\mu_1\lambda_1$, then $L = \frac{\chi_1\mu_1\lambda_1-\chi_2\mu_2\lambda_2}{\lambda_1(b+\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)}$ and

$$c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = 2\sqrt{\frac{a(b - \frac{2}{\lambda_1}(\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2))}{b - \frac{1}{\lambda_1}(\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2)}} - \frac{aD\sqrt{N}}{b - \frac{1}{\lambda_1}(\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2)}.$$

(v) If $\chi_2 = 0$, by (ii) and (iv), we have that $c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2) = 2\sqrt{\frac{a(b-2\chi_1\mu_1)}{b-\chi_1\mu_1}} - \frac{a\chi_1\mu_1\sqrt{N}}{2(b-\chi_1\mu_1)\sqrt{\lambda_1}}$. Hence in the case $\chi_2 = 0$, $\mu_1 = \lambda_1 = 1$, we obtain a better estimate on $c_-^*(\chi_1, \mu_1, \lambda_1, \chi_2, \mu_2, \lambda_2)$ than the ones obtained in [37] and [36].

Observe that, if either $\chi_1 = \chi_2 = 0$ or $\chi_1 - \chi_2 = \mu_1 - \mu_2 = \lambda_1 - \lambda_2 = 0$, the first equation in (1.4) becomes the following scalar reaction diffusion equation,

$$u_t = \Delta u + u(a - bu), \quad x \in \mathbb{R}^N, \ t > 0, \tag{1.20}$$

which is referred to as Fisher or KPP equations due to the pioneering works by Fisher ([9]) and Kolmogorov, Petrowsky, Piscunov ([25]) on the spreading properties of (1.20). It follows from the works [9,25], and [47] that c_-^* and c_+^* in Theorem C and Theorem D, respectively, can be chosen so that $c_-^* = c_+^* = 2\sqrt{a}$ ($c^* := 2\sqrt{a}$ is called the *spatial spreading speed* of (1.20) in literature), and that (1.20) has traveling wave solutions $u(t, x) = \phi(x - ct)$ connecting $\frac{a}{b}$ and 0 (i.e. $(\phi(-\infty) = \frac{a}{b}, \phi(\infty) = 0)$) for all speeds $c \ge c^*$ and has no such traveling wave solutions of slower speed. Since the pioneering works by Fisher [9] and Kolmogorov, Petrowsky, Piscunov [25], a huge amount research has been carried out toward the spreading properties of reaction diffusion equations of the form,

$$u_t = \Delta u + uf(t, x, u), \quad x \in \mathbb{R}^N, \tag{1.21}$$

where $f(t,x,u) < 0$ for $u \gg 1$, $\partial_u f(t,x,u) < 0$ for $u \geq 0$ (see [2–5,10,11,26,28,32,34,35,39,40,47,48,58], etc.).

Remark 1.5 (i) It is clear from Theorem C and Theorem D that

$$\lim_{(\chi_1,\chi_2)\to(0,0)} c_-^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) = \lim_{(\chi_1,\chi_2)\to(0,0)} c_+^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) = 2\sqrt{a}$$

and

$$\begin{aligned}
&\lim_{(\delta_1,\delta_2,\delta_3)\to(0,0,0)} c_-^*(\chi+\delta_1,\mu+\delta_2,\lambda+\delta_3,\chi,\mu,\lambda)\\
&= \lim_{(\delta_1,\delta_2,\delta_3)\to(0,0,0)} c_+^*(\chi+\delta_1,\mu+\delta_2,\lambda+\delta_3,\chi,\mu,\lambda)\\
&\lim_{(\delta_1,\delta_2,\delta_3)\to(0,0,0)} c_-^*(\chi,\mu,\lambda,\chi+\delta_1,\mu+\delta_2,\lambda+\delta_3)\\
&= \lim_{(\delta_1,\delta_2,\delta_3)\to(0,0,0)} c_+^*(\chi,\mu,\lambda,\chi+\delta_1,\mu+\delta_2,\lambda+\delta_3)\\
&= 2\sqrt{a}, \quad \forall\, \chi > 0, \mu > 0 \text{ and } \lambda > 0.
\end{aligned}$$

Hence we recover the know results in the literature when $\chi_1 = \chi_2 = 0$ or $\chi_1 - \chi_2 = \mu_1 - \mu_2 = \lambda_1 - \lambda_2 = 0$.

(ii) For every $\chi_i \geq 0,\ \mu_i > 0,\ \lambda_i > 0$, let

$$c_{up}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) = \inf\{c^* > 0 \mid (1.15) \text{ holds}\}$$

and

$$c_{low}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) = \sup\{c^* \geq 0 \mid (1.19) \text{ holds}\}.$$

$[c_{low}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2), c_{up}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)]$ is called the spreading speed interval of (1.4). Theorem C implies that if (1.7) holds, then

$$c_{up}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) \leq c_+^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) < \infty.$$

Under the hypotheses of Theorem D, we have that

$$c_{low}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) \geq c_-^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) > 0.$$

It is interesting to know the relationship between $c_{up}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)$ and $2\sqrt{a}$ as well as the relationship between $c_{low}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)$ and $2\sqrt{a}$. It is also interesting to know whether $c_{low}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) = c_{up}^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)$. We plan to study these questions in our future works.

(iii) When $\chi_2 = 0$, $\lambda_1 = \mu_1 = 1$, and $0 < \chi_1 < \frac{b}{2}$, in a very recent work [37] it was shown that there is a positive constant $c^*(\chi_1) \geq 2\sqrt{a}$ such that for every $c \geq c^*(\chi_1)$ and $\xi \in S^{N-1}$, (1.4) has a traveling wave solution $(u(x,t), v(x,t)) = (u(x\cdot\xi - ct), v(x\cdot\xi - ct))$ connecting the trivial solutions $(\frac{a}{b}, \frac{a}{b})$ and $(0,0)$ and propagating in the direction of ξ with speed c, and no such traveling wave solution exists for speed less than $2\sqrt{a}$. We plan to study these questions for (1.4) when both $\chi_1 > 0$ and $\chi_2 > 0$.

We end up the introduction with the following remarks. First, our study is based on many techniques developed in [36]. But, to apply these techniques to (1.4) with non-zeros χ_1 and χ_2, nontrivial modifications are needed and made in the current paper. The modified techniques would be useful for the further study of attraction–repulsion chemotaxis systems. Second, most results obtained in [36] for the special case $\chi_2 = 0$ are recovered and extended further in the current paper. Third, conditions explicitly depending on the sensitivity parameters χ_1 and χ_2 of the chemoattractant and chemorepellent are provided in the current paper for the global existence of classical solutions of (1.4) and stability of the nonzero constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$, and lower and upper bounds explicitly depending on χ_1 and χ_2 are established for the spreading speeds of positive solutions with compactly supported initial distributions. These conditions and lower and upper bounds would be of great practical importance.

The rest of the paper is organized as follows. Section 2 is devoted to the study of global existence of classical solutions. It is here that we prove Theorem A. In Sect. 3, we study the asymptotic stability of the constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$ and prove Theorem B. We study the spreading properties of global classical solutions of (1.4) with compactly supported initial functions and prove Theorems C and D in Sect. 4.

2 Global Existence

In this section, we discuss the existence of global/bounded classical solutions and prove Theorem A. We start with the following result which guarantees the existence of a unique local in time classical solution of (1.4) for any nonnegative bounded and uniformly continuous initial data.

