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Exponential Propagation for Fractional Reaction–Diffusion Cooperative Systems with Fast Decaying Initial Conditions

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Abstract We study the time asymptotic propagation of solutions to the reaction–diffusion cooperative systems with fractional diffusion. We prove that the propagation speed is exponential in time, and we find the precise exponent of propagation. This exponent depends on the smallest index of the fractional laplacians and on the principal eigenvalue of the matrix $DF(0)$ where F is the reaction term. We also note that this speed does not depend on the space direction.

Keywords Fractional laplacian · Nonlinear Fisher-KPP reaction–diffusion equation · Cooperative systems · Time asymptotic propagation

Mathematics Subject Classification Primary 35R11 · Secondary 35B40

1 Introduction

The reaction diffusion equation with Fisher-KPP nonlinearity

$$\partial_t u + (-\triangle)^\alpha u = f(u) \tag{1.1}$$

with $\alpha = 1$, has been the subject of intense research since the seminal work by Kolmogorov et al. [13]. Of particular interest are the results of Aronson and Weinberger [2] which describe the evolution of solution starting with compactly supported data. They showed that there exists a critical threshold $c^* = 2\sqrt{f'(0)}$ such that, for any compactly supported initial value

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u_0 in [0, 1], if $c > c^*$ then $u(t, x) \to 0$ uniformly in $\{|x| \geq ct\}$ as $t \to +\infty$ and if $c < c^*$ then $u(t, x) \to 1$ uniformly in $\{|x| \leq ct\}$ as $t \to +\infty$. This corresponds to a linear propagation of the fronts. In addition, (1.1) admits planar traveling wave solutions connecting 0 and 1.

Reaction–diffusion equations with fractional Laplacian, that is when $\alpha \in (0, 1)$ in (1.1), appear in physical models when the diffusive phenomena are better described by Lévy processes allowing long jumps, than by Brownian processes—obtained when $\alpha = 1$. The Lévy processes occur widely in physics, chemistry and biology. Recently these models have attracted much interest. In connection with the discussion given above, in the recent paper [6], Cabré and Roquejoffre showed that for any compactly supported initial condition, or more generally for initial values decaying faster than $|x|^{-d-2\alpha}$, where d is the dimension of the spatial variable, the speed of propagation becomes exponential in time. They also showed that no traveling wave exist. Their result was sharpened and extended in [7], who proposed a new (and more flexible) argument to treat models of the form (1.1). They indeed notice that diffusion only plays a role for small times, the large time dynamics being given by a simple transport equation. All these results are in great contrast with the case $\alpha = 1$.

By other hand, in the one-dimensional case, if the initial condition is assumed to be globally front-like and to decay at infinity towards the unstable steady state more slowly than any exponentially decaying function when $\alpha = 1$ and decays at infinity more slowly than a power x^{-b} with $b < 2\alpha$ when $\alpha \in (0, 1)$, [11] and [10] respectively, state that the level sets of the solutions move exponentially fast as time goes to infinity. Moreover, a quantitative estimate of motion of the level sets is obtained in terms of the decay of the initial condition.

The work on the single Eq. (1.1) can be extended to reaction–diffusion systems. The first definitions of spreading speeds for cooperative systems in population ecology and epidemic theory are due to Lui in [15]. In a series of papers, Lewis et al. [14,17,18] studied spreading speeds and travelling waves for a particular class of cooperative reaction–diffusion systems, with standard diffusion. Results on single equations in the singular perturbation framework proved by Evans and Souganidis in [9] have also been extended by Barles et al. in [3]. The viscosity solutions framework is studied in [5], with a precise study of the Harnack inequality. In these papers, the system under study is of the following form

$$\partial_t u_i - \rho_i \Delta u_i = f_i(u),$$

where, for $m \in \mathbb{N}^*$, $u = (u_i)_{i=1}^m$ is the unknown function. For all $i \in [\![1, m]\!] := \{1, \dots, m\}$, the constants ρ_i are assumed to be positive as well as the bounded, smooth and Lipschitz initial conditions, defined from $\mathbb{R}^d$ to $\mathbb{R}_+$. As the essential assumptions that concern the reaction term $F = (f_i)_{i=1}^m$, it is assumed to be smooth, to have only two zeroes, 0 and $a = (a_i)_{i=1}^m \in \mathbb{R}^m$ in $[0, a_1] \times \cdots \times [0, a_m]$, and for all $i \in [\![1, m]\!]$, each f_i is nondecreasing in all its components, with the possible exception of the ith one. The last assumption means that the system is cooperative. Under additional hypotheses, which imply that the point 0 is unstable, the limiting behavior of the solution $u = (u_i)_{i=1}^m$ is understood.

Here, we focus on similar systems, but considering that at least one diffusive term is given by a fractional Laplacian. More precisely, we focus on the large time behavior of the solution $u = (u_i)_{i=1}^m$, for $m \in \mathbb{N}^*$, to the fractional reaction–diffusion system:

$$\begin{cases} \partial_t u_i + (-\triangle)^{\alpha_i} u_i = f_i(u), \quad t > 0, x \in \mathbb{R}^d, \\ u_i(0, x) = u_{0i}(x), \; x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where

$$\alpha_i \in (0, 1] \quad \text{and} \quad \alpha := \min_{[\![1,m]\!]} \alpha_i < 1.$$

Note that when $\alpha_i = 1$, $(-\triangle)^{\alpha_i}$ is the standard Laplacian. As general assumptions, we impose, for all $i \in [\![1, m]\!]$, the initial condition u_{0i} to be nonnegative, non identically equal to 0, continuous and to satisfy

$$u_{0i}(x) = \mathrm{O}\left(|x|^{-(d+2\alpha_i)}\right) \quad \text{as} \quad |x| \to +\infty. \tag{1.3}$$

We also assume that for all $i \in [\![1, m]\!]$, the function f_i satisfies $f_i(0) = 0$ and that system (1.2) is cooperative, which means :

$$f_i \in C^1(\mathbb{R}^m) \ \text{ and } \ \partial_j f_i > 0, \ \text{ on } \mathbb{R}^m, \quad \text{for all } j \in [\![1, m]\!], \ j \neq i. \tag{1.4}$$

The aim of this paper is to understand the time asymptotic location of the level sets of solutions to (1.2). Hence, inspired by the formal analysis done in [7], taking λ_1 the principal positive eigenvalue of $DF(0)$ where $F = (f_i)_{i=1}^m$ with associated eigenvector ϕ_1, we consider the family of functions of the form

$$v(t, x) = a\left(1 + b(t)|x|^{\delta(d+2\alpha)}\right)^{-\frac{1}{\delta}} \phi_1, \tag{1.5}$$

where $b(t)$ is a continuous function asymptotically proportional to $e^{-\delta\lambda_1 t}$ with a and δ a positive constants, in addition, we note that the level sets of functions given by (1.5) spread exponentially fast in time with an exponent $\lambda_1/(d+2\alpha)$. Similarly to [7], we will prove that v serves as super and subsolutions of (1.2). The scheme of their proof will be reproduced here, but some steps - and this is why it makes system (1.2) worth studying - become more difficult. The small time study will require the manipulation of some Polya integrals, and the transport equation will also become more complex. Furthermore, since the particularity of the index α is that the fundamental solution has the slowest decay compared to the other fractional or standard laplacians, we show that the speed of propagation of solutions to (1.2) are exponential in time, with a precise exponent depending on the smallest index $\alpha := \min_{i \in [\![1,m]\!]} \alpha_i$ and on the principal eigenvalue of the matrix $DF(0)$. Also we note that this speed does not depend on the space direction.

For what follows and without loss of generality, we suppose that $\alpha_{i+1} \leq \alpha_i$ for all $i \in [\![1, m-1]\!]$ so that $\alpha = \alpha_m < 1$. Before stating the main results, we need some additional hypotheses on the nonlinearities f_i, for all $i \in [\![1, m]\!]$.

