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# Eigenfunctionals of Homogeneous Order-Preserving Maps with Applications to Sexually Reproducing Populations

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**Abstract** Homogeneous bounded maps $B$ on cones $X_+$ of ordered normed vector spaces $X$ allow the definition of a cone spectral radius which is analogous to the spectral radius of a bounded linear operator. If $X_+$ is complete and $B$ is also order-preserving, conditions are derived for $B$ to have a homogeneous order-preserving eigenfunctional $\theta : X_+ \to \mathbb{R}_+$ associated with the cone spectral radius in analogy to one part of the Krein–Rutman theorem. Since homogeneous $B$ arise as first order approximations at 0 of maps that describe the year-to-year development of sexually reproducing populations, these eigenfunctionals are an important ingredient in the persistence theory of structured populations with mating.

**Keywords** Homogeneous map · Order-preserving map · Concave map · Cone spectral radius · Eigenfunctional · Krein–Rutman type theorems · Collatz–Wielandt numbers and bound · Mating functions

## 1 Introduction

The celebrated Krein–Rutman theorem states that the spectral radius $\mathbf{r}(B)$ of a positive bounded linear map $B$ on an ordered Banach space $X$,

$$\mathbf{r}(B) = \inf_{n\in\mathbb{N}} \|B^n\|^{1/n}, \tag{1.1}$$

which equals

$$\mathbf{r}(B) = \lim_{n\in\mathbb{N}} \|B^n\|^{1/n} \tag{1.2}$$

(see the proof of Theorem 3 in [46, Sect. VIII.2]), is not only associated with of a positive eigenvector but also with a positive eigenfunctional provided that the map is compact and the cone is total [27] (see also [39, App. 2.4,2.6]). Generalizations of the Krein–Rutman theorem

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have weakened the compactness assumptions for the linear map [27,35–37]. The existence of positive eigenvectors has also been established for maps on the positive cone that are homogeneous and order-preserving but not additive [1,4,5,27,31–33]. These eigenvectors are associated with a modification of the spectral radius, the cone spectral radius. In comparison, the existence of homogeneous eigenfunctionals of homogeneous maps on cones has attracted less attention [43]. Since such eigenfunctionals play a crucial role in extending the persistence theory for structured populations [41] to models that take sexual reproduction into account [19–22], their existence will be addressed in this paper. Since duality does no longer seem to work in an effective way if the map is not additive, this will be done less in the spirit of the Krein–Rutman theorem but more in the spirit of an even older result by Krein [26] that established the existence of a positive linear eigenfunctional if the linear map is just bounded (rather than compact) but the cone is normal and solid (see also [6,27], [39, App. 2.6]) and in the spirit of a result by Bonsall [6, Thm. 1] where $X$ is an ordered vector space with an order unit and the map is linear and positive.

We mention without further elaboration that, because of the crucial threshold character of the cone spectral radius for population persistence, there is a natural interest in alternative characterizations of the cone spectral radius by other types of spectral radii and by lower and upper Collatz–Wielandt bounds [1,11,12,29,32,33,44,45].

This paper is organized as follows. In Sect. 2, we introduce the central concepts: cones and their properties, homogeneous (order-preserving) maps and their cone operator norm and cone spectral radius and orbital spectral radius.

In Sect. 3, we give a flavor of the sense in which homogeneous maps can be considered first order approximations of fully nonlinear maps, how their eigenfunctionals are involved in persistence theorems and how the cone spectral radius acts as a threshold parameter in nonlinear dynamics.

The main part of the paper is devoted to the existence of an eigenfunctional associated with $\mathbf{r}_+(B)$. This requires a few preparations [43]: a uniform boundedness principle (Sect. 4) and a left resolvent for homogeneous maps (Sect. 5). In Sect. 6, we prove three existence results concerning homogeneous eigenfunctionals. One results shows that the existence of a lower eigenvector implies the existence of an eigenfunctional under some extra conditions (Sect. 6.1). However, if $B$ is homogeneous and superadditive and the cone is normal and complete, existence of an eigenfunctional follows without assuming the existence of an eigenvector (Sect. 6.2). The third result is the observation that, if a power of $B$ has an eigenfunctional associated with its cone spectral radius, the same holds for $B$ itself.

We also derive conditions for the eigenfunctionals to be continuous and strictly positive (Sect. 6.3). We present an example that the eigenfunctional can be discontinuous even if it and the map are additive on the cone and the cone is total (but not generating).

Applications to the dynamics of structured two-sex populations are presented in Sect. 7, one with state space $L^1_+(\Omega)$, a regular cone, and one with state space $\mathrm{BC}_+(\Omega)$, a solid normal cone. In Sect. 8, we back up the relevance of concave homogeneous maps by showing that the usual mating functions $\phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ [14,15,18] are not only homogeneous but also superadditive and thus concave.

## 2 Cones, Homogeneous Maps, and the Cone Spectral Radius

For models in the biological, social, or economic sciences, there is a natural interest in solutions that are positive in an appropriate sense, i.e., they take their values in the cone of an ordered normed vector space.

### 2.1 Cones and Their Properties

A subset $X_+$ of a real vector space $X$ is called a *wedge* if

(i) $X_+$ is convex and $\alpha X_+ \subseteq X_+$ for all $\alpha \in \mathbb{R}_+$
(ii) $X_+$ is closed if $X$ is a normed vector space.

A wedge is called a *cone* if $X_+ \cap (-X_+) = \{0\}$. Nonzero points in a cone or wedge are called *positive*. A wedge is called *generating* if $X = X_+ - X_+$.

If $X_+$ is a cone in $X$, we introduce a partial order on $X$ by $x \leq y$ if $y - x \in X_+$ for $x, y \in X$ and call $X$ an *ordered normed vector space*.

Let $X$ be a normed vector space (then any wedge in $X_+$ is closed by (ii)). A wedge $X_+$ in $X$ is called *solid* if it contains interior points, and *total* if $X$ is the closure of $X_+ - X_+$.

A cone $X_+$ is called *normal*, if there exists some $c > 0$ such that

$$\|x\| \leq c\|x + z\| \text{ whenever } x \in X_+, z \in X_+. \tag{2.1}$$

An element $u \in X_+$ is called a *normal point* of $X_+$ if the set $\{x \in X_+, x \leq u\}$ is bounded; $u \in X_+$ is called a *regular point* of $X_+$ if every monotone sequence $(x_n)$ in $X_+$ with $x_n \leq u$ for all $n \in \mathbb{N}$ converges.

$X_+$ is called *regular* if any decreasing sequence in $X_+$ converges, i.e., all elements in $X_+$ are regular points.

**Proposition 2.1** *Let $X$ be an ordered normed vector space with cone $X_+$. Then every regular point of $X_+$ is a normal point of $X_+$.*

*Proof* Let $u$ be a regular point of $X_+$ that is not a normal point of $X_+$. Then, for any $n \in \mathbb{N}$, there exists some $x_n \in X_+$ with $x_n \leq u$ and $\|x_n\| \geq 4^n$. Define

$$y_m = \sum_{n=1}^{m} 2^{-n} x_n, \qquad m \in \mathbb{N}.$$

Then $(y_m)$ is an increasing sequence in $X_+$ and $y_m \leq u$ for all $n \in \mathbb{N}$. Since $u$ is a regular point in $X_+$, $(y_m)$ converges and is a Cauchy sequence. Hence $y_m - y_{m-1} = 2^{-m} x_m \to 0$. But $\|2^{-m} x_m\| \geq 2^m$, a contradiction. □

**Proposition 2.2** *Every solid cone is generating. Every complete regular cone is normal.*

See [24, Sects. 1.5, 5.2]. In function spaces, like the one formed by the bounded continuous functions, CB($\Omega$), or the various spaces of measurable functions, $L^p(\Omega)$, typical cones are formed by the nonnegative functions or their equivalence classes. The nonnegative functions in CB($\Omega$) and $L^\infty(\Omega)$ form a complete, normal, solid cone while the nonnegative functions in $L^p(\Omega)$, $1 \leq p < \infty$, form a regular, complete (and thus normal) cone. The function spaces also carry a lattice structure.

A cone $X_+$ is called an inf-*semilattice* [1] (or *minihedral* [23]) if $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in X_+$.

$X_+$ is called a sup-*semilattice* if $x \vee y = \sup\{x, y\}$ exist for all $x, y \in X_+$.

$X_+$ is called a *lattice* if $x \wedge y$ and $x \vee y$ exist for all $x, y \in X_+$.

$X$ is called a *lattice* if $x \vee y$ exist for all $x, y \in X$.

Since $x \wedge y = -((-x) \vee (-y))$, also $x \wedge y$ exist for all $x, y$ in a lattice $X$.

For more information on cones see [8,23,24,30,39].

## 2.2 Positive and Order-Preserving Maps

Throughout this paper, let $X$ be an ordered vector space with cone $X_+$. We use the notation

$$\dot{X} = X \setminus \{0\} \quad \text{and} \quad \dot{X}_+ = X_+ \setminus \{0\}.$$

**Definition 2.3** Let $X$ and $Z$ be ordered vector spaces with cones $X_+$ and $Z_+$ and $U \subseteq X$. $B : U \to Z$ is called *positive* if $B(U \cap X_+) \subseteq Z_+$.

$B$ is called *order-preserving* (or monotone or increasing) if $B(x) \leq B(y)$ whenever $x, y \in U$ and $x \leq y$.

Positive linear maps from $X$ to $Z$ are order-preserving.

## 2.3 Homogenous Maps

In the following, $X, Y$ and $Z$ are ordered vector spaces with cones $X_+, Y_+$ and $Z_+$ respectively.

**Definition 2.4** $B : X_+ \to Y$ is called *(positively) homogeneous (of degree one)*, if $B(\alpha x) = \alpha B(x)$ for all $\alpha \in \mathbb{R}_+$, $x \in X_+$.

Since we do not consider maps that are homogeneous in other ways, we will simply call them homogeneous maps. It follows from the definition that

$$B(0) = 0.$$

### *2.3.1 Cone Norms for Homogeneous Bounded Maps*

Let $X, Y, Z$ be ordered normed vector spaces.

For a homogeneous map $B : X_+ \to Y$, we define

$$\|B\|_+ = \sup\left\{\|B(x)\|; x \in X_+, \|x\| \leq 1\right\} \tag{2.2}$$

and call $B$ *bounded* if this supremum is a real number. Since $B$ is homogeneous,

$$\|B(x)\| \leq \|B\|_+ \|x\|, \qquad x \in X_+. \tag{2.3}$$

Let $H(X_+, Y)$ denote the set of bounded homogeneous maps $B : X_+ \to Y$ and $H(X_+, Y_+)$ denote the set of bounded homogeneous maps $B : X_+ \to Y_+$ and $\mathrm{HM}(X_+, Y_+)$ the set of those maps in $H(X_+, Y_+)$ that are also order-preserving.

$H(X_+, Y)$ is a real vector space and $\| \cdot \|_+$ is a norm on $H(X_+, Y)$, called the *cone operator norm*.

$H(X_+, Y_+)$ and $\mathrm{HM}(X_+, Y_+)$ are cones in $H(X_+, Y)$. We write $H(X_+) = H(X_+, X_+)$ and $\mathrm{HM}(X_+) = \mathrm{HM}(X_+, X_+)$.

It follows for $B \in H(X_+, Y_+)$ and $C \in H(Y_+, Z_+)$ that $CB \in H(X_+, Z_+)$ and

$$\|CB\|_+ \leq \|C\|_+ \|B\|_+. \tag{2.4}$$

If $X_+$ is complete and $Y_+$ is normal, homogeneous order-preserving maps from $X_+$ to $Y_+$ are automatically bounded. In particular, if $X_+$ is complete, homogeneous order-preserving functionals from $X_+$ to $\mathbb{R}_+$ are bounded.

**Proposition 2.5** *Let $X_+$ be complete, $B : X_+ \to Y_+$ be homogeneous and order-preserving, and $B(x)$ a normal point of $Y_+$ for any $x \in X_+$. Then $B$ is bounded.*

*Proof* Suppose that $B$ is not bounded. Then, for any $n \in \mathbb{N}$, there exists some $x_n \in X_+$ with $\|x_n\| \leq 1$ such that $\|B(x_n)\| \geq 4^n$. Since $X_+$ is complete, $u = \sum_{n=1}^{\infty} 2^{-n} x_n$ converges. Since $B$ is homogeneous and order-preserving, for each $n \in \mathbb{N}$, $2^{-n} B(x_n) \leq B(u)$. Since $B(u)$ is a normal point of $Y_+$, $\{\|2^{-n} B(x_n)\|, n \in \mathbb{N}\}$ is a bounded subset in $\mathbb{R}$ which contradicts $\|B(x_n)\| \geq 4^n$ for all $n \in \mathbb{N}$. □

### 2.3.2 Cone and Orbital Spectral Radius

Let $B \in H(X_+)$ and define $\phi : \mathbb{Z}_+ \to \mathbb{R}$ by $\phi(n) = \ln \|B^n\|_+$. By (2.4), $\phi(m+n) \leq \phi(m) + \phi(n)$ for all $m, n \in \mathbb{Z}_+$, and a well-known result implies the following formula for the *cone spectral radius*

$$\mathbf{r}_+(B) := \inf_{n \in \mathbb{N}} \|B^n\|_+^{1/n} = \lim_{n \to \infty} \|B^n\|_+^{1/n}, \tag{2.5}$$

which is analogous to (1.1) and (1.2).