Lemma 2.1 *For any $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $u_0 \geq 0$, there exists $T_{max} \in (0, \infty]$ such that (1.4) has a unique non-negative classical solution $(u(x, t; u_0), v_1(x, t; u_0), v_2(x, t, u_0))$ on $[0, T_{\max})$ with $\lim_{t\to 0} u(\cdot, t; u_0) = u_0$ in $C^b_{\rm uinf}(\mathbb{R}^N)$-norm. Moreover, if $T_{max} < \infty$, then*

$$\limsup_{t\to T_{max}} \|u(\cdot, t; u_0)\|_\infty = \infty. \tag{2.1}$$

Proof It follows from the similar arguments used in the proof of [36, Theorem 1.1]. □

Proof of Theorem A Let $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $u_0 \geq 0$ be given and let $(u(\cdot, \cdot; u_0), v_1(\cdot, \cdot; u_0), v_2(\cdot, \cdot; u_0))$ be the classical solution of (1.4) with initial function u_0 defined on the maximal interval $[0, T_{\max})$ of existence. Then,

$$\begin{aligned} u_t &= \Delta u - \chi_1 \nabla(u\nabla v_1) + \chi_2 \nabla(u \nabla v_2) + u(a - bu) \\ &= \Delta u + \nabla(\chi_2 v_2 - \chi_1 v_1)\nabla u + u(a - \chi_1 \Delta v_1 + \chi_2 \Delta v_2 - bu), \quad x \in \mathbb{R}^N. \end{aligned} \tag{2.2}$$

The second and third equations of (1.4) yield that $\Delta v_i = \lambda_i v_i - \mu_i u$, $i = 1, 2$. Hence equation (2.2) becomes

$$u_t = \Delta u + \nabla(\chi_2 v_2 - \chi_1 v_1)\nabla u + u\Big(a + (\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1) - (b + \chi_2\mu_2 - \chi_1\mu_1)u\Big), \quad x \in \mathbb{R}^N. \tag{2.3}$$

Let

$$C_0 := \begin{cases} \|u_0\|_\infty & \text{if } \chi_1 = a = b = 0, \\ \max\{\|u_0\|_\infty, \frac{a}{b+\chi_2\mu_2-\chi_1\mu_1-M}\} & \text{if } b + \chi_2\mu_2 - \chi_1\mu_1 - M > 0 \end{cases} \tag{2.4}$$

where M is given by (1.8). Let $T > 0$ be a given positive real number and consider $\mathcal{E}^T := C^b_{\rm unif}(\mathbb{R}^N \times [0, T])$ endowed with the norm

$$\|u\|_{\mathcal{E}^T} := \sum_{k=1}^{\infty} \frac{1}{2^k} \|u\|_{L^\infty([-k,k]\times[0,T])}. \tag{2.5}$$

We note that the convergence in $(\mathcal{E}^T, \|.\|_{\mathcal{E}^T})$ is equivalent to the uniform convergence on compact subsets on $\mathbb{R}^N \times [0, T]$. Next, we consider the subset $\mathcal{E}$ of $\mathcal{E}^T$ defined by

$$\mathcal{E} := \{u \in C^b_{\text{unif}}(\mathbb{R}^N \times [0, T]) \,|\, u(\cdot, 0) = u_0, 0 \le u(x, t) \le C_0, x \in \mathbb{R}^N, 0 \le t \le T\}.$$

It is clear that

$$\|u\|_{\mathcal{E}^T} \le C_0, \quad \forall\, u \in \mathcal{E}. \tag{2.6}$$

It readily follows from the definition of $\mathcal{E}$ and (2.6) that $\mathcal{E}$ is a closed bounded and convex subset of $\mathcal{E}^T$. We shall show that $u(\cdot, \cdot; u_0) \in \mathcal{E}$.

For every $u \in \mathcal{E}$ let us define $v_i(\cdot, \cdot; u)$, $i = 1, 2$ by

$$v_i(x, t; u) = \mu_i \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-\lambda_i s}}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|z-x|^2}{4s}} u(z, t)dzds, \quad x \in \mathbb{R}^N, \ t \in [0, T]. \tag{2.7}$$

and let $U(x, t; u)$ be the solution of the initial value problem

$$\begin{cases} U_t = \Delta U + \nabla\big(\chi_2 v_2(x, t; u) - \chi_1 v_1(x, t; u)\big)\nabla U \\ \qquad +U\Big(a + (\chi_2\lambda_2 v_2(x, t; u) - \chi_1\lambda_1 v_1(x, t; u)) - (b + \chi_2\mu_2 - \chi_1\mu_1)U\Big), \quad x \in \mathbb{R}^N \\ U(\cdot, 0, u) = u_0(\cdot). \end{cases} \tag{2.8}$$

For every $u \in \mathcal{E}$, using (2.7), we have that

$$\begin{aligned} (\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(x, t; u) &= \int_0^\infty \int_{\mathbb{R}^N} \Big[\chi_2\lambda_2\mu_2 e^{-\lambda_2 s} - \chi_1\lambda_1\mu_1 e^{-\lambda_1 s}\Big] \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z, t)dzds \\ &= (\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1) \int_0^\infty \int_{\mathbb{R}^N} e^{-\lambda_2 s} \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z, t)dzds \\ &\quad +\chi_1\mu_1\lambda_1 \int_0^\infty \int_{\mathbb{R}^N} (e^{-\lambda_2 s} - e^{-\lambda_1 s}) \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z, t)dzds \\ &\le (\chi_2\lambda_2\mu_2 - \chi_1\lambda_1\mu_1)_+ C_0 \int_0^\infty \int_{\mathbb{R}^N} e^{-\lambda_2 s} \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} dzds \\ &\quad +\chi_1\mu_1\lambda_1 C_0 \int_0^\infty \int_{\mathbb{R}^N} (e^{-\lambda_2 s} - e^{-\lambda_1 s})_+ \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} dzds \\ &= \frac{C_0}{\lambda_2}\Big((\chi_2\lambda_2\mu_2 - \chi_1\lambda_1\mu_1)_+ + \chi_1\mu_1(\lambda_1 - \lambda_2)_+\Big). \end{aligned} \tag{2.9}$$

Similarly, we have that

$$
\begin{aligned}
(\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(x,t;u) &= \int_0^\infty \int_{\mathbb{R}^N} \Big[\chi_2\lambda_2\mu_2 e^{-\lambda_2 s} - \chi_1\lambda_1\mu_1 e^{-\lambda_1 s}\Big] \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z,t)dzds \\
&= \chi_2\mu_2\lambda_2 \int_0^\infty \int_{\mathbb{R}^N} (e^{-\lambda_2 s} - e^{-\lambda_1 s}) \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z,t)dzds \\
&\quad + (\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1) \int_0^\infty \int_{\mathbb{R}^N} e^{-\lambda_1 s} \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z,t)dzds \\
&\le \chi_2\mu_2\lambda_2 C_0 \int_0^\infty \int_{\mathbb{R}^N} (e^{-\lambda_2 s} - e^{-\lambda_1 s})_+ \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} dzds \\
&\quad + (\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ C_0 \int_0^\infty \int_{\mathbb{R}^N} e^{-\lambda_1 s} \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} dzds \\
&= \frac{C_0}{\lambda_1}\Big(\chi_2\mu_2(\lambda_1 - \lambda_2)_+ + (\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+\Big).
\end{aligned} \tag{2.10}
$$

Thus, it follows from (2.9) and (2.10) that for every $u \in \mathcal{E}$, we have that

$$
(\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(x,t;u) \le MC_0 \tag{2.11}
$$

where M is given by (1.8). Thus for every $u \in \mathcal{E}$, we have that

$$
\begin{aligned}
U_t(x,t;u) \le & \Delta U(x,t;u) + \nabla(\chi_2 v_2 - \chi_1 v_1)\nabla U(x,t;u) \\
& + \underbrace{\Big(a + MC_0 - (b + \chi_2\mu_2 - \chi_1\mu_1)U(x,t;u)\Big)U(x,t;u)}_{\mathcal{L}(U)}.
\end{aligned} \tag{2.12}
$$

Note that

$$
\mathcal{L}(C_0) = \Big(a - (b + \chi_2\mu_2 - \chi_1\mu_1 - M)C_0\Big)C_0 \le 0.
$$

Thus, using comparison principle for parabolic equations, we obtain that

$$
U(x,t;u) \le C_0, \quad \forall\, x \in \mathbb{R}^N,\ \forall\, t \in [0,T],\ \forall\, u \in \mathcal{E}. \tag{2.13}
$$

Thus $U(\cdot,\cdot;u) \in \mathcal{E}$ for every $u \in \mathcal{E}$. By the arguments in [37, Lemma 4.3], the mapping $\mathcal{E} \ni u \mapsto U(\cdot,\cdot;u) \in \mathcal{E}$ is continuous and compact, and then by Schauder's fixed theorem, it has a fixed point u^*. Clearly $(u^*, v_1(\cdot,\cdot;u^*), v_2(\cdot,\cdot;u^*))$ is a classical solution of (1.4). Thus, by Lemma 2.1, we have that $T_{\max} \ge T$ and $u(\cdot,\cdot;u_0) = u^*$. Since $T > 0$ is arbitrary chosen, Theorem A follows. □

3 Asymptotic Stability of the Constant Equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$

In this section, we discuss the asymptotic stability of the constant equilibrium $(\frac{a}{b}, \frac{\mu_1}{\lambda_1}\frac{a}{b}, \frac{\mu_2}{\lambda_2}\frac{a}{b})$ of (1.4) and prove Theorem B. Throughout this section we suppose that (1.7) holds, so that for every nonnegative, bounded, and uniformly continuous initial function u_0, (1.4) has a nonnegative bounded global classical solution $(u(x,t;u_0), v_1(x,t;u_0), v_2(x,t;u_0))$.

