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The 2d Nonlinear Fully Hyperbolic Inviscid Shallow Water Equations in a Rectangle

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Dedicated to the memory of K. Kirchgässner

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Abstract We continue our study of the inviscid shallow water equations (SWE) in a rectangle. In an earlier work (Huang and Temam in Arch Ration Mech Anal 211(3):1027–1063, 2014) we studied the well-posedness for all time of the linearized inviscid SWE in a non-smooth domain. We defined and classified the different sets of boundary conditions which make these equations well-posed for all time and showed the existence and uniqueness of solutions. As we show below totally different boundary conditions are needed in the full nonlinear cases. The case of supercritical flows was investigated in Huang et al. (Asymptot Anal 93:187–218, 2015), and the case of subcritical flows in a channel was studied in Huang and Temam (Commun Pure Appl Anal **13**(5):2005–2038, 2014). We continue here and study subcritical flows in a rectangle which raises the additional issue of the compatibility of the boundary and initial conditions at $t = 0$ and of the boundary conditions between them at the corners of the rectangle.

Keywords Shallow water equations · Hyperbolic equations · Initial and boundary value problem · Non-smooth domain

1 Introduction

Motivated by the study of the well-posedness of the inviscid primitive equations (PEs), we were led in earlier works, to study the well-posedness of the inviscid shallow water equations. Indeed, as shown in [9], the inviscid shallow water equations can be seen as a single mode of the PEs. Hence studying the SWE, besides being useful by itself, can be seen also as a step toward the study of the well-posedness of the PEs.

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The inviscid shallow water equations are hyperbolic equations and there is a vast literature available concerning the initial and boundary value hyperbolic problems in a smooth domain in relation with the Kreiss–Lopatinskii conditions (see [11,14]); many results in this direction can be found in [3]. However the usual framework for the Primitive Equations is to work in a rectangle in dimension two and in a cube in dimension three. Hence we encounter the difficulty of the well-posedness of hyperbolic initial and boundary value problems for non-smooth domains and the literature is rather scarce in this case; see the discussion in [9].

In [9], we presented the different set of boundary conditions that make the inviscid linearized SWE well-posed in a rectangle and classified these boundary conditions. For that reason, we made the equations as a linear evolution equation and used the semigroup theory. The classification of the flows correspond to different properties of the underlying linear (stationary) operator. In a same way as a stationary compressible flow can be subsonic (elliptic), transonic (parabolic), or supersonic (hyperbolic), we were led in [9] to classify the linearized shallow water flows in the fully hyperbolic case and the elliptic-hyperbolic case, where the fully hyperbolic case contains four sub-cases, the supercritical case, two mixed hyperbolic case, and the fully hyperbolic subcritical case.

Going from the linearized inviscid SWE to the full nonlinear inviscid SWE is not straightforward, and truly new boundary conditions have to be derived. Besides, global existence of smooth solutions is not generally expected and we limit ourselves to the existence and uniqueness of smooth solutions for a limited time. However, we retain from [9] the classification of the (nonlinear) flows according to their initial values. When the flow is fully supercritical, the problem was studied in [8], we then considered in [10] the case where the flow was fully hyperbolic but took place in a channel $(0, 1)_x \times \mathbb{T}_y$ where $\mathbb{T}_y$ is one dimensional torus. We now consider in this article the case of a fully hyperbolic (subcritical) flow taking place in a rectangle. We then encounter the difficulty of the compatibility of the initial and boundary values. When the datas are compatible, we are able to embed the flow in a channel flow (distinct from the case in [10]) and we are able to establish the existence and uniqueness of solutions.

The inviscid fully nonlinear shallow water equations (SWE) read

$$\begin{cases} u_t + uu_x + vu_y + g\phi_x - fv = 0, \\ v_t + uv_x + vv_y + g\phi_y + fu = 0, \\ \phi_t + u\phi_x + v\phi_y + \phi(u_x + v_y) = 0, \end{cases} \tag{1.1}$$

where u and v are the two horizontal components of the velocity and ϕ is the height of the water; f and g are universal constants, standing for the Coriolis parameter and the gravitational acceleration, respectively. Setting $U = (u, v, \phi)^t$, we write (1.1) in compact form

$$U_t + \mathcal{E}_1(U)U_x + \mathcal{E}_2(U)U_y + \ell(U) = 0. \tag{1.2}$$

where $\ell(U) = (-fv, fu, 0)^t$, and

$$\mathcal{E}_1(U) = \begin{pmatrix} u & 0 & g \\ 0 & u & 0 \\ \phi & 0 & u \end{pmatrix}, \quad \mathcal{E}_2(U) = \begin{pmatrix} v & 0 & 0 \\ 0 & v & g \\ 0 & \phi & v \end{pmatrix}.$$

The Assumptions

Our objective in this article is to study the initial and boundary value problem (IBVP) for the 2d nonlinear inviscid SWE (1.1) in the *fully hyperbolic case* in a rectangular domain. In [8,10], we studied two types of problems for the 2d nonlinear inviscid SWE: the *supercritical* *case* in a rectangle and the *subcritical case* in a 2d channel with periodicity. For a classification of the 2d inviscid SWE, see [9, Sect. 5]. The *fully hyperbolic case* studied here corresponds to

$$u^2 + v^2 > g\phi, \tag{1.3}$$

and the rectangular domain has to be properly chosen according to the characteristics of the 2d inviscid SWE, as explained in Sect. 4.2 below.

In order to show the idea for studying the IBVP of 2d nonlinear inviscid SWE, we assume that the domain Ω is

$$\Omega = (0, 1)_x \times (0, 1)_y,$$

and the 2d nonlinear SWE is *supercritical* in the direction $(0, 1)$ (see Definition 4.1 below), that is

$$v^2 > g\phi. \tag{1.4}$$

Note that condition (1.4) is stronger than (1.3), while after a suitable coordinate transformation, the condition (1.3) would become (1.4) (see Sect. 4.2 below). We can also assume that $u,\ v \geq 0$ and the cases where u and, or v are negative can be treated in a similar manner.

Now, we have two sub-cases to consider according to the sign of $u^2 - g\phi$. The case when $u^2 > g\phi$, that is the 2d nonlinear SWE is also *supercritical* in the direction $(1, 0)$, is already studied in [8]. The remaining case when $u^2 < g\phi$, that is the 2d nonlinear SWE is *subcritical* in the direction $(1, 0)$, is the main goal of this article and we already termed it the *mixed hyperbolic case* in [9]. We now assume the enhanced *mixed hyperbolic condition*:

$$\begin{cases} c_0 \leq u,\ v,\ \phi \leq c_1, \\ u^2 + v^2 > g\phi, \qquad u^2 - g\phi \leq -c_2^2, \qquad v^2 - g\phi \geq c_2^2, \end{cases} \tag{1.5}$$

for some given positive constants $c_0, c_1, c_2 > 0$.

This article is dedicated to the memory of Klaus Kirchgässner, a good friend and a gentle colleague, who has made deep and lasting contributions to the theory of bifurcation and partial differential equations, and has invested much time for services to the national and international mathematical communities.

2 The Boundary Conditions

2.1 Failure of the (Linearized) Boundary Conditions

In [9, Sect. 5], the suitable boundary conditions for the linearized SWE in a rectangle have been proposed for the well-posedness and it is natural to consider these (modified) boundary conditions for the nonlinear SWE (1.1) in a rectangle. However, as we will see below, those boundary conditions are not suitable for the nonlinear problem (1.1) in the *mixed hyperbolic case* since we can not derive the suitable nonlinear boundary conditions from them. The arguments are as follows. Recall from [9, Sect. 5] that the boundary conditions for the linearized SWE around the state (u_0, v_0, ϕ_0) in the *mixed hyperbolic case* (1.5) are

$$\begin{cases} v_0 u - u_0 v + \kappa_0 \phi = u_0 u + v_0 v + g\phi = 0,\ \text{on } \{x = 0\}, \\ v_0 u - u_0 v - \kappa_0 \phi = 0,\ \text{on } \{x = 1\}, \\ u = v = \phi = 0,\ \text{on } \{y = 0\}, \end{cases} \tag{2.1}$$

where $\kappa_0 = \sqrt{g(u_0^2 + v_0^2 - g\phi_0)/\phi_0}$. If we could derive a set of nonlinear boundary conditions from (2.1), then considering the boundary conditions at $x = 1$, there must exist two non-zero functions $\Phi(u, v, \phi)$ and $\Psi(u, v, \phi)$ such that

$$(\Phi_u(u, v, \phi),\ \Phi_v(u, v, \phi),\ \Phi_\phi(u, v, \phi)) = (v,\ -u,\ -\kappa)\Psi(u, v, \phi), \tag{2.2}$$

where $\kappa = \sqrt{g(u^2 + v^2 - g\phi)/\phi}$. We now infer from (2.2) the following identities

$$\begin{cases} \Psi + v\Psi_v = \Phi_{uv} = -\Psi - u\Psi_u, \\ v\Psi_\phi = \Phi_{u\phi} = -\kappa_u\Psi - \kappa\Psi_u, \\ -u\Psi_\phi = \Phi_{v\phi} = -\kappa_v\Psi - \kappa\Psi_v. \end{cases} \tag{2.3}$$

Multiplying $(2.3)_2$ by u and $(2.3)_3$ by v, and adding the resulting identities together, we find

$$0 = (u\kappa_u + v\kappa_v)\Psi + \kappa(u\Psi_u + v\Psi_v),$$

which, together with $(2.3)_1$, shows that

$$(u\kappa_u + v\kappa_v)\Psi - 2\kappa\Psi = 0. \tag{2.4}$$

Now, we directly calculate

$$\kappa_u = \frac{gu}{\phi\kappa}, \quad \kappa_v = \frac{gv}{\phi\kappa},$$

and then deduce from (2.4) that

$$\left(gu^2/\phi + gv^2/\phi - 2\kappa^2\right)\Psi = 0,$$

that is equivalent to

$$\left(u^2 + v^2 - 2g\phi\right)\Psi = 0,$$

which is impossible for non-zero Ψ. Therefore, we conclude that the (linearized) boundary conditions proposed in [9] are not suitable for the nonlinear SWE (1.1). This fact is rather general of course.

