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Entire Solutions with Annihilating Fronts to a Nonlocal Dispersal Equation with Bistable Nonlinearity and Spatio-Temporal Delay

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Abstract This paper deals with the entire solutions to a nonlocal dispersal bistable equation with spatio-temporal delay. Assuming that the equation has a traveling wave front with non-zero wave speed, we establish the existence of entire solutions with annihilating-fronts by using the comparison principle combined with explicit constructions of sub- and super-solutions. These entire solutions constitute a two-dimensional manifold and the traveling wave fronts belong to the boundary of the manifold. We also prove the uniqueness, Liapunov stability and continuous dependence on the shift parameters of the entire solutions.

Keywords Entire solution · Traveling wave front · Nonlocal dispersal equation · Spatio-temporal delay · Bistable nonlinearity

Mathematics Subject Classification 34K30 · 47G20 · 92D25

1 Introduction

In recent years, in order to investigate the interaction effects of spatial diffusion and time delay on the evolutionary behavior of biological systems, the study of reaction-diffusion equations with spatio-temporal delay (or nonlocal delay) has drawn great attention. We refer the readers to the survey papers of Gourley et al. [10] and Ruan [23] for more results and references. For example, the following equation is a typical and important model describing the evolution of matured population of a single species (see [1,2,24]):

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$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - du(x,t) + \int_0^\tau \int_{-\infty}^{+\infty} G(x-y,s)b(u(y,t-s))dyds, \tag{1.1}$$

where $u(x,t)$ denotes the density of the adult population at location $x \in \mathbb{R}$ and time $t \geq 0$; $D > 0$ and $d > 0$ are the diffusion rate and death rate of the adult population, respectively; $b(\cdot)$ is related to the birth function. The kernel function $G(x-y,s)$ represents the probability of the population which have been born at location y and time $t-s$, and become mature at location x and time t.

The basic assumption for the model (1.1) is that the internal interaction of the species is random and local, i.e. any individual moves randomly between the adjacent spatial locations. However, in realistic world, the movements and interactions of many species in ecology and biology can occur between the non-adjacent spatial locations, see e.g. Lee et al. [14] and Murray [22]. Taking this fact into account, the authors of [34] recently introduced a nonlocal dispersal equation for a structured population with spatio-temporal delay. The governing equation is

$$\frac{\partial u}{\partial t} = D\big[(J*u)(x,t) - u(x,t)\big] - du + \int_0^\tau \int_{-\infty}^{+\infty} G(x-y,s)b(u(y,t-s))dyds, \tag{1.2}$$

where $(J*u)(x,t) - u(x,t)$ means the "nonlocal dispersal operator" and $(J*u)(x,t)$ is a "spatial convolution operator" defined by

$$(J*u)(x,t) := \int_{-\infty}^{+\infty} J(x-y)u(y,t)dy. \tag{1.3}$$

In biological and epidemiological models, the existence of traveling wave solutions is an important issue due to their significant applications. Many mathematical results related to traveling wave solutions have been established in the past decades. For example, the traveling wave solutions of reaction-diffusion equations with spatial-temporal delay and nonlocal dispersal equations have been widely studied in the literature [4,5,7,9,24,26]. On the other hand, from the viewpoint of dynamical systems, it is significant to understand the dynamical structure of the global attractor (or the maximal invariant set) which consists of *entire solutions*, i.e. solutions defined for all time variable $t \in \mathbb{R}$. It is clear that the traveling wave solution is a special type of entire solutions.

Although the traveling wave solutions constitute important parts of the global attractor, the structure of global attractor could be quite complicated. Recently, many types of front-like entire solutions have been observed for various evolution equations by mixing the traveling wave solutions and some spatially independent solutions, see [11–13,15–17,20,21,25,27–33]. For examples, Hamel and Nadirashvili [12] established three-, four- and five-dimensional manifolds of entire solutions for the Fisher-KPP equation. In [13], Hamel and Nadirashvili further obtained an infinite-dimensional manifold of entire solutions for the Fisher-KPP equation in high-dimensional spaces. Different from those entire solutions obtained in [12, 13], Morita and Ninomiya [21] further constructed other types of entire solutions for some bistable reaction-diffusion equations. As mentioned in [21], we see that such entire solutions also play important roles in some other areas, such as, transient dynamics and distinct history of two solutions, etc..

For Eq. (1.2), Wu and Ruan [34] recently established the existence and qualitative properties of entire solutions under the monostable assumption of birth functions. However, for the case of bistable birth functions, the study for entire solutions of (1.2) other than traveling wave solutions still remains open. Therefore, the purpose of this paper is to study the entire solutions of (1.2) with bistable birth functions. To this end, we make the following assumptions for the kernel functions $J(\cdot)$, $G(\cdot)$ and the birth function $b(\cdot)$.

(G) $J(-x) = J(x) \geq 0$, $G(x,t) = G(-x,t) \geq 0$, $\forall x \in \mathbb{R}$, $t \in [0,\tau]$,

$$\int_{-\infty}^{+\infty} J(y)dy = \int_0^\tau \int_{-\infty}^{+\infty} G(y,s)dyds = 1 \text{ (normalization)},$$

and for any $c, \lambda \geq 0$,

$$\int_{-\infty}^{+\infty} e^{-\lambda y} J(y)dy < +\infty \text{ and } \int_0^\tau \int_{-\infty}^{+\infty} e^{-\lambda(y+cs)} G(y,s)dyds < +\infty.$$

(B1) There exists some $K > 0$ such that $b(\cdot) \in C^2([0,K],\mathbb{R})$, $d > \max\{b'(0), b'(K)\}$, $b(0) = dK - b(K) = 0$, and $b'(u) \geq 0$ for $u \in [0,K]$.

