J Dyn Diff Equat (2018) 30:359–377
https://doi.org/10.1007/s10884-017-9574-6

Asymptotical Stability of Differential Equations Driven by Hölder Continuous Paths

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Received: 18 April 2016 / Revised: 12 December 2016 / Published online: 18 January 2017

Abstract In this manuscript, we establish local exponential stability of the trivial solution of differential equations driven by Hölder continuous paths with Hölder exponent greater than 1/2. This applies in particular to stochastic differential equations driven by fractional Brownian motion with Hurst parameter greater than 1/2. We motivate the study of local stability by giving a particular example of a scalar equation, where global stability of the trivial solution can be obtained.

Keywords Differential equations · Hölder continuous driving signal · Fractional Brownian motion · Exponential stability

Mathematics Subject Classification Primary: 37L15; Secondary: 34A34 · 34F05

1 Introduction

This article is concerned with the study of the stability of $\mathbb{R}^d$-valued differential equations driven by Hölder continuous signals ω of the form

This work was partially supported by MTM2015-63723-P, MINECO/FEDER funding (M. J. Garrido-Atienza and B. Schmalfuß).

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$$du(t) = F(u(t))dt + G(u(t))d\omega(t), \qquad t \geq 0, \tag{1}$$

where $u(0) = u_0 \in \mathbb{R}^d$ and F, G are smooth functions defined on $\mathbb{R}^d$. Our canonical example for ω is a sample path of fractional Brownian motion with Hurst index greater than 1/2.

The existence and uniqueness of solutions for differential equations driven by a Hölder continuous function with Hölder exponent greater than 1/2 is now well understood, in the context of fractional calculus, see e.g. [11,24,27], and in the context of rough path theory as well, see e.g. [8,18,19].

However, the study of the stability of such equations is, to some extent, in its beginnings, in contrast to the case when the driving signal arises from a standard Brownian motion, where the existing literature is huge. See Remarks 1 and 2 below for a discussion. The closest related work we are aware of is [25], which deals with the exponential stability of a linear delay equation additively disturbed by a Riemann–Liouville fractional Brownian motion.

Our main result can be summarized as follows: if ω is tempered, i.e. its Hölder norm over $[t, t+1]$ increases subexponentially for $t \to \pm\infty$, and F, G satisfy

$$F(x) = Ax + \mathcal{O}\left(\|x\|^2\right), \qquad G(x) = \mathcal{O}\left(\|x\|^2\right), \qquad \text{for} \quad x \to 0,$$

where the real parts of the eigenvalues of A are bounded from above by $-\lambda < 0$, then the solution u to (1) satisfies for any $\mu < \lambda$ that

$$\lim_{t\to\infty} e^{\mu t}\|u(t)\| = 0,$$

provided that u_0 is in a neighborhood of zero, which depends on μ and ω. The assumption on ω is in particular fulfilled by a fractional Brownian motion with Hurst parameter $H > 1/2$.

Remark 1 Let us mention here a few pioneering investigations related to stability for classical stochastic differential equations (SDEs). Almost sure exponential stability was considered in [16] for linear SDEs with Brownian motion as integrator using Lyapunov exponents and ergodic theory. In [2] a.s. exponential stability and uniform boundedness was proved. The multiplicative ergodic theorem of Oseledets allowed the analysis of all exponents for a stochastic flow, leading to a detailed analysis of the dynamics of random systems, see [3]. In [20] the author uses stochastic Lyapunov functions to discuss the stability of SDEs with semimartingale integrators, making use of the exponential martingale inequality, obtaining sufficient criteria for a.s. exponential stability and for polynomial stability. In [21] the same author gives a consistent account of the theory of SDEs driven by a nonlinear integrator and their exponential stability at fixed points via Lyapunov function techniques.

Remark 2 The fractional Brownian motion (FBM) B^H belongs to a family of Gaussian processes indexed by the Hurst parameter $H \in (0, 1)$. When $H = 1/2$ FBM is standard Brownian motion, but when $H \neq 1/2$ the process B^H has properties that differ sharply to those of $B^{1/2}$. In fact, B^H is not a semimartingale nor a Markov process unless $H = 1/2$, and therefore the techniques to analyse stochastic differential equations driven by fractional Brownian motion are rather different to the Brownian motion case. Previous contributions, which study the longtime behavior of SDEs driven by FBM belong mainly to two categories:

(i) Using the theory of random dynamical systems the existence of random attractors has been established in [11] for a finite-dimensional setting and in [9] for an infinite-dimensional one, in both cases under the condition $H > 1/2$. Precursors to theses works are [10,22]. In [22] the existence of exponentially attracting random fixed points was shown for linear and semilinear infinite-dimensional stochastic equations with additive FBM with $H > 1/2$. Under suitable dissipativity conditions on the drift the existence and uniqueness of a stationary solution that attracts all other solutions was obtained in [10] for finite-dimensional SDEs with additive FBM of any Hurst parameter.

(ii) In a series of articles [12–15] Hairer and coworkers studied the existence of "adapted" stationary solutions to dissipative finite-dimensional SDEs driven by FBM and their speed of convergence to the stationary state. The analysis in those papers is built on suitable extensions of Markovian notions as strong Feller property, invariant measure and adaptedness to the non-Markovian setting. These works cover the case of additive noise with any Hurst parameter and multiplicative noise with $H > 1/3$. Furthermore, the contribution by Deya et al. [5] establishes the convergence order 1/8 (in total variation norm) of the system to the stationary solution for multiplicative noise and $H > 1/3$.

This article can be seen as a first attempt to analyze the exponential stability of the solution of (1), since we are able to obtain local stability, in the sense that the initial condition u_0 must belong to a neighborhood of zero. Only in very particular situations, in which we can transform the equation into a random differential equation, we can establish global exponential stability, see Sect. 2 below. Therefore, more efforts are needed in order to cover the global stability and this will be the topic of our future research.

The remainder of this article is structured as follows: we begin with an example in Sect. 2, where we can establish global stability. In Sect. 3 we recall some basic facts about fractional Brownian motions and prove some auxiliary results, while in Sect. 4 we recall the definition and main properties of the Young integral. Section 5 concerns the analysis of the existence and uniqueness of solutions, and the study of the local exponential zero stability of our problem.

2 Global Stability for Linear Scalar Noise

In this section we want to examine the following scalar equation

$$du(t) = F(u(t))dt + \gamma u(t)d\omega(t), \qquad t \geq 0, \tag{2}$$

where $F\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative, $\gamma \in \mathbb{R}$ and $\omega : \mathbb{R}^+ \to \mathbb{R}$ is a Hölder continuous function of order $\beta' > 1/2$. Moreover, we assume that we have the splitting

$$F(x) = -\lambda x + \hat{F}(x), \qquad x \in \mathbb{R},$$

with $\lambda > 0$, and

$$|\hat{F}(x)| \leq \delta |x|, \quad x \in \mathbb{R}, \tag{3}$$

where $0 \leq \delta < \lambda$.