Mallet-Paret and Nussbaum [32,33] suggest an alternative definition of a spectral radius for homogeneous (not necessarily bounded) maps $B : X_+ \to X_+$. First, define asymptotic least upper bounds for the geometric growth factors of $B$-orbits,

$$\gamma(x, B) = \gamma_B(x) = \gamma_x(B) := \limsup_{n \to \infty} \|B^n(x)\|^{1/n}, \qquad x \in X_+, \tag{2.6}$$

and then

$$\mathbf{r}_o(B) = \sup \gamma_B(X_+). \tag{2.7}$$

Here $\gamma_B(x) := \infty$ if the sequence $(\|B^n(x)\|^{1/n})$ is unbounded and $\mathbf{r}_o(B) = \infty$ if $\gamma_B(x) = \infty$ for some $x \in X_+$ or the set $\{\gamma_B(x); x \in X_+\}$ is unbounded.

The number $\mathbf{r}_+(B)$ has been called *partial spectral radius* by Bonsall [7], $X_+$ spectral radius by Schaefer [38,39], and *cone spectral radius* by Nussbaum [35,37]. Mallet-Paret and Nussbaum [32,33] call $\mathbf{r}_+(B)$ the *Bonsall cone spectral radius* and $\mathbf{r}_o(B)$ the cone spectral radius. For $x \in X_+$, the number $\gamma_B(x)$ has been called *local spectral radius* of $B$ at $x$ by Förster and Nagy [11].

We will follow Nussbaum's older terminology which shares the spirit with Schaefer's [38] term $X_+$ *spectral radius* and stick with *cone spectral radius* for $\mathbf{r}_+(B)$. One readily checks that

$$\mathbf{r}_+(\alpha B) = \alpha \mathbf{r}_+(B), \quad \alpha \in \mathbb{R}_+, \qquad \mathbf{r}_+(B^m) = (\mathbf{r}_+(B))^m, \qquad m \in \mathbb{N}. \tag{2.8}$$

The same properties hold for $\mathbf{r}_o(B)$ though proving the second property takes some more effort [32, Prop. 2.1]. Actually, as we show in [44,45],

$$\gamma(x, B^m) = (\gamma(x, B))^m, \qquad m \in \mathbb{N}, x \in X_+, \tag{2.9}$$

which readily implies

$$\mathbf{r}_o(B^m) = (\mathbf{r}_o(B))^m, \qquad m \in \mathbb{N}. \tag{2.10}$$

The cone spectral radius and the orbital spectral radius are meaningful if $B$ is just positively homogeneous and bounded, but as in [32,33] we will be mainly interested in the case that $B$ is also order-preserving and continuous.

Though the two concepts coincide for many practical purposes, they are both useful. Gripenberg [13] gives an example for $\mathbf{r}_o(B) < \mathbf{r}_+(B)$.

**Theorem 2.6** *Let $X$ be an ordered normed vector with cone $X_+$ and $B : X_+ \to X_+$ be continuous, homogeneous and order-preserving.*

*Then* $\mathbf{r}_+(B) \geq \mathbf{r}_o(B) \geq \gamma_B(x)$, $x \in X_+$.
*Further* $\mathbf{r}_o(B) = \mathbf{r}_+(B)$ *if one of the following hold:*

(i) $X_+$ *is complete and normal.*
(ii) $B$ *is power compact.*
(iii) $X_+$ *is normal and a power of* $B$ *is uniformly order-bounded.*
(iv) $X_+$ *is complete and* $B$ *is additive* ($B(x+y) = B(x) + B(y)$ *for all* $x \in X_+$).

The inequality is a straightforward consequence of the respective definitions. For the concepts of (iii) see Sect. 2.5 and for the proof see [44,45]. The other three conditions for equality have been verified in [32, Sect. 2], (the overall assumption of [32] that $X$ is a Banach space is not used in the proofs).

### 2.4 The Space of Certain Order-Bounded Elements and Some Functionals

Let $X$ be an ordered vector space and with cone $X_+$.

**Definition 2.7** Let $x \in X$ and $u \in X_+$. Then $x$ is called *u-bounded* if there exists some $c > 0$ such that $-cu \leq x \leq cu$. If $x$ is $u$-bounded, we define

$$\|x\|_u = \inf\{c > 0; -cu \leq x \leq cu\}. \tag{2.11}$$

The set of $u$-bounded elements in $X$ is denoted by $X_u$. If $x, u \in X_+$ and $x$ is not $u$-bounded, we define

$$\|x\|_u = \infty.$$

The element $x \in X_+$ is called *u-positive* if there exists some $\epsilon > 0$ such that $x \geq \epsilon u$. So $x \in X_+$ is $u$-positive if and only if $u$ is $x$-bounded.

Two elements $v$ and $u$ in $X_+$ are called *comparable* if $v$ is $u$-bounded and $u$ is $v$-bounded, i.e., if there exist $\epsilon, c > 0$ such that $\epsilon u \leq v \leq cu$. Comparability is an equivalence relation for elements of $X_+$, and we write $u \sim v$ if $u$ and $v$ are comparable. Notice that $X_u = X_v$ if and only if $u \sim v$.

If $X$ is a space of real-valued functions on a set $\Omega$,

$$\|x\|_u = \sup\left\{\frac{|x(\xi)|}{u(\xi)}; \xi \in \Omega, u(\xi) > 0\right\}.$$

$X_u$ is a linear subspace of $X$, and $\|\cdot\|_u$ is a seminorm on $X_u$. It is a norm on $X_u$ if and only if $x = 0$ is the only $x \in X$ such that $-cu \leq x \leq cu$ for all $c > 0$.

Now assume that $X$ is an ordered normed vector space with norm $\|\cdot\|$. Then the cone $X_+$ is closed by assumption and

$$-\|x\|_u u \leq x \leq \|x\|_u u, \qquad x \in X_u. \tag{2.12}$$

$X_u$ is a linear subspace of $X$, $\|\cdot\|_u$ is a norm on $X_u$, and $X_u$, under this norm, is an ordered normed vector space with cone $X_+ \cap X_u$ which is normal, generating, and has nonempty interior. The following is proved in [44,45].

**Lemma 2.8** *Let* $u \in \dot{X}_+$. *Then* $u$ *is in the interior of* $X_+$ *if and only if* $X_u = X$ *and there exists some* $\epsilon > 0$ *such that* $\|x\| \geq \epsilon \|x\|_u$ *for all* $x \in X$.

If $X_+$ is normal, by (2.12) there exists some $M \geq 0$ such that

$$\|x\| \leq M\|x\|_u\|u\|, \qquad x \in X_u. \tag{2.13}$$

If $X_+$ is a normal and complete cone of $X$, then $X_+ \cap X_u$ is a complete subset of $X_u$ with the metric induced by the norm $\|\cdot\|_u$. For more information see [5, I.4], [23, 1.3], [24, 1.4].

We define

$$[x]_u = \sup\{\beta \geq 0; \beta u \leq x\}, \qquad x, u \in X_+. \tag{2.14}$$

Since $X_+$ is closed,

$$x \geq [x]_u u, \qquad x, u \in X_+. \tag{2.15}$$

Further $[x]_u$ is the largest number for which this inequality holds.

If $X$ is a space of real-valued functions on a set $\Omega$,

$$[x]_u = \inf\left\{\frac{|x(\xi)|}{u(\xi)}; \xi \in \Omega, u(\xi) > 0\right\}.$$

**Lemma 2.9** *Let $X_+$ be closed and $u \in \dot{X}_+$. Then the functional $\theta = [\cdot]_u : X_+ \to \mathbb{R}_+$ is homogeneous, order-preserving and concave. It is bounded with respect to the original norm on $X$,*

$$[x]_u \leq \frac{\|x\|}{d(u, -X_+)}, \qquad x \in X_+.$$

*$\theta$ is upper semicontinuous with respect to the original norm and continuous on $X_u$ with respect to the $u$-norm,*

$$\big|[y]_u - [x]_u\big| \leq \|y - x\|_u, \qquad y, x \in X_u \cap X_+.$$

*Proof* All properties except the last one have been proved in [44,45]. Let $x, y \in X_+ \cap X_u$. By (2.15),

$$[x]_u u \leq x \leq x - y + y \leq \|x - y\|_u u + y.$$

We reorganize,

$$y \geq ([x]_u - \|x - y\|_u)u.$$

By (2.14),

$$[y]_u \geq [x]_u - \|x - y\|_u.$$

We reorganize,

$$[x]_u - [y]_u \leq \|x - y\|_u.$$

By symmetry,

$$[y]_u - [x]_u \leq \|y - x\|_u = \|x - y\|_u.$$

This implies the assertion. □

In general, $[\cdot]_u$ is not continuous with respect to the original norm (Example 6.17).

### 2.5 Order-Bounded Maps

The following terminology has been adapted from various works by Krasnosel'skii and coworkers [23, Sect. 2.1.1], [24, Sec.9.4] though it has been modified.

**Definition 2.10** Let $X$ be an ordered vector space with cone $X_+$ and $B : X_+ \to X_+$, $u \in X_+$. $B$ is called *pointwise* $u$*-bounded* if, for any $x \in X_+$, there exist some $n \in \mathbb{N}$ such that $B^n(x) \in X_u$ (Definition 2.7). The point $u$ is called a *pointwise order bound* of $B$.

$B$ is called *pointwise order-bounded* if it is pointwise $u$-bounded for some $u \in X_+$.

Now let $X$ be an ordered normed vector space with cone $X_+$.

**Definition 2.11** $B$ is called *uniformly* $u$*-bounded* if there exists some $c > 0$ such that $B(x) \leq c\|x\|u$ for all $x \in X_+$. The element $u$ is called a *uniform order bound* of $B$.

$B$ is called *uniformly order-bounded* if it is uniformly $u$-bounded for some $u \in X_+$.

If $B : X_+ \to X_+$ is bounded and $X_+$ is solid, then $B$ is uniformly $u$-bounded for every interior point $u$ of $X_+$. The following result is proved in [43–45] by a Baire category argument.

**Proposition 2.12** *Let $X_+$ be a complete cone, $u \in X_+$, and $B : X_+ \to X_+$ be continuous, order-preserving and homogeneous. Then the following hold.*

(a) *$B$ is uniformly $u$-bounded if $B(X_+) \subseteq X_u$.*
(b) *If $B$ is pointwise $u$-bounded, then some power of $B$ is uniformly $u$-bounded.*

## 3 Homogeneous Maps and Nonlinear Dynamics

One of the mathematical motivations to consider homogeneous maps is that they appear as first order approximations of nonlinear maps. Let $X$ be an ordered vector space with cone $X_+$ and $Y$ a normed vector space.

**Theorem 3.1** *Let $F : X_+ \to Y$ and $u \in X_+$. Assume that the directional derivatives of $F$ at $u$ exist in all directions of the cone. Then the map $B : X_+ \to Y_+$, $B = \partial F(u, \cdot)$,*

$$B(x) = \partial F(u, x) = \lim_{t \to 0+} \frac{F(u + tx) - F(u)}{t}, \qquad x \in X_+,$$

*is homogeneous.*

*Proof* Let $\alpha \in \mathbb{R}_+$. Obviously, if $\alpha = 0$, $B(\alpha x) = 0 = \alpha B(x)$. So we assume $\alpha \in (0, \infty)$. Then

$$\frac{F(u + t[\alpha x]) - F(u)}{t} = \alpha \frac{F(u + [t\alpha]x) - F(u)}{t\alpha}.$$

As $t \to 0$, also $\alpha t \to 0$ and so the directional derivative in direction $\alpha x$ exists and

$$\partial F(u, \alpha x) = \alpha F(u, x). \qquad \square$$

Obviously, the map $B$ in Theorem 3.1 is uniquely determined whenever it exists and is called the *cone directional derivative* of $F$ at $u$.

The cone directional derivative of a homogeneous map $B$ at 0 is $B$ itself.

**Proposition 3.2** *Let $B : X_+ \to Y$ be homogeneous. Then the directional derivatives of $B$ exist at 0 in all directions of the cone and*

$$\partial B(0, x) = \lim_{t \to 0+} \frac{B(tx) - B(0)}{t} = B(x), \qquad x \in X_+.$$

More on homogeneous maps as first order approximations can be found in [19,22]. As illustration how eigenfunctionals are involved in persistence results, we present the following theorem in which a homogeneous map $B$ is a lower first order approximation of a map $F$ at 0 in an order sense [19,22].

**Theorem 3.3** *Let $X$ be an ordered normed vector space with cone $X_+$ and $F, B : X_+ \to X_+$ and $B$ be homogeneous and order preserving. Let $\theta : X_+ \to \mathbb{R}_+$ be homogeneous, bounded, and order-preserving, and let the following properties be satisfied.*

(a) *If $x \in \dot{X}_+$, then $F^n(x) \in \dot{X}_+$ for infinitely many $n \in \mathbb{N}$.*
(b) *For any $x \in \dot{X}_+$, there exists some $n \in \mathbb{N}$ such that $\theta(B^n(x)) > 0$.*
(c) *For all $\epsilon \in (0, 1)$ there exists some $\delta > 0$ such that $F(x) \geq (1 - \epsilon)B(x)$ for all $x \in X_+$ with $\|x\| \leq \delta$.*
(d) *There exists some $r > 1$ such that $\theta(B(x)) \geq r\theta(x)$ for all $x \in X_+$.*

*Then the semiflow induced by $F$ is uniformly weakly norm-persistent: There exists some $\delta > 0$ such that* $\limsup_{n\to\infty} \|F^n(x)\| \geq \delta$ *for all $x \in \dot{X}_+$.*

Notice that uniform weak norm-persistence implies instability of the origin. In [19,21], conditions are given also for the stronger notion of uniform norm persistence where the lim sup is replaced by the lim inf. In view of assumption (d), it is worth mentioning that there is an elementary way of obtaining a lower eigenfunctional $\rho$ from a lower eigenvector of $B$.

*Remark 3.4* Let $v \in \dot{X}_+$ and $r > 0$ such that $B(v) \geq rv$. Define

$$\theta(x) = [x]_v, \qquad x \in X_+.$$

Here $[\cdot]_v$ is the concave homogenous functional introduced in (2.14).

