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Time Periodic Traveling Waves for a Periodic and Diffusive SIR Epidemic Model

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Abstract In this paper, we study the time periodic traveling wave solutions for a periodic SIR epidemic model with diffusion and standard incidence. We establish the existence of periodic traveling waves by investigating the fixed points of a nonlinear operator defined on an appropriate set of periodic functions. Then we prove the nonexistence of periodic traveling via the comparison arguments combined with the properties of the spreading speed of an associated subsystem.

Keywords Periodic traveling waves · Diffusive SIR model · Time-T map · Schauder fixed point theorem

Mathematics Subject Classification 35K57 · 35B40 · 92D30

1 Introduction

The investigation on traveling wave solutions for various evolution systems arising in biology, chemistry, epidemiology and physics has received increasing interest, see, e.g., [6,16,35,36,47,51,56] and references therein. As a basic but important subject, the existence of traveling wave solutions has been widely studied. For autonomous monotone evolution systems, by standard approaches such as monotone iteration, comparison arguments or monotone semiflow, the theory of traveling wave solutions has been well developed, see, e.g., [13,14,28,42,48] and references therein. Meanwhile, there are a few results on the existence of traveling wave solutions for nonautonomous (in particular, time periodic) monotone

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systems: Alikakos et al. [1] established the existence and global stability of time periodic traveling wave solutions (see the form of (1.3)) for periodic reaction–diffusion equations with bistable nonlinearities; Liang et al. [27] extended the theory of spreading speeds and traveling waves for monotone autonomous semiflows to periodic semiflows in the monostable case; Fang and Zhao [14] developed the theory of traveling waves for monotone semiflows with bistable structure and applied it to time-periodic evolution system; Zhao and Ruan [54,55] studied the existence, uniqueness and asymptotic stability of time periodic travelling wave solutions to periodic reaction–diffusion, advection–reaction–diffusion Lotka–Volterra competition systems, respectively; For bistable periodic traveling waves of periodic and diffusive Lotka–Volterra competition system, we refer to Bao and Wang [2]. More recently, Fang et al. [15] developed the theory of traveling waves and spreading speeds for time-space periodic monotone semiflows with monostable structure and applied the abstract results to a two species competition reaction–advection–diffusion model.

It is well known that many nonlinear reaction–diffusion systems modeling interaction of multi-species, such as predator and prey, the disease transmission among the susceptible individuals and infective individuals, combustion and the chemical reaction, etc., are non-monotone. Due to the lack of the comparison principle and monotonic properties for such evolution systems, the study of traveling waves is very challenging, and the related research is very limited. In the pioneering work of Dunbar [10,11], the shooting argument was applied to prove the existence of traveling waves for a classical Lotka–Volterra predator–prey model. This method is also used in [25,30] for predator–prey systems with different functional response, and in [19,20] for classical Kermack–McKendrick SIR models. Huang [21] further developed the method in [10,11] to provide a more effective way to obtain traveling waves for a large class of predator–prey systems. Based on a fixed-point problem and the limiting argument, Ducrot and Magal [8] and Ducrot et al. [9] studied the existence of traveling waves for an infection-age structured Kermack–McKendrick model with diffusion. Motivated by the method in [8,9], there were also some works involving in traveling waves for a bio-reaction model [44], an H5N1 arian influenza model [46], and nonlocal dispersal Kermack–Mckendrick models [26,49,50]. By constructing an invariant cone and applying Schauder's fixed point theorem, Wang and Wu [45] obtained the existence of travelling waves for a class of diffusive Kermack–McKendrick SIR models with non-local and delayed disease transmission (see also [38,39]). Schauder's fixed point theorem is also applied for the existence of traveling waves for evolution systems without monotonicity, see, e.g., [24,29,33,34,37,52]. More recently, Huang [22] presented a geometrical approach to investigate the existence of traveling waves and their minimum wave speed for non-monotone reaction–diffusion systems, which include the models of predator–prey interaction, the combustion, Belousov-Zhabotinskii reaction, SI-type of disease transmission, and biological flow reactor in chemostat. Zhang et al. [53] introduced the concept of weak traveling waves and obtained the necessary and sufficient conditions for the existence of such solutions for a class of non-cooperative diffusion-reaction systems. Fu and Tsai [18] employed an iteration process to construct a set of super/sub-solutions to establish the existence of a family of traveling waves with the minimum speed.

However, there are very few investigations on the time periodic traveling wave solutions for periodic non-monotone evolution systems. The purpose of this paper is to study time periodic traveling waves for the following periodic and diffusive SIR model with standard incidence:

$$\begin{cases} \frac{\partial}{\partial t} S(t,x) = d_1 \Delta S(t,x) - \frac{\beta(t)S(t,x)I(t,x)}{S(t,x)+I(t,x)}, \\ \frac{\partial}{\partial t} I(t,x) = d_2 \Delta I(t,x) + \frac{\beta(t)S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \gamma(t)I(t,x), \\ \frac{\partial}{\partial t} R(t,x) = d_3 \Delta R(t,x) + \gamma(t)I(t,x), \end{cases} \tag{1.1}$$

where $S(t,x)$, $I(t,x)$ and $R(t,x)$ denote the densities of the susceptible, infected and removed individuals at time t and in location x, respectively. Further, d_1, d_2 and d_3 are the diffusion rates for the susceptible, infected and removed individuals, respectively. The infection rate β and the removal rate γ are positive T-periodic continuous functions of t. Here the incidence reflects the recovered individuals are removed from the population, and not involved in the contact and disease transmission, see [5,40]. Since the equation for R of system (1.1) is decoupled from the equations for S and I, it suffices to consider a two-dimensional system for S and I:

$$\begin{cases} \frac{\partial}{\partial t} S(t,x) = d_1 \Delta S(t,x) - \frac{\beta(t)S(t,x)I(t,x)}{S(t,x)+I(t,x)}, \\ \frac{\partial}{\partial t} I(t,x) = d_2 \Delta I(t,x) + \frac{\beta(t)S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \gamma(t)I(t,x). \end{cases} \tag{1.2}$$

Time periodic traveling waves to system (1.2) are defined to be solutions of the form

$$\begin{pmatrix} S(t,x) \\ I(t,x) \end{pmatrix} = \begin{pmatrix} \phi(t,x+ct) \\ \psi(t,x+ct) \end{pmatrix}, \quad \begin{pmatrix} \phi(t+T,z) \\ \psi(t+T,z) \end{pmatrix} = \begin{pmatrix} \phi(t,z) \\ \psi(t,z) \end{pmatrix} \tag{1.3}$$

satisfying

$$\begin{pmatrix} \phi(t,\pm\infty) \\ \psi(t,\pm\infty) \end{pmatrix} = \begin{pmatrix} \phi_\pm(t) \\ \psi_\pm(t) \end{pmatrix},$$

where c is called the wave speed, $z = x + ct$ is the moving coordinate, and $\begin{pmatrix} \phi_+(t) \\ \psi_+(t) \end{pmatrix}$ and $\begin{pmatrix} \phi_-(t) \\ \psi_-(t) \end{pmatrix}$ are two periodic solutions of the corresponding kinetic system:

$$\begin{cases} \frac{dS}{dt} = -\frac{\beta(t)S(t)I(t)}{S(t)+I(t)}, \\ \frac{dI}{dt} = \frac{\beta(t)S(t)I(t)}{S(t)+I(t)} - \gamma(t)I(t). \end{cases} \tag{1.4}$$

The profile (ϕ, ψ) then solves the following time periodic parabolic system:

$$\begin{cases} \phi_t(t,z) = d_1\phi_{zz}(t,z) - c\phi_z(t,z) - \frac{\beta(t)\phi(t,z)\psi(t,z)}{\phi(t,z)+\psi(t,z)}, \quad (t,z) \in \mathbb{R} \times \mathbb{R}, \\ \psi_t(t,z) = d_2\psi_{zz}(t,z) - c\psi_z(t,z) + \frac{\beta(t)\phi(t,z)\psi(t,z)}{\phi(t,z)+\psi(t,z)} - \gamma(t)\psi(t,z), \quad (t,z) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{1.5}$$

Since the periodic system (1.2) does not admit the comparison principle, the theory and methods developed for monotone periodic systems (see, e.g., [14,27,54,55]) cannot be used here. In view of the profile system (1.5), the shooting arguments (see, e.g., [10,11,21,22]) for predator–prey systems do not apply to periodic system (1.2). Although Schauder's fixed point theorem is a powerful tool to prove the existence of traveling wave solutions for autonomous evolution systems (see, e.g., [24,33,38,39,45,53]), it may not be applied directly to periodic system (1.2). Our strategy is to reduce the existence of periodic traveling waves to a fixed point problem by constructing a non-monotone operator on an appropriate convex set of periodic functions. To obtain the nonexistence of time periodic traveling waves, we combine the comparison arguments for single equations and the properties of spreading speeds for periodic and monotone systems, which is of its own interest and may apply to other non-monotone models.

This paper is organized as follows. In Sect. 2, we first construct some appropriate sub- and super-solutions to obtain an invariant convex set, then define a nonlinear non-monotone operator on it, and finally apply Schauder's fixed point theorem to get the existence of periodic traveling waves. More precisely, we prove that if the basic reproduction number $R_0 := \frac{\int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt}$ of the periodic kinetic system (1.4) is greater than unity, then there exists a $c^* > 0$ such that for any $c \in (c^*, \infty)$, system (1.2) for S and I admits a time periodic, non-trivial and non-negative traveling wave solution with speed c. Section 3 is devoted to the nonexistence of such traveling waves for two cases where $R_0 \leq 1$, or $R_0 > 1$ and $c \in (0, c^*)$.

2 The Existence of Periodic Traveling Waves

In this section, we focus on the non-trivial and time periodic travelling waves $(\phi(t,z), \psi(t,z))$ of the form (1.3). Such solutions satisfy the following system:

$$\begin{cases} \phi_t(t,z) = d_1\phi_{zz}(t,z) - c\phi_z(t,z) - \frac{\beta(t)\phi(t,z)\psi(t,z)}{\phi(t,z)+\psi(t,z)}, \\ \psi_t(t,z) = d_2\psi_{zz}(t,z) - c\psi_z(t,z) + \frac{\beta(t)\phi(t,z)\psi(t,z)}{\phi(t,z)+\psi(t,z)} - \gamma(t)\psi(t,z). \end{cases} \tag{2.1}$$

This system is posed on $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and is supplemented with the following asymptotic boundary conditions

$$\phi(t, -\infty) = S_0,\ \phi(t, \infty) = S^\infty,\ \psi(t, \pm\infty) = 0 \text{ uniformly in } t \in \mathbb{R}. \tag{2.2}$$

Here $S_0 > 0$ is a constant, and $(S_0, 0)$ is the initial disease-free steady state. The parameter $c > 0$ is the wave speed, while constant $S^\infty \geq 0$ describes the density of susceptible individuals after the epidemic. Our investigation procedure is as follows. Firstly, we construct some appropriate sub- and super-solutions that will be essential to obtain a closed and convex set $\mathcal{D}$. Note that this set contains all the bounded and uniformly continuous functions which lie between the sub- and super-solutions. Secondly, for any $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{D}$, we find a unique T-periodic solution (ϕ^*, ψ^*) to a linear integral system, and then we define a nonlinear operator $\mathcal{F}$ such that $\mathcal{F}(\tilde{\phi}, \tilde{\psi}) = (\phi^*, \psi^*)$. Finally, by applying Schauder's fixed point theorem to $\mathcal{F}$, we establish the existence of periodic traveling waves.

2.1 Construction of Sub- and Super-solutions

Linearizing system (2.1) at the disease-free steady state $(S_0, 0)$, we obtain the following equation for the infective variable:

$$J_t = d_2 J_{zz}(t,z) - cJ_z(t,z) + (\beta(t) - \gamma(t))J(t,z). \tag{2.3}$$

Denote $\overline{H} = \frac{1}{T}\int_0^T H(t)dt$ for any T-periodic function $H(\cdot)$. Define

$$\Lambda_c(\lambda) := d_2\lambda^2 - c\lambda + \kappa_0,\ \ \kappa_0 := \frac{1}{T}\int_0^T [\beta(t) - \gamma(t)]dt = \overline{\beta(\cdot) - \gamma(\cdot)},\ c \in \mathbb{R},\ \lambda \in \mathbb{R},$$

$$Q^\lambda(t) = \exp\left(\int_0^t \left[\beta(s) - \gamma(s)\right] ds - t\kappa_0\right).$$

Clearly,

$$\kappa_0 Q^\lambda(t) = \left[\beta(t) - \gamma(t)\right] Q^\lambda(t) - \frac{dQ^\lambda(t)}{dt}.$$

We also set

$$\kappa = d_2\kappa_0,\quad \lambda_c = \frac{c-\sqrt{c-4\kappa}}{2d_2}\ \text{ if } c > c^* := 2\sqrt{\kappa}.$$

In the following, we always assume that $R_0 := \frac{\int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt} > 1$, and fix $c > c^* := 2\sqrt{\kappa}$. Let $K(t) := \exp\left(\int_0^t \left[d_2\lambda_c^2 - c\lambda_c + (\beta(s)-\gamma(s))\right] ds\right)$. We define four functions as follows:

$$\phi^+(t,z) := S_0,\ \phi^-(t,z) := \max\left\{S_0\left(1-M_1e^{\epsilon_1 z}\right), 0\right\},$$
$$\psi^+(t,z) := K(t)e^{\lambda_c z},\ \psi^-(t,z) := \max\left\{K(t)e^{\lambda_c z}\left(1-M_2e^{\epsilon_2 z}\right), 0\right\},$$

where $\epsilon_1, M_1, \epsilon_2$ and M_2 are all positive constants and will be determined below. Then we have the following results.