We are going to see that the scalar and linear structure of the noise now allows us to obtain global exponential stability using the Doss–Sussman transformation. For this, define first

$$v(t) = e^{\lambda t} u(t), \qquad t \geq 0.$$

Then the usual change of variable formula, see e.g. Theorem 4.3.1 in [27], gives

$$dv(t) = b(t, v(t))dt + \gamma v(t)d\omega(t), \qquad t \geq 0, \tag{4}$$

where we have set

$$b(t,x) = e^{\lambda t} \hat{F}\left(e^{-\lambda t} x\right), \qquad t \geq 0, \; x \in \mathbb{R}. \tag{5}$$

The results by Doss, see Theorem 19 in [6], state that the solution of equation (4) can be written as

$$v(t) = h(D(t), \omega(t)), \qquad t \geq 0, \tag{6}$$

where $h: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the solution of

$$\frac{\partial}{\partial \beta} h(\alpha, \beta) = \gamma h(\alpha, \beta), \qquad h(\alpha, 0) = \alpha, \qquad \alpha, \beta \in \mathbb{R}, \tag{7}$$

i.e.

$$h(\alpha, \beta) = e^{\gamma \beta} \alpha,$$

and $D: \mathbb{R}^+ \to \mathbb{R}$ solves

$$\begin{aligned} dD(t) &= e^{-\gamma \omega(t)} b\big(t, e^{\gamma \omega(t)} D(t)\big) dt, \qquad t \geq 0, \\ D(0) &= u(0). \end{aligned} \tag{8}$$

The idea behind this representation is to assume that the solution of (2) can be written in the form (6) and to derive necessary and sufficient conditions for D and h, i.e. (7) and (8). In [6] this approach is introduced for SDEs driven by Brownian motion, but with more general diffusion coefficients, which satisfy a commutativity condition. In our context, this representation follows by the change of variable formula from Theorem 4.3.1 in [27].

Using (5) we obtain

$$dD(t) = e^{-\gamma \omega(t) + \lambda t} \hat{F} \left(e^{\gamma w(t) - \lambda t} D(t) \right) dt, \qquad t \geq 0.$$

Now set $r(t) = |D(t)|^2$. Then for $t \geq 0$ we have

$$dr(t) = 2D(t) e^{-\gamma \omega(t) + \lambda t} \hat{F}\big(e^{\gamma \omega(t) - \lambda t} D(t)\big) dt \leq 2\delta |D(t)|^2 dt = 2\delta r(t) dt,$$

using assumption (3). Therefore, Gronwall's Lemma gives

$$|D(t)|^2 = |r(t)| \leq e^{2\delta t} |u(0)|^2, \qquad t \geq 0,$$

and hence (6) implies that

$$|u(t)| \leq e^{\gamma |\omega(t)|} e^{(\delta - \lambda) t} |u(0)|, \qquad t \geq 0.$$

If

$$\lim_{t \to \infty} \frac{|\omega(t)|}{t} = 0, \tag{9}$$

then it follows

$$\lim_{t \to \infty} e^{\mu t} |u(t)| = 0,$$

for all $0 \leq \mu < \lambda - \delta$.

Condition (9) is in particular fulfilled for almost all sample paths of the fractional Brownian motion B^H, see (12) below. Hence we obtain in this case almost sure exponential stability of the zero solution for all rates smaller than $\lambda - \delta$.

So, in the particular situation of a scalar equation with a linear multiplicative noise, the above method ensures global exponential stability of the zero solution. However, this method seems not to be applicable in general when considering a multidimensional driven signal even if the diffusion coefficient is still linear.

As we have said in the Introduction, we want to consider general multidimensional noise perturbations of the type $G(u(\cdot))d\omega(\cdot)$. Our strategy here will be to deal directly with the Eq. (1) to obtain local exponential zero stability.

3 Preliminaries

From now on, we denote by $\|\cdot\|$ the norm of both spaces $\mathbb{R}^d$ and $\mathbb{R}^m$, while $|\cdot|$ represents as usual the absolute value.

For $\beta' \in (0,1)$, let us consider the space $C^{\beta'}([0,T];\mathbb{R}^m)$ of β'-Hölder continuous functions on some interval $[0,T]$ with values in $\mathbb{R}^m$. The norm in this space is given by

$$\|\omega\|_{\beta'} = \|\omega\|_{\infty,0,T} + |||\omega|||_{\beta',0,T}\,,$$

where

$$\|\omega\|_{\infty,0,T} = \sup_{r\in[0,T]} \|\omega(r)\|, \quad |||\omega|||_{\beta',0,T} = \sup_{0\le q<r\le T} \frac{\|\omega(r)-\omega(q)\|}{(r-q)^{\beta'}}.$$

A fractional Brownian motion (FBM) B^H with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with covariance function

$$R(s,t) = \frac{1}{2}Q\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad s,t \in \mathbb{R},$$

where Q is a non-negative and symmetric matrix in $\mathbb{R}^m \otimes \mathbb{R}^m$.

Let $C_0(\mathbb{R};\mathbb{R}^m)$ be the set of continuous functions which are zero at zero equipped with the compact open topology. Then let $(C_0(\mathbb{R};\mathbb{R}^m), \mathcal{B}(C_0(\mathbb{R};\mathbb{R}^m)), \mathbb{P}_H)$ be the canonical space for fractional Brownian motion, i.e. $B^H(t,\omega) = \omega(t)$, where $\mathbb{P}_H$ denotes the measure of the FBM with Hurst parameter H. On $C_0(\mathbb{R};\mathbb{R}^m)$ we can introduce the Wiener shift θ given by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}, \quad \omega \in C_0(\mathbb{R};\mathbb{R}^m). \tag{10}$$

In particular θ_t leaves $\mathbb{P}_H$ invariant.

Since the fractional Brownian motion is a Gaussian process we can show that for $m \in \mathbb{N}$ and $s,\ t \in \mathbb{R}$ we have

$$\mathbb{E}\|B^H(t) - B^H(s)\|^{2m} = c_m|t-s|^{2Hm} \quad \text{for all } m \in \mathbb{N},$$

where $c_m = \mathbb{E}\|B^{1/2}(1)\|^{2m}$. Applying Kunita [17] Theorem 1.4.1 we obtain that

$$\mathbb{E}|||B^H|||^n_{\beta',0,T} < \infty \tag{11}$$

for any $T > 0$, $\beta' < H$ and for $n \in \mathbb{N}$. This is in particular true for the canonical fractional Brownian motion. Thus, we can conclude that the set $C_0^{\beta'}(\mathbb{R};\mathbb{R}^m)$ of continuous functions, which have a finite β'-Hölder seminorm on any compact interval and which are zero at zero, has $\mathbb{P}_H$-measure one for $\beta' < H$. This set is θ-invariant.

Moreover, since

$$\mathbb{E}\|B^H\|^n_{\infty,0,T} \le C_{n,H}T^{nH}$$

for all $n \ge 1$ and all $T > 0$, see e.g. Chapter 5.1 in [23], the Borel–Cantelli lemma implies that

$$\lim_{t\to\infty} \frac{\|B^H(t)\|}{t} = 0 \qquad \text{for almost all} \quad \omega \in \Omega. \tag{12}$$

A random variable $R \in (0, \infty)$ is called tempered from above if

$$\limsup_{t\to\pm\infty} \frac{\log^+ R(\theta_t\omega)}{t} = 0 \qquad \text{for almost all} \quad \omega \in \Omega. \tag{13}$$

Therefore, temperedness from above describes the subexponential growth or decay of a stochastic stationary process $(t, \omega) \mapsto R(\theta_t\omega)$. A random variable R is called tempered from below if R^{-1} is tempered from above. In particular, if the random variable R is tempered from below and such that $t \mapsto R(\theta_t\omega)$ is continuous for all $\omega \in \Omega$, then for any $\epsilon > 0$ there exists a (random) constant $C_\epsilon(\omega) > 0$ such that

$$R(\theta_t\omega) \geq C_\epsilon(\omega)e^{-\epsilon|t|} \qquad \text{for almost all} \quad \omega \in \Omega.$$

A sufficient condition for temperedness is that

$$\mathbb{E} \sup_{t\in[0,1]} \log^+ R(\theta_t\omega) < \infty.$$

By (11) we obtain that $\|\omega\|_{\beta',0,1}$ is tempered from above because $\log^+ r \leq r$ for $r \geq 0$. Note that the set of all ω, which satisfy (13), is invariant with respect to the flow θ.

We need the following simple result.