It is well-known that a solution $u(x,t)$ of (1.2) is called a traveling wave solution connecting 0 and K with speed c, if $u(x,t) = \phi(x+ct)$, $x, t \in \mathbb{R}$, for some function $\phi(\cdot) \in C^1(\mathbb{R})$ (called wave profile) such that $\phi(-\infty) = 0$ and $\phi(+\infty) = K$. Following the above definition, we see that (c,ϕ) satisfies the following equation

$$c\phi'(\xi) = D\big[(J*\phi)(\xi) - \phi(\xi)\big] - d\phi(\xi) + \int_0^\tau \int_{-\infty}^{+\infty} G(y,s)b\big(\phi(\xi - y - cs)\big)dyds, \quad (1.4)$$

where

$$(J*\phi)(\xi) = \int_{-\infty}^{+\infty} J(y)\phi\big(\xi - y\big)dy.$$

Under the basic assumptions (G) and (B1), the following condition ensures the existence of traveling wave fronts of (1.2) connecting 0 and K

(B2) There exists an $a \in (0,K)$ such that $d < b'(a)$, $b(u) < du$ for $u \in (0,a)$ and $b(u) > du$ for $u \in (a,K)$.

In fact, under the assumptions (G), (B1)–(B2) and applying the abstract theory established by Chen [4] and Fang and Zhao [8], one can show that (1.4) has a monotone solution $U(x+ct)$ (called a traveling wave front of (1.2)) connecting 0 and K with wave speed c. Clearly, $U(-x+ct)$ is also a traveling wave front of (1.2) connecting 0 and K. Moreover, it could be verified that $b(u) = pu^2e^{-\alpha u}$ with $p > 0$ and $\alpha > 0$ satisfies the assumptions (B1)–(B2) for a wide range of the parameters p and α. Such specific birth function has been widely used in mathematical biology literature, see e.g. Ma and Zou [18] and Wang et al. [29].

Throughout this paper, we always assume that (G) and (B1) hold and (1.2) has a traveling wave front $U(x+ct)$ connecting 0 and K with speed $c \neq 0$. Using the traveling wave fronts $U(x+ct+\theta_1)$ and $U(-x+ct+\theta_2)$, where θ_1, θ_2 are the shift parameters, we first construct a pair of sub- and supersolutions of (1.2) (see Definition 2.1). Then we establish the existence of entire solutions of (1.2) by using the comparison principle combining with the sub- and supersolutions. According to our constructions, one can see that entire solutions behave as two traveling wave fronts approaching each other from both sides of the x-axis as $t \to -\infty$ and annihilating as time increases. We call such entire solutions as "*annihilating-front*" entire solutions. In addition, based on the construction of different pairs of sub- and supersolutions via the derived entire solutions (see Lemmas 4.2 and 4.3), we prove the uniqueness, Liapunov stability and continuous dependence on the shift parameters θ_1, θ_2 of the entire solutions. Here we point out that the assumption (B2) will not be needed in studying the problems on the entire solutions.

For convenience, hereinafter we denote $\Phi^{\pm}_{\theta}(x,t) := U(\pm x + ct + \theta)$ for any $\theta \in \mathbb{R}$. Our main results are stated as follows.

Theorem 1.1 *Assume that* (G) *and* (B1) *hold and* (1.2) *has a traveling wave front* $U(x+ct)$ *connecting* 0 *and* K *with speed* $c > 0$*. Then for any* $\theta_1, \theta_2 \in \mathbb{R}$*, there exists a unique entire solution* $\Phi_{\theta_1,\theta_2}(x,t)$ *of* (1.2) *which satisfies*

$$\lim_{t\to-\infty}\left\{\sup_{x\le 0}\left|\Phi_{\theta_1,\theta_2}(x,t)-\Phi^{-}_{\theta_1}(x,t)\right|+\sup_{x\ge 0}\left|\Phi_{\theta_1,\theta_2}(x,t)-\Phi^{+}_{\theta_2}(x,t)\right|\right\}=0. \tag{1.5}$$

Furthermore, the following statements hold:

(1) $\partial_t\Phi_{\theta_1,\theta_2}(x,t) > 0$ *and* $0 < \Phi_{\theta_1,\theta_2}(x,t) < K$ *for all* $(x,t)\in\mathbb{R}^2$.
(2) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|\Phi_{\theta_1,\theta_2}(x,t)-K| = 0$, $\lim_{t\to-\infty}\sup_{|x|\le N_0}\Phi_{\theta_1,\theta_2}(x,t) = 0$ *for any* $N_0\in\mathbb{R}_+$*, and* $\lim_{|x|\to+\infty}\sup_{t\ge t_0}|\Phi_{\theta_1,\theta_2}(x,t)-K| = 0$ *for any* $t_0\in\mathbb{R}$.
(3) *For any* $(x,t)\in\mathbb{R}^2$, $\Phi_{\theta_1,\theta_2}(x,t)$ *converges to* $\begin{cases}\Phi^{-}_{\theta_1}(x,t) \text{ as } \theta_2\to-\infty,\\ \Phi^{+}_{\theta_2}(x,t) \text{ as } \theta_1\to-\infty.\end{cases}$
(4) *For any* $\theta_1^*, \theta_2^*\in\mathbb{R}$*, there exists* $(x_0,t_0)\in\mathbb{R}^2$ *depending on* $\theta_1,\theta_2,\theta_1^*,\theta_2^*$ *such that* $\Phi_{\theta_1^*,\theta_2^*}(\cdot,\cdot) = \Phi_{\theta_1,\theta_2}(\cdot+x_0,\cdot+t_0)$ *on* $\mathbb{R}^2$.
(5) *For any* $(x,t)\in\mathbb{R}^2$, $\Phi_{\theta_1,\theta_2}(x,t)$ *is increasing with respect to* $(\theta_1,\theta_2)\in\mathbb{R}^2$.
(6) $\Phi_{\theta_1,\theta_2}(x,t)$ *depends continuously on* $(\theta_1,\theta_2)\in\mathbb{R}^2$.
(7) *The entire solution* $\Phi_{\theta_1,\theta_2}(x,t)$ *is Liapunov stable in the following sense:* $\forall\epsilon>0$, $\exists\,\bar\delta>0$ *such that* $\forall\varphi\in\mathcal{C}_{[0,K]}$ *(see* (2.2) *for the definition) satisfying*

$$\sup_{x\in\mathbb{R}}\left\|\varphi(x,\cdot)-\Phi_{\theta_1,\theta_2}(x+x_0,\cdot+t_0)\right\|_{L^\infty[-\tau,0]} < \bar\delta,$$

the solution $u(x,t;\varphi)$ *of* (1.2) *with initial value* φ *satisfies*

$$\left|u(x,t;\varphi)-\Phi_{\theta_1,\theta_2}(x+x_0,t+t_0)\right| < \epsilon$$

for any $x\in\mathbb{R}$ *and* $t\ge 0$*, where* $x_0, t_0\in\mathbb{R}$ *are two constants.*