For given $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$ with $u_0 \geq 0$, define

$$\underline{u} := \liminf_{t\to\infty} \inf_{x\in\mathbb{R}^N} u(x,t;u_0) \quad \text{and} \quad \overline{u} := \limsup_{t\to\infty} \sup_{x\in\mathbb{R}^N} u(x,t;u_0).$$

Using the definition of limsup and liminf, we have that for every $\varepsilon > 0$, there is $T_\varepsilon > 0$ such that

$$\underline{u} - \varepsilon \leq u(x,t;u_0) \leq \overline{u} + \varepsilon \quad \forall\, x \in \mathbb{R}^N,\ \forall\, t \geq T_\varepsilon.$$

Hence, it follows from comparison principle for elliptic equations, that

$$\mu_i(\underline{u} - \varepsilon) \leq \lambda_i v_i(x,t;u_0) \leq \mu_i(\overline{u}+\varepsilon), \forall\, x \in \mathbb{R}^N, \quad \forall\, t \geq T_\varepsilon,\ i = 1,2. \tag{3.1}$$

We first show the following important result.

Lemma 3.1 *Suppose that* (1.7) *holds. If* $\inf_{x\in\mathbb{R}^N} u_0(x) > 0$*, then*

$$\inf_{x\in\mathbb{R}^N} u(x,t;u_0) > 0, \quad \forall\, t > 0. \tag{3.2}$$

Proof Let $K := \chi_1\lambda_1 \sup_{x\in\mathbb{R}^N} v_1(x,t;u_0)$. Thus, it follows from (2.3) that

$$\begin{aligned} u_t(\cdot,\cdot;u_0) \geq & \Delta u + \nabla(\chi_2 v_2(\cdot,\cdot;u_0) - \chi_1 v_1(\cdot,\cdot;u_0))\nabla u(\cdot,\cdot;u_0) \\ & + (a - K - (b + \chi_2\mu_2 - \chi_1\mu_1)u(\cdot,\cdot;u_0))u(\cdot,\cdot;u_0). \end{aligned}$$

Hence, comparison principle for parabolic equations implies that

$$u(x,t;u_0) \geq W(t), \quad \forall\, t \geq 0,\ x \in \mathbb{R}^N,$$

where W is the solution of the ODE

$$\begin{cases} W_t = W(a - K - (b + \chi_2\mu_2 - \chi_1\mu_1)W),\ t > 0, \\ W(0) = \inf_{x\in\mathbb{R}^N} u_0(x). \end{cases}$$

Since $b + \chi_2\mu_2 - \chi_1\mu_1 > 0$, and $\inf_{x\in\mathbb{R}^N} u_0(x) > 0$, we have that $W(t)$ is defined for all time and satisfies $W(t) > 0$ for every $t \geq 0$. Hence, we obtain that $0 < W(t) \leq \inf_{x\in\mathbb{R}^N} u(x,t;u_0)$ for all $t \geq 0$.

Proof of Theorem B We divide the proof into two cases.
Case I. Assume that $b + \chi_2\mu_2 - \chi_1\mu_1 - \frac{1}{\lambda_2}\Big[|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_1\mu_1|\lambda_1 - \lambda_2|\Big] > 0.$

For every $t \geq T_\varepsilon$ (T_ϵ is such that (3.1) holds), and $x \in \mathbb{R}^N$, we have that

$$\begin{aligned} (\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(x,t;u_0) = & (\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)\int_0^\infty\int_{\mathbb{R}^N} e^{-\lambda_2 s}\frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}}u(z,t;u_0)dzds \\ & + \chi_1\mu_1\lambda_1\int_0^\infty\int_{\mathbb{R}^N}(e^{-\lambda_2 s} - e^{-\lambda_1 s})\frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}}u(z,t;u_0)dzdt \\ \leq & \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1 - \lambda_2)_+\right](\overline{u}+\varepsilon) \\ & - \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1 - \lambda_2)_-\right](\underline{u}-\varepsilon) \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
(\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(x,t;u_0) = &(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)\int_0^\infty\int_{\mathbb{R}^N} e^{-\lambda_2 s}\frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}}u(z,t;u_0)dzds \\
&+ \chi_1\mu_1\lambda_1\int_0^\infty\int_{\mathbb{R}^N}(e^{-\lambda_2 s}-e^{-\lambda_1 s})\frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}}u(z,t;u_0)dzdt \\
&\ge \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\right](\underline{u}-\varepsilon) \\
&\quad -\frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\right](\overline{u}+\varepsilon).
\end{aligned}
\tag{3.4}
$$

Hence, for every $t \ge T_\varepsilon$, $x \in \mathbb{R}^N$, it follows from (2.3), (3.1) and (3.3) that

$$
\begin{aligned}
u_t \le\ & \Delta u + \nabla(\chi_2 v_2 - \chi_1 v_1)\nabla u + \left(a + \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\right](\overline{u}+\varepsilon)\right)u \\
& -\left(\frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\right](\underline{u}-\varepsilon) + (b+\chi_2\mu_2-\chi_1\mu_1)u\right)u.
\end{aligned}
\tag{3.5}
$$

Thus, by comparison principle for parabolic equations, we have that

$$
u(x,t;u_0) \le U_\varepsilon(t), \quad \forall x \in \mathbb{R}^N,\ t \ge T_\varepsilon, \tag{3.6}
$$

where $U_\varepsilon(t)$ is the solution of the ODE

$$
\begin{cases}
\partial_t U = \left(a + \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\right](\overline{u}+\varepsilon)\right)U \\
\quad -\left(\frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\right](\underline{u}-\varepsilon) + (b+\chi_2\mu_2-\chi_1\mu_1)U\right)U \quad t > T_\varepsilon, \\
U(T_\varepsilon) = \|u(\cdot,T_\varepsilon;u_0)\|_\infty.
\end{cases}
$$

Since $b+\chi_2\mu_2-\chi_1\mu_1 > 0$ and $\|u(\cdot,T_\infty;u_0)\|_\infty > 0$, we have that $U_\varepsilon(t)$ is defined for all time $t \ge T_\varepsilon$ and satisfies

$$
\begin{aligned}
\lim_{t\to\infty} U_\varepsilon = &\frac{1}{b+\chi_2\mu_2-\chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\right](\overline{u}+\varepsilon) \\
&-\frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\right](\underline{u}-\varepsilon)\Big\}_+
\end{aligned}
$$

This combined with (3.6) yield that

$$
\begin{aligned}
\overline{u} \le &\frac{1}{b+\chi_2\mu_2-\chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\right](\overline{u}+\varepsilon) \\
&-\frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\right](\underline{u}-\varepsilon)\Big\}_+.
\end{aligned}
$$

Letting ε goes to 0 in the last inequality, we obtain that

$$
\begin{aligned}
\overline{u} \le &\frac{1}{b+\chi_2\mu_2-\chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\right]\overline{u} \\
&-\frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\right]\underline{u}\Big\}_+.
\end{aligned}
$$