Lemma 2.1 *The function $\psi^+(t,z) = K(t)e^{\lambda_c z}$ satisfies the following linear equation:*

$$\psi_t = d_2\psi_{zz} - c\psi_z + (\beta(t)-\gamma(t))\psi. \tag{2.4}$$

Lemma 2.2 *Suppose ϵ_1 is sufficiently small such that $0 < \epsilon_1 < \lambda_c$ and $M_1 > 1$ is sufficiently large. Then the function ϕ^- satisfies*

$$\phi_t - d_1\phi_{zz} + c\phi_z \le -\frac{\beta(t)\psi^+\phi}{\psi^+ + \phi} \tag{2.5}$$

for any $z \ne z_1 := -\epsilon_1^{-1}\ln M_1$.

Proof If $z > -\epsilon_1^{-1}\ln M_1$, then $\phi^-(t,z) = 0$, which implies that the inequality (2.5) holds.

If $z < -\epsilon_1^{-1}\ln M_1$, then $\phi^-(t,z) = S_0(1-M_1e^{\epsilon_1 z})$. Hence, the inequality (2.5) is equivalent to

$$d_1S_0M_1\epsilon_1^2e^{\epsilon_1 z} - cS_0M_1\epsilon_1e^{\epsilon_1 z} \le -\frac{\beta(t)S_0(1-M_1e^{\epsilon_1 z})K(t)e^{\lambda_c z}}{S_0(1-M_1e^{\epsilon_1 z}) + K(t)e^{\lambda_c z}}$$

for any $z < z_1 := -\epsilon_1^{-1}\ln M_1$. Rewriting the above inequality, we have

$$S_0M_1\epsilon_1(c-d_1\epsilon_1) \ge \frac{\beta(t)S_0(1-M_1e^{\epsilon_1 z})K(t)e^{(\lambda_c-\epsilon_1)z}}{S_0(1-M_1e^{\epsilon_1 z}) + K(t)e^{\lambda_c z}}.$$

So for $z < z_1 := -\epsilon_1^{-1}\ln M_1$, it is sufficient to verify

$$S_0M_1\epsilon_1(c-d_1\epsilon_1) \ge \beta(t)K(t)e^{-\epsilon_1^{-1}(\lambda_c-\epsilon_1)\ln M_1} = \beta(t)K(t)M_1^{-\epsilon_1^{-1}(\lambda_c-\epsilon_1)},\quad \forall t \in \mathbb{R}.$$

Note that $\beta(t)$ and $K(t)$ are positive T-periodic functions. Thus the above inequality holds true if we choose $M_1 = 1/\epsilon_1$ with $\epsilon_1 > 0$ sufficiently small. □

Lemma 2.3 *Suppose $\epsilon_2 > 0$ sufficiently small such that $\epsilon_2 < \min\{\epsilon_1, \lambda_c' - \lambda_c\}$, where $\lambda_c' := \frac{c+\sqrt{c-4\kappa}}{2d_2}$, and M_2 is sufficiently large such that $-\epsilon_2^{-1}\ln M_2 < -\epsilon_1^{-1}\ln M_1$. Then the function ψ^- satisfies*

$$\psi_t - d_2\psi_{zz} + c\psi_z \le -\gamma(t)\psi - A[\phi^-,\psi] \tag{2.6}$$

for any $z \ne z_2 := -\epsilon_2^{-1}\ln M_2$, where $A[\phi,\psi](t,z)$ is defined by

$$A[\phi,\psi](t,z) = \begin{cases} 0, & \phi(t,z)\psi(t,z) = 0,\quad \forall (t,z) \in \mathbb{R}\times\mathbb{R}, \\ \frac{\beta(t)\phi(t,z)\psi(t,z)}{\phi(t,z)+\psi(t,z)}, & \phi(t,z)\psi(t,z) \ne 0,\quad \forall (t,z) \in \mathbb{R}\times\mathbb{R}. \end{cases}$$

Proof We assume that M_2 is sufficiently large such that $-\epsilon_2^{-1}\ln M_2 < -\epsilon_1^{-1}\ln M_1$. When $z > z_2 := -\epsilon_2^{-1}\ln M_2$, we see that $\psi^-(t,z) = 0$, and hence, the inequality (2.6) holds.

Let $z < z_2 := -\epsilon_2^{-1}\ln M_2$. Then $\psi^-(t,z) = K(t)e^{\lambda_c z}(1 - M_2 e^{\epsilon_2 z})$ and $\phi^-(t,z) = S_0(1 - M_1 e^{\epsilon_1 z})$. It suffices to verify

$$\psi_t^- - d_2\psi_{zz}^- + c\psi_z^- \le -\gamma(t)\psi^- - \frac{\beta(t)\phi^-\psi^-}{\phi^- + \psi^-} \le (\beta(t) - \gamma(t))\,\psi^- - \frac{\beta(t)(\psi^-)^2}{\phi^- + \psi^-}. \tag{2.7}$$

In view of the expression of $K(t)$ and $\psi^-(t,z)$, we have that

$$\begin{aligned}
&\psi_t^- - d_2\psi_{zz}^- + c\psi_z^- - (\beta(t) - \gamma(t))\,\psi^- \\
&= K'(t)e^{\lambda_c z}\left(1 - M_2 e^{\epsilon_2 z}\right) - d_2\Big[\lambda_c^2 K(t)e^{\lambda_c z}\left(1 - M_2 e^{\epsilon_2 z}\right) - \lambda_c\epsilon_2 M_2 K(t)e^{(\lambda_c+\epsilon_2)z} \\
&\quad -(\lambda_c + \epsilon_2)\epsilon_2 M_2 K(t)e^{(\lambda_c+\epsilon_2)z}\Big] + c\left[\lambda_c K(t)e^{\lambda_c z}\left(1 - M_2 e^{\epsilon_2 z}\right) - \epsilon_2 M_2 K(t)e^{(\lambda_c+\epsilon_2)z}\right] \\
&\quad -[\beta(t) - \gamma(t)]K(t)e^{\lambda_c z}\left(1 - M_2 e^{\epsilon_2 z}\right) \\
&= e^{\lambda_c z}\left\{K'(t) - d_2\lambda_c^2 K(t) + c\lambda_c K(t) - [\beta(t) - \gamma(t)]K(t)\right\} \\
&\quad -M_2 e^{(\lambda_c+\epsilon_2)z}\left\{K'(t) - d_2(\lambda_c + \epsilon_2)^2 K(t) + c(\lambda_c + \epsilon_2)K(t) - [\beta(t) - \gamma(t)]K(t)\right\} \\
&= -M_2 e^{(\lambda_c+\epsilon_2)z}K(t)\left\{\left[d_2\lambda_c^2 - c\lambda_c\right] - \left[d_2(\lambda_c + \epsilon_2)^2 - c(\lambda_c + \epsilon_2)\right]\right\}. \\
&= M_2 e^{(\lambda_c+\epsilon_2)z}K(t)\cdot\Lambda_c(\lambda_c + \epsilon_2)
\end{aligned}$$

Then the inequality (2.7) is equivalent to

$$M_2 e^{(\lambda_c+\epsilon_2)z}K(t)\cdot\Lambda_c(\lambda_c + \epsilon_2) \le -\frac{\beta(t)K(t)e^{2\lambda_c z}(1 - M_2 e^{\epsilon_2 z})^2}{S_0(1 - M_1 e^{\epsilon_1 z}) + K(t)e^{\lambda_c z}(1 - M_2 e^{\epsilon_2 z})}. \tag{2.8}$$

Since $\epsilon_1 < \lambda_c' - \lambda_c$, it follows that $\lambda_c + \epsilon_2 \in (\lambda_c, \lambda_c')$, and hence

$$\Lambda_c(\lambda_c + \epsilon_2) = d_2(\lambda_c + \epsilon_2)^2 - c(\lambda_c + \epsilon_2) + \kappa_0 < 0.$$

Due to the positivity and periodicity of both $K(t)$ and $\beta(t)$ in $\mathbb{R}$, we see that the inequality (2.8) is satisfied if and only if

$$\begin{aligned}
&-M_2\Lambda_c(\lambda_c + \epsilon_2)\left[S_0\left(1 - M_1 e^{\epsilon_1 z}\right) + K(t)e^{\lambda_c z}\left(1 - M_2 e^{\epsilon_2 z}\right)\right] \\
&\quad \ge \beta(t)e^{(\lambda_c-\epsilon_2)z}\left(1 - M_2 e^{\epsilon_2 z}\right)^2
\end{aligned}$$

for all $t \in [0,T]$. In terms of $z < -\epsilon_2^{-1}\ln M_2$, we only need to show

$$-M_2\Lambda_c(\lambda_c + \epsilon_2)S_0\left(1 - M_1 M_2^{-\epsilon_1/\epsilon_2}\right) \ge \beta(t)M_2^{-(\lambda_c-\epsilon_2)/\epsilon_2} \quad \text{for all } t \in [0,T].$$

Since $\lambda_c - \epsilon_2 > \lambda_c - \epsilon_1 > 0$, when M_2 tends to infinity, the right-hand side of the last inequality tends to zero and the left-hand side of the last inequality tends to infinity, which means the last inequality holds true for large M_2. □

2.2 Reduction to a Fixed Point Problem

Let $X = BUC(\mathbb{R},\mathbb{R})$ be the Banach space of all bounded uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}$ with the usual supremum norm $\|\cdot\|_X$. Let

$$X^+ = \{w \in X : w(x) \ge 0, x \in \mathbb{R}\}.$$

Then X is a Banach lattice under the partial ordering induced by X^+. It follows from [7, Theorem 1.5] that the X-realization $d\Delta_X$ of $d\Delta$ generates a strongly continuous analytic semigroup $T(t)$ on X and $T(t)X^+ \subset X^+$ for $t \geq 0$. In addition, we have

$$(T(t)w)(x) = \frac{1}{\sqrt{4\pi dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4dt}} w(y)dy, \quad t > 0,\ x \in \mathbb{R},\ w(\cdot) \in X. \tag{2.9}$$

For a given positive constant μ, denote the functional space $B_\mu\left([0,T]\times\mathbb{R},\mathbb{R}^2\right)$ by

$$B_\mu\left([0,T]\times\mathbb{R},\mathbb{R}^2\right) := \left\{ u=(u_1,u_2) : \begin{array}{l} u_i \in BUC([0,T]\times\mathbb{R},\mathbb{R}), \ \sup\limits_{t\in[0,T],x\in\mathbb{R}} e^{-\mu|x|}|u_i(t,x)| < \infty, \\ u_i(0,x) = u_i(T,x),\ x\in\mathbb{R},\ i=1,2. \end{array} \right\}$$

equipped with the norm

$$\|u\|_\mu := \max\left\{ \sup_{t\in[0,T],x\in\mathbb{R}} e^{-\mu|x|}|u_1(t,x)|, \ \sup_{t\in[0,T],x\in\mathbb{R}} e^{-\mu|x|}|u_2(t,x)| \right\}.$$