Following the same discussions in Hamel and Nadirashvili [12], we see that the entire functions $\Phi_{\theta_1,\theta_2}(x,t)$ established by Theorem 1.1 constitute a two-dimensional manifold $\mathcal{M}_2$. In addition, (1.2) possesses two one-dimensional manifolds $\mathcal{M}_1^-$ and $\mathcal{M}_1^+$ of entire solutions of traveling wave type, namely $\Phi^{-}_{\theta_1}(x,t)$ and $\Phi^{+}_{\theta_2}(x,t)$ respectively. Then, from (3) of Theorem 1.1, we know that $\mathcal{M}_1^-$ (or $\mathcal{M}_1^+$) belongs to the boundary of $\mathcal{M}_2$ by taking the limit $\theta_2\to-\infty$ (or $\theta_1\to-\infty$).

Similar to Theorem 1.1, when $c<0$, we can obtain the following results.

Theorem 1.2 *Assume that* (G) *and* (B1) *hold and* (1.2) *has a traveling wave front* $U(x+ct)$ *connecting* 0 *and* K *with speed* $c<0$*. Then for any* $\theta_1,\theta_2\in\mathbb{R}$*, there exists a unique entire solution* $\tilde\Phi_{\theta_1,\theta_2}(x,t)$ *of* (1.2) *which satisfies*

$$\lim_{t\to-\infty}\left\{\sup_{x\le 0}\left|\tilde\Phi_{\theta_1,\theta_2}(x,t)-\Phi^{+}_{\theta_1}(x,t)\right|+\sup_{x\ge 0}\left|\tilde\Phi_{\theta_1,\theta_2}(x,t)-\Phi^{-}_{\theta_2}(x,t)\right|\right\}=0. \tag{1.6}$$

Moreover, the assertions (4)–(7) *in* Theorem 1.1 *and the following statements hold:*

(1)′ $\partial_t\tilde\Phi_{\theta_1,\theta_2}(x,t) < 0$ *and* $0 < \tilde\Phi_{\theta_1,\theta_2}(x,t) < K$ *for all* $(x,t)\in\mathbb{R}^2$.

(2)′ $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\tilde{\Phi}_{\theta_1,\theta_2}(x,t)=0$, $\lim_{t\to-\infty}\sup_{|x|\le N_1}|\tilde{\Phi}_{\theta_1,\theta_2}(x,t)-K|=0$ *for any* $N_1\in\mathbb{R}_+$ *and* $\lim_{|x|\to+\infty}\sup_{t\ge t_1}\tilde{\Phi}_{\theta_1,\theta_2}(x,t)=0$ *for any* $t_1\in\mathbb{R}$.

(3)′ *For any* $(x,t)\in\mathbb{R}^2$, $\tilde{\Phi}_{\theta_1,\theta_2}(x,t)$ *converges to* $\begin{cases}\Phi^+_{\theta_1}(x,t) \text{ as } \theta_2\to+\infty,\\ \Phi^-_{\theta_2}(x,t) \text{ as } \theta_1\to+\infty.\end{cases}$

Remark 1.3 (1) Here we note that Theorem 1.2 is a consequence of Theorem 1.1. In fact, let us denote $\tilde{c}:=-c>0$ and $\tilde{U}(x+\tilde{c}t):=K-U(-(x+\tilde{c}t))=K-U(-x+ct)$. Then, $\tilde{U}(-\infty)=0$, $\tilde{U}(+\infty)=K$, and $\tilde{U}(x+\tilde{c}t)$ is an increasing traveling wave solution of the following equation

$$\frac{\partial v}{\partial t}=D[(J*v)(x,t)-v(x,t)]-dv+\int_0^\tau\int_{-\infty}^{+\infty}G(x-y,s)\tilde{b}(v(y,t-s))dyds, \quad (1.7)$$

where $\tilde{b}(v):=b(K)-b(K-v)$. Clearly, $\tilde{b}(\cdot)$ satisfies the condition (B1). Then it follows from Theorem 1.1 that there exists an entire solution $W(x,t)$ of (1.7) such that

$$\lim_{t\to-\infty}\{\sup_{x\le0}|W(x,t)-\tilde{U}(-x+\tilde{c}t-\theta_1)|+\sup_{x\ge0}|W(x,t)-\tilde{U}(x+\tilde{c}t-\theta_2)|\}=0. \quad (1.8)$$

Denote $\tilde{\Phi}_{\theta_1,\theta_2}(x,t):=K-W(x,t)$. Since $\Phi^+_{\theta_1}(x,t)=K-\tilde{U}(-x+\tilde{c}t-\theta_1)$ and $\Phi^-_{\theta_2}(x,t)=K-\tilde{U}(x+\tilde{c}t-\theta_2)$, according to (1.8), we see that $\tilde{\Phi}_{\theta_1,\theta_2}(x,t)$ is an entire solution of (1.2) which satisfies the statement of Theorem 1.2. Therefore, in the following of this work, we only prove Theorem 1.1.

(2) We prove the main results under the assumption that (1.2) has a bistable traveling wave front with non-zero wave speed. Due to the non-zero wave speed, we can establish the entire solutions by constructing an appropriate pair of sub- and supersolution of (1.2) (see Lemmas 3.2 and 3.3). However, when the wave speed is zero, there occurs the propagation failure or pinning phenomenon for the wave front of (1.2). This fact causes the construction of sub- and supersolutions becoming very difficult. We will consider this problem in future research. Moreover, it is also an interesting and important problem to consider the sign of wave speed of the bistable traveling wave front of (1.2).