Lemma 3 *Let $(R_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$ be sequences such that $R_i \geq C_\varepsilon e^{-\varepsilon i}$ for any $0 < \varepsilon < \sigma$, $i \in \mathbb{N}$, and $v_i \leq v_0 e^{-\sigma i}$ for any $i \in \mathbb{N}$, respectively. Then for sufficiently small $v_0 > 0$ we have*

$$v_i \leq R_i, \qquad i \in \mathbb{N}.$$

If for instance we assume that a random variable $R > 0$ is tempered from below, then we can find a random variable C_ϵ such that $v_i < R(\theta_i\omega)$ holds for $v_0 < C_\varepsilon(\omega)$, where v_i satisfies the assumptions of the previous Lemma.

We finish this section with several technical results that will be applied to establish exponential decay of sequences in further sections.

Lemma 4 *Let $\rho > 0$ and $\mathcal{T}$ be a function from $\bar{B}_{\mathbb{R}^d}(0, \rho)$ into a Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, which is continuously differentiable, and zero at zero. Consider the balls $\bar{B}_{\mathcal{B}}(0, R)$, $\bar{B}_{\mathbb{R}^d}(0, \hat{R})$, with $\hat{R} = \hat{R}(R) \leq \rho$, such that the latter is the largest centered ball such that*

$$\bar{B}_{\mathbb{R}^d}(0, \hat{R}) \subset \mathcal{T}^{-1}\left(\bar{B}_{\mathcal{B}}(0, R)\right),$$

that is,

$$\hat{R} = \max\left\{\hat{r} \in [0, \rho] \colon f_{\mathcal{T}}(v) \leq R \text{ for all } v \in \bar{B}_{\mathbb{R}^d}\left(0, \hat{r}\right)\right\},$$

where $f_{\mathcal{T}} : \bar{B}_{\mathbb{R}^d}(0, \rho) \to \mathbb{R}^+$ is defined by $f_{\mathcal{T}}(v) = \|\mathcal{T}(v)\|_{\mathcal{B}}$. Then there exists $\kappa \in (0, \infty)$ such that

$$\liminf_{R\to 0} \frac{\hat{R}(R)}{R} \geq \kappa.$$

Proof For every sufficiently small R there is an element v_R in $\partial\bar{B}_{\mathbb{R}^d}(0, \hat{R})$ such that

$$\|\mathcal{T}(v_R)\|_{\mathcal{B}} = R.$$

To see the existence of such an element notice that the continuous function $f_{\mathcal{T}}$ is such that $f_{\mathcal{T}}(0) = 0$ and $R \leq \max_{v\in\bar{B}_{\mathbb{R}^d}(0,\rho)} f_{\mathcal{T}}(v)$. Then we have

$$\begin{aligned} f_{\mathcal{T}}^{-1}(\{R\}) &= f_{\mathcal{T}}^{-1}([0, R] \cap [R, \infty)) = f_{\mathcal{T}}^{-1}([0, R]) \cap f_{\mathcal{T}}^{-1}([R, \infty)) \\ &\supset f_{\mathcal{T}}^{-1}([0, R]) \cap \overline{f_{\mathcal{T}}^{-1}([0, R])^c} \end{aligned}$$

and hence all arguments from the boundary of $f_{\mathcal{T}}^{-1}([0, R])$ have the value R with respect to $f_{\mathcal{T}}$. Note that the largest radius $\hat{R}$ is given by the infimum of the distances between the boundary $\partial\mathcal{T}^{-1}(\bar{B}_{\mathcal{B}}(0, R)) = \partial f_{\mathcal{T}}^{-1}([0, R])$ and zero. Since this boundary is a compact set in $\bar{B}_{\mathbb{R}^d}(0, \rho)$ and the mapping $v \to \|v\|$ is continuous, we have an element $v_R \in \partial f_{\mathcal{T}}^{-1}([0, R])$ such that

$$\hat{R} = \inf_{v\in\partial f_{\mathcal{T}}^{-1}([0,R])} \|v\| = \|v_R\|.$$

Finally, applying the mean value theorem it follows

$$\frac{\|v_R\|}{R} = \frac{\|v_R\|}{\|\mathcal{T}(v_R)\|_{\mathcal{B}}} \geq \frac{1}{\sup_{\xi\in\bar{B}_{\mathbb{R}^d}(0,\rho)} \|D\mathcal{T}(\xi)\|_{\mathcal{L}(\mathbb{R}^d,\mathcal{B})}} > 0.$$

□

For completeness, we state the following measurability result.

Lemma 5 *Let $\mathcal{H} : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and non decreasing. Then the function*

$$\mathcal{J}: \mathbb{R}^+ \to \mathbb{R}^+, \qquad \mathcal{J}(x) = \max\{r \in \mathbb{R}^+ : \mathcal{H}(r) \leq x\}$$

is Borel-measurable.

Proof Let $\alpha \geq 0$ and consider the set

$$\mathcal{M}(\alpha) = \left\{x \in \mathbb{R}^+ : \mathcal{J}(x) < \alpha\right\}.$$

Then we have to check that $\mathcal{M}(\alpha)$ belongs to $\mathcal{B}(\mathbb{R}^+)$. Clearly, $\mathcal{M}(0) = \emptyset$, so assume $\alpha > 0$. By definition and since $\mathcal{H}$ is non decreasing, $\mathcal{J}(x) < \alpha$ implies $\mathcal{H}(\alpha) > x$. Vice versa $\mathcal{H}(\alpha) > x$ implies $\mathcal{J}(x) < \alpha$. Hence

$$\mathcal{M}(\alpha) = \left\{x \in \mathbb{R}^+ : x < \mathcal{H}(\alpha)\right\} = [0, \mathcal{H}(\alpha)) \in \mathcal{B}\left(\mathbb{R}^+\right).$$

□

Now we investigate the Hölder norm of a finite-dimensional semigroup $e^{A\cdot}$ generated by an operator A, whose estimates will be used below. The main assumption is that the spectrum of A has a negative real part.

Lemma 6 *Let $e^{A\cdot}$ be the fundamental solution to*

$$du(t) = Au(t), \qquad t \geq 0.$$

Let $\operatorname{Re}\sigma(A) < -\lambda < 0$. *Then there exists an* $M \geq 1$ *such that*

$$\|e^{At}\| \leq Me^{-\lambda t}, \qquad t \geq 0. \tag{14}$$

In addition, for $0 \leq s < t$ *we have*

$$\|e^{At} - e^{As}\| \leq M\|A\|(t-s)e^{-\lambda s}, \qquad \|e^{A(t-s)} - \mathrm{id}\| \leq M\|A\|(t-s), \tag{15}$$

where $\|A\|$ is the Euclidean norm of A.

The proof follows easily by the mean value theorem and Amann [1] Chapter 13. As a consequence, for $0 < s < t$ we have

$$\left\|\left|e^{A(t-\cdot)}\right|\right\|_{\beta,0,t} = \sup_{0\leq r_1<r_2<t} \frac{\|e^{A(t-r_2)} - e^{A(t-r_1)}\|}{(r_2-r_1)^\beta} \leq M\|A\|t^{1-\beta} \tag{16}$$

and

$$\left\| e^{A(t-\cdot)} - e^{A(s-\cdot)} \right\|_{\beta,0,s} = \sup_{0\le r_1<r_2<s} \frac{\| e^{A(t-r_2)} - e^{A(s-r_2)} - \left(e^{A(t-r_1)} - e^{A(s-r_1)}\right) \|}{(r_2-r_1)^\beta}$$
$$= \sup_{0\le r_1<r_2<s} \frac{\| \left(e^{A(t-s)} - \mathrm{id}\right)\left(e^{A(s-r_1)} - e^{A(s-r_2)}\right) \|}{(r_2-r_1)^\beta}$$
$$\le M^2\|A\|^2(t-s)s^{1-\beta}. \tag{17}$$

To finish this section, we introduce a discrete Gronwall-like lemma:

