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Chaos Induced by Sliding Phenomena in Filippov Systems

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Abstract In this paper we provide a full topological and ergodic description of the dynamics of Filippov systems nearby a sliding Shilnikov orbit Γ. More specifically we prove that the first return map, defined nearby Γ, is topologically conjugate to a Bernoulli shift with infinite topological entropy. In particular, we see that for each $m \in \mathbb{N}$ it has infinitely many periodic points with period m. We also study the perturbed system and obtain similar results.

Keywords Filippov systems · Shilnikov sliding orbits · Bernoulli shifts · Chaos

Mathematics Subject Classification 34A36 · 37C29 · 34C28 · 37B10

1 Introduction

Real world problems have been the main motivation on discontinuous differential systems. They are very useful to model phenomena presenting abrupt switches such as electronic relays, mechanical impact, mitosis of living cells, and Neuronal networks. That is one of the reasons for this area to have such a variety of rich examples [2,8,9,22,25]. Therefore the further we understand discontinuous differential systems the more one is prepared to analyse real world problems.

When facing a discontinuous differential system, defining a consistent concept of solution is the first natural issue one has to deal with. One important paradigm to tackle this issue is due to Filippov. In his famous book [11] Filippov studied these systems taking advantage of

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the well developed theory of differential inclusions [1]. Then, for a class of discontinuous vector fields Z, he provided a branch of rules for what would be a local trajectory of $\dot{u} = Z(u)$ nearby a point of discontinuity. For instance, consider

$$Z(u) = \begin{cases} X(u), & \text{if} \quad g(u) > 0, \\ Y(u), & \text{if} \quad g(u) < 0, \end{cases} \tag{1}$$

where $u \in K$, being K a closure of an open subset of $\mathbb{R}^n$, X, Y are $\mathcal{C}^r$ vector fields, and $g : K \to \mathbb{R}$ has 0 as a regular value. The rules stated by Filippov may be applied to establish the notion of local solution of the discontinuous differential system $\dot{u} = Z(u)$ at a point of discontinuity $\xi \in M = g^{-1}(0)$. Nowadays these rules are known as the Filippov's conventions, and it turns out that for many physical models these conventions are the ones which have physical meaning [9]. Accordingly, discontinuous differential systems rulled by Filippov's conventions are called *Filippov systems*. Due to their importance, not only from the mathematical point of view as well as the physical point of view, we shall assume the Fillipov's conventions throughout the paper. Under this convention the switching manifold M can be generically decomposed in three regions with distinct dynamical behaviours, namely: *crossing* M^c, *sliding* M^s, and *escaping* M^e. Concisely, the system $\dot{u} = Z(u)$ may admit solutions either side of the discontinuity M that can be joined continuously, forming a solution that *crosses* $M^c \subset M$. Alternatively, solutions might be found to impinge upon M, after which they join continuously to solutions that *slide* inside $M^{s,e} = M^s \cup M^e \subset M$. See items (i)–(v) of Sect. 2.1 for the precise definition of the Filippov's conventions.

Nonlinear systems may present intricate and complex behaviours such as chaotic motions. Roughly speaking, chaos can be understood as the existence of an invariant compact set Λ of initial conditions for which their trajectories are transitive and exhibit sensitive dependence on Λ [10,16,26]. Each phenomenon from the ordinary theory of differential systems finds its analogous in discontinuous differential systems. However Filippov systems admit a richer variety of behaviours. New chaotic modes rising in discontinuous differential systems have been recently investigated. For instance, in [6,7] it was studied chaotic set-valued trajectories (nondeterministic chaos), and in [3,4] the Melnikov ideas were applied to determine the existence of chaos in nonautonomous Filippov systems. Here we shall study deterministic chaos in autonomous Filippov systems.

In this article we analyse 3D Filippov systems admitting a *sliding Shilnikov orbit* Γ (see Fig. 1), which is an entity inherent to Filippov systems. It was first studied in [17]. In the classical theory a *Shilnikov homoclinic orbit* of a smooth vector field is a trajectory connecting a hyperbolic saddle–focus equilibrium to itself, bi-asymptotically. It is well known that a chaotic behaviour may rise when the Shilnikov homoclinic orbit is perturbed [14,18,20,21]. In the Filippov context pseudo-equilibria are special points contained in $M^{s,e}$ that must be distinguished and treated as typical singularities (see Definition 1). These singularities give rise to the definition of the sliding homoclinic orbit, that is a trajectory, in the Filippov sense, connecting a pseudo-equilibrium to itself in an infinity time at least by one side, forward or backward. Particularly a sliding Shilnikov orbit (see Definition 2) is a sliding homoclinic orbit connecting a hyperbolic pseudo saddle–focus $p_0 \in M^s$ to it self. This trajectory intersects the boundary ∂M^s of M^s at a point q_0 (see Fig. 1).

In dynamics one is concerned to get the most possible complexity from a given system, that is why often for systems which exhibit some complexity one is able to find a Bernoulli shift as a factor. For smooth dynamical systems one may benefit from certain geometrical structures (hyperbolicity) which imply the existence of stable and unstable manifolds. That has been the case in many studies, we mention two classical works done by Bowen [5] and Tresser [24]. The systems we shall study in this paper do not benefit from these geometrical

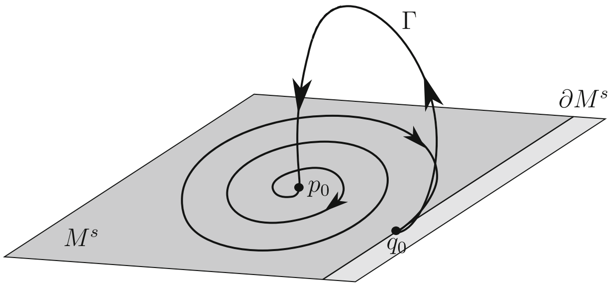

Fig. 1 The point $p_0 \in M^s$ is a hyperbolic pseudo saddle–focus. The trajectory Γ, called Shilnikov sliding orbit, connects p_0 to itself passing through the point $q_0 \in \partial M^s$. We note that the flow leaving q_0 reaches the point p_0 in a finite positive time, and approaches backward to p_0, asymptotically

structures, so we have to use the properties of the Filippov systems themselves to overcome this lacking of structures (e.g. the well-posedness of the η_* function, see Sect. 4).