Then $\theta(B(x)) \geq r\theta(x)$ for all $x \in X_+$. Moreover, $\theta$ satisfies (b) from Theorem 3.3 if for any $x \in X_+$ there exists some $n \in \mathbb{N}$ such that $B^n(x)$ is $v$-positive.

In some instances this construction can even lead to an eigenfunctional (Example 6.17).

*Proof* Let $x \in X_+$. By (2.15), $x \geq [x]_v v$. Since $B$ is order-preserving and homogeneous, $B(x) \geq [x]_v B(v) \geq [x]_v rv$. By (2.14),

$$\theta(B(x)) = [B(x)]_v \geq r[x]_v = r\theta(x).$$

The second statement is immediate. □

Persistence results are one motivation for finding homogeneous order-preserving eigenfunctionals $\theta : X_+ \to \mathbb{R}_+$, $\theta \circ B = r\theta$ with $r = \mathbf{r}_+(B)$. The cone spectral radius is of particular interest for the role of an eigenvalue in order to find a sharp threshold between stability and instability of the origin. For the following result is also valid. Its formulation is a little different from the one in [19,21], but the proof is essentially the same after a tiny modification at the beginning.

**Theorem 3.5** *Let $X_+$ be the normal cone of an ordered normed vector space. Let $F, B : X_+ \to X_+$ and let $B$ be homogeneous, bounded and order-preserving, $r = \mathbf{r}_+(B) < 1$.*

*Assume that for each $\eta > 0$ there exists some $\delta > 0$ such that $F(x) \le (1+\eta)B(x)$ for all $x \in X_+$ with $\|x\| \le \delta$. Then $F$ is locally asymptotically stable in the following sense:*

*For each $\alpha \in (r, 1)$, there exist some $\delta_0 > 0$ and $M \ge 1$ such that $\|F^n(x)\| \le M\alpha^n \|x\|$ for all $n \in \mathbb{N}$ and all $x \in X_+$ with $\|x\| \le \delta_0$.*

## 4 A Uniform Boundedness Principle

The next theorem, which will help us to show the existence of eigenfunctionals, has been proved in [7, L.3.2] for maps that are also additive. The Baire category argument still works if additivity of the map is replaced by normality of the cone [43].

**Theorem 4.1** *Let $X$ and $Z$ be ordered normed vector spaces with a complete cone $X_+$ and a normal cone $Z_+$. Let $\{B_j;\ j \in J\}$ be an indexed family of continuous, homogeneous, order preserving maps $B_j : X_+ \to Z_+$. Assume that, for each $x \in X_+$, $\{B_j(x);\ j \in J\}$ is a bounded subset of $Z_+$. Then $\{\|B_j\|_+;\ j \in J\}$ is a bounded subset of $\mathbb{R}$.*

*Proof* By assumption,

$$X_+ = \bigcup_{n \in \mathbb{N}} M_n \qquad M_n = \bigcap_{j \in J} \tilde{M}_{n,j}, \qquad \tilde{M}_{n,j} = \big\{x \in X_+;\ \|B_j(x)\| \le n\big\}.$$

Since each $B_j$ is continuous, $\tilde{M}_{n,j}$ is a closed subset of $X_+$ for all $n, j \in \mathbb{N}$. Then $M_n$ is a closed subset of $X_+$ as an intersection of closed sets. Since $X_+$ is complete by assumption, by the Baire category theorem, there exists some $n \in \mathbb{N}$ such that $M_n$ contains an interior point $z \in X_+$. So there exists some $z \in X_+$ and $\epsilon > 0$ such that

$$z + \epsilon y \in M_n \qquad \text{whenever} \quad y \in X, z + \epsilon y \in X_+, \|y\| \le 1.$$

Since $z + \epsilon y \in X_+$ if $y \in X_+$,

$$\|B_j(z + \epsilon y)\| \le n, \qquad y \in X_+, \|y\| \le 1, j \in J.$$

Let $y \in X_+$, $\|y\| \le 1$, $j \in J$. Since $B_j$ is homogeneous and order preserving,

$$\epsilon B_j(y) = B_j(\epsilon y) \le B_j(z + \epsilon y).$$

Since $Z_+$ is normal, there exists some $c \ge 0$ (independent of $y$ and $j$) such that

$$\|\epsilon B_j(y)\| \le c\|B_j(z + \epsilon y)\| \le cn.$$

Thus

$$\|B_j(y)\| \le \frac{cn}{\epsilon}, \qquad y \in X_+, \|y\| \le 1, j \in J.$$

By definition of $\|\cdot\|_+$,

$$\|B_j\|_+ \le \frac{cn}{\epsilon}, \qquad j \in J.$$

□

## 5 A Left Resolvent

There is one remnant from the usual relations between the spectral radius and the spectrum of a bounded linear operator that also holds in the homogeneous case, namely that real numbers larger than the spectral radius are in the resolvent set. However, in the homogeneous case, there only exists a left resolvent.

Let $X$ be an ordered normed vector space and the closed cone $X_+$ be complete and $B : X_+ \to X_+$ be homogeneous, bounded and order preserving.

For $\lambda > \mathbf{r}_o(B)$, we introduce $R_\lambda : X_+ \to X_+$,

$$R_\lambda(x) = \sum_{n=0}^{\infty} \lambda^{-n-1} B^n(x), \qquad x \in X_+. \tag{5.1}$$

The convergence of the series follows from the completeness of the cone. If $\lambda > \mathbf{r}_+(B)$, by the Weierstraß majorant test, the convergence of the series is uniform for $x$ in bounded subsets of $X_+$. With this in mind, the following is easily shown.

**Lemma 5.1** *For $\lambda > \mathbf{r}_o(B)$, $R_\lambda$ is defined, homogeneous, and order preserving. Further $R_\lambda$ acts as a left resolvent,*

$$\begin{aligned} R_\lambda(B(x)) &= \sum_{n=0}^{\infty} \lambda^{-(n+1)} B^{n+1}(x) = \sum_{n=1}^{\infty} \lambda^{-n} B^n(x) \\ &= \lambda R_\lambda(x) - x = R_\lambda(\lambda x) - x, \qquad x \in X_+. \end{aligned} \tag{5.2}$$

*If $\lambda > \mathbf{r}_+(B)$ and $B$ is continuous, $R_\lambda$ is a continuous map.*

**Proposition 5.2** *Let $B$ be continuous and $\{p_j;\ j \in J\}$ be a family of homogeneous, convex, order-preserving, continuous functionals $p_j : X_+ \to \mathbb{R}_+$ such that $\sup_{j\in J} \|p_j\|_+ < \infty$. Then, for each $t > \mathbf{r}_o(B)$,*

$$\sup_{j\in J,\, \lambda\geq t} \|p_j \circ R_\lambda\|_+ < \infty.$$

*Proof* Let $t > \mathbf{r}_o(B)$. Choose some $s \in (\mathbf{r}_o(B), t)$. For each $x \in X_+$, there exists some $c_x > 0$ such that $s^{-n}(p_j \circ B^n)(x) \leq c_x$. Since $s^{-n}(p_j \circ B^n) : X_+ \to \mathbb{R}_+$ are homogeneous, order-preserving and continuous, by the uniform boundedness principle in Theorem 4.1, there exists some $c > 0$ such that $\|s^{-n}(p_j \circ B^n)\|_+ \leq c$. For each $x \in X_+$ with $\|x\| \leq 1$ and each $\lambda \geq t$, since the $p_j$ are continuous, homogeneous and subadditive,

$$(p_j \circ R_\lambda)(x) \leq \sum_{n=0}^{\infty} \lambda^{-(n+1)} p_j(B^n(x)) \leq \sum_{n=0}^{\infty} t^{-(n+1)} c s^n = \frac{c}{t} \frac{1}{1-(s/t)}.$$

Thus

$$\|p_j \circ R_\lambda\|_+ \leq \frac{c}{t-s}, \quad \lambda \geq t,\ j \in J.$$

□

This result should be compared with the next one. $B$ is called *superadditive* if

$$B(x+y) \geq B(x) + B(y), \qquad x, y \in X_+. \tag{5.3}$$

Notice that every superadditive map on $X_+$ is order-preserving. Recall that $\mathbf{r}_+(B) = \mathbf{r}_o(B)$ if $X_+$ is complete and normal.

**Theorem 5.3** *Let $X_+$ be a normal complete cone. Let $B$ be homogeneous, bounded and superadditive. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$. Then $\{\|R_\lambda\|_+; \lambda > \mathbf{r}_+(B)\}$ is unbounded.*

Since $B$ is homogeneous, the superadditivity assumption for $B$ is equivalent to $B$ being concave: $B((1-\alpha)x + \alpha y) \geq (1-\alpha)B(x) + \alpha B(y)$ for $\alpha \in (0, 1)$. The proof is adapted from [7] and allows $B$ to be concave rather than additive [43].

*Proof* $R_\lambda$ inherits superadditivity from $B$,

$$R_\lambda(x+y) \geq R_\lambda(x) + R_\lambda(y), \qquad x, y \in X_+. \tag{5.4}$$

Suppose that the assertion is false. Then there exists some $M \geq 0$ such that $\|R_\lambda\|_+ \leq M$ for all $\lambda > \mathbf{r}$. Let $0 < \mu < \mathbf{r} < \lambda$ and $(\lambda - \mu)M < 1$. Since $X_+$ is complete.

$$E_\mu(x) = \sum_{k=1}^{\infty} (\lambda - \mu)^{k-1} R_\lambda^k(x)$$

converges for each $x \in X_+$. By (5.2),

$$E_\mu(\lambda x) = \sum_{k=1}^{\infty} (\lambda - \mu)^{k-1} R_\lambda^k(\lambda x) = \sum_{k=1}^{\infty} (\lambda - \mu)^{k-1} R_\lambda^{k-1}(R_\lambda(Bx) + x).$$

Since $R_\lambda$ is superadditive,

$$\begin{aligned} E_\mu(\lambda x) &\geq \sum_{k=1}^{\infty} (\lambda - \mu)^{k-1} R_\lambda^k(Bx) + \sum_{k=1}^{\infty} (\lambda - \mu)^{k-1} R_\lambda^{k-1}(x) \\ &= E_\mu(Bx) + x + (\lambda - \mu) E_\mu(x). \end{aligned}$$

Since $E_\mu$ is homogeneous,

$$E_\mu(x) \geq \frac{1}{\mu} x + \frac{1}{\mu} E_\mu(B(x)), \qquad x \in X_+.$$

By iteration and induction,

$$E_\mu(x) \geq \sum_{k=0}^{n} \frac{1}{\mu^{k+1}} B^k(x) + \frac{1}{\mu^{n+1}} E_\mu B^{n+1}(x), \qquad x \in X_+, n \in \mathbb{N}.$$

This implies that

$$E_\mu(x) \geq \frac{1}{\mu^{k+1}} B^k(x), \qquad x \in X_+, k \in \mathbb{N}.$$

Since $X_+$ is normal, there exists some $\tilde{M} > 0$ (which depends on $\mu$) such that

$$\|B^k(x)\| \leq \tilde{M} \mu^k \|x\|, \qquad k \in \mathbb{N}, x \in X_+.$$

Thus

$$\|B^k\|_+ \leq \tilde{M} \mu^k, \qquad k \in \mathbb{N}.$$

Since $\mu < \mathbf{r} = \mathbf{r}_+(B)$, this is a contradiction. □

In the next results, $X_+^*$ denotes the wedge of bounded linear positive functionals on $X$.

**Corollary 5.4** *Let $X$ be an ordered normed vector space with complete normal cone $X_+$. Let $B : X_+ \to X_+$ be homogeneous, order-preserving, continuous and $B(x+y) \geq B(x)+B(y)$ for all $x, y \in X_+$. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$. Then there exist some $x \in X_+$ and $x^* \in X_+^*$ such that $x^*(R_\lambda(x)) \to \infty$ as $\lambda \to \mathbf{r}+$.*

In order to reconcile this result with Proposition 5.2, recall that $\mathbf{r}_o(B) = \mathbf{r}_+(B)$ if $X_+$ is normal and complete (Theorem 2.6).

*Proof* By the uniform boundedness principle in Theorem 4.1, there exists some $x \in X_+$ such that $\{\|R_\lambda(x)\|; \lambda > \mathbf{r}_+(B)\}$ is unbounded. By the usual uniform boundedness principle, applied to the Banach space $X^*$, there exists some $x^* \in X^*$ such that $\{|x^* R_\lambda(x)|; \lambda > \mathbf{r}_+(B)\}$ is unbounded. Since $X_+$ is normal, the wedge $X_+^*$ of positive bounded linear functionals is generating [2, Thm. 2.26], [39, p. 218], and the unboundedness holds for some $x^* \in X_+^*$. Since $x^* R_\lambda(x)$ is a decreasing function of $\lambda$, $x^* R_\lambda(x) \to \infty$ as $\lambda \to \mathbf{r}_+(B)$. □

We define the lower Collatz–Wielandt bound of $B$ as

$$cw(B) = \sup \big\{[B(x)]_x; x \in \dot{X}_+\big\} = \sup \big\{\lambda \geq 0;\ \exists x \in \dot{X}_+ : B(x) \geq \lambda x\big\}. \tag{5.5}$$

Notice that $cw(B)$ is defined if $X$ is just an ordered vector space. If it is a normed ordered vector space, the lower Collatz–Wielandt bound relates to the orbital spectral radius (see [44,45]) as

$$cw(B) \leq \mathbf{r}_o(B). \tag{5.6}$$

**Proposition 5.5** *If $cw(B) > 0$, then for every $\mu < cw(B)$ there exists some $x_\mu^* \in X_+^*$ with $\|x_\mu^*\| = 1$ and*

$$\|x_\mu^* \circ R_\lambda\|_+ \geq \frac{1}{\lambda - \mu}, \qquad \lambda > \mathbf{r}_o(B).$$

*Proof* For any $\mu < cw(B)$, there exists some $x_\mu \in \dot{X}_+$ such that $B(x_\mu) \geq \mu x_\mu$. Let $\psi(x) = d(x, -X_+)$, $x \in X$, be the monotone companion half-norm on $X$ [44,45]. By the Hahn-Banach theorem, there exists some $x_\mu^* \in X^*$ such that $x_\mu^* x_\mu = \psi(x_\mu)$ and $-\psi(-x) \leq x_\mu^* x \leq \psi(x)$ for all $x \in X_+$ [46, IV.6]. Since $\psi(-X_+) = \{0\}$, $x_\mu^* \in X_+^*$. Let $\sharp x \sharp = \max\{\psi(x), \psi(-x)\}$ be the monotone companion norm on $X$ [44,45]. Then $\sharp x_\mu^* \sharp = 1$. Since positive bounded linear functionals have the same operator norm with respect to the original norm and the monotone companion norm, $\|x_\mu^*\| = 1$.