2.2 The Nonlinear Boundary Conditions

Under the *mixed hyperbolic condition* (1.5), all the eigenvalues of the matrix $\mathcal{E}_2$ are positive and hence the y-direction could be viewed as a time-like direction, and we only need to impose the boundary conditions at $y = 0$, that is

$$(u, v, \phi) = (g_1, g_2, g_3), \quad \text{on} \quad y = 0. \tag{2.5}$$

For the boundary conditions in the x-direction, as in [10] where we studied the channel domain $(0, 1)_x \times \mathbb{T}$ with periodicity in the y-direction, we can take the following nonlinear boundary conditions:

$$(u + 2\sqrt{g\phi}, v) = (\pi_1, \pi_2), \quad \text{on} \quad x = 0, \quad u - 2\sqrt{g\phi} = \pi_3, \quad \text{on} \quad x = 1. \tag{2.6}$$

Here, (g_1, g_2, g_3) and (π_1, π_2, π_3) are given boundary data.

We remark that although the linearized form of the boundary conditions (2.5)–(2.6) may not lead to the L^2-well-posedness of the linearized SWE in the rectangle Ω, these boundary conditions (2.5)–(2.6) will yield local well-posedness of the nonlinear SWE since we consider smooth solutions for the nonlinear problem (see Theorem 3.1).

3 The Fully Nonlinear Shallow Water System

In this section, we aim to investigate the well-posedness for Eq. (1.1) in the rectangular domain $\Omega = (0,1)_x \times (0,1)_y$ associated with initial and boundary conditions. The fully nonlinear shallow water system reads in compact form

$$U_t + \mathcal{E}_1(U)U_x + \mathcal{E}_2(U)U_y + \ell(U) = 0. \tag{3.1}$$

3.1 Stationary Solutions

We want to study system (3.1) near a stationary solution, and we start by constructing such a stationary solution $U = U_s$, that is $(u, v, \phi) = (u_s, v_s, \phi_s)$. These functions are independent of time and satisfy

$$\mathcal{E}_1(U)U_x + \mathcal{E}_2(U)U_y + \ell(U) = 0. \tag{3.2}$$

In the following, we construct a y-independent stationary solution U_s of (3.2) satisfying the subcritical conditions (1.5). Thus U_s satisfies (see Subsec. 2.1 in [7] for a stationary solution in the supercritical case):

$$\begin{cases} uu_x + g\phi_x - fv = 0, \\ uv_x + fu = 0, \\ (u\phi)_x = 0. \end{cases} \tag{3.3}$$

We infer from (3.3) that

$$\begin{cases} u\phi = \kappa_1, \\ v = -fx + \kappa_2, \\ u^2 + 2g\phi = -f^2x^2 + 2f\kappa_2 x + \kappa_3, \end{cases}$$

where $\kappa_1, \kappa_2, \kappa_3$ are constants. We first choose $\kappa_1 = 1$, $\kappa_2 = 2g + f$, and then we have $\phi = u^{-1}$, $v = 2g + f - fx$, and

$$u^2 + \frac{2g}{u} = -f^2x^2 + 2f(2g+f)x + \kappa_3, \quad x \in (0,1). \tag{3.4}$$

We then set $\Psi(u) = u^2 + \frac{2g}{u}$ and $\psi(x) = -f^2x^2 + 2f(2g+f)x + \kappa_3$ and we can easily deduce that

$$\kappa_3 \le \psi(x) \le \kappa_3 + 4gf + f^2, \qquad \forall x \in (0,1).$$

Note that the Coriolis parameter $f \ll 1$ and the gravitational constant $g \approx 9.8$. Hence, $\Psi(\frac{1}{2g}) - \Psi(\frac{1}{g}) > g^2$ and we can choose κ_3 such that

$$\Psi\Big(\frac{1}{2g}\Big) > \kappa_3 + 4gf + f^2 \ge \psi(x) \ge \kappa_3 > \Psi\Big(\frac{1}{g}\Big).$$

Then for any $x \in (0,1)$ one solution (in u) of (3.4) is between $1/2g$ and $1/g$. We choose such a solution u, and therefore $\phi = u^{-1}$ satisfies $g \le \phi \le 2g$, and furthermore $u^2 - g\phi \le 1/g^2 - g^2 < 0$ and by the implicit function theorem, such a solution u is unique and smooth. Since $x \in (0,1)$, we have $v \ge 2g$ and hence $v^2 > g\phi$. All these calculations mean that we can choose the stationary solution u_s, v_s, ϕ_s satisfying the *mixed hyperbolic conditions*

$$u,\ v,\ \phi > 0, \qquad u^2 - g\phi < 0, \qquad v^2 - g\phi > 0. \tag{3.5}$$

Therefore, we choose $\kappa_{0,1}, \kappa_{0,2}, \kappa_{0,3} > 0$ and $\delta > 0$ such that

$$\begin{cases} c_0 \le \kappa_{0,1} \pm c_3\delta < c_1, \quad c_0 \le \kappa_{0,2} \pm c_3\delta < c_1, \quad c_0 \le \kappa_{0,3} \pm c_3\delta < c_1 \\ (\kappa_{0,1} + c_3\delta)^2 - g(\kappa_{0,3} - c_3\delta) \le -c_2^2, \quad (\kappa_{0,2} + c_3\delta)^2 - g(\kappa_{0,3} - c_3\delta) \ge c_2^2, \end{cases} \tag{3.6}$$

where $c_0, c_1, c_2 > 0$ are given positive constants and $c_0 < c_1$, and c_3 is given by Lemma 3.2 below.

Note that the stationary solution we constructed for (3.2) is independent of y as described above, or saying in other way, exists for all $y \in \mathbb{R}_y$. More generally, we assume that a stationary solution $U_s(x, y)$ exists for all $(x, y) \in (0, 1)_x \times \mathbb{R}_y$ and satisfies

$$\mathcal{E}_1(U_s)U_{s,x} + \mathcal{E}_2(U_s)U_{s,y} + \ell(U_s) = 0, \qquad \forall\,(x, y) \in (0, 1)_x \times \mathbb{R}_y. \tag{3.7}$$

The reason why we assume U_s exists for all $y \in \mathbb{R}_y$ instead of $y \in (0, 1)_y$ is that we are going to use the extension method below by extending the problem into the channel domain $(0, 1)_x \times \mathbb{R}_y$ and the assumption that U_s exists for all $y \in \mathbb{R}_y$ will simplify our presentation.

In what follows, we think of the stationary solution U_s in a more general form (i.e. U_s depends on both x and y), and we choose $U_s = (u_s, v_s, \phi_s)$ such that

$$|u_s - \kappa_{0,1}| \le \delta/4, \quad |v_s - \kappa_{0,2}| \le \delta/4, \quad |\phi_s - \kappa_{0,3}| \le \delta/4, \tag{3.8}$$

and by (3.6), U_s satisfies the *mixed hyperbolic condition* (1.5). For convenience, we write

$$|U_s - \boldsymbol{\kappa}_0| \le \delta/4, \qquad \forall\,(x, y) \in (0, 1)_x \times \mathbb{R}_y. \tag{3.9}$$

to stand for (3.8), where $\boldsymbol{\kappa}_0 = (\kappa_{0,1}, \kappa_{0,2}, \kappa_{0,3})$, and the $\kappa_{0,i}$ $(i = 1, 2, 3)$ are positive constants satisfying (3.6).

We set $U = U_s + \widetilde{U}$ and substitute these values into (3.1); we obtain a new system for $\widetilde{U}$, and dropping the tildes, our new system reads:

$$L_{U_s+U}\, U = -L_{U_s+U}\, U_s, \tag{3.10}$$

where the operator L is defined by

$$L_W\, U = U_t + \mathcal{E}_1(W)U_x + \mathcal{E}_2(W)U_y + \ell(U).$$

We supplement (3.10) with the following initial and boundary conditions (see (2.5)–(2.6)):

$$U = U_0(x, y),\ \text{on } t = 0, \quad U = \mathsf{G}(x, t),\ \text{on } y = 0, \quad b(U_s + U) = \Pi(y, t), \tag{3.11}$$

where

$$b(U_s + U) = \begin{cases} u + u_s + 2\sqrt{g(\phi + \phi_s)} = \pi_1(y, t), & \text{on } x = 0, \\ v + v_s = \pi_2(y, t), & \text{on } x = 0, \\ u + u_s - 2\sqrt{g(\phi + \phi_s)} = \pi_3(y, t), & \text{on } x = 1, \end{cases} \quad \Pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}, \quad \mathsf{G} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

We regard the initial condition $U_0 = U_s + \widetilde{U_0}$ as a small perturbation of the stationary solution, and after dropping the tilde, we choose the small perturbation U_0 satisfying

$$|U_0| \le \epsilon\delta, \tag{3.12}$$

for some $0 < \epsilon < 1$ small enough.

3.2 Compatibility Conditions on the Data

In order to be able to solve the system (3.10) we need to introduce some technical conditions (see [3, Sect. 11.1.2]). First we require that $U = 0$ is a solution of the special IBVP (3.10) with zero initial data and boundary data $\Pi(y, t = 0)$ and $\mathsf{G}(x, t = 0)$, which amounts to asking that the following compatibility conditions are satisfied by U_s:

$$b(U_s) = \begin{cases} u_s + 2\sqrt{g\phi_s} = \pi_1(y,0), & \text{on } x = 0, \\ v_s = \pi_2(y,0), & \text{on } x = 0, \\ u_s - 2\sqrt{g\phi_s} = \pi_3(y,0), & \text{on } x = 1, \end{cases} \quad U_s = \mathsf{G}(x,0), \text{ on } y = 0. \tag{3.13}$$