Let Z be a 3D discontinuous vector field like (1) defined on $K \subset \mathbb{R}^3$. Assume that the Filippov system $\dot{u} = Z(u)$ admits a sliding Shilnikov orbit Γ connecting a hyperbolic pseudo saddle–focus $p_0 \in M^s$ to it self and intersecting the curve ∂M^s at a point q_0. Consider a small neighbourhood I of q_0 in ∂M^s, and denote by $\Lambda \subset I$ the points which return infinitely often by the forward orbit of the flow to this neighbourhood of q_0. If π denote the first return map defined on a subset $\mathcal{U}$ of I, then Λ is the maximal set in $\mathcal{U}$ which is π invariant (see Sect. 2.2 for more information). As we shall see, this set is non vanishing. The complexity of a flow is interpreted as the complexity of its returning map $\pi|\Lambda$. In this context our main result (see Theorem A in Sect. 3) states that π can be as much chaotic as one wishes by using symbolic dynamics.

This paper is organized as follows. On Sect. 2 we present some basic notions on Filippov systems and symbolic dynamics. On Sect. 3 we state our main result (Theorem A) and some of their consequences (Corollaries A, B and C). The main result is proved on Sect. 4. Some final words appear on Sect. 5 as well as a brief discussion of further directions.

2 Basic Notions and Preliminary Results

This section is devoted to present some basic notions needed to state our main result. On Sect. 2.1 we introduce the basic concepts of Filippov systems as well the definition of *sliding Shilnikov orbit*. Then on Sect. 2.2 we look carefully at the first return map defined nearby the sliding Shilnikov orbit. Finally, on Sect. 2.3, we present the basic notions about symbolic dynamics.

2.1 Filippov System and the Sliding Shilnikov Orbit

We remark that a major part of this section is constituted by a well known theory and may be found in other works (see for instance [11,12,17]).

Let K be the closure of an open subset of $\mathbb{R}^n$, and let $X, Y \in \mathcal{C}^r(K, \mathbb{R}^3)$, be $\mathcal{C}^r$ vector fields defined on $K \subset \mathbb{R}^3$. We denote by $\Omega^r_g(K, \mathbb{R}^3)$ the space of piecewise vector fields

$$Z(u) = \begin{cases} X(u), & \text{if} \quad g(u) > 0, \\ Y(u), & \text{if} \quad g(u) < 0, \end{cases} \tag{2}$$

defined on K, being 0 a regular value of the differentiable function $g : K \to \mathbb{R}$. As usual, system (2) is denoted by $Z = (X, Y)$ and the surface of discontinuity $g^{-1}(0)$ by M . So $\Omega_g^r(K, \mathbb{R}^3) = \mathcal{C}^r(K, \mathbb{R}^3) \times \mathcal{C}^r(K, \mathbb{R}^3)$ is endowed with the product topology, while $\mathcal{C}^r(K, \mathbb{R}^3)$ is endowed with the $\mathcal{C}^r$ topology. We concisely denote $\Omega_g^r(K, \mathbb{R}^3)$ and $\mathcal{C}^r(K, \mathbb{R}^3)$ only by Ω^r and $\mathcal{C}^r$, respectively.

In order to understand the Filippov's conventions for the discontinuous differential system $\dot{u} = Z(u)$ we need to distinguish some regions on M. The points on M where both vectors fields X and Y simultaneously point outward or inward from M define, respectively, the *escaping* M^e or *sliding* M^s regions, and the interior of its complement in M defines the *crossing region* M^c. The complementary of the union of those regions is the set of *tangency* points between X or Y with M.

The points in M^c satisfy $Xg(\xi) \cdot Yg(\xi) > 0$, where $Xg(\xi) = \langle \nabla g(\xi), X(\xi) \rangle$. The points in M^s (resp. M^e) satisfy $Xg(\xi) < 0$ and $Yg(\xi) > 0$ (resp. $Xg(\xi) > 0$ and $Yg(\xi) < 0$). Finally, the tangency points of X (resp. Y) satisfy $Xg(\xi) = 0$ (resp. $Yg(\xi) = 0$).

Now we define the *sliding vector field*

$$\widetilde{Z}(\xi) = \frac{Yg(\xi)X(\xi) - Xg(\xi)Y(\xi)}{Yg(\xi) - Xg(\xi)}. \tag{3}$$

Definition 1 A point $\xi^* \in M^{s,e}$ is called a *pseudo-equilibrium* of Z if it is a singularity of the sliding vector field, i.e. $\widetilde{Z}(\xi^*) = 0$. When ξ^* is a hyperbolic singularity of $\widetilde{Z}$, it is called a *hyperbolic pseudo-equilibrium*. Particularly if $\xi^* \in M^s$ (resp. $\xi^* \in M^e$) is an unstable (resp. stable) hyperbolic focus of $\widetilde{Z}$ then we call ξ^* a *hyperbolic pseudo saddle–focus*.

Let φ_W denotes the flow of a smooth vector field W. The local trajectory $\varphi_Z(t, p)$, $t \in I_p \subset \mathbb{R}$, of $\dot{u} = Z(u)$ passing through a point $p \in \mathbb{R}^3$ is given by the Filippov's conventions (see [11,12]). Here $0 \in I_p \subset \mathbb{R}$ denotes a interval of definition of $\varphi_Z(t, p)$. Following straightly [12], the Filippov's conventions is summarized as:

- (i) for $p \in \mathbb{R}^3$ such that $g(p) > 0$ (resp. $g(p) < 0$) and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t, p) = \varphi_X(t, p)$ (resp. $\varphi_Z(t, p) = \varphi_Y(t, p)$) for $t \in I_p$.
- (ii) for $p \in M^c$ such that $(Xg)(p), (Yg)(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t, p) = \varphi_Y(t, p)$ for $t \in I_p \cap \{t < 0\}$ and $\varphi_Z(t, p) = \varphi_X(t, p)$ for $t \in I_p \cap \{t > 0\}$. For the case $(Xg)(p), (Yg)(p) < 0$ the definition is the same reversing time.
- (iii) for $p \in M^{s,e}$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t, p) = \varphi_{\widetilde{Z}}(t, p)$ for $t \in I_p$.
- (iv) For $p \in \partial M^c \cup \partial M^s \cup \partial M^e$ such that the definitions of trajectories for points in M in both sides of p can be extended to p and coincide, the trajectory through p is this limiting trajectory. These points are called *regular tangency* points.
- (v) any other point is called *singular tangency* points and $\varphi_Z(t, p) = p$ for all $t \in \mathbb{R}$.