- (H1) The principal eigenvalue λ_1 of the matrix $DF(0)$ is positive,
- (H2) F is globally Lipschitz on $\mathbb{R}^m$,
- (H3) There exists $\Lambda > 1$ such that, for all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \geq \Lambda$, we have $f_i(s) \leq 0$,
- (H4) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \leq \Lambda$, $Df_i(0)s - f_i(s) \geq c_{\delta_1} {s_i}^{1+\delta_1}$,
- (H5) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \leq \Lambda$, $Df_i(0)s - f_i(s) \leq c_{\delta_2} |s|^{1+\delta_2}$,

where the constants c_{δ_1} and c_{δ_2} are positive and independent of $i \in [\![1, m]\!]$, and for all $j \in \{1, 2\}$

$$\delta_j \geq \frac{2}{d + 2\alpha} \tag{1.6}$$

Hence, in order to study the spread speed of solutions to (1.2), assumption (H1) guarantees that 0 is an unstable state, (H2) and (H3) are needed to state algebraically upper and lower bounds of the solution to (1.2), finally, (H4) and (H5) are technical assumptions that are not general but enable us to understand the long time behavior of a class of monotone systems, moreover, (1.6) guarantees enough regularity on the super and subsolutions we construct in our proofs.

Before going further on, let us state at least one example of nonlinearity F satisfying all the assumptions (1.4) and (H1)–(H5). Let $A = (a_{ij})_{i,j=1}^m$ be a matrix, with positive non diagonal entries and with positive principal eigenvalue. For a constant $\Lambda > 1$, for all $i \in [\![1, m]\!]$ and all $s \in \mathbb{R}^m$, we define

$$f_i(s) = (As)_i - \phi_i(s),$$

where

$$\phi_i(s) = \begin{cases} s_i \, |s_i|^\delta \, \chi_1(s), & \text{if } |s| \le \Lambda - 1, \\ \chi_2(s), & \text{if } \Lambda - 1 \le |s| \le \Lambda, \\ C_i \, |s| \, , & \text{if } |s| \ge \Lambda, \end{cases}$$

with $\delta \ge \frac{2}{d+2\alpha}$, C_i is a positive constant large enough, χ_1 and χ_2 two smooth functions defined in $\mathbb{R}^m$, chosen so that $\phi_i \in \mathcal{C}^1(\mathbb{R}^m)$ and for $i \neq j$, $\partial_j \phi_i(0) = 0$, which implies $f_i \in \mathcal{C}^1(\mathbb{R}^m)$. These choices easily ensure (1.4), (H1) and (H2) since $DF(0) = A$. Moreover, for all $s \in \mathbb{R}_+^m$ such that $|s| \ge \Lambda$, we have, for C_i large enough

$$f_i(s) = \sum_{j=1}^m a_{ij} s_j - C_i |s| \le 0,$$

which proves that (H3) is satisfied. The assumptions (H4) and (H5) are easily fulfilled taking $\delta_1 = \delta_2 = \delta$,

$$c_{\delta_1} = \min \left(\min_{\Lambda - 1 \le |\widetilde{s}| \le \Lambda} \frac{\chi_2(\widetilde{s})}{\Lambda^{1+\delta}}, \min_{\mathbb{R}^m} \chi_1 \right)$$

and

$$c_{\delta_2} = \max \left(\max_{\frac{\Lambda - 1}{2} \le |\widetilde{s}| \le \Lambda} \frac{\chi_2(\widetilde{s})}{(\Lambda - 1)^{1+\delta}}, \max_{\mathbb{R}^m} \chi_1 \right).$$

We are now in a position to state our main theorem, which show that the solution to (1.2) move exponentially fast in time.

Theorem 1 *Let $d \ge 1$ and assume that F satisfies (1.4) and (H1) to (H5). Let u be the solution to (1.2) with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3). Then there exists $\tau > 0$ large enough such that for all $i \in [\![1, m]\!]$, the following two facts are satisfied:*

(a) *For every $\mu_i > 0$, there exists a constant $c > 0$ such that,*

$$u_i(t, x) < \mu_i, \quad \textit{for all } t \ge \tau \textit{ and } |x| > c e^{\frac{\lambda_1}{d+2\alpha} t}.$$

(b) *There exist constants $\varepsilon_i > 0$ and $C > 0$ such that,*

$$u_i(t, x) > \varepsilon_i, \quad \textit{for all } t \ge \tau \textit{ and } |x| < C e^{\frac{\lambda_1}{d+2\alpha} t}.$$

The plan to set Theorem 1 is organized as follows. First, in the short Sect. 2, we state a local existence and uniqueness result of solutions for cooperative systems involving fractional diffusion and we state a comparison principle for this type of solutions which, although standard, is crucial for the sequel. In Sect. 3 we deal with finite time and large x decay estimates, which imply the global existence in time of solutions and will be the first step to construct super and subsolutions of the form (1.5), which are needed to prove Theorem 1. The end of this paper, Sect. 4 is devoted to the proof of Theorem 1, in which we state that the front position moves exponentially in time.

2 Local Existence and Comparison Principle

Recall that the operator $A = -\text{diag}((-\Delta)^{\alpha_1}, \ldots, (-\Delta)^{\alpha_m})$ is sectorial (see [12]) in $(L^2(\mathbb{R}^d))^m$, with domain $D(A) = H^{2\alpha_1}(\mathbb{R}^d) \times \cdots \times H^{2\alpha_m}(\mathbb{R}^d)$. If now u_0 satisfies the assumptions of Theorem 1, it is in $(L^2(\mathbb{R}^d))^m$, so that the Cauchy Problem (1.2) has a unique maximal solution, defined on an interval of the form $[0, t_{max})$; moreover the L^2-norm of u blows up as $t \to t_{max}$ if $t_{max} < +\infty$. Finally, we have $u \in C((0, t_{max}), D(A)) \cap C([0, t_{max}), (L^2(\mathbb{R}^d))^m)$ and $\frac{du}{dt} \in C((0, t_{max}), (L^2(\mathbb{R}^d))^m)$. A standard iteration argument and Sobolev embeddings then yield

$$u \in C^p((0, t_{max}), (H^q(\mathbb{R}^d))^m)$$

for every integer p and q.

Before to continue, we state the following notation, if $x = (x_i)_{i=1}^m$ and $y = (y_i)_{i=1}^m$ belong to $\mathbb{R}^m$, we denote $[x, y]$ as the rectangle in $\mathbb{R}^m$ given by $[x_1, y_1] \times \cdots \times [x_m, y_m]$, also, we say that $x \le y$ if $x_i \le y_i$ for all $i \in [\![1, m]\!]$. Now, we are in conditions to state the Comparison Principle to our system.