Define a convex cone $\mathcal{D}$ as

$$\mathcal{D} = \left\{ (\tilde{\phi},\tilde{\psi}) \in B_\mu\left([0,T]\times\mathbb{R},\mathbb{R}^2\right) : \ \phi^- \leq \tilde{\phi} \leq \phi^+,\ \psi^- \leq \tilde{\psi} \leq \min\{\psi^+,\Lambda\} \right\},$$

where $\Lambda > 0$ is sufficiently large such that $\frac{\beta(t)S_0}{S_0+\Lambda} - \gamma(t) < 0$ for $t \in [0,T]$. For any given $(\tilde{\phi},\tilde{\psi}) \in \mathcal{D}$, define maps

$$f_1[\tilde{\phi},\tilde{\psi}](t,z) = \alpha_1\tilde{\phi}(t,z) - A[\tilde{\phi},\tilde{\psi}](t,z)$$

and

$$f_2[\tilde{\phi},\tilde{\psi}](t,z) = \alpha_2\tilde{\psi}(t,z) + A[\tilde{\phi},\tilde{\psi}](t,z) - \gamma(t)\tilde{\psi}(t,z),$$

where the functional A is defined as in Lemma 2.3, α_1 and α_2 are positive constants and satisfy $\alpha_1 > \max_{t\in[0,T]}\beta(t)$ and $\alpha_2 > \max_{t\in[0,T]}\gamma(t)$, respectively. Fix a $(\tilde{\phi},\tilde{\psi}) \in \mathcal{D}$. Consider the following parabolic initial value problem:

$$\begin{cases} \phi_t - d_1\phi_{zz} + c\phi_z + \alpha_1\phi = f_1[\tilde{\phi},\tilde{\psi}](t,z), \ 0 < t \leq T,\ z\in\mathbb{R}, \\ \psi_t - d_1\psi_{zz} + c\psi_z + \alpha_2\psi = f_2[\tilde{\phi},\tilde{\psi}](t,z), \ 0 < t \leq T,\ z\in\mathbb{R}, \\ \phi(0,z) = \phi_0(z),\ \psi(0,z) = \psi_0(z), \quad z\in\mathbb{R}. \end{cases} \tag{2.10}$$

Rewrite (2.10) as an integral system:

$$\begin{cases} \phi(t,z) = (T_1(t)\phi_0)(z) + \int_0^t \left(T_1(t-s)f_1[\tilde{\phi},\tilde{\psi}](s)\right)(z)ds, \\ \psi(t,z) = (T_2(t)\psi_0)(z) + \int_0^t \left(T_2(t-s)f_2[\tilde{\phi},\tilde{\psi}](s)\right)(z)ds, \end{cases} \tag{2.11}$$

where $T_i(t)$ is the analytic semigroup (see, e.g., [7], [31]) generated by the linear differential operator $A_i : D(A_i) \to C(\mathbb{R})$ defined by

$$D(A_i) = \left\{ \bigcap_{1\leq p<\infty} W^{2,p}_{\text{loc}} \ : \ A_i u = d_i u_{zz} - cu_z - \alpha_i u \in C(\mathbb{R}) \right\}, \quad i=1,2.$$

Moreover, $\overline{D(A)} = UC(\mathbb{R})$ (see [31, Chapter 5]), and following from (2.9), it is not difficult to obtain that

$$(T_i(t)w)(x) = e^{-\alpha_i t}\frac{1}{\sqrt{4\pi dt}}\int_{\mathbb{R}} e^{-\frac{(x-ct-y)^2}{4dt}} w(y)dy, \quad t > 0,\ x \in \mathbb{R},\ w(\cdot) \in X. \tag{2.12}$$

We note that the solution of (2.11) is the mild solution of linear system (2.10).

In what follows, we intend to prove that for any given $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{D}$, there exists a unique $(\phi^*, \psi^*) \in \mathcal{D}$ satisfying

$$\begin{cases} \phi^*(t) = T_1(t)\phi^*(0) + \int_0^t T_1(t-s) f_1[\tilde{\phi}, \tilde{\psi}](s)ds, \\ \psi^*(t) = T_2(t)\psi^*(0) + \int_0^t T_2(t-s) f_2[\tilde{\phi}, \tilde{\psi}](s)ds. \end{cases} \tag{2.13}$$

Given a positive number η, denote a functional space $\tilde{B}_\eta(\mathbb{R}, \mathbb{R}^2)$ by

$$\tilde{B}_\eta\left(\mathbb{R}, \mathbb{R}^2\right) := \left\{ v = (v_1, v_2) : v_i \in X,\ \sup_{z\in\mathbb{R}} e^{-\eta|z|}|v_i(z)| < \infty,\ z \in \mathbb{R},\ i = 1, 2. \right\}$$

equipped with the norm

$$|v|_\eta := \max\left\{ \sup_{z\in\mathbb{R}} e^{-\eta|z|}|v_1(z)|,\ \sup_{z\in\mathbb{R}} e^{-\eta|z|}|v_2(z)| \right\}.$$

Define

$$\tilde{\mathcal{D}} := \left\{ (\phi_0(\cdot), \psi_0(\cdot)) \in \tilde{B}_\mu\left(\mathbb{R}, \mathbb{R}^2\right) : \begin{array}{l} \phi^-(0, z) \le \phi_0(z) \le \phi^+(0, z) \\ \psi^-(0, z) \le \psi_0(z) \le \min\{\psi^+(0, z), \Lambda\} \end{array} \right\}.$$

Clearly, $\tilde{\mathcal{D}}$ is convex and closed. For a given $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{D}$, $f_i[\tilde{\phi}, \tilde{\psi}](t, \cdot), i = 1, 2$ belong to $C([0, T]; C(\mathbb{R}))$. Moreover, $f_1[\tilde{\phi}, \tilde{\psi}]$ and $f_2[\tilde{\phi}, \tilde{\psi}]$ admit uniform bounds with respect to $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{D}$, respectively, uniformly for $(t, x) \in [0, T] \times \mathbb{R}$. Thus, with the aid of [31, Theorem 5.1.2], for any $(\phi_0, \psi_0) \in \tilde{D}$, it follows that (ϕ, ψ) defined by (2.11) belongs to $C([0, T] \times \mathbb{R}, \mathbb{R}) \cap C^{\theta, 2\theta}([\epsilon, T] \times \mathbb{R}, \mathbb{R})$ for every $\epsilon \in (0, T)$ and $\theta \in (0, 1)$, and there are $C_1(\epsilon, \theta) > 0$, $C_2(\epsilon, \theta) > 0$ such that

$$\|\phi(T, \cdot)\|_{C^{2\theta}(\mathbb{R})} \le C_1(\epsilon, \theta)\Big(\epsilon^{-\theta}\|\phi_0\|_\infty + \|f_1[\tilde{\phi}, \tilde{\psi}]\|_\infty\Big) \tag{2.14}$$

and

$$\|\psi(T, \cdot)\|_{C^{2\theta}(\mathbb{R})} \le C_2(\epsilon, \theta)\Big(\epsilon^{-\theta}\|\psi_0\|_\infty + \|f_2[\tilde{\phi}, \tilde{\psi}]\|_\infty\Big). \tag{2.15}$$

In view of Lemma 2.1, we have the following integral equality for the function $\psi^+(t, z)$:

$$\psi^+(t) = T_2(t)\psi^+(0) + \int_0^t T_2(t-s)\Big[\alpha_2\psi^+(s) + (\beta(s) - \gamma(s))\psi^+(s)\Big]ds. \tag{2.16}$$

By Lemmas 2.2 and 2.3, and similar arguments to [43, Lemma 3.2], we further show the integral inequalities for $\phi^-(t, z)$ and $\psi^-(t, z)$.

Lemma 2.4 *The following inequalities for ϕ^- and ψ^-*

$$\phi^-(t) \le T_1(t)\phi^-(0) + \int_0^t T_1(t-s) f_1[\phi^-, \psi^+](s)ds \tag{2.17}$$

and

$$\psi^-(t) \le T_2(t)\psi^-(0) + \int_0^t T_2(t-s) f_2[\phi^-, \psi^-](s)ds \tag{2.18}$$

are valid, respectively.

Proof Let $\hat{\phi}^-(t,z) = \phi^-(t, z+ct)$ and $\hat{\psi}^+(t,z) = \psi^+(t, z+ct)$ for any $(t,z) \in [0,T]\times\mathbb{R}$. Then for any $t \in [0,T]$, by Lemma 2.2,

$$\hat{\phi}_t^-(t,z) - d_1\hat{\phi}_{zz}^-(t,z) + \alpha_1\hat{\phi}^-(t,z) - f_1[\hat{\phi}^-, \hat{\psi}^+](t,z) \le 0$$

for any $z \neq z^-(t) = \frac{1}{\epsilon_1}(-\ln M_1 - c\epsilon_1 t)$. Clearly,

$$\frac{\partial\hat{\phi}^-(t, z^-(t_0)-0)}{\partial z} = \lim_{z\to z^-(t_0)-0}\left\{-S_0 M_1\epsilon_1 e^{\epsilon_1(z+ct)}\right\} = -\epsilon_1 S_0 < 0.$$

Define

$$G(t,z) := -\hat{\phi}_t^-(t,z) + d_1\hat{\phi}_{zz}^-(t,z) - \alpha_1\hat{\phi}^-(t,z) + f_1[\hat{\phi}^-, \hat{\psi}^+](t,z) \ge 0$$

and

$$H(\hat{\phi}^-)(t,z,r) := \frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\hat{\phi}^-(r,y)dy.$$

Then, by a direct calculation, we have

$$\begin{aligned}
\frac{\partial}{\partial r}H(\hat{\phi}^-)(t,z,r) = {} & \frac{\alpha_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\hat{\phi}^-(r,y)dy\\
&+\frac{e^{-\alpha_1(t-r)}}{2(t-r)\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\hat{\phi}^-(r,y)dy\\
&-\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} \frac{(z-y)^2}{4d_1(t-r)^2}e^{-\frac{(z-y)^2}{4d_1(t-r)}}\hat{\phi}^-(r,y)dy\\
&+\frac{d_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\frac{\partial^2\hat{\phi}^-(r,y)}{\partial y^2}dy\\
&-\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\alpha_1\hat{\phi}^-(r,y)dy\\
&+\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\Big[f_1[\hat{\phi}^-, \hat{\psi}^+](r,y) - G(r,y)\Big]dy.
\end{aligned}$$

Furthermore, integration by parts yields

$$\begin{aligned}
&\frac{d_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\frac{\partial^2\hat{\phi}^-(r,y)}{\partial y^2}dy\\
&=\frac{d_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{z^-(r)} e^{-\frac{(z-y)^2}{4d_1(t-r)}}\frac{\partial^2\hat{\phi}^-(r,y)}{\partial y^2}dy\\
&=\frac{d_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}e^{-\frac{(z-z^-(r))^2}{4d_1(t-r)}}\frac{\partial\hat{\phi}^-(r, z^-(r)-0)}{\partial z}\\
&-\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{z^-(r)} \frac{1}{2(t-r)}e^{-\frac{(z-y)^2}{4d_1(t-r)}}\hat{\phi}^-(r,y)dy\\
&+\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{z^-(r)} \frac{(z-y)^2}{4d_1(t-r)^2}e^{-\frac{(z-y)^2}{4d_1(t-r)}}\hat{\phi}^-(r,y)dy.
\end{aligned}$$

Here we have used the fact that $\hat{\phi}^-(t,z)=0,\ \forall z>z^-(t)$. In view of $\frac{\partial\hat{\phi}^-(r,z^-(r)-0)}{\partial z}=-\epsilon_1 S_0$, we have

$$\frac{\partial}{\partial r}H(\hat{\phi}^-)(t,z,r)=-\epsilon_1 S_0\frac{d_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}e^{-\frac{(z-z^-(r))^2}{4d_1(t-r)}}$$
$$+\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty}e^{-\frac{(z-y)^2}{4d_1(t-r)}}\Big[f_1[\hat{\phi}^-,\hat{\psi}^+](r,y)-G(r,y)\Big]dy.$$

Since

$$\frac{d_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\exp\left\{-\frac{(z-z^-(r))^2}{4d_1(t-r)}\right\}\frac{\partial\hat{\phi}^-(r,z^-(r)-0)}{\partial z}$$

is integrable in $r\in[0,t)$, $\frac{\partial}{\partial r}H(\hat{\phi}^-)(t,z,r)$ is continuous in $r\in[0,t)$, and

$$\lim_{r\to t-0}\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty}e^{-\frac{(z-y)^2}{4d_1(t-r)}}\hat{\phi}^-(r,y)dy=\hat{\phi}^-(t,z),$$

we conclude that

$$\begin{aligned}\hat{\phi}^-(t,z)&=\lim_{\eta\to0+0}H(\hat{\phi}^-)(t,z,t-\eta)\\&=H(\hat{\phi}^-)(t,z,0)+\lim_{\eta\to0+0}\int_0^{t-\eta}\frac{\partial}{\partial r}H(\hat{\phi}^-)(t,z,r)dr\\&=\frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}}\int_{-\infty}^{\infty}e^{-\frac{(z-y)^2}{4d_1 t}}\hat{\phi}^-(0,y)dy\\&\quad-\epsilon_1 S_0\int_0^t\frac{d_1 e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}e^{-\frac{(z-z^-(r))^2}{4d_1(t-r)}}dr\\&\quad+\int_0^t\frac{e^{-\alpha_1(t-r)}}{\sqrt{4\pi d_1(t-r)}}\int_{-\infty}^{\infty}e^{-\frac{(z-y)^2}{4d_1(t-r)}}\Big[f_1[\hat{\phi}^-,\hat{\psi}^+](r,y)-G(r,y)\Big]dydr.\end{aligned}$$

With the aid of $G(r,y)\ge0$, we see that

$$\hat{\phi}^-(t)\le\hat{T}_1(t)\hat{\phi}^-(0)+\int_0^t\hat{T}_1(t-r)f_1[\hat{\phi}^-,\hat{\psi}^+](r)dr,\ \ t\in(0,T],$$

where $\hat{T}_1(t)$ is defined by

$$\left(\hat{T}_1(t)\phi\right)(x)=\frac{e^{-\alpha_1 t}}{\sqrt{4\pi d_1 t}}\int_{-\infty}^{\infty}e^{-\frac{(z-y)^2}{4d_1 t}}\phi(y)dy.$$

Hence, it is not difficult to obtain the inequality (2.17) for ϕ^-. Similarly, we can show that the inequality (2.18) for ψ^- holds. □

On the basis of the above integral equation and integral inequalities, we shall show the invariance for integral equations (2.11) (see, e.g., [32]).