Examples of regular tangency points are the regular-fold points. A tangency point $p \in M$ is called a *visible fold* of X (resp. Y) if $X^2g(p) > 0$ (resp. $Y^2g(p) < 0$). Analogously, reversing the inequalities, we define an *invisible fold*. A fold p of X (resp. Y), visible or invisible, such that $Yg(p) \neq 0$ (resp. $Xg(p) \neq 0$) is called a *regular-fold* point. The next result provides the dynamics of the sliding vector field near regular-fold points. A proof of that can be find in [23].

Proposition 1 ([23]) *Given $Z = (X, Y) \in \Omega^r$ if $p \in \partial M^{e,s}$ is a fold-regular point of Z then the sliding vector field $\widetilde{Z}$ (3) is transverse to ∂M at p.*

The above conventions provide the unicity of the trajectories passing through a point. This property plays an important whole in establishing the notion of local equivalence between two Filippov systems (see [12]). However if one consider, for instance, a point $p \in \Sigma^s \cup \Sigma^e$, besides the trajectory defined above, there are two other trajectories (of X and Y) which arrive to p in finite time. Therefore in the study of global behavior the matching of these distinct trajectories must be taken into account.

In [17] it has been introduced the concept of *sliding Shilnikov orbits*, and some of their properties were studied. In what follows we give the definition of this object and two results. The first one is about the co-dimension of the sliding Shilnikov orbit Ω^r and the second one is about the existence of sliding periodic orbits nearby a sliding Shilnikov orbit.

Definition 2 (*Sliding Shilnikov orbit*) Let $Z = (X, Y)$ be a 3D discontinuous vector field having a hyperbolic pseudo saddle–focus $p_0 \in M^s$ (resp. $p_0 \in M^e$). We assume that there exists a tangential point $q_0 \in \partial M^s$ (resp. $q_0 \in \partial M^e$) which is a visible fold point of the vector field X such that

- (j) the orbit passing through q_0 following the sliding vector field $\widetilde{Z}$ converges to p_0 backward in time (resp. forward in time);
- (jj) the orbit starting at q_0 and following the vector field X spends a time $t_0 > 0$ (resp. $t_0 < 0$) to reach p_0.

So through p_0 and q_0 a sliding loop Γ is characterized. We call Γ a *sliding Shilnikov orbit* (see Fig. 1). Accordingly we denote $\Gamma^+ = \Gamma \cap \{u \in K : g(u) > 0\}$ and $\Gamma^s = \Gamma \cap M^s$.

Theorem 1 ([17]) *Assume that $Z_0 = (X_0, Y_0) \in \Omega^r$ (with $r \geq 1$) has a sliding Shilnikov orbit Γ_0 and let $W \subset \Omega^r$ be a small neighbourhood of Z_0. Then there exists a $\mathcal{C}^1$ function $g : W \to \mathbb{R}$ having 0 as a regular value such that $Z \in W$ has a sliding Shilnikov orbit Γ if and only if $g(Z) = 0$.*

Theorem 2 ([17]) *Assume that $Z_0 = (X_0, Y_0) \in \Omega^r$ (with $r \geq 0$) has a sliding Shilnikov orbit Γ_0 and let $Z_\alpha = (X_\alpha, Y_\alpha) \in \Omega^r$ be an 1-parameter family of Filippov systems breaking the sliding Shilnikov orbit for $|\alpha| \neq 0$, this family is called a Splitting of Γ_0 (Fig. 2). Then the following statements hold:*

- (a) *for $\alpha = 0$ every neighbourhood $G \subset \mathbb{R}^3$ of Γ_0 contains countable infinitely many sliding periodic orbits of Z_0;*
- (b) *for every neighbourhood $G \subset \mathbb{R}^3$ of Γ_0 there exists $|\alpha_0| \neq 0$ sufficiently small such that G contains a finite number $N_G(\alpha_0) > 0$ of sliding periodic orbits of Z_α. Moreover $N_G(\alpha) \to \infty$ when $\alpha \to 0$.*

In what follows we present an example of a 2-parameter family of discontinuous piecewise linear differential system $Z_{a,b}$ admitting, for every positive real numbers α and b, a sliding Shilnikov orbit $\Gamma_{\alpha,\beta}$. This family was studied in [17].

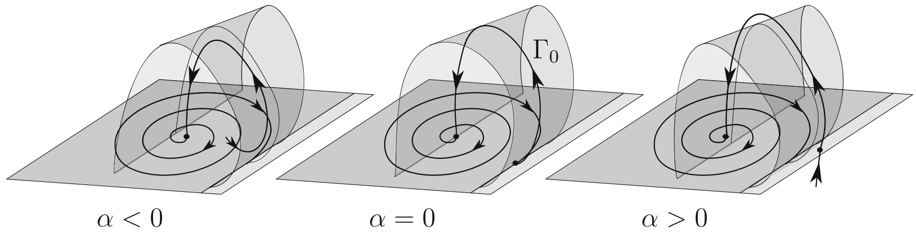

Fig. 2 Representation of a Splitting of Γ_0, $Z_\alpha \in \Omega^r$

$$Z_{a,b}(x,y,z) = \begin{cases} X_{a,b}(x,y,z) = \begin{pmatrix} -a \\ x-b \\ y - \dfrac{3b^2}{8a} \end{pmatrix} & \text{if} \quad z > 0, \\ Y_{a,b}(x,y,z) = \begin{pmatrix} a \\ \dfrac{3a}{b} y + b \\ \dfrac{3b^2}{8a} \end{pmatrix} & \text{if} \quad z < 0. \end{cases} \tag{4}$$

The switching manifold for system (4) is given $M = \{z = 0\}$, and decomposed as $M = \overline{M^c} \cup \overline{M^s} \cup \overline{M^e}$ being

$$M^c = \left\{(x,y,0):\ y > \frac{3\beta^2}{8a}\right\},\ M^s = \left\{(x,y,0):\ y < \frac{3\beta^2}{8a}\right\} \text{ and } M^e = \emptyset.$$

The origin $p_0 = (0,0,0)$ is a hyperbolic pseudo saddle–focus of system $Z_{a,b}$ (4) in such way that its projection onto M is an unstable hyperbolic focus of the sliding vector field $\widetilde{Z}_{a,b}$ (3) associated with (4). Moreover there exists a sliding Shilnikov orbit $\Gamma_{a,b}$ connecting $p_0 = (0,0,0)$ to itself and passing through the fold-regular point $q_0 = \big(3b/2, 3b^2/(8a), 0\big)$.