Theorem 2 *Consider $T > 0$, and let $u = (u_i)_{i=1}^m$ and $v = (v_i)_{i=1}^m$ such that: $u \in C((0, T], D(A)) \cap C([0, T], (L^2(\mathbb{R}^d))^m) \cap C^1((0, T), (L^2(\mathbb{R}^d))^m)$; and $v \in C([0, T] \times \mathbb{R}^d) \cap C^1((0, T) \times \mathbb{R}^d)$. Assume that, for all $i \in [\![1, m]\!]$, we have*

$$\partial_t u_i + (-\triangle)^{\alpha_i} u_i \le f_i(u), \quad \partial_t v_i + (-\triangle)^{\alpha_i} v_i \ge f_i(v),$$

where f_i satisfies (1.4). If for all $i \in [\![1, m]\!]$ and $x \in \mathbb{R}^d$, $u_i(0, x) \le v_i(0, x)$ we have

$$u(t, x) \le v(t, x) \quad \text{for all} \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

Proof of Theorem 2 Let us define for all $i \in [\![1, m]\!]$, $w_i = u_i - v_i$. Then w_i satisfies $w_i(0, x) \le 0$ and

$$\begin{aligned} \partial_t w_i + (-\triangle)^{\alpha_i} w_i \le f_i(u) - f_i(v) &= \int_0^1 \nabla f_i(\sigma u + (1-\sigma)v) \cdot (u - v) d\sigma \\ &= \int_0^1 \nabla f_i(\zeta_\sigma) \cdot w \, d\sigma, \end{aligned} \tag{2.1}$$

where $\zeta_\sigma = \sigma u + (1 - \sigma)v$. Notice now that the positive part of the function w_i denoted by w_i^+ belongs to $C((0, T), H^{2\alpha_i}(\mathbb{R}^d)) \cup W^{1,\infty}((0, T), L^2(\mathbb{R}^d))$. So, taking the scalar product of (2.1) with the vector function $(w_i^+)_{i=1}^m$ and integrating over $\mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} w_i^+ \partial_t w_i dx + \int_{\mathbb{R}^d} w_i^+ (-\triangle)^{\alpha_i} w_i dx \le \int_{\mathbb{R}^d} w_i^+ \int_0^1 \nabla f_i(\zeta_\sigma) \cdot w \, d\sigma dx \tag{2.2}$$

Recall that $\int_{\mathbb{R}^d} w_i^+(-\triangle)^{\alpha_i} w_i dx \geq 0$. So we have, since $\partial_j f_i(\zeta_\sigma) \geq 0$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{R}^d}(w_i^+)^2dx\right] &\leq \int_{\mathbb{R}^d}\int_0^1 \partial_i f_i(\zeta_\sigma)d\sigma(w_i^+)^2dx\\ &\quad+\sum_{j=1,j\neq i}^m\int_{\mathbb{R}^d}\int_0^1\partial_j f_i(\zeta_\sigma)d\sigma\, w_i^+w_j^+dx\\ &\leq C\sum_{j=1}^m\int_{\mathbb{R}^d}(w_j^+)^2dx,\end{aligned}$$

where C is a constant that depends on m. Doing this procedure for each $i \in [\![1, m]\!]$ and adding, we get for $t \in [0, T]$

$$\frac{d}{dt}\left[\sum_{i=1}^m\int_{\mathbb{R}^d}(w_i^+)^2dx\right] \leq C\sum_{i=1}^m\int_{\mathbb{R}^d}(w_i^+)^2dx.$$

So, by Gronwall's inequality, we have $w_i \leq 0$ in $[0, T] \times \mathbb{R}^d$. □

3 Finite Time Bounds and Global Existence

From hypothesis (H3), we deduce that the positive vector $M = \Lambda\mathbf{1}$, where $\mathbf{1}$ is the vector of size m with all entries equal to 1, is a supersolution to (1.2), if the initial condition $u_0 = (u_{0i})_{i=1}^m$ is smaller than M. So, from Theorem 2, we have $0 \leq u(t, x) \leq M$. In the next subsections, we obtain pointwise estimates which are needed to construct super and subsolution in order to prove Theorem 1 in Sect. 4, explicitly, the upper and lower estimates given in Lemmas 3 and 5 respectively, will be use at the moment to locate the front position in Lemmas 7 and 8. Also, these estimates imply locally finite L^2 bounds and so, global existence of solutions.

Now, we are in position to establish an algebraic upper bound for the solutions of (1.2). From (H2), we know that, for $i \in [\![1, m]\!]$ and $j \in [\![1, m]\!]$

$$\left|\partial_j f_i(s)\right| \leq l, \quad \text{for all } s \in \mathbb{R}^m,$$

where $l = Lip(f)$ is the Lipschitz constant of f. Thus, we have for all $s = (s_i)_{i=1}^m \geq 0$

$$f_i(s) = \int_0^1 \nabla f_i(\sigma s)\cdot s\, d\sigma \leq \left|\sum_{j=1}^m s_j\int_0^1\frac{\partial f_i}{\partial s_j}(\sigma s)d\sigma\right| \leq l\sum_{j=1}^m s_j. \tag{3.1}$$

Let us consider $v = (v_i)_{i=1}^m$ the solution of the following system

$$\begin{cases}\partial_t v + Lv = Bv,\ t > 0, x \in \mathbb{R}^m\\ \quad v(0,\cdot) = u_0,\ \ \mathbb{R}^m,\end{cases} \tag{3.2}$$

where $L = \mathrm{diag}((-\triangle)^{\alpha_1}, \ldots, (-\triangle)^{\alpha_m})$, $B = (b_{ij})_{i,j=1}^m$ is a matrix with $b_{ij} = l$ for all $i, j \in [\![1, m]\!]$. By (3.1) and Theorem 2, we conclude that $u \leq v$ in $[0, +\infty) \times \mathbb{R}^d$. A finite time upper bound for u is given by the following lemma.

Lemma 3 *Let $d \geq 1$ and let $u = (u_i)_{i=1}^m$ be the solution of system (1.2), with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3), and reaction* *term* $F = (f_i)_{i=1}^m$ *satisfying* (1.4) *and (H1) to (H3). Then, for all* $i \in [\![1, m]\!]$*, there exists a locally bounded function* $C_1 : (0, +\infty) \to \mathbb{R}_+$ *such that for all* $t > 0$ *and* $|x|$ *large enough, we have*

$$u_i(t, x) \le \frac{C_1(t)}{|x|^{d+2\alpha}}.$$

Taking Fourier transforms in each term of system (3.2), we have

$$\begin{cases} \partial_t \mathfrak{F}(v) = (A(|\xi|) + B)\mathfrak{F}(v), & \xi \in \mathbb{R}^d, t > 0 \\ \mathfrak{F}(v)(0, \cdot) = \mathfrak{F}(u_0), & \text{on } \mathbb{R}^d, \end{cases}$$

where $A(|\xi|) = \text{diag}(-|\xi|^{2\alpha_1}, \dots, -|\xi|^{2\alpha_m})$. Thus, we have that

$$\mathfrak{F}(v)(t, \xi) = e^{(A(|\cdot|)+B)t}\, \mathfrak{F}(u_0)(\xi)$$

and then, for all $x \in \mathbb{R}^d$ and $t \ge 0$:

$$u(t, x) \le v(t, x) = \mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t}) * u_0(x). \tag{3.3}$$

The following lemma is a crucial tool in the proof of Lemma 3, in which is stated that we can rotate the integration line of a small angle $\varepsilon > 0$ in the expression of $\mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})$. For the next results, we consider the matrix norm

$$\|A\| = \sup\left\{ \frac{|Av|}{|v|} \;:\; v \in \mathbb{C}^m \; with \; v \neq 0 \right\}$$

with $|\cdot|$ the Euclidean norm in $\mathbb{C}^m$.

