J Dyn Diff Equat (2015) 27:743–762
DOI 10.1007/s10884-015-9470-x

Steady States of Fokker–Planck Equations: II. Non-existence

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Received: 17 December 2013 / Revised: 9 June 2015 / Published online: 25 July 2015

Abstract This is the second paper in a series concerning the study of steady states, including stationary solutions and measures, of a Fokker–Planck equation in a general domain in $\mathbb{R}^n$ with L^p_{loc} drift term and $W^{1,p}_{loc}$ diffusion term for any $p > n$. In this paper, we obtain some non-existence results of stationary measures under conditions involving anti-Lyapunov type of functions associated with the stationary Fokker–Planck equation. When combined with the existence results showed in part I of the series (Huang et al. in J. Dyn Differ Equ 10.1007/s10884-015-9454-x, 2015) contained in the same volume, not only will these results yield necessary and sufficient conditions for the existence of stationary measures, but also they

Dedicated to the memory of Professor K. Kirchgaessner.

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provide a useful tool for one to study noise perturbations of systems of ordinary differential equations, especially with respect to problems of stochastic bifurcations, as demonstrated in some examples contained in this paper. Our analysis is based on the level set method, in particular the integral identity, and measure estimates contained in our work (Huang et al. in Ann Probab 43:1712–1730, 2015).

Keywords Fokker–Planck equation · Non-existence · Stationary solution · Stationary measure · Level set method

Mathematics Subject Classification Primary 35Q84 · 60J60 · 37B25 · Secondary 60H10 · 37H20

1 Introduction

In this paper, we investigate the problem of non-existence for stationary measures of a Fokker–Planck equation defined in a general domain in $\mathbb{R}^n$.

To be more precise, let $\mathcal{U} \subset \mathbb{R}^n$ be a connected open set which can be bounded, unbounded, or the entire space $\mathbb{R}^n$. We consider the *stationary Fokker–Planck equation* on $\mathcal{U}$:

$$\begin{cases} Lu(x) =: \partial_{ij}^2(a^{ij}(x)u(x)) - \partial_i(V^i(x)u(x)) = 0, \quad x \in \mathcal{U}, \\ u(x) \geq 0, \quad \int_{\mathcal{U}} u(x)\mathrm{d}x = 1, \end{cases} \tag{1.1}$$

where L is the *Fokker–Planck operator*, $A = (a^{ij})$ is an everywhere positive semi-definite matrix, called *diffusion matrix*, and $V = (V^i)$ is a vector field on $\mathcal{U}$, called the *drift field*. This equation is in fact the one satisfied by stationary solutions of the *Fokker–Planck equation*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = Lu(x,t), \quad x \in \mathcal{U},\ t > 0, \\ u(x,t) \geq 0, \quad \int_{\mathcal{U}} u(x,t)\mathrm{d}x = 1. \end{cases} \tag{1.2}$$

In the above and also through the rest of the paper, we use short notations $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \ldots, n$ whenever applicable.

Following [4–6,8,15], we make the following standard hypothesis:

(**A**) $a^{ij} \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$, $V^i \in L^p_{\mathrm{loc}}(\mathcal{U})$ for all $i, j = 1, \ldots, n$, where $p > n$ is fixed.

Under the regularity condition (A), it is necessary to consider *weak stationary solutions* of the Fokker–Planck equation (1.2), i.e., continuous functions u satisfying the following weak form of the stationary Fokker–Planck equation:

$$\begin{cases} \displaystyle\int_{\mathcal{U}} \mathcal{L}f(x)u(x)\mathrm{d}x = 0, \quad \text{for all } f \in C_0^\infty(\mathcal{U}), \\ u(x) \geq 0, \quad \int_{\mathcal{U}} u(x)\mathrm{d}x = 1, \end{cases} \tag{1.3}$$

where

$$\mathcal{L} = a^{ij}\partial_{ij}^2 + V^i\partial_i$$

is the adjoint Fokker–Planck operator and $C_0^\infty(\mathcal{U})$ denotes the space of C^∞ functions on $\mathcal{U}$ with compact supports. More generally, one considers *stationary measures* of the Fokker–Planck Eq. (1.2), i.e., Borel probability measures μ satisfying

$$V^i \in L^1_{loc}(\mathcal{U}, \mu), \quad i = 1, 2, \ldots, n, \text{ and,} \tag{1.4}$$

$$\int_{\mathcal{U}} \mathcal{L} f(x) \mathrm{d}\mu(x) = 0, \qquad \text{for all } f \in C_0^\infty(\mathcal{U}). \tag{1.5}$$

A stationary measure μ of (1.2) is called *regular* if it admits a continuous density function u with respect to the Lebesgue measure, i.e., $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$. For given continuous non-negative function u on $\mathcal{U}$, it is clear that the probability measure μ given by $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$ is a stationary measure of (1.2) if and only if u is a weak stationary solution of (1.2). In fact, under the condition (A), it follows from a regularity theorem in [7] (also recalled in Theorem 2.1 below) that if (a^{ij}) is everywhere positive definite in $\mathcal{U}$, then any stationary measure μ of (1.2) must be regular with a positive density function $u \in W^{1,p}_{loc}(\mathcal{U})$.

The Fokker–Planck equation (1.2) naturally arises in the white noise perturbation of the system of ordinary differential equations (ODE's)

$$\dot{x} = V(x), \qquad x \in \mathcal{U} \subset \mathbb{R}^n. \tag{1.6}$$

Under the white noise perturbation $G(x)\dot{W}$, where $G = (g^{ij})$ is an $n \times m$ matrix-valued function on $\mathcal{U}$ for some positive integer m, called the *noise matrix*, and W is the standard m-dimensional Brownian motion, one obtains the following system of Itô stochastic differential equations

$$\mathrm{d}x = V(x)\mathrm{d}t + G(x)\mathrm{d}W, \qquad x \in \mathcal{U} \subset \mathbb{R}^n. \tag{1.7}$$

Under the assumption that the stochastic differential equation (1.7) generates a diffusion process in $\mathcal{U}$, there is a well-defined transition probability function associated with this process. If the transition probability function admits a transition density function, then the density function is actually a fundamental solution of the Fokker–Planck Eq. (1.2) with $A(x) = \frac{G(x)G^\top(x)}{2}$. When $\mathcal{U} = \mathbb{R}^n$ and (1.7) generates a global in time diffusion process in $\mathbb{R}^n$, it is well-known that any invariant measure of the diffusion process is necessarily a stationary measure of the corresponding Fokker–Planck Eq. (1.2) in $\mathbb{R}^n$, and vice versa under some conditions (see [6,8,10] for more details).

We recall from [15] that a non-negative function $U \in C^2(\mathcal{U})$ is a *Lyapunov function* (resp. *weak Lyapunov function*) with respect to (1.1) if it is a *compact function* in $\mathcal{U}$ (see Sect. 2 for definition) satisfying

$$\limsup_{x \to \partial\mathcal{U}} \mathcal{L}U(x) = \limsup_{x \to \partial\mathcal{U}} (a^{ij}(x)\partial^2_{ij}U(x) + V^i(x)\partial_i U(x)) \le -\gamma \tag{1.8}$$

for some constant $\gamma > 0$ (resp. $\mathcal{L}U \le 0$ near $\partial\mathcal{U}$). We note that when $\mathcal{U}$ is unbounded, the notion $\partial\mathcal{U}$ and the limit $x \to \partial\mathcal{U}$ in the above (and also in the below) should be understood under the topology of the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ where $\partial\mathbb{R}^n$ is the set of the infinity elements x_*^∞ of the ray through $x_* \in \mathbb{S}^{n-1}$ (see Sect. 2 for details). Consequently, if $\mathcal{U} = \mathbb{R}^n$, then $x \to \partial\mathbb{R}^n$ under this topology simply means $x \to \infty$.

In part I of the series [15] contained in the same volume, we have obtained various new existence results for stationary measures of (1.2) with non-Lipschitz drift field and diffusion coefficients, generalizing those of [1–11,18–21]. In particular, the following result is proved.

Theorem A^0. ([15]) *Assume that* (A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *If there exists a Lyapunov function with respect to* (1.1) *in* $\mathcal{U}$, *then* (1.2) *admits a stationary measure* μ *which is regular with positive density lying in the space* $W^{1,p}_{loc}(\mathcal{U})$.

In [15], we also showed the following existence result of stationary measures when only a weak Lyapunov function with respect to (1.1) in $\mathcal{U}$ is available, provided that the weak

Lyapunov function is of the *class of* $\mathcal{B}_*(A)$. As defined in [15, Section 2.1], the class $\mathcal{B}_*(A)$ consists of compact functions U for which the decay rates of $a^{ij}\partial_i U \partial_j U$ near $\partial\mathcal{U}$ can be controlled in some way.

Theorem B^0. ([15]) *Assume that* (A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *If there exists a weak Lyapunov function with respect to* (1.1) *in* $\mathcal{U}$ *which is of the class* $\mathcal{B}_*(A)$, *then* (1.2) *admits a stationary measure* μ *which is regular with positive density lying in the space* $W^{1,p}_{loc}(\mathcal{U})$.