2.2 The First Return Map

The behaviour of a system close to a sliding Shilnikov orbit can be understood by studying the first return map in a small neighbourhood $I \subset \partial M^s$ of q_0, wherever it is defined. In what follows we shall define this map.

Let $Z_0 \in \Omega^r$ be a Filippov system admitting a sliding Shilnikov orbit Γ_0, and let Z_α be a splitting of Γ_0. For sake of simplicity we shall denote $Z = Z_0$ and $\Gamma = \Gamma_0$.

For $\xi \in M^s$ and $z \in \mathbb{R}^3$, let the functions $\varphi_s(t,\xi)$ and $\varphi_X(t,z)$ denote the solutions of the differential systems induced by $\widetilde{Z}$ and X, respectively, such that $\varphi_s(0,\xi) = \xi$ and $\varphi_X(0,z) = z$.

Take $\gamma_r := \overline{B_r(q_0) \cap \partial M^s}$. Here $B_r(q_0) \subset M$ is the planar ball with center at q_0 and radius r. Of course γ_r is a branch of the fold line contained in the boundary of the sliding region ∂M^s. From Definition 2, $\varphi_X(t_0, q_0) = p_0 \in M^s$, moreover, the intersection between Γ^+ and M at p_0 is transversal. So taking $r > 0$ sufficiently small, we find a function $\tau(\xi) > 0$, defined for $\xi \in \gamma_r$, such that $\tau(q_0) = t_0$ and $\varphi_s(\tau(\xi), \xi) \in M^s$ for every $\xi \in \gamma_r$.

The forward saturation of γ_r through the flow of X meets M in a curve μ_r, that is $\mu_r = \{\varphi_X(\tau(\xi), \xi) : \xi \in \gamma_r\}$. So let $\theta : \gamma_r \to \mu_r$ denote the diffeomorphism $\theta(\xi) = \varphi_s(\tau(\xi), \xi)$. A diffeomorphism $\theta_\alpha : \gamma_r \to \mu_r^\alpha$ can be constructed in a similar way, but now the pseudo saddle-focus is not contained in μ_r^α.

Now, Proposition 1 implies that the intersection between Γ^s and ∂M^s at q_0 is transversal. So in addition, taking $r > 0$ small enough, the backward saturation S_r of γ_r through the flow of $\widetilde{Z}$ converges to p_0. Therefore

$$S_r \cap \mu_r = \bigcup_{i=1}^{\infty} J_i,$$

where $J_i \cap J_j = \emptyset$ if $i \neq j$ and $J_i \to \{p_0\}$. For each $i = 1, 2, \ldots$, we take $I_i = \theta^{-1}(J_j) \subset \gamma_r$. Clearly $I_i \cap I_j = \emptyset$ if $i \neq j$ and $I_i \to \{q_0\}$. Set $I = \gamma_r$ and

$$\mathcal{U}_r = \bigcup_{i=1}^{\infty} I_i \subset I. \tag{5}$$

Therefore the first return map $\pi : \mathcal{U}_r \to I$ is well defined. Moreover it can be taken as

$$\pi(\xi) = \varphi_s\big(\tau_s(\varphi_X(\tau(\xi), \xi)), \varphi_X(\tau(\xi), \xi)\big), \tag{6}$$

where, for each $\xi' \in \mu_r \setminus \{p_0\}$, $\tau_s(\xi') > 0$ denotes the time such that $\varphi_s(\tau_s(\xi'), \xi') \in I$.

For $|\alpha| \neq 0$ sufficiently small one could proceed as above to construct a first return map $\pi_\alpha : \mathcal{U}_r^\alpha \to I$, with respect to the system Z_α. We notice that, since the backward saturation of I through the flow of $\widetilde{Z}_\alpha$ intersects μ_r in a finite number $n_\alpha < \infty$ of connected components J_i^α, the set $\mathcal{U}_r^\alpha$ will be given by a union of n_α intervals $I_i^\alpha \subset I$:

$$\mathcal{U}_r^\alpha = \bigcup_{i=1}^{n_\alpha} I_i^\alpha \subset I,$$

where $I_i^\alpha = \theta_\alpha^{-1}(J_i^\alpha)$.

The next result estimates the derivative of the first return map.

Proposition 2 *Consider $\mathcal{U}_r$ as defined in* (5)*. There exists $r > 0$ sufficiently small such that $|\pi'(\xi)| > 1$ for every $\xi \in \mathcal{U}_r$. Consequently, for $|\alpha| \neq 0$ sufficiently small, $|\pi'_\alpha(\xi)| > 1$ for every $\xi \in \mathcal{U}_r^\alpha$.*

Proof For each $R > 0$, the focus $p_0 \in M^s$ of the sliding vector field $\widetilde{Z}$ is contained in μ_R which is traversal to the flow of $\widetilde{Z}$. So there exists $R_0 > 0$ such that, for every $0 < R \leq R_0$, it is well defined a first return map from μ_R into $\mu_{\overline{R}}$, for some big $\overline{R} > 0$, which we denote by $\rho : \mu_R \to \mu_{\overline{R}}$. Since p_0 is a hyperbolic unstable fixed point of ρ, ρ admits a C^1 linearization in a neighborhood of p_0 (see [13,19]), that is, there exists a neighborhood $U \subset \mu_{R_0}$ and a $\mathcal{C}^1$ diffeomorphism $H : U \to U$ such that $\rho(\zeta) = H(\lambda H^{-1}(\zeta))$, with $|\lambda| > 1$. So choose $R > 0$ sufficiently small such that $\mu_R \subset U$. Therefore if $\rho^{k-1}(\zeta) \in U$ then $\rho^k(\zeta) = H(\lambda^k H^{-1}(\zeta))$

The backward saturation of γ_R through the flow of $\widetilde{Z}$ intersects U many times, indeed it converges to p_0. So denote by S the first connected component of this intersection which is entirely contained in U. The flow of $\widetilde{Z}$ induces a diffeomorphism $\widetilde{\rho}$ between S and γ_R. Moreover the flow of X induces a diffeomorphism ρ_X between γ_R and μ_R.