Lemma 7 *Let* $(y_n)_{n\ge0}$ *and* $(g_n)_{n\ge0}$ *be non negative sequences and* $c > 0$ *a non negative constant. If*

$$y_n \le c + \sum_{j=0}^{n-1} g_j y_j, \qquad n = 0, 1, \dots,$$

then

$$y_n \le c \prod_{j=0}^{n-1}(1+g_j) \qquad n = 0, 1, \dots .$$

Proof Lemma 100 in [7] states that the inequalities

$$y_n \le c + \sum_{j=0}^{n-1} g_j y_j, \qquad n = 0, 1, \dots,$$

imply that

$$y_n \le c + c\sum_{j=0}^{n-1} g_j \prod_{k=j+1}^{n-1}(1+g_k) \qquad n = 0, 1, \dots .$$

Using that

$$g_j \prod_{k=j+1}^{n-1}(1+g_k) = \prod_{k=j}^{n-1}(1+g_k) - \prod_{k=j+1}^{n-1}(1+g_k),$$

we obtain the assertion through a telescoping sum argument. □

4 Integrals for a Hölder Continuous Integrator with Hölder Exponent greater than 1/2

In this section, we present the Young integral having a Hölder continuous function with Hölder exponent greater than 1/2 as integrator. To be more precise, let $T > 0$ and consider a mapping

$$g : [0, T] \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$$

such that $g \in C^\beta([0, T]; \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$. Assuming that $\beta + \beta' > 1$ we can define the Young integral with integrand g and integrator $\omega \in C^{\beta'}([0, T]; \mathbb{R}^m)$

$$\int_s^t g d\omega$$

for $0 \leq s < t \leq T$, see [26]. Furthermore, one can represent this integral in terms of fractional derivatives: for $\alpha \in (0,1)$ we define

$$D^{\alpha}_{s+}g[r] = \frac{1}{\Gamma(1-\alpha)}\left(\frac{g(r)}{(r-s)^{\alpha}} + \alpha\int_s^r \frac{g(r)-g(q)}{(r-q)^{1+\alpha}}dq\right),$$
$$D^{1-\alpha}_{t-}\omega_{t-}[r] = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{\omega(r)-\omega(t)}{(t-r)^{1-\alpha}} + (1-\alpha)\int_r^t \frac{\omega(r)-\omega(q)}{(q-r)^{2-\alpha}}dq\right),$$

where $\omega_{t-}(\cdot) = \omega(\cdot) - \omega(t)$. Under the condition $\beta + \beta' > 1$, there exists an α such that $\alpha < \beta$, $\alpha + \beta' > 1$, and these inequalities ensure that the above operators are well defined. Then the Young integral can be expressed as

$$\int_s^t g d\omega = (-1)^{\alpha}\int_s^t D^{\alpha}_{s+}g[r]D^{1-\alpha}_{t-}\omega_{t-}[r]dr, \tag{18}$$

see for instance [27]. Taking into account the definition of the fractional derivatives, it is easy to derive the following estimate

$$\left\| \int_s^t g d\omega \right\| \leq C_{\alpha,\beta,\beta',T} \,|||\omega|||_{\beta',s,t} \left(\|g\|_{\infty,s,t} + (t-s)^{\beta} \,|||g|||_{\beta,s,t} \right)(t-s)^{\beta'}, \tag{19}$$

for $s, t \in [0,T]$, which in particular implies that

$$[0,T] \ni t \mapsto \int_0^t g d\omega \in C^{\beta'}([0,T];\mathbb{R}^d),$$

with

$$\left\| \int_0^t g d\omega \right\|_{\beta',0,T} \leq C_{\alpha,\beta,\beta',T} \|g\|_{\beta,0,T} \,|||\omega|||_{\beta',0,T}\,.$$

For $T \leq 1$ we shall denote $C_{\alpha,\beta,\beta',T}$ by $C_{\alpha,\beta,\beta'}$ in the following.

We also know that the integral is additive: let $s \leq \tau \leq t$, then

$$\int_s^{\tau} g d\omega + \int_{\tau}^t g d\omega = \int_s^t g d\omega,$$

see [27], and for any linear operator $L : \mathbb{R}^d \to \mathbb{R}^m$

$$L\int_s^t g d\omega = \int_s^t (Lg) d\omega = \int_s^t Lg d\omega$$

because $LD^{\alpha}_{s+}g = D^{\alpha}_{s+}Lg$.

Finally, for the Wiener shift flow $\theta = (\theta_t)_{t\in\mathbb{R}}$ given by (10) the following shift property of the integral holds:

Lemma 8 *For any given $T > 0$, let $g \in C^{\beta}([0,T];\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))$, $\omega \in C^{\beta'}([0,T];\mathbb{R}^m)$ such that $\beta + \beta' > 1$. Then for $0 \leq s + \tau < t + \tau \leq T$ we have*

$$\int_{s+\tau}^{t+\tau} g d\omega = \int_s^t g(\cdot + \tau) d\theta_{\tau}\omega.$$

Proof For $1-\beta' < \alpha < \beta$ we have

$$\begin{aligned} D_{t-}^{1-\alpha}(\theta_\tau\omega)_{t-}[r] =& \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{\theta_\tau\omega(r)-\theta_\tau\omega(t)}{(t-r)^{1-\alpha}}+(1-\alpha)\int_r^t\frac{\theta_\tau\omega(r)-\theta_\tau\omega(q)}{(q-r)^{2-\alpha}}dq\right)\\ =& \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{\omega(r+\tau)-\omega(t+\tau)}{(t-r)^{1-\alpha}}+(1-\alpha)\int_r^t\frac{\omega(r+\tau)-\omega(q+\tau)}{(q-r)^{2-\alpha}}dq\right)\\ =& \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{\omega(r+\tau)-\omega(t+\tau)}{(t+\tau-(r+\tau))^{1-\alpha}}+(1-\alpha)\int_{r+\tau}^{t+\tau}\frac{\omega(r+\tau)-\omega(q)}{(q-(r+\tau))^{2-\alpha}}dq\right)\\ =& D_{(t+\tau)-}^{1-\alpha}\omega_{(t+\tau)-}[r+\tau] \end{aligned}$$

and similar for $D_{s+}^{\alpha}g(\cdot+\tau)[r] = D_{(s+\tau)+}^{\alpha}g[r+\tau]$. It suffices now to apply the variable transform $r\mapsto r+\tau$ in (18). □

The Young integral introduced above can be applied to define pathwise stochastic integrals for the fractional Brownian motion B^H with Hurst parameter $H\in(1/2,1)$. In particular B^H can be replaced by the canonical fractional Brownian motion which is Hölder continuous with $\mathbb{P}_H$ probability one, see Sect. 3.

5 Local Exponential Stability

Let $F:\mathbb{R}^d\to\mathbb{R}^d$ and $G:\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)$, and let ω be a noisy input, considered as a function from $\mathbb{R}^+$ to $\mathbb{R}^m$. Then, for $T>0$ consider the equation

$$du(t)=F(u(t))dt+G(u(t))d\omega(t),\qquad t\in[0,T], \tag{20}$$

with initial condition $u(0)=u_0\in\mathbb{R}^d$. This equation is interpreted as

$$u(t)=u_0+\int_0^t F(u(r))dr+\int_0^t G(u(r))d\omega(r),\quad t\in[0,T], \tag{21}$$

where the first integral is defined as a standard Riemann integral while the second one is defined as the Young integral introduced in Sect. 4.

We will assume the following regularity for F and G:

(**A1**) $F:\mathbb{R}^d\to\mathbb{R}^d$ is continuously differentiable with bounded derivative,
(**A2**) $G:\mathbb{R}^d\to\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)$ is twice continuously differentiable with bounded derivatives.