The existence of a stationary measure of (1.2) resembles that of a global attractor for the ODE system (1.6). Under the assumption that the ODE system (1.6) generates a local flow in $\mathcal{U}$, it is well-known that if (1.6) admits a Lyapunov function U in $\mathcal{U}$, i.e.,

$$\limsup_{x\to\partial\mathcal{U}} V(x)\cdot\nabla U(x) = \limsup_{x\to\partial\mathcal{U}} V^i(x)\partial_i U(x) \le -\gamma \tag{1.9}$$

for some constant $\gamma > 0$, then (1.6) admits a global attractor in $\mathcal{U}$ (see Theorem 5.1 in the Appendix). In this sense, Theorem A^0 may be regarded as a stochastic counterpart of Theorem 5.1.

By simply reversing time, Theorem 5.1 also implies that if the ODE system (1.6) admits an anti-Lyapunov function U in $\mathcal{U}$, i.e.,

$$\liminf_{x\to\partial\mathcal{U}} V(x)\cdot\nabla U(x) = \liminf_{x\to\partial\mathcal{U}} V^i(x)\partial_i U(x) \ge \gamma \tag{1.10}$$

for some constant $\gamma > 0$, then it admits no global attractor in $\mathcal{U}$ (see Remark 5.1). Given the similarity between Theorem 5.1 and Theorem A^0, a natural question is whether one can obtain a non-existence result of stationary measures of (1.2) by having a stochastic version of *anti-Lyapunov function* with respect to (1.1) in $\mathcal{U}$, i.e., a non-negative compact function $U \in C^2(\mathcal{U})$ such that

$$\liminf_{x\to\partial\mathcal{U}} \mathcal{L}U(x) = \liminf_{x\to\partial\mathcal{U}} (a^{ij}(x)\partial^2_{ij}U(x) + V^i(x)\partial_i U(x)) \ge \gamma \tag{1.11}$$

for some constant $\gamma > 0$. It turns out that, unlike the case of the deterministic ODE system (1.6) satisfying (1.10), the condition (1.11) alone is not sufficient to guarantee the non-existence of stationary measures of (1.2) due to the impact of noise (see Remark 3.2 (2)). In fact, in addition to (1.11), such non-existence requires that the noise does not become too large near $\partial\mathcal{U}$, for otherwise large noise could force the existence of a stationary measure (see Remark 4.2 (2) in [15] for discussions and an example concerning large noise stabilization). To be more precise, an additional assumption for the non-existence is that U is of the *class* $\mathcal{B}^*(A)$ - a condition controlling the growth rates of $a^{ij}\partial_i U \partial_j U$ near $\partial\mathcal{U}$ (see Sect. 2 for details).

Our main results of this paper are as follows.

Theorem A *Assume that* (A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *If there exists an anti-Lyapunov function in* $\mathcal{U}$ *with respect to* (1.1) *which is of the class* $\mathcal{B}^*(A)$, *then* (1.2) *admits no stationary measure in* $\mathcal{U}$.

When $\gamma = 0$ in (1.11), in particular $\mathcal{L}U \ge 0$ near $\partial\mathcal{U}$, we obtain a *weak anti-Lyapunov function* in $\mathcal{U}$ with respect to (1.1). Comparable to Theorem B^0, we will show the following non-existence result involving a weak anti-Lyapunov function U that is of the *class* $\mathcal{B}(A)$ - a condition controlling both decay and growth rates of $a^{ij}\partial_i U \partial_j U$ near $\partial\mathcal{U}$ (see Sect. 2 for details).

Theorem B *Assume that* (A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *If there exists a weak anti-Lyapunov function in* $\mathcal{U}$ *with respect to* (1.1) *which is of the class* $\mathcal{B}(A)$, *then* (1.2) *admits no stationary measure in* $\mathcal{U}$.

Like the proof of Theorems A^0, B^0, the one of Theorems A, B uses the level set method based on the integral identity and the derivative formula which we derived in [14] (see also Sect. 3.1). As to be seen in the paper, the integral identity and the derivative formula will play crucial roles in capturing information of a weak stationary solution in each sublevel set of an anti-Lyapunov type of function from its boundary.

Using Theorems A^0, A above, we are able to derive a necessary and sufficient condition for the existence of stationary measures of (1.2) when the diffusion is sufficiently small in amplitude. The following result is a special case of Corollary 3.4 in this paper when $U(x) = |x|^2/2$.

Corollary *Assume that* (A) *holds in* $\mathbb{R}^n$, $A = (a^{ij})$ *is bounded under the sup-norm and uniformly positive definite in* $\mathbb{R}^n$, *and the limit*

$$\lim_{x\to\infty} V(x)\cdot x =: \nu$$

exists. Then $\nu < 0$ *is a necessary condition for the existence of a stationary measure of the Fokker–Planck Eq.* (1.2) *in* $\mathbb{R}^n$, *and, it is also sufficient for the existence if* $\limsup_{x\to\infty} |A(x)| < -\frac{\nu}{\sqrt{n}}$.

To make a comparison of the above result with the corresponding ones for the deterministic ODE system (1.6) when $\mathcal{U} = \mathbb{R}^n$, let us assume that (1.6) generates a flow φ^t on $\mathbb{R}^n$ and V is continuous. It follows from Theorem 5.1 and Remark 5.1 that (1.6) admits a global attractor in $\mathbb{R}^n$ when $\nu < 0$ and admits no global attractor in $\mathbb{R}^n$ when $\nu > 0$. When $\nu = 0$, the existence of a global attractor of (1.6) is undetermined. To the contrary, for the stochastic case with $\nu = 0$, the Corollary guarantees the non-existence of stationary measures of the Fokker–Planck Eq. (1.2). This indicates a special role played by noise perturbations.

The existence and non-existence results above will play important roles in studying stochastic bifurcation problems in which Theorems A, A^0 are typically used at regular parameter values for the existence or non-existence of stationary measures, while Theorems B, B^0 should play the same roles but at the critical parameter values. We refer the reader to the examples in Sect. 5 for details. For many stochastic systems, there are also bifurcations occurring due to dramatic "dynamical changes" with respect to parameters, or alternatively due to the "structural changes" of stationary measures of the corresponding Fokker–Planck equations. We will study these bifurcation phenomena in separate works.

We remark that if the Eq. (1.6) is defined on $\mathcal{U} \times M$, where M is a smooth, compact manifold without boundary, then one can modify the definitions of anti-Lyapunov type of functions in this paper in an obvious way by replacing the domain $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{U} \times M$. Then the proofs in later sections can be modified accordingly so that Theorems A, B also hold with respect to such generalized domains.

This paper is organized as follows. Section 2 is a preliminary section in which we review the notions of boundary, compact and Lyapunov-like functions for a general domain, define classes $\mathcal{B}^*(A)$, $\mathcal{B}(A)$ of compact functions, introduce anti-Lyapunov functions and their weak forms, and also review the regularity Theorem in [7]. In Sect. 3, we study the non-existence of stationary measures under anti-Lyapunov conditions. We also give necessary and sufficient conditions for the existence of stationary measures under small diffusion. Similar non-existence problem is considered in Sect. 4 under weak anti-Lyapunov conditions. In

Sect. 5, we give some examples of applications of the existence and non-existence results to problems of stochastic bifurcations. In the Appendix at the end, we summarize some basic properties of dissipative dynamical systems in a general domain of $\mathbb{R}^n$.

Through the rest of the paper, for simplicity, we will use the same symbol $|\cdot|$ to denote the absolute value of a number as well as the Euclidean norm of a vector or the Frobenious norm of a matrix.

2 Preliminary

2.1 Compact Functions

As in [15], in the case that $\mathcal{U}$ is unbounded, its boundary $\partial\mathcal{U}$ is defined by considering the extended Euclidean space

$$\mathbb{E}^n =: \mathbb{R}^n \cup \partial\mathbb{R}^n, \qquad \partial\mathbb{R}^n =: \{x_*^\infty : x_* \in \mathbb{S}^{n-1}\},$$

where for each $x_* \in \mathbb{S}^{n-1}$, x_*^∞ denotes the infinity element of the ray through x_*. To be more precise, let

$$h(x) = \begin{cases} \frac{x}{1+|x|}, & x \in \mathbb{R}^n; \\ x_*, & x = x_*^\infty \in \partial\mathbb{R}^n, \end{cases}$$

which identifies $\mathbb{E}^n$ with the closed unit ball $\bar{\mathbb{B}}^n = \{x \in \mathbb{R}^n : |x| \le 1\}$. We call $\Gamma =: \partial\mathcal{U} \subset \mathbb{E}^n$ the *boundary* of $\mathcal{U}$ if $h(\Gamma)$ is the boundary of $h(\mathcal{U})$ in $\bar{\mathbb{B}}^n$. We define the topology of $\mathbb{E}^n$ as the one inherited from this identification. Then h clearly becomes a homeomorphism. We note that when $\mathcal{U} = \mathbb{R}^n$, $x \to \partial\mathcal{U}$ under the topology described above simply means $x \to \infty$ in the usual sense.