Since $\widetilde{\rho}$ and ρ_X are diffeomorphism, there exists $\widetilde{\alpha} > 0$ and $\alpha_X > 0$ such that $\widetilde{\alpha} = \min\{|\widetilde{\rho}'(\zeta)| : \zeta \in S\}$ and $\alpha_X = \min\{|\rho'_X(\xi)| : \xi \in \gamma_R\}$.

Now given $k_0 \in \mathbb{N}$, there exists a sufficiently small $r \in (0, R)$ such that $\rho^k(\mu_r) \cap S = \emptyset$ for every $0 < k < k_0$. In particular, we can assume that $\alpha_X \widetilde{\alpha} |\lambda|^{k_0} > 1$.

Finally take $\mathcal{U}_r$ as defined in (5). For $\xi \in \mathcal{U}_r$, let $\overline{k}$ be a positive integer such that $\rho^{\overline{k}}\big(\rho_X(\xi)\big) \in S$. From the continuity of the map ρ, there exists a neighborhood $W \subset \mathcal{U}_r$ of ξ

such that $\rho^{\overline{k}}\big(\rho_X(w)\big) \in S \subset U$ for every $w \in W$. Since $\rho_X(\xi) \in \mu_r, \overline{k} \geq k_0$. Therefore, for every $w \in W$, the first return map reads

$$\begin{aligned}\pi(w) &= \widetilde{\rho} \circ \rho^{\overline{k}} \circ \rho_X(w) \\ &= \widetilde{\rho} \circ H(\lambda^{\overline{k}} H^{-1}(\rho_X(w))).\end{aligned}$$

Hence $|\pi'(\xi)| \geq \alpha_X \widetilde{\alpha} |\lambda|^{\overline{k}} \geq \alpha_X \widetilde{\alpha} |\lambda|^{k_0} > 1$. □

2.3 Basic Facts on Bernoulli Shifts

On what follows the reader may find a good introduction to the subject on [15] and the references therein.

Let $(X, \mathcal{A}, \mu)$ be a probability space and $f : X \to X$ be a measurable function. We say that a measurable set $B \subset X$ is f-invariant if

$$f^{-1}(B) = B \bmod 0,$$

where mod 0 means that except a measure zero set both sets are equal. We say that f preserves the measure μ, or that f is μ-invariant, when

$$\mu(f^{-1}(B)) = \mu(B)$$

for every measurable set $B \subset X$.

Given a measurable preserving function $f : X \to X$ in a probability space (X, μ), we say that f is ergodic if, and only if, for every f-invariant measurable set $B \subset X$ we have

$$\mu(B) = 0 \text{ or } \mu(B) = 1.$$

Ergodicity is a very important property in Dynamical Systems and it roughly means that the dynamics cannot be broken in smaller simple dynamics. Hence it actually implies a certain type of chaos for a system with respect to a given measure. Bernoulli shifts have a fairly simple description and still amazingly they are the most chaotic possible examples. We now describe the Bernoulli shifts.

Throughout this paper we denote $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$. Given any natural number $k \in \mathbb{N}^*$, we define the space of all sequences of natural numbers between 0 and $k-1$ by

$$\Sigma_k = \{0, 1, \ldots, k-1\}^{\mathbb{N}}.$$

Due to a more intuitive approach in the proof of our main result (see Theorem A of Sect. 3) we will need the set

$$\Sigma_k^* = \{1, \ldots, k\}^{\mathbb{N}},$$

which we point out it is not a standard notation on symbolic dynamics.

These are a countable product space where each coordinate is a discrete compact space. By Tychonoff's theorem Σ_k (respectively Σ_k^*) is compact with the product topology induced by the discrete topology of $\{0, 1, \ldots, k-1\}$ (respectively $\{1, \ldots, k\}$). A metric in this space, which generates the product topology, is given by

$$\begin{aligned} d : \Sigma_k \times \Sigma_k &\to \mathbb{R} \\ d(\alpha, \beta) &= \begin{cases} 0 & \text{se } \alpha = \beta \\ \left(\frac{1}{2}\right)^n, & n = max\{a \in \mathbb{N} : \alpha(i) = \beta(i), |i| \leq a\} \end{cases} \end{aligned}$$

Acting on Σ_k we have the so called one-sided Bernoulli shift $\sigma : \Sigma_k \to \Sigma_k$, which simply operates a left-translation on each sequence, that is, given any sequence $(x_n)_{n\in\mathbb{N}}$ the image if this sequence is the sequence

$$\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}.$$

Definition 3 Given $n \in \mathbb{N}$ and m values $a_1, a_2, .., a_m \in \{0, 1, \dots, k-1\}$. We denote by $C(n; a_1, a_2, \dots, a_m)$ the set defined by

$$C(n; a_1, a_2, \dots, a_m) = \{(x_i)_{i\in\mathbb{N}} : x_{n+1} = a_1, x_{n+2} = a_2, \dots, x_{n+m} = a_m\}.$$

The sets of this form are called cylinders.

Let $\mathfrak{C}$ be the family of all cylinders in Σ_k. This family generates a σ-algebra $\mathcal{C}$, which will be the standard σ-algebra to work with on Σ_k.