The second condition is that the initial and boundary data should satisfy some additional compatibility conditions and these conditions are very natural for smooth solutions, which we are looking for. Let us first rewrite (3.10) as

$$U_t = H(U + U_s) - \mathcal{E}_1(U + U_s)U_x - \mathcal{E}_2(U + U_s)U_y - \ell(U), \tag{3.14}$$

where we denote by $H(U + U_s)$ the right-hand side of (3.10), that is $-L_{U_s+U}U_s$. Now, if U is continuous, then necessarily at $t = 0$, there should holds

$$b(U_s + U_0) = \Pi(y, 0), \qquad \mathsf{G}(x, 0) = U_0|_{y=0}; \tag{3.15}$$

and if U is $\mathcal{C}^1$ up to the boundary, then at $t = 0$,

$$\begin{aligned} \partial_t \Pi(y,0) &= \mathrm{d}b(U_s + U_0) \cdot \partial_t U(x,0) \\ &= \mathrm{d}b(U_s + U_0) \cdot \big(H(U_0 + U_s) - \mathcal{E}_1(U_0 + U_s)U_{0,x} - \mathcal{E}_2(U_0 + U_s)U_{0,y} - \ell(U_0)\big), \\ \partial_t \mathsf{G}(x,0) &= \partial_t U(x,0) = H(U_0 + U_s) - \mathcal{E}_1(U_0 + U_s)U_{0,x} - \mathcal{E}_2(U_0 + U_s)U_{0,y} - \ell(U_0), \end{aligned}$$

where $\mathrm{d}b(U_s + U)$ is a matrix-valued function, the gradient of the function $b(U_s + U)$ with respect to the variable U. More generally, if U is $\mathcal{C}^{m-1}$ up to the boundary, then at $t = 0$,

$$\begin{aligned} \partial_t^p \Pi(y,0) &= \mathsf{C}_{p,U_0}(V_0, \ldots, V_p), && \forall p \in \{1, \ldots, m-1\}, \\ \partial_t^p \mathsf{G}(x,0) &= V_p\big|_{y=0}, && \forall p \in \{1, \ldots, m-1\}, \end{aligned} \tag{3.16}$$

where the complicated nonlinear function C_{p,U_0} is given by

$$\mathsf{C}_{p,U_0}(V_0, \ldots, V_p) = \sum_{k=1}^{p} \sum_{j_1+\cdots+j_k=p} c_{j_1,\ldots,j_k} \mathrm{d}^k b(U_s + U_0) \cdot (V_{j_1}, \ldots, V_{j_k}),$$

and the functions V_i ($i = 0, \ldots, m$) are defined by induction by (with U being replaced by U_0)

$$\begin{aligned} V_0 &= U, \\ V_1 &= \partial_t U = H(U + U_s) - \mathcal{E}_1(U + U_s)U_x - \mathcal{E}_2(U + U_s)U_y - \ell(U), \end{aligned} \tag{3.17}$$

and for all $i = 1, \ldots, m-1$,

$$\begin{aligned} V_{i+1} = \partial_t^{i+1} U = &\sum_{k=1}^{i} \sum_{j_1+\cdots+j_k=i} c_{j_1,\ldots,j_k} (\mathrm{d}^k H(U+U_s)) \cdot (V_{j_1}, \ldots, V_{j_k}) \\ &- \sum_{l=1}^{i} \binom{i}{l} \sum_{k=1}^{l} \sum_{j_1+\cdots+j_k=l} c_{j_1,\ldots,j_k} (\mathrm{d}^k \mathcal{E}_1(U+U_s)) \cdot (V_{j_1}, \ldots, V_{j_k}) V_{i-l,x} \\ &- \sum_{l=1}^{i} \binom{i}{l} \sum_{k=1}^{l} \sum_{j_1+\cdots+j_k=l} c_{j_1,\ldots,j_k} (\mathrm{d}^k \mathcal{E}_2(U+U_s)) \cdot (V_{j_1}, \ldots, V_{j_k}) V_{i-l,y} \\ &- \mathcal{E}_1(U+U_s) V_{i,x} - \mathcal{E}_2(U+U_s) V_{i,y} - \ell(V_i). \end{aligned} \tag{3.18}$$

Here the coefficients $c_{j_1,\ldots,j_k}$ are derived from the Faà di Bruno's formula, see [4,5]. The conditions (3.15)–(3.16) express the classical compatibility conditions which are necessary for the solution U of (3.10) to be $\mathcal{C}^{m-1}$ near $t = 0$; see e.g. [15,17,19].

We also need to express the compatibility conditions at $y = 0$. For this reason, we rewrite (3.10) as

$$\begin{aligned} U_y &= \mathcal{E}_2(U+U_s)^{-1} \big(H(U+U_s) - \mathcal{E}_1(U+U_s) U_x - U_t - \ell(U) \big) \\ &= \widetilde{H}(U+U_s) - \widetilde{\mathcal{E}}_1(U+U_s) U_x - \mathcal{E}_2(U+U_s)^{-1} (U_t - \ell(U)), \end{aligned} \tag{3.19}$$

where

$$\widetilde{H}(U+U_s) = \mathcal{E}_2(U+U_s)^{-1} H(U+U_s), \qquad \widetilde{\mathcal{E}}_1(U+U_s) = \mathcal{E}_2(U+U_s)^{-1} \mathcal{E}_1(U+U_s).$$

Similar to the definition of the V_i's, we now define the functions W_i ($i = 0, \ldots, m$) by induction by setting (with U being replaced by G)

$$\begin{aligned} &W_0 = U, \\ &W_1 = \partial_y U = \widetilde{H}(U+U_s) - \widetilde{\mathcal{E}}_1(U+U_s) U_x - \mathcal{E}_2(U+U_s)^{-1} U_t - \mathcal{E}_2(U+U_s)^{-1} \ell(U), \end{aligned} \tag{3.20}$$

and for all $i = 1, \ldots, m-1$,

$$\begin{aligned} W_{i+1} = \partial_y^{i+1} U = &\sum_{k=1}^{i} \sum_{j_1+\cdots+j_k=i} c_{j_1,\ldots,j_k} (\mathrm{d}^k \widetilde{H}(U+U_s)) \cdot (W_{j_1}, \ldots, W_{j_k}) \\ &- \sum_{l=1}^{i} \binom{i}{l} \sum_{k=1}^{l} \sum_{j_1+\cdots+j_k=l} c_{j_1,\ldots,j_k} (\mathrm{d}^k \widetilde{\mathcal{E}}_1(U+U_s)) \cdot (W_{j_1}, \ldots, W_{j_k}) W_{i-l,x} \\ &- \sum_{l=1}^{i} \binom{i}{l} \sum_{k=1}^{l} \sum_{j_1+\cdots+j_k=l} c_{j_1,\ldots,j_k} (\mathrm{d}^k \mathcal{E}_2(U+U_s)^{-1}) \cdot (W_{j_1}, \ldots, W_{j_k}) (W_{i-l,t} - \ell(W_{i-l})) \\ &- \widetilde{\mathcal{E}}_1(U+U_s) W_{i,x} - \mathcal{E}_2(U+U_s)^{-1} (W_{i,t} - \ell(W_i)). \end{aligned} \tag{3.21}$$

Now, if U is continuous, then necessarily at $y = 0$, there should holds

$$b(U_s + \mathsf{G}) = \Pi(0,t), \qquad U_0|_{y=0} = \mathsf{G}(x,0). \tag{3.22}$$

More generally, if U is $\mathcal{C}^{m-1}$ up to the boundary, then at $y = 0$,

$$\begin{aligned} &\partial_y^p \Pi(0,t) = \mathsf{C}_{p,\mathsf{G}}(W_0 + U_s, \ldots, W_p + \partial_y^p U_s), \qquad \forall p \in \{1, \ldots, m-1\}, \\ &\partial_y^p U_0|_{y=0} = W_p|_{t=0}, \end{aligned} \tag{3.23}$$

where $\mathsf{C}_{p,\mathsf{G}}$ is defined in the same fashion as C_{p,U_0}, and the reason why we have the additional term $\partial_y^p U_s$ in $\mathsf{C}_{p,\mathsf{G}}$ is because U_s is independent of t but generally depends on y. The conditions (3.22)–(3.23) express the classical compatibility conditions which are necessary for the solution U of (3.10) to be $\mathcal{C}^{m-1}$ near $y = 0$.

We remark that the compatibility conditions between the boundary data G at $y = 0$ and the initial data U_0 which are expressed either near $t = 0$ or near $y = 0$ are equivalent.

3.3 Approximate Solutions

The disadvantage of this new formulation (3.10)–(3.11) is that the boundary conditions in both the x-and y-directions and the initial condition U_0 are generally non-zero. To overcome these difficulties, we will use two approximate solutions lifting the boundary data G at $y = 0$ and the initial data U_0 at $x = 0$. The approximate lifting solutions U_{g} of the boundary data G is given by the following lemma.

Lemma 3.1 *We are given $m \geq 3$, the stationary solution $U_s \in H^{m+1}(\Omega)$, the initial data $U_0 = (u_0, v_0, \phi_0)$ belonging to $H^{m+1/2}(\Omega)$, the boundary data $\mathsf{G} = (g_1, g_2, g_3)$ belonging to $H^{m+1/2}((0,1)_x \times (0,T))$ and $\Pi = (\pi_1, \pi_2, \pi_3)$ belonging to $H^{m+1/2}((0,1)_y \times (0,T))$. Then there exists a function $U_{\mathsf{g}} \in H^{m+1}(\Omega \times (0,T))$ such that $U_{\mathsf{g}}|_{y=0} = \mathsf{G}$, and*

$$\|U_{\mathsf{g}}\|_{H^{m+1}(\Omega\times(0,T))} \leq C \|\mathsf{G}\|_{H^{m+1/2}((0,1)_x\times(0,T))} \|U_s\|_{H^{m+1}(\Omega\times(0,T))}, \tag{3.24}$$

for some constant $C > 0$ depending on m and Ω, independent of G and U_s.

If we let $U_0^0 = U_0 - U_{\mathsf{g}}|_{t=0}$, $\Pi^0 = -b(U_{\mathsf{g}} + U_s) + \Pi$, and

$$\widetilde{F}^0 = -\partial_y U_{\mathsf{g}} + \mathcal{E}_2(U_{\mathsf{g}} + U_s)^{-1}\big(H(U_{\mathsf{g}} + U_s) - \mathcal{E}_1(U_{\mathsf{g}} + U_s)U_{\mathsf{g},x} - U_{\mathsf{g},t} - \ell(U_{\mathsf{g}})\big),$$

then $U_0^0 \in H^{m+1/2}(\Omega)$, $\widetilde{F}^0 \in H^m(\Omega \times (0,T))$, $\Pi^0 \in H^{m+1/2}((0,1)_y \times (0,T))$, and

$$\partial_y^j \widetilde{F}^0 = 0, \quad \partial_y^j U_0^0 = 0, \quad \partial_y^j \Pi^0 = 0, \text{ on } y = 0, \qquad \forall j \in \{0, \dots, m-1\}. \tag{3.25}$$

Proof Similar to [3, Lemma 11.1], we can construct $\{W_i = W_i|_{y=0}\}_{i=0,\dots,m-1}$ with $U|_{y=0} = \mathsf{G}$ satisfying (3.20)–(3.21) and

$$W_i \in H^{m+1/2-i}((0,1)_x \times (0,T)), \qquad \forall i \in \{0, \dots, m-1\}.$$

Then, by the lifting result in Proposition 5.3, we find $U_{\mathsf{g}} \in H^{m+1}(\Omega \times (0,T))$ such that

$$(\partial_y^i U_{\mathsf{g}})|_{y=0} = W_i, \qquad \forall j \in \{0, \dots, m-1\},$$

and by the classical inequalities for the Sobolev spaces (see for example [18, Chapter 13], [3, Appendix C], or [10, Lemma B.1]):

$$\begin{aligned} \|U_{\mathsf{g}}\|_{H^{m+1}(\Omega\times(0,T))} &\lesssim \sum_{i=0}^{m} \|W_i\|_{H^{m+1/2-i}((0,1)_x\times(0,T))} \\ &\lesssim \|\mathsf{G}\|_{H^{m+1/2}((0,1)_x\times(0,T))} \|U_s\|_{H^{m+1}(\Omega\times(0,T))}, \end{aligned} \tag{3.26}$$

where $\lesssim$ means $\leq$ up to a multiplicative absolute constant C.