Recall from [15] that a non-negative continuous function U in $\mathcal{U}$ is a *compact function* if

(i) $U(x) < \rho_M$, $x \in \mathcal{U}$; and
(ii) $\lim_{x\to\partial\mathcal{U}} U(x) = \rho_M$,

where $\rho_M = \sup_{x\in\mathcal{U}} U(x)$ is called the *essential upper bound of* U. It is clear from the above definition that an unbounded, non-negative function $U \in C(\mathcal{U})$ is a compact function in $\mathcal{U}$ if and only if

$$\lim_{x\to\partial\mathcal{U}} U(x) = +\infty.$$

Below, we define two new classes of compact functions. Let $A = (a^{ij})$ be an everywhere positive semi-definite, $n \times n$ matrix-valued function on $\mathcal{U}$ and $U \in C^1(\mathcal{U})$ be a compact function with essential upper bound ρ_M. Then is is clear that there are non-negative, locally bounded functions $H_1 \le H_2$ on $[0, \rho_M)$ such that

$$H_1(\rho) \le a^{ij}(x)\partial_i U(x)\partial_j U(x) \le H_2(\rho), \qquad x \in U^{-1}(\rho), \quad \rho \in [0, \rho_M), \tag{2.1}$$

where $U^{-1}(\rho) = \{x \in \mathcal{U} : U(x) = \rho\}$ denotes the ρ-level set of U. For instance, $H_1(\rho)$, respectively $H_2(\rho)$ can be taken as the infimum, respectively as the supremum, of $a^{ij}(x)$ $\partial_i U(x)\partial_j U(x)$ on $U^{-1}(\rho)$.

Definition 2.1 Let $U \in C^1(\mathcal{U})$ be a compact function with essential upper bound ρ_M.

(i) U is said to be *of the class* $\mathcal{B}^*(A)$ if there exist $\rho_m \in (0, \rho_M)$ and a non-negative function H_2 satisfying (2.1) such that

$$\nabla U(x) \neq 0, \quad \forall x \in U^{-1}(\rho) \text{ for } a.e.\ \rho \in [\rho_m, \rho_M), \tag{2.2}$$

$$\int_{\rho_0}^{\rho_M} \frac{1}{H_2(\rho)} \mathrm{d}\rho = +\infty, \quad \forall \rho_0 \in (\rho_m, \rho_M). \tag{2.3}$$

(ii) U is said to be *of the class* $\mathcal{B}(A)$ if there exist $\rho_m \in (0, \rho_M)$ and positive functions H_1, H_2 satisfying (2.1) for $\rho \in [\rho_m, \rho_M)$ such that

$$\int_{\rho_0}^{\rho_M} \left(H_2(\rho) \int_{\rho_m}^{\rho} H_1^{-1}(s)\mathrm{d}s \right)^{-1} \mathrm{d}\rho = +\infty, \quad \forall \rho_0 \in (\rho_m, \rho_M). \tag{2.4}$$

Remark 2.1 (1) We note that (2.2) means the set of regular values of the function U is of full Lebesgue measure in $[\rho_m, \rho_M)$. By Sard's Theorem [17], if $U \in C^n(\mathcal{U})$, then the set of regular values of U is of full Lebesgue measure in $[0, \rho_M)$. Consequently, (2.2) is satisfied by any compact function $U \in C^n(\mathcal{U})$.

(2) It is clear that the class $\mathcal{B}^*(A)$ contains the class $\mathcal{B}(A)$. But the two classes are not the same.

(3) To give simple examples of functions of these classes, consider $\mathcal{U} = \mathbb{R}^n$, $U(x) = |x|^q$ for some $q > 0$ as $|x| \gg 1$. Then when $q \leq 2$, U is of both classes $\mathcal{B}^*(A)$ and $\mathcal{B}(A)$ if A is bounded under the sup-norm and uniformly positive definite in $\mathbb{R}^n$.

(4) When $\mathcal{U} = \mathbb{R}^n$, conditions for a C^2 function being of the class $\mathcal{B}^*(A)$ can be explicitly given (see Lemma 3.2).

2.2 Anti-Lyapunov Type of Functions

Let U be a compact function on $\mathcal{U}$ with essential upper bound ρ_M. For each $\rho \in [0, \rho_M)$, we denote by Ω_ρ the ρ-sublevel set of U, i.e.,

$$\Omega_\rho = \{x \in \mathcal{U} : U(x) < \rho\}.$$

We recall from [15] that U is called a *Lyapunov function* (resp. *weak Lyapunov function*) in $\mathcal{U}$ with respect to (1.1) or (1.7) if it is of the class C^2 and there exists $\rho_m \in (0, \rho_M)$, called *essential lower bound of* U, such that

$$\mathcal{L}U(x) \leq -\gamma, \quad x \in \tilde{\mathcal{U}} =: \mathcal{U} \backslash \bar{\Omega}_{\rho_m}, \tag{2.5}$$

for some constant $\gamma > 0$ (resp. $\gamma = 0$), where $\tilde{\mathcal{U}}$ is referred to as the *essential domain of* U.

We now define counterparts of these functions as follows.

Definition 2.2 Let U be a C^2 compact function in $\mathcal{U}$ with essential upper bound ρ_M.

1. U is called an *anti-Lyapunov function* in $\mathcal{U}$ with respect to (1.1) or (1.7) if there exist $\rho_m \in (0, \rho_M)$, called *essential lower bound of* U, and a constant $\gamma > 0$, called *anti-Lyapunov constant* of U, such that

$$\mathcal{L}U(x) \geq \gamma, \quad x \in \tilde{\mathcal{U}} =: \mathcal{U} \backslash \bar{\Omega}_{\rho_m}. \tag{2.6}$$

   We again refer to $\tilde{\mathcal{U}}$ as the *essential domain of* U.
2. U is called a *weak anti-Lyapunov function* with respect to (1.1) or (1.7) in $\mathcal{U}$ if it satisfies (2.6) in an *essential domain* $\tilde{\mathcal{U}} = \mathcal{U} \backslash \bar{\Omega}_{\rho_m}$ with $\gamma = 0$. We still refer to such ρ_m as an *essential lower bound of* U.

2.3 Regularity of Stationary Measures

The following regularity result for stationary measures of Fokker–Planck equations is proved in [7].

Theorem 2.1 (Bogachev-Krylov-Röckner [7]) *Assume that* (A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *Then any stationary measure* μ *of* (1.2) *admits a positive density function* $u \in W^{1,p}_{loc}(\mathcal{U})$.

3 Non-existence of Stationary Measure Under Anti-Lyapunov Condition

In this section, we will prove a result more general than Theorem A by allowing degeneracy of (a^{ij}) in $\mathcal{U}$, followed by some special non-existence results with more explicit conditions. Using Theorems A, A^0, we will also give a necessary and sufficient condition for the existence of stationary measures in the case of small diffusions.

3.1 Measure Estimates via Level Set Method

The level set method introduced in [14] contains two main ingredients: an integral identity and a derivative formula, both play important roles to the measure estimates in our study of non-existence of stationary measures.

Under the condition that (A) holds in a domain $\Omega \subset \mathbb{R}^n$, the *integral identity* proved in [14, Theorem 2.1] reads

$$\int_{\Omega'} (\mathcal{L}F)u \, \mathrm{d}x = \int_{\partial\Omega'} (a^{ij}\partial_i F \nu_j)u \, \mathrm{d}s, \tag{3.1}$$

where $u \in W^{1,p}_{loc}(\Omega)$ is a weak stationary solution of (1.2) in Ω, $\Omega' \subset\subset \Omega$ is a generalized Lipschitz sub-domain, F is a C^2 function on $\bar{\Omega}'$ which assumes a constant value on $\partial\Omega'$, and $(\nu_j(x))$ is the unit outward normal vector of $\partial\Omega'$ at x for a.e. $x \in \partial\Omega'$.

For a compact function $U \in C^1(\mathcal{U})$ and a function $u \in C(\mathcal{U})$, consider the function

$$y(\rho) := \int_{\Omega_\rho} u \, \mathrm{d}x, \qquad \rho \in (0, \rho_M)$$

where ρ_M is the essential upper bound of U and Ω_ρ is the ρ-sublevel set of U for each $\rho \in (0, \rho_M)$. Then as shown in [14, Theorem 2.2], y is a C^1 function on the open set

$$\mathcal{I} =: \{\rho \in (0, \rho_M) : \nabla U(x) \neq 0, \ x \in U^{-1}(\rho)\},$$

and satisfies the following *derivative formula*:

$$y'(\rho) = \int_{\partial\Omega_\rho} \frac{u}{|\nabla U|} \, \mathrm{d}s, \qquad \rho \in \mathcal{I}. \tag{3.2}$$

The following measure estimate is proved in [14] via the level set method.

Lemma 3.1 ([14, Theorem B(a)]) *Assume that* (A) *holds and let* U *be an anti-Lyapunov function in* $\mathcal{U}$ *with respect to* (1.1) *satisfying* (2.2) *on* $[\rho_m, \rho_M)$, *where* ρ_m, ρ_M, *are essential lower, upper bounds of* U, *respectively. Denote* γ *as an anti-Lyapunov constant of* U *and* Ω_ρ *as the* ρ-*sublevel set of* U *for each* $\rho \in [\rho_m, \rho_M)$. *Then for any non-negative, locally* *bounded function H_2 satisfying* (2.1)*, any weak stationary solution $u \in W^{1,p}_{loc}(\mathcal{U})$ of* (1.2)*, and any $\rho_0 \in (\rho_m, \rho_M)$, we have*

$$\mu(\Omega_\rho \backslash \Omega^*_{\rho_m}) \geq \mu(\Omega_{\rho_0} \backslash \Omega^*_{\rho_m}) \mathrm{e}^{\gamma \int_{\rho_0}^{\rho} \frac{1}{H_2(t)} \mathrm{d}t}, \qquad \rho \in (\rho_0, \rho_M),$$

where μ is the measure with density function u, i.e., $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$, *and* $\Omega^*_{\rho_m} = \Omega_{\rho_m} \cup U^{-1}(\rho_m) = \{x \in \mathcal{U} : U(x) \leq \rho_m\}$.