Lemma 2.5 *Let* $(\phi(t,z;\phi_0,\psi_0),\psi(t,z;\phi_0,\psi_0))$ *be the solutions of the system* (2.11) *with the initial value* $(\phi_0,\psi_0)\in\tilde{D}$. *Then*

$$\begin{aligned}&\phi^-(t,z)\le\phi(t,z;\phi_0,\psi_0)\le\phi^+(t,z),\\&\psi^-(t,z)\le\psi(t,z;\phi_0,\psi_0)\le\min\{\psi^+(t,z),\Lambda\}\end{aligned}$$

for $(t,z)\in[0,T]\times\mathbb{R}$.

Proof Recall that $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{D}$, $(\phi_0, \psi_0) \in \tilde{\mathcal{D}}$ and (ϕ, ψ) satisfies the system (2.11). Moreover,

$$\phi^-(t, z) \le \tilde{\phi}(t, z) \le \phi^+(t, z), \quad \psi^-(t, z) \le \tilde{\psi}(t, z) \le \min\{\psi^+(t, z), \Lambda\}, \quad (t, z) \in [0, T] \times \mathbb{R}$$

and

$$\phi^-(0, z) \le \phi_0(z) \le \phi^+(0, z), \quad \psi^-(0, z) \le \psi_0(z) \le \min\{\psi^+(0, z), \Lambda\}, \quad (t, z) \in [0, T] \times \mathbb{R}.$$

Since $\phi^+(t, z) \equiv S_0$, it is easy to see that

$$\phi^+(t) = T_1(t)\phi^+(0) + \alpha_1 \int_0^t T_1(t-s)\phi^+(s)ds. \tag{2.19}$$

Due to the positivity of semigroup $T_1(\cdot)$, we have

$$\int_0^t T_1(t-s) f_1[\tilde{\phi}, \tilde{\psi}](s)ds \le \alpha_1 \int_0^t T_1(t-s)\tilde{\phi}(s)ds$$

for any $t \in (0, T]$. By (2.19), it follows that

$$\int_0^t T_1(t-s) f_1[\tilde{\phi}, \tilde{\psi}](s)ds \le \phi^+(t) - T_1(t)\phi^+(0) \le \phi^+(t) - T_1(t)\phi_0$$

for any $t \in (0, T]$, which implies that $\phi(t, z) \le \phi^+(t, z)$ for any $t \in [0, T]$ and $z \in \mathbb{R}$. Let $w(t, z) = \phi(t, z) - \phi^-(t, z)$, $\forall (t, z) \in [0, T] \times \mathbb{R}$. By (2.17), we have

$$\begin{aligned} w(t) =& T_1(t)[\phi_0 - \phi^-(0)] \\ &+ \int_0^t T_1(t-s) \left\{\alpha_1[\tilde{\phi}(s) - \phi^-(s)] - A[\tilde{\phi}, \tilde{\psi}](s) + A[\phi^-, \psi^+](s)\right\} ds \\ &\ge T_1(t)[\phi_0 - \phi^-(0)] \\ &+ \int_0^t T_1(t-s) \left\{\alpha_1[\tilde{\phi}(s) - \phi^-(s)] - \frac{\beta(s)\tilde{\phi}(s)\psi^+(s)}{\tilde{\phi}(s) + \psi^+(s)} + \frac{\beta(s)\phi^-(s)\psi^+(s)}{\tilde{\phi}(s) + \psi^+(s)}\right\} ds \\ =& T_1(t)[\phi_0 - \phi^-(0)] + \int_0^t T_1(t-s) \left[\alpha_1 - \frac{\beta(s)\psi^+(s)}{\tilde{\phi}(s) + \psi^+(s)}\right] [\tilde{\phi}(s) - \phi^-(s)]ds \\ \ge& T_1(t)[\phi_0 - \phi^-(0)] + \int_0^t T_1(t-s)[\alpha_1 - \beta(s)][\tilde{\phi}(s) - \phi^-(s)]ds. \end{aligned}$$

Since $\alpha_1 > \max_{t\in[0,T]} \beta(t)$, it follows that $w(t) \ge 0$, $\forall t \in [0, T]$, which implies that

$$\phi(t, z) \ge \phi^-(t, z), \quad \forall (t, z) \in [0, T] \times \mathbb{R}.$$

In the following, we consider $\psi(t, z; \phi_0, \psi_0)$ for $t \in [0, T]$, $z \in \mathbb{R}$. It is easy to see that

$$\begin{aligned} \int_0^t T_2(t-s) f_2[\tilde{\phi}, \tilde{\psi}](s)ds &= \int_0^t T_2(t-s) \left[\alpha_2\tilde{\psi}(s) + A[\tilde{\phi}, \tilde{\psi}](s) - \gamma(s)\tilde{\psi}(s)\right] ds \\ &\le \int_0^t T_2(t-s) \left[\alpha_2\psi^+(s) + (\beta(s) - \gamma(s))\psi^+(s)\right] ds \end{aligned}$$

for any $t \in (0, T]$. By virtue of (2.16), we have

$$\int_0^t T_2(t-s) f_2[\tilde{\phi}, \tilde{\psi}](s)ds \le \psi^+(t) - T_2(t)\psi^+(0) \le \psi^+(t) - T_2(t)\psi_0$$

for all $t \in (0, T]$. In addition, $\psi(0, z) = \psi_0(z) \le \psi^+(0, z)$ for any $z \in \mathbb{R}$. It then follows that

$$\psi(t, z; \tilde{\phi}, \tilde{\psi}) \le \psi^+(t, z), \quad \forall t \in [0, T], \ z \in \mathbb{R}.$$

Recall that $\Lambda > 0$ satisfies $\frac{\beta(t)S_0}{S_0+\Lambda} - \gamma(t) < 0$ for $t \in [0, T]$. It is not difficult to prove that $\psi_\Lambda^+(t, z) \equiv \Lambda$ satisfies that

$$\psi_\Lambda^+(t, z) = T_2(t)\psi_\Lambda^+(0, z) + \alpha_2 \int_0^t T_2(t-s)\psi_\Lambda^+(s, z)ds.$$

By a similar argument to the proof for $\phi^+(t, z)$, we can prove that

$$\psi(t, z; \tilde{\phi}, \tilde{\psi}) \le \psi_\Lambda^+(t, z) \equiv \Lambda, \quad \forall t \in [0, T], \ z \in \mathbb{R}.$$

Since $T_2(\cdot)$ is positive and $\alpha_2 > \max_{t\in[0,T]} \gamma(t)$, it follows that

$$\int_0^t T_2(t-s) f_2[\tilde{\phi}, \tilde{\psi}](s)ds \ \ge \ \int_0^t T_2(t-s) f_2[\phi^-, \psi^-](s)ds$$

for any $t \in (0, T]$. According to (2.18), we have

$$\begin{aligned}\int_0^t T_2(t-s) f_2[\tilde{\phi}, \tilde{\psi}](s)ds &\ge \psi^-(t) - T_2(t)\psi^-(0) \\ &\ge \psi^-(t) - T_2(t)\psi, \quad \forall t \in (0, T],\end{aligned}$$

which yields

$$\psi(t) = T_2(t)\psi_0 + \int_0^t T_2(t-s) f_2[\tilde{\phi}, \tilde{\psi}](s)ds \ge \psi^-(t), \ \forall t \in (0, T].$$

Additionally, $\psi(0, z) = \psi_0(z) \ge \psi^-(0, z)$, $\forall z \in \mathbb{R}$. Consequently, we have proved that

$$\psi(t, z; \tilde{\phi}, \tilde{\psi}) \ge \psi^-(t, z), \quad \forall t \in [0, T], \ z \in \mathbb{R}.$$

This completes the proof. □

For any given $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{D}$, we denote the time-$T$ map of system (2.11): $(\phi_0(z), \psi_0(z)) \mapsto (\phi(T, z; \phi_0, \psi_0), \psi(T, z; \phi_0, \psi_0))$ by

$$F_{(\tilde{\phi},\tilde{\psi})}(\phi_0(\cdot), \psi_0(\cdot)) = (\phi(T, \cdot; \phi_0, \psi_0), \psi(T, \cdot; \phi_0, \psi_0))\,.$$

Thus, any fixed point of the T-map $F_{(\tilde{\phi},\tilde{\psi})}$ gives a T-periodic solution of system (2.11).

Theorem 2.6 *For any given $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{D}$, there exists a unique $(\phi^*, \psi^*) \in \mathcal{D}$ such that* (2.13) *holds.*

Proof In view of Lemma 2.5 and the definitions of $\phi^\pm$ and $\psi^\pm$, we assert that $F_{(\tilde{\phi},\tilde{\psi})}$ maps $\tilde{\mathcal{D}}$ into $\tilde{\mathcal{D}}$. For any compact interval $I \subset \mathbb{R}$, due to the estimates (2.14) and (2.15), we can conclude that $\{(\phi(T, \cdot; \phi_0, \psi_0), \psi(T, \cdot; \phi_0, \psi_0)) : (\phi_0, \psi_0) \in \tilde{\mathcal{D}}\}$ is compact on $C(I, \mathbb{R}^2)$. We can further show that $F_{(\tilde{\phi},\tilde{\psi})} : \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ is compact with respect to $|\cdot|_\mu$. In addition, it is not difficult to see that $F_{(\tilde{\phi},\tilde{\psi})} : \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ is continuous with respect to $|\cdot|_\mu$. Thus, the Schauder's fixed point theorem implies that $F_{(\tilde{\phi},\tilde{\psi})}$ admits a fixed point $(\phi_0^*, \psi_0^*) \in \tilde{\mathcal{D}}$. As a result, $\big(\phi(t, z; \phi_0^*, \psi_0^*), \psi(t, z; \phi_0^*, \psi_0^*)\big)$ satisfies $\phi(T, z; \phi_0^*, \psi_0^*) = \phi_0^*(z)$ and $\psi(T, z; \phi_0^*, \psi_0^*) = \psi_0^*(z)$, $\forall z \in \mathbb{R}$. Furthermore, we claim that such a fixed point (ϕ_0^*, ψ_0^*) is unique. Suppose that there exists $(\phi_0^{**}, \psi_0^{**}) \in \tilde{\mathcal{D}}$ such that $\big(\phi(t, z; \phi_0^{**}, \psi_0^{**}), \psi(t, z; \phi_0^{**}, \psi_0^{**})\big)$ satisfies (2.11), and $\phi(T, z; \phi_0^{**}, \psi_0^{**}) = \phi_0^{**}(z)$, $\psi(T, z; \phi_0^{**}, \psi_0^{**}) = \psi_0^{**}(z)$, $\forall z \in \mathbb{R}$. Then

$$\begin{aligned}\left|\phi(T, z; \phi_0^*, \psi_0^*) - \phi(T, z; \phi_0^{**}, \psi_0^{**})\right| &\le e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{e^{-\frac{(z-x-cT)^2}{4d_1T}}}{\sqrt{4\pi d_1 T}} \left|\phi_0^*(x) - \phi_0^{**}(x)\right| dx \\ &\le \left\|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\right\|_{L^\infty} e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{e^{-\frac{(z-x-cT)^2}{4d_1T}}}{\sqrt{4\pi d_1 T}} dx \\ &= e^{-\alpha_1 T} \left\|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\right\|_{L^\infty}.\end{aligned}$$

On the other hand, $\phi(T, \cdot; \phi_0^*, \psi_0^*) = \phi_0^*(\cdot)$ and $\phi(T, \cdot; \phi_0^{**}, \psi_0^{**}) = \phi_0^{**}(\cdot)$, we then see that

$$\left\|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\right\|_{L^\infty} \le e^{-\alpha_1 T} \left\|\phi_0^*(\cdot) - \phi_0^{**}(\cdot)\right\|_{L^\infty}.$$

Since $e^{-\alpha_1 T} < 1$, we have $\phi_0^*(\cdot) \equiv \phi_0^{**}(\cdot)$. Similarly, we can also obtain $\psi_0^*(\cdot) \equiv \psi_0^{**}(\cdot)$. Hence, there exists a unique (ϕ^*, ψ^*) satisfying (2.13). □

Let $(\phi^*(t, z), \psi^*(t, z)) = \big(\phi(t, z; \phi_0^*, \psi_0^*), \psi(t, z; \phi_0^*, \psi_0^*)\big)$, where $(\phi_0^*, \psi_0^*) \in \tilde{\mathcal{D}}$ is the unique fixed point of the operator $F_{(\tilde{\phi}, \tilde{\psi})}$. In view of Theorem 2.6, we can define an operator $\mathcal{F} : \mathcal{D} \to B_\mu$ by $\mathcal{F}(\tilde{\phi}, \tilde{\psi}) = (\phi^*, \psi^*)$. Thus, the existence of periodic traveling waves is reduced to the existence of a fixed point of the operator $\mathcal{F}$.