Regarding the existence of solutions, the next result follows by [24], although with a slight modification of the phase spaces that appear in that reference; see also [4], but notice that in this last article a delay equation is considered, and therefore in our setting we should take the delay equal to zero.

Theorem 9 *Suppose* (**A1**) *and* (**A2**). *If* $\omega\in C^{\beta'}([0,T];\mathbb{R}^m)$ *with* $\beta'>\beta>1/2$, *then* (21) *has a unique solution* $u\in C^\beta([0,T];\mathbb{R}^d)$ *for any* $T>0$.

In what follows, we would like to consider mild solutions of (20). To this end, define a matrix $A\in\mathbb{R}^{d\times d}$ and a function $\hat F:\mathbb{R}^d\to\mathbb{R}^d$ by

$$A=DF(0),\qquad \hat F(x)=F(x)-Ax,\quad x\in\mathbb{R}^d.$$

Then $x\mapsto\hat F(x)$ and $x\mapsto D\hat F(x)=DF(x)-DF(0)$ are continuous, and $D\hat F(0)=0$.

We also will need further assumptions:

(**A3**) $F(0) = 0, G(0) = 0$,
(**A4**) $DG(0) = 0$.

We consider then the following equation

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-r)}\hat{F}(u(r))dr + \int_0^t e^{A(t-r)}G(u(r))d\omega(r), \quad t \in [0, T], \tag{22}$$

where the last integral is understood as in Sect. 4.

Lemma 10 *Let $T > 0$ and assume* (**A1**) *and* (**A2**). *If $\omega \in C^{\beta'}([0, T]; \mathbb{R}^m)$ with $\beta' > \beta > 1/2$, then Eq.* (22) *has a unique solution $u \in C^{\beta}([0, T]; \mathbb{R}^d)$ that also coincides with* (21). *Furthermore, if we also assume* (**A3**), *Eq.* (22) *possesses the trivial solution $u = 0$.*

Proof In view of the regularity of $\hat{F}$ and G, the existence and uniqueness of a solution to (22) follows by [11]. Now we want to prove that such a solution coincides with the solution of (21). To achieve such a result, notice that when ω is a sufficiently smooth path, then (21) and (22) are the same solutions using classical calculus.

Now it suffices to follow an approximation argument. To be more precise, consider (22) for a sequence of driving paths $(\omega^n)_{n\in\mathbb{N}}$, which are given by the piecewise linear interpolation of w with stepsize $T2^{-n}$. Then the sequence $(u^n)_{n\in\mathbb{N}}$ related to these piecewise linear paths converges to the solution of (21) and (22) as well, being both of them driven by ω, see Chapter 10 in [8]. Therefore both solutions are the same. □

Note that a sufficient condition for the convergence of $(\omega^n)_{n\in\mathbb{N}}$ to ω and of $(u^n)_{n\in\mathbb{N}}$ to u is that $\omega \in C^{\beta''}([0, T]; \mathbb{R}^m)$ for $\beta' < \beta'' \le 1$.

In the following, we will focus on the study of the asymptotic behavior of (22). First of all, we introduce the notion of stability that we are interested in:

Definition 11 We say that the solution u of (22) is locally exponentially zero stable with exponential rate $\mu > 0$, if there exists a neighborhood $U(\omega, \mu)$ of zero such that

$$u_0 \in U(\omega, \mu) \quad \Longrightarrow \quad \lim_{t\to\infty} e^{\mu t}\|u(t)\| = 0.$$

The strategy that we will carry out to prove exponential stability is as follows:

(i) Since the norm of any solution of (22) depends on the norm of ω, we will use a cut-off argument, by which the functions $\hat{F}$ and G appearing in (22) are only required to be defined on $\bar{B}_{\mathbb{R}^d}(0, \rho)$, for some $\rho > 0$. Indeed, we will take a composition of the locally defined functions with a cut-off function depending on a variable $\hat{R}$.
(ii) With these compositions we construct a sequence $(u^n)_{n\in\mathbb{N}}$ such that each element u^n is a solution of a modified differential equation of the type (22), defined on [0, 1] and driven by $\theta_n\omega$, where the norm of each u^n depends now on the magnitude of $\theta_n\omega$ but also on a new variable R related to $\hat{R}$. By a suitable choice of these variables (which depend on the fixed ω) we can apply the discrete Gronwall-like Lemma 7 to obtain a subexponential estimate of every element of the sequence.
(iii) Thanks to the temperedness of R and $\hat{R}$ we will end up proving that $(u^n)_{n\in\mathbb{N}}$ describes the solution of (22), and that it is exponentially zero stable as described in Definition 11.

Consequently, we begin by restricting the mappings F and G to be defined on some neighborhood of zero. For $\rho > 0$ assume:

(**A1**)' $F\colon \bar{B}_{\mathbb{R}^d}(0, \rho) \to \mathbb{R}^d$ is continuously differentiable with bounded derivative, (**A2**)' $G : \bar{B}_{\mathbb{R}^d}(0,\rho) \to \mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)$ is twice continuously differentiable with bounded derivatives.

We also define χ to be the cut-off function

$$\chi : \mathbb{R}^d \to \bar{B}_{\mathbb{R}^d}(0,1) \quad \text{where} \quad \chi(u) = \begin{cases} u & \text{if} \quad \|u\| \le \frac{1}{2} \\ 0 & \text{if} \quad \|u\| \ge 1 \end{cases}.$$

In particular the norm of $\chi(u)$ is bounded by 1. Let us assume that χ is twice continuously differentiable with bounded derivatives $D\chi$ and $D^2\chi$ and let us denote by $L_{D\chi}$, $L_{D^2\chi}$ the bounds for those derivatives. Now for $u \in \mathbb{R}^d$ and some $0 < \hat{R} \le \rho$ we set

$$\chi_{\hat{R}}(u) = \hat{R}\,\chi(u/\hat{R}) \in \bar{B}_{\mathbb{R}^d}(0,\hat{R}).$$

Then it is not difficult to see that the first derivative $D\chi_{\hat{R}}$ is bounded by $L_{D\chi}$, while the second derivative $D^2\chi_{\hat{R}}$ is bounded by $L_{D^2\chi}/\hat{R}$.

Define the functions

$$\hat{F}_{\hat{R}} := \hat{F} \circ \chi_{\hat{R}} : \mathbb{R}^d \to \mathbb{R}^d, \qquad G_{\hat{R}} := G \circ \chi_{\hat{R}} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^m,\mathbb{R}^d).$$

Now we construct the aforementioned sequence $(u^n)_{n\in\mathbb{N}}$, defined on $[0,1]$, with driving path $\theta_n\omega$ and coefficients $\hat{F}_{\hat{R}}$ and $G_{\hat{R}}$, where $\hat{R}$ also depends on $\theta_n\omega$:

$$\begin{aligned} u^n(t) &= e^{At}u^n(0) + \int_0^t e^{A(t-r)}\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))dr \\ &\quad + \int_0^t e^{A(t-r)}G_{\hat{R}(\theta_n\omega)}(u^n(r))d\theta_n\omega(r), \qquad t \in [0,1]. \end{aligned}$$

Recall here that $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$. We set $u^0(0) = u_0$ and $u^n(0) = u^{n-1}(1)$ for $n \in \mathbb{N}$. Under (**A1**)' and (**A2**)', the functions $\hat{F}_{\hat{R}}$, $G_{\hat{R}}$ satisfy the conditions of Lemma 10, so for any $n \in \mathbb{N}$ each one of the above problems has a unique solution $u^n \in C^\beta([0,1];\mathbb{R}^d)$.

In order to estimate the norm of each u^n, we need the following result, that gives us suitable estimates of the localized coefficients $\hat{F}_{\hat{R}}$ and $G_{\hat{R}}$.