Proof The original proof of [14, Theorem B(a)]) involves a complicated partition of $[\rho_m, \rho_M)$ so that the integral identity (3.1) and the derivative formula (3.2) are applicable on each partitioning sub-interval. To highlight the applications of the integral identity and the derivative formula, we give the proof below in the special case that $\nabla U \neq 0$ everywhere in the essential domain $\mathcal{U} \backslash \bar{\Omega}_{\rho_m}$ of U. In this case, Ω_ρ for each $\rho \in (\rho_m, \rho_M)$ is a C^2 domain whose outward unit normal vector $\nu(x)$ is well-defined and equals $\frac{\nabla U(x)}{|\nabla U(x)|}$ for each $x \in \partial\Omega_\rho$.

Let $\eta_* \in (\rho_m, \rho_M)$ and $\eta \in (\eta_*, \rho_M)$ be arbitrarily chosen. Applying (3.1) with $F = U$ on $\Omega' = \Omega_\eta, \Omega_{\eta_*}$, respectively, we have

$$\int_{\partial\Omega_{\eta_*}} u a^{ij} \frac{\partial_i U \partial_j U}{|\nabla U|} \,\mathrm{d}s + \int_{\Omega_\eta \backslash \Omega_{\eta_*}} (a^{ij} \partial^2_{ij} U + V^i \partial_i U) u \,\mathrm{d}x = \int_{\partial\Omega_\eta} u a^{ij} \frac{\partial_i U \partial_j U}{|\nabla U|} \,\mathrm{d}s.$$

In the above identity, since the first term in the left hand side is non-negative, applications of the definition of anti-Lyapunov function and (2.1) yield that for $\epsilon > 0$

$$\gamma \int_{\Omega_\eta \backslash \Omega_{\eta_*}} u \,\mathrm{d}x \leq (H_2(\eta) + \epsilon) \int_{\partial\Omega_\eta} \frac{u}{|\nabla U|} \,\mathrm{d}s. \tag{3.3}$$

Consider the function

$$y(\eta) = \mu(\Omega_\eta \backslash \Omega_{\eta_*}) = \int_{\Omega_\eta \backslash \Omega_{\eta_*}} u \,\mathrm{d}x, \qquad \eta \in (\eta_*, \rho_M).$$

Then at each $\eta \in (\eta_*, \rho_M)$, $y(\eta)$ is of the class C^1 and by (3.2),

$$y'(\eta) = \int_{\partial\Omega_\eta} \frac{u}{|\nabla U|} \,\mathrm{d}s.$$

Hence (3.3) becomes

$$y'(\eta) - \frac{\gamma}{H_2(\eta) + \epsilon} y(\eta) \geq 0, \qquad \eta \in (\eta_*, \rho_M).$$

For any $\rho_0 \in (\eta_*, \rho_M)$, $\rho \in (\rho_0, \rho_M)$, a direct integration of the above on $[\rho_0, \rho)$ yields that

$$\mu\left(\Omega_\rho \backslash \Omega_{\eta_*}\right) \geq \mu\left(\Omega_{\rho_0} \backslash \Omega_{\eta_*}\right) \mathrm{e}^{\gamma \int_{\rho_0}^{\rho} \frac{1}{H_2(t)+\epsilon} \mathrm{d}t},$$

from which the lemma simply follows by taking limits $\epsilon \to 0$ and $\eta_* \searrow \rho_m$.

Remark 3.1 As shown in a separate work [16], the measure estimates contained in the above lemma will also be useful in characterizing local de-concentration of a family of stationary measures associated with a so-called null family of diffusion matrices, as noises tend to zero.

3.2 Non-existence of Stationary Measures

Theorem A in Sect. 1 follows from the following result.

Theorem 3.1 *Assume that* (A) *holds in* $\mathcal{U}$ *and there exists an anti-Lyapunov function with respect to* (1.1) *in* $\mathcal{U}$ *which is of the class* $\mathcal{B}^*(A)$. *Then the Fokker–Planck Eq.* (1.2) *has no regular stationary measure with positive density function lying in* $W^{1,p}_{loc}(\mathcal{U})$. *Moreover, if* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$, *then* (1.2) *admits no stationary measure in* $\mathcal{U}$.

Proof Let U be the anti-Lyapunov function with respect to (1.1) in $\mathcal{U}$ which is of the class $\mathcal{B}^*(A)$. Then there exists $\rho_m \in (0, \rho_M)$, where ρ_M denotes the essential upper bound of U, such that U satisfies (2.2) and (2.3) with respect to a non-negative, locally bounded function H_2 on $[\rho_m, \rho_M)$. Without loss of generality, we assume that ρ_m is an essential lower bound of U.

Suppose for contradiction that (1.2) admits a regular stationary measure μ with positive density function $u \in W^{1,\bar{p}}_{loc}(\mathcal{U})$. Then for a fixed $\rho_0 \in (\rho_m, \rho_M)$, $\mu(\Omega_{\rho_0}\backslash\Omega^*_{\rho_m}) > 0$, where $\Omega^*_{\rho_m}$ is as in Lemma 3.1. It follows from Lemmas 3.1 and (2.3) that

$$\mu\left(\Omega_\rho\backslash\Omega^*_{\rho_m}\right) \geq \mu\left(\Omega_{\rho_0}\backslash\Omega^*_{\rho_m}\right) \mathrm{e}^{\gamma\int_{\rho_0}^{\rho}\frac{1}{H_2(t)}\mathrm{d}t} \to +\infty, \quad \text{as } \rho \to \rho_M.$$

But for any $\rho < \rho_M$, $\mu(\Omega_\rho\backslash\Omega^*_{\rho_m}) \leq \mu(\Omega_\rho) \leq 1$, a contradiction.

Now assume that (a^{ij}) is everywhere positive definite in $\mathcal{U}$. If (1.2) admits a stationary measure μ on $\mathcal{U}$, then by Theorem 2.1, there is a positive density function $u \in W^{1,p}_{loc}(\mathcal{U})$ such that $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$, which is impossible as shown above. □

Remark 3.2 (1) In the case that the diffusion matrix is degenerate in the domain, stationary measures of the corresponding Fokker–Planck equation may actually exist though by Theorem 3.1 regular stationary measures with positive density cannot exist under the conditions of this theorem. As a simple example, consider $\mathcal{U} = \mathbb{R}^1$, $A \equiv 0$, and $V(x) = x$, $x \in R^1$. Then it is easy to see by applying Theorem 3.1 to the anti-Lyapunov function $U(x) = x^2$, which is clearly of the class $\mathcal{B}^*(0)$, that the corresponding Fokker–Planck equation admits no weak stationary solution at all. However, it is clear that the Dirac measure at the origin is a stationary measure of the corresponding Fokker–Planck equation.

(2) As shown in Examples 5.1, 5.2 in Sect. 5, the condition that U is of the class $\mathcal{B}^*(A)$ cannot be removed in Theorem 3.1, i.e, having an anti-Lyapunov function alone does not necessarily guarantee the non-existence of stationary measures in the domain.

We now consider the special case $\mathcal{U} = \mathbb{R}^n$. In this case, conditions for a C^2 function being of the class $\mathcal{B}^*(A)$ can be explicitly verified as follows.

Lemma 3.2 *Let* $U \in C^2(\mathbb{R}^n)$ *be a function such that the Hessian matrix* D^2U *is bounded under the sup-norm and uniformly positive definite on* $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ *for some* $r_0 > 0$. *Then the following properties hold.*

(a) *There is a constant* $c \geq 0$ *such that* $U + c$ *is an unbounded compact function in* $\mathbb{R}^n$.
(b) *There exists a constant* $\rho_m \gg 1$ *such that* $\nabla U(x) \neq 0$ *for all* $x \in \mathbb{R}^n$ *with* $U(x) > \rho_m$.
(c) $U + c$ *is of the class* $\mathcal{B}^*(A)$ *with respect to any* $n \times n$ *matrix-valued function* A *which is bounded under the sup-norm.*

Proof (a) Let $\lambda > 0$, $\Lambda > 0$ be constants such that $D^2U \geq \lambda I$ and $|D^2U| \leq \Lambda$ on $\{x \in \mathbb{R}^n : |x| \geq r_0\}$. For any $x \in \mathbb{R}^n$ with $|x| \geq r_0$, we have by considering integrals along line segments joining x and $x_0 =: \frac{x}{|x|}r_0$ that

$$\begin{aligned} &\lambda|x - x_0|^2 \leq U(x) - U(x_0) - \nabla U(x_0) \cdot (x - x_0), \\ &\lambda|x - x_0| \leq |\nabla U(x) - \nabla U(x_0)| \leq \Lambda|x - x_0|. \end{aligned} \tag{3.4}$$

It follows that there are constants $C_1, C_2 > 0$, depending only on r_0, λ, Λ, and $|U|_{C^1(\{|x|=r_0\})}$, such that for any $x \in \mathbb{R}^n$ with $|x| \gg r_0$,

$$|x|^2 \leq C_1\,(1 + U(x))\,, \tag{3.5}$$

$$|\nabla U(x)| \leq C_2\,(1 + |x|)\,. \tag{3.6}$$

Since (3.5) implies that $\inf_{x\in\mathbb{R}^n} U(x) > -\infty$, $U + c$ is a non-negative function for some constant $c \geq 0$. Therefore $U + c$ becomes a compact function in $\mathbb{R}^n$ with $\rho_M = +\infty$.