2.3 The Periodic Traveling Waves

In this section, we prove the existence of periodic traveling waves. As discussed in Sect. 2.2, we need to study the existence of fixed points of the operator $\mathcal{F}$. We start with the properties of $\mathcal{F}$. In view of Lemma 2.5, $\mathcal{F}$ maps $\mathcal{D}$ into $\mathcal{D}$.

Lemma 2.7 *The map $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is continuous with respect to the norm $\|\cdot\|_\mu$ in $B_\mu([0, T] \times \mathbb{R}, \mathbb{R}^2)$.*

Proof For any $(\tilde{\phi}_1, \tilde{\psi}_1) \in \mathcal{D}$ and $(\tilde{\phi}_2, \tilde{\psi}_2) \in \mathcal{D}$, let $(\phi_i^*(t, z; \tilde{\phi}_i, \tilde{\psi}_i), \psi_i^*(t, z; \tilde{\phi}_i, \tilde{\psi}_i)) = \mathcal{F}(\tilde{\phi}_i, \tilde{\psi}_i)$, $i = 1, 2$. From the first equation of system (2.13) and (2.12), we see that

$$\begin{aligned}\phi_i^*(T, z; \tilde{\phi}_i, \tilde{\psi}_i) = e^{-\alpha_1 T} &\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 T}} e^{-\frac{(z-y-cT)^2}{4d_1T}} \phi_i^*(0, y) dy \\ &+ \int_0^T e^{-\alpha_1 s} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{(z-y-cs)^2}{4d_1 s}} f_1[\tilde{\phi}_i, \tilde{\psi}_i](T-s, y) dy ds.\end{aligned}$$

Let $\tilde{\beta} = \max_{t\in[0,T]} \beta(t)$ and choose μ sufficiently small such that $e^{d_1 T\mu^2 + cT\mu - \alpha_1 T} \le \frac{1}{4}$. Consequently,

$$\begin{aligned}&\left|\phi_1^*(T, z; \tilde{\phi}_1, \tilde{\psi}_1) - \phi_2^*(T, z; \tilde{\phi}_2, \tilde{\psi}_2)\right| e^{-\mu|z|} \\ &\le e^{-\alpha_1 T} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 T}} e^{-\frac{(z-y-cT)^2}{4d_1T}} \left|\phi_1^*(0, y) - \phi_2^*(0, y)\right| dy e^{-\mu|z|} \\ &+ \int_0^T e^{-\alpha_1 s} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1 s}} e^{-\frac{(z-y-cs)^2}{4d_1 s}} \Big(\left|\tilde{\phi}_1(T-s, y) - \tilde{\phi}_2(T-s, y)\right| \\ &+ \left|\tilde{\psi}_1(T-s, y) - \tilde{\psi}_2(T-s, y)\right|\Big) (\alpha_1 + \tilde{\beta}) dy ds e^{-\mu|z|}\end{aligned}$$

$$\le e^{-\alpha_1 T}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 T}}e^{-\frac{(z-y-cT)^2}{4d_1T}}\left|\phi_1^*(0,y)-\phi_2^*(0,y)\right|e^{-\mu|y|}e^{\mu|y-z|}dy$$
$$+\int_0^T e^{-\alpha_1 s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}e^{-\frac{(z-y-cs)^2}{4d_1s}}\left(\left|\tilde\phi_1(T-s,y)-\tilde\phi_2(T-s,y)\right|e^{-\mu|y|}\right.$$
$$\left.+\left|\tilde\psi_1(T-s,y)-\tilde\psi_2(T-s,y)\right|e^{-\mu|y|}\right)e^{\mu|y-z|}(\alpha_1+\tilde\beta)dyds$$
$$\le e^{-\alpha_1T+\mu cT}|\phi_1^*(0)-\phi_2^*(0)|_\mu\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}e^{-\frac{(z-y-cT)^2}{4d_1T}}e^{\mu|z-y-cT|}dy$$
$$+(\alpha_1+\tilde\beta)\left(\left\|\tilde\phi_1-\tilde\phi_2\right\|_\mu+\left\|\tilde\psi_1-\tilde\psi_2\right\|_\mu\right)$$
$$\times\int_0^T e^{-\alpha_1 s}e^{\mu cs}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}e^{-\frac{(z-y-cs)^2}{4d_1s}}e^{\mu|z-y-cs|}dyds$$
$$=e^{-\alpha_1T+\mu cT}|\phi_1^*(0)-\phi_2^*(0)|_\mu\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1T}}e^{-\frac{y^2}{4d_1T}}e^{\mu|y|}dy$$
$$+(\alpha_1+\tilde\beta)\left(\left\|\tilde\phi_1-\tilde\phi_2\right\|_\mu+\left\|\tilde\psi_1-\tilde\psi_2\right\|_\mu\right)$$
$$\times\int_0^T e^{-\alpha_1 s}e^{\mu cs}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}e^{-\frac{y^2}{4d_1s}}e^{\mu|y|}dyds$$
$$\le 2e^{(d_1\mu^2+c\mu-\alpha_1)T}|\phi_1^*(0)-\phi_2^*(0)|_\mu$$
$$+(\alpha_1+\tilde\beta)\left(\left\|\tilde\phi_1-\tilde\phi_2\right\|_\mu+\left\|\tilde\psi_1-\tilde\psi_2\right\|_\mu\right)\int_0^T 2e^{(d_1\mu^2+c\mu-\alpha_1)s}ds$$
$$\le 2e^{(d_1\mu^2+c\mu-\alpha_1)T}|\phi_1^*(0)-\phi_2^*(0)|_\mu$$
$$+\frac{2(\alpha_1+\tilde\beta)\left(e^{(d_1\mu^2+c\mu-\alpha_1)T}-1\right)}{d_1\mu^2+c\mu-\alpha_1}\left(\left\|\tilde\phi_1-\tilde\phi_2\right\|_\mu+\left\|\tilde\psi_1-\tilde\psi_2\right\|_\mu\right)$$
$$\le\frac{1}{2}|\phi_1^*(0)-\phi_2^*(0)|_\mu$$
$$+\frac{2(\alpha_1+\tilde\beta)\left(e^{(d_1\mu^2+c\mu-\alpha_1)T}-1\right)}{d_1\mu^2+c\mu-\alpha_1}\left(\left\|\tilde\phi_1-\tilde\phi_2\right\|_\mu+\left\|\tilde\psi_1-\tilde\psi_2\right\|_\mu\right)$$

Let

$$L:=\frac{4(\alpha_1+\tilde\beta)\left(e^{(d_1\mu^2+c\mu-\alpha_1)T}-1\right)}{d_1\mu^2+c\mu-\alpha_1}.$$

Since $\phi_i^*(T,z;\tilde\phi_i,\tilde\psi_i)=\phi_i^*(0,z)$, $i=1,2$, we obtain from the above inequalities that

$$|\phi_1^*(0)-\phi_2^*(0)|_\mu\le L(\|\tilde\phi_1-\tilde\phi_2\|_\mu+\|\tilde\psi_1-\tilde\psi_2\|_\mu).$$

On the other hand, $\phi_i^*(t,z;\tilde\phi_i,\tilde\psi_i)$ satisfies that

$$\phi_i^*(t,z;\tilde\phi_i,\tilde\psi_i)=e^{-\alpha_1 t}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 t}}e^{-\frac{(z-y-ct)^2}{4d_1t}}\phi_i^*(0,y)dy$$
$$+\int_0^t e^{-\alpha_1 s}\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d_1 s}}e^{-\frac{(z-y-cs)^2}{4d_1s}}f_1[\tilde\phi_i,\tilde\psi_i](t-s,y)dyds.$$

Hence, by similar arguments to above, it is not difficult to conclude that $\phi^*(t,z;\tilde{\phi},\tilde{\psi})$ is continuous in $(\tilde{\phi},\tilde{\psi})$ with respect to the norm $\|\cdot\|_\mu$. Similarly, we can prove that $\psi^*(t,z;\tilde{\phi},\tilde{\psi})$ is continuous in $(\tilde{\phi},\tilde{\psi})$ with respect to the norm $\|\cdot\|_\mu$. □

Lemma 2.8 *The map $\mathcal{F}:\mathcal{D}\to\mathcal{D}$ is compact with respect to the norm $\|\cdot\|_\mu$ in $B_\mu([0,T]\times\mathbb{R},\mathbb{R}^2)$.*

Proof For any $(\tilde{\phi},\tilde{\psi})\in\mathcal{D}$, let $(\phi^*,\psi^*)=\mathcal{F}(\tilde{\phi},\tilde{\psi})$, where $(\phi^*(t,z),\psi^*(t,z)), t\in[0,T], z\in\mathbb{R}$ is the solution of system (2.13). In particular, it follows from the estimates (2.14) and (2.15) that there exists $K'(\theta)>0$ independent of $(\tilde{\phi},\tilde{\psi})$ such that $\|\phi^*(0)\|_{C^{2\theta}(\mathbb{R})}=\|\phi^*(T)\|_{C^{2\theta}(\mathbb{R})}\le K'$ and $\|\psi^*(0)\|_{C^{2\theta}(\mathbb{R})}=\|\psi^*(T)\|_{C^{2\theta}(\mathbb{R})}\le K'$. Moreover, $f_1[\tilde{\phi},\tilde{\psi}]$ and $f_2[\tilde{\phi},\tilde{\psi}]$ admit uniform bounds with respect to $(\tilde{\phi},\tilde{\psi})\in\mathcal{D}$, respectively, uniformly for $(t,x)\in[0,T]\times\mathbb{R}$. Thanks to [31, Theorem 5.1.2], it follows that $\phi^*,\psi^*\in C^{\theta,2\theta}([0,T]\times\mathbb{R},\mathbb{R})$ with some $\theta\in(0,1)$, and there exists $C_i(\theta)>0, i=1,2$ and $\tilde{K}(\theta)>0$ such that

$$\|\phi^*\|_{C^{\theta,2\theta}([0,T]\times\mathbb{R})}\le C_1\left(\|\phi^*(0)\|_{C^{2\theta}(\mathbb{R})}+\|f_1[\tilde{\phi},\tilde{\psi}]\|_\infty\right)\le\tilde{K}(\theta) \tag{2.20}$$

and

$$\|\psi^*\|_{C^{\theta,2\theta}([0,T]\times\mathbb{R})}\le C_2\left(\|\psi^*(0)\|_{C^{2\theta}(\mathbb{R})}+\|f_2[\tilde{\phi},\tilde{\psi}]\|_\infty\right)\le\tilde{K}(\theta). \tag{2.21}$$

Let $(\phi_n^*,\psi_n^*)=\mathcal{F}(\tilde{\phi}_n,\tilde{\psi}_n)$. Since ϕ_n^* and ψ_n^* satisfy the estimations (2.20) and (2.21), respectively, there is a subsequence of $\{(\phi_n^*,\psi_n^*)\}$, without loss of generality, still labeled by $\{(\phi_n^*,\psi_n^*)\}$, such that it converges in $C_{\mathrm{loc}}([0,T]\times\mathbb{R},\mathbb{R}^2)$ to a function $(\phi^{**},\psi^{**})\in C([0,T]\times\mathbb{R},\mathbb{R}^*)$, that is, for any $N\in\mathbb{R}^+$,

$$\lim_{n\to\infty}\|\left(\phi_n^*,\psi_n^*\right)-\left(\phi^{**},\psi^{**}\right)\|_{C([0,T]\times[-N,N],\mathbb{R}^2)}=0. \tag{2.22}$$

Clearly, $(\phi^{**},\psi^{**})\in\mathcal{D}$.