Lemma 12 *Assume* (**A1**)*',* (**A2**)*',* (**A3**) *and* (**A4**)*. Then for every $R > 0$ there exists a positive $\hat{R} \le \rho$ such that for $u, z \in \mathbb{R}^d$ we have*

$$\|\hat{F}_{\hat{R}}(u)\| \le RL_{D\chi}\|u\|, \tag{23}$$

$$\|G_{\hat{R}}(u)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)} \le RL_{D\chi}\|u\|, \tag{24}$$

$$\|G_{\hat{R}}(u) - G_{\hat{R}}(z)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)} \le RL_{D\chi}\|u - z\|. \tag{25}$$

Proof Since $D\hat{F} : \bar{B}_{\mathbb{R}^d}(0,\rho) \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^d) =: \mathcal{L}(\mathbb{R}^d)$ and $DG : \bar{B}_{\mathbb{R}^d}(0,\rho) \to \mathcal{L}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))$ are continuous with $D\hat{F}(0) = 0$ and $DG(0) = 0$, for any $R > 0$ we can choose an $\hat{R} \le \rho$ such that

$$\sup_{\|v\|\le\hat{R}} \|D\hat{F}(v)\|_{\mathcal{L}(\mathbb{R}^d)} \le R \qquad \text{and} \qquad \sup_{\|v\|\le\hat{R}} \|DG(v)\|_{\mathcal{L}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))} \le R.$$

Then, since $\hat{F}_{\hat{R}}(0) = 0$, for $u \in \mathbb{R}^d$ we have

$$\begin{aligned}\|\hat{F}_{\hat{R}}(u)\| &\leq \sup_{z\in\mathbb{R}^d} \|D(\hat{F}(\chi_{\hat{R}}(z)))\|_{\mathcal{L}(\mathbb{R}^d)}\|u\| \\ &\leq \sup_{\|v\|\leq\hat{R}} \|D\hat{F}(v)\|_{\mathcal{L}(\mathbb{R}^d)} \sup_{z\in\mathbb{R}^d} \|D\chi_{\hat{R}}(z)\|_{\mathcal{L}(\mathbb{R}^d)}\|u\| \\ &\leq \sup_{\|v\|\leq\hat{R}} \|D\hat{F}(v)\|_{\mathcal{L}(\mathbb{R}^d)} L_{D\chi}\|u\| \leq RL_{D\chi}\|u\|,\end{aligned}$$

and we obtain (23). Due to the fact that $G(0) = 0$ we can follow the same steps to prove (24).

Finally, due to the regularity of G, we have

$$\begin{aligned}\|G_{\hat{R}}(u) - G_{\hat{R}}(z)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)} &\leq \sup_{\|v\|\leq\hat{R}} \|DG(v)\|_{\mathcal{L}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))}\|\chi_{\hat{R}}(u) - \chi_{\hat{R}}(z)\| \\ &\leq L_{D\chi} \sup_{\|v\|\leq\hat{R}} \|DG(v)\|_{\mathcal{L}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))}\|u - z\| \\ &\leq RL_{D\chi}\|u - z\|.\end{aligned}$$

□

In what follows we want to estimate the Hölder norm of each solution u^n on [0, 1]. The next assumption we need is:

(**A5**) Assume $\mathrm{Re}\,\sigma(A) < -\lambda < 0$ for $A = DF(0)$.

Regarding the standard Riemann integral, by (14) and (23) we have

$$\left\| \int_0^{\cdot} e^{A(\cdot-r)}\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))dr \right\|_{\infty,0,1} \leq MR(\theta_n\omega)L_{D\chi}\|u^n\|_{\infty,0,1}.$$

Furthermore, for the Hölder seminorm, thanks to (15), we have

$$\begin{aligned}&\left|\left|\left| \int_0^{\cdot} e^{A(\cdot-r)}\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))dr \right|\right|\right|_{\beta,0,1} \\ &= \sup_{0\leq s<t\leq 1} \frac{\left\| \int_s^t e^{A(t-r)}\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))dr + \int_0^s \left(e^{A(t-r)} - e^{A(s-r)}\right)\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))dr \right\|}{(t-s)^\beta} \\ &\leq \sup_{0\leq s<t\leq 1}\left((t-s)^{1-\beta} \sup_{r\in[s,t]}\left(\|e^{A(t-r)}\|\|\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))\|\right)\right) \\ &\quad + \sup_{0\leq s<t\leq 1}\left(\frac{s}{(t-s)^\beta} \sup_{r\in[0,s]}\left(\|e^{A(t-r)} - e^{A(s-r)}\|\|\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))\|\right)\right) \\ &\leq MR(\theta_n\omega)L_{D\chi}\|u^n\|_{\infty,0,1} + M\|A\|R(\theta_n\omega)L_{D\chi}\|u^n\|_{\infty,0,1} \\ &\leq M(1 + \|A\|)R(\theta_n\omega)L_{D\chi}\|u^n\|_{\infty,0,1},\end{aligned}$$

and therefore

$$\left\| \int_0^{\cdot} e^{A(\cdot-r)}\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r))dr \right\|_{\beta,0,1} \leq M(2 + \|A\|)R(\theta_n\omega)L_{D\chi}\|u^n\|_{\beta,0,1}. \tag{26}$$

Now we estimate the β-Hölder norm of the integral containing $G_{\hat{R}}$. Choose α such that $0 < \alpha < 1/2$, $\alpha + \beta' > 1$ and assume that $0 \le s < t \le 1$. Then from (19) it follows

$$\begin{aligned}
&\left\| \int_0^t e^{A(t-r)} G_{\hat{R}(\theta_n\omega)}(u^n(r)) d\theta_n\omega(r) - \int_0^s e^{A(s-r)} G_{\hat{R}(\theta_n\omega)}(u^n(r)) d\theta_n\omega(r) \right\| \\
&\quad \le \left\| \int_s^t e^{A(t-\cdot)} G_{\hat{R}(\theta_n\omega)}(u^n(r)) d\theta_n\omega(r) \right\| \\
&\qquad + \left\| \int_0^s (e^{A(t-r)} - e^{A(s-r)}) G_{\hat{R}(\theta_n\omega)}(u^n(r)) d\theta_n\omega(r) \right\| \\
&\quad \le C_{\alpha,\beta,\beta'} \, \| \theta_n\omega \|_{\beta'} \, \| e^{A(t-\cdot)} G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\beta,0,t} (t-s)^{\beta'} \\
&\qquad + C_{\alpha,\beta,\beta'} \, \| \theta_n\omega \|_{\beta'} \, \| (e^{A(t-\cdot)} - e^{A(s-\cdot)}) G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\beta,0,s} s^{\beta'}.
\end{aligned}$$

Here we have written $\| \theta_n\omega \|_{\beta'}$ instead of $\| \theta_n\omega \|_{\beta',0,1}$ for notational simplicity. Since for any two β-Hölder continuous functions f, g we have

$$\| fg \|_{\beta,0,t} \le \| f \|_{\infty,0,t} \| g \|_{\beta,0,t} + \| g \|_{\infty,0,t} \, \| f \|_{\beta,0,t} \,,$$

it follows

$$\begin{aligned}
\| e^{A(t-\cdot)} G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\beta,0,t} &\le \| e^{A(t-\cdot)} \|_{\infty,0,t} \| G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\beta,0,t} \\
&\quad + \| G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\infty,0,t} \| e^{A(t-\cdot)} \|_{\beta,0,t}.
\end{aligned}$$