(b) By (3.4), it is clear that there is a $r_1 \geq r_0$, depending only on r_0, λ, and $|U|_{C^1(\{|x|=r_0\})}$, such that $\nabla U(x) \neq 0$ when $|x| \geq r_1$. Let $\rho_m = \max_{|x|\leq r_1} U(x)$. We then have $\nabla U(x) \neq 0$ for all x with $U(x) > \rho_m$.

(c) If follows from (a) that $U + c$ is a compact function on $\mathbb{R}^n$ and from (b) that $U + c$ satisfies (2.2) for $\rho \in (\rho_m + c, +\infty)$. Let $A = (a^{ij})$ be a matrix-valued, bounded function under the sup-norm and denote $\Lambda_* = \sup_{x\in\mathbb{R}^n} |A(x)|$. We have by (3.5), (3.6) that

$$a^{ij}(x)\partial_i(U(x) + c)\partial_j(U(x) + c) \leq \Lambda_*|\nabla U(x)|^2 \leq \Lambda^*(1 + U(x) + c), \qquad \forall U(x) > \rho_m,$$

where $\Lambda^* > 0$ is a constant depending only on Λ_*, C_1, and C_2. Hence (2.3) is satisfied with $H_2(\rho) = \Lambda^*(1 + \rho)$. It follows that $U + c$ is of the class $\mathcal{B}^*(A)$. □

Corollary 3.1 *Consider* $\mathcal{U} = \mathbb{R}^n$. *Assume* (A) *and that* (a^{ij}) *is bounded under the sup-norm. Also assume that there is a function* $U \in C^2(\mathbb{R}^n)$ *such that*

$$\liminf_{x\to\infty} \mathcal{L}U(x) > 0, \tag{3.7}$$

and that the Hessian matrix D^2U *is bounded under the sup-norm and uniformly positive definite in* $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ *for some constant* $r_0 > 0$. *Then the Fokker–Planck Eq.* (1.2) *has no regular stationary measure with positive density function lying in* $W^{1,\bar{p}}_{loc}(\mathbb{R}^n)$. *Moreover, if* (a^{ij}) *is everywhere positive definite in* $\mathbb{R}^n$, *then* (1.2) *admits no stationary measure in* $\mathbb{R}^n$.

Proof By Lemma 3.2, for some constant $c \geq 0$, $U + c$ is a compact function in $\mathbb{R}^n$ which is of the class $\mathcal{B}^*(A)$. The compactness of $U + c$ and (3.7) also imply that $U + c$ is an anti-Lyapunov function in $\mathbb{R}^n$. The corollary now follows from Theorem 3.1. □

Corollary 3.2 *Consider* $\mathcal{U} = \mathbb{R}^n$. *Assume* (A) *and that* $A = (a^{ij})$ *is bounded under the sup-norm and uniformly positive definite in* $\mathbb{R}^n$. *Also assume that there exists a function* $U \in C^2(\mathbb{R}^n)$ *such that*

$$\liminf_{x\to\infty} V(x) \cdot \nabla U(x) \geq 0, \tag{3.8}$$

and that the Hessian matrix D^2U *is bounded under the sup-norm and uniformly positive definite in* $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ *for some constant* $r_0 > 0$. *Then* (1.2) *admits no stationary measure in* $\mathbb{R}^n$.

Proof Let $\lambda_1, \lambda_2 > 0$ be constants such that

$$A(x) \geq \lambda_1 I, \qquad D^2U(x) \geq \lambda_2 I, \qquad |x| \geq r_0,$$

where I denotes the identity matrix. Then

$$a^{ij}(x)\partial^2_{ij}U(x) = \mathrm{Tr}(A(x)D^2U(x)) \geq n\lambda_1\lambda_2, \qquad |x| \geq r_0.$$

This, together with (3.8), implies that U satisfies (3.7). The corollary now follows from Corollary 3.1. □

An application of Corollary 3.2 using $U(x) = |x|^2$, $x \in \mathbb{R}^n$, immediately yields the following result.

Corollary 3.3 *Consider* $\mathcal{U} = \mathbb{R}^n$. *Assume* (A) *and that* (a^{ij}) *is bounded under the sup-norm and uniformly positive definite in* $\mathbb{R}^n$. *If*

$$\liminf_{x\to\infty} V(x)\cdot x \geq 0,$$

then (1.2) *admits no stationary measure in* $\mathbb{R}^n$.

Remark 3.3 As to be shown in Example 5.2 in Sect. 5, Corollaries 3.1–3.3 can fail when the matrix (a^{ij}) is unbounded under the sup-norm.

3.3 Necessary and Sufficient Conditions for the Existence of Stationary Measures

In the case of small diffusions, Theorem A^0 and Corollary 3.2 together can yield a necessary and sufficient condition for the existence of stationary measures. The following result implies the Corollary stated in Sect. 1 simply by taking $U(x) = |x|^2/2$, $x \in \mathbb{R}^n$.

Corollary 3.4 *Consider* $\mathcal{U} = \mathbb{R}^n$. *Assume* (A) *and that* $A = (a^{ij})$ *is bounded under the sup-norm and uniformly positive definite in* $\mathbb{R}^n$. *Assume further that there is a function* $U \in C^2(\mathbb{R}^n)$ *satisfying the following conditions:*

(i) *The limit*

$$\lim_{x\to\infty} V(x)\cdot\nabla U(x) =: \nu$$

exists;

(ii) *The Hessian matrix* D^2U *is bounded under the sup-norm and uniformly positive definite in* $\{x \in \mathbb{R}^n : |x| \geq r_0\}$ *for some* $r_0 > 0$.

Then $\nu < 0$ *is a necessary condition for the existence of a stationary measure in* $\mathbb{R}^n$ *of the Fokker–Planck Eq.* (1.2)*, and, it is also sufficient for the existence if* $\limsup_{x\to\infty}|A(x)| < -\frac{\nu}{C_0}$, *where* $C_0 = \limsup_{x\to\infty}|D^2U(x)|$.

Proof When $\nu \geq 0$, it follows immediately from Corollary 3.2 that the corresponding Fokker–Planck has no stationary measures. The necessity is shown.

For sufficiency, consider the case $\nu < 0$. We note by Lemma 3.2 that there is a constant $c \geq 0$ such that $U + c$ is an unbounded compact function in $\mathbb{R}^n$. Since

$$\limsup_{x\to\infty} a^{ij}(x)\partial^2_{ij}U(x) \leq \limsup_{x\to\infty}|D^2U(x)|\limsup_{x\to\infty}|A(x)| = C_0\limsup_{x\to\infty}|A(x)| < -\nu,$$

we have

$$\limsup_{x\to\infty}\mathcal{L}(U(x)+c) = \limsup_{x\to\infty}\mathcal{L}U(x) < 0.$$

It follows that there are positive constants γ, r_* such that $\mathcal{L}(U(x)+c) \leq -\gamma$ as $|x| > r_*$, i.e., $U + c$ is a Lyapunov function in $\mathbb{R}^n$. The existence of a stationary measure of the corresponding Fokker–Planck equation now follows from Theorem A^0. □

4 Non-existence of Stationary Measures Under Weak Anti-Lyapunov Condition

In this section, we will prove a non-existence result slightly more general than Theorem B stated in Sect. 1.

4.1 Measure Estimates via Level Set Method

The following measure estimate result is proved in [14] using level set method.

Lemma 4.1 ([14, Theorem B(b)]) *Assume that* (A) *holds and there exists a weak anti-Lyapunov function U with respect to* (1.1) *in $\mathcal{U}$ with essential lower, upper bound ρ_m, ρ_M, respectively. Let $H_1 \le H_2$ be two functions as in* (2.1) *with H_1 being positive and continuous in $[\rho_m, \rho_M)$. Then for any weak stationary solution $u \in W^{1,p}_{loc}(\mathcal{U})$ of* (1.2) *in $\mathcal{U}$ and any $\rho_0 \in (\rho_m, \rho_M)$, we have*

$$\mu\left(\Omega_\rho \backslash \Omega_{\rho_m}\right) \ge \mu\left(\Omega_{\rho_0} \backslash \Omega_{\rho_m}\right) \mathrm{e}^{\int_{\rho_0}^{\rho} \frac{1}{\tilde{H}(t)} \mathrm{d}t}, \qquad \rho \in [\rho_0, \rho_M),$$

where μ is the measure such that $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$, Ω_ρ denotes the ρ-sublevel set of U for each $\rho \in [\rho_m, \rho_M)$, and

$$\tilde{H}(\rho) = H_2(\rho) \int_{\rho_m}^{\rho} H_1^{-1}(s)\mathrm{d}s, \qquad \rho \in [\rho_0, \rho_M).$$

4.2 Proof of Theorem B

Theorem B follows from the following result.