For each $\lambda > \mathbf{r}_o(B)$, by Lemma 5.1, since $R_\lambda$ is homogeneous and order-preserving,

$$\lambda R_\lambda(x_\mu) = R_\lambda(B(x_\mu)) + x \geq R_\lambda(\mu x_\mu) + x = \mu R_\lambda(x_\mu) + x_\mu.$$

So $R_\lambda(x_\mu) \geq \frac{1}{\lambda-\mu} x_\mu$. We apply the positive linear function $x_\mu^*$,

$$x_\mu^* R_\lambda(x_\mu) \geq \frac{1}{\lambda - \mu} \psi(x_\mu).$$

This implies that $\sharp x_\mu^* R_\lambda \sharp_+ \geq \frac{1}{\lambda-\mu}$. Since $\sharp x_\mu^* R_\lambda \sharp_+ = \|x_\mu^* R_\lambda\|_+$ [44,45], this implies the statement. □

## 6 Eigenfunctionals

The celebrated Krein–Rutman theorem does not only state the existence of a positive eigenvector but also of a positive eigenfunctional of a positive linear map on an ordered Banach space provided that the map is compact and the cone is total or that the cone is normal and solid [27] (see also [39, App. 2.4,2.6]).

We explore what still can be done if the additivity of the operator is dropped. Some ideas from the linear case are adopted [6], [9, IV.Prop. 1.10]. Recall the left resolvents $R_\lambda$, $\lambda > \mathbf{r}_+(B) > 0$, in Sect. 5. As before, let $X$ be an ordered normed vector space with cone $X_+$. Throughout this section, we assume that $X_+$ is complete.

**Proposition 6.1** *Let $B : X_+ \to X_+$ be homogeneous, order preserving, and uniformly order-bounded. Assume that $\mathbf{r} = \mathbf{r}_o(B) > 0$ and that there exist a sequence $(\lambda_n)$ in $(\mathbf{r}, \infty)$ with $\lambda_n \to \mathbf{r}$ as $n \to \infty$ and a sequence $p_n : X_+ \to \mathbb{R}_+$ of homogenous, convex, order-preserving functionals with*

$$\|p_n\|_+ \le 1, \quad n \in \mathbb{N}, \qquad \|p_n \circ R_{\lambda_n}\|_+ \to \infty, \quad n \to \infty.$$

*Then there exists a homogeneous, order-preserving, bounded nonzero eigenfunctional $\theta : X_+ \to \mathbb{R}_+$ such that $\theta \circ B = \mathbf{r}\theta$.*

*If $B$ is subadditive, so is $\theta$. If the $p_n$ are additive and $B$ is additive, so is $\theta$. If the $p_n$ are additive and $B$ is superadditive, $\theta$ is superadditive.*

Our proof will not provide continuity of $\theta$.

*Proof* Let $(\lambda_n)$ and $(p_n)$ be as above. Define $\psi_n : X_+ \to \mathbb{R}_+$ by

$$\psi_n(x) = p_n(R_{\lambda_n}(x)), \qquad n \in \mathbb{N}, x \in X_+.$$

The functionals $\psi_n$ are homogeneous, order preserving, and bounded, $\|\psi_n\|_+ \to \infty$ as $n \to \infty$. By (5.2),

$$p_n(R_{\lambda_n}(\lambda_n x)) = p_n\big(R_{\lambda_n}(B(x)) + x\big).$$

Since the $p_n$ are order-preserving, subadditive, and homogeneous,

$$\psi_n(B(x))) \le \psi_n(\lambda_n x) = \lambda_n \psi_n(x) \le \psi_n((B(x)) + p_n(x).$$

We set $\theta_n = \psi_n/\|\psi_n\|_+$. The $\theta_n$ are homogeneous and order-preserving, $\|\theta_n\|_+ = 1$ and

$$0 \le \lambda_n \theta_n(x) - \theta_n(B(x)) \le \frac{p_n(x)}{\|\psi_n\|_+} \to 0, \qquad n \to \infty, x \in X_+. \tag{6.1}$$

By Tychonoff's compactness theorem for topological products, there exists some

$$\theta \in \bigcap_{m\in\mathbb{N}} \overline{B_m}, \qquad B_m = \{\theta_n; n \ge m\},$$

where the closure is taken in the topology of pointwise convergence on $\{x \in X_+; \|x\| \le 1\}$. Notice that all $\theta_n$ are order-preserving, bounded and homogeneous. $\theta$ inherits these properties. For instance, let $x_1, x_2 \in X_+$ and $x_1 \le x_2$. Then there exist a strictly increasing sequence $(n_j)$ of natural numbers such that $\theta_{n_j}(x_i) \to \theta(x_i)$ as $j \to \infty$, $i = 1, 2$. Since $\theta_{n_j}(x_1) \le \theta_{n_j}(x_2)$ for all $j \in \mathbb{N}$, also $\theta(x_1) \le \theta(x_2)$.

Similarly, for $x \in X_+$, there exists a strictly increasing sequence $(n_j)$ of natural number such that $\theta_{n_j}(x) \to \theta(x)$ and $\theta_{n_j}(B(x)) \to \theta(B(x))$ as $j \to \infty$. By (6.1), $\theta(B(x)) = \mathbf{r}\theta(x)$.

The inheritance of various properties by $\theta$ from $B$ and the functionals $p_n$ follows similarly.

We need to rule out that $\theta$ is the zero functional. Since $B$ is uniformly order-bounded, there is some $u \in X_+$, $\|u\| = 1$, such that $B$ is uniformly $u$-bounded: There exist some $c \ge 0$ such that $B(x) \le c\|x\|u$ for all $x \in X_+$.

Let $x \in X_+$, $\|x\| \le 1$. Since each $\theta_n$ is order-preserving, by (6.1),

$$\lambda_n \theta_n(x) \le \theta_n(B(x)) + \frac{p_n(x)}{\|\psi_n\|_+} \le \theta_n(c\|x\|u) + \frac{p_n(x)}{\|\psi_n\|_+} \le c\theta_n(u) + \frac{\|p_n\|_+}{\|\psi_n\|_+}.$$

Since this holds for all $x \in X_+$, $\|x\| \le 1$, and since $\|p_n\|_+ \le 1$,

$$\lambda_n = \lambda_n \|\theta_n\|_+ \le c\theta_n(u) + \frac{1}{\|\psi_n\|_+}.$$

Since $\theta_{n_j}(u) \to \theta(u)$ for some strictly increasing sequence $(n_j)$ in $\mathbb{N}$ and $\|\psi_{n_j}\|_+ \to \infty$ and $\lambda_{n_j} \to \mathbf{r}$, we have $0 < \mathbf{r} \le c\theta(u)$. □

**Definition 6.2** Let $u \in X_+$ and $B : X_+ \to X_+$. Then $B$ is called *pointwise u-positive* if for any $x \in \dot{X}_+$ there exists some $n \in \mathbb{N}$ such that $B^n(x)$ is $u$-positive.

Recall Definition 2.7.

*Remark 6.3* Let $u \in \dot{X}_+$ and $\theta : X_+ \to \mathbb{R}_+$ be a homogeneous, order-preserving nonzero eigenfunctional of $B$, $\theta \circ B = r\theta$ with some $r > 0$.

If $B$ is pointwise $u$-bounded, then $\theta(u) > 0$. If $B$ is pointwise $u$-bounded and pointwise $u$-positive, $\theta(x) > 0$ for all $x \in \dot{X}_+$.

*Proof* Suppose $\theta(u) = 0$. Let $x \in X_+$. Then there exists some $n \in \mathbb{N}$ and $c > 0$ such that $B^n(x) \le cu$. Since $\theta$ is order-preserving and homogeneous,

$$0 = c\theta(u) = \theta(cu) \ge \theta(B^n(x)) = r^n \theta(x).$$

So $\theta$ is the zero functional.

By contraposition, since $\theta$ is a nonzero functional, $\theta(u) > 0$.

Assume that $B$ is pointwise $u$-positive and $x \in \dot{X}_+$. Then there exist some $n \in \mathbb{N}$ and $\epsilon > 0$ such that $B^n(x) \ge \epsilon u$. Since $\theta$ is order-preserving and homogeneous,

$$0 < \epsilon\theta(u) = \theta(\epsilon u) \le \theta(B^n x) = r^n \theta(x).$$

Since $r > 0$, $\theta(x) > 0$. □

### 6.1 Eigenfunctionals for Maps with CW and Lower KR Property

A homogeneous order-preserving map $B : X_+ \to X_+$ on the cone $X_+$ of an ordered normed vectors space is said to have the *CW (Collatz-Wielandt) property* if $0 < \mathbf{r}_o(B) < \infty$ implies $cw(B) = \mathbf{r}_o(B)$, i.e., by (5.5) and (5.6), if for any $\mu < \mathbf{r}_o(B)$ there exists some $x \in \dot{X}_+$ with $B(x) \ge \mu x$.

**Theorem 6.4** *Let $B : X_+ \to X_+$ be homogeneous, order preserving, and uniformly order bounded. Assume that $\mathbf{r} = \mathbf{r}_o(B) > 0$ and $B$ has the CW property.*

*Then there exists a homogeneous, order preserving, bounded eigenfunctional $\theta : X_+ \to \mathbb{R}$ such that $\theta \circ B = \mathbf{r}\theta$ and $\theta(u) > 0$ for any uniform order bound of $B$. $\theta$ inherits the following properties from $B$: additive, subadditive, and superadditive.*

*Proof* Choose sequences $(\lambda_n)$ and $(\mu_n)$ with $\mu_n < \mathbf{r} < \lambda_n$ and $\lambda_n, \mu_n \to \mathbf{r}$ as $n \to \infty$. By Proposition 5.5, we have a sequence $(x_n^*)$ in $X_+^*$ with $\|x_n^*\| = 1$ and $\|x_n^* \circ R_{\lambda_n}\|_+ \to \infty$ as $n \to \infty$. Apply Proposition 6.1. □

If $X_+$ is a sup-semilattice, the following condition implies the CW property.

**Lemma 6.5** *Let $X_+$ be a* sup*-semilattice. Then a homogeneous order-preserving map $B : X_+ \to X_+$ has the CW property if for any $\mu \in (0, \mathbf{r}_o(B))$ there exists some $n \in \mathbb{N}$ and some $v \in \dot{X}_+$ such that $B^n(v) \geq \mu^n v$.*

*Proof* Let $\mu \in (0, \mathbf{r}_o(B))$, $n \in \mathbb{N}$ and $v \in \dot{X}_+$ such that $B^n(v) \geq \mu^n v$. Following [1], we define $w = \sup_{j=0}^{n-1} \mu^{-j} B^j v$. For $j = 0, \dots, n-1$, $B(w) \geq B(\mu^{-j} B^j(v)) = \mu^{-j} B^{j+1}(v) = \mu\mu^{-(j+1)} B^{j+1}(v)$. Since $B^n(v) \geq \mu^n v$, $B(w) \geq \mu^{-n+1} B^n(v) \geq \mu v$. Thus $B(w) \geq \mu\mu^{-j} B^j(v)$ for $j = 0, \dots, n-1$, and so $B(w) \geq \mu w$. This implies that $cw(B) \geq \mu$ for any $\mu \in (0, \mathbf{r}_o(B))$. □

A homogeneous bounded order-preserving map $B : X_+ \to X_+$ is said to have the *KR property* (Krein–Rutman property) if $r := \mathbf{r}_+(B) > 0$ implies that there exists some $v \in \dot{X}_+$ such that $Bv = \mathbf{r}v$. $B$ is said to have the *lower KR property* if $r := \mathbf{r}_+(B) > 0$ implies that there exists some $v \in \dot{X}_+$ such that $Bv \geq \mathbf{r}v$. Obviously, the lower KR property implies the CW property. An example, where $B$ has the CW property but not the lower KR property, is given in Example 6.17.

*Remark 6.6* (a) $B$ has the KR property if $B = K + A$ where $K : X_+ \to X_+$ is homogeneous, compact, continuous and order-preserving and $A : X \to X$ is linear, positive and bounded and

(i) $\|A\| < \mathbf{r}_+(B)$ [32], or (ii) $\mathbf{r}(A) < \mathbf{r}_+(B)$ and $X_+$ is normal [33].

(b) $B$ has the lower KR property if

(i) some power of $B$ has the KR property and $X_+$ is a sup-semilattice [1], or
(ii) $X_+$ is an inf-semilattice, $B$ is continuous, and there is some $u \in \dot{X}_+$ such that $B$ is uniformly $u$-bounded and $B(u)$ is a regular point of $X_+$.

More general classes of maps with lower KR property can be found in [44,45].