Thanks to (24) and (25) we obtain

$$\begin{aligned}
\| G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\beta,0,t} &= \sup_{s\in[0,t]} \| G_{\hat{R}(\theta_n\omega)}(u^n(s)) \|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\quad + \sup_{0\le r_1<r_2\le t} \frac{\| G_{\hat{R}(\theta_n\omega)}(u^n(r_2)) - G_{\hat{R}(\theta_n\omega)}(u^n(r_1)) \|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)}}{(r_2-r_1)^\beta} \\
&\le R(\theta_n\omega) L_{D\chi} \left(\| u^n \|_{\infty,0,t} + \sup_{0\le r_1<r_2\le t} \frac{\| u^n(r_2) - u^n(r_1) \|}{(r_2-r_1)^\beta} \right) \\
&= R(\theta_n\omega) L_{D\chi} \| u^n \|_{\beta,0,t},
\end{aligned}$$

hence, taking into account (14) and (16), it follows that

$$\begin{aligned}
\| e^{A(t-\cdot)} G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\beta,0,t} &\le M R(\theta_n\omega) L_{D\chi} \| u^n \|_{\beta,0,1} + M \| A \| R(\theta_n\omega) L_{D\chi} \| u^n \|_{\infty,0,1} \\
&\le M R(\theta_n\omega) L_{D\chi} (1 + \| A \|) \| u^n \|_{\beta,0,1}.
\end{aligned}$$

In a similar way, using (15) and (17) we obtain

$$\begin{aligned}
\| (e^{A(t-\cdot)} - e^{A(s-\cdot)}) G_{\hat{R}(\theta_n\omega)}(u^n(\cdot)) \|_{\beta,0,s} &\le M \| A \| R(\theta_n\omega) L_{D\chi} \\
&\quad \times (1 + M \| A \|) \| u^n \|_{\beta,0,1} (t-s),
\end{aligned}$$

and therefore

$$\begin{aligned}
\left\| \int_0^\cdot e^{A(\cdot-r)} G_{\hat{R}(\theta_n\omega)}(u^n(r)) d\theta_n\omega(r) \right\|_{\beta,0,1} &\le C_{\alpha,\beta,\beta'} \, \| \theta_n\omega \|_{\beta'} \, M R(\theta_n\omega) L_{D\chi} \\
&\quad \times \left(1 + 2\| A \| + M \| A \|^2 \right) \| u^n \|_{\beta,0,1}.
\end{aligned}$$

Using the same kind of calculations we get

$$\left\| \int_0^{\cdot} e^{A(\cdot - r)} G_{\hat{R}(\theta_n \omega)}(u(r)) d\theta_n \omega(r) \right\|_{\infty,0,1} \leq C_{\alpha,\beta,\beta'} \, |||\theta_n \omega|||_{\beta'} \, M R(\theta_n \omega) L_{D\chi} \times (1 + \|A\|) \|u^n\|_{\beta,0,1}. \tag{27}$$

Collecting these estimates we have

$$\left\| \int_0^{\cdot} e^{A(\cdot - r)} G_{\hat{R}(\theta_n \omega)}(u(r)) d\theta_n \omega(r) \right\|_{\beta,0,1} \leq K \, |||\theta_n \omega|||_{\beta',0,1} \, R(\theta_n \omega) \|u^n\|_{\beta,0,1}, \tag{28}$$

where

$$K = \max\{1, C_{\alpha,\beta,\beta'}\} M^2 L_{D\chi} \left(2 + 3\|A\| + \|A\|^2\right) \tag{29}$$

using that $M \geq 1$. Note that the constant K is also an upper bound for the constant $M L_{D\chi}(2 + \|A\|)$ in (26), i.e. we have

$$\left\| \int_0^{\cdot} e^{A(\cdot - r)} \hat{F}_{\hat{R}(\theta_n \omega)}(u^n(r)) dr \right\|_{\beta,0,1} \leq K R(\theta_n \omega) \|u^n\|_{\beta,0,1}. \tag{30}$$

For $n \in \mathbb{N}$, define now the function

$$u(t) = u^n(t - n) \quad \text{if } t \in [n, n+1]. \tag{31}$$

On account of Lemma 8, for $t \in [n, n+1]$ we have

$$\begin{aligned}
u(t) &= e^{A(t-n)} u(n) + \int_n^t e^{A(t-r)} \hat{F}_{\hat{R}(\theta_n \omega)}(u(r)) dr + \int_n^t e^{A(t-r)} G_{\hat{R}(\theta_n \omega)}(u(r)) d\omega(r) \\
&= e^{At} u_0 + \sum_{j=0}^{n-1} e^{A(t-j-1)} \int_j^{j+1} e^{A(j+1-r)} \hat{F}_{\hat{R}(\theta_j \omega)}(u(r)) dr \\
&\quad + \sum_{j=0}^{n-1} e^{A(t-j-1)} \int_j^{j+1} e^{A(j+1-r)} G_{\hat{R}(\theta_j \omega)}(u(r)) d\omega(r) \\
&\quad + \int_n^t e^{A(t-r)} \hat{F}_{\hat{R}(\theta_n \omega)}(u(r)) dr + \int_n^t e^{A(t-r)} G_{\hat{R}(\theta_n \omega)}(u(r)) d\omega(r) \\
&= e^{At} u_0 + \sum_{j=0}^{n-1} e^{A(t-j-1)} \int_0^1 e^{A(1-r)} \hat{F}_{\hat{R}(\theta_j \omega)}(u^j(r)) dr \\
&\quad + \sum_{j=0}^{n-1} e^{A(t-j-1)} \int_0^1 e^{A(1-r)} G_{\hat{R}(\theta_j \omega)}(u^j(r)) d\theta_j \omega(r) \\
&\quad + \int_0^{t-n} e^{A(t-n-r)} \hat{F}_{\hat{R}(\theta_n \omega)}(u^n(r)) dr + \int_0^{t-n} e^{A(t-n-r)} G_{\hat{R}(\theta_n \omega)}(u^n(r)) d\theta_n \omega(r).
\end{aligned}$$

Note that the β-Hölder norm of the last two terms of the above expression can be estimated as above, i.e. by (28) and (30). The terms under the sum can be estimated in a different way, since

$$\begin{aligned}
&\left\| e^{A(\cdot - j - 1)} \int_0^1 e^{A(1-r)} G_{\hat{R}(\theta_j \omega)}(u^j(r)) d\theta_j \omega(r) \right\|_{\beta,n,n+1} \\
&\quad \leq \|e^{A(\cdot - j - 1)}\|_{\beta,n,n+1} \left\| \int_0^{\cdot} e^{A(\cdot - r)} G_{\hat{R}(\theta_j \omega)}(u^j(r)) d\theta_j \omega(r) \right\|_{\infty,0,1},
\end{aligned}$$

and from Lemma 6, it is easy to obtain that

$$\|e^{A(\cdot-j-1)}\|_{\beta,n,n+1} \leq M(1+\|A\|)e^{-\lambda(n-j-1)},$$

giving us

$$\begin{aligned}
&\left\| e^{A(\cdot-j-1)} \int_0^1 e^{A(1-r)} G_{\hat{R}(\theta_j\omega)}(u^j(r))d\theta_j\omega(r) \right\|_{\beta,n,n+1} \\
&\quad \leq M^2(1+\|A\|)^2 e^{-\lambda(n-j-1)} C_{\alpha,\beta,\beta'} |||\theta_j\omega|||_{\beta'} R(\theta_j\omega) L_{D\chi} \|u^j\|_{\beta,0,1} \\
&\quad \leq K e^{-\lambda(n-j-1)} |||\theta_j\omega|||_{\beta'} R(\theta_j\omega) \|u^j\|_{\beta,0,1},
\end{aligned}$$

where the constant K has been introduced in (29) and we have used (27). Following similar steps we have

$$\begin{aligned}
&\left\| e^{A(\cdot-j-1)} \int_0^1 e^{A(1-r)} \hat{F}_{\hat{R}(\theta_j\omega)}(u^j(r))dr \right\|_{\beta,n,n+1} \\
&\quad \leq M^2(1+\|A\|) L_{D\chi} e^{-\lambda(n-j-1)} R(\theta_j\omega) \|u^j\|_{\beta,0,1} \\
&\quad \leq K e^{-\lambda(n-j-1)} R(\theta_j\omega) \|u^j\|_{\beta,0,1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|u^n\|_{\beta,0,1} \leq \|u_0\| \|e^{A\cdot}\|_{\beta,n,n+1} &+ K \sum_{j=0}^{n-1} R(\theta_j\omega) \left(1+|||\theta_j\omega|||_{\beta',0,1}\right) \|u^j\|_{\beta,0,1} e^{-\lambda(n-j-1)} \\
&+ K R(\theta_n\omega) \left(1+|||\theta_n\omega|||_{\beta',0,1}\right) \|u^n\|_{\beta,0,1}.
\end{aligned}$$