Lemma 4 *For all* $z \in \left\{z \in \mathbb{C} \mid 0 \le \arg(z) < \frac{\pi}{4\alpha_1}\right\}$ *and* $t \ge 0$*, we have*

$$\left\| e^{(A(z)+B)t} \right\| \le m e^{(\|B\| - |z|^{2\alpha_1} \cos(2\alpha_1 \arg(z)))t} + e^{(\|B\| - |z|^{2\alpha} \cos(2\alpha_1 \arg(z)))t}, \tag{3.4}$$

and if

$$I_t(z) := \int_0^t e^{(t-s)(A(z)+B)}[e^{sB}, A(z)]e^{sA(z)} ds, \tag{3.5}$$

where $[e^{sB}, A(z)] = e^{sB}A(z) - A(z)e^{sB}$*, then there exists* $C_2 : (0, \infty) \to \mathbb{R}_+$ *a locally bounded function such that*

$$\|I_t(z)\| \le C_2(t)\left(|z|^{2\alpha}\, e^{-|z|^{2\alpha}\cos(2\alpha_1 \arg(z))t} + |z|^{2\alpha_1}\, e^{-|z|^{2\alpha_1}\cos(2\alpha_1 \arg(z))t} \right). \tag{3.6}$$

Proof of Lemma 4 Let z be in $\{z \in \mathbb{C} \mid 0 \le \arg(z) < \frac{\pi}{4\alpha_1}\}$. For any $j \in [\![1, m]\!]$, we consider the system

$$\begin{cases} \partial_t w = (A(z) + B)w, & z \in \mathbb{C}, t > 0, \\ w(0, z) = e_j & z \in \mathbb{C}, \end{cases} \tag{3.7}$$

where e_j is the jth vector of the canonical basis of $\mathbb{R}^m$. Thus, we have

$$w(t, z) = e^{(A(z)+B)t} e_j$$

Multiply (3.7) by the conjugate transpose $\overline{w}$ and take the real part to get

$$\frac{1}{2}\partial_t |w|^2 + \sum_{k=1}^m \cos(2\alpha_k \arg(z))\, |z|^{2\alpha_k}\, |w_k|^2 = Re(Bw \cdot \overline{w}) \le \|B\|\, |w|^2.$$

The choice of $\arg(z)$ and Gronwall's Lemma end the proof.

To prove (3.6), it is sufficient to notice that, for $s \in [0, t]$, we have

$$\left\| e^{sA(|z|e^{i\arg(z)})} \right\| \leq m e^{-|z|^{2\alpha}\cos(2\alpha_1 \arg(z))s} + e^{-|z|^{2\alpha_1}\cos(2\alpha_1\arg(z))s},$$

$$\left\| [e^{sB}, A(|z|\, e^{i\arg(z)})] \right\| \leq C(t)(|z|^{2\alpha} + |z|^{2\alpha_1}),$$

where $C : (0, +\infty) \to \mathbb{R}_+$ is a locally bounded function, and due to (3.4), we also have

$$\left\| e^{(A(|z|e^{i\arg(z)})+B)(t-s)} \right\| \leq m e^{(\|B\|-|z|^{2\alpha_1}\cos(2\alpha_1\arg(z)))(t-s)} + e^{(\|B\|-|z|^{2\alpha}\cos(2\alpha_1\arg(z)))(t-s)}.$$

□

In what follows, we prove that for each time $t > 0$, the solution of (1.2) decays as $|x|^{-d-2\alpha}$ for large values of $|x|$. Due to the decay of u_0 at infinity, we only need to prove that the entries of $\mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})$ have the desired decay. Indeed, defining by $\eta(t, \cdot)$ any component of $\mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t})$, if we assume that

$$|\eta(t, x)| \leq \frac{C(t)}{1 + |x|^{d+2\alpha}}, \quad \forall\, t > 0,\ |x| > R$$

for some $R > 0$ and $C(\cdot)$ a locally positive bounded function in $(0, +\infty)$, taking $R > 0$ large if necessary, there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^d} \frac{1}{1 + |y|^{d+2\alpha}} \frac{1}{1 + |x-y|^{d+2\alpha}} \leq \frac{c}{|x|^{d+2\alpha}}, \quad \text{if } |x| \geq 2R \tag{3.8}$$

Hence, for all $t > 0$, $|x| \geq 2R$ and $i \in [\![1, m]\!]$, by (1.3), there is a constant $c_i > 0$ such that

$$|\eta(t, \cdot) * u_{0i}(x)| \leq \int_{|y|<R} \frac{c_i |\eta(t, y)|}{1 + |x-y|^{d+2\alpha}} dy + \int_{|y|\geq R} \frac{C(t)}{1 + |y|^{d+2\alpha}} \frac{c_i}{1 + |x-y|^{d+2\alpha}} dy$$

Now, if $|y| < R$, we have that $|x|/2 \geq R > |y|$ and then $|x - y| \geq |x| - |y| \geq |x|/2$, thus, by Lemma 4, the first integral of the right side has the desired decay. The bound for the second integral follows directly from (3.8).

To continue, we split the proof of Lemma 3 into two cases. First, for the sake of simplicity, we consider the one space dimension case to underline the idea of the proof. The higher space dimension case is treated after and requires the use of Whittaker functions.

Proof of Lemma 3 Case $d = 1$. In this proof, we denote by $C : (0, +\infty) \to \mathbb{R}_+$ a locally bounded function. From (3.3), we only have to find an upper bound to $\mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})$. First, we consider for $t \geq 0$ and $z \in \mathbb{C}$, $w(t, z) := e^{tB} e^{tA(z)}$. Thus, w satisfies the Cauchy problem

$$\begin{cases} \partial_t w = (A(z) + B)w + [e^{tB}, A(z)]e^{tA(z)}, & t > 0, z \in \mathbb{C} \\ w(0, z) = Id, & z \in \mathbb{C}, \end{cases}$$

By Duhamel's formula, we get for all $z \in \mathbb{C}$ and $t \geq 0$:

$$e^{t(A(z)+B)} = e^{tB} e^{tA(z)} - \int_0^t e^{(t-s)(A(z)+B)} [e^{sB}, A(z)] e^{sA(z)} ds. \tag{3.9}$$

Thus, for all $t > 0$ and all $x \in \mathbb{R}$, we have

$$\begin{aligned} \mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})(x) &= \int_{\mathbb{R}} e^{ix\xi} e^{(A(|\xi|)+B)t} d\xi \\ &= \int_{\mathbb{R}} e^{ix\xi} e^{tB} e^{tA(|\xi|)} d\xi - \int_{\mathbb{R}} e^{ix\xi} I_t(|\xi|) d\xi \\ &= e^{tB}\, \mathrm{diag}(p_{\alpha_1}(t,x), \ldots, p_{\alpha_m}(t,x)) - \int_{\mathbb{R}} e^{ix\xi} I_t(|\xi|) d\xi, \end{aligned} \tag{3.10}$$

where for $i \in [\![1, m]\!]$, p_{α_i} is the heat kernel of the operator $(-\Delta)^{\alpha_i}$ in $\mathbb{R}$, that satisfies for $x \in \mathbb{R}$ and $t > 0$

$$\begin{cases} p_{\alpha_i}(t,x) = \dfrac{e^{-\frac{|x|^2}{4t}}}{\sqrt{4\pi t}}, & \text{if } \alpha_i = 1; \\ \dfrac{B^{-1}t}{t^{\frac{1}{2\alpha_i}+1} + |x|^{1+2\alpha_i}} \le p_{\alpha_i}(t,x) \le \dfrac{Bt}{t^{\frac{1}{2\alpha_i}+1} + |x|^{1+2\alpha_i}}, & \text{if } \alpha_i \in (0,1). \end{cases}$$

Since $\alpha = \min_{i \in [\![1,m]\!]} \alpha_i \in (0, 1)$, for large values of $|x|$, we clearly have

$$\left\| e^{tB}\, \mathrm{diag}(p_{\alpha_1}(t,x), \ldots, p_{\alpha_m}(t,x)) \right\| \le \frac{C(t)}{|x|^{1+2\alpha}}. \tag{3.11}$$

It remains to bound from above the following quantity :

$$\int_{\mathbb{R}} e^{ix\xi} I_t(|\xi|) d\xi = 2\int_0^\infty \cos(xr) I_t(r) dr = 2\Re\mathrm{e}\left(\int_0^\infty e^{ixr} I_t(r) dr\right).$$

We use the following two facts. First, for all $t \ge 0$, the function $z \mapsto e^{ixz} I_t(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Second, for $\delta > 0$ (respectively $R > 0$), on the arc $\{\pm\delta e^{i\theta}, \theta \in [0, \varepsilon]\}$ (respectively $\{\pm R e^{i\theta}, \theta \in [0, \varepsilon]\}$), the outcomes of I_t tends to 0 as δ tends to 0 (respectively R tends to $+\infty$, due to Lemma 4). Consequently, we can rotate the integration line of a small angle $\varepsilon \in \left(0, \frac{\pi}{4\alpha_1}\right)$ and the quantity we have to bound from above becomes $\int_0^\infty e^{ixre^{i\varepsilon}} I_t(re^{i\varepsilon}) dr$, with

$$I_t(re^{i\varepsilon}) = \int_0^t e^{(t-s)(A(re^{i\varepsilon})+B)} [e^{sB}, A(re^{i\varepsilon})] e^{sA(re^{i\varepsilon})} ds.$$