The rest of the paper is organized as follows. In Sect. 2, we first establish the existence and comparison principle for solutions of the Cauchy problem of (1.2). Then we investigate the asymptotic behavior of the traveling wave fronts at $\pm\infty$. Sect. 3 is devoted to the construction of a pair of sub- and supersolution of (1.2). Using the sub- and supersolutions and comparison principle, we first prove the existence and qualitative properties of entire solutions in Sect. 4. Based on the construction of different pairs of sub- and supersolutions via the derived entire solutions, the uniqueness, Liapunov stability and continuous dependence on the shift parameters of the entire solutions are then proved.

2 Preliminaries

We first establish the existence and comparison principle for solutions of the Cauchy problem of (1.2). Then we investigate the asymptotic behavior of the traveling wave fronts at $\pm\infty$. It could be seen that the asymptotic decay rates of the traveling wave fronts play an important role in the constructions of sub- and supersolutions of (1.2) (see Lemma 3.2).

Let us define $\hat{b}(\cdot):[0,2K]\to\mathbb{R}$ by $\hat{b}(u):=\begin{cases}b(u), & u\in[0,K],\\ b(K)+b'(K)(u-K), & u\in[K,2K].\end{cases}$

Obviously, $\hat{b}(u)$ is an extension of $b(u)$, $\hat{b}'(u) \geq 0$ and

$$\left|\hat{b}'(u_1) - \hat{b}'(u_2)\right| \leq \max_{u\in[0,K]} \left|b''(u)\right||u_1 - u_2|, \quad \text{for } u_1, u_2 \in [0, 2K].$$

For the sake of convenience, we still denote $\hat{b}(u)$ by $b(u)$ in the remainder of this paper.

2.1 Cauchy problem and comparison principle

Let X be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}$ with the supremum norm $\|\cdot\|_X$ and $\mathcal{C} = C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm. Then we denote the following spaces:

$$X_{[0,K]} := \{\varphi \in X : \varphi(x) \in [0, K], x \in \mathbb{R}\}, \tag{2.1}$$

$$\mathcal{C}_{[0,K]} := \{\varphi \in \mathcal{C} : \varphi(x, s) \in [0, K], x \in \mathbb{R}, s \in [-\tau, 0]\}. \tag{2.2}$$

As usual, we identify an element $\varphi \in \mathcal{C}$ as a function from $\mathbb{R} \times [-\tau, 0]$ into $\mathbb{R}$ defined by $\varphi(x, s) = \varphi(s)(x)$. For any continuous function $u(\cdot) : [-\tau, \ell) \to X$, $\ell > 0$, we define $u^t \in \mathcal{C}$, $t \in [0, \ell)$ by $u^t(s) = u(t + s)$, $s \in [-\tau, 0]$. Then $t \to u^t(\cdot)$ is a continuous function from $[0, \ell)$ to $\mathcal{C}$. Define $F[\cdot] : \mathcal{C}_{[0,K]} \to X$ by

$$F[\varphi](x) := \big(J * \varphi\big)(x, 0) + \int_0^\tau \int_{-\infty}^{+\infty} G(x - y, s)b\big(\varphi(y, -s)\big)dyds.$$

It is easy to see that $F[\cdot] : \mathcal{C}_{[0,K]} \to X$ is globally Lipschitz continuous and $T(t) := e^{-(D+d)t}$ is a linear semigroup on X. Then the definitions of super- and subsolutions of (1.2) are given as follows.

Definition 2.1 A continuous function $u(\cdot) : [-\tau, \ell) \to X_{[0,K]}$, $\ell > 0$, is called a supersolution (or a subsolution) of (1.2) on $[0, \ell)$ if

$$u(t) \geq (or \leq)\ T(t - s)u(s) + \int_s^t T(t - r)F[u^r]dr$$

for any $0 \leq s < t < \ell$.

Applying the theory of abstract functional differential equations [19, Corollary5], we have the following result (see also [34]).

Lemma 2.2 *Assume that* (G) *and* (B1) *hold and* $\varphi(\cdot)$ *is the Cauchy data of Eq.* (1.2)*. We have the following results.*

(1) *For any* $\varphi \in \mathcal{C}_{[0,K]}$, (1.2) *has a classical and unique solution* $u(x, t; \varphi)$ *satisfying* $0 \leq u(x, t; \varphi) \leq K$ *for* $(x, t) \in \mathbb{R} \times (0, \infty)$.
(2) *Let* $u^-(x, t)$ *and* $u^+(x, t)$ *be a pair of sub- and supersolutions of* (1.2) *on* $\mathbb{R} \times [-\tau, \infty)$ *such that* $u^-(x, s) \leq u^+(x, s)$ *for* $(x, s) \in \mathbb{R} \times [-\tau, 0]$*, then* $0 \leq u^-(x, t) \leq u^+(x, t) \leq K$ *for* $(x, t) \in \mathbb{R} \times [0, \infty)$.

2.2 Asymptotic behavior of traveling wave fronts

By elementary computations, the characteristic functions of the profile equation (1.4) with respect to the equilibria 0, K can be represented by

$$\triangle_1(\lambda) := c\lambda - D\big[\mathcal{J}(\lambda) - 1\big] + d - b'(0)\mathcal{G}(\lambda),$$

$$\triangle_2(\lambda) := c\lambda - D\big[\mathcal{J}(\lambda) - 1\big] + d - b'(K)\mathcal{G}(\lambda),$$

respectively, where

$$\mathcal{J}(\lambda) := \int_{-\infty}^{+\infty} e^{-\lambda y} J(y)dy \text{ and } \mathcal{G}(\lambda) := \int_0^{\tau}\int_{-\infty}^{+\infty} G(y,s)e^{-\lambda(y+cs)}dyds.$$

Since $d > \max\{b'(0), b'(K)\}$, one can easily obtain the following result.