The following result is well-known for vectors rather than functionals if $B$ is linear [24, Thm.9.3], [37, Thm. 2.2].

**Lemma 6.7** *Let $B : X_+ \to X_+$. Let $\mathbf{r} > 0$, $p \in \mathbb{N}$, and $\phi : X_+ \to \mathbb{R}$ with $\phi \circ B^p = \mathbf{r}^p \phi$. Set*

$$\theta = \sum_{k=0}^{p-1} \mathbf{r}^{-k} \phi \circ B^k.$$

*Then $\theta \circ B = \mathbf{r}\theta$ and $\theta(x) \geq \phi(x)$ for all $x \in X_+$.*

**Corollary 6.8** *Let $B : X_+ \to X_+$ be homogeneous, bounded, pointwise order bounded and order-preserving. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$ and that some power of $B$ has the lower KR property. Then there exists a homogeneous, order-preserving, bounded nonzero functional $\theta : X_+ \to \mathbb{R}$ such that $\theta \circ B = \mathbf{r}\theta$. $\theta$ inherits the following properties from $B$: additive, subadditive, and superadditive.*

*Proof* There exists some $m, \ell \in \mathbb{N}$ such that $B^m$ has the lower KR property and $K^\ell$ is uniformly order-bounded. Recall our overall assumption that $X_+$ is complete and Theorem 2.12. Set $p = m\ell$. Then $B^p v \geq \mathbf{r}^p v$ for some $v \in \dot{X}_+$, and $B^p$ is uniformly order-bounded. By Theorem 6.4, there exists some homogeneous, order preserving, bounded nonzero functional $\phi : X_+ \to \mathbb{R}$ such that $\phi(B^p(x)) = \mathbf{r}^p \phi(x)$ for all $x \in X_+$. Apply the previous lemma and notice that $\theta$ inherits the desired properties from $\phi$ and $B$. □

If $B$ has the lower KR property, order boundedness can be replaced by a growth condition for the (left) resolvent [37,40]. The following result extends [37, Thm. 2.1] to nonadditive maps.

**Theorem 6.9** *Let $B : X_+ \to X_+$ be homogeneous, bounded, and order-preserving. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$ and that $B$ has the lower KR property and satisfies the following resolvent growth condition:*

*For any $x^* \in X_+^*$ and $x \in X_+$, there exists some $\epsilon > 0$ such that*

$$\{(\lambda - \mathbf{r})\|x^* R_\lambda(x)\|; \mathbf{r} < \lambda < \mathbf{r} + \epsilon\} \text{ is bounded}.$$

*Then there exists a homogeneous, order-preserving, bounded nonzero functional $\theta : X_+ \to \mathbb{R}$ such that $\theta \circ B = \mathbf{r}\theta$. $\theta$ inherits the following properties from $B$: additive, subadditive, and superadditive.*

*Proof* Since $B$ has the lower KR property, there exists some $v \in \dot{X}_+$ such that $B(v) \geq \mathbf{r}v$. The same proof as for Proposition 5.5 provides an $x^* \in X_+^*$ with $x^* R_\lambda(v) \geq \frac{1}{\lambda - \mathbf{r}} d(v, -X_+) > 0$ for $\lambda > r$. By the resolvent growth condition and the uniform boundedness principle (Theorem 4.1), there exists some $c > 0$ such that $\|x^* \circ R_\lambda\|_+ \leq \frac{c}{\lambda - \mathbf{r}}$ for $\lambda \in (\mathbf{r}, \mathbf{r} + \epsilon)$. Choose a sequence $(\lambda_n)$ in $(\mathbf{r}, \mathbf{r} + \epsilon)$ such that $\lambda_n \to \mathbf{r}$. Now, in the proof of Proposition 6.1, set $\psi_n = x^* \circ R_{\lambda_n}$ and $\theta_n = \psi_n / \|\psi_n\|_+$. The arguments now proceed as in this proof, except for showing toward the end that $\theta$ is not the zero functional. From our estimates above,

$$\theta_n(v) = \frac{x^* \circ R_{\lambda_n}(v)}{\|x^* \circ R_{\lambda_n}\|_+} \geq \frac{d(v, -X_+)}{c} > 0.$$

This estimate is inherited by $\theta$. □

Similarly as in [37], one notices that the resolvent growth condition is satisfied if, for any $x^* \in X_+^*$ and $x \in X_+$, there is some $c > 0$ such that $\|x^* B^n(x)\|_+ \leq c\mathbf{r}^n \|x\|$ for all $n \in \mathbb{N}$.

### 6.2 Eigenfunctionals for Homogeneous Concave Maps

If $B$ is homogeneous and superadditive and thus concave, we do not need to assume that $B$ or some power of $B$ has the lower Collatz–Wielandt property.

**Theorem 6.10** *Let $X_+$ be a normal complete cone $X_+$. Let $B$ be homogeneous, superadditive, continuous, and pointwise order bounded. Assume that $\mathbf{r} = \mathbf{r}_+(B) > 0$. Then there exists a homogeneous, superadditive, bounded eigenfunctional $\theta : X_+ \to \mathbb{R}$ such that $\theta(B(x)) = \mathbf{r}\theta(x)$ for all $x \in X_+$. If $B$ is additive, $\theta$ is additive.*

The normality of the cone cannot be dropped in general even if $B$ is a bounded linear positive map on $X$ and $X_+$ is solid [7, Sect. 2(iv)].

*Proof* By Theorem 2.12, we can assume that some power of $B$ is uniformly order bounded. By Lemma 6.7, we can assume that $B$ itself is uniformly order bounded. By Corollary 5.4, there exists some $x^* \in X_+^*$ and some $x \in X_+$ such that $x^*(R_\lambda(x)) \to \infty$ as $\lambda \to \mathbf{r}+$. Apply Proposition 6.1. □

The next result generalizes [6, Thm. 2] for an ordered vector space without an a priori norm.

**Theorem 6.11** *Let $X$ be an ordered vector space with cone $X_+$ and $B : X_+ \to X_+$ be homogeneous and superadditive.*

*Let $X$ have an order unit $u \in \dot{X}_+$, i.e., $X = X_u$. Further let $B$ and $u$ satisfy the following property:*

$\diamond$ *For each $\epsilon > 0$ there exists some $\delta > 0$ such $B(x + \delta u) \leq B(x) + \epsilon u$ for all $x \in X_+$ with $x \leq u$.*

*Define $\eta_u = \inf_{n \in \mathbb{N}} \|B^n\|_u^{1/n}$ where $\|B\|_u := \|B(u)\|_u$. Then $\eta_u = \lim_{n\to\infty} \|B^n\|_u^{1/n}$ and, if $\eta_u > 0$, there exists a concave homogeneous $\theta : X_+ \to \mathbb{R}_+$ with $\theta(u) > 0$ and $\theta \circ B = \eta_u \theta$. If $B$ is additive, $\theta$ is additive.*

Here $\|\cdot\|_u$ is the seminorm introduced in (2.11). Notice that if $X = X_u = X_v$ for $u, v \in \dot{X}_+$, then $\eta_u = \eta_v$.

*Proof* Let

$$J = \{x \in X;\ \forall \xi > 0 : -\xi u \leq x \leq \xi u\} = \{x \in X : \|x\|_u = 0\}. \tag{6.2}$$

We first assume that $J = \{0\}$; equivalently, $\|\cdot\|_u$ is a norm on $X = X_u$. One readily sees that $B$ is uniformly $u$-bounded and bounded and the cone operator norm of $B$ is given by $\|B(u)\|_u =: \|B\|_u$. So $\eta_u$ is the cone spectral radius of $B$ with respect to the norm $\|\cdot\|_u$.

We claim that $B$ is uniformly continuous on every bounded set. Since $B$ is homogeneous, it is sufficient to show that $B$ is continuous on the set $\{x \in X_+; x \leq u\}$. Let $\epsilon > 0$. Choose $\delta > 0$ according to $\diamond$ and $x, y \in X_+, x, y \leq u$, $\|x - y\|_u \leq \delta$. Then $-\delta u \leq x - y \leq \delta u$ and $x \leq y + \delta u$. Since $B$ is order-preserving, $B(x) \leq B(y + \delta u) \leq B(y) + \epsilon u$. Since $y \leq x + \delta u$, $B(y) \leq B(x) + \epsilon u$ by the same argument. So $\|B(x) - B(y)\|_u \leq \epsilon$.

Now let $\tilde{X}$ be the completion of $X$ under $\|\cdot\|_u$ and $\tilde{X}_+$ the closure of $X_+$ in $\tilde{X}$. We identify $X$ and $X_+$ with their isometric embeddings in $\tilde{X}$. By the proof of [6, L.7], $\tilde{X}_+$ is a normal solid cone and $u$ is an order-unit of $\tilde{X}$. Since $B$ is uniformly continuous on bounded sets, it preserves Cauchy sequences in $X_+$. So $B$ can be extended to a map on $\tilde{X}_+$,

$$\tilde{B}(\tilde{x}) = \lim_{n\to\infty} B(x_n), \qquad \tilde{x} \in \tilde{X}_+,$$

for any sequence $(x_n)$ in $X_+$ with $x_n \to \tilde{x}$. We write $\tilde{x} \leq \tilde{y}$ if $\tilde{y} - \tilde{x} \in \tilde{X}_+$. $\tilde{B}$ inherits $\diamond$ and is uniformly continuous on bounded subsets of $\tilde{X}_+$. $\tilde{B}$ is concave and homogeneous, and $\eta_u$ is the cone spectral radius of $\tilde{B}$.

By Theorem 6.10, there exists a concave and homogeneous functional $\tilde{\theta} : \tilde{X}_+ \to \mathbb{R}_+$ such that $\tilde{\theta} \circ \tilde{B} = \eta_u \tilde{\theta}$. Restricting $\tilde{\theta}$ to $X_+$ provides the desired functional $\theta$.

We now drop the assumption that $J = \{0\}$. Recall (6.2). As pointed out in the proof of [6, Thm. 1], $J$ is an ideal in $X$, $u \notin J$, and the factor space $X/J = \{x + J; x \in X\}$ is an ordered vector space with cone $X_+ + J$ and unit $u + J$. Further $\|x + J\|_{u+J} = \|x\|_u$ provides a norm on $X/J$. Differently from [6], $B$ does not leave $J$ invariant, but we have from $\diamond$ that

$$x, y \in X_+, x - y \in J \implies B(x) - B(y) \in J.$$

The arguments are very similar to those that provided the continuity of $B$ before.

We can now define $B_0(x + J) = B(x) + J$, $x \in X_+$.

By our previous considerations, there exists a concave homogeneous functional $\theta_0 : X_+ + J \to \mathbb{R}_+$ such that $\theta_0 \circ B_0 = \eta_u \theta_0$ where

$$\eta_u = \inf_{n\in\mathbb{N}} \|B_0^n(u + J)\|_{u+J}^{1/n} = \inf_{n\in\mathbb{N}} \|B^n(u) + J\|_{u+J}^{1/n} = \inf_{n\in\mathbb{N}} \|B^n(u)\|_u^{1/n}.$$

We define $\theta(x) = \theta_0(x + J)$ for $x \in X_+$. One easily checks that $\phi$ is homogeneous and concave and the same proof as in [6] shows that $\theta \circ B = \eta_u \theta$. $\square$

Recall $X_u$, the set of $u$-bounded elements in $X$ (Definition 2.7). The point $u \in \dot{X}_+$ is an order unit if $X = X_u$. If $u$ is not an order unit but $B(u)$ is $u$-bounded, we introduce

$$X_u^B = \big\{x \in X_+;\ \exists n \in \mathbb{Z}_+ : B^n(x) \in X_u\big\}. \tag{6.3}$$

We note that $X_u^B$ is a cone which is mapped into itself by $B$ and that $X_u^B = X$ if and only if $B$ is pointwise $u$-bounded (Definition 2.10).

**Corollary 6.12** *Let $X$ be an ordered vector space with cone $X_+$ and $u \in \dot{X}_+$. Assume that $B : X_+ \to X_+$ is homogeneous and superadditive and satisfies $\diamond$ in Theorem 6.11.*

*Let $\eta_u = \inf_{n\in\mathbb{N}} \|B^n\|_u^{1/n}$ where $\|B\|_u := \|B(u)\|_u$.*

*The following hold:*

(a) $\eta_u = \lim_{n\to\infty} \|B^n\|_u^{1/n}$.
(b) *If $u, v \in \dot{X}_+$ and $X_u^B = X_v^B$, then $\eta_v > 0$ implies $\eta_u > 0$ and $\eta_u = \eta_v$.*
(c) *If $\eta_u > 0$, there exists a concave homogeneous $\theta : X_u^B \to \mathbb{R}_+$ with $\theta(u) > 0$ and $\theta \circ B = \eta_u \theta$ on $X_u^B$.*
(d) *If $B$ is additive, $\theta$ is additive.*

*Proof* It follows from $\diamond$ with $x = 0$ and the homogeneity of $B$ that $B(u)$ is $u$-bounded and $B$ maps $X_+ \cap X_u$ into $X_+ \cap X_u$. By construction, $u$ is an order unit of $X_u$. Part (a) follows from Theorem 6.11.

For Part (b), assume $X_u^B = X_v^B$ for $u, v \in \dot{X}_+$ and $\eta_v > 0$. There exist $k, m \in \mathbb{Z}_+$ such that $B^k(v) \in X_u$ and $B^m(u) \in X_v$. So $B_k(v) \le cu$ for some $c > 0$. For all $n \in \mathbb{N}$, since $B$ is order-preserving and homogeneous,

$$B^{n+k}(v) \le cB^n(u) \le c\|B^n\|_u u.$$

By the same token, for some $\tilde{c} > 0$,

$$B^{n+k+m}(v) \le c\|B^n\|_u B^m(u) \le c\tilde{c}\|B^n\|_u v, \qquad n \in \mathbb{N}.$$

Hence $\|B^{n+k+m}\|_v \le c\tilde{c}\|B^n\|_u$ for all $n \in \mathbb{N}$ and

$$\big(\|B^{n+k+m}\|_v^{1/(n+k+m)}\big)^{(n+m+k)/n} \le c\tilde{c}\|B^n\|_u^{1/n}, \qquad n \in \mathbb{N}.$$

We take the limit as $n \to \infty$ and obtain $\eta_v \le \eta_u$. Equality follows by symmetry.