To define a measure on Σ_k we take any probability vector $p = (p_0, \dots, p_{k-1})$ (i.e. $p_i \in [0, 1]$ and $\sum_i p_i = 1$). The probability vector p defines, in a trivial way, a measure p on $\{0, 1, \dots, k-1\}$. Thus, we can take μ as the product measure $\mu = p^{\mathbb{N}}$ on Σ_k. This measure is characterized by its values on cylinders. Given a cylinder $C(n; a_1, a_2, \dots, a_m)$, one can easily see that

$$\mu(C(n; a_1, a_2, \dots, a_m)) = p_{a_1} \cdot p_{a_2} \cdot \dots \cdot p_{a_m}.$$

The measure μ is called a *Bernoulli measure*. It is easy to see that μ is σ invariant for any Bernoulli shift $\sigma : \Sigma_k \to \Sigma_k$. Also, the system (σ, μ) is ergodic. A measurable automorphism $f : X \to X$ of a probability space (X, μ) is called a *Bernoulli automorphism* if it is isomorphic to a Bernoulli shift $\sigma : \Sigma_k \to \Sigma_k$ for some $k \in \mathbb{N}$. By an isomorphism we mean a bimeasurable function that conjugates the dynamics and takes μ to a Bernoulli measure. That is a Bernoulli automorphism preserves all ergodic properties of a Bernoulli shift.

The following proposition is a very well-known fact from the theory of Bernoulli shifts (e.g. [15]).

Proposition 3 *For any $k \in \mathbb{N}$, the Bernoulli shift $\sigma : \Sigma_k \to \Sigma_k$ has periodic orbits of all periods and the set of transitive points is a residual set.*

Along the paper we will work with the spaces $\Sigma_2 \times \Sigma_k^*$. For each $k \in \mathbb{N}^*$, on each $\Sigma_2 \times \Sigma_k^*$ we have the shift on two-coordinates

$$\sigma((x_n)_n, (y_m)_m) = ((x_{n+1})_n, (y_{m+1})_m).$$

This shift on two coordinates is isomorphic to a standard shift on $\Sigma_{2\cdot k}$, so it is also a Bernoulli automorphism. It is also a direct fact that the two-coordinates shift above is topologically conjugate to the standard shift on $\Sigma_{2\cdot k}$, thus the conclusions of Proposition 3 are also true for the two-coordinates shift.

Let us define

$$\Sigma^b := \bigcup_{k\in\mathbb{N}^*} \Sigma_k^* = \{\{x_i\}_i | \exists L \in \mathbb{R} \text{ s.t. } |x_i| \le L, x_i \in \mathbb{N}^* \ \forall i\}.$$

We consider the two-coordinates shift

$$\sigma : \Sigma_2 \times \Sigma^b \to \Sigma_2 \times \Sigma^b.$$

To make notations easier, we will denote by σ_k the restriction of σ to the space $\Sigma_2 \times \Sigma_k^*$. Hence $\sigma_k : \Sigma_2 \times \Sigma_k^* \to \Sigma_2 \times \Sigma_k^*$.

The space of sequences $\Sigma_2 \times \Sigma_k^*$ is naturally endowed with the product topology, which is the coarsest topology for which the cylinders are open set. And the topology on $\Sigma_2 \times \Sigma^b$ is the coarset topology having the cylinders of $\Sigma_2 \times \Sigma_k^*$ as open sets $\forall k \in \mathbb{N}^*$. We note that σ_k is defined on a compact space while σ is not.

One of the most useful invariants on Dynamical Systems and Ergodic Theory is the topological entropy. One may think of topological entropy as a measurement of chaos. Topological entropy has a not so straight definition (which we recomend the reader to take a look [15]) but fortunately to our context it can be associated to the growth of periodic points. Therefore to our purpose we consider the topological entropy as follows. For a compact π-invariant set $\Omega \subset \Sigma_2 \times \Sigma^b$ we define the topological entropy of $\pi|\Omega$ as

$$h_{\sigma|\Omega} := \lim_{n\to\infty} \frac{1}{n} \# Per_n(\sigma|\Omega),$$

where $\# Per_n(f)$ means the number of periodic point of period n. It is not difficult to prove that $h_{\sigma_k} = log(2k)$.

3 Statement of Results

Using the same notations as above, our main result is:

Theorem A *Let $Z = (X, Y) \in \Omega^r$ be a Filippov system (2). Assume that Z admits a sliding Shilnikov orbit Γ and let π denote the first return map (6) defined on $\mathcal{U}_r$ nearby $q_0 = \Gamma \cap \partial M^s$. Then, for $r > 0$ sufficiently small, there is a set $\Lambda \subset \mathcal{U}_r$ such that:*

(a) *for each $k \in \mathbb{N}$ there exists a π-invariant cantor set $\Lambda_k \subset \Lambda$ such that $\pi|\Lambda_k$ is conjugate to the shift on $\Sigma_2 \times \Sigma_k^*$, that is*

$$h_k \circ \sigma_k = \pi \circ h_k$$

where $h_k : \Sigma_2 \times \Sigma_k^* \to \Lambda_k$ is a homeomorphism. In particular the dynamics on Λ_k is transitive, sensitive to initial conditions and has dense periodic points.

(b) *There is a homeomorphism $h : \Sigma_2 \times \Sigma^b \to \Lambda := \bigcup_k \Lambda_k$ such that h conjugates the dynamics of σ and π and $\Lambda \cup \{q_0\}$ is a compact set. In particular the topological entropy of π is infinite.*

Hence, given any natural number $m \geq 1$ we can find infinitely many periodic points for the first return map with period m and, consequently, infinity many closed orbits of $\dot{u} = Z(u)$ nearby Γ. Indeed, given $k \geq 1$, each periodic point of period m for σ_k is mapped by h_k in a periodic point of period m for π, thus varying $k \geq 1$ we obtain infinitely many periodic points of a fixed period m for π.

We also obtain another two consequences from Theorem A. The first one states that we are able to understand any compact invariant set for the first returning map (or flow) by some dynamics on a symbolic dynamics.