From Lemma 4, taking

$$\eta_t = \left\| \int_0^\infty e^{ixre^{i\varepsilon}} I_t(re^{i\varepsilon}) dr \right\|$$

we get, for large values of $|x|$

$$\begin{aligned} \eta_t &\le C(t) \int_0^\infty e^{-xr\sin(\varepsilon)} \left(r^{2\alpha} e^{-r^{2\alpha}\cos(2\alpha_1\varepsilon)t} + r^{2\alpha_1} e^{-r^{2\alpha_1}\cos(2\alpha_1\varepsilon)t} \right) dr \\ &\le \frac{C(t)}{|x|^{1+2\alpha}} \int_0^\infty e^{-\tilde{r}\sin(\varepsilon)} \left(\tilde{r}^{2\alpha} e^{-\frac{\tilde{r}^{2\alpha}}{|x|^{2\alpha}}\cos(2\alpha_1\varepsilon)t} + \tilde{r}^{2\alpha_1} e^{-\frac{\tilde{r}^{2\alpha}}{|x|^{2\alpha}}\cos(2\alpha_1\varepsilon)t} \right) d\tilde{r} \\ &\le \frac{C(t)}{|x|^{1+2\alpha}}. \end{aligned} \tag{3.12}$$

With (3.10), (3.11) and (3.12), we conclude that for large values of $|x|$ and for all $t \geq 0$

$$\left\| \mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})(x) \right\| \leq \frac{C_1(t)}{|x|^{1+2\alpha}},$$

which concludes the proof. □

Now, we state the proof of Lemma 3 in the higher space dimension case, i.e. when $d > 1$.

Proof of Lemma 3 Case $d > 1$. As previously, from (3.3), we only need to bound from above the function $\mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})$. Let $t > 0$ and $x \in \mathbb{R}^d$, the matrix $e^{(A(|\cdot|)+B)t}$ is split into two pieces as done in (3.9), thus, similarly to (3.10), we have

$$\mathfrak{F}^{-1}\left(e^{(A(|\cdot|)+B)t}\right)(x) = e^{tB}\,\mathrm{diag}(p_{\alpha_1}(t,x),\ldots,p_{\alpha_m}(t,x)) - \int_{\mathbb{R}^d} e^{ix\xi} I_t(|\xi|)d\xi,$$

where I_t has been defined in (3.5). Since for $x \in \mathbb{R}^d$ and $t > 0$

$$\begin{cases} p_{\alpha_i}(t,x) = \dfrac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}}, & \text{if } \alpha_i = 1; \\ \dfrac{B^{-1}t}{t^{\frac{d}{2\alpha_i}+1} + |x|^{d+2\alpha_i}} \leq p_{\alpha_i}(t,x) \leq \dfrac{Bt}{t^{\frac{d}{2\alpha_i}+1} + |x|^{d+2\alpha_i}}, & \text{if } \alpha_i \in (0,1). \end{cases}$$

the first term of the right hand side has the correct algebraic decay, it remains to bound the second term. Therefore, taking $t > 0$ and $|x| > 1$, using the spherical coordinates system in dimension $d > 1$ and Whittaker function $W_{0,\frac{d}{2}-1}$ (defined in [8] for example), we have

$$\begin{aligned} \int_{\mathbb{R}^d} e^{ix\xi} I_t(|\xi|)d\xi &= C_d \int_0^\infty \int_{-1}^1 I_t(r)\cos(|x|rs) r^{d-1}\left(1-s^2\right)^{\frac{d-3}{2}} ds dr \\ &= \frac{C_d}{|x|^{\frac{d-1}{2}}\sqrt{2\pi}} \Re\mathrm{e}\left(\int_0^\infty I_t(r) e^{\frac{d-1}{4}i\pi} W_{0,\frac{d}{2}-1}(2i|x|r) r^{\frac{d-1}{2}} dr\right) \\ &= \frac{C_d}{|x|^d\sqrt{2\pi}} \Re\mathrm{e}\left(\int_0^\infty I_t(\tilde{r}|x|^{-1}) e^{\frac{d-1}{4}i\pi} W_{0,\frac{d}{2}-1}(2i\tilde{r}) \tilde{r}^{\frac{d-1}{2}} d\tilde{r}\right) \end{aligned}$$

where C_d is a positive constant depending on d.

As done in the one dimension case, since the Whittaker function is bounded, we can rotate the integration line of a small angle $\varepsilon \in \left(0, \frac{\pi}{4\alpha_1}\right)$. Thus, using (3.6), we have the result if we prove that the following integral

$$\int_0^\infty \left| W_{0,\frac{d}{2}-1}(2i\tilde{r}e^{i\varepsilon}) \right| \tilde{r}^{\frac{d-1}{2}} (\tilde{r}^{2\alpha} + \tilde{r}^{2\alpha_1}) d\tilde{r}$$

is convergent. From [1], $W_{0,\frac{d}{2}-1}$ has the following asymptotic expressions, thus $W_{0,\frac{d}{2}-1}(z) \underset{|z|\to+\infty}{\sim} e^{-\frac{z}{2}}$ and

$$W_{0,\frac{d}{2}-1}(z) \underset{|z|\to 0}{\sim} \begin{cases} -\Gamma(\frac{d-1}{2})^{-1}\left(\ln(z) + \dfrac{\Gamma'(\frac{d-1}{2})}{\Gamma\left(\frac{d-1}{2}\right)}\right) z^{\frac{d-1}{2}}, & \text{if } d = 2 \\ \dfrac{\Gamma(d-2)}{\Gamma\left(\frac{d-1}{2}\right)} z^{\frac{3-d}{2}}, & \text{if } d \geq 3. \end{cases}$$

□

3.1 Lower Bound

The following result is important and needed to prove Theorem 1. It sets an algebraically lower bound for the solutions of the cooperative system (1.2). This result is valid for any dimension $d \in \mathbb{N}^*$. Moreover, since for all $i \in [\![1, m]\!]$, $f_i(0) = 0$, we have for all $s = (s_i)_{i=1}^m \in \mathbb{R}^m$ with $0 \le s \le M$

$$f_i(s) = \int_0^1 \nabla f_i(\sigma s) \cdot s \, d\sigma = \sum_{j=1}^m s_j \int_0^1 \frac{\partial f_i}{\partial s_j}(\zeta_\sigma) d\sigma$$

where $\zeta_\sigma = \sigma s \in [0, M]$ and $\frac{\partial f_i}{\partial s_j} : [0, M] \to \mathbb{R}$ is continuous for all $i, j \in [\![1, m]\!]$, since the system is cooperative, there exist constants $\gamma_{ij} > 0$ such that for all $i \in [\![1, m]\!]$ and $j \in [\![1, m]\!]$:

$$|\partial_i f_i(\zeta_\sigma)| \le \gamma_{ii} \quad \text{and} \quad \gamma_{ij} \le \partial_j f_i(\zeta_\sigma). \tag{3.13}$$

Lemma 5 *Let $u = (u_i)_{i=1}^m$ be the solution of the system (1.2), with non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3) and with reaction term $F = (f_i)_{i=1}^m$ satisfying (1.4), (H1), (H2) and (H3). Then, for all $i \in [\![1, m]\!]$ and $x \in \mathbb{R}^d$, there exists $\tau_1 > 0$ such that*

$$u_i(t, x) \ge \frac{\underline{c}\, t\, e^{-\gamma t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \tag{3.14}$$

for all $t \ge \tau_1$, where $\underline{c}$ and γ are positive constants.