(c) We apply Theorem 6.12 to $X_u$ and the restriction of $B$ to $X_+ \cap X_u$. This provides an concave homogenous $\tilde{\theta} : X_+ \cap X_u \to \mathbb{R}_+$ with $\tilde{\theta} \circ B = \eta_u \theta$. We extend $\tilde{\theta}$ to $X_u^B$ as follows: If $B^n(x) \in X_+$ for some $n \in \mathbb{Z}_+$ we set $\theta(x) = \eta_u^{-n} \tilde{\theta}(B^n(x))$. This definition does not depend on $n \in \mathbb{N}$ and has the desired properties. $\square$

*Remark 6.13* If $X$ is an ordered normed vector space, $u \in \dot{X}_+$, $B : X_+ \to X_+$ is homogeneous, bounded, superadditive, and satisfies $\diamond$, the question arises how $\eta_u(B)$ is related to $\gamma_u(B)$ and $\mathbf{r}_+(B)$. If some power of $B$ is uniformly $u$-bounded, $\eta_u(B) \le \gamma_u(B)$ and, if $u$ is a normal point of $X_+$ in addition, $\mathbf{r}_+(B) \le \eta_u(B)$ and all three numbers are equal among themselves and to $\mathbf{r}_o(B)$ [44,45]. If $u$ is not a normal point of $X_+$, $\eta_u(B) < \gamma_u(B) = \mathbf{r}_+(B)$ may occur even if $B$ is a positive bounded linear map on an ordered Banach space $X$ and $X_+$ is a solid cone [7, Sect. 2(iv)].

*Example 6.14* Let $X = \mathbb{R}^{\mathbb{N}}$ with cone $\mathbb{R}_+^{\mathbb{N}}$ and $B$ the right-shift operator $B : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$. Then $cw(B) = 0$. For $\alpha > 0$, let $X_{(\alpha)}$ be the normed vector space of sequences in $\mathbb{R}^{\mathbb{N}}$ that are $u$-bounded for $u = (\alpha^n)$. Then $B$ maps $X_{(\alpha)}$ into itself and $\eta_u = 1/\alpha$. By [26] (or Corollary 6.12), there exists a linear positive eigenfunctional of $B$ on $X_{(\alpha)}$ associated with $1/\alpha$.

If $\alpha = 1$, $u$ is the constant sequence with all terms being 1 and $X_u$ can be identified with $\ell^\infty$ with the supremum norm and $\eta_u = 1$. There is an explicit superadditive eigenfunctional of $B$ on $\ell^\infty$, namely, the limit inferior of the sequences. If we restrict $B$ to $c$, the closed linear subspace of convergent sequences, the limits of sequences form an explicit linear positive eigenfunctional. However, no linear positive eigenfunctional that is associated with 1, the spectral radius of $B$, exists on $c_0$, the closed linear subspace of sequences that converge to 0. The same is the case for $\ell^p \subseteq c_0$ with $p > 1$ with the $p$-norm which shows that regularity of the cone is not suffucient for the existence of an additive eigenfunctional. On $\ell^1$, with the sum-norm, $(x_n) \mapsto \sum_{n=1}^\infty x_n$ is a bounded linear positive eigenfunctional of the right-shift operator associated with one. Notice that the existence of this eigenfunctional does not follow from any of our results or the Krein–Rutman theorem and holds without uniform order-boundedness of the map.

### 6.3 Continuity of Homogeneous Order-Preserving Eigenfunctionals

Our existence proofs do not provide that the eigenfunctional $\theta$ is continuous and Example 6.17 will show that this cannot be expected in general even if $B$ is additive. So some extra conditions will be needed for continuity.

**Proposition 6.15** *Let $\theta : X_+ \to \mathbb{R}_+$ be homogeneous and order-preserving and $u \in \dot{X}_+$. Then $\theta : X_+ \cap X_u \to \mathbb{R}_+$ is continuous with respect to the $u$-norm at every $u$-comparable point $v \in X_+$. In fact,*

$$|\theta(x) - \theta(v)| \le \|x - v\|_u \|u\|_v \theta(v), \qquad x \in X_+ \cap X_u.$$

*Proof* Let $x \in X_+ \cap X_u$. Set $\epsilon = \|x - v\|_u$. Then

$$-\epsilon u \le x - v \le \epsilon u \qquad \text{and} \qquad v - \epsilon u \le x \le v + \epsilon u.$$

So

$$(1 - \epsilon \|u\|_v) v \le x \le (1 + \epsilon \|u\|_v) v.$$

Since $\theta$ is homogeneous and order-preserving,

$$\theta(x) \le (1 + \epsilon \|u\|_v)\theta(v).$$

If $(1 - \epsilon \|u\|_v) \ge 0$, then

$$\theta(x) \ge (1 - \epsilon \|u\|_v)\theta(v)$$

by the same token. If $(1 - \epsilon \|u\|_v) \le 0$, the previous inequality is true anyway. So

$$-\epsilon \|u\|_v \theta(v) \le \theta(x) - \theta(v) \le \epsilon \|u\|_v \theta(v)$$

and the assertion follows. □

**Theorem 6.16** *Let $B : X_+ \to X_+$ be homogeneous, continuous and order-preserving and $\theta : X_+ \to \mathbb{R}_+$ be homogeneous, bounded, and order-preserving. Moreover let $r \in (0, \infty)$ and $\theta(B(x)) = r\theta(x)$ for all $x \in X_+$.*

*Let $u \in \dot{X}_+$ and $B$ be pointwise $u$-positive. Further assume that there exists some $k \in \mathbb{N}$ such that $B^k$ maps $X_+$ with the original norm continuously into $X_+ \cap X_u$ with the $u$-norm.*

*Then $\theta$ is continuous on $X_+$ with respect to the original norm.*

*Proof* Since $B$ is continuous, for all $p \geq k$, $B^p$ maps $X_+$ with the original norm continuously into $X_+ \cap X_u$ with the $u$-norm. Let $w \in \dot{X}_+$. Since $B$ is pointwise $u$-positive, there exists $m \in \mathbb{N}$ and $\epsilon > 0$ such that $B^m(w) \geq \epsilon u$. Further there exists some $q \in \mathbb{N}$ and $\delta > 0$ such that $B^q(u) \geq \delta u$. Then $B^{jq+m}(w) \geq \epsilon \delta^j u$ for all $j \in \mathbb{N}$.

By choosing $n \in \mathbb{N}$ large enough, we can assume that $B^n(w)$ is $u$-comparable and $B^n$ maps $X_+$ with the original norm continuously into $X_+ \cap X_u$ with the $u$-norm. Set $v = B^n(w)$. By Proposition 6.15,

$$r^n|\theta(x) - \theta(w)| = |\theta(B^n(x) - \theta(B^n(w))| \leq \|B^n(x) - B^n(w)\|_u \|u\|_v \theta(v).$$

So $\theta$ is continuous at $w \in \dot{X}_+$. Since $\theta$ is homogeneous and bounded, $\theta$ is continuous at 0. □

Even if $B$ is additive and $X_+$ is complete and normal, there may exist an additive eigenfunctional associated with the cone spectral radius that is not continuous. The following example [7] has served for other counterexamples as well [31].

*Example 6.17* Let $X = C_0(0, 1]$ be the Banach space of continuous functions $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ endowed with the supremum norm. Let $X_+$ be the cone of nonnegative convex functions in $X$. $X_+$ is complete and normal but neither solid nor regular. As shown in [7, Sect. 2], $X_+$ is also total, but not generating. Let $u \in X_+$ be given by $u(t) = t$ for all $t \in [0, 1]$. For any $f \in X_+$, $f(t)/t$ is an increasing function of $t$, and so $f \leq f(1)u$, and $u$ is an order unit of $X_+$. As in [7, Sect. 4], we consider the continuous linear positive map $B$ on $X$ given by

$$B(f)(t) = f(t/2), \qquad 0 \leq t \leq 1, f \in X.$$

It is shown in [7, Sect. 4], that $B$ is compact on $X_+$ but not on $X_+ - X_+$, that $\mathbf{r}_+(B) = 1/2$ and $B(u) = (1/2)u$. Recall that for $f \in X_+$, $f(t)/t$ is an increasing function of $t \in (0, 1]$. So each $f \in X_+$ is differentiable at 0 and

$$\theta(f) := [f]_u = \lim_{t \to 0+} f(t)/t = f'(0) \leq f(1), \qquad f \in X_+.$$

The functional $\theta$ is homogeneous, bounded, and additive, and an eigenfunctional of the restriction of $B$ to $X_+$ satisfying $\theta(Bx) = (1/2)\theta(x)$ for all $x \in X_+$. The eigenfunctional $\theta$ is not continuous at $u$ because $\theta(u) = 1$ while $\theta(f_\alpha) = 0$ with $f_\alpha(t) = t^\alpha$, $\alpha > 1$.

To see that there is no eigenfunctional $\theta$ on $X_+$ that is continuous at $u$, we consider $g_\alpha \in X_+$, $\alpha \in [0, 1]$, given by

$$g_\alpha(t) = \begin{cases} 0, & 0 \leq t \leq \alpha, \\ t - \alpha, & \alpha < t \leq 1. \end{cases}$$

If $\alpha \in (0, 1]$, $B^n(g_\alpha) = 0$ for sufficiently large $n \in \mathbb{N}$. So any eigenfunctional of $B$ associated with a positive eigenvalue satisfies $\theta(g_\alpha) = 0$ if $\alpha > 0$. If $\theta$ were continuous at $u$, then also $\theta(u) = 0$ because $\|g_\alpha - u\|_\infty \to 0$ as $\alpha \to 0$. But this would imply that $\theta = 0$ because $B$ is uniformly $u$-bounded.

Notice that $\theta = [\cdot]_u$ is continuous with respect to the $u$-norm $\|\cdot\|_u$ by Lemma 2.9 (an information not provided by our existence theorems).

We could also have considered the cone of $f \in X$ such that $f(t)/t$ is an increasing function of $t \in (0, 1]$. It is no longer clear whether $B$ is compact on that cone.

The functional $[\cdot]_u$ is also defined for the cone of nonnegative functions in $X$, $C_{0+}(0,1]$, but it is only a lower eigenfunctional associated with 1/2 of the restriction of $B$ to $C_{0+}(0,1]$ as one can check directly and as it follows from Remark 3.4 and $Bu = (1/2)u$. If $f_\alpha(t) = t^\alpha$ with $0 < \alpha < 1$, then $Bf_\alpha = (1/2)^\alpha f_\alpha$.

This shows that the cone spectral radius of $B$ with respect to $C_{0+}(0,1]$ is one. It also shows that $B$ has the CW property. $B$ has not the lower KR property. Indeed, if $B(f) \geq f \in C_{0+}(0,1]$, then $f(t) \leq \sup f([0, 2^{-n}])$ for all $n \in \mathbb{N}$ and $t \in [0,1]$. Since $f$ is continuous at $0 = f(0)$, $f \equiv 0$.

Although $B$ is additive and has the CW property, there is no additive homogeneous eigenfunctional on $C_{0+}(0,1]$ because this cone is generating and any such eigenfunctional could be extended to a linear positive eigenfunctional on the Banach space $X$ that would be automatically continuous. Since the continuous functions with compact support in $(0,1]$ are dense in $X$ and $B^n f = 0$ for any such function if $n$ is large enough, the eigenfunctional would be zero. This shows that, in general, uniform order boundedness cannot be omitted as a condition in Theorem 6.4 or 6.10.

On $C[0,1]$, there is a positive linear eigenfunctional of $B$ associated with 1, namely $f \mapsto f(0)$.

## 7 Spatially Distributed Two-Sex Populations

The population we consider has spatially distributed individuals of both sexes which form pairs in order to reproduce. Most two-sex population models are formulated in continuous time [14–18] but some discrete time models are also considered [3,19–21,34]. Here we consider the case that the mating occurs once a year and that the mating season is short which makes a discrete-time model more appropriate. We also assume that individuals do not live to see two mating seasons. Between mating seasons, both male and females move in space.

The spatial habitat of the population is represented by a nonempty set $\Omega$. If $f : \Omega \to \mathbb{R}_+$, the value $f(\xi)$, $\xi \in \Omega$, represents the number of newborns at $\xi \in \Omega$.

We will first consider the state space $X_+ = L^1_+(\Omega)$ where $\Omega$ is a Borel subset of $\mathbb{R}^m$. $L^1_+(\Omega)$ is a complete regular (and thus normal) cone and a lattice, and we will apply Corollary 6.8 in conjunction with Remark 6.6.

We will also consider the state spaces $X_+ = \mathrm{BM}_+(\Omega)$ and $X_+ = \mathrm{BC}_+(\Omega)$ of bounded measurable (continuous) functions on a topological Hausdorff space $\Omega$. These are normal solid complete cones and we will apply Theorem 6.10 for the existence of an eigenfunctional and Theorem 6.16 for its continuity.

In this context, $B : X_+ \to X_+$ maps the spatial distribution of this year's offspring to the spatial distribution of next year's offspring. $B$ will be the composition of a homogeneous (sometimes concave) mating and reproduction map and linear migration maps for female and male individuals. In this model, there is no density-dependence of per capita mortality rates or per pair reproduction rates. The map $B$ is a lower homogeneous first order approximation at the origin in the sense of Theorem 3.3 (c) of a map in an analogous fully density-dependent model [22].