Theorem 4.1 *Assume that* (A) *holds and there is a weak anti-Lyapunov function U with respect to* (1.1) *in $\mathcal{U}$ which is of the class $\mathcal{B}(A)$. Then the Fokker–Planck Eq.* (1.2) *has no regular stationary measure with positive density function lying in $W^{1,p}_{loc}(\mathcal{U})$. Moreover, if (a^{ij}) is everywhere positive definite in $\mathcal{U}$, then* (1.2) *admits no stationary measure in $\mathcal{U}$.*

Proof Using the argument in the proof of Theorem 3.1, it suffices to only show that (1.2) does not admit any regular stationary measure with positive density function lying in $W^{1,p}_{loc}(\mathcal{U})$.

Suppose for contradiction that (1.2) admits a positive weak stationary solution $u \in W^{1,p}_{loc}(\mathcal{U})$. Let μ be the stationary measure of (1.2) with density u. Since U is of the class $\mathcal{B}(A)$, there are two locally bounded functions $H_1 \le H_2$ satisfying (2.1), (2.4) such that H_1 is positive in $[\rho_m, \rho_M)$, where ρ_M is the essential upper bound of U and ρ_m is the constant as in Definition 2.1. Without loss of generality, we assume that the constant ρ_m is an essential lower bound of U. Let

$$h_1(\rho) = \min_{x \in U^{-1}(\rho)} \left(a^{ij}(x)\partial_i U(x)\partial_j U(x)\right), \qquad \rho \in [\rho_m, \rho_M).$$

Since $H_1(\rho) > 0$, $\nabla U \ne 0$ for $x \in U^{-1}(\rho)$, $\rho \in [\rho_m, \rho_M)$. Then it is clear that h_1 is continuous and satisfies $0 < H_1(\rho) \le h_1(\rho) \le a^{ij}(x)\partial_i U(x)\partial_j U(x)$ for $x \in U^{-1}(\rho)$, i.e., (2.1) is satisfied with h_1 replacing H_1 for $\rho \in [\rho_m, \rho_M)$. It follows from Lemma 4.1 that

$$\mu\left(\Omega_\rho \backslash \Omega_{\rho_m}\right) \ge \mu\left(\Omega_{\rho_0} \backslash \Omega_{\rho_m}\right) \mathrm{e}^{\int_{\rho_0}^{\rho} \frac{1}{\tilde{H}(t)} \mathrm{d}t}, \qquad \rho \in [\rho_0, \rho_M), \tag{4.1}$$

where $\rho_0 \in (\rho_m, \rho_M)$ is fixed and

$$\tilde{H}(\rho) = H_2(\rho) \int_{\rho_m}^{\rho} h_1^{-1}(s)\mathrm{d}s, \qquad \rho \in [\rho_0, \rho_M).$$

Since $H_1 \leq h_1$, the condition (2.4) implies that $\int_{\rho_0}^{\rho_M} \frac{1}{\tilde{H}(t)}\mathrm{d}t = +\infty$. Now since $\mu(\Omega_{\rho_0} \backslash \Omega_{\rho_m}) > 0$, letting $\rho \to \rho_M$ in (4.1) yields that $\mu(\mathcal{U} \backslash \Omega_{\rho_m}) = \infty$, a contradiction to the fact that $\mu(\mathcal{U}) = 1$. □

5 Examples

In this section, we give some examples illustrating the applications of our existence and non-existence results. In particular, we show that the existence of Lyapunov, anti-Lyapunov functions and their weak forms in a domain plays an important role in characterizing stochastic bifurcations with respect to the existence and non-existence of stationary measures, for which Lyapunov and anti-Lyapunov functions are useful when the parameters are non-critical while weak Lyapunov and weak anti-Lyapunov functions are useful when the parameters are at critical values. The general stochastic bifurcations concerning the change of behaviors of stationary measures can be much more complicated than what are described in the examples below. We will explore such bifurcation phenomena further in separate works.

These examples also give justifications to Remark 3.2 (2), Remark 3.3, and [15, Remark 5.1 (2)].

Example 5.1 (Bifurcation w.r.t. drift parameter) Consider

$$\mathrm{d}x = -\frac{bx}{1+x^2}\mathrm{d}t + \sqrt{2}\,\mathrm{d}W, \qquad x \in \mathbb{R}^1. \tag{5.1}$$

When $b > 1$, it is already shown in Remark 4.2 (1) of [15] that the Fokker–Planck equation associated with (5.1) admits a unique stationary measure in $\mathbb{R}^1$.

Consider $U = x^{2q}$ for some $q \geq 1$. Then

$$\mathcal{L}U = 2qx^{2q-2}\left(2q - 1 - \frac{bx^2}{1+x^2}\right). \tag{5.2}$$

It follows that any $U = x^{2q}$ with $q > \frac{b+1}{2}$ is an anti-Lyapunov function with respect to (5.1) in $\mathbb{R}^1$. But the Fokker–Planck equation associated with (5.1) does admit a stationary measure when $b > 1$. This justifies Remark 3.2 (2) that having an anti-Lyapunov function alone is not sufficient to guarantee the non-existence of stationary measures.

When $b < 1$, we see that $U(x) = x^2$, $x \in \mathbb{R}^1$, satisfies (3.7). Since $A \equiv 1$ in this case, Corollary 3.1 asserts that the Fokker–Planck equation associated with (5.1) admits no stationary measure in $\mathbb{R}^1$ in this parameter range.

For the critical value $b = 1$, (5.2) shows that $U = x^2$ is a weak anti-Lyapunov function with respect to (5.1) in $\mathbb{R}^1$. Since U is of the class $\mathcal{B}(A)$ (see Remark 2.1 (3)), it follows from Theorem B that the Fokker–Planck equation associated with (5.1) admits no stationary measure in $\mathbb{R}^1$.

We note that the case $b = -1$ precisely corresponds to the example in [15, Remark 5.1(2)].

Example 5.2 (Bifurcation w.r.t. diffusion parameter) Consider the stochastic differential equation

$$\mathrm{d}x = bx\mathrm{d}t + \sqrt{2\sigma(x^2+1)}\,\mathrm{d}W, \qquad x \in \mathbb{R}^1, \tag{5.3}$$

where $b > 0$ is a fixed constant and $\sigma > 0$ is parameter.

When $\sigma > b$, it is shown in Remark 4.2 (2) of [15] that the Fokker–Planck equation associated with (5.3) admits a unique stationary measure. So the diffusion stabilizes the unperturbed ODE in which the origin is repelling.

Now take $U(x) = x^2$. It is easy to check that $\mathcal{L}U(x) = 2\sigma(x^2+1) + 2bx^2 \geq 2\sigma > 0$, $V(x) \cdot \nabla U = 2V(x) \cdot x = 2bx^2 \geq 0$ everywhere. We note that except the boundedness of diffusion, all other conditions in Corollaries 3.1–3.3 are satisfied. But stationary measures do exist as mentioned above. This gives a justification of Remark 3.3 and another justification of Remark 3.2 (2).

When $0 < \sigma \leq b$, we claim that the associated Fokker–Planck equation has no stationary measure. Indeed, let $U(x) = \log(x^2+1)$. Then $\rho_M = +\infty$ and

$$\mathcal{L}U(x) = -2 \cdot \frac{(\sigma - b)x^2 - \sigma}{x^2 + 1}. \tag{5.4}$$

It is readily seen from (5.4) that U is an anti-Lyapunov function with respect to (5.3) in $\mathbb{R}^1$ when $\sigma < b$, and becomes a weak anti-Lyapunov function with respect to (5.3) in $\mathbb{R}^1$ in the critical case $\sigma = b$. Moreover, U is of the class $\mathcal{B}(A)$ because $H_1(\rho) \equiv 3\sigma$ and $H_2(\rho) \equiv 4\sigma$ in (2.1) satisfy the condition (2.4). It then follows from Theorem B that the associated Fokker–Planck equation has no stationary measure when σ lies in the range above.

Example 5.3 (Bifurcations with degenerate diffusion on the boundary) Consider

$$\mathrm{d}x = bx\mathrm{d}t + \sqrt{2}\left(1 - x^2\right)\mathrm{d}W, \qquad x \in \mathcal{U} = (-1, 1). \tag{5.5}$$

We note that the noise coefficient vanishes at the boundary of $\mathcal{U}$.

Let $U(x) = -\log(1-x^2)$, $x \in (-1,1)$. It is clear that $U(x) \to +\infty$, as $|x| \to 1$, i.e., $\rho_M = +\infty$, and

$$\mathcal{L}U(x) = 2 + 2x^2 + \frac{2bx^2}{1-x^2}, \qquad x \in (-1, 1). \tag{5.6}$$

When $b < 0$, it is shown in Remark 4.1 of [15] that the Fokker–Planck equation associated with (5.5) admits a unique stationary measure.