If

$$
\begin{aligned}
&\Big\{a + \frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\right]\overline{u} \\
&-\frac{1}{\lambda_2}\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\right]\underline{u}\Big\}_+ = 0,
\end{aligned}
$$

then $\overline{u} = \underline{u} = 0$. This in turn yields that

$$0 = \Big\{a + \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big]\overline{u} - \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big]\underline{u}\Big\}_+ = a,$$

which is impossible, since $a > 0$. Hence

$$\overline{u} \le \frac{1}{b+\chi_2\mu_2-\chi_1\mu_1}\Big[a + \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big]\overline{u} - \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big]\underline{u}\Big]. \tag{3.7}$$

On the other hand, for every $t \ge T_\varepsilon$, $x \in \mathbb{R}^N$, it follows from (2.3), (3.1) and (3.3) that

$$u_t \ge \Delta u + \nabla(\chi_2 v_2 - \chi_1 v_1)\nabla u + \Big(a + \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big](\underline{u}-\varepsilon)\Big)u - \Big(\frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big](\overline{u}+\varepsilon) + (b+\chi_2\mu_2-\chi_1\mu_1)u\Big)u. \tag{3.8}$$

Thus, by comparison principle for parabolic equations, we have that

$$u(x,t;u_0) \ge U^\varepsilon(t), \quad \forall x \in \mathbb{R}^N,\ t \ge T_\varepsilon, \tag{3.9}$$

where $U^\varepsilon(t)$ is the solution of the ODE

$$\begin{cases} \partial_t U = \Big(a + \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big](\underline{u}-\varepsilon)\Big)U \\ \quad -\Big(\frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big](\overline{u}+\varepsilon) + (b+\chi_2\mu_2-\chi_1\mu_1)U\Big)U, \quad t > T_\varepsilon \\ U(T_\varepsilon) = \inf_{x\in\mathbb{R}^N} u(x,T_\varepsilon). \end{cases}$$

But, by Lemma 3.1 we have that $\inf_{x\in\mathbb{R}^N} u(x,T_\varepsilon;u_0) > 0$. Since $b+\chi_2\mu_2-\chi_1\mu_1 > 0$, we have that $U^\varepsilon(t)$ is defined for all time $t \ge T_\varepsilon$ and satisfies

$$\lim_{t\to\infty} U^\varepsilon = \frac{1}{b+\chi_2\mu_2-\chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big](\underline{u}-\varepsilon) - (\frac{1}{\lambda_2}\big[\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big](\overline{u}+\varepsilon)\Big\}_+.$$

This combined with (3.9) yield that

$$\underline{u} \ge \frac{1}{b+\chi_2\mu_2-\chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big](\underline{u}-\varepsilon) - (\frac{1}{\lambda_2}\big[\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big](\overline{u}+\varepsilon)\Big\}_+.$$

Letting ε goes to 0 in the last inequality, we obtain that

$$\underline{u} \ge \frac{1}{b+\chi_2\mu_2-\chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big]\underline{u} - (\frac{1}{\lambda_2}\big[\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big]\overline{u}\Big\}_+. \tag{3.10}$$

It follows from inequalities (3.7) and (3.10) that

$$\Big(b+\chi_2\mu_2-\chi_1\mu_1 - \frac{1}{\lambda_2}\big[|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_1\mu_1|\lambda_1-\lambda_2|\big]\Big)(\overline{u}-\underline{u}) \le 0. \tag{3.11}$$

In this case, it follows from inequality (3.11) that $\overline{u} = \underline{u}$. Combining this with (3.7) and (3.10), we obtain that $\overline{u} = \underline{u} = \frac{a}{b}$. This ends the first case.

Case II. Assume that $b + \chi_2\mu_2 - \chi_1\mu_1 - \frac{1}{\lambda_1}\big[|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_2\mu_2|\lambda_1 - \lambda_2|\big] > 0$.

Rewrite $\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1$ in the form

$$(\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(x, t; u_0) = \chi_2\mu_2\lambda_2 \int_0^\infty \int_{\mathbb{R}^N} (e^{-\lambda_2 s} - e^{-\lambda_1 s}) \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z, t; u_0)dzds$$
$$+(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1) \int_0^\infty \int_{\mathbb{R}^N} e^{-\lambda_1 s} \frac{e^{-\frac{|x-z|^2}{4s}}}{(4\pi s)^{\frac{N}{2}}} u(z, t; u_0)dzds. \tag{3.12}$$

It follows from the arguments used to establish inequalities (3.7) and (3.10) that

$$\overline{u} \le \frac{1}{b + \chi_2\mu_2 - \chi_1\mu_1}\Big\{a + \frac{1}{\lambda_1}\big[(\chi_2\mu_2(\lambda_1 - \lambda_2)_+ + \chi_2\mu_2(\lambda_1 - \lambda_2)_+\big]\overline{u}$$
$$- \frac{1}{\lambda_1}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_2\mu_2(\lambda_1 - \lambda_2)_+\big]\underline{u}\Big\} \tag{3.13}$$

and

$$\underline{u} \ge \frac{1}{b + \chi_2\mu_2 - \chi_1\mu_1}\Big\{a + \frac{1}{\lambda_1}\big[(\chi_2\mu_2(\lambda_1 - \lambda_2)_+ + \chi_2\mu_2(\lambda_1 - \lambda_2)_+\big]\underline{u}$$
$$- \frac{1}{\lambda_1}\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_2\mu_2(\lambda_1 - \lambda_2)_+\big]\overline{u}\Big\} \tag{3.14}$$

hold respectively. It follows from (3.13) and (3.14) that

$$\Big(b + \chi_2\mu_2 - \chi_1\mu_1 - \frac{1}{\lambda_1}\big[|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_2\mu_2|\lambda_1 - \lambda_2|\big]\Big)(\overline{u} - \underline{u}) \le 0. \tag{3.15}$$

Since $b + \chi_2\mu_2 - \chi_1\mu_1 - \frac{1}{\lambda_1}\left[|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_2\mu_2|\lambda_1 - \lambda_2|\right] > 0$, it follows from inequality (3.15) that $\overline{u} = \underline{u}$. Combining this with (3.13) and (3.14), we obtain that $\overline{u} = \underline{u} = \frac{a}{b}$. This end the second case.

Therefore, it follows from the results of cases I and II that if

$$b + \chi_2\mu_2 - \chi_1\mu_1 > \min\Big\{\frac{1}{\lambda_2}\left[|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_1\mu_1|\lambda_1 - \lambda_2|\right],$$
$$\frac{1}{\lambda_1}\left[|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_2\mu_2|\lambda_1 - \lambda_2|\right]\Big\},$$

then $\overline{u} = \underline{u} = \frac{a}{b}$. Thus Theorem B follows. □

4 Spreading Properties of Classical Solutions

In this section we study how fast the mobiles species spread over time and prove Theorems C and D. Throughout this section, we always suppose that $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $u_0(x) \ge 0$ has compact and nonempty support. The next three lemmas will be useful in the subsequent.

Lemma 4.1 *Let $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $u_0 \ge 0$, and $(u(\cdot, \cdot; u_0), v_1(\cdot, \cdot; u_0), v_2(\cdot, \cdot; u_0))$ be the classical solution of* (1.4) *with $u(\cdot, 0; u_0) = u_0$. Then for every $i \in \{1, \cdots, N\}$, we have* *that*

$$\begin{aligned}\|\partial_{x_i}(\chi_2 v_2 - \chi_1 v_1)(\cdot, t; u_0)\|_\infty \\ \le \min\Big\{ \frac{|\chi_2\mu_2 - \chi_1\mu_1|}{2\sqrt{\lambda_2}} + \frac{\chi_1\mu_1|\sqrt{\lambda_1} - \sqrt{\lambda_2}|}{2\sqrt{\lambda_1\lambda_2}}, \frac{|\chi_1\mu_1 - \chi_2\mu_2|}{2\sqrt{\lambda_1}} \\ + \frac{\chi_2\mu_2|\sqrt{\lambda_2} - \sqrt{\lambda_1}|}{2\sqrt{\lambda_1\lambda_2}} \Big\} \|u(\cdot, t; u_0)\|_\infty \end{aligned} \tag{4.1}$$