Lemma 2.3 *Assume* (G) *and* (B1). *The equation* $\triangle_j(\lambda) = 0$ $(j = 1, 2)$ *has two real roots* $\lambda_{j1} := \lambda_{j1}(c) < 0$ *and* $\lambda_{j2} := \lambda_{j2}(c) > 0$ *such that* $\triangle_j(\lambda) > 0$ *if* $\lambda \in (\lambda_{j1}, \lambda_{j2})$, *and* $\triangle_j(\lambda) < 0$ *if* $\lambda \in \mathbb{R} \setminus [\lambda_{j1}, \lambda_{j2}]$.

To establish the asymptotic behavior of traveling wave fronts at $\pm\infty$, we first recall the following Ikehara's Theorem, see e.g. [3,6].

Theorem 2.4 *Let* $u(\xi)$ *be a positive decreasing function and* $F(\Lambda) := \int_0^{+\infty} e^{-\Lambda\xi}u(\xi)d\xi$. *If* $F(\Lambda)$ *can be written as* $F(\Lambda) = H(\Lambda)(\Lambda + \Lambda_0)^{-(k+1)}$, *where* $k > -1$, $\Lambda_0 > 0$ *are two constants and* $H(\Lambda)$ *is analytic in the strip* $-\Lambda_0 \le \mathrm{Re}\Lambda < 0$, *then*

$$\lim_{\xi\to+\infty} u(\xi)e^{\Lambda_0\xi}/\xi^k = H(-\Lambda_0)/\Gamma(\Lambda_0 + 1).$$

Here $\Gamma(\cdot)$ *means the gamma-function.*

For convenience, we denote

$$\big(G \star \phi\big)(\xi) := \int_0^{\tau}\int_{-\infty}^{+\infty} G(y,s)\phi\big(\xi - y - cs\big)dyds,\ \forall\phi \in C\big(\mathbb{R}, [0, 2K]\big). \tag{2.3}$$

Since $G(x,t) = G(-x,t) \ge 0$, $\forall x \in \mathbb{R}$, $t \in [0,\tau]$, it is clear that

$$\big(G \star \phi\big)(\xi) = \int_0^{\tau}\int_{-\infty}^{+\infty} G(y,s)\phi\big(\xi + y - cs\big)dyds. \tag{2.4}$$

By Lemma 2.3 and Theorem 2.4, we have the following results.

Lemma 2.5 *Assume* (G) *and* (B1). *Let* $U(x+ct)$ *be a traveling wave front of* (1.2) *connecting 0 and* K *with* $c \in \mathbb{R}$. *Then,*

$$\lim_{\xi\to-\infty} U(\xi)e^{-\lambda_{12}\xi} = a_1,\ \lim_{\xi\to-\infty} U'(\xi)e^{-\lambda_{12}\xi} = a_1\lambda_{12}, \tag{2.5}$$

$$\lim_{\xi\to-\infty} \big(G \star U^m\big)^{\frac{1}{m}}(\xi)e^{-\lambda_{12}\xi} = a_1\mathcal{G}^{\frac{1}{m}}\big(m\lambda_{12}\big),\ m = 1, 2, \tag{2.6}$$

$$\lim_{\xi\to+\infty} \big(K - U(\xi)\big)e^{-\lambda_{21}\xi} = b_1,\ \lim_{\xi\to+\infty} U'(\xi)e^{-\lambda_{21}\xi} = -b_1\lambda_{21}, \tag{2.7}$$

$$\lim_{\xi\to+\infty} \big[K - (G \star U^m)^{\frac{1}{m}}(\xi)\big]e^{-\lambda_{21}\xi} = b_1\mathcal{G}^{\frac{1}{m}}\big(m\lambda_{21}\big),\ m = 1, 2, \tag{2.8}$$

where a_1 *and* b_1 *are positive constants.*

Proof The proof is similar to those of [27, Theorem 3.5] and [3, Theorem 1]. For the sake of completeness and reader's convenience, we sketch the outline for assertions (2.5) and (2.6) in the following three steps. Note that the other assertions can be considered by the same way.

Step 1. We show that $U(\xi)$ is integrable on $(-\infty, \xi']$ for some $\xi' \in \mathbb{R}$.

Step 2. We show that $U(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$ for some $\gamma > 0$. We obtain this assertion by showing that there exists a $\gamma > 0$ such that $V(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$, where $V(\xi) := \int_{-\infty}^{\xi} U(s)ds$.

Step 3. For $0 < \mathrm{Re}\lambda < \gamma$, let us define a two-sided Laplace transformation of U by

$$F(\lambda) := \int_{-\infty}^{+\infty} U(\xi)e^{-\lambda\xi}d\xi.$$

The first part of assertion (2.5) follows from Lemma 2.3, Theorem 2.4 and a property of Laplace transformation. In addition, it follows that (2.6) and the second part of (2.5) hold. The proof is complete. □

3 Construction of sub- and supersolutions

According to Remark 1.3, we may assume $c > 0$ in the following of this work. By Lemma 2.5, we know that there exist positive constants k, L, η, μ such that

$$ke^{\lambda_{12}\xi} \le U(\xi) \le Le^{\lambda_{12}\xi},\ \mu U(\xi) \le U'(\xi),\ \xi \le 0, \tag{3.1}$$

$$ke^{\lambda_{12}\xi} \le \big(G \star U^m\big)^{\frac{1}{m}}(\xi) \le Le^{\lambda_{12}\xi},\ \mu\big(G \star U^m\big)^{\frac{1}{m}}(\xi) \le U'(\xi),\ \xi \le 0,\ m = 1, 2, \tag{3.2}$$

$$\mu\eta e^{\lambda_{21}\xi} \le \mu\big(K - U(\xi)\big),\ \mu\big[K - (G \star U^m)^{\frac{1}{m}}(\xi)\big] \le U'(\xi),\ \xi \ge 0,\ m = 1, 2. \tag{3.3}$$

In order to construct appropriate sub- and superolutions of (1.2), we give the following definitions and then introduce two important functions $p_1(t)$ and $p_2(t)$.