In the following, we are ready to prove that

$$\lim_{n\to\infty}\|\left(\phi_n^*,\psi_n^*\right)-\left(\phi^{**},\psi^{**}\right)\|_\mu=0.$$

Note that $\mathcal{D}$ is uniformly bounded with respect to the norm $\|\cdot\|_\mu$. Accordingly, the norm $\|\left(\phi_n^*,\psi_n^*\right)-(\phi^{**},\psi^{**})\|_\mu$ is uniformly bounded for all $n\in\mathbb{N}$. Given any $\rho>0$, it is not difficult to find an $M^*>0$ such that

$$e^{-\mu|z|}|\left(\phi_n^*(t,z),\psi_n^*(t,z)\right)-\left(\phi^{**}(t,z),\psi^{**}(t,z)\right)|<\rho$$

for any $t\in[0,T]$, $|z|>M^*$ and $n\in\mathbb{N}$. On the other hand, by virtue of (2.22), there exists $H\in\mathbb{N}$ such that

$$e^{-\mu|z|}|\left(\phi_n^*(t,z),\psi_n^*(t,z)\right)-\left(\phi^{**}(t,z),\psi^{**}(t,z)\right)|<\rho$$

for any $t\in[0,T]$, $z\in[-M^*,M^*]$ and $n>H$. As a consequence, it follows from the above two inequalities that $\left(\phi_n^*(t,z),\psi_n^*(t,z)\right)\to(\phi^{**}(t,z),\psi^{**}(t,z))$ with respect to the norm $\|\cdot\|_\mu$. □

To complete the proof of this section, we also need the following powerful lemma on the Harnack inequalities of cooperative parabolic systems, which is from Földes and Poláčik [17] (see also [41,54]).

Lemma 2.9 ([17]) *Let the differential operators*

$$\mathbf{L}_k := \sum_{i,j=1}^{n} a_{i,j}^k(t,x)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}} + \sum_{i=1}^{n} b_i^k \frac{\partial}{\partial_{x_i}} - \frac{\partial}{\partial t}, \quad k = 1, 2, \ldots, l,$$

be uniformly parabolic in an open domain $(\tau, M) \times \Omega$ of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, that is, there is $\alpha_0 > 0$ such that $a_{i,j}^k(t,x)\xi_i\xi_j \geq \alpha_0 \sum_{i=1}^n \xi_i^2$ for any n-tuples of real numbers $(\xi_1, \xi_2, \ldots, \xi_n)$, where $-\infty < \tau < M \leq +\infty$ and Ω is open and bounded. Suppose that $a_{i,j}^k, b_{i,j}^k \in C((\tau, M) \times \Omega, \mathbb{R})$ and

$$\max_{(t,\mathbf{x})\in(\tau,M)\Omega} |b_i^k(t,\mathbf{x})| + |a_{ij}^k(t,\mathbf{x})| \leq \beta_0$$

for some $\beta_0 > 0$. Assume that

$$\mathbf{w} = (w_1, w_2, \ldots, w_l) \in C((\tau, M) \times \bar{\Omega}, \mathbb{R}^l) \cap C^{1,2}((\tau, M) \times \Omega, \mathbb{R}^l)$$

satisfies

$$\sum_{s=1}^{l} c^{k,s}(t,\mathbf{x})w_s + \mathbf{L}_k w_k \leq 0, \quad (t,\mathbf{x}) \in (\tau, M) \times \Omega, \ k = 1, 2, \ldots, l, \tag{2.23}$$

where $c^{k,s} \in C((\tau, M) \times \Omega, \mathbb{R})$ and $c^{k,s} \geq 0$ if $k \neq s$, and

$$\max_{t,\mathbf{x}\in(\tau,M)\times\Omega} |c^{k,s}(t,\mathbf{x})| \leq \gamma_0$$

(k,s=1,2,...,l) for some $\gamma_0 > 0$. Let D and U be domains in Ω such that $D \subset\subset U$, $\operatorname{dist}(\bar{D}, \partial U) > \varrho$, and $|D| > \epsilon$ for certain positive constants ϱ and ϵ. Let θ be a positive constant with $\tau + 4\theta < M$. Then there exist positive constants p, ω_1 and ω_2 determined only by $\alpha_0, \beta_0, \gamma_0, \varrho, \epsilon, n, \operatorname{diam}\Omega$ and θ, such that

$$\inf_{(\tau+3\theta,\tau+4\theta)\times D} w_k \geq \omega_1 \|(w_k)^+\|_{L^p((\tau+\theta,\tau+2\theta)\times D)} - \omega_2 \max_{j=1,2,\ldots,k} \sup_{\partial_P((\tau,\tau+4\theta)\times U)} (w_j)^-.$$

Here $(w_k)^+ = \max\{w_k, 0\}$, $(w_k)^- = \max\{-w_k, 0\}$ and $\partial_P((\tau, \tau+4\theta) \times U) = \tau \times U \cup [\tau, \tau+4\theta) \times \partial U$. Moreover, if all inequalities in (2.23) are replaced by equalities, then the conclusion holds with $p = \infty$ and with ω_1, ω_2 independent of ϵ.

Now we are ready to prove the main result of this section.

Theorem 2.10 *Assume that $R_0 > 1$. For any $c > c^*$, system (1.2) admits a time periodic travelling wave solution (ϕ^*, ψ^*) satisfying (2.2). Furthermore, $0 < \frac{1}{T}\int_0^T \psi^*(t,z)dt \leq S_0 - S^\infty$ for any $z \in \mathbb{R}$, and*

$$\frac{1}{T}\int_{-\infty}^{\infty}\int_0^T \gamma(t)\psi^*(t,z)dtdz = \frac{1}{T}\int_{-\infty}^{\infty}\int_0^T \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dtdz = c[S_0 - S^\infty].$$

Proof In view of Lemmas 2.7 and 2.8, the operator $\mathcal{F}$ is continuous and compact on $\mathcal{D}$ with respect to the norm $\|\cdot\|_\mu$. Additionally, it is easy to verify that $\mathcal{D}$ is closed and convex. Then, the Schauder's fixed point theorem implies that $\mathcal{F}$ has a fixed point $(\phi^*, \psi^*) \in \mathcal{D}$. Moreover, $(\phi^*(T,\cdot), \psi^*(T,\cdot)) = (\phi^*(0,\cdot), \psi^*(0,\cdot))$ and (ϕ^*, ψ^*) satisfies that

$$\begin{cases} \phi^*(t) = T_1(t)\phi^*(0) + \int_0^t T_1(t-s)f_1[\phi^*, \psi^*](s)ds, \\ \psi^*(t) = T_2(t)\psi^*(0) + \int_0^t T_2(t-s)f_2[\phi^*, \psi^*](s)ds \end{cases} \tag{2.24}$$

for $t \in [0, T]$. Define $(\hat{\phi}^*(t,z), \hat{\psi}^*(t,z)) = (\phi^*(t-kT,z), \psi^*(t-kT,z))$ for any $t \in \mathbb{R}$ and $z \in \mathbb{R}$, where $k \in \mathbb{Z}$ satisfies $kT \leq t < (k+1)T$. Then we get $(\hat{\phi}^*(t+T,z), \hat{\psi}^*(t+T,z)) = (\hat{\phi}^*(t,z), \hat{\psi}^*(t,z)), \forall (t,z) \in \mathbb{R} \times \mathbb{R}$. Since $(\phi^*, \psi^*) \in C^{\theta,2\theta}([0,T] \times \mathbb{R}, \mathbb{R}^2)$ for some $\theta \in (0,1)$, we have $(\hat{\phi}^*, \hat{\psi}^*) \in C^{\theta,2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$. Due to the T-periodicity of $\hat{\phi}^*$ and $\hat{\psi}^*$, we see that $(\hat{\phi}^*, \hat{\psi}^*)$ satisfies

$$\begin{cases} \hat{\phi}^*(t) = T_1(t)\hat{\phi}^*(0) + \int_0^t T_1(t-s) f_1[\hat{\phi}^*, \hat{\psi}^*](s)ds, \\ \hat{\psi}^*(t) = T_2(t)\hat{\psi}^*(0) + \int_0^t T_2(t-s) f_2[\hat{\phi}^*, \hat{\psi}^*](s)ds \end{cases} \tag{2.25}$$

for $t \in \mathbb{R}$. Denote $(\hat{\phi}^*, \hat{\psi}^*)$ by (ϕ^*, ψ^*) again. It follows from [31, Theorem 5.1.2, 5.1.3 and 5.1.4] that $(\phi^*, \psi^*) \in C^{1,2+2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ satisfies

$$\begin{cases} \phi_t^*(t,z) = d_1\phi_{zz}^*(t,z) - c\phi_z^*(t,z) - \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}, & t \in \mathbb{R},\ z \in \mathbb{R}, \\ \psi_t^*(t,z) = d_2\psi_{zz}^*(t,z) - c\psi_z^*(t,z) + \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)} - \gamma(t)\psi^*(t,z), & t \in \mathbb{R},\ z \in \mathbb{R} \end{cases} \tag{2.26}$$

and

$$\|\phi^*\|_{C^{1,2+2\theta}(\mathbb{R}\times\mathbb{R},\mathbb{R})} + \|\psi^*\|_{C^{1,2+2\theta}(\mathbb{R}\times\mathbb{R},\mathbb{R})} < \infty \tag{2.27}$$

for some $\theta \in (0,1)$.

Next, we need to verify that (ϕ^*, ψ^*) satisfies the boundary conditions (2.2). By the definitions of $\phi^{\pm}$ and $\psi^{\pm}$, it follows that $\phi^*(t,z) \to S_0$ and $\psi^*(t,z) \to 0$ uniformly for $t \in \mathbb{R}$, as $z \to -\infty$. On the other hand, by the estimate (2.27) and Landau type inequalities (see, e.g., [23] or [4]), we have

$$\left|\phi_z^*\right|_{L^\infty([0,T]\times(-\infty,M])} \leq 2\left|\phi^* - S_0\right|^{\frac{1}{2}}_{L^\infty([0,T]\times(-\infty,M])} \left|\phi_{zz}^*\right|^{\frac{1}{2}}_{L^\infty([0,T]\times(-\infty,M])}$$

and

$$\left|\psi_z^*\right|_{L^\infty([0,T]\times(-\infty,M])} \leq 2\left|\psi^*\right|^{\frac{1}{2}}_{L^\infty([0,T]\times(-\infty,M])} \left|\psi_{zz}^*\right|^{\frac{1}{2}}_{L^\infty([0,T]\times(-\infty,M])}.$$

As a result,

$$\lim_{z\to-\infty} (\phi_z^*(t,z), \psi_z^*(t,z)) = (0,0) \ \text{ uniformly for } t \in \mathbb{R}.$$

We further discuss the asymptotic behavior of ϕ_{zz}^* and ψ_{zz}^* when z tends to $-\infty$. By the (strong) maximum principle, it follows that $\phi^*(t,x) > 0, \psi^*(t,x) > 0, \forall t > 0, x \in \mathbb{R}$. Differentiating two side of the first equation of (2.26) with respect to z yields

$$(\phi_z^*)_t = d_1(\phi_z^*)_{zz} - c(\phi_z^*)_z - \frac{\beta(t)\phi_z^*(\psi^*)^2 + \psi_z^*(\phi^*)^2}{(\phi^* + \psi^*)^2}, \quad t > 0,\ z \in \mathbb{R}. \tag{2.28}$$

Since $\phi_z^* \in C^{\theta,2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ for some $\theta \in (0,1)$, it follows from the T-periodicity of ϕ^* and [31, Theorems 5.1.3 and 5.1.4] that $\phi_z^* \in C^{1,2+2\theta}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and

$$\|\phi_z^*\|_{C^{1,2+2\theta}(\mathbb{R}\times\mathbb{R},\mathbb{R})} < \infty$$

for some $\theta \in (0,1)$. By a similar argument to ϕ^*, we can conclude from the Landau type inequality that

$$\lim_{z\to-\infty} \phi_{zz}^*(t,z) = 0 \ \text{ uniformly for } t \in \mathbb{R}.$$

Similarly, we have

$$\lim_{z\to-\infty} \psi_{zz}^*(t,z) = 0 \ \text{ uniformly for } t \in \mathbb{R}.$$

Consequently, we can see from the system (2.26) that

$$\lim_{z\to-\infty}(\phi_t^*(t,z),\psi_t^*(t,z))=(0,0)\ \text{ uniformly for } t\in\mathbb{R}.$$

Define $\Phi(z)=\frac{1}{T}\int_0^T\phi^*(t,z)dt$. Clearly, $\Phi_z(z)\to 0$ as $z\to-\infty$. It then follows from the first equation of system (2.26) that

$$c\Phi_z=d_1\Phi_{zz}-\frac{1}{T}\int_0^T\frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dt. \tag{2.29}$$