### 7.1 The Mating and Reproduction Map

The mating and reproduction map, $G : \mathbb{R}^\Omega_+ \times \mathbb{R}^\Omega_+ \to \mathbb{R}^\Omega_+$, is defined by

$$G(f,g)(\xi) = \phi(\xi, f(\xi), g(\xi)), \qquad f, g \in \mathbb{R}^\Omega_+, \quad \xi \in \Omega. \tag{7.1}$$

Here $\mathbb{R}_+^\Omega$ is the set of functions on $\Omega$ with values in $\mathbb{R}_+$, and $\phi : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is the local mating and reproduction function. If there are $x_1$ females and $x_2$ males at location $\xi$ in $\Omega$, $\phi(\xi, x)$ with $x = (x_1, x_2)$ is the amount of offspring produced at $\xi$. We equip $\mathbb{R}_+^2$ with the standard order and make the following assumptions which hold throughout the rest of this paper:

**Assumption 7.1** The mating and reproduction function $\phi$ has the following properties:

(a) $\phi(\xi, \cdot)$ is order preserving on $\mathbb{R}_+^2$ for each $\xi \in \Omega$.
(b) $\phi(\xi, \cdot)$ is homogeneous for each $\xi \in \Omega$,

$$\phi(\xi, \alpha x) = \alpha\phi(\xi, x), \qquad \alpha \geq 0, \xi \in \Omega, x \in \mathbb{R}_+^2.$$

(c) $\phi(\xi, \cdot)$ is continuous for each $\xi \in \Omega$.
(d) The function $\psi : \Omega \to \mathbb{R}_+$ defined by $\psi(\xi) = \phi(\xi, 1, 1)$ is bounded.

One example is given by the harmonic mean

$$\phi(\xi, x) = \beta(\xi)\frac{x_1 x_2}{x_1 + x_2}, \quad x = (x_1, x_2) \in \mathbb{R}_+ \setminus \{(0, 0)\}.$$

Here $\beta : \Omega \to \mathbb{R}_+$ is Borel measurable, and $\beta(\xi)$ is the per pair birth rate at $\xi$. Another example is

$$\phi(\xi, x) = \min\left\{\beta_1(\xi)x_1, \beta_2(\xi)x_2\right\}$$

with two Borel measurable functions $\beta_1, \beta_2 : \Omega \to \mathbb{R}_+$. More examples can be found in [14,15,18] and in Sect. 8 where we will show that $\phi(\xi, \cdot)$ is concave on $\mathbb{R}_+^2$ for the standard mating and reproduction functions.

Notice that $G : \mathbb{R}_+^\Omega \times \mathbb{R}_+^\Omega \to \mathbb{R}_+^\Omega$ is homogeneous and order-preserving. Here $\mathbb{R}_+^\Omega$ is equipped with the pointwise order $f \leq g$ if $f(\xi) \leq g(\xi)$ for all $\xi \in \Omega$. $G$ has only weak positivity and order-preserving properties: If can happen that $f, g$ are not identically zero but $G(f, g)$ is zero if the supports of $f$ and $g$ have empty intersections. $G$ is the Nemytskii or substitution operator associated with $\phi$.

For $x = (x_1, x_2) \in \mathbb{R}_+^2$,

$$\phi(\xi, x_1, x_2) \leq \phi(\xi, x_1 + x_2, x_1 + x_2) = \psi(\xi)(x_1 + x_2), \qquad \psi(\xi) := \phi(\xi, 1, 1). \tag{7.2}$$

So

$$G(f, g) \leq (f + g)\psi, \qquad f, g \in \mathbb{R}_+^\Omega, \qquad \psi(\xi) = \phi(\xi, 1, 1). \tag{7.3}$$

### 7.2 The Migration and Next Year Offspring Maps: State Space $L_+^1(\Omega)$

In order to take account of the movements of individuals over the year, we consider integral operators $K_j$, $j = 1, 2$,

$$(K_j f)(\xi) = \int_\Omega k_j(\xi, \eta) f(\eta) d\eta, \qquad f \in M_+(\Omega), \xi \in \Omega, \ j = 1, 2, \tag{7.4}$$

and assume that $\Omega$ is a Borel measurable subset of $\mathbb{R}^m$. Here $k_j(\xi, \eta) \geq 0$ gives the rate at which individuals that are born at $\eta$ are female ($j = 1$) or male ($j = 2$) and will be at $\xi$ in the year after. $M_+(\Omega)$ denotes the set of Borel measurable functions on $\Omega$ with values in $[0, \infty]$.

The maps $K_j$ are well defined if $k_j : \Omega^2 \to \mathbb{R}_+$ are Borel measurable.

### 7.3 The Next Year Offspring Map

Our first state space of choice is $X_+ = L^1_+(\Omega)$, the cone of $X = L^1(\Omega)$. To make $K_j$ bounded linear maps on $X = L^1(\Omega)$ we assume that the $k_j$ are Borel measurable from $\mathbb{R}^2_+$ to $\mathbb{R}_+$. We also make the following assumptions.

**Assumption 7.2** Assume that $\phi(\cdot, x)$ is Borel measurable on $\Omega$ for each $x \in \mathbb{R}^2_+$.

Assume $k_j : \Omega^2 \to \mathbb{R}_+$ are Borel measurable and that there exists a function $u \in L^1(\Omega)$ such that

$$k_1(\xi, \eta) + k_2(\xi, \eta) \leq u(\xi)$$

for a.a. $(\xi, \eta) \in \Omega^2$ with respect to the $2m$-dimensional Lebesgue measure.

The next year offspring map is formally given by

$$B(f) = G(K_1 f, K_2 f), \quad f \in L^1_+(\Omega). \tag{7.5}$$

By (7.2),

$$B(f) \leq (K_1 f + K_2 f)\psi, \qquad f \in L^1_+(\Omega). \tag{7.6}$$

We will establish that the $K_j$ map $X$ into $X \cap X_u$ and $B$ maps $X_+$ into $X_u \cap X_+$ where $X_u$ is defined as in Definition 2.7.

**Lemma 7.3** *The $K_j$ map $X = L^1(\Omega)$ into $X_u$ and $B$ maps $X_+ = L^1_+(\Omega)$ into $X_u \cap X_+$.*

*Further $\{\|Bf\|_u;\ f \in X_+, \|f\|_1 \leq 1\}$ is bounded and $B$ is uniformly $u$-bounded. Finally $B$ is continuous.*

*Proof* For all $g \in L^\infty_+(\Omega)$, by our assumption and Tonelli's theorem,

$$\begin{aligned}\int_{\Omega^2} g(\xi)|k_j(\xi,\eta) f(\eta)| d\xi d\eta &= \int_{\Omega^2} g(\xi) k_j(\xi,\eta)|f(\eta)| d\xi d\eta \\ &\leq \int_{\Omega^2} g(\xi) u(\xi) |f(\eta)| d\xi d\eta \\ &= \int_{\Omega} g(\xi) u(\xi) d\xi \ \|f\|_1.\end{aligned}$$

This shows that $K_j(f)$ is defined for a.a. $\xi \in \Omega$ and that $K_j(f) \leq \|f\|_1 u$ a.e. on $\Omega$. So $K_j$ maps $X$ into $X_u$ and $\|K_j f\|_u \leq \|f\|_1$. We also see that if $\|f_n\|_1 \to 0$, $K_j(f_n)(\xi) \to 0$ as $n \to \infty$ for a.a. $\xi \in \Omega$.

Further, for a.a. $\xi \in \Omega$, since $\phi(\xi, \cdot)$ is homogeneous,

$$\phi\big(\xi, K_1(f)(\xi), K_2(f)(\xi)\big) \leq \phi\big(\xi, \|f\|_1 u(\xi), \|f\|_1 u(\xi)\big) = \psi(\xi)\|f\|_1 u(\xi),$$

with $\psi(\xi) = \phi(\xi, 1, 1, )$ being a bounded function of $\xi$ by assumption. Hence

$$B(f) \leq \sup \psi(\Omega)\|f\|_1 u.$$

□

To show that $B$ is continuous let $(f_n)$ be a sequence in $L^1_+(\Omega)$ and $f \in L^1_+(\Omega)$ such that $\|f_n - f\|_1 \to 0$. By our previous considerations, $\|K_j(f_n) - K_j(f)\|_u \leq \|f_n - f\|_1 \to 0$ as $n \to \infty$ and $K_j(f_n) \to K_j(f)$ as $n \to \infty$ a.e. on $\Omega$. Since $\phi(\xi, \cdot)$ is continuous by assumption,

$$\phi\big(\xi, K_1(f_n)(\xi), K_2(f_n)(\xi)\big) \to \phi\big(\xi, K_1(f)(\xi), K_2(f)(\xi)\big) \text{ for a.a. } \xi \in \Omega.$$

Since $\phi(\xi, \cdot)$ is increasing,

$$\phi\big(\xi, K_1(f_n)(\xi), K_2(f_n)(\xi)\big) \le \phi\big(\xi, \|f_n\|_1 u(\xi), \|f_n\|_1 u(\xi)\big) \le \psi(\xi)\|f_n\|_1 u(\xi).$$

By the a.e. version of the dominated convergence theorem, $\|B(f_n) - B(f)\|_1 \to 0$.

Since $X_+$ is regular and a lattice, we have the following result from Remark 6.6 (b)(ii) and Theorem 6.8.

**Theorem 7.4** *Let the Assumptions 7.1 and 7.2 hold and* $\mathbf{r}_+(B) > 0$.

*Then there exists some* $f \in L^1_+(\Omega)$, $f \ne 0$, *such that* $B(f) \ge \mathbf{r}_+(B) f$. *There also is some homogeneous, order-preserving* $\theta : L^1_+(\Omega) \to \mathbb{R}_+$ *such that* $\theta \circ B = \mathbf{r}_+(B)\theta$. *If* $\phi(\xi, \cdot)$ *is superadditive on* $\mathbb{R}^2_+$ *for all* $\xi \in \Omega$, $\theta$ *is a superadditive functional.*

### 7.4 The State Space of Bounded Measurable Functions

The continuity of eigenfunctionals can be more easily studied for cones with nonempty interior. We assume that $(\Omega, \Sigma)$ is a measurable space with a set $\Omega$ and a $\sigma$-algebra $\Sigma$ of subsets of $\Omega$. Let $\mathcal{M}(\Omega)$ denote the finite (signed) measures on $\Sigma$ and $\mathcal{M}_+(\Omega)$ the finite (nonnegative) measures on $\Sigma$. Further let $X = \mathrm{BM}(\Omega)$ denote the vector space of bounded measurable functions with supremum norm $\|f\| = \sup_{\xi\in\Omega} |f(\xi)|$. $X_+ = \mathrm{BM}_+(\Omega)$, the cone of nonnegative bounded measurable functions, is a solid normal cone.

The migration maps are based on measure-kernels.

**Assumption 7.5** Let $\Gamma_j : \Omega \times \Sigma \to \mathbb{R}_+$, $j = 1, 2$, be measure kernels, i.e.,

(a) $\Gamma_j(\cdot, S) \in \mathrm{BM}(\Omega)$ for all $S \in \Sigma$ and
(b) $\Gamma_j(\xi, \cdot) \in \mathcal{M}_+(\Omega)$ for all $\xi \in \Omega$.

Then we have migration maps $K_j : \mathrm{BM}(\Omega) \to \mathrm{BM}(\Omega)$ defined by

$$(K_j f)(\xi) = \int_\Omega \Gamma_j(\xi, d\eta) f(\eta), \qquad \xi \in \Omega,\ f \in \mathrm{BM}(\Omega), \tag{7.7}$$

and the bounded linear maps $K_j$ map $X_+ = \mathrm{BM}_+(\Omega)$ into itself.

**Assumption 7.6** The mating and reproduction function $\phi$ has the following properties:

(a) $\phi(\xi, \cdot)$ is concave on $\mathbb{R}^2_+$ for each $\xi \in \Omega$.
(b) $\{\phi(\xi, \cdot); \xi \in \Omega\}$ is equicontinuous on $\mathbb{R}^2_+$.
(c) $\phi(\cdot, x)$ is measurable on $\Omega$ for each $x \in \mathbb{R}^2_+$.

The property $(b)$ means that for each $x \in \mathbb{R}^2_+$ and each $\epsilon > 0$ there exist some $\delta > 0$ such that $|\phi(\xi, x) - \phi(\xi, y)| < \epsilon$ for all $\xi \in \Omega$ and all $y \in \mathbb{R}^2_+$ with $\|y - x\| < \delta$.

This property together with the homogeneity of $\phi$ implies that $\phi$ is bounded on $\Omega \times S$ for any bounded subset $S$ of $\mathbb{R}^2_+$. It also implies that the substitution map $G$ associated with $\phi$ by (7.1) is continuous on $\mathrm{BM}(\Omega)^2$.

We define $B : \mathrm{BM}_+(\Omega) \to \mathrm{BM}_+(\Omega)$ by

$$B(f) = G(K_1 f, K_2 f), \qquad f \in \mathrm{BM}_+(\Omega). \tag{7.8}$$

Assumption 7.6 implies that $B$ is a concave, continuous map on $\mathrm{BM}_+(\Omega)$.

We have the following result from Theorem 6.10.

**Theorem 7.7** *Let the Assumptions 7.1, 7.5 and 7.6 hold. If* $\mathbf{r} = \mathbf{r}_+(B) > 0$, *then there exists a homogeneous, concave, bounded* $\theta : \mathrm{BM}_+(\Omega) \to \mathbb{R}_+$ *such that* $\theta \circ B = \mathbf{r}_+\theta$.