When $b \geq 0$, we claim that the associated Fokker–Planck equation admits no stationary measure in $\mathcal{U} = (-1, 1)$ (though it always has stationary measures in any proper sub-interval of $(-1, 1)$). By (5.6), we immediately see that $\mathcal{L}U(x) \to 4$ when $b = 0$, and $\mathcal{L}U \to +\infty$ when $b > 0$, as $|x| \to 1$, and hence U is an anti-Lyapunov function with respect to (5.5) in $\mathcal{U}$. It is also easy to check that U is of the class $\mathcal{B}^*(A)$ when taking $H_2(\rho) \equiv 4$, $\rho \in [0, +\infty)$, in (2.1). The conclusion now follows from Theorem A.

Acknowledgments The first author was partially supported by NSFC Grants 11225105, 11431012. The second author was partially supported by NSFC Innovation Grant 10421101. The third author was partially supported by NSFC Grant 11271151 and the startup fund of Dalian University of Technology. The fourth author was partially supported by NSF Grants DMS0708331 and DMS1109201, NSERC discovery Grant 1257749, a faculty development grant from University of Alberta, and a Scholarship from Jilin University.

Appendix

We summarize some basic properties of a dissipative system of ordinary differential equations in a general domain $\mathcal{U} \subset \mathbb{R}^n$. As most of these properties are usually stated for the entire space, we sketch proofs of the results below for a general domain, following the approaches in the unpublished lecture notes [12]. For other general properties of dissipative dynamical systems, we refer the reader to [13].

Consider (1.6) for a continuous V. Throughout the section, we assume that solutions of (1.6) are locally unique (e.g., V is locally Lipschitz continuous). For each $\xi \in \mathcal{U}$, denote $\varphi^t(\xi)$ as the solution starting at ξ of (1.6) for t lying in the maximal interval $\mathcal{I}_\xi$ of existence. Then (1.6) generates a (continuous) *local flow* φ^t on $\mathcal{U}$, i.e., $u(\xi, t) =: \varphi^t(\xi)$ is continuous in its domain of definition, and,

(i) $\varphi^0(\xi) = \xi, \xi \in \mathcal{U}$;
(ii) $\varphi^{t+s}(\xi) = \varphi^t \circ \varphi^s(\xi), \xi \in \mathcal{U}, t, s, t+s \in \mathcal{I}_\xi$.

We call $B \subset \mathcal{U}$ an *invariant set* (resp. *positively invariant set*, resp. *negatively invariant set*) of φ^t if $\mathcal{I}_\xi = \mathbb{R}$ (resp. $\mathcal{I}_\xi \supset \mathbb{R}_+$, resp. $\mathcal{I}_\xi \supset \mathbb{R}_-$) for all $\xi \in B$, and $\varphi^t(B) = B$ for all $t \in \mathbb{R}$ (resp. $\varphi^t(B) \subset B$ for all $t \in \mathbb{R}_+$, resp. $\varphi^t(B) \subset B$ for all $t \in \mathbb{R}_-$). If $\mathcal{U}$ itself is an *invariant set* (resp. *positively invariant set*, resp. *negatively invariant set*) of φ^t, φ^t is called a *flow* (resp. *positive semiflow*, resp. *negative semiflow*). We note that if φ^t is a flow, then for each $t \in \mathbb{R}$, $\phi^t : \mathcal{U} \to \mathcal{U}$ is a homeomorphism.

Let $E \subseteq \mathbb{R}^n$. A subset $K \subset E$ is said to be *pre-compact in* E if the closure of K in $\mathbb{R}^n$ is a compact set contained in E.

5.1 ω- and α-Limit Sets

For any set $B \subset \mathcal{U}$ such that $\cup_{t\in\mathbb{R}_+}\varphi^t(B)$ (resp. $\cup_{t\in\mathbb{R}_-}\varphi^t(B)$) exists and is pre-compact in $\mathcal{U}$, the *ω-limit set of B* (resp. *α-limit set of B*) is defined as

$$\omega(B) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\varphi^t(B)} \qquad \left(\text{resp. } \alpha(B) = \cap_{\tau\leq 0}\overline{\cup_{t\leq\tau}\varphi^t(B)}\right).$$

Let $\operatorname{dist}(A, B) = \sup_{a\in A}\inf_{b\in B}|a-b|$ denote the Hausdorff semi-distance from a set A to a set B. Then the following property is well-known (see, e.g. [13]).

Proposition 5.1 *Let $B \subset \mathcal{U}$ be a set such that $\cup_{t\in\mathbb{R}_+}\varphi^t(B)$ (resp. $\cup_{t\in\mathbb{R}_-}\varphi^t(B)$) exists and is pre-compact in $\mathcal{U}$. Then the following holds.*

(a) *$\omega(B)$ (resp. $\alpha(B)$) is compact and invariant. It is connected if B is a connected set.*
(b) *$\omega(B)$ (resp. $\alpha(B)$) attracts (resp. repels) B in the sense that* $\operatorname{dist}(\varphi^t(B), \omega(B)) \to 0$ (*resp.* $\operatorname{dist}(\varphi^t(B), \alpha(B)) \to 0$) *as $t \to +\infty$ (resp. as $t \to -\infty$).*

5.2 Dissipation and Attractor

We first give the following definitions of dissipation and attractors which are parallel to those in the case $\mathcal{U} = \mathbb{R}^n$.

Definition 5.1 Let $\Omega \subset \mathcal{U}$ be a connected, positively (resp. negatively) invariant open set of φ^t.

(1) φ^t is said to be *dissipative* (resp. *anti-dissipative*) *in* Ω if there exists a compact subset C of Ω with the property that for any $\xi \in \Omega$ there exists $t_0(\xi) > 0$ such that $\varphi^t(\xi) \in C$ as $t \geq t_0(\xi)$ (resp. $t \leq -t_0(\xi)$).
(2) A compact invariant set $\mathcal{J} \subset \Omega$ (resp. $\mathcal{R} \subset \Omega$) of φ^t is said to be a *maximal attractor* (resp. *maximal repeller*) *of φ^t in* Ω if $\mathcal{J}$ attracts (resp. $\mathcal{R}$ repels) any pre-compact subset K of Ω, i.e., $\omega(K) \subset \mathcal{J}$ (resp. $\alpha(K) \subset \mathcal{R}$), or equivalently, $\lim_{t\to+\infty}\operatorname{dist}(\varphi^t(K), \mathcal{J}) = 0$ (resp. $\lim_{t\to-\infty}\operatorname{dist}(\varphi^t(K), \mathcal{R}) = 0$).

(3) Suppose that the maximal attractor $\mathcal{J}$ (resp. maximal repeller $\mathcal{R}$) of φ^t exists in Ω. If $\Omega = \mathcal{U}$, then we call $\mathcal{J}$ (resp. $\mathcal{R}$) the *global attractor* (resp. *global repeller*) *of* φ^t. Otherwise, it is called a *local attractor* (resp. *local repeller*) *of* φ^t.

It is clear that the maximal attractor or repeller of φ^t in Ω, if exists, must be unique. Consequently, the global attractor or repeller of φ^t is unique.

Proposition 5.2 *Let $\Omega \subset \mathcal{U}$ be a connected, positively* (*resp. negatively*) *invariant open set of φ^t.*

(1) *φ^t is dissipative* (*resp. anti-dissipative*) *in Ω if and only if there exists a maximal attractor* (*resp. repeller*) *of φ^t in Ω.*
(2) *If φ^t is dissipative* (*resp. anti-dissipative*) *in Ω, then the maximal attractor $\mathcal{J}$* (*resp. maximal repeller $\mathcal{R}$*) *of φ^t in Ω equals*

$$\mathcal{J} = \bigcup_{B \subset \Omega \text{ pre-compact}} \omega(B) \quad \left(resp.\ \mathcal{R} = \bigcup_{B \subset \Omega \text{ pre-compact}} \alpha(B) \right).$$

Proof The result essentially follows from the standard theory of dissipative dynamical systems. We sketch the proof for the necessity part of 1) for the reader's convenience.

Let φ^t be dissipative in Ω, C be a compact subset of Ω satisfying the property in Definition 5.1 (1), and $D \subset \Omega$ be a pre-compact open subset of Ω containing C. Denote $K =: \overline{D}$. For each $\xi \in K$, the property of C implies that there exists $s(\xi) > 0$ such that $\varphi^{s(\xi)}(\xi) \in C$. Since D is open, there exists an open neighborhood O_ξ of ξ in Ω such that $\varphi^{s(\xi)}(O_\xi) \subset D$. By the compactness of K, there are finitely many points $\{\xi_1, \xi_2, \ldots, \xi_k\} \subset K$ such that $\{O_{\xi_i} : 1 \le i \le k\}$ is a finite open cover of K. Let $T(K) = \max\{s(\xi_i) : 1 \le i \le k\}$ and $\widetilde{K} = \bigcup_{0 \le t \le T(K)} \varphi^t(K)$. It is clear that $\widetilde{K}$ is a compact subset of Ω.

We now *Claim* that $\tilde{K}$ is a positively invariant set of φ^t. In virtual of the definition of $\tilde{K}$, this amounts to show that $\varphi^t(K) \subset \tilde{K}$ for any $t > T(K)$.