That $\widetilde{F}^0$ is in H^m follows in a classical way from the inequalities in [10, Lemma B.1] (see also [18, Chapter 13] and [3, Appendix C]) and that U_0^0 and Π^0 are in $H^{m+1/2}$ follows from the trace theorem (see Proposition 5.3). The vanishing of $\partial_y^j \widetilde{F}^0$ at $y = 0$ follows from the construction of U_{g} and the vanishing of $\partial_y^j \Pi^0$ and U_0^0 at $y = 0$ are consequences of the compatibility conditions in (3.22)–(3.23). □

If we let $F^0 = \mathcal{E}_2(U_{\mathsf{g}} + U_s)\widetilde{F}^0$, then $F^0 = -L_{U_{\mathsf{g}}+U_s}(U_{\mathsf{g}} + U_s)$ and F^0 has the same properties as $\widetilde{F}^0$, that is

$$F^0 \in H^m(\Omega \times (0,T)), \qquad \partial_y^j F^0 = 0, \text{ on } y = 0, \qquad \forall\, j \in \{0,\ldots,m-1\}. \tag{3.27}$$

We recall that we have set $U = U_s + \widetilde{U}$ and that we have dropped the tilde in the above. Now let us reintroduce the tilde and set $\widetilde{U} = U_{\mathsf{g}} + \overline{U}$, so that $U = U_s + U_{\mathsf{g}} + \overline{U}$. We then substitute this expression into the system (3.10) and in (3.12) (where $U_0 = \widetilde{U_0}$) and drop the bars. Then the new system for $U = \overline{U}$ becomes the following initial and boundary value problem (IBVP):

$$\begin{cases} L_{U_{\mathsf{g}}+U_s+U} U = -L_{U_{\mathsf{g}}+U_s+U}(U_{\mathsf{g}} + U_s), \\ U|_{t=0} = U_0 - U_{\mathsf{g}}, \\ U|_{y=0} = 0, \\ b(U_{\mathsf{g}} + U_s + U) = \Pi. \end{cases} \tag{3.28}$$

If U is a solution of the IBVP (3.28), then by use of the induction method we can show that

$$\partial_y^j U|_{y=0} = 0, \quad \forall\, j \in \{0,\ldots,m-1\}. \tag{3.29}$$

Indeed, applying ∂_y^j to the equation (3.19) with U_s replaced by $U_s + U_{\mathsf{g}}$ and $H(U + U_s)$ replaced by $-L_{U_{\mathsf{g}}+U_s+U}(U_{\mathsf{g}} + U_s)$, we find

$$\begin{aligned} \partial_y^{j+1} U = &\sum_{k=0}^{j} \partial_y^{j-k} (\mathcal{E}_2(U + U_s + U_{\mathsf{g}})^{-1} \big[\partial_y^k(-L_{U_{\mathsf{g}}+U_s+U}(U_{\mathsf{g}} + U_s)) - \partial_y^k U_t - \partial_y^k \ell(U)\big] \\ &+ \sum_{k=0}^{j} \partial_y^{j-k} (\mathcal{E}_2(U + U_s + U_{\mathsf{g}})^{-1} \mathcal{E}_1(U + U_s + U_{\mathsf{g}}))\partial_y^k U_x. \end{aligned} \tag{3.30}$$

The result (3.29) then follows from the above identity and $U|_{y=0} = 0$. The vanishing property (3.29) points to the fact that we may extend the system to the smooth domain $(0,1)_x \times \mathbb{R}_y$, which we will now do. We point out that the assumption that the 2d nonlinear SWE is supercritical in the direction $(0, 1)$ enables us to prescribe all the boundary conditions at $y = 0$ in the y-direction and hence by the lifting Lemma 3.1, the boundary conditions at $y = 0$ are lifted to 0, which yields the vanishing property (3.29) at $y = 0$.

3.4 The Extension Problem

We now aim to extend the problem (3.28) to the channel (smooth) domain $\mathcal{Q} := (0,1)_x \times \mathbb{R}_y$ and for this reason, we need the following extension result.

Lemma 3.2 (Extension Theorem) *There exists a continuous linear operator* $P = P_m$ *from* $H^m(\Omega \times [0,T])$ *into* $H^m(\mathcal{Q} \times [0,T])$ *such that for all* $u \in H^m(\Omega \times [0,T])$, *the restriction of* Pu *to* $\Omega \times [0,T]$ *is* u *itself, i.e.*

$$Pu|_{\Omega\times[0,T]} = u,$$

and furthermore Pu *has compact support in the* y*-direction (i.e. in* $\mathcal{Q} \times [0,T]$*) and satisfies the estimate*

$$\|Pu\|_{L^\infty(\mathcal{Q}\times[0,T])} \le c_3 \|u\|_{L^\infty(\Omega\times[0,T])}, \quad \|Pu\|_{H^m(\mathcal{Q}\times[0,T])} \le c_4 \|u\|_{H^m(\Omega\times[0,T])}.$$

where $c_3 > 1, c_4 > 1$ *only depend on* m*, and are independent of* u.

Furthermore, we have the following vanishing properties:

(*1*) *if u vanishes on $x = 0$ (resp. $x = 1$), then Pu also vanishes on $x = 0$ (resp. $x = 1$);*
(2) *if $\partial_t^p u$ vanishes on $t = 0$ for all $p = 0, \ldots, m-1$, then $\partial_t^p Pu$ also vanishes on $t = 0$ for all $p = 0, \ldots, m-1$.*

See [12, Chapter 2] for a detailed proof of Lemma 3.2, and using the Babitch extension procedure, the L^∞-estimate and the vanishing properties come from the reflection formula (4.8) in [12, Chapter 2]. See also [1,6].

We now describe how to extend the initial data U_0 and the boundary data Π to make sure that the extended data still satisfy the compatibility condition stated in Sect. 3.2. We first extend the initial data U_0 and we find from (3.13) and (3.15) that

$$b(U_s) = \Pi(y, 0) = b(U_s + U_0), \quad \text{on} \quad t = 0;$$

specifically, we have

$$\begin{cases} u_s + 2\sqrt{g\phi_s} = \pi_1(y, 0) = u_s + u_0 + 2\sqrt{g(\phi_s + \phi_0)}, & \text{on } x = 0, \\ v_s = \pi_2(y, 0) = v_s + v_0, & \text{on } x = 0, \\ u_s - 2\sqrt{g\phi_s} = \pi_3(y, 0) = u_s + u_0 - 2\sqrt{g(\phi_s + \phi_0)}, & \text{on } x = 1, \end{cases} \tag{3.31}$$

which is equivalent to

$$\begin{cases} u_0 + 2\sqrt{g(\phi_s + \phi_0)} - 2\sqrt{g\phi_s} = 0, & \text{on } x = 0, \\ v_0 = 0, & \text{on } x = 0, \\ u_0 - 2\sqrt{g(\phi_s + \phi_0)} + 2\sqrt{g\phi_s} = 0, & \text{on } x = 1. \end{cases} \tag{3.32}$$

We now set

$$\begin{cases} \xi = u_0 + 2\sqrt{g(\phi_s + \phi_0)} - 2\sqrt{g\phi_s}, \\ \eta = v_0, \\ \zeta = u_0 - 2\sqrt{g(\phi_s + \phi_0)} + 2\sqrt{g\phi_s}, \end{cases}$$

and we have for all $y \in (0, 1)$:

$$\xi = \eta = 0, \text{ on } x = 0, \qquad \zeta \text{ on } x = 1.$$

Using Lemma 3.2, we can extend (ξ, η, ζ) to $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ in the channel domain $\mathcal{Q}$ such that for all $y \in \mathbb{R}$:

$$\hat{\xi} = \hat{\eta} = 0, \text{ on } x = 0, \qquad \hat{\zeta} \text{ on } x = 1.$$

Note that the stationary solution U_s exists in the channel domain $(0, 1)_x \times \mathbb{R}_y$, then the extended initial data $\widehat{U}_0 = (\hat{u}_0, \hat{v}_0, \hat{\phi}_0)$ are now given by the following equations

$$\begin{cases} \hat{u}_0 + 2\sqrt{g(\phi_s + \hat{\phi}_0)} - 2\sqrt{g\phi_s} = \hat{\xi}, \\ \hat{v}_0 = \hat{\eta}, \\ \hat{u}_0 - 2\sqrt{g(\phi_s + \hat{\phi}_0)} + 2\sqrt{g\phi_s} = \hat{\zeta}. \end{cases}$$

We remark that as long as U_0 is small enough in the sense of the L^∞-norm, then (ξ, η, ζ) will also be small enough and hence the extended data $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ and the extended initial data $\widehat{U}_0$.

We are now going to describe how to extend the boundary data Π. We first construct $\{\widehat{V}_i\}|_{i=0,\dots,m-1}$ with $U=\widehat{U}_0$ by (3.17) and (3.18). We now set

$$\Xi(y,t)=\Pi(y,t)-b(\widehat{U}_0+U_s)-\sum_{p=1}^{m-1}\frac{t^p}{p!}\mathsf{C}_{p,\widehat{U}_0}(\widehat{V}_0,\dots,\widehat{V}_p),\qquad \forall\, y\in(0,1),$$

and we find from the compatibility conditions (3.15) and (3.16) that

$$\partial_t^p\,\Xi(y,t=0)=0,\qquad \forall\, p=0,\cdots,m-1,\quad \forall\, y\in(0,1).$$

Using Lemma 3.2, we can extend Ξ to $\widehat{\Xi}$ in the domain $\mathbb{R}_y\times(0,T)$ such that

$$\partial_t^p\,\widehat{\Xi}(y,t=0)=0,\quad \forall\, p=0,\dots,m-1,\quad \forall\, y\in\mathbb{R}_y.$$

The extended boundary data $\widehat{\Pi}$ are now given by

$$\widehat{\Pi}(y,t)=\widehat{\Xi}(y,t)+b(\widehat{U}_0+U_s)+\sum_{p=1}^{m-1}\frac{t^p}{p!}\mathsf{C}_{p,\widehat{U}_0}(\widehat{V}_0,\dots,\widehat{V}_p),\qquad \forall\, y\in\mathbb{R}_y.$$

By the construction of $\widehat{U}_0$ and $\widehat{\Pi}$, we can see that $\widehat{U}_0$ and $\widehat{\Pi}$ satisfy the compatibility conditions (3.15) and (3.16) for all $y\in\mathbb{R}_y$.

