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Steady States of Fokker–Planck Equations: I. Existence

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Abstract This is the first paper in a series concerning the study of steady states of a Fokker–Planck equation in a general domain in $\mathbb{R}^n$ with L^p_{loc} drift term and $W^{1,p}_{loc}$ diffusion term for any $p > n$. In this paper, by using the level set method especially the integral identity which we introduced in Huang et al. (Ann Probab, 2015), we obtain several new existence results of steady states, including stationary solutions and measures, of the Fokker–Planck equation with non-degenerate diffusion under Lyapunov-like conditions. As applications of these results, we give some examples on the noise stabilization of an unstable equilibrium and the existence and uniqueness of steady states subject to boundary degeneracy of diffusion in a bounded domain.

Dedicated to the memory of Professor K. Kirchgaessner.

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1 Introduction

We consider a system of ordinary differential equations (ODE's)

$$\dot{x} = V(x), \qquad x \in \mathcal{U} \subset \mathbb{R}^n \tag{1.1}$$

under white noise perturbation $G(x)\dot{W}$, where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set which can be bounded, unbounded, or the entire space $\mathbb{R}^n$, $V = (V^i)$ is a vector field on $\mathcal{U}$, called the *drift field*, $G = (g^{ij})$ is an $n \times m$ matrix-valued function on $\mathcal{U}$ for some positive integer m, called the *noise matrix*, and W is the standard m-dimensional Brownian motion. This leads to the following system of Itô stochastic differential equations

$$\mathrm{d}x = V(x)\mathrm{d}t + G(x)\mathrm{d}W, \qquad x \in \mathcal{U} \subset \mathbb{R}^n. \tag{1.2}$$

A fundamental issue is the impact of noises on the basic dynamics of (1.1). On one hand, as all differential equations are idealized models of physical motions which are actually subjected to noise perturbations, one would like to know to which extent or under what conditions basic dynamics of a differential equation are robust under small noise perturbations. On the other hand, it has also been observed that large noises can destroy otherwise deterministically robust or unstable dynamics in a system, leading to interesting dynamical phenomena such as random destabilization or stabilization.

Noise impacts on dynamics of differential equations have been extensively studied using a "trajectory based approach" within the framework of random dynamical systems which are skew-product flows over measurable, ergodic base flows. Due to its similarity with the study of a deterministic skew-product flow, this approach has been proven to be very useful in studying random perturbations with respect to problems such as random attractors, random invariant manifolds, and random bifurcations etc. We refer the reader to [3] and references therein for many interesting studies on these problems. However, when a physical system either has sufficiently high complexity or contains intrinsic uncertainties, "trajectory based" model and study would not provide much information to its dynamical description. Instead, a "distribution based approach" using Fokker–Planck equations seems necessary to synthesize the typical patterns of dynamics. The later approach is significantly different from the former one not only because of distinct dynamical objects considered but also because completely different natures of noise perturbations they adopt.

In a sequence of papers including [20], we will develop a theoretical framework, from a distribution point of view, for one to analyze the impact of Itô white noises on compact invariant sets and invariant measures of the ODE system (1.1), and we will link this development to the study of steady states of Fokker–Planck equations associated with the stochastic differential equations (1.2). This paper and [18,19] in the same series, devoting to the existence and non-existence of these steady states, thus serve as the foundation for such a development.

In the case that both V and G are locally Lipschitz-continuous in $\mathcal{U}$, the stochastic differential equation (1.2) generates a local (in time) diffusion process in $\mathcal{U}$ whose *transition probability density function* $p(t, \xi, x)$, if exists, is actually a fundamental solution of the Fokker–Planck equation associated with (1.2). More precisely, denote $A = (a^{ij}) = \frac{GG^\top}{2}$, the *diffusion matrix*. Then for any bounded, non-negative, measurable function f in $\mathcal{U}$ with $\int_{\mathcal{U}} f(x)\mathrm{d}x = 1$,

$$u(x,t) = \int_{\mathcal{U}} p(t,\xi,x) f(\xi)\mathrm{d}\xi$$

(formally) satisfies the *Fokker–Planck equation* (also called *Kolmogorov forward equation*):

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = Lu(x,t), & x \in \mathcal{U},\ t > 0, \\ u(x,t) \geq 0, & \int_{\mathcal{U}} u(x,t)\mathrm{d}x = 1 \end{cases} \tag{1.3}$$

with $u(x,0) = f(x)$, where L is the Fokker–Planck operator defined as

$$Lg(x) = \partial^2_{ij}(a^{ij}(x)g(x)) - \partial_i(V^i(x)g(x)), \qquad g \in C^2(\mathcal{U}).$$

Among solutions of the Fokker–Planck equations (1.3), of particular interest are the *stationary solutions*, i.e., solutions $u(x)$ of the *stationary Fokker–Planck equation*

$$\begin{cases} Lu(x) = 0, & x \in \mathcal{U} \\ u(x) \geq 0, & \int_{\mathcal{U}} u(x)\mathrm{d}x = 1. \end{cases} \tag{1.4}$$

We note that, since $A = (a^{ij})$ is everywhere positive semi-definite, L is an elliptic operator.

In the above and also through the rest of the paper, we use short notations $\partial_i = \frac{\partial}{\partial x_i}$, $\partial^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, and we also adopt the usual summation convention on $i, j = 1, 2, \ldots, n$ whenever applicable.

For the generality of our theory, we will assume less regularity conditions on both the drift field and the diffusion matrix. Following [8–13] and others, we make the following standard hypothesis:

(A) $a^{ij} \in W^{1,p}_{\mathrm{loc}}(\mathcal{U})$, $V^i \in L^p_{\mathrm{loc}}(\mathcal{U})$ for all $i, j = 1, \ldots, n$, where $p > n$ is fixed.

There are many situations arising in applications in which the drift field is only continuous, or piece-wise continuous, or even measurable. Also, a white noise perturbed differential inclusion can often lead to a stochastic selection of form (1.2) in which V is of the class L^q_{loc} for some $q \geq 1$ in general (see e.g., [14]). Even when a Lipschitz drift field is considered, there is no physical reason for a multiplicative (i.e., spatially non-homogenous) white noise perturbation to have sufficiently smooth coefficients.

Due to the weak regularity condition (A), it is necessary to consider *weak stationary solutions* of the Fokker–Planck equation (1.3) or *weak solutions* of the stationary Fokker–Planck equation (1.4), i.e., continuous functions satisfying the following weak form of the stationary Fokker–Planck equation:

$$\begin{cases} \displaystyle\int_{\mathcal{U}} \mathcal{L}f(x)u(x)\mathrm{d}x = 0, & \text{for all } f \in C_0^\infty(\mathcal{U}) \\ u(x) \geq 0, & \int_{\mathcal{U}} u(x)\mathrm{d}x = 1, \end{cases} \tag{1.5}$$

where

$$\mathcal{L} = a^{ij}\partial^2_{ij} + V^i\partial_i$$

is the adjoint Fokker–Planck operator and $C_0^\infty(\mathcal{U})$ denotes the space of C^∞ functions on $\mathcal{U}$ with compact supports.

In the absence of weak solutions satisfying (1.5), one further considers Borel probability measure-valued solutions μ satisfying

$$V^i \in L^1_{loc}(\mathcal{U}, \mu), \qquad i = 1, 2, \ldots, n, \text{ and,} \tag{1.6}$$

$$\int_{\mathcal{U}} \mathcal{L} f(x) \mathrm{d}\mu(x) = 0, \qquad \text{for all } f \in C_0^\infty(\mathcal{U}). \tag{1.7}$$

We refer to a Borel probability measure μ satisfying (1.6), (1.7) as a *stationary measure* of the Fokker–Planck equation (1.3) or a *measure solution* of the stationary Fokker–Planck equation (1.4). We call a stationary measure μ of (1.3) *regular* if μ admits a continuous density function u with respect to the Lebesgue measure, i.e., $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$. It is clear that such a density u is necessarily a weak stationary solution of the Fokker–Planck equation (1.3), and vice versa. In fact, under the condition (A) and the condition that (a^{ij}) is everywhere positive definite in $\mathcal{U}$, it follows from a regularity theorem due to Bogachev–Krylov–Röckner ([6], also recalled in Theorem 2.2 below) that all stationary measures of (1.3) are regular with density functions $u \in W^{1,p}_{loc}(\mathcal{U})$. If $a^{ij} \in C^{2,\alpha}_{loc}(\mathcal{U})$, $V^i \in C^{1,\alpha}_{loc}(\mathcal{U})$, $i, j = 1, \ldots, n$, for some $\alpha \in (0, 1)$, then it follows from the standard Schauder theory that the density functions become classical solutions of (1.4).

The study of stationary measures of (1.3) is closely related to that of invariant measures of the diffusion process generated from (1.2). In the case that $\mathcal{U} = \mathbb{R}^n$, V, G are locally Lipschitz-continuous, and (1.2) generates a global (in time) diffusion process in $\mathbb{R}^n$, it is well-known that any invariant measure of the diffusion process is necessarily a stationary measure of the Fokker–Planck equation (1.3), and vice versa under some suitable conditions (see e.g. [12, Thm. 2.12 and Prop. 2.9]). In the case that (1.2) fails to generate a diffusion process due to low regularity of V, G etc, under certain conditions a stationary measure of (1.3) is an invariant measure of certain generalized diffusion process associated to (1.2) (see [7,13,23] for some interesting discussions in this regard, in particular with respect to the uniqueness of stationary measures and their invariance).