for every $t \ge 0$*.*

Proof For every $i \in \{1, \cdots, N\}$ and $k \in \{1, 2\}$, we have that

$$\begin{aligned} &\partial_{x_i}(\chi_k v_k)(x, t; u_0) \\ &= \frac{\chi_k\mu_k}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^N} \frac{e^{-\lambda_k s}}{\sqrt{s}} z_i e^{-|z|^2} u(x + 2\sqrt{s}z, t; u_0) dz ds \\ &= \frac{\chi_k\mu_k}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{e^{-\lambda_k s} e^{-|y|^2}}{\sqrt{s}} \left[\int_{\mathbb{R}} \tau e^{-\tau^2} u(x + 2\sqrt{s}\tau e_i + 2\sqrt{s}\pi_i^{-1}(y), t; u_0) d\tau \right] dy ds \\ &= \frac{\chi_k\mu_k}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{e^{-\lambda_k s} e^{-|y|^2}}{\sqrt{s}} \left[\int_0^\infty \tau e^{-\tau^2} u(x + 2\sqrt{s}\tau e_i + 2\sqrt{s}\pi_i^{-1}(y), t; u_0) d\tau \right] dy ds \\ &- \frac{\chi_k\mu_k}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{e^{-\lambda_k s} e^{-|y|^2}}{\sqrt{s}} \left[\int_0^\infty \tau e^{-\tau^2} u(x - 2\sqrt{s}\tau e_i + 2\sqrt{s}\pi_i^{-1}(y), t; u_0) d\tau \right] dy ds, \end{aligned}$$

where $e_i = (\delta_{1i}, \delta_{2i}, \cdots, \delta_{Ni})$ with $\delta_{ij} = 0$ if $i \ne j$ and $\delta_{ii} = 1$ for $i, j = 1, 2, \cdots, N$, and $\pi_i^{-1}(y) = (y_1, y_2, \cdots, y_{i-1}, 0, y_i, \cdots, y_{N-1})$. Hence,

$$\begin{aligned} &\partial_{x_i}(\chi_2 v_2 - \chi_1 v_1)(x, t; u_0) \\ &= \frac{(\chi_2\mu_2 - \chi_1\mu_1)}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{e^{-\lambda_2 s} e^{-|y|^2}}{\sqrt{s}} \left[\int_0^\infty \tau e^{-\tau^2} u(x + 2\sqrt{s}\tau e_i + 2\sqrt{s}\pi_i^{-1}(y), t; u_0) d\tau \right] dy ds \\ &+ \frac{\chi_1\mu_1}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{(e^{-\lambda_2 s} - e^{-\lambda_1 s}) e^{-|y|^2}}{\sqrt{s}} \left[\int_0^\infty \tau e^{-\tau^2} u(x + 2\sqrt{s}\tau e_i + 2\sqrt{s}\pi_i^{-1}(y), t; u_0) d\tau \right] dy ds \\ &+ \frac{(\chi_1\mu_1 - \chi_2\mu_2)}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{e^{-\lambda_2 s} e^{-|y|^2}}{\sqrt{s}} \left[\int_0^\infty \tau e^{-\tau^2} u(x - 2\sqrt{s}\tau e_i + 2\sqrt{s}\pi_i^{-1}(y), t; u_0) d\tau \right] dy ds \\ &+ \frac{\chi_1\mu_1}{\pi^{\frac{N}{2}}} \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{(e^{-\lambda_1 s} - e^{-\lambda_2 s}) e^{-|y|^2}}{\sqrt{s}} \left[\int_0^\infty \tau e^{-\tau^2} u(x - 2\sqrt{s}\tau e_i + 2\sqrt{s}\pi_i^{-1}(y), t; u_0) d\tau \right] dy ds. \end{aligned} \tag{4.2}$$

Using the fact that $\int_0^\infty \frac{e^{-\lambda_k s}}{\sqrt{s}} ds = \frac{\sqrt{\pi}}{\sqrt{\lambda_k}}$, $\int_0^\infty \tau e^{-\tau^2} d\tau = \frac{1}{2}$, $\int_{\mathbb{R}^{N-1}} e^{-|y|^2} dy = \pi^{\frac{N-1}{2}}$, it follows from (4.2) that for every $x \in \mathbb{R}^N$, $t \ge 0$, we have

$$|\partial_{x_i}(\chi_2 v_2 - \chi_1 v_1)(x, t; u_0)| \le \left[\frac{|\chi_2\mu_2 - \chi_1\mu_1|}{2\sqrt{\lambda_2}} + \frac{\chi_1\mu_1|\sqrt{\lambda_1} - \sqrt{\lambda_2}|}{2\sqrt{\lambda_1\lambda_2}} \right] \|u(\cdot, t; u_0)\|_\infty.$$

Similarly, we have that

$$|\partial_{x_i}(\chi_1 v_1 - \chi_2 v_2)(x, t; u_0)| \le \left[\frac{|\chi_1\mu_1 - \chi_2\mu_2|}{2\sqrt{\lambda_1}} + \frac{\chi_2\mu_2|\sqrt{\lambda_2} - \sqrt{\lambda_1}|}{2\sqrt{\lambda_1\lambda_2}} \right] \|u(\cdot, t; u_0)\|_\infty.$$

The lemma thus follows. □

Lemma 4.2 *Suppose that (1.7) holds. Let $u_0 \in C^b_{\rm unif}(\mathbb{R}^N)$, $u_0 \geq 0$, and $(u(\cdot,\cdot;u_0), v_1(\cdot,\cdot;u_0), v_2(\cdot,\cdot;u_0))$ be the classical solution of (1.4) with $u(\cdot,0;u_0) = u_0$. Then we have that*

$$\limsup_{t\to\infty} \|u(\cdot,t;u_0)\|_\infty \leq \frac{a}{b+\chi_2\mu_2-\chi_1\mu_1-M}, \tag{4.3}$$

where M is given by (1.8).

Proof It follows from inequalities (3.7) and (3.13) that

$$\overline{u} \leq \frac{a+\frac{1}{\lambda_2}\left[((\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+)\right]\overline{u}}{b+\chi_2\mu_2-\chi_1\mu_1}$$

and

$$\overline{u} \leq \frac{a+\frac{1}{\lambda_1}\left[((\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_2\mu_2(\lambda_1-\lambda_2)_+)\right]\overline{u}}{b+\chi_2\mu_2-\chi_1\mu_1}.$$

Which is equivalent to

$$(b+\chi_2\mu_2-\chi_1\mu_1)\overline{u} \leq a + \frac{1}{\lambda_2}\left[((\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+)\right]\overline{u}$$

and

$$(b+\chi_2\mu_2-\chi_1\mu_1)\overline{u} \leq a + \frac{1}{\lambda_1}\left[((\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_2\mu_2(\lambda_1-\lambda_2)_+)\right]\overline{u}.$$

Hence

$$(b+\chi_2\mu_2-\chi_1\mu_1)\overline{u} \leq a + M\overline{u}.$$

The lemma thus follows. □

Lemma 4.3 *1) If there is a positive constant $c_-^*(\chi_1,\mu_1,\lambda_2,\chi_2,\mu_2,\lambda_2)$ such that*

$$\lim_{t\to\infty}\sup_{|x|\leq ct} |u(x,t;u_0)-\frac{a}{b}| = 0 \quad \forall\, 0\leq c < c_-^*(\chi_1,\mu_1,\lambda_2,\chi_2,\mu_2,\lambda_2), \tag{4.4}$$

then for every $i = 1,2$ we have

$$\lim_{t\to\infty}\sup_{|x|\leq ct} |\lambda_i v_i(x,t;u_0)-\frac{a}{b}\mu_i| = 0 \quad \forall\, 0\leq c < c_-^*(\chi_1,\mu_1,\lambda_2,\chi_2,\mu_2,\lambda_2). \tag{4.5}$$

2) If there is a positive constant $c_+^*(\chi_1,\mu_1,\lambda_2,\chi_2,\mu_2,\lambda_2)$ such that

$$\lim_{t\to\infty}\sup_{|x|\geq ct} u(x,t;u_0) = 0 \quad \forall\, c > c_+^*(\chi_1,\mu_1,\lambda_2,\chi_2,\mu_2,\lambda_2), \tag{4.6}$$

then for each $i = 1,2$ we have that

$$\lim_{t\to\infty}\sup_{|x|\geq ct} v_i(x,t;u_0) = 0 \quad \forall\, c > c_+^*(\chi_1,\mu_1,\lambda_2,\chi_2,\mu_2,\lambda_2). \tag{4.7}$$

The proof of Lemma 4.3 follows from the proof of Lemma 5.5 [36].