Integrating two sides of (2.29) from y to z and letting $y\to-\infty$ yield

$$d_1\Phi_z(z)=c\left[\Phi(z)-S_0\right]+\frac{1}{T}\int_{-\infty}^z\int_0^T\frac{\beta(t)\phi^*(t,y)\psi^*(t,y)}{\phi^*(t,y)+\psi^*(t,y)}dtdy. \tag{2.30}$$

Due to the uniform boundedness of $\phi^*(t,z)$ and $\phi_z^*(t,z)$, it is easy to see that $\Phi(z)=\frac{1}{T}\int_0^T\phi^*(t,z)dt$ and $\Phi_z(z)=\frac{1}{T}\int_0^T\phi_z^*(t,z)dt$ are uniformly bounded, respectively, and hence, $\frac{1}{T}\int_0^T\frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dt$ is integrable on $\mathbb{R}$. From (2.29), we have

$$\left(e^{-cz/d_1}\Phi_z\right)_z=e^{-cz/d_1}\left(\Phi_{zz}-c\Phi_z/d_1\right)=\frac{e^{-cz/d_1}}{d_1T}\int_0^T\frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dt.$$

For the above equality, an integration from z to ∞ gives

$$e^{-cz/d_1}\Phi_z(z)=-\frac{1}{d_1T}\int_z^\infty e^{-cy/d_1}\int_0^T\frac{\beta(t)\phi^*(t,y)\psi^*(t,y)}{\phi^*(t,y)+\psi^*(t,y)}dtdy,$$

which implies that $\Phi_z(z)<0$ for $z\in\mathbb{R}$. It follows that $\Phi(+\infty)$ exists and $\Phi(+\infty)<\Phi(-\infty)=S_0$. With the aid of Barbălat's lemma (see, e.g., [3,12]), we have $\Phi_z(z)\to 0$ as $z\to\infty$. Furthermore, letting $z\to\infty$ in (2.30) yields

$$\frac{1}{T}\int_{-\infty}^\infty\int_0^T\frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dtdz=c[S_0-\Phi(\infty)]=c[S_0-S^\infty],$$

where $S^\infty:=\Phi(\infty)<S^0$.

In the following, we explore the asymptotic behavior of $\psi^*(t,z)$ as $z\to\infty$. Let $\hat{\gamma}:=\min_{t\in[0,T]}\gamma(t)$ and $\tilde{\gamma}=\max_{t\in[0,T]}\gamma(t)$, and define $\Psi(z)=\frac{1}{T}\int_0^T\psi^*(t,z)dt$. Then $\Psi(z)$ satisfies

$$-d_2\Psi_{zz}+c\Psi_z+\hat{\gamma}\Psi=\frac{1}{T}\int_0^T\frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dt-\frac{1}{T}\int_0^T\left(\gamma(t)-\hat{\gamma}\right)\psi^*(t,z)dt. \tag{2.31}$$

Denote by

$$\hat{\lambda}^\pm:=\frac{c\pm\sqrt{c^2+4d_2\hat{\gamma}}}{2d_2}$$

the two roots of the characteristic equation

$$-d_2\lambda^2+c\lambda+\hat{\gamma}=0.$$

In addition, denote

$$\hat{\rho}:=d_2\left(\hat{\lambda}^+-\hat{\lambda}^-\right)=\sqrt{c^2+4d_2\hat{\gamma}}.$$

It is easy to see that $\hat{\lambda}^- < 0 < \hat{\lambda}^+$. Since $\frac{1}{T}\int_0^T \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dt \le \frac{S_0}{T}\int_0^T \beta(t)dt = \overline{\beta}S_0$, we see from (2.31) that

$$\begin{aligned}
\Psi(z) &= \frac{1}{\hat{\rho}T}\int_{-\infty}^{z} e^{\hat{\lambda}^-(z-y)}\left[\int_0^T \frac{\beta(t)\phi^*(t,y)\psi^*(t,y)}{\phi^*(t,y)+\psi^*(t,y)} - \int_0^T \left(\gamma(t)-\hat{\gamma}\right)\psi^*(t,z)\right]dtdy \\
&\quad + \frac{1}{\hat{\rho}T}\int_{z}^{\infty} e^{\hat{\lambda}^+(z-y)}\left[\int_0^T \frac{\beta(t)\phi^*(t,y)\psi^*(t,y)}{\phi^*(t,y)+\psi^*(t,y)} - \int_0^T \left(\gamma(t)-\hat{\gamma}\right)\psi^*(t,z)\right]dtdy \\
&\le \frac{1}{\hat{\rho}T}\int_{-\infty}^{z} e^{\hat{\lambda}^-(z-y)}\int_0^T \frac{\beta(t)\phi^*(t,y)\psi^*(t,y)}{\phi^*(t,y)+\psi^*(t,y)}dtdy \\
&\quad + \frac{1}{\hat{\rho}T}\int_{z}^{\infty} e^{\hat{\lambda}^+(z-y)}\int_0^T \frac{\beta(t)\phi^*(t,y)\psi^*(t,y)}{\phi^*(t,y)+\psi^*(t,y)}dtdy \\
&= \frac{1}{\hat{\rho}T}\int_{0}^{\infty} e^{\hat{\lambda}^- y}\int_0^T \frac{\beta(t)\phi^*(t,z-y)\psi^*(t,z-y)}{\phi^*(t,z-y)+\psi^*(t,z-y)}dtdy \\
&\quad + \frac{1}{\hat{\rho}T}\int_{-\infty}^{0} e^{\hat{\lambda}^+ y}\int_0^T \frac{\beta(t)\phi^*(t,z-y)\psi^*(t,z-y)}{\phi^*(t,z-y)+\psi^*(t,z-y)}dtdy.
\end{aligned}$$

Integrating $\Psi(z)$ from ζ to ξ, we obtain

$$\begin{aligned}
\int_\zeta^\xi \Psi(z)dz &= \frac{1}{\hat{\rho}T}\int_{0}^{\infty} e^{\hat{\lambda}^- y}\int_\zeta^\xi\int_0^T \frac{\beta(t)\phi^*(t,z-y)\psi^*(t,z-y)}{\phi^*(t,z-y)+\psi^*(t,z-y)}dtdzdy \\
&\quad + \frac{1}{\hat{\rho}T}\int_{-\infty}^{0} e^{\hat{\lambda}^+ y}\int_\zeta^\xi\int_0^T \frac{\beta(t)\phi^*(t,z-y)\psi^*(t,z-y)}{\phi^*(t,z-y)+\psi^*(t,z-y)}dtdzdy.
\end{aligned}$$

Note that $\int_0^T \frac{\beta(t)\phi^*\psi^*}{\phi^*+\psi^*}dt$ is integrable on $\mathbb{R}$. It then follows from Fubini's theorem that $\Psi(z)$ is integral on $\mathbb{R}$, and

$$\int_{-\infty}^{\infty} \Psi(z)dz \le \frac{1}{\hat{\gamma}T}\int_{-\infty}^{\infty}\int_0^T \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dtdz.$$

In view of (2.27), it is easy to see that $\Psi_z(z)$ is uniformly bounded on $\mathbb{R}$, and hence, Barbălat's lemma guarantees that $\Psi(z) \to 0$ as $z \to \infty$. On the other hand, for the second equation of system (2.26), applying Lemma 2.9 with $\tau = -T, \theta = T$ and $D := D_z = (z-\frac{1}{4}, z+\frac{1}{4}), U = (z-\frac{1}{2}, z+\frac{1}{2}), \Omega = (z-1, z+1)$ with $z \in \mathbb{R}$, we have

$$\begin{aligned}
\sup_{(0,T)\times D} \psi^*(t,y) &\le C_0 \inf_{(2T,3T)\times D} \psi^*(t,z) \\
&= C_0 \min_{[2T,3T]\times \overline{D}} \psi^*(t,y) \\
&\le C_0 \min_{\overline{D}} \psi^*(0,y),
\end{aligned}$$

where C_0 is a positive constant independent of D. Due to the periodicity of ψ^* in time t, we see that $\psi^*(t,z) \to 0$ uniformly for $t \in \mathbb{R}$, as $z \to \infty$.

We further prove that $\phi^*(t,z) \to S^\infty$ uniformly for $t \in \mathbb{R}$, as $z \to \infty$. On the basis of the T-periodicity of ϕ^*, it suffices to show

$$\limsup_{z\to\infty}\max_{t\in[0,T]} \phi^*(t,z) =: S_+^\infty = S^\infty = S_-^\infty := \liminf_{z\to\infty}\min_{t\in[0,T]} \phi^*(t,z).$$

It is clear that there exist $\{t_n\}$ and $\{z_n\}$ satisfying $\{t_n\} \subset [0, T]$ and $z_n \to \infty$ (as $n \to \infty$), respectively, such that

$$\lim_{n\to\infty} \phi^*(t_n, z_n) = S_+^\infty.$$

Let $\phi_n(t, z) = \phi^*(t + t_n, z + z_n)$, $\psi_n(t, z) = \psi^*(t + t_n, z + z_n)$, $\forall n \in \mathbb{N}, t \in \mathbb{R}, z \in \mathbb{R}$. Due to the estimation (2.27), there exists a subsequence of $(\phi_n(t, z), \psi_n(t, z))$, still denoted by $(\phi_n(t, z), \psi_n(t, z))$, converging to $(\phi_*(t, z), 0)$ in $C_{\text{loc}}^{\theta,2\theta}(\mathbb{R} \times \mathbb{R})$ for some $\theta \in (0, 1)$, as $n \to \infty$. In particularly, we have $\phi_*(0, 0) = S_+^\infty$ and

$$\phi_*(t + T, z) = \phi_*(t, z), \quad \phi_*(t, z) \le S_+^\infty, \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}.$$

Since $\{t_n\} \subset [0, T]$, without loss of generality, let $t_n \to t^* \in [0, T]$. Then $\phi_*^+(t, z) = \phi_*(t - t^*, z)$ satisfies

$$\begin{aligned}\phi_*^+(t) =&T_1(t)\phi_*^+(0) + \int_0^t T_1(t - s) f_1[\phi_*^+, 0](s)ds\\ =&T_1(t)\phi_*^+(0) + \int_0^t T_1(t - s)\alpha_1\phi_*^+(s)ds.\end{aligned}$$

Consequently, $\phi_*^+(t, z)$ satisfies

$$\partial_t\phi_*^+(t, z) = d_1\partial_{zz}\phi_*^+(t, z) - c\partial_z\phi_*^+(t, z), \quad (t, z) \in \mathbb{R} \times \mathbb{R}.$$

Since $\phi_*^+(t^*, 0) = S_+^\infty$ and $\phi_*^+(t, z) \le S_+^\infty$, it follows from the maximum principle that $\phi_*^+(t, z) \equiv S_+^\infty$ for $t < t^*$. By the T-periodicity of $\phi_*^+(\cdot, z)$, we have $\phi_*^+(t, z) \equiv S_+^\infty$, $\forall t \in \mathbb{R}$, and hence $\Phi_*^+(z) := \frac{1}{T}\int_0^T \phi_*^+(t, z)dt \equiv S_+^\infty$. On the other hand,

$$\begin{aligned}\Phi_*^+(z) =&\frac{1}{T}\int_0^T \phi_*^+(t, z)dt = \frac{1}{T}\int_0^T \phi_*(t - t^*, z)dt\\ =&\lim_{n\to\infty}\frac{1}{T}\int_0^T \phi_n(t - t^*, z)dt\\ =&\lim_{n\to\infty}\frac{1}{T}\int_0^T \phi^*(t - t^* + t_n, z + z_n)dt\\ =&S^\infty,\end{aligned}$$

which implies $S_+^\infty = S^\infty$. Therefore, $\limsup_{z\to\infty}\max_{t\in[0,T]}\phi^*(t, z) = S^\infty$. Similarly, we can prove $\liminf_{z\to\infty}\min_{t\in[0,T]}\phi^*(t, z) = S^\infty$. This implies that $\phi_*^+(t, z)$ converges to S^∞ uniformly in $t \in \mathbb{R}$ as $z \to \infty$.