It is well-known that smooth stationary solutions of a Fokker–Planck equation defined on a compact manifold without boundary always exist and are unique when (a^{ij}) is a C^2-smooth, everywhere positive definite matrix on the manifold and the drift field is everywhere smooth (see e.g. [29] for an argument using Perron–Frobenius method). In a bounded domain $\Omega \subset \mathbb{R}^n$, with $(a^{ij}) \in W^{1,p}(\Omega)$ being uniformly elliptic and $(V^i) \in L^p(\Omega)$, the existence of weak stationary solutions of the associated Fokker–Planck equation in Ω follows from classical theory of linear elliptic equations (see [8, Proof of Thm. 1.2] which is recalled as Lemma 2.1 below). These solutions depend on the imposed boundary conditions and thus they need not be unique.

The existence of stationary solutions of Fokker–Planck equations (1.3) in an unbounded domain necessarily requires certain "dissipation" conditions, for otherwise the Laplacian equation in $\mathbb{R}^n$ provides a simple counter example. Some "dissipation" conditions are also needed for the existence of stationary solutions of Fokker–Planck equations (1.3) even in a bounded domain when (a^{ij}) becomes degenerate on the boundary. For instance, taking $a(x) = x^2$, $\mathcal{U} = (0, 1)$, and $V \equiv 0$, the corresponding stationary Fokker–Planck equation $(x^2 u(x))'' = 0$, $x \in (0, 1)$, admits no solution that is non-negative and integrable.

In this paper, we will consider a general domain $\mathcal{U} \subset \mathbb{R}^n$, which can be bounded, unbounded or $\mathbb{R}^n$, and focus on the existence of stationary measures of (1.3) in $\mathcal{U}$ under the condition (A) and that (a^{ij}) is everywhere positive definite in $\mathcal{U}$, by considering "dissipation" conditions which are compatible with those in deterministic ODE systems. One of such "dissipation" conditions is the existence of a *Lyapunov function* $U \in C^2(\mathcal{U})$ with respect to

(1.4) which is a so-called *compact function* in $\mathcal{U}$ (see Sect. 2.1) satisfying the "dissipation" property that

$$\limsup_{x\to\partial\mathcal{U}} \mathcal{L}U(x) = \limsup_{x\to\partial\mathcal{U}} (a^{ij}(x)\partial^2_{ij}U(x) + V^i(x)\partial_i U(x)) \le -\gamma \tag{1.8}$$

for some constant $\gamma > 0$, called a *Lyapunov constant*. We note that when $\mathcal{U}$ is unbounded, the notion of $\partial\mathcal{U}$ and the limit $x \to \partial\mathcal{U}$ in the above should be understood under the topology of the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ which identifies $\mathbb{R}^n$ with the unit ball $\mathbb{B}^n$ and $\partial\mathbb{R}^n$ with the unit sphere $\mathbb{S}^{n-1}$, and in particular, identifies each $x_* \in \mathbb{S}^{n-1}$ with the infinity element $x_*^\infty \in \partial\mathbb{R}^n$ of the ray through x_* (see Sect. 2 for details). Consequently, if $\mathcal{U} = \mathbb{R}^n$, then $x \to \partial\mathbb{R}^n$ under this topology simply means $x \to \infty$ and (1.8) becomes

$$\limsup_{x\to\infty} \mathcal{L}U(x) = \limsup_{x\to\infty} (a^{ij}(x)\partial^2_{ij}U(x) + V^i(x)\partial_i U(x)) \le -\gamma. \tag{1.9}$$

In the case $\mathcal{U} = \mathbb{R}^n$, it was first shown by Has'minskiǐ [15,16] for locally Lipschitz-continuous drift field (V^i) and noise matrix (g^{ij}) that if there exists a non-negative function $U \in C^2(\mathbb{R}^n)$ with

$$\lim_{x\to\infty} U(x) = +\infty \tag{1.10}$$

and

$$\lim_{x\to\infty} \mathcal{L}U(x) = -\infty, \tag{1.11}$$

then the diffusion process generated from (1.2) admits an invariant measure, and consequently (1.3) admits a stationary measure because an invariant measure of the diffusion process is necessary a stationary measure of (1.3). Has'minskiǐ also remarked in [16] that same holds when the the condition (1.11) is replaced by (1.9). Recently, a vast amount of attentions have been paid to the existence of stationary measures of (1.3) for less regular coefficients under various "dissipation" conditions (see e.g., [1,4–13,22,26–28]). In particular, Bogachev-Röckner [8] showed the existence of a regular stationary measure of (1.3) with density function lying in the space $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ when the condition (A) holds and there exists a non-negative function $U \in C^2(\mathbb{R}^n)$ satisfying both (1.10) and (1.11). In [10], this result is shown to actually hold when U satisfies both (1.9) and (1.10). We remark that a non-negative function $U \in C^2(\mathbb{R}^n)$ is an unbounded Lyapunov function in $\mathbb{R}^n$ iff it satisfies both (1.9) and (1.10) (see Proposition 2.1).

In fact, if V is continuous on $\mathbb{R}^n$, then it is also shown in [8] that stationary measures of (1.3) in $\mathbb{R}^n$ still exist even when (a^{ij}) is degenerate in $\mathbb{R}^n$. We will leave more discussions on the degenerate case of (a^{ij}) to part III of the series [19].

One of our main results of this paper is as follows.

Theorem A *Assume that* (A) *holds in* $\mathcal{U}$ *and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$*. If there exists a Lyapunov function with respect to* (1.4) *in* $\mathcal{U}$*, then* (1.3) *admits a stationary measure in* $\mathcal{U}$ *which is regular with positive density lying in the space* $W^{1,p}_{loc}(\mathcal{U})$*. If, in addition, the Lyapunov function is unbounded, then stationary measures are unique in* $\mathcal{U}$*.*

A Lyapunov function with respect to (1.4) may be regarded as a stochastic counterpart of a Lyapunov function defined for a dissipative dynamical system. Recall that a smooth Lyapunov function U for an ODE system (1.1) is a compact function such that

$$\limsup_{x\to\partial\mathcal{U}} V(x)\cdot\nabla U(x) = \limsup_{x\to\partial\mathcal{U}} V^i(x)\partial_i U(x) \le -\gamma \tag{1.12}$$

for some constant $\gamma > 0$. On one hand, it is well-known in the theory of dissipative dynamical systems that the existence of such a smooth Lyapunov function for the deterministic ODE system (1.1) with locally Lipschitz-continuous V implies the existence of a (compact) global attractor of (1.1) in $\mathcal{U}$ (see the Appendix in part II of the series [18] contained in the same volume). On the other hand, in the case that both V and G are locally Lipschitz-continuous in $\mathcal{U}$, it can be shown using [16, Thm. 4.3] that if there exists an unbounded Lyapunov function with respect to (1.4), then the stationary measure μ of (1.3) (which must be invariant and unique in this case) attracts all orbits of the semi-flow defined on the space of probability measures on $\mathcal{U}$ which is generated by the diffusion process of (1.2). In this sense, one may conclude that a stationary measure of (1.3) obtained from Theorem A resembles a global attractor in the deterministic case, though for the existence of a stationary measure the corresponding deterministic counterpart need not be always dissipative (see example 4.12).

When the Lyapunov constant $\gamma = 0$ in (1.8), in particular $\mathcal{L}U \le 0$ near $\partial\mathcal{U}$, the compact function U is referred to as a *weak Lyapunov function*. Such a weak Lyapunov function is insufficient to yield a "dissipation" condition, just like the case of an ODE system when $\gamma = 0$ in (1.12). It turns out that a proper "dissipation" condition requires that U is of the *class of* $\mathcal{B}_*(A)$—a condition controlling the decay rates of $a^{ij}\partial_i U \partial_j U$ near $\partial\mathcal{U}$ (see Sect. 2.1 for details), which is purely stochastic and unable to be satisfied by an ODE system. Under this additional condition, we are able to obtain the following result.

Theorem B *Assume that* (A) *holds and* (a^{ij}) *is everywhere positive definite in* $\mathcal{U}$. *If there exists a weak Lyapunov function with respect to* (1.4) *in* $\mathcal{U}$ *which is of the class* $\mathcal{B}_*(A)$, *then* (1.3) *admits a stationary measure in* $\mathcal{U}$ *which is regular with positive density lying in the space* $W^{1,p}_{loc}(\mathcal{U})$.

By assuming weaker Lyapunov-like conditions, Theorems A, B allow a much broader class of applications. First of all, in many situations the commonly adopted or physical Lyapunov-like functions are often compact functions and associated with a finite (positive or zero) Lyapunov constant, which are not necessarily unbounded. As to be demonstrated in a separate work [20], such finiteness of Lyapunov constants are important in studying stochastic bifurcation problems in a general domain because it allows to define a function from the parameter space to Lyapunov constants. Secondly, still as to be seen in [20], when a family of positive definite diffusion matrices are considered, problems of concentration and limit behaviors of stationary measures crucially depend on the existence of Lyapunov functions associated with the family that can be merely of a finite Lyapunov constant. Thirdly but not lastly, Theorems A, B, when combined with the non-existence results in part II of the series [18] contained in the same volume, provide a very useful tool for one to study stochastic bifurcations from the existence to the non-existence of stationary measures with respect to parameters lying in either the drift term or the diffusion term. In such stochastic bifurcations problems, Theorem A is typically used at the non-critical parameter values and Theorem B should be useful at the critical parameter values.