Definition 3.1 (1) Let k, L, η and μ be the constants stated in (3.1)–(3.3), we denote

$$L_1 := \max_{u\in[0,2K]} b'(u),\ L_2 := \max_{u\in[0,K]} |b''(u)|,$$
$$N := \max\{\mu^{-1}k^{-1}L_2L^2,\ \mu^{-1}\eta^{-1}L_2LK\}.$$

(2) For any $\rho_1 \in (-\infty, 0]$, we denote the function

$$\omega(\rho_1) := \rho_1 - \frac{1}{\lambda_{12}} \ln\left(1 + \frac{N}{c}e^{\lambda_{12}\rho_1}\right) \quad \text{and} \quad \bar{\omega} := \omega(0) < 0. \tag{3.4}$$

Since $\omega(\rho_1)$ is increasing in $\rho_1 \in (-\infty, 0]$, we may denote its inverse function by $\rho_1 = \rho_1(\omega) : (-\infty, \bar{\omega}] \to (-\infty, 0]$. Then, for any $(\omega, \tilde{\omega}) \in (-\infty, \bar{\omega}]^2$, we further define

$$\rho_2(\omega, \tilde{\omega}) := \tilde{\omega} + \frac{1}{\lambda_{12}} \ln\left(1 + \frac{N}{c}e^{\lambda_{12}\rho_1(\omega)}\right),$$
$$\tilde{p}_1(t;\omega) := \rho_1(\omega) + ct - \frac{1}{\lambda_{12}} \ln\left\{1 + \frac{N}{c}e^{\lambda_{12}\rho_1(\omega)}(1 - e^{c\lambda_{12}t})\right\},\quad \text{for } t \le 0,$$
$$\tilde{p}_2(t;\omega,\tilde{\omega}) := \rho_2(\omega,\tilde{\omega}) + ct - \frac{1}{\lambda_{12}} \ln\left\{1 + \frac{N}{c}e^{\lambda_{12}\rho_1(\omega)}(1 - e^{c\lambda_{12}t})\right\},\quad \text{for } t \le 0.$$

Elementary computations show that $\tilde{p}_1(t;\omega)$ and $\tilde{p}_2(t;\omega,\tilde{\omega})$ satisfy the problems:

$$\begin{cases} \tilde{p}_1'(t;\omega) = c + Ne^{\lambda_{12}\tilde{p}_1(t;\omega)}, \\ \tilde{p}_1(0;\omega) = \rho_1(\omega), \end{cases} \quad \text{and} \quad \begin{cases} \tilde{p}_2'(t;\omega,\tilde{\omega}) = c + Ne^{\lambda_{12}\tilde{p}_1(t;\omega)}, \\ \tilde{p}_2(0;\omega,\tilde{\omega}) = \rho_2(\omega,\tilde{\omega}). \end{cases} \tag{3.5}$$

Obviously, we have $\tilde{\omega} - \omega = \rho_2(\omega, \tilde{\omega}) - \rho_1(\omega)$,

$$\tilde{p}_2(t; \omega, \tilde{\omega}) - \tilde{p}_1(t; \omega) = \rho_2(\omega, \tilde{\omega}) - \rho_1(\omega) = \tilde{\omega} - \omega, \tag{3.6}$$

$$\tilde{p}_1(t; \omega) - ct - \omega = \tilde{p}_2(t; \omega, \tilde{\omega}) - ct - \tilde{\omega} = -\frac{1}{\lambda_{12}} \ln \left(1 - \frac{r}{1+r} e^{c\lambda_{12} t}\right), \tag{3.7}$$

for $t \le 0$, where $r := Nc^{-1} e^{\lambda_{12}\rho_1(\omega)}$.

Moreover, given any $(\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$, we set

$$\begin{aligned} & p_1(t) = p_1(t; \omega_1, \omega_2) := \tilde{p}_1(t; \omega_1),\ p_2(t) = p_2(t; \omega_1, \omega_2) \\ & \quad := \tilde{p}_2(t; \omega_1, \omega_2),\ \text{if } \omega_2 \le \omega_1; \\ \text{or} \quad & p_1(t) = p_1(t; \omega_1, \omega_2) := \tilde{p}_2(t; \omega_2, \omega_1),\ p_2(t) = p_2(t; \omega_1, \omega_2) \\ & \quad := \tilde{p}_1(t; \omega_2),\ \text{if } \omega_1 \le \omega_2. \end{aligned}$$

Then, $p_2(t) \le p_1(t) \le 0$ when $\omega_2 \le \omega_1$; and $p_1(t) \le p_2(t) \le 0$ when $\omega_1 \le \omega_2$. By (3.7), there exists a positive constant R_0, independent of ω_1 and ω_2, such that

$$0 < p_1(t) - ct - \omega_1 = p_2(t) - ct - \omega_2 \le R_0 e^{c\lambda_{12} t}, \quad \text{for } t \le 0.$$

Using $p_1(t)$ and $p_2(t)$, we are ready to establish the supersolution of (1.2). For simplicity, we denote

$$(G \star v)(x, t) := \int_0^\tau \int_{-\infty}^{+\infty} G(y, s) v(x - y, t - s) dy ds, \quad \forall v \in C\big(\mathbb{R}^2, [0, 2K]\big).$$

Lemma 3.2 *For any $(\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$, there exists a $T < 0$ such that the function $\overline{u}(x, t)$ defined by*

$$\overline{u}(x, t) = U(x + p_1(t)) + U(-x + p_2(t))$$

is a supersolution of (1.2) on $\mathbb{R} \times (-\infty, T)$.