Proof of Lemma 5 We split the proof into three steps: first, we prove the result for $i = m$, which serves as an initiation of the process. In an intermediate step, for all $i \in [\![1, m-1]\!]$, $t \ge 1$ and $s \in [0, t-1]$, we find a lower bound of $p_{\alpha_i}(\cdot, t-s) * (s^{\frac{d}{2\alpha}+1} + |\cdot|^{d+2\alpha})^{-1}$, that decays like $|x|^{-(d+2\alpha)}$ for large values of $|x|$. In a third step, for all $i \in [\![1, m-1]\!]$, $t \ge 1$ and $s \in [0, t-1]$, we prove that $u_i(t, \cdot)$ can be bounded from below by an expression that only depends on the integral $\displaystyle\int_0^t p_{\alpha_i}(\cdot, t-s) * (s^{\frac{d}{2\alpha}+1} + |\cdot|^{d+2\alpha})^{-1} ds$.

Step 1 We take $\gamma \ge \max_{j \in [\![1,m]\!]}(\gamma_{jj} + 1)$ with γ_{jj} defined in (3.13). Thus, we have for all $x \in \mathbb{R}^d$ and $t > 0$:

$$\partial_t u_m + (-\triangle)^{\alpha_m} u_m = f_m(u) \ge \int_0^1 \partial_m f_m(\zeta_\sigma) d\sigma u_m \ge -\gamma u_m,$$

By the maximum principle of reaction diffusion equations, we have for all $t \ge 0$

$$u_m(t, x) \ge e^{-\gamma t}(p_{\alpha_m}(t, \cdot) * u_{0m})(x),$$

Since $u_{0m}(\cdot) \not\equiv 0$ is continuous and nonnegative, we can find $\xi \in \mathbb{R}^d$ fixed, such that $u_{0m}(y) \ge C$ for all $y \in B_R(\xi)$ for some $R > 0$ and $C > 0$. If $|x| > R$, $t \ge 1$ and using that $\alpha := \alpha_m < 1$, we get

$$\begin{aligned}(p_{\alpha_m}(t, \cdot) * u_{0m})(x) &\ge C \int_{B_R(\xi)} \frac{B^{-1} t}{t^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dy \\ &= C \int_{B_R(0)} \frac{B^{-1} t}{t^{\frac{d}{2\alpha}+1} + |x-\xi-z|^{d+2\alpha}} dz.\end{aligned}$$

We also have $|x-\xi-z| \leq \left(2+\frac{|\xi|}{R}\right)|x|$. Thus

$$t^{\frac{d}{2\alpha}+1}+|x-\xi-z|^{d+2\alpha} \leq \left(2+\frac{|\xi|}{R}\right)^{d+2\alpha} t^{\frac{d}{2\alpha}+1}+\left(2+\frac{|\xi|}{R}\right)^{d+2\alpha} |x|^{d+2\alpha}.$$

Then

$$\begin{aligned}(p_{\alpha_m}(t,\cdot)*u_{0m})(x) &\geq \frac{CB^{-1}}{\left(2+\frac{|\xi|}{R}\right)^{d+2\alpha}} \int_{B_R(0)} \frac{t}{t^{\frac{d}{2\alpha}+1}+|x|^{d+2\alpha}} dz \\ &= \frac{\widetilde{C}t}{t^{\frac{d}{2\alpha}+1}+|x|^{d+2\alpha}},\end{aligned}$$

where $\tilde{C}$ is a positive constant. If $|x| \leq R$ and $t \geq 1$,

$$\begin{aligned}(p_{\alpha_m}(t,\cdot)*u_{0m})(x) &\geq \int_{B_R(\xi)} \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1}+|x-y|^{d+2\alpha}} u_{0m}(y)dy \\ &\geq \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1}+(2R+|\xi|)^{d+2\alpha}} \int_{B_R(\xi)} u_{0m}(y)dy \\ &\geq \frac{\overline{C}t}{t^{\frac{d}{2\alpha}+1}} \geq \frac{\overline{C}t}{t^{\frac{d}{2\alpha}+1}+|x|^{d+2\alpha}},\end{aligned}$$

for some small constant $\overline{C} > 0$. Then, there exist $C_m > 0$ such that for all $x \in \mathbb{R}^d$ and $t \geq 1$

$$u_m(t,x) \geq \frac{C_m t e^{-\gamma t}}{t^{\frac{d}{2\alpha}+1}+|x|^{d+2\alpha}}. \tag{3.15}$$

Step 2 By similar computations as done in Step 1, it is possible to find a constant $C > 0$ such that for all $x \in \mathbb{R}^d$, $t > 1$ and $s \in [0, t-1]$:

- if $\alpha_i = 1$ then

$$\begin{aligned}&p_{\alpha_i}(t-s,\cdot)*\left(s^{\frac{d}{2\alpha}+1}+|\cdot|^{d+2\alpha}\right)^{-1}(x) \\ &\qquad \geq \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{s^{\frac{d}{2\alpha}+1}+|x-y|^{d+2\alpha}} dy \\ &\qquad \geq \frac{1}{(4\pi(t-s))^{\frac{d}{2}}\left(s^{\frac{d}{2\alpha}+1}+|x|^{d+2\alpha}\right)},\end{aligned}$$

- if $\alpha_i \in (0,1)$ then

$$\begin{aligned}&p_{\alpha_i}(t-s,\cdot)*\left(s^{\frac{d}{2\alpha}+1}+|\cdot|^{d+2\alpha}\right)^{-1}(x) \\ &\quad \geq \int_{\mathbb{R}^d} \frac{1}{\left((t-s)^{\frac{d}{2\alpha_i}+1}+|y|^{d+2\alpha_i}\right)\left(s^{\frac{d}{2\alpha}+1}+|x-y|^{d+2\alpha}\right)} dy \\ &\quad \geq \frac{(t-s)^{-\frac{d}{2\alpha_i}}}{s^{\frac{d}{2\alpha}+1}+|x|^{d+2\alpha}}.\end{aligned}$$

Step 3 For $i \in [\![1, m-1]\!]$, we have for all $x \in \mathbb{R}^d$ and $t \geq 0$

$$\partial_t u_i + (-\triangle)^{\alpha_i} u_i \geq \int_0^1 \partial_m f_i(\zeta_\sigma) d\sigma u_m + \int_0^1 \partial_i f_i(\zeta_\sigma) d\sigma u_i \geq \gamma_{im} u_m - \gamma u_i,$$

where $\zeta_\sigma = \sigma u$. Then, by the maximum principle of reaction diffusion equations and Duhamel's formula, we have for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$u_i(t,x) \geq e^{-\gamma t}(p_{\alpha_i}(t,\cdot) * u_{0i})(x) + \gamma_{im} e^{-\gamma t} \int_0^t \int_{\mathbb{R}^d} p_{\alpha_i}(t-s, y) u_m(s, x-y) e^{\gamma s} dy ds.$$

So, taking $t \geq \tau_1$ with at least $\tau_1 \geq 3$, and using (3.15), we get

$$u_i(t,x) \geq C_m \gamma_{im} e^{-\gamma t} \int_1^{t-1} \int_{\mathbb{R}^d} p_{\alpha_i}(t-s, y) \frac{s e^{(\gamma - \gamma_{mm})s}}{s^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dy ds$$

Using Step 2, we get the following lower bound, for all $x \in \mathbb{R}^d$ and $t \geq \tau_1$ with t_1 large if necessary:

$$\begin{aligned} u_i(t,x) &\geq C_i e^{-\gamma t} \int_1^{t-1} \frac{s e^{(\gamma-\gamma_{mm})s}}{(t-s)^{\frac{d}{2\alpha}} \left(s^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}\right)} ds \\ &\geq C_i \frac{e^{-\gamma t}}{t^{\frac{d}{2\alpha}}} \int_1^{t-1} \frac{e^s}{s^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}} ds \\ &\geq C_i \frac{e^{-\gamma t}(e^{t-1} - e)}{t^{\frac{d}{2\alpha}} \left(t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}\right)} \\ &\geq \frac{C_i t e^{-\gamma t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}. \end{aligned}$$

□

4 Proof of Theorem 1

Inspired by the formal analysis done in [7], we construct an explicit supersolution (respectively subsolution) of the form

$$v(t,x) = a \left(1 + b(t)|x|^{\delta(d+2\alpha)}\right)^{-\frac{1}{\delta}} \phi_1, \tag{4.1}$$

where $b(t)$ is a time continuous function asymptotically proportional to $e^{-\delta\lambda_1 t}$, $\phi_1 = (\phi_{1,i})_{i=1}^m \in \mathbb{R}^m$ is the normalized (positive) principal eigenvector of $DF(0)$ associated to the principal eigenvalue λ_1, and δ is equal to δ_1 (respectively δ_2) defined in (H4) (respectively (H5)).