Generality of the domain considered in the above results does allow a wide range of applications. For instance, in a population model describing the time evolution of n-species, the biologically meaningful domain is always taken to be $\mathcal{U} = \mathbb{R}^n_+ =: \{x = (x_i) \in \mathbb{R}^n : x_i > 0,\ i = 1, 2, \ldots, n\}$. Even for a system defined in the entire space $\mathbb{R}^n$, one can also apply the results to any domain $\mathcal{U} \subset \mathbb{R}^n$ which is either the entire space or a bounded open set, to obtain a global or a local measure, resembling a global or a local attractor of the deterministic system, respectively. Though stationary measures need not exist in general when $\mathcal{U}$ admits a usual boundary and (a^{ij}) becomes degenerate on the boundary of $\mathcal{U}$, Theorem A says that they exist if a Lyapunov function exists in $\mathcal{U}$, and moreover, stationary measures are also unique when the Lyapunov function is unbounded. Such uniqueness due to degeneracy of noise on the boundary is particularly interesting when applying to a bounded domain because if (a^{ij}) is uniformly elliptic in the domain, then stationary measures are always non-unique.

We remark that if the system (1.1) is defined on $\mathcal{U} \times M$, where $\mathcal{U} \subset \mathbb{R}^n$ is a connected open set and M is a smooth, compact manifold without boundary, then one can modify the definitions of Lyapunov and weak Lyapunov functions in Sect. 2 in an obvious way by replacing the domain $\mathcal{U} \subset \mathbb{R}^n$ with $\mathcal{U} \times M$. Then the proofs in later sections can be modified accordingly so that Theorems A, B still hold with respect to such a generalized domain.

The proof of our results uses a level set method which crucially relies on the integral identity we derived in [17] (see also Theorem 3.1 below). Such a method, by overcoming limitations of the usual Lyapunov function method, the traditional large deviation theory, and the classical PDE estimates, has the advantage of allowing more delicate measure estimates than those made using classical methods (see [21] for some discussions in this regard), and moreover, unlike existing methods, it works the same in any domain: bounded, unbounded, or the entire space $\mathbb{R}^n$. Indeed, the integral identity reveals fundamental natures of stationary Fokker–Planck equations and plays a similar role as the Pohozaev Identity does to semi-linear elliptic equations. As in [17] and the present series, it enables one to obtain useful measure estimates for a stationary measure in a sub-domain by making use of information of noise distributions on the boundary of the domain. Besides their usefulness in the study of existence and non-existence of stationary measures, these estimates will also play important roles in studying problems like the concentration and limit behaviors of stationary measures when diffusions tend to zero, as what we will explore in separate works (see e.g., [20]).

This paper is organized as follows. Section 2 is a preliminary section in which we introduce the notions of boundary $\partial\mathcal{U}$, compact functions, and Lyapunov-like functions, for a general domain $\mathcal{U} \subset \mathbb{R}^n$. We will also review an existence result of stationary solutions of (1.3) in a bounded domain from [8], a Harnack inequality from [24], and a regularity theorem from [6]. In Sect. 3, we recall the integral identity from [17], which is of fundamental importance to the level set method to be adopted in this paper and other parts of the series. Theorem A and Theorem B will be proved respectively in Sects. 4 and 5. The proof uses the level set method and measure estimates for stationary measures contained in [17] through a Lyapunov or a weak Lyapunov function. Some examples and discussions concerning the noise stabilization of an unstable equilibrium and the existence and uniqueness of stationary measures subject to boundary degeneracy of diffusion in a bounded domain are also given in Sect. 4. In the Appendix at the end, we give a characterization of Lyapunov functions in one dimension.

For measure estimates which we recall from [17], we will also provide proofs in some special cases. Besides doing so for the reader's convenience, our purpose is also to highlight the deep inside of these estimates by avoiding complicated technical details.

Through the rest of the paper, we let $\mathcal{U} \subset \mathbb{R}^n$ be a connected open set which can be bounded, unbounded, or the entire space $\mathbb{R}^n$. For simplicity, we will use the same symbol $|\cdot|$ to denote absolute value of a number, cardinality or Lebesgue measure of a set, and norm of a vector or a matrix.

2 Preliminary

2.1 Compact Functions

To unify both cases when $\mathcal{U}$ is bounded and unbounded, we consider the extended Euclidean space $\mathbb{E}^n = \mathbb{R}^n \cup \partial\mathbb{R}^n$ which is identified with the closed unit ball $\bar{\mathbb{B}}^n = \mathbb{B}^n \cup \partial\mathbb{B}^n$, through the homeomorphism $h : \mathbb{E}^n \to \bar{\mathbb{B}}^n$ to be defined as follows. Let $\partial\mathbb{R}^n = \{x_*^\infty : x_* \in \mathbb{S}^{n-1}\}$, where for each $x_* \in \mathbb{S}^{n-1}$, x_*^∞ denotes the infinity element of the ray through x_*. Define $h : \mathbb{E}^n \to \bar{\mathbb{B}}^n$:

$$h(x) = \begin{cases} \frac{x}{1+|x|}, & x \in \mathbb{R}^n; \\ x_*, & x = x_*^\infty \in \partial\mathbb{R}^n. \end{cases}$$

Then h clearly identifies $\mathbb{R}^n$ with $\mathbb{B}^n$ and $\partial\mathbb{R}^n$ with $\mathbb{S}^{n-1}$, and it becomes a homeomorphism when the topology of $\partial\mathbb{R}^n$ is defined as the one inherited from this identification.

Definition 2.1 We call $\Gamma =: \partial\mathcal{U} \subset \mathbb{E}^n$ the *boundary* of $\mathcal{U}$ if $h(\Gamma)$ is the boundary of $h(\mathcal{U})$ in $\bar{\mathbb{B}}^n$.

For instance, when $\mathcal{U} = \mathbb{R}^n$, $\mathcal{U}$ has only one boundary component $\partial\mathbb{R}^n$, and when $\mathcal{U} = \mathbb{R}^n_+$—the first octant of $\mathbb{R}^n$ for $n \geq 2$, $\mathcal{U}$ also has only one boundary component Γ which is the union of all non-negative, coordinate hyperplanes and the portion of $\partial\mathbb{R}^n$ lying in between these hyperplanes.

Definition 2.2 Let $U \in C(\mathcal{U})$ be a non-negative function and denote $\rho_M = \sup_{x\in\mathcal{U}} U(x)$, the *essential upper bound of* U. U is said to be a *compact function in* $\mathcal{U}$ if

(i) $U(x) < \rho_M$, $x \in \mathcal{U}$; and
(ii) $\lim_{x\to\partial\mathcal{U}} U(x) = \rho_M$.

According to the topology we described earlier, $x \to \partial\mathcal{U}$ in the above means that the Hausdorff semi-distance $\mathrm{dist}(h(x), h(\partial\mathcal{U})) \to 0$. Therefore, if Γ^* is a bounded boundary component of $\partial\mathcal{U}$, $x \to \Gamma^*$ under this topology is equivalent to $\mathrm{dist}(x, \Gamma^*) \to 0$. Moreover, if $\mathcal{U} = \mathbb{R}^n$, then $x \to \partial\mathcal{U}$ under this topology simply means $x \to \infty$ in the usual sense.

From the definition, we immediately have the following result.

Proposition 2.1 *An unbounded, non-negative function $U \in C(\mathcal{U})$ is a compact function in $\mathcal{U}$ iff*

$$\lim_{x\to\partial\mathcal{U}} U(x) = +\infty.$$

Consequently, an unbounded, non-negative function $U \in C(\mathbb{R}^n)$ is a compact function in $\mathbb{R}^n$ iff it satisfies (1.10).

For a non-negative function $U \in C(\mathcal{U})$ and each $\rho \in [0, \rho_M)$, where ρ_M is the essential upper bound of U, we denote $\Omega_\rho = \{x \in \mathcal{U} : U(x) < \rho\}$ as the *ρ-sublevel set of U*.

Proposition 2.2 *A continuous, non-negative function U is a compact function in $\mathcal{U}$ iff the following holds:*

(a) *$\Omega_\rho \subset\subset \mathcal{U}$ for any $0 \leq \rho < \rho_M$;*
(b) *$h(\partial\Omega_\rho) \to h(\partial\mathcal{U})$ in Hausdorff metric as $\rho \to \rho_M$.*

Proof Let (i), (ii) be as in Definition 2.2. It is clear that (ii) implies (a). We note that (b) is equivalent to the following:

- ($*$) For any neighborhood $\mathcal{N}$ of $\partial\mathcal{U}$, there exists a $\rho_{\mathcal{N}} \in (0, \rho_M)$ such that $\partial\Omega_\rho \subset \mathcal{N}$ for all $\rho \in [\rho_{\mathcal{N}}, \rho_M)$;
- ($**$) For any $x \in \partial\mathcal{U}$ and any sequence $\rho_k \to \rho_M$, there is a sequence of points $x_k \in \partial\Omega_{\rho_k}$ such that $x_k \to x$.

Now, (i) implies ($*$) and (ii) implies ($**$) by continuity of U, i.e., (i) and (ii) together imply (b).

Conversely, it is easy to see that (a) and (b) together imply (i), and (a) implies (ii). □

Definition 2.3 Let $A = (a^{ij})$ be an everywhere positive semi-definite, $n \times n$ matrix-valued function on $\mathcal{U}$. A compact function $U \in C^1(\mathcal{U})$ with essential upper bound ρ_M is said to be *of the class* $\mathcal{B}_*(A)$ if there exist a $\rho_m \in (0, \rho_M)$ and a positive function H on $[\rho_m, \rho_M)$ such that

$$H(\rho) \le a^{ij}(x)\partial_i U(x)\partial_j U(x), \qquad x \in U^{-1}(\rho), \quad \rho \in [\rho_m, \rho_M), \text{ and} \tag{2.1}$$

$$\int_{\rho_m}^{\rho_M} \frac{1}{H(\rho)} \mathrm{d}\rho < +\infty. \tag{2.2}$$

2.2 Lyapunov-Like Functions

Below, we introduce two types of Lyapunov-like functions with respect to the stationary Fokker–Planck equation (1.4) or the adjoint Fokker–Planck operator $\mathcal{L}$. Each type will play an important role to the existence of stationary measures.