In order to find conditions for the eigenfunctional $\theta$ to be strictly positive, we make the following assumptions.

**Assumption 7.8** $\Omega$ is a connected topological Hausdorff space and the measure kernels have the following properties.

(a) For any $\xi \in \Omega$ there exists an open subset $U$ with $\xi \in U \subseteq \Omega$ such that $\Gamma_j(\eta, V) > 0$ for all $\eta \in U$ and all nonempty open subsets $V$ of $U$.
(b) There exist subsets $\Omega_1, \ldots, \Omega_k$ of $\Omega$ such that $\Omega_1 = \Omega$, $\Omega_k$ is compact and

$$\inf_{x \in \Omega_j} \Gamma_j(x, \Omega_{j+1}) > 0, \qquad j = 1, \ldots, k-1.$$

(c) The measure-kernels $\Gamma_i$ have the Feller property, i.e., the maps $K_i$ map $Y$ =BC($\Omega$) into itself [42, p. 54].

Assumption (a) makes sure that females and males spread locally and (b) makes sure that they finally get everywhere. Examples of measure kernels that have the Feller property but to not satisfy the positivity assumption are certain Dirac kernels, i.e.,

$$(K_j f)(\xi) = g_j(\xi) f(\xi), \qquad f \in \mathrm{BM}(\Omega), \xi \in \Omega,$$

with continuous functions $g_j : \Omega \to \mathbb{R}_+$. Criteria for the Feller property can be found in [28, App. A].

We also make the following positivity and continuity assumptions for the mating function $\phi$.

**Assumption 7.9** The mating and reproduction function $\phi$ has the following additional properties.

(a) $\phi$ is continuous on $\Omega \times \mathbb{R}^2_+$.
(b) $\phi(\xi, 1, 1) > 0$ for all $\xi \in \Omega$.

**Proposition 7.10** *Let the assumptions of Theorem 7.7 be satisfied and let the Assumptions 7.8 and 7.9 also hold. Then $B$ is pointwise $u$-positive on* $\mathrm{BC}_+(\Omega)$ *with $u$ being the constant function with value 1.*

*Proof* For $f \in \mathrm{BC}_+(\Omega)$, $f \not\equiv 0$, we define

$$\Omega_n(f) = \big\{x \in \Omega;\, B^n(f)(x) > 0\big\}, \qquad n \in \mathbb{Z}_+.$$

Since $B^n(f)$ is continuous, the sets $\Omega_n(f)$ form a sequence of open subsets of $\Omega$. By Assumption 7.8 (a), this sequence is increasing with respect to the subset relation. Then $\tilde\Omega = \bigcup_{n \in \mathbb{Z}_+} \Omega_n(f)$ is an open subset of $\Omega$. To show that $\tilde\Omega$ is closed, let $\xi$ be a limit point of $\tilde\Omega$. By Assumption 7.8 (a), there exists an open set $U$ with $\xi \in U \subseteq \Omega$ such that $\Gamma_i(\eta, V) > 0$ for all $\eta \in U$ and all nonempty open subsets $V$ of $U$. Since $x$ is a limit point of $\tilde\Omega$, $U \cap \tilde\Omega \neq \emptyset$ and $U \cap \Omega_n(f) \neq \emptyset$ for some $n \in \mathbb{Z}_+$. Since $\Omega_n(f) = \bigcup_{m \in \mathbb{N}} \{\zeta \in \Omega;\, B^n(f)(\zeta) > 1/m\}$, there exists a nonempty open set $V$ in $U$ such that $B^n(f)(\zeta) > 1/m$ for all $\zeta \in V$. So

$$K_i(B^n(f))(\eta) \geq (1/m)\Gamma_i(\eta, V) > 0, \qquad \eta \in U,$$

and $B^{n+1}(f)(\eta) > 0$ for all $\eta \in U$ by Assumption 7.9 (b) and the homogeneity of $\phi(\xi, \cdot)$. Since $\xi \in U$, $\xi \in \tilde\Omega$.

So $\tilde\Omega$ is also closed and thus equals $\Omega$ because $\Omega$ is connected. Let $\Omega_1, \ldots, \Omega_k$ like in Assumption 7.8 (b). Since $\Omega_k$ is compact, $\Omega_k \subseteq \Omega_n(f)$ for some $n \in \mathbb{N}$ and $\inf B^n(f)(\Omega_k) > 0$. It follows successively that $\inf B_{n+i}(f)(\Omega_{k-i}) > 0$, $i = 1, \ldots, k-1$. Hence $B^{n+k}(f)$ is $u$-positive where $u$ is the constant function with value 1. □

Since $Y = Y_u$, $B$ continuously maps $Y_+$ into $Y_u$. So we have the following result from Theorem 6.16, Theorem 7.7, and Proposition 7.10.

**Theorem 7.11** *Let the assumptions of Theorem 7.7 be satisfied and let the Assumptions 7.8 and 7.9 also hold. If* $\mathbf{r} = \mathbf{r}_+(B) > 0$, *$B$ has a homogeneous, concave, bounded eigenfunctional* $\theta : \mathrm{BM}_+(\Omega) \to \mathbb{R}_+$ *with* $\theta \circ B = \mathbf{r}\theta$ *which is continuous and strictly positive on* $\mathrm{BC}_+(\Omega)$.

## 8 The Usual Mating Functions are Concave

We will prove the following conjecture in the case that $\phi : \mathbb{R}^2_+ \to \mathbb{R}$ is twice continuously differentiable on $(0, \infty)^2$.

**Conjecture** *Assume that* $\phi : \mathbb{R}^2_+ \to \mathbb{R}$ *is homogeneous and, for all* $y \in \mathbb{R}_+$, $\phi(\cdot, y)$ *or* $\phi(y, \cdot)$ *are concave on* $\mathbb{R}_+$. *Then* $\phi$ *is concave.*

We first show that concave dependence on the single variables is necessary.

**Proposition 8.1** *Let* $\phi : \mathbb{R}^2_+ \to \mathbb{R}$ *be concave,* $\phi(0) = 0$. *Then, for all* $y \in \mathbb{R}_+$, $\phi(\cdot, y)$ *and* $\phi(y, \cdot)$ *are concave on* $\mathbb{R}_+$ *and* $\phi(tz)$ *is a concave function of* $t \geq 0$ *for all* $z \in \mathbb{R}^2$.

*Proof* Let $y \in \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\psi(x) = \phi(x, y)$. Then, for $t \in (0, 1)$,

$$\psi((1-t)x + t\tilde{x}) = \phi((1-t)(x, y) + t(\tilde{x}, y)) \geq (1-t)\phi(x, y) + t\phi(\tilde{x}, y) = (1-t)\psi(x) + t\psi(\tilde{x}).$$

This implies that $\phi(\cdot, y)$ is concave. The concavity of $\phi(y, \cdot)$ follows in the same way. For $z \in \mathbb{R}^2_+$ and $t, s \geq 0$, $r \in (0, 1)$,

$$\phi((1-r)tz + rsz) \geq (1-r)\phi(tz) + r\phi(sz).$$

□

The assumption that $\phi$ is concave in the separate variables seems natural. If the number of individuals of males is kept fixed, the number of matings should depend almost linearly on the number of females as long as this number is small but reach a plateau as the number of females becomes very large.

Proposition 8.1 shows that the homogeneity of $\phi$ cannot be dropped in general for our conjecture to be true. The function $\phi(x, y) = x^\gamma y^\theta$ is concave in $x$ and in $y$ if $\gamma, \theta \in (0, 1]$, but $\phi(t, t) = t^{\gamma+\theta}$ is not concave if $\gamma + \theta > 1$.

### 8.1 The Twice Differentiable Case

We assume that $\phi$ is twice continuously differentiable. Further, we assume that for all $y \in \mathbb{R}_+$, $\phi(\cdot, y)$ or $\phi(y, \cdot)$ are concave on $\mathbb{R}_+$ and $\phi$ is homogeneous.

We will show that $\phi$ is concave.

Fix $z \in \mathbb{R}^2_+$ and set $\psi(t) = \phi(tz)$ for $t \in (0, 1)$. Then, if $z = (x, y)$, since $\psi(t) = t\phi(z)$,

$$\phi(z) = \psi'(t) = \partial_1\phi(tx, ty)x + \partial_2\phi(tx, ty)y.$$

We set $t = 1$ and obtain

$$\phi(x, y) = x\partial_x\phi(x, y) + y\partial_y\phi(x, y).$$

We differentiate once more,

$$\partial_x \phi(x, y) = \partial_x \phi(x, y) + x \partial_x^2 \phi(x, y) + y \partial_x \partial_y \phi(x, y)$$

and

$$\partial_y \phi(x, y) = x \partial_y \partial_x \phi(x, y) + \partial_y \phi(x, y) + y \partial_y^2 \phi(x, y).$$

Let $xy > 0$. Then this simplifies to

$$\partial_x \partial_y \phi(x, y) = -\frac{x}{y} \partial_x^2 \phi(x, y), \qquad \partial_y \partial_x \phi(x, y) = -\frac{y}{x} \partial_y^2 \phi(x, y). \tag{8.1}$$

We actually need that $\phi$ is twice continuously differentiable; then $\partial_x \partial_y \phi = \partial_y \partial_x \phi$. By one-dimensional calculus and our assumptions, $\partial_x^2 \phi(x, y) \le 0$ or $\partial_y^2 \phi(x, y) \le 0$. But then both $\partial_x^2 \phi(x, y) \le 0$ and $\partial_y^2 \phi(x, y) \le 0$ and $\partial_x \partial_y \phi(x, y) \ge 0$.

For the determinant of the Hessian matrix of $\phi$, (8.1) implies

$$\det H = (\partial_x^2 \phi)(\partial_y^2 \phi) - (\partial_x \partial_y \phi)(\partial_y \partial_x \phi) = 0.$$

We check whether the Hessian matrix is negative semi-definite on $(0, \infty)^2$: Let $r, s \ge 0$. Then

$$\begin{aligned} &r^2 \partial_x^2 \phi(x, y) + rs(\partial_x \partial_y \phi(x, y) + \partial_y \partial_x \phi(x, y)) + s^2 \partial_y \phi(x, y) \\ &\quad = \partial_x \partial_y \phi(x, y) \Big[ - r^2 \frac{y}{x} + rs \Big] + \partial_y \partial_x \phi(x, y) \Big[ - s^2 \frac{x}{y} + rs \Big] \le 0. \end{aligned}$$

Now let $z = (x, y)$, $\tilde{z} = (\tilde{x}, \tilde{y}) \in (0, \infty)^2$. Set $\psi(t) = \phi((1 - t)z + t\tilde{z})$. Then

$$\psi'(t) = \partial_1 \phi\big((1 - t)z + t\tilde{z}\big)(\tilde{x} - x) + \partial_2 \phi\big((1 - t)z + t\tilde{z}\big)(\tilde{y} - y)$$

and

$$\psi''(t) = (\partial_1^2 \phi)(\tilde{x} - x)^2 + 2(\partial_1 \partial_2 \phi)(\tilde{x} - x)(\tilde{y} - y) + (\partial_2^2 \phi)(\tilde{y} - y)^2.$$

Since $H$ is negative semidefinite, $\psi'' \le 0$ and $\psi$ is concave. This implies that

$$\psi(t) = \psi((1 - t)0 + t1) \ge (1 - t)\psi(0) + t\psi(1) = (1 - t)\phi(z) + t\phi(\tilde{z})$$

and $\phi$ is concave on $(0, \infty)^2$. Concavity on $\mathbb{R}_+^2$ now follows by an approximation argument.

We remark that, for the relation between concavity and the Hessian matrix, one can refer to Theorem 4 in [10, Sect. 2–4].

One important class of standard mating functions is

$$\phi(s, t) = (ps^\gamma + qt^\gamma)^{1/\gamma}, \quad s, t > 0, \tag{8.2}$$

where $\gamma < 0$ and $p, q > 0$, $p + q = 1$ [14,15]. Set $\beta = -\gamma$. Then

$$\phi(s, t) = \frac{st}{(qt^\beta + ps^\beta)^{1/\beta}}. \tag{8.3}$$

Notice that, for this form of $\phi$, $\phi(s, t) = 0$ if either $s$ or $t = 0$, and

$$\min\{s, t\} \le \phi(s, t) \le \max\left\{p^{1/\gamma}, q^{1/\gamma}\right\} \min\{s, t\}, \qquad s, t \ge 0. \tag{8.4}$$

We conclude that $\phi(s, t) \to 0$ as $(s, t) \to 0$ and set

$$\phi(t, s) = 0, \quad t, s \ge 0, st = 0. \tag{8.5}$$

With this definition, $\phi$ becomes continuous on $\mathbb{R}^2_+$. One readily checks that $\phi$ is homogeneous and that $\phi$ is a concave function of the first variable.

Another class is

$$\phi(s,t) = s^p t^q, \qquad s,t \geq 0, \tag{8.6}$$

with $p, q > 0$ and $p + q = 1$. Here it is even more obvious that $\phi$ is a concave function of each separate variable. Actually, this $\phi$ is the limit of (8.2) as $\gamma \to 0$ [14]. Since pointwise limits of concave functions are concave, the concavity of this $\phi$ follows from the concavity of the previous one. By the same token, the function

$$\phi(x,y) = \min\{x,y\} \tag{8.7}$$

is concave because it is the limit of (8.2) as $\gamma \to -\infty$ by (8.4).

Notice new homogeneous superadditive functions can be obtained from known ones by setting

$$\tilde{\phi}(x,y) = \beta\phi(\alpha x, \tilde{\alpha} y), \qquad x,y \geq 0$$

where $\alpha, \tilde{\alpha}, \beta \geq 0$.

As for the general conjecture, we have tried to use standard mollifying techniques, but they failed to preserve positive homogeneity.

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## References

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