Now, we are ready to prove Theorem C.

Proof of Theorem C Combining inequalities (4.1) and (4.3), we obtain that

$$\|\nabla(\chi_2 v_2 - \chi_1 v_1)(\cdot, t; u_0)\|_\infty \le \frac{aD\sqrt{N}}{b + \chi_2\mu_2 - \chi_1\mu_1 - M} + D\varepsilon\sqrt{N}, \quad \forall\, t \ge T_\varepsilon \tag{4.8}$$

where D is given by (1.14) and M given by (1.8). Let

$$K_\varepsilon := \sup_{0\le t\le T_\varepsilon} \|\nabla(\chi_2 v_2 - \chi_1 v_1)(\cdot, t; u_0)\|_\infty \quad \text{and} \quad K_{\varepsilon,2} =: \sup_{0\le t\le T_\varepsilon} \|\frac{\chi_2\lambda_2}{\sqrt{a}} v_2(\cdot, t; u_0)\|_\infty.$$

Choose $C > 0$ such that

$$u_0(x) \le Ce^{-\sqrt{a}|x|}, \quad \forall\, x \in \mathbb{R}^N.$$

Let $\xi \in S^{N-1}$ be given and consider

$$\overline{U}(x, t; \xi) = Ce^{-\sqrt{a}(x\cdot\xi - (2\sqrt{a} + K_\varepsilon + K_{\varepsilon,2})t)}.$$

We have that

$$\begin{aligned}
&\overline{U}_t - \Delta\overline{U} - \nabla((\chi_2 v_2 - \chi_1 v_1)(\cdot, \cdot; u_0))\nabla\overline{U} - (a + (\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(\cdot, \cdot; u_0)\\
&\quad - (b + \chi_2\mu_2 - \chi_1\mu_1)\overline{U})\overline{U}\\
&= \Big(\sqrt{a}(2\sqrt{a} + K_\varepsilon + K_{\varepsilon,2}) - a + \sqrt{a}\nabla((\chi_2 v_2 - \chi_1 v_1)(\cdot, \cdot; u_0)) \cdot \xi\Big)\overline{U}\\
&\quad - \Big(a + (\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(\cdot, \cdot; u_0) - (b + \chi_2\mu_2 - \chi_1\mu_1)\overline{U}\Big)\overline{U}\\
&= \Big(\sqrt{a}\big(K_\varepsilon + \nabla((\chi_2 v_2 - \chi_1 v_1)(\cdot, \cdot; u_0)) \cdot \xi\big) + (\sqrt{a}K_{\varepsilon,2} - \chi_2\lambda_2 v_2(\cdot, \cdot; u_0))\Big)\overline{U}\\
&\quad + \Big(\chi_1\lambda_1 v_1(\cdot, \cdot; u_0) + (b + \chi_2\mu_2 - \chi_1\mu_1)\overline{U}\Big)\overline{U}\\
&\ge 0 \quad \forall x \in \mathbb{R}^N,\ 0 < t \le T_\varepsilon,\ \forall\, \xi \in S^{N-1}.
\end{aligned} \tag{4.9}$$

Since $\overline{U}(x, 0; \xi) = Ce^{-\sqrt{a}x\cdot\xi} \ge Ce^{-\sqrt{a}|x|} \ge u_0(x)$, by comparison principle for parabolic equations, we obtain that

$$u(x, t; u_0) \le \overline{U}(x, t; \xi), \quad \forall\, x \in \mathbb{R}^N,\ 0 \le t \le T_\varepsilon\ \forall\, \xi \in S^{N-1}. \tag{4.10}$$

Next, consider

$$\overline{W}(x, t; \xi) = Ce^{-\sqrt{a}(x\cdot\xi - (2\sqrt{a} + L_\varepsilon + L_{\varepsilon,2})(t - T_\varepsilon))} e^{\sqrt{a}(2\sqrt{a} + K_\varepsilon + K_{\varepsilon,2})T_\varepsilon}, \quad x \in \mathbb{R}^N,\ t \ge T_\varepsilon, \tag{4.11}$$

where

$$L_\varepsilon := \frac{aD\sqrt{N}}{b + \chi_2\mu_2 - \chi_1\mu_1 - M} + D\varepsilon\sqrt{N}$$

and

$$L_{\varepsilon,2} := \frac{\sqrt{a}\chi_2\mu_2}{b + \chi_2\mu_2 - \chi_1\mu_1 - M} + \frac{\chi_2\mu_2\varepsilon}{\sqrt{a}}.$$

It follows from (3.1), (4.3) and (4.8) that for any $x \in \mathbb{R}^N$ and $t \ge T_\epsilon$,

$$\begin{aligned}
&\overline{W}_t - \Delta\overline{W} - \nabla((\chi_2 v_2 - \chi_1 v_1)(\cdot, \cdot; u_0))\nabla\overline{W} - (a + (\chi_2\lambda_2 v_2 - \chi_1\lambda_1 v_1)(\cdot, \cdot; u_0)\\
&\quad -(b + \chi_2\mu_2 - \chi_1\mu_1)\overline{W})\overline{W} \ge 0.
\end{aligned}$$

Observe that $\overline{W}(\cdot, T_\varepsilon; \xi) = \overline{U}(\cdot, T_\varepsilon; \xi) \geq u(\cdot, T_\varepsilon)$. Hence, comparison principle for parabolic equations implies that

$$0 \leq u(x,t;u_0) \leq \overline{W}(x,t;\xi), \quad x \in \mathbb{R}^N,\ t \geq T_\varepsilon,\ \xi \in S^{N-1}. \tag{4.12}$$

Hence, for every $c > 2\sqrt{a} + L_\varepsilon + L_{\varepsilon,2}$, and $t > T_\varepsilon$, we have

$$\begin{aligned}\sup_{|x|\geq ct} u(x,t;u_0) &\leq \sup_{|x|\geq ct} \overline{W}\left(x,t,\frac{1}{|x|}x\right)\\ &\leq \sup_{|x|\geq ct} Me^{-\sqrt{a}(c-(2\sqrt{a}+L_\varepsilon+L_{\varepsilon,2})(t-T_\varepsilon))}e^{\sqrt{a}(2\sqrt{a}+K_\varepsilon+K_{\varepsilon,2})T_\varepsilon} \to 0\end{aligned}$$

as $t \to \infty$. Thus by taking

$$c_+^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2) := 2\sqrt{a} + \lim_{\varepsilon\to 0^+}(L_\varepsilon + L_{\varepsilon,2}) = 2\sqrt{a} + \frac{\sqrt{a}(D\sqrt{Na}+\chi_2\mu_2)}{b+\chi_2\mu_2-\chi_1\mu_1-M}, \tag{4.13}$$

and using Lemma 4.3, the result of Theorem C follows.

In order to prove Theorem D, we first establish the following important Lemma.