Definition 2.4 Let U be a C^2 compact function in $\mathcal{U}$ with essential upper bound ρ_M.

1. U is called a *Lyapunov function* in $\mathcal{U}$ with respect to (1.4) or $\mathcal{L}$, if there is a $\rho_m \in (0, \rho_M)$, called *essential lower bound of* U, and a constant $\gamma > 0$, called *Lyapunov constant* of U, such that

$$\mathcal{L}U(x) \le -\gamma, \qquad x \in \tilde{\mathcal{U}} =: \mathcal{U} \setminus \bar{\Omega}_{\rho_m}, \tag{2.3}$$

   where $\tilde{\mathcal{U}}$ is called the *essential domain of* U.
2. U is called a *weak Lyapunov function* in $\mathcal{U}$ with respect to (1.4) or $\mathcal{L}$, if it satisfies (2.3) in an essential domain $\tilde{\mathcal{U}} = \mathcal{U} \setminus \bar{\Omega}_{\rho_m}$ with $\gamma = 0$. We still refer to such ρ_m as an *essential lower bound of* U.

2.3 Stationary Solutions in a Bounded Domain

Let Ω be a bounded domain in $\mathbb{R}^n$ and consider the stationary Fokker–Planck equation (1.4) in Ω. Assume that $A = (a^{ij})$ is *uniformly elliptic* in Ω, i.e., there are positive constants λ, Λ such that

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \xi \in \mathbb{R}^n, \ x \in \Omega. \tag{2.4}$$

As shown in [8, Thm. 1.2], by imposing Dirichlet boundary condition $u|_{\partial\Omega} = 1$, positive weak solutions of (1.4) in Ω can be obtained from classical works of Trudinger on the existence of weak solutions [24, Thm. 3.2] and weak maximal principle [25, Thm. 7] of linear elliptic equations with measurable coefficients. More precisely, the following holds.

Lemma 2.1 [8] *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Assume that* $A = (a^{ij})$ *is uniformly elliptic in* Ω *and* $a^{ij} \in W^{1,p}(\Omega)$, $V^i \in L^p(\Omega)$, $i, j = 1, \ldots, n$, *for some* $p > n$. *Then* (1.4) *admits a positive weak solution* $u \in W^{1,p}(\Omega)$, *with* $\Omega \equiv \mathcal{U}$.

Remark 2.1 We note that if Ω is a C^2 domain and $\{a^{ij}\} \subset C^2(\bar{\Omega})$, $\{V^i\} \subset C(\bar{\Omega})$ in the operator L, then the above lemma can be alternatively proved by imposing a homogeneous Robin boundary condition and by using a result of Amann [2] concerning the principle eigenvalues of general elliptic operators. In fact, consider the equation

$$Lu = \partial_i(a^{ij}(x)\partial_j u + b^i(x)u) = 0, \qquad x \in \Omega,$$

with the boundary condition

$$\mathcal{B}u =: \partial_j(a^{ij}u)\nu_i - (V^i\nu_i)u = 0, \qquad x \in \partial\Omega, \tag{2.5}$$

where $b^i = \partial_j a^{ij} - V^i$, $i = 1, \ldots, n$, and (ν_i) denotes the field of unit outward normal vectors on $\partial\Omega$. Since, for each $x \in \partial\Omega$, $(a^{ij}(x)\nu_i(x))$ is transversal to the tangent vector of $\partial\Omega$ at x, the boundary condition (2.5) is of Robin type. Let

$$T : \mathrm{Dom}(T) \hookrightarrow L^p(\Omega) \longrightarrow L^p(\Omega) : \quad Tu = Lu,$$

be the linear operator with domain

$$\mathrm{Dom}(T) = \left\{u \in W^{2,p}(\Omega) : \mathcal{B}u|_{\partial\Omega} = 0\right\}.$$

By [2, Thm. 12.1], T has a principle eigenvalue λ_0, i.e, λ_0 is a simple eigenvalue with positive eigenfunction in $\bar{\Omega}$. Let $u_0 \in \mathrm{Dom}(T)$ be such an eigenfunction. Then

$$\lambda_0 \int_\Omega u_0(x)\,\mathrm{d}x = \int_\Omega Lu_0(x)\,\mathrm{d}x = \int_{\partial\Omega} \mathcal{B}u_0\,\mathrm{d}s = 0.$$

Hence $\lambda_0 = 0$, i.e., u_0 is a positive solution of the equation $Lu = 0$. Thus, (1.4) with Ω in place of $\mathcal{U}$ admits a solution $u =: u_0(\int_\Omega u_0 \mathrm{d}x)^{-1} \in W^{2,p}(\Omega)$.

2.4 Harnack Inequality

Consider the divergence operators of the form

$$Lu := \partial_i\left(a^{ij}(x)\partial_j u + b^i(x)u\right) + c^i(x)\partial_i u + d(x)u,$$

where $a^{ij}, b^i, c^i, d, i, j = 1, \ldots, n$, are measurable functions on a bounded domain $\Omega \subset \mathbb{R}^n$, $A = (a^{ij})$ is almost everywhere positive definite in Ω and satisfies

$$\lambda(x)|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda(x)|\xi|^2, \qquad \xi \in \mathbb{R}^n,\ x \in \Omega,$$

for some positive, integrable functions λ^{-1}, Λ on Ω.

Let

$$g(x) := b_{ij}(x)(b^i(x)b^j(x) + c^i(x)c^j(x)) + |d(x)|,$$

where $(b_{ij}) = (a^{ij})^{-1}$ and assume that g is integrable on Ω. Denote $\mathcal{H}^1(A, \Omega)$ as the completion of $C^\infty(\Omega)$ under the inner product

$$\langle u, v\rangle = \int_\Omega a^{ij}u_{x_i}v_{x_j}\mathrm{d}x, \qquad u, v \in C^\infty(\Omega).$$

We note by [24, Prop. 1.3] that if $\lambda^{-1}, \frac{\Lambda}{\lambda} \in L^\infty(\Omega)$, then $\mathcal{H}^1(A,\Omega) = W^{1,2}(A,\Omega) = H^1(\Omega)$, where $W^{1,2}(A,\Omega)$ denotes the class of strongly differentiable functions u in Ω with $\langle u, u\rangle < \infty$.

Theorem 2.1 (Harnack Inequality, Trudinger [24, Cor. 5.3]) *Let $u \in \mathcal{H}^1(A,\Omega)$ be a non-negative solution of $Lu = 0$ in Ω. Assume that*

$$\lambda^{-2} \in L^p(\Omega), \quad \Lambda, g \in L^q(\Omega), \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{n}.$$

Then for any $\Omega' \subset\subset \Omega$,

$$\sup_{x\in\Omega'} u(x) \le C \inf_{x\in\Omega'} u(x),$$

where $C > 0$ is a constant depending only on n, p, q, $|\lambda^{-1}|_{L^p(\Omega)}$, $|\Lambda|_{L^q(\Omega)}$, $|g|_{L^q(\Omega)}$ and $\mathrm{dist}(\Omega', \partial\Omega)$.

2.5 Regularity of Stationary Measures

The following regularity result on stationary measures of Fokker–Planck equations is proved in [6].

Theorem 2.2 (Bogachev–Krylov–Röckner [6]) *Assume that* (A) *holds and* (a^{ij}) *is everywhere positive definite in $\mathcal{U}$. Then any stationary measure μ of* (1.3) *admits a positive density function $u \in W^{1,p}_{loc}(\mathcal{U})$, i.e.,* $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$.

3 Integral Identity and Derivative Formula

In this section, we recall from [17] a fundamental integral identity to be used in the level set method for conducting measure estimates of stationary measures of (1.3). This identity plays a crucial role in capturing information of a weak stationary solution in each sublevel set of a Lyapunov-like function from its boundary.

Theorem 3.1 (Integral Identity, [17, Thm. 2.1]) *Assume that* (A) *holds and let $u \in W^{1,p}_{loc}(\mathcal{U})$ be a weak solution of* (1.4)*. Then for any generalized Lipschitz domain $\Omega' \subset\subset \mathcal{U}$ and any function $F \in C^2(\bar{\Omega}')$ with $F|_{\partial\Omega'}$ =constant,*

$$\int_{\Omega'} (\mathcal{L}F)u \,\mathrm{d}x = \int_{\partial\Omega'} (a^{ij}\partial_i F \nu_j)u \,\mathrm{d}s, \tag{3.1}$$

where for a.e. $x \in \partial\Omega'$, $(\nu_j(x))$ denotes the unit outward normal vector of $\partial\Omega'$ at x.

Proof For the reader's convenience, we sketch the proof.