Finally, we also use Lemma 3.2 to extend U_{g} to $\widehat{U_{\mathsf{g}}}$ in the domain $\mathcal{Q}\times(0,T)$.

Now, to solve the IBVP (3.28), we first consider the following extension problem, that is we look for a solution $\widehat{U}$ satisfying

$$\begin{cases} L_{\widehat{U_{\mathsf{g}}}+U_s+\widehat{U}}\widehat{U}=-L_{\widehat{U_{\mathsf{g}}}+U_s+\widehat{U}}(\widehat{U_{\mathsf{g}}}+U_s), & (x,y)\in\mathcal{Q}=(0,1)_x\times\mathbb{R}_y,\\ \widehat{U}|_{t=0}=\widehat{U_0}-\widehat{U_{\mathsf{g}}}, \\ b(\widehat{U_{\mathsf{g}}}+U_s+\widehat{U})=\widehat{\Pi}. \end{cases}\tag{3.33}$$

We are going to apply [3, Theorem 11.1] to the IBVP (3.33) and the use of [3] is legitimate since the domain $(0,1)_x\times\mathbb{R}_y$ is smooth. In order to exactly fit the statements in [3, Theorem 11.1], we search for a solution $V:=\widehat{U_{\mathsf{g}}}+\widehat{U}$ satisfying

$$\begin{cases} L_{V+U_s}V=-L_{V+U_s}U_s, & (x,y)\in\mathcal{Q}=(0,1)_x\times\mathbb{R}_y,\\ V|_{t=0}=\widehat{U_0}, \\ b(V+U_s)=\widehat{\Pi}. \end{cases}\tag{3.34}$$

The corresponding functions h, b, and $\underline{b}$ in [3, Theorem 11.1] are as follows:

$$h(V):=-\ell(V)-L_{V+U_s}U_s,\qquad b(V):=b(V+U_s),\qquad \underline{b}=\widehat{\Pi},\tag{3.35}$$

and we choose $\mathscr{U}$ to be the open ball in the space $H^m(\mathcal{Q}\times(0,T))$ with radius $\delta/(4\nu_m)$, where ν_m denotes the norm of the Sobolev embedding $H^m(\mathcal{Q}\times(0,T))\hookrightarrow L^\infty(\mathcal{Q}\times(0,T))$. Hence, if $V\in\mathscr{U}$, then

$$\|V\|_{L^\infty(\mathcal{Q}\times(0,T))}\le\delta/4,$$

and then U_s+V satisfies the *mixed hyperbolic condition* (1.5). Therefore, it is not hard to verify that the conditions (**CH**), (**T**), (**NC**$_b$), (**N**$_b$), (**UKL**$_b$) in [3, pp. 317–319] hold in this context. The compatibility conditions hold true for the extended data $\widehat{U_0}$ and $\widehat{\Pi}$ by our construction. We apply [3, Theorem 11.1] with $\Omega=\mathcal{Q}$, and the condition $h(0)=0$ follows from (3.7) and the condition $b(0)\equiv\underline{b}(\cdot,t=0)$ follows from (3.13) and the construction of $\widehat{\Pi}$; we arrive at the local well-posedness of the system (3.33) if the initial data $\widehat{U_0} \in \mathscr{U}$ belongs to $H^{m+1/2}(\mathcal{Q})$.

Once we have a unique solution V for the system (3.34) and hence a unique solution $\widehat{U} = V - \widehat{U_{\mathsf{g}}}$ for the extension IBVP (3.33), we now set $U = \widehat{U}|_{\Omega\times(0,T)}$. Then U satisfies the IBVP (3.28) except that we need to verify that $U|_{y=0} = 0$. In order to show $U|_{y=0} = 0$, restricting the extension IBVP (3.33) to $y = 0$ yields

$$\begin{cases} L_{\widehat{U_{\mathsf{g}}}+U_s+\widehat{U}|_{y=0}}\widehat{U}|_{y=0} = -L_{\widehat{U_{\mathsf{g}}}+U_s+\widehat{U}|_{y=0}}(\widehat{U_{\mathsf{g}}} + U_s), & \text{on } y = 0, \\ \widehat{U}|_{t=0,y=0} = (\widehat{U_0} - \widehat{U_{\mathsf{g}}})|_{y=0}, \\ b(\widehat{U_{\mathsf{g}}} + U_s + \widehat{U}|_{y=0}) = \widehat{\Pi}, & \text{on } y = 0. \end{cases} \tag{3.36}$$

We observe from (3.22) and (3.27) that $V = \widehat{U_{\mathsf{g}}}|_{y=0} = \mathsf{G}$ is a solution of the following system

$$\begin{cases} L_{V+U_s|_{y=0}}V = -L_{V+U_s|_{y=0}}U_s|_{y=0}, & x \in (0,1)_x, \\ V|_{t=0} = \widehat{U_0}|_{y=0} = U_0|_{y=0}, \\ b(V + U_s|_{y=0}) = \widehat{\Pi}|_{y=0} = \Pi|_{y=0}, \end{cases} \tag{3.37}$$

and the uniqueness result in [3, Theorem 11.1] implies that $V = \widehat{U_{\mathsf{g}}}|_{y=0}$ is the unique solution of (3.37), and hence we can conclude from (3.36) that $\widehat{U}|_{y=0} = 0$. Therefore, $U = \widehat{U}|_{\Omega\times(0,T)}$ satisfies the IBVP (3.28) and consequently, the system (3.10)–(3.11) admits a solution $U_{\mathsf{g}} + U$.

3.5 The Main Result

We now conclude by stating the main result proved in the previous subsections.

Theorem 3.1 *We are given a rectangular domain $\Omega = (0,1)_x \times (0,1)_y$, a real number $T > 0$, an integer $m \geq 3$*[1]*, the stationary solution $U_s \in H^{m+1}(\Omega)$ satisfying (3.8) (i.e. the* mixed hyperbolic condition *(1.5)), the initial data $U_0 = (u_0, v_0, \phi_0)$ belonging to $H^{m+1/2}(\Omega)$, the boundary data $\mathsf{G} = (g_1, g_2, g_3)$ belonging to $H^{m+1/2}((0,1)_x \times (0,T))$ and $\Pi = (\pi_1, \pi_2, \pi_3)$ belonging to $H^{m+1/2}((0,1)_y \times (0,T))$. We assume the condition (3.13) and the compatibility conditions (3.15)–(3.16) and (3.22)–(3.23), which are necessary to obtain a smooth solution in $H^m(\Omega \times (0,T))$. We also assume that the initial data U_0 is small enough in the space $H^m(\Omega)$ so that the extended function $\widehat{U_0}$ in Sect. 3.4 belongs to the ball $\mathscr{U}$. Then there exists $T^* > 0$ ($T^* \leq T$) such that the system (3.10)–(3.11) admits a unique solution $U \in H^m(\Omega \times (0,T^*))$.*

Proof The existence part is already proved in the previous subsections by considering the extension problem. We are now going to prove the uniqueness part. Suppose there are two solutions U_1 and U_2 belonging to $H^m(\Omega \times (0,T^*))$ that satisfy the system (3.10)–(3.11), and set $W = U_1 - U_2$. Then W satisfies

[1] The assumption $m \geq 3$ allows us to control the L^∞-norm of ∇U by the Sobolev embedding theorem, see also [3, Theorem 11.1].

$$\begin{cases} L_{U_s+U_1}W=(L_{U_s+U_2}-L_{U_s+U_1})(U_2+U_s),\\ W=0, \quad \text{on } t=0,\\ W=0, \quad \text{on } y=0,\\ \mathrm{d}b(U_s+U_1)\cdot W=b(U_s+U_1)-b(U_s+U_2)-\mathrm{d}b(U_s+U_1)\cdot W, \quad \text{on } x=0,1. \end{cases} \tag{3.38}$$

In order to obtain the L^2-estimate for W, as in [10], we set $S_0=\mathrm{diag}(1,1,g/(\phi_s+\phi_1)$, which is positive-definite; we denote by $\langle\cdot,\cdot\rangle$ the L^2-inner product in $L^2(\Omega)$. Multiplying $(3.38)_1$ with S_0 and taking the inner product in $L^2(\Omega)$ with W, we obtain that

$$\begin{aligned} &\langle S_0W_t,W\rangle+\langle S_0\mathcal{E}_1(U_s+U_1)W_x,W\rangle+\langle S_0\mathcal{E}_2(U_s+U_1)W_y,W\rangle+\langle S_0\ell(W),W\rangle\\ &\quad=\langle(L_{U_s+U_2}-L_{U_s+U_1})(U_2+U_s),W\rangle. \end{aligned} \tag{3.39}$$

We are now going to estimate the terms in (3.39). Direct calculation and integration by parts give

$$\begin{aligned} &\langle S_0\ell(W),W\rangle=0,\\ &\langle S_0W_t,W\rangle=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle S_0W,W\rangle-\frac{1}{2}\langle(S_0)_tW,W\rangle. \end{aligned} \tag{3.40}$$

Note that U_1 satisfies the mixed hyperbolic condition (1.5), hence, the matrix $S_0\mathcal{E}_2(U_s+U_1)$ is positive definite. Using the boundary conditions at $y=0$ and integrating by parts yield

$$\begin{aligned} \langle S_0\mathcal{E}_2(U_s+U_1)W_y,W\rangle&=\frac{1}{2}\langle S_0\mathcal{E}_2(U_s+U_1)W,W\rangle_{L^2((0,1)_x)}\big|_{y=1}\\ &\quad-\frac{1}{2}\langle(S_0\mathcal{E}_2(U_s+U_1))_yW,W\rangle\\ &\ge-\frac{1}{2}\langle(S_0\mathcal{E}_2(U_s+U_1))_yW,W\rangle; \end{aligned} \tag{3.41}$$

integrating by parts also yields

$$\begin{aligned} \langle S_0\mathcal{E}_1(U_s+U_1)W_x,W\rangle&=\frac{1}{2}\langle S_0\mathcal{E}_1(U_s+U_1)W,W\rangle_{L^2((0,1)_y)}\big|_{x=0}^{x=1}\\ &\quad-\frac{1}{2}\langle(S_0\mathcal{E}_1(U_s+U_1))_xW,W\rangle. \end{aligned} \tag{3.42}$$