Let $\epsilon < 1$ and consider

$$R(\omega) = \frac{\epsilon}{2K\left(1+|||\omega|||_{\beta',0,1}\right)}. \tag{32}$$

With this choice, the coefficient in front of $\|u^n\|_{\beta,0,1}$ on the right hand side of the above expression is less than or equal 1/2, since $\epsilon < 1$. As a consequence,

$$\frac{1}{2}\|u^n\|_{\beta,0,1} \leq \|u_0\| \|e^{A\cdot}\|_{\beta,n,n+1} + \frac{\epsilon}{2} \sum_{j=0}^{n-1} e^{-\lambda(n-j-1)} \|u^j\|_{\beta,0,1}$$

and hence

$$\|u^n\|_{\beta,0,1} \leq 2M(1+\|A\|)\|u_0\| e^{-\lambda n} + \epsilon \sum_{j=0}^{n-1} e^{-\lambda(n-j-1)} \|u^j\|_{\beta,0,1}.$$

Taking $y_j = e^{\lambda j}\|u^j\|_{\beta,0,1}$, $c = 2M(1+\|A\|)\|u_0\|$ and $g_j = \epsilon e^{\lambda}$, Lemma 7 ensures that

$$y_n \leq 2M(1+\|A\|)\|u_0\|(1+\epsilon e^{\lambda})^n$$

and thus

$$\|u^n\|_{\beta,0,1} \leq 2M(1+\|A\|)\|u_0\| \left(\epsilon + e^{-\lambda}\right)^n = 2M\left(1+\|A\|\right)\|u_0\| e^{n\log\left(\epsilon+e^{-\lambda}\right)}. \tag{33}$$

We can state a first result regarding the function u defined by (31), for which we require a bit more regularity for F:

(**A1**)” $F\colon \bar{B}_{\mathbb{R}^d}(0,\rho) \to \mathbb{R}^d$ is twice continuously differentiable with bounded derivatives.

Lemma 13 *Let $\beta' > 1/2$, $\epsilon + e^{-\lambda} < 1$ and assume that ω is such that*

$$\lim_{t\to\pm\infty} \frac{\log^+ \|\!|\theta_t\omega|\!\|_{\beta',0,1}}{t} = 0. \tag{34}$$

Then, under (**A1**)*”,* (**A2**)*’,* (**A3**)*–*(**A5**)*, u defined by* (31) *solves* (22) *on any interval* $[0, T]$.

Proof First of all, for $R(\omega)$ given by (32) define $\hat{R}(\omega)$ by

$$\hat{R}(\omega) = \max\Bigg\{\hat{r} \in [0,\rho] : \|D\hat{F}(v)\|_{\mathcal{L}(\mathbb{R}^d)} + \|DG(v)\|_{\mathcal{L}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))} \le R(\omega), \\ \text{for all } v \in \bar{B}_{\mathbb{R}^d}\left(0,\hat{r}\right)\Bigg\}.$$

We apply Lemma 4 taking the space $\mathcal{B} = \mathcal{L}(\mathbb{R}^d) \times \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ equipped with the norm $\|(f,g)\|_{\mathcal{B}} = \|f\|_{\mathcal{L}(\mathbb{R}^d)} + \|g\|_{\mathcal{L}(\mathbb{R}^d,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d))}$. We also take $\mathcal{T} = (D\hat{F}, DG)$. Then, $\mathcal{T}$ is continuously differentiable (for which we have required F to be twice differentiable) and $\mathcal{T}(0) = 0$. Hence, $\hat{R}$ is well-defined,

$$\liminf_{R\to 0} \frac{\hat{R}(R)}{R} \ge \kappa \in (0,\infty)$$

and, in virtue of Lemma 5, is measurable.

Furthermore, thanks to (34) and the continuity of the mapping $t \to \|\!|\omega|\!\|_{\beta',0,t}$ (see [9]), for a sufficiently small $\varepsilon > 0$ there exists $C_\varepsilon(\omega)$ such that

$$\hat{R}(\theta_t\omega) \ge \frac{\kappa}{2} R(\theta_t\omega) \ge \frac{\kappa}{2} C_\varepsilon(\omega) e^{-\varepsilon|t|}$$

for sufficiently large $|t|$.

On the other hand, due to Lemma 3 and $\epsilon + e^{-\lambda} < 1$, we can find a zero neighborhood depending on ω such that for u_0 contained in this neighborhood we have

$$\|u^n(t)\| \le \|u^n\|_{\beta,0,1} \le \frac{\hat{R}(\theta_n\omega)}{2} \quad \text{for all } n \in \mathbb{Z}^+,\ t \in [0,1].$$

Then it holds

$$\hat{F}_{\hat{R}(\theta_n\omega)}(u^n(r)) = \hat{F}(u^n(r)), \quad G_{\hat{R}(\theta_n\omega)}(u^n(r)) = G(u^n(r))$$

for $r \in [0,1]$ and $n \in \mathbb{Z}^+$, and so we see that u defined by (31) solves (22) on any interval $[0, T]$. □

Condition (34) holds in particular if ω is a sample path of the canonical fractional Brownian motion with Hurst parameter $H > 1/2$ defined on the probability space introduced in Sect. 3. In that case, both R and $\hat{R}$ are tempered from below.

Finally, we can state and prove our main result:

Theorem 14 *Suppose that $\omega \in C_0^{\beta'}(\mathbb{R};\mathbb{R}^m)$ with $\beta' > 1/2$ and that* (34)*,* (**A1**)*”,* (**A2**)*’,* (**A3**)*–*(**A5**) *hold. Then for every $\epsilon \in (0, 1 - e^{-\lambda})$ the solution of* (22) *is locally exponentially zero stable with an exponential rate $\mu < -\log(\epsilon + e^{-\lambda})$.*

Proof Take $t \in [n, n+1]$. Then, due to the choice of ϵ we can easily derive that

$$e^{n\log\left(\epsilon+e^{-\lambda}\right)} \le e^{-\log\left(\epsilon+e^{-\lambda}\right)} e^{t\log\left(\epsilon+e^{-\lambda}\right)},$$

thus, from (33) we have

$$e^{\mu t}\|u(t)\| \le 2M(1+\|A\|)e^{-\log\left(\epsilon+e^{-\lambda}\right)}\|u_0\|e^{t\left(\mu+\log\left(\epsilon+e^{-\lambda}\right)\right)},$$

and therefore

$$\lim_{t\to\infty} e^{\mu t}\|u(t)\| = 0$$

since $\mu + \log(\epsilon + e^{-\lambda}) < 0$. □

Remark 15 The solution of (22) is locally exponentially zero stable with any rate less than λ. Indeed, for any arbitrary $\mu < \lambda$ we can choose $\epsilon \in (0, 1 - e^{-\lambda})$ sufficiently small such that

$$\lambda > \lambda - \log\left(1 + \epsilon e^{\lambda}\right) = -\log\left(\epsilon + e^{-\lambda}\right) > \mu.$$

Remark 16 The assumptions on ω of Theorem 14 are in particular satisfied by $\mathbb{P}_H$-almost all sample paths of the canonical fractional Brownian motion with $H > 1/2$.

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