The following result allow us to understand the behavior of the fractional laplacian $(-\triangle)^{\alpha_i}$ on the function v_i defined by (4.1) for all $i \in [\![1, m]\!]$. The estimate obtained in Lemma 6 is crucial at the moment to prove that the function v given by (4.1) serves as super and subsolution in Lemmas 7 and 8, respectively.

Lemma 6 *Let v be defined as in (4.1). Then, there exist a constant $D > 0$ such that for all $i \in [\![1, m]\!]$, $t > 0$ and $x \in \mathbb{R}^d$*

$$| (-\triangle)^{\alpha_i} v_i(t, x) | \leq Db(t)^{\frac{2\alpha_i}{\delta(d+2\alpha)}} v_i(t, x),$$

where $\alpha_i \in (0, 1]$.

Proof of Lemma 6 The case $\alpha_i = 1$ is trivial. For $\alpha_i \in (0, 1)$ and $\delta \geq \dfrac{2}{d+2\alpha}$, since $(-\Delta)^{\alpha_i}$ is $2\alpha_i$-homogeneous, we only need to prove

$$| (-\triangle)^{\alpha_i} w(x) | \leq Dw(x)$$

where $w(x) = (1 + |x|^{\delta(d+2\alpha)})^{-\frac{1}{\delta}}$.

We consider the following decomposition, which is the central part of the proof :

$$\begin{aligned}(-\triangle)^{\alpha_i} w(x) = &\int_{|y|>3|x|/2} \frac{w(x) - w(y)}{|x - y|^{d+2\alpha_i}} dy + \int_{B_{|x|/2}(x)} \frac{w(x) - w(y)}{|x - y|^{d+2\alpha_i}} dy \\ &+ \int_{\{|x|\leq 2|y|\leq 3|x|\}\setminus B_{|x|/2}(x)} \frac{w(x) - w(y)}{|x - y|^{d+2\alpha_i}} dy \\ &+ \int_{|y|\leq|x|/2} \frac{w(x) - w(y)}{|x - y|^{d+2\alpha_i}} dy.\end{aligned}$$

Each piece is easily bounded, as in [4] for instance. □

In what follows, we will use the results of previous sections to obtain appropriate sub and super solutions to (1.2) of the form (4.1). We divide the proof of Theorem 1 in two lemmas.

Lemma 7 *Assume that F satisfies (1.4), (H1), (H2), (H3) and (H4). Let u be the solution to (1.2) with u_0 satisfying the assumptions of Theorem 1. Then, for every $\mu = (\mu_i)_{i=1}^m > 0$, there exists $c > 0$ such that, for all $t > \tau$, with $\tau > 0$ large enough*

$$\left\{x \in \mathbb{R}^d \mid |x| > ce^{\frac{\lambda_1}{d+2\alpha}t}\right\} \subset \left\{x \in \mathbb{R}^d \mid u(t, x) < \mu\right\}.$$

Proof of Lemma 7 We consider the function $\overline{u}$ given by (4.1) with $\delta = \delta_1$ as in (H4). The idea is to adjust $a > 0$ and $b(t)$ so that the function $\overline{u}$ serves as supersolution of (1.2).

In the sequel, a is any positive constant satisfying

$$a \geq \left(\frac{D + \lambda_1}{c_{\delta_1}}\right)^{\frac{1}{\delta_1}} \max_{i\in[\![1,m]\!]} \left(\frac{1}{\phi_{1,i}}\right),$$

where c_{δ_1} is defined in (H4). For any constant $B \in (0, (1 + D\lambda_1^{-1})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}})$, where $D > 0$ is given in Lemma 6, we consider the following ordinary differential equation

$$b'(t) + \delta_1 Db(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}+1} + \delta_1\lambda_1 b(t) = 0 \tag{4.2}$$

with the initial condition $b(0) = (-D\lambda_1^{-1} + B^{-\frac{2\alpha}{\delta_1(d+2\alpha)}})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}}$, whose solution is given by

$$b(t) = \left(-D\lambda_1^{-1} + B^{-\frac{2\alpha}{\delta_1(d+2\alpha)}} e^{\frac{2\alpha\lambda_1}{d+2\alpha}t}\right)^{-\frac{\delta_1(d+2\alpha)}{2\alpha}}$$

For all $t \geq 0$, we have $b(t) \geq 0$ and more precisely

$$Be^{-\lambda_1 \delta_1 t} \leq b(t) \leq b(0) \leq 1$$

Defining

$$\mathcal{L}(\overline{u}_i) = \partial_t \overline{u}_i + (-\triangle)^{\alpha_i} \overline{u}_i - f_i(\overline{u})$$

and using Lemma 6, we have for all $i \in [\![1, m]\!]$

$$\begin{aligned}\mathcal{L}(\overline{u}_i) &= \partial_t \overline{u}_i + (-\triangle)^{\alpha_i} \overline{u}_i - Df_i(0)\overline{u} + [Df_i(0)\overline{u} - f_i(\overline{u})] \\ &\geq \frac{a\phi_{1,i}}{\delta_1 \left(1+b(t)|x|^{\delta_1(d+2\alpha)}\right)^{\frac{1}{\delta_1}+1}} \left\{-b'(t) - \delta_1 D b(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}+1} - \delta_1 \lambda_1 b(t)\right\} |x|^{\delta_1(d+2\alpha)} \\ &\quad + \frac{a\phi_{1,i}}{\left(1+b(t)|x|^{\delta_1(d+2\alpha)}\right)^{\frac{1}{\delta_1}+1}} \left\{-Db(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}} - \lambda_1 + c_h \phi_{1,i}^{\delta_1} a^{\delta_1}\right\} \geq 0.\end{aligned}$$

Due to Lemma 3, for a fixed $t_0 > 0$, there exists $t_1 \geq 0$ such that for all $x \in \mathbb{R}^d$ and all $i \in [\![1, m]\!]$, we have $\overline{u}_i(t_1, x) \geq u_i(t_0, x)$. Thus, for any $(\mu_i)_{i=1}^m > 0$, we define for $i \in [\![1, m]\!]$ the constants

$$c_i^{d+2\alpha} := a\phi_{1,i} e^{\lambda_1(t_1 - t_0)} [\mu_i B^{\frac{1}{\delta_1}}]^{-1}.$$

and we set $c = \max_{i \in [\![1,m]\!]} c_i$.