We now recall relation (7) in [10, Sect. 2.1], which will be useful for handling the boundary terms at $x=0$ and $x=1$ in (3.42). The relation (7) in [10, Sect. 2.1] can be restated as the following: there exists $\epsilon_0>0$ and $C_0>0$ such that

$$\big(S_0\mathcal{E}_1(U_s+U_1)W,W\big)\big|_{x=0}^{x=1}\ge\epsilon_0|W|^2-C_0|\mathrm{d}b(U_s+U_1)W|^2,\ \forall\,W\in\mathbb{R}^3, \tag{3.43}$$

where $(\cdot,\cdot)$ (resp. $|\cdot|$) denotes the standard inner product (resp. norm) on $\mathbb{R}^3$. Taking (3.43) into account, we deduce from (3.42) that

$$\begin{aligned} \langle S_0\mathcal{E}_1(U_s+U_1)W_x,W\rangle&\ge\epsilon_0\|W\|^2_{L^2((0,1)_y)}-C_0\|\mathrm{d}b(U_s+U_1)W\|^2_{L^2((0,1)_y)}\\ &\quad-\frac{1}{2}\langle(S_0\mathcal{E}_1(U_s+U_1))_xW,W\rangle, \end{aligned} \tag{3.44}$$

where $\epsilon_0, C_0>0$ only depend on c_0, c_1, c_2, g.

Combining these estimates, we first obtain from (3.39) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle S_0 W, W\rangle + \epsilon_0\|W\|^2_{L^2((0,1)_y)}\\
&\quad\le \frac{1}{2}\langle (S_0)_t W, W\rangle + \frac{1}{2}\langle (S_0\mathcal{E}_2(U_s+U_1))_y W, W\rangle\\
&\qquad + \frac{1}{2}\langle (S_0\mathcal{E}_1(U_s+U_1))_x W, W\rangle + \langle (L_{U_s+U_2} - L_{U_s+U_1})(U_2+U_s), W\rangle\\
&\qquad + C_0\|\mathrm{d}b(U_s+U_1)W\|^2_{L^2((0,1)_y)}.
\end{aligned} \tag{3.45}$$

As a preliminary, since ϕ_1 satisfies the mixed hyperbolic condition (1.5), we first have

$$\langle W, W\rangle = \langle S_0^{-1}S_0 W, W\rangle \le \|S_0^{-1}\|_{L^\infty}\langle S_0 W, W\rangle \le \max(1, g/c_0)\langle S_0 W, W\rangle.$$

We now estimate the right-hand side of (3.45) term by term.

$$\begin{aligned}
\langle (S_0)_t W, W\rangle &= \langle (S_0)_t W, W\rangle \le \|(S_0)_t\|_{L^\infty}\langle W, W\rangle\\
&\le C(\|(U_s+U_1)_t\|_{L^\infty})\langle W, W\rangle \le C(\|U_s\|_{H^3}, \|U_1\|_{H^3})\langle S_0 W, W\rangle,
\end{aligned} \tag{3.46}$$

where we have used the Sobolev embedding theorem and similarly, we have

$$\begin{aligned}
\langle (S_0\mathcal{E}_2(U_s+U_1))_y W, W\rangle \le C(\|U_s\|_{H^3}, \|U_1\|_{H^3})\langle S_0 W, W\rangle,\\
\langle (S_0\mathcal{E}_1(U_s+U_1))_x W, W\rangle \le C(\|U_s\|_{H^3}, \|U_1\|_{H^3})\langle S_0 W, W\rangle,
\end{aligned} \tag{3.47}$$

where $C(\cdots) > 0$ only depends on the parameters in its parenthesis and may vary from line to line. By the mean-value theorem, we obtain $c' > 0$, depending only on the L^∞-norms of $U_s + U_1$ and $U_s + U_2$ such that

$$\begin{aligned}
&\langle (L_{U_s+U_2} - L_{U_s+U_1})(U_2+U_s), W\rangle\\
&\quad= \langle (\mathcal{E}_1(U_s+U_2) - \mathcal{E}_1(U_s+U_1))(U_2+U_s)_x, W\rangle\\
&\qquad+ \langle (\mathcal{E}_2(U_s+U_2) - \mathcal{E}_2(U_s+U_1))(U_2+U_s)_y, W\rangle\\
&\quad\le c'(\|(U_2+U_s)_x\|_{L^\infty} + \|(U_2+U_s)_y\|_{L^\infty})\langle W, W\rangle\\
&\quad\le C(\|U_s\|_{H^3}, \|U_1\|_{H^3}, \|U_2\|_{H^3})\langle S_0 W, W\rangle.
\end{aligned} \tag{3.48}$$

By the second-order Taylor expansion of b, we obtain that

$$\begin{aligned}
\|\mathrm{d}b(U_s+U_1)W\|^2_{L^2((0,1)_y)} &= \|b(U_s+U_1) - b(U_s+U_2) - \mathrm{d}b(U_s+U_1)\cdot W\|^2_{L^2((0,1)_y)}\\
&\le C(\|U_s\|_{L^\infty}, \|U_1\|_{L^\infty}, \|U_2\|_{L^\infty})\|W^2\|^2_{L^2((0,1)_y)}\\
&\le C(\|U_s\|_{H^3}, \|U_1\|_{H^3}, \|U_2\|_{H^3})\|W\|^2_{L^\infty((0,1)_y)}\|W\|^2_{L^2((0,1)_y)};
\end{aligned} \tag{3.49}$$

Using the Cauchy-Schwarz inequality and noting that $W = 0$ at $t = 0$, we find

$$\|W\|_{L^\infty((0,1)_y)} \le \|W\|^2_{L^\infty} \le T^2\|W_t\|^2 \le T^2(\|U_1\|_{H^1} + \|U_2\|_{H^1})^2.$$

Therefore, upon diminishing T again such that

$$T^2 C(\|U_s\|_{H^3}, \|U_1\|_{H^3}, \|U_2\|_{H^3})(\|U_1\|_{H^1} + \|U_2\|_{H^1})^2 \le \epsilon_0,$$

the boundary term in the right-hand side of (3.45) is less than the left-hand side of (3.45), and hence we can derive from (3.45) the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S_0 W, W\rangle \le C(\|U_s\|_{H^3}, \|U_1\|_{H^3}, \|U_2\|_{H^3})\langle S_0 W, W\rangle.$$

Since the solutions U_1, U_2 and the stationary solution U_s belong to H^3, and together with the initial condition $W = 0$ at $t = 0$, the Gronwall lemma implies that $\langle S_0 W, W\rangle \equiv 0$ and hence $W \equiv 0$. We thus completed the proof of Theorem 3.1. □

3.6 An Example of the Compatibility Conditions

Since the compatibility conditions stated in Sect. 3.2 are very dense and technical, we now aim to present those compatibility conditions explicitly in the (least) case when $m = 3$[2].

Recall that $H(U+U_s)$, $\widetilde{H}(U+U_s)$, and $\widetilde{\mathcal{E}}_1(U+U_s)$ are already defined in Sect. 3.2. The compatibility conditions at $t = 0$ are (3.15) and (3.16), which can be written explicitly as

$$\begin{aligned}
\Pi(y,0) &= b(U_s+U_0), & \mathsf{G}(x,0) &= U_0|_{y=0},\\
\partial_t\Pi(y,0) &= \mathrm{d}b(U_s+U_0)\cdot V_1, & \partial_t\mathsf{G}(x,0) &= V_1|_{y=0},\\
\partial_{tt}\Pi(y,0) &= (\mathrm{d}^2b(U_s+U_0)\cdot V_1)V_1 + \mathrm{d}b(U_s+U_0)\cdot V_2, & \partial_{tt}\mathsf{G}(x,0) &= V_2|_{y=0},
\end{aligned} \tag{3.50}$$

where V_1 and V_2 are defined by

$$\begin{aligned}
V_1 &= H(U_0+U_s) - \mathcal{E}_1(U_0+U_s)U_{0,x} - \mathcal{E}_2(U_0+U_s)U_{0,y} - \ell(U_0),\\
V_2 &= \mathrm{d}H(U_0+U_s)\cdot V_1 - (\mathrm{d}\mathcal{E}_1(U_0+U_s)\cdot V_1)U_{0,x} - (\mathrm{d}\mathcal{E}_2(U_0+U_s)\cdot V_1)U_{0,y}\\
&\quad - \mathcal{E}_1(U_0+U_s)V_{1,x} - \mathcal{E}_2(U_0+U_s)V_{1,y} - \ell(V_1).
\end{aligned}$$

The compatibility conditions at $y = 0$ are (3.22) and (3.23) and can also be written explicitly as

$$\begin{aligned}
\Pi(0,t) &= b(U_s+\mathsf{G}), \quad U_0|_{y=0} = \mathsf{G}|_{t=0},\\
\partial_y\Pi(0,t) &= \mathrm{d}b(U_s+\mathsf{G})\cdot(U_{s,y}+W_1), \quad \partial_y U_0|_{y=0} = W_1|_{t=0},\\
\partial_{yy}\Pi(0,t) &= (\mathrm{d}^2b(U_s+\mathsf{G})\cdot(U_{s,y}+W_1))(U_{s,y}+W_1)\\
&\quad + \mathrm{d}b(U_s+\mathsf{G})\cdot(U_{s,yy}+W_2), \quad \partial_{yy}U_0|_{y=0} = W_2|_{t=0},
\end{aligned} \tag{3.51}$$

where W_1 and W_2 are defined by

$$\begin{aligned}
W_1 &= \widetilde{H}(\mathsf{G}+U_s) - \widetilde{\mathcal{E}}_1(\mathsf{G}+U_s)\mathsf{G}_x - \mathcal{E}_2(\mathsf{G}+U_s)^{-1}(\mathsf{G}_t+\ell(\mathsf{G})),\\
W_2 &= \mathrm{d}\widetilde{H}(\mathsf{G}+U_s)\cdot(W_1+U_{s,y}) - (\mathrm{d}\widetilde{\mathcal{E}}_1(\mathsf{G}+U_s)\cdot(W_1+U_{s,y}))\mathsf{G}_x\\
&\quad - (\mathrm{d}(\mathcal{E}_2(\mathsf{G}+U_s)^{-1})\cdot(W_1+U_{s,y}))(\mathsf{G}_t+\ell(\mathsf{G}))\\
&\quad - \widetilde{\mathcal{E}}_1(\mathsf{G}+U_s)W_{1,x} - \mathcal{E}_2(\mathsf{G}+U_s)^{-1}(W_{1,t}+\ell(W_1)).
\end{aligned}$$

Note that written component by component, the first compatibility condition $(3.50)_1$ is equivalent to (3.31).