Corollary A *Given a compact set $K \subset \mathcal{U}_r$ π-invariant, then $\pi|K$ is conjugate for some k to a $\sigma_k|\Omega$ where $\Omega \subset \Sigma_2 \times \Sigma_k^*$ is an σ_k-invariant set.*

Proof Notice that for $\xi \in \Lambda$, there exists a positive number $M_\xi < \infty$ such that $n_*(\xi) < M_\xi$ for $* \in \{0, 1\}$. By continuity of π there is an neighbourhood U_ξ of ξ such that $\forall z \in U_\xi$ $n_*(z) \leq M_\xi$. Since K is a compact set take a finite cover of K and consider the maximum of n_* for these finite cover. Let k_0 be this maximum. This means that for all points in K if we catalogue its trajectory it has to be given by a sequence in $\Sigma_2 \times \Sigma_{k_0}^*$. Proving the corollary. □

We are able to fully characterize the ergodic properties of the system:

Corollary B *For $r > 0$ sufficiently small if (π, μ) is ergodic, then there exist $k \in \mathbb{N}$ such that*

- $\mu(\Lambda_k) = 1$;
- *there exist a measure ν which is σ_k-invariant for which $(\pi|\Lambda_k), \mu)$ is isomorphic to (σ_k, ν).*

Proof We know that $\Lambda_k \subset \Lambda_{k+1}$ and $\Lambda = \bigcup_{k=1}^{\infty} \Lambda_k$ and Λ_k is π-invariant. Hence, by ergodicity $\mu(\Lambda_k) \in \{0, 1\}$, if $\mu(\Lambda_k) = 0$, $\forall k \in \mathbb{N}$ then $\mu(\Lambda) = 0$ which is an absurd. Therefore, there exist k_0 such that $\mu(\Lambda_{k_0}) = 1$. Since there is a conjugacy from $\pi|\Lambda_{k_0}$ to σ_{k_0} the theorem is done with $\nu := (h_{k_0})_*\mu$. □

Corollary C *Let $k \in \mathbb{N}$ be fixed. Then for any sufficiently small α the first return map π_α is defined in a neighborhood of Λ_k and there is a cantor set $\Lambda_{k,\alpha}$ for which $\pi|_{\Lambda_k}$ and $\pi_\alpha|_{\Lambda_{k,\alpha}}$ are topologically conjugate. In particular one has that any sufficiently small α the system Z_α exhibits infinitely many periodic orbits.*

Proof The first return map π is defined on the open (in the line topology) set $\mathcal{U}_r$, given k as in the corollary we know that $\Lambda_k \subset \mathcal{U}_r$. Since Λ_k is compact we know that we may restrict to a smaller neighborhood of Λ_k let us call $\mathcal{V}_r$ for which for a sufficiently small parameter α the first return map of the perturbation is also defined on $\mathcal{V}_r$. Hence, because Λ_k is a hyperbolic repeller, it is structurally stable (e.g. [15, Chapter 18]), therefore there exists $\Lambda_{k,\alpha}$ which is π_α invariant such that $\pi|_{\Lambda_k}$ and $\pi_\alpha|_{\Lambda_{k,\alpha}}$ are topologically conjugate. In particular, it has infinitely many periodic orbits. □

4 Proof of Theorem A

Consider the Filippov system $\dot{u} = Z(u) = (X, Y)(u)$ given by (2), and denote by $\varphi(t, v)$ its (Filippov) solution such that $\varphi(0, v) = v$. Assume that Z contains a sliding Shilnikov orbit Γ, and let $p_0 \in M^s$ and $q_0 \in \partial M^s$ be as in Definition 2. We consider a neighbourhood $I \subset \partial M^s$ of q_0 for which the first return map π is well defined. We assume that I has end points q_1 and q_2 and we denote $I = [q_1, q_2]$. The forward saturation of I through the flow of X intersects M in a curve J.

Let us call by Λ the set of points in I which return infinitely often to I through the forward flow of Z. That is

$$\Lambda = \{\xi \in I \mid \exists \{t_n\}_{n \in \mathbb{N}}, t_n \to \infty, \varphi(t_n, \xi) \in I\}.$$

We note that Theorem 2 guarantees that $\Lambda \neq \emptyset$.

Call $I_0 := [q_1, q_0]$, $I_1 := [q_0, q_2]$, J_0 and J_1 denote the intersection of M with the forward saturation of I_0 and I_1, respectively. Given a point $\xi \in \Lambda$ we denote by $\eta_*(\xi)$, $* \in \{0, 1\}$ the number of intersections that the forward flow orbit of ξ has with J_* before returning to $\Lambda \subset I$, that is,

$$\eta_*(\xi) := \#\{\varphi(t, \xi) \cap J_* : 0 < t < t_\xi\}.$$

where t_ξ is the first return time of ξ on Λ. The intersections detected by the function $\eta_*(\xi)$ are occurring on the sliding region M^s of the switching surface M. That means, for a point $\xi \in \partial M^s$ sufficiently close to q_0, the flow starting at ξ travels forward in time following the vector field X. After a finite time it reaches transversally the switching surface at a point of $J \subset M^s$ close to p_0. Then the flow follows the sliding vector field $\widetilde{Z}$ [see (3)], spiralling outward around p_0 until reaching the fold line ∂M^s. Since the curve J is transversal to the sliding vector field $\widetilde{Z}$ and contains the pseudo saddle–focus p_0, the number $\eta_*(\xi)$ is well defined. Notice that $\eta_*(\xi)$ counts the amount of times that the flow of $\widetilde{Z}$ intersects J_*, it could, of course, turn around p_0 several times more before reaching Λ without intersect J.

We will construct a map

$$h_k : \Sigma_2 \times \Sigma_k^* \to \Lambda$$

that will conjugate the dynamics of σ_k with π (i.e. $h_k \circ \sigma_k = \pi \circ h_k$), where $\Sigma_k^* = \{1, 2, \ldots, k\}^{\mathbb{N}}$.

Fix a natural number $k > 0$ and take a point

$$(X, N) = ((x_i)_{i \in \mathbb{N}}, (n_i)_{i \in \mathbb{N}}) \in \Sigma_2 \times \Sigma_k^*$$

We will define $h_k((X, N))$ through a limit process.

Define $P_0(X, N)$ as the points which are in I_{x_0}, that is $P_0(X, N) = I_{x_0}$. Define $P_1(X, N)$ as the points which are in I_{x_0} and before arriving by the first return maps to I_{x_1} touches n_0 times the segment J_{x_1}, that is:

$$P_1(X, N) = \{\xi \in P_0(X, N) \mid \eta_{x_1}(\xi) = n_0, \pi(\xi) \in I_{x_1}\}.$$

In general we define

$$P_{m+1}(X, N) = \{\xi \in P_m(X, N) | \eta_{x_{m+1}}(\pi^m(\xi)) = n_m,\ \pi^{m+1}(\xi) \in I_{x_{m+1}}\}.$$

Now consider the following set

$$P(X, N) := \bigcap_{i \in \mathbb{N}} P_i(X, N). \tag{7}$$

Notice that $P_i(X, N) \subset P_{i-1}(X, N)$ and each $P_i(X, N)$ is a closed interval. Hence $P(X, N)$ is a point or a non-degenerated interval, we want to rule out the non-degenerate interval case. We start with the following.