Proof We only consider the case $\omega_1 \le \omega_2$, since the other case can be discussed in the same way. In this case, $p_1(t) \le p_2(t)$ and $p_i'(t) = c + Ne^{\lambda_{12} p_2(t)}$, $i = 1, 2$, for $t \le 0$. By direct computations, we have

$$\begin{aligned} \mathcal{F}(\overline{u})(x, t) :=& \overline{u}_t - D\big[(J * \overline{u})(x, t) - \overline{u}(x, t)\big] + d\overline{u} - \big(G \star b(\overline{u})\big)(x, t) \\ =& p_1'(t) U'(x + p_1) + p_2'(t) U'(-x + p_2) \\ & - D\big[(J * U)(x + p_1) + (J * U)(-x + p_2) - U(x + p_1) - U(-x + p_2)\big] \\ & + d\big[U(x + p_1) + U(-x + p_2)\big] - \big(G \star b(\overline{u})\big)(x, t) \\ =& \big(p_1'(t) - c\big) U'(x + p_1) + \big(p_2'(t) - c\big) U'(-x + p_2) - H(x, t) \\ =& \big[U'(x + p_1) + U'(-x + p_2)\big]\big[Ne^{\lambda_{12} p_2(t)} - R(x, t)\big], \end{aligned}$$

where

$$\text{and} \quad \begin{aligned} & R(x, t) := H(x, t) / \big[U'(x + p_1) + U'(-x + p_2)\big] \\ & H(x, t) := \big(G \star b(\overline{u})\big)(x, t) - \big(G \star b(U)\big)(x + p_1(t)) - \big(G \star b(U)\big)(-x + p_2(t)). \end{aligned}$$

Clearly, $p_i(t - s) \le p_i(t) - cs$ for $s \in [0, \tau], i = 1, 2$. Note that $|b'(u_1) - b'(u_2)| \le L_2 |u_1 - u_2|$ for $u_1, u_2 \in [0, 2K]$. For any $v_1, v_2 \in [0, K]$, we have

$$\left|b(v_1+v_2)-b(v_1)-b(v_2)\right| = \left|\int_0^1 v_2\big[b'(v_1+\theta v_2)-b'(\theta v_2)\big]d\theta\right| \le L_2 v_1 v_2.$$

Then, using Cauchy–Schwarz inequality and (2.3) and (2.4), we obtain

$$\begin{aligned} H(x,t) =& \int_0^\tau \int_{-\infty}^{+\infty} G(y,s)\big[b\big(U(x-y+p_1(t-s))+U(-x+y+p_2(t-s))\big) \\ &- b(U(x+p_1(t)-y-cs)) - b(U(-x+p_2(t)+y-cs))\big]dyds \\ \le& \int_0^\tau \int_{-\infty}^{+\infty} G(y,s)\big[b\big(U(x+p_1(t)-y-cs)+U(-x+p_2(t)+y-cs)\big) \\ &- b(U(x+p_1(t)-y-cs)) - b(U(-x+p_2(t)+y-cs))\big]dyds \\ \le& L_2 \int_0^\tau \int_{-\infty}^{+\infty} G(y,s)U(x+p_1(t)-y-cs)U(-x+p_2(t)+y-cs)dyds \\ \le& L_2\big(G\star U^2\big)^{\frac12}(x+p_1(t))\big(G\star U^2\big)^{\frac12}(-x+p_2(t)). \end{aligned} \tag{3.8}$$

Now we estimate $R(x,t)$ by dividing $\mathbb{R}$ into the following 3 regions:

$$(1)\ p_2(t)\le x\le -p_1(t) \quad (2)\ x\ge -p_1(t), \quad (3)\ x\le p_2(t).$$

(1) By (3.1), (3.2) and (3.8), we have $H(x,t)\le L_2L^2e^{\lambda_{12}(p_1+p_2)}$ and

$$\begin{aligned} U'(x+p_1)+U'(-x+p_2) &\ge \mu\big[U(x+p_1)+U(-x+p_2)\big] \\ &\ge \mu k\big[e^{\lambda_{12}(x+p_1)}+e^{\lambda_{12}(-x+p_2)}\big] \ge 2\mu k e^{\lambda_{12}p_1}. \end{aligned}$$

Hence, it follows that

$$R(x,t)\le 2^{-1}\mu^{-1}k^{-1}L_2L^2e^{\lambda_{12}p_2}. \tag{3.9}$$

(2) In this case, we further consider two sub-cases:

$$(2\text{-}1) b'(0)\le b'(K) \text{ and } (2\text{-}2) b'(K)<b'(0).$$

(2-1) Let us denote

$$\triangle_3(\lambda) := \triangle_2(-\lambda) = -c\lambda - D\big[\mathcal{J}(\lambda)-1\big]+d-b'(K)\mathcal{G}(-\lambda).$$

It is clear that $\triangle_3(\lambda)$ has exactly one positive zero $-\lambda_{21}$. Moreover,

$$\triangle_1(\lambda)-\triangle_3(\lambda) = 2c\lambda + b'(K)\mathcal{G}(-\lambda) - b'(0)\mathcal{G}(\lambda)\ge 0,\ \forall \lambda\ge 0.$$

By the properties of $\triangle_1(\lambda)$ and $\triangle_3(\lambda)$, we see that $\lambda_{12}\ge -\lambda_{21}$. Then it follows from (3.2), (3.3) and (3.8) that

$$\begin{aligned} R(x,t) &\le L_2K\big(G\star U^2\big)^{\frac12}(-x+p_2)/U'(x+p_1) \le \mu^{-1}\eta^{-1}L_2LKe^{\lambda_{12}(-x+p_2)}e^{-\lambda_{21}(x+p_1)} \\ &= \mu^{-1}\eta^{-1}L_2LKe^{\lambda_{12}p_2}e^{-(\lambda_{12}+\lambda_{21})x}e^{-\lambda_{21}p_1} \le \mu^{-1}\eta^{-1}L_2LKe^{\lambda_{12}p_2}. \end{aligned} \tag{3.10}$$

(2-2) A direct computation shows that

$$\begin{aligned} H(x,t) \le& \int_0^\tau \int_{-\infty}^{+\infty} G(y,s)\big[b\big(U(x+p_1(t)-y-cs)+U(-x+p_2(t)+y-cs)\big) \\ &- b(U(x+p_1(t)-y-cs)) - b(U(-x+p_2(t)+y-cs))\big]dyds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\tau \int_{-\infty}^{+\infty} G(y,s)\big[b'\big(U(x+p_1(t)-y-cs)+\theta_1 U(-x+p_2(t)+y-cs)\big) \\
&\quad - b'(\theta_2 U(-x+p_2(t)+y-cs))\big]U(-x+p_2(t)+y-cs)dyds, \qquad (3.11)
\end{aligned}
$$

where $\theta_1, \theta_2 \in (0,1)$. Let $\epsilon := b'(0) - b'(K) > 0$. Noting that $b'(u) = b'(K)$ for $u \in [K, 2K]$, there exists a $\delta > 0$ such that

$$
b'(u) < b'(0) - \epsilon/2, \quad \text{for all } u \in [K-\delta/2, 2K]. \qquad (3.12)
$$