Let $F|_{\partial\Omega'} = c$ and Ω_* be a smooth domain such that $\Omega' \subset\subset \Omega_* \subset\subset \mathcal{U}$. Consider the function

$$\tilde{F}(x) = \begin{cases} F(x) - c, & x \in \Omega', \\ 0, & x \in \mathcal{U} \setminus \Omega' \end{cases}$$

and its regularizations $\tilde{F}_h$, $0 < h < 1$. Then $\tilde{F}_h \in C_0^\infty(\mathcal{U})$, $\mathrm{supp}(\tilde{F}_h) \subset \bar{\Omega}_*$ as $0 < h \ll 1$, and $\tilde{F}_h \to \tilde{F}$ in $W^{1,q}(\Omega_*)$, as $h \to 0$, for any $0 < q < \infty$. Since u is a weak solution of

(1.4) in $\mathcal{U}$,

$$\int_{\mathcal{U}} (a^{ij}\partial^2_{ij}\tilde{F}_h + V^i\partial_i\tilde{F}_h)u \,\mathrm{d}x = 0, \qquad \text{as } 0 < h \ll 1.$$

Since (A) holds and $u \in W^{1,p}_{loc}(\mathcal{U})$, one can show that, as $h \to 0$,

$$\int_{\mathcal{U}} (a^{ij}\partial^2_{ij}\tilde{F}_h)u \,\mathrm{d}x \to \int_{\Omega'} ua^{ij}\partial^2_{ij}F \,\mathrm{d}x - \int_{\partial\Omega'} ua^{ij}\partial_i F\nu_j \,\mathrm{d}s,$$
$$\int_{\mathcal{U}} (V^i\partial_i\tilde{F}_h)u \,\mathrm{d}x \to \int_{\Omega'} uV^i\partial_i F \,\mathrm{d}x.$$

Hence (3.1) holds. □

Remark 3.1 1. We note that the theorem does not require (a^{ij}) to be even positive semi-definite. It also holds for less regular (a^{ij}), (V^i), and u, as long as $a^{ij}u \in W^{1,\alpha}_{loc}(\mathcal{U})$ and $V^iu \in L^{\alpha}_{loc}(\mathcal{U})$, $\forall i, j, = 1, 2, \ldots, n$, for some $\alpha > 1$.
2. We recall from [17] that a bounded open set Ω in $\mathbb{R}^n$ is called a *generalized Lipschitz domain* if (i) Ω is a disjoint union of finitely many Lipschitz sub-domains; and (ii) intersections of boundaries among these Lipschitz sub-domains only occur at finitely many points. A generalized Lipschitz domain need not be a Lipschitz domain in the usual sense because the boundary need not be orientable. As a simple example, consider the function $U : \mathbb{R}^2 \to \mathbb{R}$:

$$U(u,z) = \begin{cases} -u^2 + z^2 + 1, & |u| \le 1, \\ (u-1)^2 + z^2, & u > 1, \\ (u+1)^2 + z^2, & u < -1. \end{cases} \tag{3.2}$$

Let $\Omega \equiv \Omega_1$ be the sublevel set of U corresponding to $\rho = 1$. Then Ω has only one boundary component $U^{-1}(1)$ which is the union of two closed Lipschitz curves intersecting at 0 (i.e., the boundary self-intersects at 0). We note that $\partial\Omega = U^{-1}(1)$ encloses two disjoint connected open sets, and hence Ω itself is not a single Lipschitz domain but rather a generalized Lipschitz domain.

4 Stationary Measures Under Lyapunov Condition

We will show Theorem A in this section. In order to apply general Lyapunov functions as defined in Definition 2.4, our approach differs from existing works on the existence of stationary measures of (1.3). In particular, the integral identity (3.1) contained in Theorem 3.1 will play an important role in our study.

4.1 Measure Estimates via Level Set Method

We first recall from [17] the following measure estimates with respect to a regular stationary measure of (1.3) in the essential domain of a Lyapunov function. As to be seen in a separate work [20], such estimates will also be useful in showing the relative sequential compactness of a family of stationary measures associated with a so-called null family of diffusion matrices, as well as in characterizing the concentration of these measures as noises tend to zero.

Lemma 4.1 [17, Thm. A (a)] *Assume that* (A) *holds and let* U *be a Lyapunov function in* $\mathcal{U}$ *with Lyapunov constant* γ *and essential lower bound* ρ_m *and upper bound* ρ_M*. Then for any weak solution* $u \in W^{1,p}_{loc}(\mathcal{U})$ *of* (1.4) *and any* $\rho_0 \in (\rho_m, \rho_M)$*,*

$$\mu(\mathcal{U} \setminus \Omega_{\rho_0}) \le \gamma^{-1} C_{\rho_m,\rho_0} |A|_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})} |\nabla U|^2_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})} \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}), \tag{4.1}$$

where μ *is the measure with density function* u*, i.e.,* $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$*, the constant* $C_{\rho_m,\rho_0} > 0$ *depends only on* ρ_m, ρ_0*, and* Ω_ρ *denotes the* ρ*-sublevel set of* U *for each* $\rho \in [\rho_m, \rho_M)$*.*

Proof The lemma follows immediately from [17, Thm. A (a)] by taking

$$H(\rho) = \sup_{x \in U^{-1}(\rho)} (a^{ij}(x)\partial_i U(x)\partial_j U(x))$$

there. The original proof of [17, Thm. A (a)] uses the approximation of U by a sequence of Morse functions because its sublevel sets Ω_ρ, $\rho \in (\rho_m, \rho_M)$, need not be generalized Lipschitz domains, though by Proposition 2.2 we always have $\Omega_\rho \subset\subset \mathcal{U}$, $\rho \in (\rho_m, \rho_M)$. In order to highlight the main ideas of using the integral identity (3.1), we give the proof below for the special case that $\nabla U \neq 0$ everywhere in the essential domain $\mathcal{U} \setminus \bar{\Omega}_{\rho_m}$ of U. For this special case, we note that each $\partial\Omega_\rho$ agrees with the level set $U^{-1}(\rho)$ which is also a C^2 hypersurface with $\frac{\nabla U(x)}{|\nabla U(x)|}$, $x \in \partial\Omega_\rho$, being the unit outward normal vectors.

For given $\rho_0 \in (\rho_m, \rho_M)$, consider a fixed monotonically increasing function $\phi \in C^2(\mathbb{R}_+)$ satisfying

$$\phi(t) = \begin{cases} 0, & \text{if } t \in [0, \rho_m]; \\ t, & \text{if } t \in [\rho_0, +\infty). \end{cases}$$

Then $\phi'(t) = 0$, $t \in [0, \rho_m]$ and $\phi''(t) = 0$, $t \in [0, \rho_m] \cup [\rho_0, +\infty)$.

For any $\rho \in (\rho_0, \rho_M)$, an application of Theorem 3.1 with $F = \phi \circ U$ and $\Omega' = \Omega_\rho$ yields that

$$\int_{\Omega_\rho} (a^{ij}\partial^2_{ij}\phi(U) + V^i \partial_i \phi(U))u \,\mathrm{d}x = \int_{\partial\Omega_\rho} u a^{ij} \partial_i \phi(U) \nu_j \,\mathrm{d}s,$$

i.e.,

$$\begin{aligned} &\int_{\Omega_\rho} \phi'(U)(\mathcal{L}U)u \,\mathrm{d}x + \int_{\Omega_\rho} \phi''(U)(a^{ij}\partial_i U \partial_j U)u \,\mathrm{d}x \\ &\quad = \int_{\partial\Omega_\rho} \phi'(U) u a^{ij} \partial_i U \nu_j \,\mathrm{d}s = \int_{\partial\Omega_\rho} u a^{ij} \partial_i U \nu_j \,\mathrm{d}s, \end{aligned} \tag{4.2}$$

where $(\nu_j) = \nu = \nu(x)$, being the unit outward normal vector at $x \in \partial\Omega_\rho$, equals $\frac{\nabla U(x)}{|\nabla U(x)|}$. Since

$$a^{ij}(x)\partial_i U(x)\nu_j(x) \ge 0, \qquad x \in \partial\Omega_\rho,$$

(4.2) yields that

$$\int_{\Omega_\rho} \phi'(U)(\mathcal{L}U)u \,\mathrm{d}x + \int_{\Omega_\rho} \phi''(U)(a^{ij}\partial_i U \partial_j U)u \,\mathrm{d}x \ge 0.$$

By the definition of ϕ, it follows that

$$\int_{\Omega_\rho \setminus \Omega_{\rho m}} \phi'(U)(\mathcal{L}U)u \,\mathrm{d}x \ge -\int_{\Omega_\rho \setminus \Omega_{\rho m}} \phi''(U)(a^{ij}\partial_i U \partial_j U)u \,\mathrm{d}x.$$

Letting $\rho \to \rho_M$ in the above, we obtain

$$\int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi'(U)(\mathcal{L}U)u\, dx \geq -\int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi''(U)(a^{ij}\partial_i U\partial_j U)u\, dx. \tag{4.3}$$

Using the facts $\phi'(t) \geq 0$, $\phi'(t) = 1$ as $t \geq \rho_0$, and that U is a Lyapunov function, we have

$$\int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi'(U)(\mathcal{L}U)u\, dx \leq -\gamma \int_{\mathcal{U}\setminus\Omega_{\rho m}} \phi'(U)u\, dx \leq -\gamma\mu(\mathcal{U}\setminus\Omega_{\rho_0}). \tag{4.4}$$

Denote $C_{\rho_m,\rho_0} = \max_{\rho_m\leq\rho\leq\rho_0} |\phi''(\rho)|$. We also have

$$\int_{\mathcal{U}\setminus\Omega_{\rho m}} |\phi''(U)|(a^{ij}\partial_i U\partial_j U)u\, dx = \int_{\Omega_{\rho_0}\setminus\Omega_{\rho m}} |\phi''(U)|(a^{ij}\partial_i U\partial_j U)u\, dx$$
$$\leq C_{\rho_m,\rho_0}|A|_{C(\Omega_{\rho_0}\setminus\Omega_{\rho m})}|\nabla U|^2_{C(\Omega_{\rho_0}\setminus\Omega_{\rho m})}\mu(\Omega_{\rho_0}\setminus\Omega_{\rho m}). \tag{4.5}$$

The lemma now follows from (4.3) to (4.5). □

4.2 Proof of Theorem A

To prove the theorem, we let U be a Lyapunov function in $\mathcal{U}$ with Lyapunov constant γ, essential lower bound ρ_m and upper bound ρ_M. We also denote by Ω_ρ the ρ-sublevel set of U.