Lemma 4.4 *Let L be given by* (1.18). *Then,*

$$\begin{aligned}&\lim_{R\to\infty}\inf_{|x|\geq R,\ T\geq R}\left(4(a+(\chi_2\lambda_2 v_2-\chi_1\lambda_1 v_1)(x,t;u_0)-|\nabla(\chi_2 v_2-\chi_1 v_1)(x,t;u_0)|^2\right)\\ &\geq 4a(1-L)-\frac{Na^2D^2}{(b+\chi_2\mu_2-\chi_1\mu_1)^2}.\end{aligned} \tag{4.14}$$

Proof Using inequalities (3.4) and (4.1), we have that for every $t \geq T_\varepsilon,\ x \in \mathbb{R}^N$,

$$\begin{aligned}&4\big(a+(\chi_2\lambda_2 v_2-\chi_1\lambda_1 v_1)(x,t;u_0)\big)-|\nabla(\chi_2 v_2(x,t;u_0)-\chi_1 v_1(x,t;u_0))|^2\\ &\geq 4\Big(a+\frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big](\underline{u}-\varepsilon)\Big)\\ &-\frac{4}{\lambda_2}\big[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big](\overline{u}+\varepsilon)-ND^2(\overline{u}+\varepsilon)^2.\end{aligned} \tag{4.15}$$

Letting first $R \to \infty$ and then $\varepsilon \to 0$, it follows from (4.15) that

$$\begin{aligned}&\lim_{R\to\infty}\inf_{|x|\geq R,\ T\geq R}\left(4(a+(\chi_2\lambda_2 v_2-\chi_1 v_1\lambda_1)(x,t;u_0)-|\nabla(\chi_2 v_2-\chi_1 v_1)(x,t;u_0)|^2\right)\\ &\geq 4\Big(a+\frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big]\underline{u}\Big)\\ &-\frac{4}{\lambda_2}\big[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big]\overline{u}-ND^2\overline{u}^2.\end{aligned} \tag{4.16}$$

But Theorem C implies that $\underline{u} = 0$. Hence, inequality (4.16) implies that

$$\begin{aligned}&\lim_{R\to\infty}\inf_{|x|\geq R,\ T\geq R}\left(4(a+(\chi_2\lambda_2 v_2-\chi_1 v_1\lambda_1)(x,t;u_0)-|\nabla(\chi_2 v_2-\chi_1 v_1)(x,t;u_0)|^2\right)\\ &\geq 4\Big(a-\frac{1}{\lambda_2}\big[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big]\overline{u}\Big)-ND^2\overline{u}^2.\end{aligned} \tag{4.17}$$

Thus, it follows from (4.17) and (4.3) that

$$\lim_{R\to\infty}\inf_{|x|\geq R,\ T\geq R}\left(4(a+(\chi_2\lambda_2v_2-\chi_1v_1\lambda_1)(x,t;u_0)-|\nabla(\chi_2v_2-\chi_1v_1)(x,t;u_0)|^2\right)$$
$$\geq 4\big(a-\frac{a\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_-+\chi_1\mu_1(\lambda_1-\lambda_2)_-\right]}{\lambda_2(b+\chi_2\mu_2-\chi_1\mu_1-M)}\big)-\frac{ND^2a^2}{(b+\chi_2\mu_2-\chi_1\mu_1-M)^2}. \tag{4.18}$$

Similarly, by rewriting $(\chi_2\mu_2v_2-\chi_1\mu_1v_1)(x,t;u_0)$ in the form given by (3.12), same arguments as above yield that

$$\lim_{R\to\infty}\inf_{|x|\geq R,\ T\geq R}\left(4(a+(\chi_2\lambda_2v_2-\chi_1v_1\lambda_1)(x,t;u_0)-|\nabla(\chi_2v_2-\chi_1v_1)(x,t;u_0)|^2\right)$$
$$\geq 4\big(a-\frac{a\left[(\chi_2\mu_2\lambda_2-\chi_1\mu_1\lambda_1)_-+\chi_2\mu_2(\lambda_1-\lambda_2)_-\right]}{\lambda_1(b+\chi_2\mu_2-\chi_1\mu_1-M)}\big)-\frac{ND^2a^2}{(b+\chi_2\mu_2-\chi_1\mu_1-M)^2}. \tag{4.19}$$

The Lemma thus follows. □

Proof of Theorem D The arguments used in this proof generalize some of the arguments used in the proof of Theorem 9(i) [36]. Hence some details might be omitted. We refer the reader to [36] for the proofs of the estimates stated below.

Since (1.17) holds, we have

$$c_-^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2):=2\sqrt{a(1-L)}-\frac{aD\sqrt{N}}{b+\chi_2\mu_2-\chi_1\mu_1-M}>0,$$

where M, D and L are given by (1.8) and (1.14) and (1.18) respectively. We first note that, it follows from Lemma 4.4 and the proof of Lemma 5.4 [36] that for every $0\leq c<c_-^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)$ we have

$$\liminf_{t\to\infty}\inf_{|x|\leq ct}u(x,t;u_0)>0. \tag{4.20}$$

It suffices to prove the following claim.

Claim. For every $0\leq c<c_-^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)$, we have that

$$\lim_{t\to\infty}\sup_{|x|\leq ct}|u(x,t;u_0)-\frac{a}{b}|=0. \tag{4.21}$$

Suppose that the claim is not true. Then there is $0\leq c<c_-^*(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2)$, $\delta>0$, a sequence $\{x_n\}_{n\geq1}$, a sequence of positive numbers $\{t_n\}_{n\geq1}$ with $t_n\to\infty$ as $n\to\infty$ such that

$$|x_n|\leq ct_n,\quad \forall\, n\geq1, \tag{4.22}$$

and

$$|u(x_n,t_n;u_0)-\frac{a}{b}|\geq\delta,\qquad \forall\, n\geq1. \tag{4.23}$$

For every $n\geq1$, let us define

$$u_n(x,t)=u(x+x_n,t+t_n;u_0),\quad v_{kn}(x,t)=v_k(x+x_n,t_n;u_0)\ \ (k=1,2) \tag{4.24}$$

for all $x\in\mathbb{R}^N$ and $t\geq -t_n$.

We first show that there is a subsequence of $\{(u_n, v_{1n}, v_{2n})\}$ which converges locally uniformly. To this end, let $\{T(t)\}_{t\geq 0}$ denote the analytic semigroup generated by the closed linear operator $(\Delta - I)u$ on $C^b_{\text{unif}}(\mathbb{R}^N)$. Then the variation of constant formula yield that

$$\begin{aligned} u(x,t;u_0) =& T(t)u_0 + \int_0^t T(t-s)\nabla\cdot((\chi_2 u\nabla v_2 - \chi_1 u\nabla v_1)(\cdot,s;u_0))ds \\ &+ \int_0^t T(t-s)(((a+1)u - bu^2)(\cdot,s;u_0))ds. \end{aligned} \tag{4.25}$$

Let $0 < \alpha < \frac{1}{2}$ be fixed and let X^α denotes the fractional powers associated to the semigroup $\{T(t)\}_{t\geq 0}$. Thus, there is a constant C_α(see [15]) depending only on α and the dimension N such that

$$\begin{aligned} \|u_n(\cdot,0)\|_{X^\alpha} \leq & C_\alpha t_n^{-\alpha}\|u_0\|_\infty + C_\alpha \int_0^{t_n} e^{-(t_n-s)}(t_n-s)^{-\frac{1}{2}-\alpha}\|(\chi_2 u\nabla v_2 - \chi_1 u\nabla v_1)(\cdot,s;u_0)\|_\infty ds \\ &+ C_\alpha \int_0^{t_n} e^{-(t_n-s)}(t_n-s)^{-\alpha}\|((a+1)u - bu^2)(\cdot,s;u_0)\|_\infty ds. \end{aligned} \tag{4.26}$$

Using the facts that $\sup_{t\geq 0}\|u(\cdot,t)\|_\infty < \infty$, $t_n \to \infty$ as $n\to\infty$, $\int_0^\infty e^{-\tau}\tau^{-\frac{1}{2}-\alpha}d\tau = \Gamma(\frac{1}{2}-\alpha) < \infty$ and $\int_0^\infty e^{-\tau}\tau^{-\alpha}d\tau = \Gamma(1-\alpha) < \infty$, it follows from (4.26) that

$$\sup_{n\geq 1}\|u_n(\cdot,0)\|_{X^\alpha} < \infty. \tag{4.27}$$

Similar arguments as those used in the proof of Theorem 1.1 [36] yield that the functions $u_n : [-T\ ,\ T] \to X^\alpha$ are equicontinuous for every $T > 0$. Hence Arzela-Ascili's Theorem and Theorem 15 (page 80 of [12]) imply that there is a function $(\tilde{u}, \tilde{v}_1, \tilde{v}_2) \in \left[C^{2,1}(\mathbb{R}^N\times\mathbb{R})\right]^3$ and a subsequence $\{(u_{n'}, v_{1n'}, v_{2n'})\}_{n\geq 1}$ of $\{(u_n, v_{1n}, v_{2n})\}_{n\geq 1}$ such that $(u_{n'}, v_{1n'}, v_{2n'}) \to (\tilde{u}, \tilde{v}_1, \tilde{v}_2)$ in $C^{1+\delta',\delta'}_{loc}(\mathbb{R}^N\times\mathbb{R})$ for some $\delta' > 0$. Moreover $\mu_i\tilde{u} = (\lambda_i I - \Delta)\tilde{v}_i$ for every $i = 1, 2$. Note that

$$\tilde{u}(x,t) = \lim_{n\to\infty} u(x + x_{n'}, t + t_{n'}; u_0), \quad \forall\, x\in\mathbb{R}^N, \quad t\in\mathbb{R}.$$