Finally, by Theorem 2 we have, for all $t \geq t_0$, all $x \in \mathbb{R}^d$ and all $i \in [\![1, m]\!]$: $\overline{u}_i(t + t_1 - t_0, x) \geq u_i(t, x)$. Moreover, if $|x| > ce^{\frac{\lambda_1}{d+2\alpha} t}$, then, for all $t > \tau := t_0$ and all $i \in [\![1, m]\!]$

$$u_i(t, x) \leq \overline{u}_i(t + t_1 - t_0, x) = \frac{a\phi_{1,i}}{\left(1 + b(t + t_1 - t_0)|x|^{\delta_1(d+2\alpha)}\right)^{\frac{1}{\delta_1}}} < \mu_i.$$

□

Lemma 8 *Let $d \geq 1$ and assume that F satisfies (1.4), (H1), (H2), (H3) and (H5). Let u be the solution to (1.2) with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3). Then, for all $i \in [\![1, m]\!]$, there exist constants $\varepsilon_i > 0$ and $C > 0$ such that,*

$$u_i(t, x) > \varepsilon_i, \quad \textit{for all} \quad t \geq t_1 \;\; \textit{and} \;\; |x| < Ce^{\frac{\lambda_1}{d+2\alpha} t},$$

with $t_1 > 0$ large enough.

Proof of Lemma 8 As in the previous proof, we consider the function $\underline{u}$ given by (4.1) with $\delta = \delta_2$ defined in (H5). Since, $\underline{u}_i(0, \cdot) \leq u_{0i}$ may not hold for all $i \in [\![1, m]\!]$, we look for a time $t_1 > 0$ such that $\underline{u}_i(0, \cdot) \leq u_i(t_1, \cdot)$ for all $i \in [\![1, m]\!]$. Indeed, let L be a constant greater than $\max\{D, \lambda_1\}$, where D is given by Lemma 6. We choose $t_1 \geq \max(\tau_1, 2D\lambda_1^{-1})$ large enough, where $\tau_1 > 0$ was obtained in Lemma 5, so that if we set

$$a = \frac{\min_{i \in [\![1,m]\!]} C_i \, e^{-\gamma t_1}}{2 \max_{i \in [\![1,m]\!]} \phi_{1,i} \, t_1^{\frac{d}{2\alpha}}} \quad \text{and} \quad B = \left(\frac{2}{t_1}\right)^{\frac{(d+2\alpha)}{2\alpha}\delta_2}, \tag{4.3}$$

then

$$a \leq \left(\frac{\min\limits_{i \in [\![1,m]\!]} \phi_{1,i} \, \lambda_1}{2c_{\delta_2}} \right)^{\frac{1}{\delta_2}} \quad \text{and} \quad B \leq (D\lambda_1^{-1})^{-\frac{(d+2\alpha)}{2\alpha}\delta_2},$$

where c_{δ_2} is defined in (H5). Then we set

$$b(t) = \left(D\lambda_1^{-1} + B^{-\frac{2\alpha}{\delta_2(d+2\alpha)}} e^{\frac{2\alpha\lambda_1}{d+2\alpha}t} \right)^{-\frac{(d+2\alpha)}{2\alpha}\delta_2}.$$

Using Lemma 6 and (H5), similarly to the previous proof, we can state that, for all $i \in [\![1, m]\!]$,

$$\partial_t \underline{u}_i + (-\triangle)^{\alpha_i} \underline{u}_i - f_i(\underline{u}) \leq 0, \quad \text{in } (0, +\infty) \times \mathbb{R}^d.$$

From Lemma 5, we know that for all $i \in [\![1, m]\!]$ and all $x \in \mathbb{R}^d$

$$u_i(t_1, x) \geq \underline{c} \frac{t_1 e^{-\gamma t_1}}{t_1^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}.$$

By (4.3), we deduce

$$\underline{c} t_1 e^{-\gamma t_1} \left(1 + b(0)|x|^{\delta_2(d+2\alpha)}\right)^{\frac{1}{\delta_2}} \geq \frac{\underline{c}}{2} t_1 e^{-\gamma t_1} \left(1 + b(0)^{\frac{1}{\delta_2}} |x|^{d+2\alpha}\right)$$
$$\geq a\phi_i \left(t_1^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}\right).$$

Therefore, we get, for all $i \in [\![1, m]\!]$, $u_i(t_1, \cdot) \geq \underline{u}_i(0, \cdot)$ in $\mathbb{R}^d$, and by Theorem 2, we have for all $t \geq t_1$

$$u_i(t, \cdot) \geq \underline{u}_i(t - t_1, \cdot), \quad \text{in } \mathbb{R}^d$$

Finally we choose

$$\varepsilon_i = \frac{a\phi_{1,i}}{2^{\frac{1}{\delta_2}}} \quad \text{and} \quad C^{d+2\alpha} = e^{-\lambda_1 t_1} B^{-\frac{1}{\delta_2}}.$$

If $t \geq \tau := t_1$ and $|x| \leq Ce^{\frac{\lambda_1}{d+2\alpha}t}$, we have

$$u_i(t, x) \geq \underline{u}_i(t - t_1, x) = \frac{a\phi_{1,i}}{\left(1 + b(t - t_1)|x|^{\delta_2(d+2\alpha)}\right)^{\frac{1}{\delta_2}}} \geq \frac{a\phi_{1,i}}{2^{\frac{1}{\delta_2}}} = \varepsilon_i.$$

□

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References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables. Dover Publications, New York (1972)

2. Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusions arising in population genetics. Adv. Math. **30**, 33–76 (1978)
3. Barles, G., Evans, L.C., Souganidis, P.E.: Wavefront propagation for reaction-diffusion systems of PDE. Duke Math. J. **61**(3), 835–858 (1990)
4. Bonforte, M., Vazquez, J.: Quantitative local and global a priori estimates for fractional nonlinear diffusion equations. Adv. Math. **250**, 242–284 (2014)
5. Busca, J., Sirakov, B.: Harnack type estimates for nonlinear elliptic systems and applications. Ann. Inst. H. Poincaré Anal. Non Linéaire **21**, 543–590 (2004)
6. Cabré, X., Roquejoffre, J.: The influence of fractional diffusion in Fisher-KPP equation. Commun. Math. Phys. **320**, 679–722 (2013)
7. Cabré, X., Coulon, A.C., Roquejoffre, J.M.: Propagation in Fisher-KPP type equations with fractional diffusion in periodic media. C. R. Math. Acad. Sci. Paris **350**(19–20), 885–890 (2012)
8. Erdélyi, A.: Higher Transcendental Functions, vol. I. McGraw-Hill Book Company, Inc., New York (1953)
9. Evans, L.C., Souganidis, P.E.: A PDE approach to geometric optics for certain semilinear parabolic equations. Indiana Univ. Math. J. **45**(2), 141–172 (1989)
10. Felmer, P., Yangari, M.: Fast propagation for fractional KPP equations with slowly decaying initial conditions. SIAM J. Math. Anal. **45**(2), 662–678 (2013)
11. Hamel, F., Roques, L.: Fast propagation for KPP equations with slowly decaying initial conditions. J. Differ. Equ. **249**, 1726–1745 (2010)
12. Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, vol. 840. Springer-Verlag, New York (1981)
13. Kolmogorov, A.N., Petrovsky, I.G., Piskunov, N.S.: Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. Bull. Univ. État Moscou Sér. Inter. A **1**, 1–26 (1937)
14. Lewis, M., Li, B., Weinberger, H.: Spreading speed and linear determinacy for two-species competition models. J. Math. Biol. **45**, 219–233 (2002)
15. Lui, R.: Biological growth and spread modeled by systems of recursions. I. Mathematical theory. Math. Biosci. **93**(2), 269–295 (1989)
16. Stan, D., Vázquez, L.J.: The Fisher-KPP equation with nonlinear fractional diffusion. SIAM J. Math. Anal. **46**(5), 3241–3276 (2014)
17. Weinberger, H.F., Lewis, M., Li, B.: Anomalous spreading speeds of cooperative recursion systems. J. Math. Biol. **55**, 207–222 (2007)
18. Weinberger, H.F., Lewis, M., Li, B.: Analysis of linear determinacy for spread in cooperative models. J. Math. Biol. **45**, 183–218 (2002)