Let $\rho^k \in (\rho_m, \rho_M)$, $k = 1, 2, \ldots$, be an increasing sequence such that $\rho^k \to \rho_M$ as $k \to \infty$. We denote $\Omega_k = \Omega_{\rho^k}$ and $\Gamma_k = \partial\Omega_{\rho^k}$, $k = 1, 2, \ldots$. Since $\Gamma_k \to \partial\mathcal{U}$ by Proposition 2.2, for any domain $\Omega_* \subset\subset \mathcal{U}$ we have that $\Omega_* \subset \Omega_k$ as $k \gg 1$.

Since (a^{ij}) is continuous and everywhere positive definite in $\mathcal{U}$ and $\Omega_k \subset\subset \mathcal{U}$ by Proposition 2.2 for each k, (a^{ij}) is uniformly elliptic in Ω_k. Hence by Lemma 2.1, for each k, the stationary Fokker–Planck equation (1.4) with Ω_k in place of $\mathcal{U}$ admits a positive solution $u_k \in W^{1,p}(\Omega_k) \hookrightarrow C(\bar{\Omega}_k)$. For each k, we extend u_k to $\mathcal{U}$ by setting $u_k \equiv 0$ in $\mathcal{U}\setminus\Omega_k$, and still denote it by u_k.

For any fixed domain $\Omega_* \subset\subset \mathcal{U}$, we let λ, Λ, and η be positive constants such that

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j, \qquad \xi \in \mathbb{R}^n,\ x \in \Omega_*,$$
$$|a^{ij}|_{W^{1,p}(\Omega_*)} \leq \Lambda,\ |V^i|_{L^p(\Omega_*)} \leq \eta, \quad i, j = 1, \ldots, n.$$

Then for any two domains B_*, B^* with $B_* \subset\subset B^* \subset\subset \Omega_*$, an application of Harnack inequality (Theorem 2.1) to $\Omega = \Omega_*$, $\Omega' = B^*$ with $2q = p$, $b^i = \partial_j a^{ij} - V^i$, $c_i = d \equiv 0$, $i = 1, \ldots, n$, yields that

$$\sup_{B^*} u_k \leq C_1 \inf_{B^*} u_k,$$

for some constant $C_1 > 0$ depending only on n, p, λ, Λ, η, and $\mathrm{dist}(B^*, \partial\Omega_*)$, in particular, not on k. For $k \gg 1$, since

$$|B^*| \inf_{B^*} u_k \leq \int_{B^*} u_k\, dx \leq \int_{\Omega_k} u_k\, dx \equiv 1,$$

where $|B^*|$ stands for the volume of B^*, we have

$$\inf_{B^*} u_k \leq |B^*|^{-1}.$$

This implies that

$$\sup_{B^*} u_k \le C_1|B^*|^{-1}, \qquad k \gg 1. \tag{4.6}$$

For each $k \gg 1$, we also have by Hölder estimate that $u_k \in C^\alpha(B_*)$ and

$$|u_k|_{C^\alpha(B_*)} \le C_2|u_k|_{L^2(B^*)} \tag{4.7}$$

for some constants $0 < \alpha < 1, C_2 > 0$ which depend only on $n, p, \lambda, \Lambda, \eta$, and $\text{dist}(B_*, \partial B^*)$. It follows from (4.6), (4.7) that

$$|u_k|_{C^\alpha(B_*)} \le C_1C_2|B^*|^{-\frac{1}{2}}, \qquad k \gg 1.$$

Since B_*, B^*, Ω_* are arbitrary, the sequence $\{u_k\}$ is equi-continuous on any compact subset of $\mathcal{U}$. By taking a subsequence if necessary, we assume without loss of generality that $\{u_k\}$ converges, as $k \to \infty$, to some non-negative function $\tilde{u} \in C(\mathcal{U})$ under the compact-open topology of $C(\mathcal{U})$. For any given $f \in C_0^\infty(\mathcal{U})$, we let $k \gg 1$ such that $\text{supp}(f) \subset \Omega_k$, i.e., $f \in C_0^\infty(\Omega_k)$. Then

$$\int_{\mathcal{U}} \mathcal{L}f(x)u_k(x)\,\mathrm{d}x = 0, \qquad \text{as } k \gg 1.$$

Since $u_k \to \tilde{u}$ uniformly on $\text{supp}(f)$ as $k \to \infty$, we have

$$\int_{\mathcal{U}} \mathcal{L}f(x)\tilde{u}(x)\,\mathrm{d}x = 0.$$

Moreover, since $\int_{\mathcal{U}} u_k \mathrm{d}x = \int_{\Omega_k} u_k\,\mathrm{d}x \equiv 1$ and $u_k \ge 0$, it follows from Fatou's Lemma that

$$\int_{\mathcal{U}} \tilde{u}\mathrm{d}x = \int_{\mathcal{U}} \lim_{k\to\infty} u_k\mathrm{d}x \le \underline{\lim}_{k\to\infty} \int_{\mathcal{U}} u_k\mathrm{d}x = 1.$$

Thus, if $\tilde{u}$ is not identically zero in $\mathcal{U}$, then μ with $\mathrm{d}\mu(x) = \frac{\tilde{u}(x)}{\int_{\mathcal{U}}\tilde{u}\mathrm{d}x}\mathrm{d}x$ is a stationary measure of (1.3), and we have by Theorem 2.2 that μ admits a positive density function $u \in W^{1,p}_{loc}(\mathcal{U})$.

We now show that $\tilde{u} \not\equiv 0$ in $\mathcal{U}$. For $k \ge 1$, since the set Ω_k contains $\bar{\Omega}_{\rho_m}$, U is a Lyapunov function in Ω_k with respect to $\mathcal{L}$ having the essential domain $\Omega_k \setminus \bar{\Omega}_{\rho_m}$, Lyapunov constant γ, and essential lower bound ρ_m and upper bound ρ^k. For fixed $\rho_0 \in (\rho_m, \rho_1) \subset (\rho_m, \rho^k)$, an application of Lemma 4.1 to μ_k with $\mathrm{d}\mu_k(x) = u_k\mathrm{d}x$ on Ω_k for every $k \ge 1$ yields that

$$\mu_k(\Omega_k \setminus \Omega_{\rho_0}) \le \gamma^{-1}C_*\mu_k(\Omega_{\rho_0} \setminus \Omega_{\rho_m}) \le \gamma^{-1}C_*|\Omega_{\rho_0} \setminus \Omega_{\rho_m}||u_k|_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})},$$

where $C_* = C_{\rho_m,\rho_0}|A|_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})}|\nabla U|^2_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})}$ for a positive constant C_{ρ_m,ρ_0} depending only on ρ_m, ρ_0 as in Lemma 4.1. It follows that

$$\begin{aligned} 1 \equiv \mu_k(\Omega_k) &= \mu_k(\Omega_k \setminus \Omega_{\rho_0}) + \mu_k(\Omega_{\rho_0}) \\ &\le \gamma^{-1}C_*|\Omega_{\rho_0} \setminus \Omega_{\rho_m}||u_k|_{C(\Omega_{\rho_0}\setminus\Omega_{\rho_m})} + |\Omega_{\rho_0}||u_k|_{C(\Omega_{\rho_0})}. \end{aligned} \tag{4.8}$$

If $\tilde{u} \equiv 0$ in $\mathcal{U}$, then $u_k \to 0$ as $k \to \infty$ uniformly on Ω_{ρ_0}, leading to a contradiction to (4.8).

The uniqueness follows from [13, Example 5.1], where it is shown that if there exists an unbounded function $U \in C^2(\mathcal{U})$ satisfying: (i) each set $\Omega_\rho \cup U^{-1}(\rho)$, $\rho > 0$, is compact; (ii) for a constant $\alpha > 0$ and a compact set K,

$$\mathcal{L}U(x) \le \alpha U(x), \qquad a.e.\ x \in \mathcal{U} \setminus K, \tag{4.9}$$

then (1.4) admits at most one stationary measure in $\mathcal{U}$. Thus, by taking U as the Lyapunov function when it is unbounded and also taking K as $\bar{\Omega}_{\rho_m}$, we obtain the uniqueness result. □

Remark 4.1 When dealing with a stochastic system over a bounded domain, Theorem A is particularly useful in showing the existence of stationary measures when the diffusion matrix becomes degenerate on the boundary, for which Lemma 2.1 fails to apply. For example, consider

$$\mathrm{d}x = bx\mathrm{d}t + \sqrt{2}(1 - x^2)\mathrm{d}W, \qquad x \in \mathcal{U} = (-1, 1), \tag{4.10}$$

where $b < 0$ is a constant. We note that the noise coefficient vanishes at the boundary of $\mathcal{U}$.

Consider $U(x) = -\log(1 - x^2)$, $x \in (-1, 1)$. It is clear that $U(x) \to +\infty$, as $|x| \to 1$, and

$$\mathcal{L}U(x) = 2 + 2x^2 + \frac{2bx^2}{1 - x^2}, \qquad x \in (-1, 1).$$

Hence, $\mathcal{L}U(x) \to -\infty$, as $|x| \to 1$, i.e., U is in fact an unbounded Lyapunov function with respect to the stationary Fokker–Planck equation associated with (4.10). It follows from Theorem A that the Fokker–Planck equation associated with (4.10) has a unique stationary measure in $\mathcal{U}$.