4 An Invariance Property for the Shallow Water Equations and Application

In this section, the goal is to show that we are able to solve the IBVP for the 2d inviscid SWE under the *fully hyperbolic condition* (1.3) for a more general orientation of the rectangular domain Ω as long as we choose the rectangular domain properly. In order to achieve this goal, we first prove an *invariance* property for the fluid equations (in particular for the 2d inviscid

[2] Recall that we require $m \geq 3$ in Theorem 3.1.

SWE) and then show how to choose the domain. The results of Sect. 4.1 are essentially well-known but necessary to classify the notations.

4.1 An Invariance Property for the Fluid Equations

The partial differential equations arising from geophysical fluid dynamics are generally derived from physical laws, in particular, the conservation of mass and conservation of momentum. A basic principle in physics is that the physical laws should be independent of the reference frame chosen. Hence, we expect that the fluid equations are also independent of the coordinate system chosen and we call that the *invariance* property for the fluid equations. We first prove the *invariance* property for the 2d inviscid SWE and then extend it to more general fluid equations. We recall the 2d inviscid SWE (1.1) as

$$\begin{cases} u_t + uu_x + vu_y + g\phi_x - fv = 0, \\ v_t + uv_x + vv_y + g\phi_y + fu = 0. \\ \phi_t + u\phi_x + v\phi_y + \phi(u_x + v_y) = 0. \end{cases}$$

For the purpose of unifying notations in this section, we set $\boldsymbol{u} = (u, v)^t$ and $\boldsymbol{x} = (x_1, x_2) = (x, y)$; then we can rewrite the 2d inviscid SWE as

$$\begin{cases} \phi_t + (\boldsymbol{u} \cdot \nabla)\phi + \phi \nabla \cdot \boldsymbol{u} = 0, \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + g\nabla\phi + \mathcal{F}\boldsymbol{u} = 0, \end{cases} \tag{4.1}$$

where

$$\mathcal{F} = \begin{pmatrix} 0 & -f \\ f & 0 \end{pmatrix}.$$

We are going to show that the equations in (4.1) are invariant under a coordinate transformation and a variable change of the velocity (adapted to the coordinate transformation). We adopt the convention that the vectors in $\mathbb{R}^2$ are viewed as column vectors and the dot product on $\mathbb{R}^2$ are defined as

$$\boldsymbol{y} \cdot \boldsymbol{z} = \boldsymbol{y}^t \boldsymbol{z}, \qquad \forall\, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^2.$$

Let T be a 2×2 orthogonal matrix, that is $T^t T = I_2$, where I_2 is the 2×2 identity matrix. Since T is orthogonal, we have

$$T\boldsymbol{y} \cdot T\boldsymbol{z} = T^t\boldsymbol{y} \cdot T^t\boldsymbol{z} = \boldsymbol{y} \cdot \boldsymbol{z}, \qquad \forall\, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^2. \tag{4.2}$$

We now introduce the new coordinate system $\boldsymbol{x}'$ by setting

$$\boldsymbol{x}' = T\boldsymbol{x}, \qquad \boldsymbol{x} = T^t\boldsymbol{x}',$$

and the new variables $\boldsymbol{u}'$ (adapted to the coordinate transformation) by setting

$$\boldsymbol{u}' = T\boldsymbol{u}, \qquad \boldsymbol{u} = T^t\boldsymbol{u}'. \tag{4.3}$$

Writing the gradient ∇ as a column vector $\nabla = (\partial_{x_1}, \partial_{x_2})^t$, direct computations show that

$$\begin{aligned} &\nabla' = T\nabla, \qquad \nabla = T^t\nabla', \\ &\Delta' = \nabla' \cdot \nabla' = T\nabla \cdot T\nabla = \nabla \cdot \nabla = \Delta, \end{aligned} \tag{4.4}$$

where ∇' and Δ' are the gradient and laplacian in the new coordinate system $\boldsymbol{x}'$.

The *invariance* property for the 2d inviscid SWE (4.1) reads

Proposition 4.1 *In the new coordinate system $\boldsymbol{x}'$, the variables $(\boldsymbol{u}', \phi')$ defined by $\boldsymbol{u}' = \boldsymbol{u}'(x') = T\boldsymbol{u}(x')$ and $\phi' = \phi(x')$ satisfy the same set of equations (4.1) as $(\boldsymbol{u}, \phi)$, that is*

$$\begin{cases} \phi'_t + (\boldsymbol{u}' \cdot \nabla')\phi' + \phi'\nabla' \cdot \boldsymbol{u}' = 0, \\ \boldsymbol{u}'_t + (\boldsymbol{u}' \cdot \nabla')\boldsymbol{u}' + g\nabla'\phi' + \mathcal{F}\boldsymbol{u}' = 0. \end{cases} \tag{4.5}$$

Proof We first show that $(\boldsymbol{u}', \phi')$ satisfies the first equation $(4.5)_1$. Using (4.2)–(4.4), we compute

$$(\boldsymbol{u}' \cdot \nabla')\phi' = (T\boldsymbol{u} \cdot T\nabla)\phi' = (\boldsymbol{u} \cdot \nabla)\phi,$$

and

$$\nabla' \cdot \boldsymbol{u}' = T\nabla \cdot T\boldsymbol{u} = \nabla \cdot \boldsymbol{u},$$

which, together with $(4.1)_1$, implies that in the new coordinate system $\boldsymbol{x}'$, the new variables $(\boldsymbol{u}', \phi')$ satisfy $(4.5)_1$.

For the second equation $(4.5)_2$, in the new variables $(\boldsymbol{u}', \phi')$, we infer from $(4.1)_2$ that

$$T^t\partial_t\boldsymbol{u}' + (T^t\boldsymbol{u}' \cdot \nabla)T^t\boldsymbol{u}' + g\nabla\phi' + \mathcal{F}T^t\boldsymbol{u}' = 0,$$

which, together with (4.4), reads in the new coordinate system $\boldsymbol{x}'$:

$$T^t\partial_t\boldsymbol{u}' + (T^t\boldsymbol{u}' \cdot T^t\nabla')T^t\boldsymbol{u}' + gT^t\nabla'\phi' + \mathcal{F}T^t\boldsymbol{u}' = 0. \tag{4.6}$$

Observe that any 2×2 orthogonal matrix is of the form

$$T = \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix}, \quad \text{for some } \alpha, \quad \beta \in \mathbb{R}, \ \alpha^2 + \beta^2 = 1,$$

and direct calculations show the commutation relation $\mathcal{F}T^t = T^t\mathcal{F}$. We then can simplify (4.6) as

$$T^t\partial_t\boldsymbol{u}' + T^t(\boldsymbol{u}' \cdot \nabla')\boldsymbol{u}' + gT^t\nabla'\phi' + T^t\mathcal{F}\boldsymbol{u}' = 0.$$

which, multiplying by T on both sides, is $(4.5)_2$. We thus completed the proof. □

We now extend Proposition 4.1 to more general fluid equations, which read

$$\begin{cases} \partial_t\rho + \boldsymbol{u} \cdot \nabla\rho + \rho\nabla \cdot \boldsymbol{u} = 0, \\ \partial_t\boldsymbol{u} - \mu\Delta\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \Phi(\rho)\nabla p = 0, \end{cases} \tag{4.7}$$

where the gradient ∇ and Laplacian Δ are with respect to $\boldsymbol{x} = (x_1, \cdots, x_d)^t \in \mathbb{R}^d$, and $\rho \in \mathbb{R}$ is the mass-like quantity (e.g. density), $\boldsymbol{u} = (u_1, \cdots, u_d) \in \mathbb{R}^d$ the velocity, $p \in \mathbb{R}$ the pressure, μ the viscosity, and $\Phi(\rho) \in \mathbb{R}$ a scalar function of ρ. The first equation in (4.7) generally comes from the conservation of mass and the second equation in (4.7) from the conservation of momentum. Again, the vectors in $\mathbb{R}^d$ are viewed as column vectors and the dot product on $\mathbb{R}^d$ are defined as

$$\boldsymbol{y} \cdot \boldsymbol{z} = \boldsymbol{y}^t\boldsymbol{z}, \qquad \forall\, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d.$$

Let T be a $d \times d$ orthogonal matrix, that is $T^tT = I_d$, where I_d is the $d \times d$ identity matrix. Since T is orthogonal, we have

$$T\boldsymbol{y} \cdot T\boldsymbol{z} = T^t\boldsymbol{y} \cdot T^t\boldsymbol{z} = \boldsymbol{y} \cdot \boldsymbol{z}, \qquad \forall\, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d. \tag{4.8}$$

We now introduce the new coordinate system $\boldsymbol{x}'$ by setting

$$\boldsymbol{x}' = T\boldsymbol{x}, \qquad \boldsymbol{x} = T^t\boldsymbol{x}',$$

and the new variables $\boldsymbol{u}'$ (adapted to the coordinate transformation) by setting

$$\boldsymbol{u}' = T\boldsymbol{u}, \qquad \boldsymbol{u} = T^t\boldsymbol{u}'. \tag{4.9}$$

Then the *invariance* property for the fluid equations (4.7) reads

Proposition 4.2 *In the new coordinate system* $\boldsymbol{x}'$*, the variables* $(\boldsymbol{u}', \rho')$ *defined by* $\boldsymbol{u}' = \boldsymbol{u}'(x') = T\boldsymbol{u}(x')$ *and* $\rho' = \rho(x')$ *satisfy the same set of equations* (4.7) *as* $(\boldsymbol{u}, \rho)$*, that is*

$$\begin{cases} \partial_t\rho' + \boldsymbol{u}' \cdot \nabla'\rho' + \rho'\nabla' \cdot \boldsymbol{u}' = 0, \\ \partial_t\boldsymbol{u}' - \mu\Delta'\boldsymbol{u}' + (\boldsymbol{u}' \cdot \nabla')\boldsymbol{u}' + \Phi(\rho')\nabla' p = 0. \end{cases} \tag{4.10}$$

The proof of Proposition 4.2 is similar to that of Proposition 4.1, we thus omit the details here.