Lemma 1 *If* $P(X, N) \cap P(X', N') \neq \emptyset$, *then* $P(X, N) = P(X', N')$. *In particular* $(X, N) = (X', N')$.

Proof That comes directly from the definition of the sets, because $P(X, N)$ is solely defined by stating what the orbit of a point "behaves", hence if a point is also on $P(X', N')$ that means the sets are the same. □

Lemma 2 $\pi(P(X, N))$ *is of the form* $P(X', N')$

Proof In fact one have $\pi(P(X, N)) = P(\sigma(X, N))$, this comes once again from the definition of the set. □

Lemma 3 *If* $\pi : \Lambda \to \Lambda$ *is such that* $|\pi'(\xi)| > 1$ *for all* $\xi \in \Lambda$, *then* $P(X, N)$ *is a point* $\forall (X, N) \in \Sigma_2 \times \Sigma_k^*$.

Proof Let us consider l as the length measure. Notice that if $l(P(X, N)) > 0$, then $l(\pi(P(X, N))) > l(P(X, N))$, but since $\pi^n(P(X, N)) \subset I$ and $l(I) < \infty$ the family $\{\pi^n(P(X, N)))\}_{n \in \mathbb{N}}$ cannot be pairwise disjoint, otherwise one would have

$$\infty > l(I) \geq l(\cup_n \pi^n(P(X, N))) = \sum_n \pi^n(P(X, N)) > \sum_n l(P(X, N)) = \infty,$$

which is an absurd. Therefore, there must exist n_1 and n_2 such that

$$\pi^{n_1}(P(X,N)) \cap \pi^{n_2}(P(X,N)) \neq \emptyset,$$

the above lemmas imply that $\pi^{n_2-n_1}P(X,N) = P(X,N)$, which cannot happen since $|\pi'| > 1$. Hence $P(X,N)$ is a point. □

By Proposition 2 we know that if q_1 and q_2 are sufficiently close to q_0, then $|\pi'| > 1$ on I. This means that the functions h_k are well defined by the above lemma as

$$\begin{aligned} h_k : \Sigma_2 \times \Sigma_k^* &\to \Lambda \\ (X,N) &\mapsto P(X,N). \end{aligned}$$

Since the domain of h_{k+1} contain the domain of h_k and the two functions by construction coincide on the domain of h_k, the function h

$$\begin{aligned} h : \Sigma_2 \times \Sigma^b &\to \Lambda \\ (X,N) &\mapsto h_k(X,N), \text{ if } (X,N) \in \Sigma_2 \times \Sigma_k^*. \end{aligned}$$

is well defined. Recall that $\pi(P(X,N)) = P(\sigma(X,N))$, which implies $\pi \circ h_k = h_k \circ \sigma_k$ as well as $\pi \circ h = h \circ \sigma$.

Lemma 4 *The maps h_k and h are continuous.*

Proof Let $(X,N) \in \Sigma_2 \times \Sigma_k^*$ and $\epsilon > 0$ be given. From (7) and (8) we know that

$$h_k(X,N) = \bigcap_{n\in\mathbb{N}} P_n(X,N),$$

hence consider n_ϵ such that $P_{n_\epsilon}(X,N) \subset (-\epsilon + h_k(X,N), h_k(X,N) + \epsilon) \subset I$.

Let $\mathcal{V}_{(X,N)}$ be a neighborhood of (X,N) in $\Sigma_2 \times \Sigma_k^*$ given by the cylinder

$$\mathcal{V}_{(X,N)} := \{(Y,M) | y_i = x_i,\ m_{i+1} = n_{i+1}\ i, j \in \{0, 1, \ldots, n_\epsilon\}\}.$$

And the continuity follows, since

$$h_k(\mathcal{V}_{(X,N)}) \subset (-\epsilon + h_k(X,N), h_k(X,N) + \epsilon).$$

The same proof serves for h. □

Lemma 5 *The map h_k and h are homeomorphisms onto their image.*

Proof We prove h_k is a homeomorphism onto its image, the case for h is analogous. Notice that h_k is injective by Lemma 1. To see that the inverse is continuous, consider a point $h_k(X,N)$ and a neighborhood $\mathcal{U}_{(X,N)}$ of (X,N), therefore $(Y,M) \in \mathcal{U}_{(X,N)}$ means that the the first digits of both sequences (X,N) and (Y,M) coincide, using the continuity of the flow we get that for points close enough to $h_k(X,N)$ they must have this predefined trajectory and the continuity follows. □

The above lemmas imply Theorem A, where $\Lambda_k := h_k(\Sigma_2 \times \Sigma_k^*)$. □

5 Final Comments

5.1 Conclusion and Further Directions

In this paper we studied Filippov systems admitting a sliding Shilnikov orbit Γ, which is a homoclinic connection inherent to Filippov systems. This connection has been firstly studied in [17]. Using the well known theory of Bernoulli shifts, we were able to provide a full topological and ergodic description of the dynamics of Filippov systems nearby a sliding Shilnikov orbit Γ, answering then some inquiries made in [17]. As our main result, we established the existence of a set $\Lambda \subset \partial M^s$ such that the restriction to Λ of the the first return map π, defined nearby Γ, is topologically conjugate to a Bernoulli shift with infinite topological entropy. This ensures π, consequently the flow, to be as much chaotic as one wishes. In particular, given any natural number $m \geq 1$ one can find infinitely many periodic points of the first return map with period m and, consequently, infinitely many closed orbits nearby Γ of the Filippov system.

As it has already been observed in [17], a possible direction for further investigations is to consider higher dimensional vector fields, since in higher dimension it is allowed the existence of many other kinds of sliding homoclinic connections. We feel that the techniques applied in this paper may be straightly followed to obtain similar results in higher dimensions.

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