Given $x \in K$ and $t > T(K)$, there exists $1 \le i_1 \le k$ such that $x \in O_{\xi_{i_1}}$. We denote $x_1 = \varphi^{s(\xi_{i_1})}(x)$ and $t_1 = t - s(\xi_{i_1})$. Then $x_1 \in D$, $0 \le t_1 < t$ and $\varphi^t(x) = \varphi^{t_1}(x_1)$. If $t_1 \le T(K)$, then $\varphi^t(x) = \varphi^{t_1}(x_1) \in \varphi^{t_1}(K) \subset \widetilde{K}$, which already proves the *Claim*. If $t_1 > T(K)$, then we further let $1 \le i_2 \le k$ be such that $x_1 \in O_{\xi_{i_2}}$ and denote $x_2 = \varphi^{s(\xi_{i_2})}(x_1)$ and $t_2 = t_1 - s(\xi_{i_2})$. Clearly, $x_2 \in D$, $0 \le t_2 < t_1$, and $\varphi^{t_1}(x_1) = \varphi^{t_2}(x_2)$. By repeating the above process and using the fact that $\min\{s(\xi_i) : 1 \le i \le k\} > 0$, we find finite sequences $\{i_1, \ldots, i_m\}$, $\{x_1, \ldots, x_m\}$, and $\{t_1, \ldots, t_m\}$, such that $x_j = \varphi^{s(\xi_{i_j})}(x_{j-1}) \in D$ for all $1 \le j \le m$, $t_j = t_{j-1} - s(\xi_{i_j}) > T(K)$ when $j \le m-1$, but $0 \le t_m = t_{m-1} - s(\xi_{i_m}) \le T(K)$. Therefore,

$$\varphi^t(x) = \varphi^{t_1}(x_1) = \cdots = \varphi^{t_{m-1}}(x_{m-1}) = \varphi^{t_m}(x_m) \in \tilde{K}.$$

This proves the *Claim*. The *Claim* implies that $\omega(\widetilde{K})$ exists.

Now for any pre-compact subset H of Ω, as in the above we find a finite cover $\{O_1, \ldots, O_r\}$ of H and $0 \le s_1, s_2, \ldots, s_r < +\infty$ such that $\varphi^{s_i}(O_i) \subset D$ for all $1 \le i \le r$. Let $T(H) = \max\{s_i : 1 \le i \le r\}$. Then by the *Claim*,

$$\varphi^t(H) \subset \bigcup_{i=1}^{r} \varphi^t(O_i) \subset \bigcup_{i=1}^{r} \varphi^{t-s_i}(D) \subset \widetilde{K}$$

for all $t \ge T(H)$. It follows that $\omega(H) \subset \omega(\widetilde{K})$. Since, by Proposition 5.1 (a), $\omega(\widetilde{K})$ is a compact invariant set, it is the maximal attractor of φ^t in Ω.

The proof for the case when φ^t is anti-dissipative in Ω is similar. □

Definition 5.2 Let $\Omega \subset \mathcal{U}$ be a connected open set and $U \in C^1(\Omega)$ be a compact function in Ω with essential upper bound ρ_M.

(1) U is called a *weak Lyapunov function* (resp. *weak anti-Lyapunov function*) *of* (1.6) *in* Ω if

$$V(x) \cdot \nabla U(x) \leq 0 \qquad (\text{resp. } \geq 0), \quad x \in \tilde{\Omega} = \Omega \backslash \bar{\Omega}_{\rho_m}, \tag{5.7}$$

where $\rho_m \in (0, \rho_M)$ is a constant, called *essential lower bound of* U, and $\tilde{\Omega}$ is called *essential domain of* U *in* Ω.

(2) U is a *Lyapunov function* (resp. *anti-Lyapunov function*) *of* (1.6) *in* Ω if there exists a constant $\gamma > 0$, called a *Lyapunov constant of* U (resp. *anti-Lyapunov constant of* U), such that

$$V(x) \cdot \nabla U(x) \leq -\gamma \qquad (\text{resp. } \geq \gamma), \quad x \in \tilde{\Omega} = \Omega \backslash \bar{\Omega}_{\rho_m}, \tag{5.8}$$

where $\rho_m \in (0, \rho_M)$ is a constant, called *essential lower bound of* U, and $\tilde{\Omega}$ is called *essential domain of* U *in* Ω.

Proposition 5.3 *Consider the system* (1.6) *and the local flow* φ^t *generated by it. Let* $\Omega \subset \mathcal{U}$ *be a connected open set.*

(a) *If* (1.6) *admits a weak Lyapunov* (*resp. anti-Lyapunov*) *function in* Ω*, then* Ω *must be a positively* (*resp. negatively*) *invariant set of* φ^t.

(b) *If* (1.6) *admits a Lyapunov* (*resp. anti-Lyapunov*) *function in* Ω*, then it must be dissipative* (*resp. anti-dissipative*) *in* Ω*, with the maximal attractor* (*resp. repeller*) *in* Ω *being* $\omega(\Omega_{\rho_m})$ (*resp.* $\alpha(\Omega_{\rho_m})$)*, where* ρ_m *is the essential lower bound of the Lyapunov* (*resp. anti-Lyapunov*) *function.*

Proof We let U be the weak Lyapunov function of (1.6) in Ω in the case (a) and the Lyapunov function of (1.6) in Ω in the case (b). Denote ρ_m, respectively ρ_M, as the essential lower, respectively upper, bound of U, and $\tilde{\Omega}$ as the essential domain of U.

(a) For any $x_0 \in \tilde{\Omega}$, we have by (5.7) that

$$U(\varphi^t(x_0)) - U(x_0) = \int_0^t \frac{\mathrm{d}(U(\varphi^s(x_0)))}{\mathrm{d}s}\, \mathrm{d}s = \int_0^t \nabla U \cdot V(\varphi^s(x_0))\, \mathrm{d}s \leq 0, \tag{5.9}$$

whenever $t > 0$, $\varphi^s(x_0)$ exists and lies in $\tilde{\Omega}$ for any $s \in [0, t]$. It then follows from (5.9) and properties of compact functions that any local forward orbit starting in $\tilde{\Omega}$ must be bounded in forward time, hence it can neither blow-up in finite time nor approach $\partial\Omega$. Also, any local forward orbit starting in $\Omega \backslash \tilde{\Omega} = \bar{\Omega}_{\rho_m}$ can neither blow-up in finite time nor approach $\partial\Omega$, because if it does then it has to go through $\tilde{\Omega}$. Thus Ω is a positively invariant set of φ^t.

(b) By (5.8) and the continuity of V, U is also a Lyapunov function of (1.6) in Ω_{ρ_m} with $\frac{\gamma}{2}$ being a Lyapunov constant, hence Ω_{ρ_m} is a positively invariant set of φ^t by (a).

For any $x \in \Omega \backslash \Omega_{\rho_m}$, it follows that

$$U(\varphi^t(x)) - U(x) = \int_0^t \frac{\mathrm{d}(U(\varphi^s(x)))}{\mathrm{d}s}\, \mathrm{d}s = \int_0^t \nabla U \cdot V(\varphi^s(x))\, \mathrm{d}s \leq -\frac{\gamma t}{2} \tag{5.10}$$

whenever $t > 0$ such that $\varphi^s(x) \in \Omega \backslash \Omega_{\rho_m}$, $\forall s \in [0, t]$. Hence there exists $t_0(x) > 0$ such that $\varphi^t(x) \in \Omega_{\rho_m}$ for all $t \geq t_0(x)$. By taking $C = \bar{\Omega}_{\rho_m}$, we have by Definition 5.1 that φ^t is dissipative in Ω.

By Proposition 5.2 (1), there exists a maximal attractor $\mathcal{J}$ of φ^t in Ω. On one hand, we have by Proposition 5.2 (2) that $\omega(\Omega_{\rho_m}) \subset \mathcal{J}$. On the other hand, for any pre-compact subset B of Ω, it is not hard to see from the positive invariance of Ω_{ρ_m} and (5.10) that there exists $T \geq 0$ such that $\varphi^t(B) \subset \Omega_{\rho_m}$ for all $t \geq T$. Then $\varphi^{t+T}(B) = \varphi^t(\varphi^T(B)) \subset \varphi^t(\Omega_{\rho_m})$, which implies that $\omega(B) \subset \omega(\Omega_{\rho_m})$. It now follows from Proposition 5.2 (2) that $\mathcal{J} \subset \omega(\Omega_{\rho_m})$. Thus, $\mathcal{J} = \omega(\Omega_{\rho_m})$.

Similar arguments hold when (1.6) admits a weak anti-Lyapunov function in Ω in the case (a) and an anti-Lyapunov function in Ω in the case (b). □

Propositions 5.2, 5.3 immediately yield the following result.

Theorem 5.1 *If* (1.6) *admits a Lyapunov* (*resp. anti-Lyapunov*) *function in* $\mathcal{U}$*, then* φ^t *is a positive* (*resp. negative*) *semiflow which is dissipative* (*resp. anti-dissipative*) *in* $\mathcal{U}$*, hence it admits a global attractor* (*resp. global repeller*) *in* $\mathcal{U}$.

Remark 5.1 We note that since global attractor and repeller cannot co-exist, Theorem 5.1 implies that φ^t does not admit a global attractor in $\mathcal{U}$ if (1.6) admits an anti-Lyapunov function in $\mathcal{U}$.

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