Moreover, since $\Psi(z)$ satisfies

$$-d_2\Psi_{zz} + c\Psi_z = \frac{1}{T}\int_0^T \frac{\beta(t)\phi^*(t, z)\psi^*(t, z)}{\phi^*(t, z) + \psi^*(t, z)}dt - \frac{1}{T}\int_0^T \gamma(t)\psi^*(t, z)dt, \tag{2.32}$$

by making an integration of (2.32) on $\mathbb{R}$, we get

$$\frac{1}{T}\int_{-\infty}^{\infty}\int_0^T \gamma(t)\psi^*(t, z)dtdz = \frac{1}{T}\int_{-\infty}^{\infty}\int_0^T \frac{\beta(t)\phi^*(t, z)\psi^*(t, z)}{\phi^*(t, z) + \psi^*(t, z)}dtdz = c[S_0 - S^\infty].$$

It remains to prove that $0 < \frac{1}{T}\int_0^T \psi^*(t, z)dt \le S_0 - S^\infty$. In order to achieve this, we shall use a similar argument to the proof of [45, Theorem 2.9]. First, by similar arguments to the proof of the asymptotic behavior of $\phi_z^*(t, z)$ and $\phi_{zz}^*(t, z)$ as $z \to -\infty$, we can show that

$$\lim_{z\to\infty}\psi_z^*(t, z) = \lim_{z\to\infty}\psi_{zz}^*(t, z) = 0$$

uniformly for $t \in \mathbb{R}$. Thus, we have

$$\lim_{z\to\pm\infty} \psi_z^*(t,z) = \lim_{z\to\pm\infty} \psi_{zz}^*(t,z) = 0 \tag{2.33}$$

uniformly for $t \in \mathbb{R}$. For any $z \in \mathbb{R}$, we define a function

$$\Psi^*(z) = \frac{1}{cT}\int_{-\infty}^{z}\int_0^T \gamma(t)\psi^*(t,y)dtdy + \frac{1}{cT}\int_z^\infty e^{c/d_2(z-y)}\int_0^T \gamma(t)\psi^*(t,y)dtdy. \tag{2.34}$$

It is easy to see that $\Psi^*(z)$ satisfies the following equation:

$$c\Psi_z^*(z) = d_2\Psi_{zz}^*(z) + \frac{1}{T}\int_0^T \gamma(t)\psi^*(t,y)dt, \quad \forall z \in \mathbb{R}.$$

By means of (2.33) and L'Hôpital's rule, it follows that

$$\lim_{z\to-\infty} \Psi^*(z) = 0, \; \lim_{z\to\infty} \Psi^*(z) = \frac{1}{cT}\int_{-\infty}^{\infty}\int_0^T \gamma(t)\psi^*(t,y)dy = S_0 - S^\infty$$

and

$$\lim_{z\to\pm\infty} \Psi_z^*(z) = 0.$$

Recall that $\Psi(z) = \frac{1}{T}\int_0^T \psi^*(t,z)dt$. We further introduce a function

$$\hat{\Psi}(z) := \Psi(z) + \Psi^*(z), \quad \forall z \in \mathbb{R}.$$

Consequently, it is not difficult to obtain from (2.32) and (2.34) that

$$c\hat{\Psi}_z(z) = d_2\hat{\Psi}_{zz}(z) + \frac{1}{T}\int_0^T \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dt, \quad \forall z \in \mathbb{R}.$$

Multiplying two sides of the above equation by $e^{-c/d_2 z}$ and integrating from z to ∞, we have

$$\hat{\Psi}_z(z) = \frac{1}{d_2 T}\int_z^\infty e^{c/d_2(z-y)}\int_0^T \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{\phi^*(t,z)+\psi^*(t,z)}dt.$$

This implies that $\hat{\Psi}(z)$ is non-decreasing in $\mathbb{R}$. Note that $\lim_{z\to\infty}\hat{\Psi}(z) = S_0 - S^\infty$. Hence, $\hat{\Psi}(z) \le S_0 - S^\infty$ for all $z \in \mathbb{R}$. In view of the definition of $\hat{\Psi}(z)$ and $\Psi^*(z)$, we conclude that $\Psi(z) \le \hat{\Psi}(z) \le S_0 - S^\infty$ for all $z \in \mathbb{R}$, that is, $0 \le \frac{1}{T}\int_0^T \psi^*(t,z)dt \le S_0 - S^\infty$ for any $z \in \mathbb{R}$. □

3 The Nonexistence of Periodic Traveling Waves

In this section, we prove the nonexistence of time periodic traveling waves for two cases. In the case where $R_0 \le 1$, there is no time periodic traveling wave. In the case where $R_0 > 1$ and $c < c^*$, there is no time periodic, non-trivial and non-negative travelling waves.

Theorem 3.1 *Assume that* $R_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T \gamma(t)dt} \le 1$. *Then for any* $c \ge 0$, *there is no time periodic traveling wave solutions* (ϕ, ψ) *satisfying*

$$\phi(t,-\infty) = S_0,\ \phi(t,\infty) = S^\infty < S_0,\ \psi(t,\pm\infty) = 0 \text{ uniformly in } t \in \mathbb{R}. \tag{3.1}$$

Proof Suppose, by way of contradiction, that there exists a time periodic, non-trivial and non-negative solution $(\phi(t,z), \psi(t,z))$ of (2.1) with (3.1). Then there exists a positive constant b such that $0 \le \phi(t,z) \le b$, $\forall t \ge 0$, $x \in \mathbb{R}$, and hence,

$$\psi_t(t,z) = d_2\psi_{zz}(t,z) - c\psi_z(t,z) + \frac{\beta(t)\phi(t,z)\psi(t,z)}{\phi(t,z)+\psi(t,z)} - \gamma(t)\psi(t,z)$$
$$\leq d_2\psi_{zz}(t,z) - c\psi_z(t,z) + \left[\frac{b\beta(t)}{b+\psi(t,z)} - \gamma(t)\right]\psi(t,z).$$

for any $t > 0$ and $z \in \mathbb{R}$. Let $\eta := \sup_{z\in\mathbb{R}} \psi(0,z) < \infty$. Then $\psi(0,z) \leq \eta, \forall z \in \mathbb{R}$. By the comparison principle, we have

$$\psi(t,z) \leq v(t;\eta), \quad \forall t > 0,\ z \in \mathbb{R},$$

where $v(t;\eta)$ is the solution of the following ordinary differential equation:

$$\begin{cases} v'(t) = \left[\frac{b\beta(t)}{b+v(t)} - \gamma(t)\right] v(t), & t > 0, \\ v(0) = \eta. \end{cases}$$

Since $R_0 \leq 1$, we have $\frac{1}{T}\int_0^T (\beta(t)-\gamma(t))\,dt \leq 0$. Set

$$p(t,v) = \frac{b\beta(t)}{b+v(t)} - \gamma(t).$$

Then we have

$$\int_0^T p(t,0)dt = \frac{1}{T}\int_0^T (\beta(t)-\gamma(t))\,dt \leq 0.$$

Hence, [56, Theorem 3.1.2] implies that $\lim_{t\to\infty} v(t;\eta) = 0$. It follows that $\lim_{t\to\infty} \psi(t,z) = 0, \forall z \in \mathbb{R}$, which contradicts to the time periodicity of $\psi(t,\cdot)$ in t. □

Next, we prove the non-existence of periodic traveling waves for the case where $R_0 > 1$ and $c < c^*$. We first consider the following scalar periodic reaction–diffusion equation:

$$\frac{\partial u}{\partial t} = du_{xx} + f(t,u), \quad t > 0,\ x \in \mathbb{R}, \tag{3.2}$$

where $d > 0$, $f \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and $f(t,\cdot)$ is T-periodic in t for some $T > 0$. Assume that

(A1) $f(t,0) = 0$ for $t \geq 0$, and there is a real number $H > 0$ such that $f(t,H) \leq 0$, and for each $t \geq 0$, $f(t,\cdot)$ is strictly subhomogeneous on $[0,H]$ in the sense that $f(t,\alpha u) > \alpha f(t,u)$ whenever $\alpha \in (0,1)$, $u \in (0,H]$.

(A2) $\overline{f_u(t,0)} := \frac{1}{T}\int_0^T \frac{\partial f(t,0)}{\partial u} dt > 0$.

By [56, Theorem 3.1.2], it follows that the periodic ordinary differential equation

$$\frac{du}{dt} = f(t,u), \quad t \geq 0 \tag{3.3}$$

has a unique positive T-periodic solution $q(t)$ with $q(t) \in [0,H], \forall t \in [0,T]$, and $q(t)$ is globally asymptotically stable in $(0,H]$. By the same arguments as in [27, Sect. 4] (just letting $\tau = 0$ in Theorems 4.1 and 4.2), we have the following two results.

Proposition 3.2 *Assume that (A1) and (A2) hold. Let $c^* = 2\sqrt{d\cdot\overline{f_u(t,0)}}$ and $u(t,x,\varphi)$ be the solution of equation (3.2) with the initial data φ. Then the following statements are valid:*

(1) *For any* $c > c^*$, *if* $\varphi \in C_{q(0)} = \{\varphi \in C(\mathbb{R}, \mathbb{R}) : 0 \le \varphi(x) \le q(0), \forall x \in \mathbb{R}\}$ *with* $\varphi(x) < q(0), \forall x \in \mathbb{R}$, *and* $\varphi(x) = 0$ *for* x *outside a bounded interval, then* $\lim_{t\to\infty, |x|\ge ct} u(t, x, \varphi) = 0$.
(2) *For any* $c < c^*$, *if* $\varphi \in C_{q(0)}$ *with* $\varphi \not\equiv 0$, *then* $\lim_{t\to\infty, |x|\le ct} (u(t, x, \varphi) - q(t)) = 0$.

Proposition 3.3 *Assume that* $(A1)$ *and* $(A2)$ *hold. Let* c^* *be defined as in Proposition* 3.2*. Then* c^* *is the minimal wave speed for the monotone periodic traveling waves* $U(t, x + ct)$ *of equation* (3.2) *connecting* $q(t)$ *to* 0.

Now we are in a position to prove the non-existence of periodic traveling wave solutions in the case where $R_0 > 1$ and $0 < c < c^*$.

Theorem 3.4 *Assume that* $R_0 > 1$ *and* $0 < c < c^*$. *Then there is no time-periodic traveling waves* (ϕ, ψ) *satisfying*

$$\phi(t, -\infty) = S_0,\ \phi(t, \infty) = S^\infty > 0,\ \psi(t, \pm\infty) = 0 \text{ uniformly in } t \in \mathbb{R}. \tag{3.4}$$

Proof Suppose, by contradiction, that there exists such a traveling wave satisfying (3.4) for some $c < c^*$. Then there exists $a > 0$ such that $\phi(t, x + ct) \ge a > 0, \forall t \ge 0, x \in \mathbb{R}$. It follows that $v(t, x) := \psi(t, x + ct)$ satisfies

$$v_t \ge d_2 v_{xx} + \frac{a\beta(t)}{a + v(t, x)} v(t, x) - \gamma(t) v(t, x),\quad t \ge 0,\ x \in \mathbb{R}.$$

Note that $R_0 > 1$ implies that $(A2)$ holds. Let $q^a(t)$ be the unique positive T-periodic solution of

$$u'(t) = -\gamma(t)u(t) + \frac{a\beta(t)}{a + u(t)} u(t),\quad t > 0$$

and choose a continuous function $\psi_0(x)$ such that $0 \le \psi_0(x) \le q^a(0)$ and $\psi_0(x) \le \psi(0, x), \forall x \in \mathbb{R}$, and $\psi_0 \not\equiv 0$. Then the comparison principle implies that

$$v(t, x) = \psi(t, x + ct) \ge u(t, x, \psi_0),\quad \forall t \ge 0,\ x \in \mathbb{R}, \tag{3.5}$$

where $u(t, x, \psi_0)$ is the unique solution of the following scalar reaction–diffusion equation

$$\begin{cases} u_t = d_2 u_{xx} + \frac{a\beta(t)}{a+u(t,x)} u(t, x) - \gamma(t) u(t, x), & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = \psi_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.6}$$

By Proposition 3.2, $c^* = 2\sqrt{d_2 \cdot \overline{\beta(t) - \gamma(t)}}$ is the spreading speed of system (3.6). Fix a real number $\bar{c} \in (c, c^*)$. It then follows from Proposition 3.2(2) that

$$\lim_{t\to\infty, |x|\le \bar{c}t} \left(u(t, x, \psi_0) - q^a(t)\right) = 0. \tag{3.7}$$

Since $q^a(t)$ is T-periodic, letting $t = nT$, $x = -\bar{c}t$ in (3.7), we obtain

$$\lim_{n\to\infty} u(nT, -\bar{c}nT, \psi_0) = q^a(0).$$

In view of (3.5), we have

$$\psi(nT, (c - \bar{c})nT) \ge u(nT, -\bar{c}nT, \psi_0),\quad \forall n \ge 1.$$

It then follows that

$$\psi(0, -\infty) = \lim_{n\to\infty} \psi(0, (c - \bar{c})nT) = \lim_{n\to\infty} \psi(nT, (c - \bar{c})nT) \ge q^a(0) > 0,$$

which contradicts $\psi(0, -\infty) = 0$. □

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