4.3 The Case of $\mathbb{R}^n$

Applying Theorem A to $\mathcal{U} = \mathbb{R}^n$, we obtain the following result which generalizes the one contained in [10].

Corollary 4.1 *Assume that* (A) *holds in* $\mathbb{R}^n$ *and* (a^{ij}) *is everywhere positive definite in* $\mathbb{R}^n$. *Then the following holds.*

(a) *If there is a compact function* $U \in C^2(\mathbb{R}^n)$ *satisfying* (1.9)*, then the Fokker–Planck equation* (1.3) *has a regular stationary measure in* $\mathbb{R}^n$ *with positive density function lying in the space* $W^{1,p}_{loc}(\mathbb{R}^n)$.
(b) *If there is a function* U *satisfying both* (1.9) *and* (1.10)*, then a stationary measure as in* (a) *exists and is also unique.*

Proof Part (a) follows immediately from Theorem A.

Part (b) is precisely the main result stated in [10]. By (1.10), we let c be a constant such that $U + c \geq 0$ on $\mathbb{R}^n$. Then by Proposition 2.1, $U + c$ becomes a compact function on $\mathbb{R}^n$. Since $U + c$ clearly satisfies (1.9), (b) follows from (a). □

Remark 4.2 (1) In applying Theorem A or Corollary 4.1 to a particular stochastic system, one often need to construct a Lyapunov function. Just like in the case of deterministic systems, such a Lyapunov function is usually constructed (for the sake of using undetermined coefficients technique) as a polynomial which often ends up with a finite Lyapunov constant. To give an example, consider

$$\mathrm{d}x = -\frac{bx}{1 + x^2}\mathrm{d}t + \sqrt{2}\,\mathrm{d}W, \qquad x \in \mathbb{R}, \tag{4.11}$$

where $b > 1$ is a constant.

We find that the functions $I_{\pm}(x)$ defined in the Appendix (Proposition 6.1) are constant multiples of $(1 + x^2)^{-\frac{b}{2}}$ in the present situation. It is clear that (6.2) is satisfied and hence by Proposition 6.1 there exists an unbounded Lyapunov function U with respect to the stationary Fokker–Planck equation associated with (4.11). Therefore, by Theorem A or Corollary 4.1, the Fokker–Planck equation associated with (4.11) admits a unique regular stationary measure.

Now let $b \in (1, 3)$. If one would prefer to have a polynomial Lyapunov function U, then the Lyapunov constant is always finite. To see this, we note that for U being a polynomial Lyapunov function, it has to be of even degree with a positive leading coefficient because it need to satisfy (1.10). Therefore, we may assume without loss of generality that $U = x^{2p} + P(x)$, $x \in \mathbb{R}$, where $p \geq 1$ is a natural number and P is a polynomial of degree $k < 2p$. A direct calculation yields that

$$\mathcal{L}U = 2px^{2p-2}\left(2p - 1 - \frac{bx^2}{1+x^2}\right) + O\left(|x|^{k-2}\right), \qquad |x| \gg 1.$$

As $x \to \infty$, the coefficient of x^{2p-2} tends to $2p(2p-1-b)$, which must be non-positive for U being a Lyapunov function. Since $b \in (1, 3)$, it cannot be zero, and it is negative only when $p = 1$, in which case we see that $\gamma =: b - 1$ is a Lyapunov constant.

(2) It is already known that a sufficiently large noise can stabilize an unstable equilibrium of a deterministic system. By choosing an appropriate Lyapunov function, one can in fact quantify the optimal lower bound of amplitudes of diffusions needed for the stabilization process. For example, consider the stochastic differential equation

$$\mathrm{d}x = bx\mathrm{d}t + \sqrt{2\sigma(x^2+1)}\,\mathrm{d}W, \qquad x \in \mathbb{R}, \tag{4.12}$$

where $b > 0$ is a fixed constant. We note that the unperturbed ODE has a unique equilibrium $\{0\}$ which is unstable and repelling.

When $\sigma > b$, we claim that the Fokker–Planck equation associated with (4.12) has a unique stationary measure. Hence in this parameter range the diffusion stabilizes the unperturbed ODE. To show the claim, we consider $U(x) = \log(x^2 + 1)$. Then

$$\mathcal{L}U(x) = -2 \cdot \frac{(\sigma - b)x^2 - \sigma}{x^2 + 1}.$$

Hence $\mathcal{L}U(x) < -(\sigma - b)$ as $|x| \gg 1$. Therefore, U is an unbounded Lyapunov function with respect to the stationary Fokker–Planck equation associated with (4.12). The claim now follows from Corollary 4.1.

When $0 < \sigma \leq b$, in fact, it can be shown that (4.12) admits no stationary measure (see [18]), i.e., b is the optimal lower bound of amplitudes of diffusions needed for this particular stabilization process.

5 Stationary Measures Under Weak Lyapunov Condition

In this section, we will study the existence of regular stationary measures under a weak Lyapunov condition. Theorem B will be proved.

5.1 Measure Estimates w.r.t. Weak Lyapunov Function

We recall the following measure estimate from [17].

Lemma 5.1 [17, Thm. A (c)] *Let* (A) *hold in* $\mathcal{U}$. *Assume that* (1.4) *admits a weak Lyapunov function* U *in* $\mathcal{U}$ *with essential lower bound* ρ_m *and upper bound* ρ_M *such that*

$$H_1(\rho) \le a^{ij}(x)\partial_i U(x)\partial_j U(x) \le H_2(\rho), \qquad x \in U^{-1}(\rho), \quad \rho \in [\rho_m, \rho_M) \tag{5.1}$$

for some positive function H_1 *and continuous function* H_2 *on* $[\rho_m, \rho_M)$. *Then for any weak solution* $u \in W^{1,p}_{loc}(\mathcal{U})$ *of* (1.4) *and any* $\rho_0 \in (\rho_m, \rho_M)$,

$$\mu(\mathcal{U} \setminus \Omega_{\rho_m}) \le \mu(\Omega_{\rho_0} \setminus \Omega_{\rho_m}) \mathrm{e}^{\int_{\rho_0}^{\rho_M} \frac{1}{\tilde{H}(\rho)}\mathrm{d}\rho}, \tag{5.2}$$

where μ *is the probability measure with density function* u, *i.e.*, $\mathrm{d}\mu(x) = u(x)\mathrm{d}x$, Ω_ρ *denotes the* ρ*-sublevel set of* U *for each* $\rho \ge 0$, *and* $\tilde{H}(\rho) = H_1(\rho)\int_{\rho_m}^{\rho} \frac{1}{H_2(s)}\mathrm{d}s$, $\rho \in [\rho_m, \rho_M)$.

This lemma is proved using Theorem 3.1 and a derivative formula contained in [17, Thm. 2.2]. We refer the reader to [17] for details.

5.2 Proof of Theorem B

To prove the theorem, we let U denote the weak Lyapunov function in $\mathcal{U}$ with respect to (1.4). Without loss of generality, we may assume that for U the essential lower bound ρ_m and the constant ρ_m appearing in Definition 2.3 are the same. Let $\{\rho_k\} \subset (\rho_m, \rho_M)$ be a sequence such that $\rho_k \to \rho_M$ as $k \to \infty$, where ρ_M denotes the essential upper bound of U. As in the proof of Theorem A, we obtain a sequence of $\{u_k\}$ such that, for each k, u_k is a weak solution of (1.4) in the domain $\Omega_k =: \Omega_{\rho_k}$, and, as $k \to \infty$, u_k converges under the compact-open topology of $C(\mathcal{U})$ to a non-negative function $\tilde{u} \in C(\mathcal{U})$ which satisfies $\int_{\mathcal{U}} \tilde{u}(x)\,\mathrm{d}x \le 1$ and

$$\int_{\mathcal{U}} \mathcal{L}f(x)\tilde{u}(x)\,\mathrm{d}x = 0, \qquad \text{for all } f \in C_0^\infty(\mathcal{U}).$$

As argued in the proof of Theorem A, we only need to show that $\tilde{u}$ is not identically zero in $\mathcal{U}$.

For a given $\rho_0 \in (\rho_m, \rho_M)$, we let $\bar{k}$ be such that $\rho_k \in (\rho_0, \rho_M)$ as $k \ge \bar{k}$. For each $k \ge \bar{k}$, U remains a weak Lyapunov function with respect to (1.4) in the domain Ω_k with essential upper bound ρ_k. Since U is of the class $\mathcal{B}_*(A)$, there is a positive function H which satisfies the properties (2.1) and (2.2) in Definition 2.3 on $[\rho_m, \rho_M)$. Clearly (5.1) is satisfied with

$$\begin{aligned} H_1(\rho) &=: \min_{x\in U^{-1}(\rho)} \left(a^{ij}(x)\partial_i U(x)\partial_j U(x)\right), \\ H_2(\rho) &=: \max_{x\in U^{-1}(\rho)} \left(a^{ij}(x)\partial_i U(x)\partial_j U(x)\right), \qquad \rho \in [\rho_m, \rho_M). \end{aligned}$$