We now consider some specific fluid equations, where the form of these equations is slightly different from (4.7).

Example 1—Navier–Stokes equations The famous (incompressible) Navier-Stokes equations read

$$\begin{cases} \boldsymbol{u}_t - \mu\Delta\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \\ \nabla \cdot \boldsymbol{u} = 0, \end{cases} \tag{NSE}$$

where $\boldsymbol{u}$ is the velocity, μ the viscosity, and p is the pressure. We can infer from the proofs of Propositions 4.1–4.2, that the incompressible Navier-Stokes equations have the *invariance* property.

Example 2—Euler equations The motion of a compressible, inviscid fluid in the absence of heat convection is governed by the Euler equations:

$$\begin{cases} \partial_t\rho + \boldsymbol{u} \cdot \nabla\rho + \rho\nabla \cdot \boldsymbol{u} = 0, \\ \partial_t\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \rho^{-1}\nabla p = 0, \\ \partial_t e + \boldsymbol{u} \cdot \nabla e + \rho^{-1} p\nabla \cdot \boldsymbol{u} = 0. \end{cases} \tag{EE}$$

where ρ is the density, $\boldsymbol{u}$ the velocity, e the internal energy, and p the pressure. The equation of state (pressure law) reads

$$p = p(\rho, e).$$

The last equation in (EE) representing energy conservation law is similar to the first equation in (EE) representing the mass conservation law so that from the proofs of Propositions 4.1–4.2, we can deduce that the Euler equations have the *invariance* property.

4.2 The Choice of the Domain

We return to the inviscid SWE and first introduce a notion to express the intrinsic structure of the 2d inviscid SWE and then show that the proper domain, which will lead to well-posedness result, is related to the intrinsic structure of the 2d inviscid SWE.

Definition 4.1 The 2d nonlinear inviscid SWE are said to be *supercritical* (resp. *subcritical*) in the direction $\vec{l} = (\alpha, \beta)$ with $\alpha^2 + \beta^2 = 1$ (α, β are constants) if the following holds

$$(u\alpha + v\beta)^2 \ > (resp. <) \ g\phi. \tag{4.11}$$

We observe that in the *fully hyperbolic case*, we are able to construct a *supercritical* direction for the 2d nonlinear inviscid SWE. We first rewrite the *fully hyperbolic* condition (1.3) as:

$$\left(u \cdot \frac{u}{\sqrt{u^2+v^2}} + v \cdot \frac{v}{\sqrt{u^2+v^2}}\right)^2 > g\phi,$$

and since in this article we consider local smooth solutions, hence we could choose two constants $\bar{u}, \bar{v}$ such that in a short time interval the differences $|u-\bar{u}|$ and $|v-\bar{v}|$ are sufficiently enough so that in the *fully hyperbolic case*, the 2d nonlinear inviscid SWE are *supercritical* in the direction $\frac{(\bar{u},\bar{v})}{\sqrt{\bar{u}^2+\bar{v}^2}}$. Therefore, without loss of generality, we can assume that

$$\text{The 2d nonlinear inviscid SWE is } \textit{supercritical} \text{ in the direction } \vec{l}, \tag{4.12}$$

where the direction $\vec{l} = (\alpha, \beta)$ with $\alpha^2 + \beta^2 = 1$. We then choose the domain Ω to be a rectangle with one side parallel to the direction $\vec{l} = (\alpha, \beta)$. As we already saw in Sect. 3, the assumption (4.12) enables us to extend the rectangular domain to a channel (smooth) domain, which allows us to apply the general results from [3] for the IBVP of the first-order hyperbolic equations in smooth domains. First, by the *invariance* property of the nonlinear SWE, we see that if we introduce the coordinate transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and the corresponding variables change

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

then we know that in the new coordinate (x', y') system, the variables (u', v', ϕ') defined by $u' = u'(x', y')$, $v' = v'(x', y')$, and $\phi' = \phi(x', y')$ satisfy the same set of equations (1.1) as (u, v, ϕ) and the new domain Ω' denoting the image of Ω under the coordinate transformation is a rectangular domain with one side parallel to the direction (0, 1). We now observe that the original assumption (4.12) becomes that the 2d inviscid SWE satisfied by (u', v', ϕ') are *supercritical* in the direction (0, 1) in the new coordinate (x', y') system, that is

$$v'^2 > g\phi'.$$

Clearly, we have

$$u'^2 + v'^2 = (u', v') \cdot (u', v')^t = (u, v)T^t T(u, v)^t = (u, v) \cdot (u, v)^t = u^2 + v^2 > g\phi = g\phi'.$$

Therefore, in the new coordinate system (x', y'), we are going back to the assumptions made in (1.3)–(1.4) for the 2d inviscid SWE satisfied by (u', v', ϕ') and hence the local well-posedness result could be achieved.

Remark 4.1 In order to solve the IBVP associated to the SWE system (1.1), we need to properly choose the domain Ω according to the intrinsic structure of the SWE system. Specifically, in the *fully hyperbolic case*, we know that the SWE system (1.1) must be *supercritical* in some direction (e.g. the direction $\vec{l} = (0, 1)$) and we choose the domain Ω to be a rectangle with one side parallel to the direction $\vec{l}$. We remark that we could also choose a curvilinear polygonal domain as long as such a domain could be extended to a curvilinear channel (smooth) domain in the direction $\vec{l}$ with periodicity. For example, Fig. 1 below provides a curvilinear polygonal domain that could be extended to a curvilinear channel (smooth) domain in the

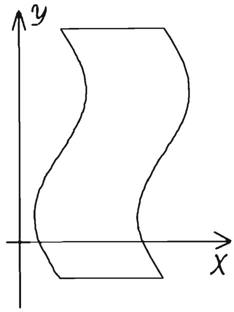

Fig. 1 The curvilinear polygonal domain

direction $\vec{l} = (0, 1)$. For the sake of simplicity, we consider the rectangular domain in this article in order to simplify the presentation.

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Interpolation and Trace Theorems

In this appendix, we extend the classical interpolation and trace results (see [1,13]) in smooth domains to Lipschitz domains by using the results in [6].

Let d be a positive integer, we first recall a well known extension result (see e.g. [6, Theorem 1.4.3.1]).

Proposition 5.1 (Extension theorem) *Let $\mathcal{U}$ be a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary and let $s > 0$. Then there exists a continuous linear operator P_s from $H^s(\mathcal{U})$ into $H^s(\mathbb{R}^n)$ such that for all $u \in H^s(\mathcal{U})$, the restriction of $P_s u$ to $\mathcal{U}$ is u itself, that is*

$$P_s u|_{\mathcal{U}} = u.$$

By Proposition 5.1, each function $u \in H^s(\mathcal{U})$ is the restriction of a function $P_s u \in H^s(\mathbb{R}^d)$. Note that the extension operator P_s can be chosen independently of s (see [2,16]).

From [13, Chapter I, Sect. 7.1], we have the following interpolation result:

$$H^s(\mathbb{R}^d) = [H^m(\mathbb{R}^d), L^2(\mathbb{R}^d)]_\theta, \quad 0 \le \theta \le 1, \;\; s = (1-\theta)m, \tag{5.1}$$

and more generally

$$H^s(\mathbb{R}^d) = [H^{s_1}(\mathbb{R}^d), H^{s_2}(\mathbb{R}^d)]_\theta, \quad 0 \le \theta \le 1, \;\; s_1, s_2 \in \mathbb{R}, \;\; s = (1-\theta)s_1 + \theta s_2. \tag{5.2}$$

From [13, Chapter I, Section 9], the interpolation equalities (5.1)–(5.2) have been extended to the Sobolev spaces with bounded smooth domains. Combining the proof of [13, Chapter I, Theorem 9.1] and Proposition 5.1, we can conclude that the interpolation equalities (5.1)–(5.2) also hold for the Sobolev spaces for bounded Lipschitz domains.

Proposition 5.2 (Interpolation theorem) *Let $\mathcal{U}$ be a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary. Then there holds*

$$H^s(\mathcal{U}) = [H^m(\mathcal{U}), L^2(\mathcal{U})]_\theta, \quad 0 \le \theta \le 1, \quad s = (1-\theta)m, \quad m > 0 \textit{ integer},$$

and

$$H^s(\mathcal{U}) = [H^{s_1}(\mathcal{U}), H^{s_2}(\mathcal{U})]_\theta, \quad 0 \le \theta \le 1, \quad s_1, s_2 > 0, \quad s = (1-\theta)s_1 + \theta s_2.$$

Proof To prove the first interpolation equality, we temporally denote by $\widetilde{H}^s(\mathcal{U})$ the right-hand side of the first interpolation equality. Thanks to the extension result Proposition 5.1, similar arguments for [13, Chapter I, Theorem 9.1] show that the space $\widetilde{H}^s(\mathcal{U})$ coincides with the space of restrictions to $\mathcal{U}$ of the elements of $H^s(\mathbb{R}^d)$, which is $H^s(\mathcal{U})$. This shows the first interpolation equality. The second interpolation equality follows from the first one and the reiteration theorem [13, Chapter I, Theorem 6.1]. □

We remark that the reason why we need either $m > 0$ or $s_1, s_2 > 0$ in Proposition 5.2 unlike (5.2) is that we only have the extension result for $s > 0$ in Proposition 5.1.

The trace theorem [13, Chapter I, Theorem 3.2] with $X = H^m(\mathcal{U})$ and $Y = L^2(\mathcal{U})$ reads

Proposition 5.3 (Trace theorem) *Let $u \in W^m(0,\infty)$, where*

$$W^m(0,\infty) = \left\{ u \mid u \in L^2(0,\infty; H^m(\mathcal{U}), \frac{\mathrm{d}^m u}{\mathrm{d}t^m} \in L^2(0,\infty; L^2(\mathcal{U})) \right\}.$$

Then

$$\frac{\mathrm{d}^j u}{\mathrm{d}t^j}(0) \in [H^m(\mathcal{U}), L^2(\mathcal{U})]_{(j+1/2)/m} = H^{m-j-1/2}(\mathcal{U}), \quad 0 \le j \le m-1,$$

and the mapping

$$u \mapsto \left\{ \frac{\mathrm{d}^j u}{\mathrm{d}t^j}(0) \mid 0 \le j \le m-1 \right\} \quad \textit{of} \quad W^m(0,\infty) \quad \textit{into} \quad \prod_{j=0}^{m-1} H^{m-j-1/2}(\mathcal{U})$$

is onto.

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