Hence

$$|\tilde{u}(0,0) - \frac{a}{b}| \geq \delta. \tag{4.28}$$

Choose $\tilde{c} \in (c\ ,\ c^*_-(\chi_1,\mu_1,\lambda_1,\chi_2,\mu_2,\lambda_2))$. For every $x\in\mathbb{R}^N$, $t\in\mathbb{R}$ and $t_{n'} \geq \frac{|x|+\tilde{c}|t|}{\tilde{c}-c}$, we have

$$|x + x_{n'}| \leq |x| + ct_{n'} \leq \tilde{c}(t_{n'} + t).$$

It follows from last inequality and (4.20) that

$$\tilde{u}(x,t) = \lim_{n\to\infty} u(x + x_{n'}, t + t_{n'}; u_0) \geq \liminf_{s\to\infty}\inf_{|y|\leq\tilde{c}s} u(y,s;u_0) > 0, \quad \forall\, x\in\mathbb{R}^N, \quad t\in\mathbb{R}.$$

Hence $\inf_{(x,t)\in\mathbb{R}^{N+1}}\tilde{u}(x,t) > 0$.

Next, we claim that $\tilde{u}(x,t) = \frac{a}{b}$ for every $x\in\mathbb{R}^N$, $t\in\mathbb{R}$. Indeed, let $\underline{u}_0 = \inf_{(x,t)\in\mathbb{R}^{N+1}}\tilde{u}(x,t)$ and $\overline{u}_0(x,t) = \sup_{(x,t)\in\mathbb{R}^{N+1}}\tilde{u}(x,t)$. For every $t_0\in\mathbb{R}$, let $\overline{U}(t,t_0)$ and $\underline{U}(t,t_0)$ be the solution of the ODEs

$$\begin{cases} \overline{U}_t = \big(a - (b + \chi_2\mu_2 - \chi_1\mu_1)\overline{U}\big)\overline{U} \\ \qquad + \frac{1}{\lambda_2}\Big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1-\lambda_2)_+\big]\overline{u}_0 \\ \qquad - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1-\lambda_2)_-\big]\underline{u}_0\Big)\overline{U}, \quad t > t_0 \\ \overline{U}(t_0,t_0) = \overline{u}_0 \end{cases}$$

and

$$\begin{cases} \underline{U}_t = \big(a - (b + \chi_2\mu_2 - \chi_1\mu_1)\underline{U}\big)\underline{U} \\ \qquad + \frac{1}{\lambda_2}\Big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1 - \lambda_2)_+\big]\underline{u}_0 \\ \qquad - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1 - \lambda_2)_-\big]\overline{u}_0\Big)\underline{U}, \quad t > t_0 \\ \underline{U}(t_0, t_0) = \underline{u}_0 \end{cases}$$

respectively. It follows from the arguments used to establish (3.6) and (3.9) that

$$\underline{U}(t - t_0, 0) = \underline{U}(t, t_0) \le \tilde{u}(x, t), \quad \forall\, x \in \mathbb{R}^N, \ t \ge t_0 \tag{4.29}$$

and

$$\overline{U}(t - t_0, 0) = \overline{U}(t, t_0) \ge \tilde{u}(x, t), \quad \forall\, x \in \mathbb{R}^N, \ t \ge t_0 \tag{4.30}$$

respectively. Note that for every $t \in \mathbb{R}$ fixed, we have that

$$\lim_{t_0 \to -\infty} \overline{U}(t, t_0) = \frac{1}{b + \chi_2\mu_2 - \chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1 - \lambda_2)_+\big]\overline{u}_0 \\ - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1 - \chi_2)_-\big]\underline{u}_0\big)\Big\} \tag{4.31}$$

and

$$\lim_{t_0 \to -\infty} \underline{U}(t, t_0) = \frac{1}{b + \chi_2\mu_2 - \chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1 - \lambda_2)_+\big]\underline{u}_0 \\ - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1 - \chi_2)_-\big]\overline{u}_0\big)\Big\}. \tag{4.32}$$

Combining (4.29) and (4.32), we have that

$$\frac{1}{b + \chi_2\mu_2 - \chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1 - \lambda_2)_+\big]\underline{u}_0 \\ - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1 - \chi_2)_-\big]\overline{u}_0\big)\Big\} \le \underline{u}_0. \tag{4.33}$$

Combining (4.30) and (4.31), we have that

$$\frac{1}{b + \chi_2\mu_2 - \chi_1\mu_1}\Big\{a + \frac{1}{\lambda_2}\big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_1\mu_1(\lambda_1 - \lambda_2)_+\big]\overline{u}_0 \\ - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_1\mu_1(\lambda_1 - \chi_2)_-\big]\underline{u}_0\big)\Big\} \ge \overline{u}_0. \tag{4.34}$$

Thus, it follows from inequalities (4.33) and (4.34) that

$$\Big(b + \chi_2\mu_2 - \chi_1\mu_1 - \frac{1}{\lambda_2}\Big(|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_1\mu_1|\lambda_1 - \lambda_2|\Big)\Big)(\overline{u}_0 - \underline{u}_0) \le 0. \tag{4.35}$$

Similarly, for every $t_0 \in \mathbb{R}$, by considering $\overline{V}(t, t_0)$ and $\underline{V}(t, t_0)$ to be the solutions of the ODEs

$$\begin{cases} \overline{V}_t = \big(a - (b + \chi_2\mu_2 - \chi_1\mu_1)\overline{V}\big)\overline{V} \\ \qquad + \frac{1}{\lambda_1}\Big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_2\mu_2(\lambda_1 - \lambda_2)_+\big]\overline{u}_0 \\ \qquad - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_2\mu_2(\lambda_1 - \lambda_2)_-\big]\underline{u}_0\Big)\overline{V}, \quad t > t_0 \\ \overline{V}(t_0, t_0) = \overline{u}_0 \end{cases}$$

and

$$\begin{cases} \underline{V}_t = \big(a - (b + \chi_2\mu_2 - \chi_1\mu_1)\underline{V}\big)\underline{V} \\ \qquad + \frac{1}{\lambda_1}\Big(\big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_+ + \chi_2\mu_2(\lambda_1 - \lambda_2)_+\big]\underline{u}_0 \\ \qquad - \big[(\chi_2\mu_2\lambda_2 - \chi_1\mu_1\lambda_1)_- + \chi_2\mu_2(\lambda_1 - \lambda_2)_-\big]\overline{u}_0\Big)\underline{V}, \quad t > t_0 \\ \underline{V}(t_0, t_0) = \underline{u}_0 \end{cases}$$

respectively. Using systems (4.36) and (4.36), similar arguments used to establish (4.35) yield that

$$\left(b + \chi_2\mu_2 - \chi_1\mu_1 - \frac{1}{\lambda_1}\Big(|\chi_1\mu_1\lambda_1 - \chi_2\mu_2\lambda_2| + \chi_2\mu_2|\lambda_1 - \lambda_2|\Big)\right)(\overline{u}_0 - \underline{u}_0) \le 0. \tag{4.36}$$

It follows from inequalities (4.35) and (4.36) that

$$(b + \chi_2\mu_2 - \chi_1\mu_1 - K)(\overline{u}_0 - \underline{u}_0) \le 0. \tag{4.37}$$

Since (1.10) holds, it follows from the last inequality that $\overline{u}_0 = \underline{u}_0$. Combining this with inequalities (4.33) and (4.34) we obtain that $\overline{u}_0 = \underline{u}_0 = \frac{a}{b}$. Hence, we have that $\tilde{u}(x, t) = \frac{a}{b}$ for every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. In particular, we have that $\tilde{u}(0, 0) = \frac{a}{b}$, which contradicts (4.28).

Hence the claim is true and Theorem D is thus proved. □

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