By (2.1), $H \le H_1$. It is also clear that H_1 and H_2 are positive and continuous. For each $k \ge \bar{k}$, applying (5.2) in Lemma 5.1 with u_k, Ω_k, ρ_k in place of $u, \mathcal{U}, \rho_M$ respectively, yields that

$$\int_{\Omega_k \setminus \Omega_{\rho_m}} u_k(x)\mathrm{d}x \le \mathrm{e}^{\int_{\rho_0}^{\rho_k} \frac{1}{\tilde{H}(\rho)}\mathrm{d}\rho} \int_{\Omega_{\rho_0}\setminus\Omega_{\rho_m}} u_k(x)\mathrm{d}x,$$

i.e.,

$$1 = \int_{\Omega_k} u_k(x)\mathrm{d}x \le \int_{\Omega_{\rho_m}} u_k(x)\mathrm{d}x + \mathrm{e}^{\int_{\rho_0}^{\rho_k} \frac{1}{\tilde{H}(\rho)}\mathrm{d}\rho} \int_{\Omega_{\rho_0}\setminus\Omega_{\rho_m}} u_k(x)\mathrm{d}x, \tag{5.3}$$

where

$$\tilde{H}(\rho) = H_1(\rho)\int_{\rho_m}^{\rho} \frac{1}{H_2(s)}\mathrm{d}s, \qquad \rho \ge \rho_0.$$

Let $C =: \int_{\rho_m}^{\rho_0} \frac{1}{H_2(s)} \mathrm{d}s$. Then C is a positive constant and $CH_1(\rho) \le \tilde{H}(\rho)$, $\rho \in [\rho_0, \rho_M)$. By (2.2) and the fact that $H_1 \ge H$, $\int_{\rho_m}^{\rho_M} \frac{1}{H_1(\rho)} \mathrm{d}\rho < +\infty$. It follows that

$$\int_{\rho_0}^{\rho_M} \frac{1}{\tilde{H}(\rho)} \mathrm{d}\rho < +\infty. \tag{5.4}$$

Now, $\tilde{u} \not\equiv 0$, for otherwise, (5.4) would lead to a contradiction when taking $k \to \infty$ in (5.3). □

Remark 5.1 (1) From the above proof, we see that (5.4), which appears weaker than (2.2), is sufficient to conclude Theorem B. But we remark that condition (2.2) is actually equivalent to (5.4). To show (5.4) implies (2.2), we note that $\tilde{H}(\rho) \le H_1(\rho) \int_{\rho_m}^{\rho} \frac{1}{H_1(s)} \mathrm{d}s$, i.e.,

$$y'(\rho) \le \frac{1}{\tilde{H}(\rho)} y(\rho), \qquad \rho \in (\rho_m, \rho_M),$$

where $y(\rho) = \int_{\rho_m}^{\rho} \frac{1}{H_1(s)} \mathrm{d}s$. It follows that, for any $\rho_0 \in (\rho_m, \rho_M)$,

$$\int_{\rho_0}^{\rho_M} \frac{1}{\tilde{H}(\rho)} \mathrm{d}\rho \ge \log\left(\int_{\rho_m}^{\rho_M} \frac{1}{H_1(s)} \mathrm{d}s\right) - \log\left(\int_{\rho_m}^{\rho_0} \frac{1}{H_1(s)} \mathrm{d}s\right).$$

Since H_1 is positive and continuous on $[\rho_m, \rho_0]$, we have by (5.4) that (2.2) is satisfied with $H = H_1$.

(2) The condition that U is of the class $\mathcal{B}_*(A)$ cannot be removed in Theorem B. As an example, consider

$$\mathrm{d}x = \frac{x}{1+x^2}\mathrm{d}t + \sqrt{2}\,\mathrm{d}W, \qquad x \in \mathbb{R}. \tag{5.5}$$

Let $\gamma(x)$ be a given everywhere positive function satisfying

$$\int_{-\infty}^{+\infty} \gamma(x)(1+x^2)^{\frac{1}{2}} \mathrm{d}x < \infty.$$

Then by the proof of Proposition 6.1 below, there is a non-negative C^2 function U in $\mathbb{R}$ satisfying

$$\mathcal{L}U(x) = U''(x) + \frac{x}{1+x^2}U'(x) = -\gamma(x) < 0, \qquad |x| \gg 1 \tag{5.6}$$

and $U(x) \to +\infty$ as $x \to \infty$. Hence by Proposition 2.1, U is a compact function in $\mathbb{R}$. It follows from (5.6) that U is a weak Lyapunov function with respect to (5.5) in $\mathbb{R}$, and in fact a strict weak Lyapunov function in the sense that $\mathcal{L}U(x) < 0$ in its essential domain. But, as to be seen in [18, Example 5.1], the Fokker–Planck equation associated with (5.5) admits no stationary measure in $\mathbb{R}$.

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Appendix

When $\mathcal{U} = \mathbb{R}$, Lyapunov functions can be explicitly classified in the case of non-degenerate and bounded diffusion. Consider

$$\mathrm{d}x = V(x)\mathrm{d}t + \sqrt{2a(x)}\mathrm{d}W, \qquad x \in \mathbb{R}, \tag{6.1}$$

where V, a are continuous functions, and $0 < a(x) \le \Lambda$ in $\mathbb{R}$ for some constant Λ. We denote the adjoint Fokker–Planck operator associated with (6.1) by $\mathcal{L}_{V,a}$.

Proposition 6.1 *$\mathcal{L}_{V,a}$ admits a Lyapunov function in $\mathbb{R}$ iff there is a continuous function $\gamma(x)$ such that for some constant $K > 0$, $\inf_{|x|\ge K} \gamma(x) > 0$, and*

$$\int_{-\infty}^{-K} I_-(t)q(t)\mathrm{d}t + \int_K^{+\infty} I_+(t)q(t)\mathrm{d}t < \infty, \tag{6.2}$$

where

$$I_\pm(x) = \mathrm{e}^{\int_{\pm K}^x p(s)\mathrm{d}s}, \quad p(x) = \frac{V(x)}{a(x)}, \qquad q(x) = \frac{\gamma(x)}{a(x)}.$$

Moreover, if $\mathcal{L}_{V,a}$ admits a Lyapunov function in $\mathbb{R}$, then it also admits an unbounded Lyapunov function in $\mathbb{R}$.

Proof Suppose $U \in C^2(\mathbb{R})$ is a Lyapunov function with respect to $\mathcal{L}_{V,a}$. Denote

$$\gamma(x) =: -\mathcal{L}_{V,a}U(x) = -a(x)U''(x) - V(x)U'(x), \qquad x \in \mathbb{R}. \tag{6.3}$$

Then $\gamma(x)$ is continuous in $\mathbb{R}$ and there exists a constant $K > 0$ such that $\inf_{|x|\ge K} \gamma(x) > 0$. We note that (6.3) is equivalent to

$$U'' + p(x)U' = -q(x). \tag{6.4}$$

A direct integration of (6.4) yields that

$$U(x) = \begin{cases} U_-(x), & x \le -K, \\ U_+(x), & x \ge K, \end{cases} \tag{6.5}$$

where

$$U_\pm(x) = \int_{\pm K}^x U'_\pm(u)\mathrm{d}u + U_\pm(\pm K), \tag{6.6}$$

and

$$U'_\pm(x) = I_\pm(x)^{-1}\left(-\int_{\pm K}^x I_\pm(t)q(t)\mathrm{d}t + U'_\pm(\pm K)\right).$$

Since U, as a Lyapunov function, must satisfy $0 \le U(x) < U(\infty) = U(-\infty)$, $x \in \mathbb{R}$, there are sequences $x_i^\pm \to \pm\infty$ such that $\pm U'_\pm(x_i^\pm) > 0$ for all i. It follows that

$$\int_K^{+\infty} I_+(t)q(t)\mathrm{d}t \le U'_+(K), \tag{6.7}$$

$$\int_{-\infty}^{-K} I_-(t)q(t)\mathrm{d}t \le -U'_-(-K), \tag{6.8}$$

i.e., (6.2) holds.

Conversely, suppose that there is a continuous γ satisfying (6.2) with $\inf_{|x|\geq K}\gamma(x)>0$ for some constant $K>0$. Let $U'_{\pm}(\pm K)$ be chosen such that both (6.7) and (6.8) are valid as strict inequalities. For arbitrarily fixed $U_{\pm}(\pm K)>0$, we define positive functions $U_{\pm}$ through the formula in (6.6) and let U be a positive C^2 function in $\mathbb{R}$ which is defined by (6.5) when $|x|\geq K$. Then U satisfies the equation (6.3) when $|x|>K$, hence it is a Lyapunov function in $\mathbb{R}$ with respect to $\mathcal{L}_{V,a}$ if it is also a compact function.

We now claim that $U(+\infty)=U(-\infty)=+\infty$, which implies that U is an unbounded Lyapunov function. Suppose for contradiction that U_+ is bounded. Since $U'_+(x)>0$ as $x\geq K$, $U_+(+\infty)$ exists. It follows that

$$\begin{aligned}\infty &> \int_K^\infty I_+(x)^{-1}\left(-\int_K^x I_+(u)q(u)\mathrm{d}u+U'(K)\right)\mathrm{d}x\\ &\geq \int_K^\infty I_+(x)^{-1}\mathrm{d}x\left(U'(K)-\int_K^\infty I_+(u)q(u)\mathrm{d}u\right)=:C\int_K^\infty I_+(x)^{-1}\mathrm{d}x.\end{aligned}$$

Since $a(t)$ is bounded from above and $\gamma(t)$ is bounded away from 0, $q(t)$ is bounded away from 0. It follows from (6.2) that

$$\int_K^\infty I_+(x)\mathrm{d}x<\infty.$$

Then by Hölder's inequality,

$$\infty=\int_K^\infty I_+(x)^{-\frac{1}{2}}I_+(x)^{\frac{1}{2}}\mathrm{d}x\leq\left(\int_K^\infty I_+(x)^{-1}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_K^\infty I_+(x)\mathrm{d}x\right)^{\frac{1}{2}}<\infty,$$

which leads to a contradiction. Similarly, $U_-(-\infty)=+\infty$. □

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