By Lemma 2.5, we have

$$
\lim_{\xi\to-\infty} U'(\xi)/U(\xi) = \lambda_{12} \quad \text{and} \quad \lim_{\xi\to+\infty} U'(\xi)/U(\xi) = 0.
$$

Then, we can choose $\beta > 0$ such that

$$
\frac{d}{d\xi}\Big[U(\xi)e^{-\beta\xi}\Big] = e^{-\beta\xi}U(\xi)\left[\frac{U'(\xi)}{U(\xi)} - \beta\right] \le 0, \quad \forall \xi \in \mathbb{R},
$$

that is, $U(\xi)e^{-\beta\xi}$ is decreasing in $\mathbb{R}$. Noting that

$$
\lim_{\xi\to-\infty} U(\xi)/(G\star U)(\xi) = 1/\mathcal{G}(\lambda_{12}) \quad \text{and} \quad p_i(-\infty) = -\infty, \; i = 1, 2,
$$

then there exists a $T < 0$ such that

$$
U(-x+p_2(t)) \le \frac{2}{\mathcal{G}(\lambda_{12})}(G\star U)(-x+p_2(t)), \quad \text{for } x \ge -p_1(t) \text{ and } t \le T.
$$

By assumption (G), we can choose $B > c\tau$ such that

$$
\frac{2L_1}{\mathcal{G}(\lambda_{12})}\left\{\int_0^\tau \int_{-\infty}^{-B} + \int_0^\tau \int_B^{+\infty} e^{\beta(y-cs)}\right\} G(y,s)dyds \le \frac{\epsilon}{2}.
$$

Thus, for $x \ge -p_1(t)$ and $t \le T$, we have

$$
\begin{aligned}
&\left\{\int_0^\tau \int_{-\infty}^{-B} + \int_0^\tau \int_B^{+\infty}\right\} G(y,s)b'\big(U(x+p_1(t)-y-cs)+\theta_1 U(-x+p_2(t)+y-cs)\big) \\
&\qquad \times U(-x+p_2(t)+y-cs)dyds \\
&\le L_1 \left\{\int_0^\tau \int_{-\infty}^{-B} + \int_0^\tau \int_B^{+\infty}\right\} G(y,s)U(-x+p_2(t)+y-cs)dyds \\
&\le L_1 \left\{\int_0^\tau \int_{-\infty}^{-B} + \int_0^\tau \int_B^{+\infty} e^{\beta(y-cs)}\right\} G(y,s)dydsU(-x+p_2(t)) \\
&\le \frac{2L_1}{\mathcal{G}(\lambda_{12})}\left\{\int_0^\tau \int_{-\infty}^{-B} + \int_0^\tau \int_B^{+\infty} e^{\beta(y-cs)}\right\} G(y,s)dyds(G\star U)(-x+p_2(t)) \\
&\le \frac{\epsilon}{2}\big(G\star U\big)(-x+p_2(t)). \qquad (3.13)
\end{aligned}
$$

Since $U(+\infty) = K$, we may assume $U(\xi) > K - \delta/2$ for $\xi \ge -B - c\tau$ by translations if necessary. If $x \ge -p_1(t)$ and $t \le T$, then it follows from (3.12) and (3.13) that

$$
\begin{aligned}
&\int_0^\tau \int_{-\infty}^{+\infty} G(y,s)b'\big(U(x+p_1(t)-y-cs) \\
&\qquad + \theta_1 U(-x+p_2(t)+y-cs)\big)U(-x+p_2(t)+y-cs)dyds
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ \int_0^\tau \int_{-\infty}^{-B} + \int_0^\tau \int_B^{+\infty} + \int_0^\tau \int_{-B}^{B} \right\} G(y,s) b'\big(U(x + p_1(t) - y - cs) \\
&\qquad + \theta_1 U(-x + p_2(t) + y - cs)\big) U(-x + p_2(t) + y - cs) dy ds \\
&\le \frac{\epsilon}{2} \big(G \star U\big)(-x + p_2(t)) + \left(b'(0) - \frac{\epsilon}{2}\right) \big(G \star U\big)(-x + p_2(t)) \\
&= b'(0) \big(G \star U\big)(-x + p_2(t)).
\end{aligned} \tag{3.14}
$$

Using (3.11) and (3.14), for $x \ge -p_1(t)$ and $t \le T$, we conclude that

$$
\begin{aligned}
H(x,t) &\le \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) \big[b'(0) - b'(\theta_2 U(-x + p_2(t) + y - cs))\big] \\
&\qquad \times U(-x + p_2(t) + y - cs) dy ds \\
&\le L_2 \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) U^2(-x + p_2(t) + y - cs) dy ds \\
&= L_2 \big(G \star U^2\big)(-x + p_2(t)).
\end{aligned} \tag{3.15}
$$

Therefore, by (3.1), (3.2) and (3.15), we have

$$
R(x,t) \le \frac{L_2\big(G \star U^2\big)(-x + p_2(t))}{U'(-x + p_2)} \le \frac{L_2 L^2 e^{2\lambda_{12}(-x+p_2)}}{\mu k e^{\lambda_{12}(-x+p_2)}} \le \mu^{-1} k^{-1} L_2 L^2 e^{\lambda_{12} p_2}. \tag{3.16}
$$

(3) Similar to the discussion of case (2), we can also derive

$$
R(x,t) \le \max\left\{\mu^{-1} k^{-1} L_2 L^2,\ \mu^{-1} \eta^{-1} L_2 L K\right\} e^{\lambda_{12} p_2}. \tag{3.17}
$$

Thus, combining (3.9), (3.10), (3.16) and (3.17), we have $\mathcal{F}(\overline{u})(x,t) \ge 0$, that is $\overline{u}(x,t)$ is a supersolution of (1.2) on $\mathbb{R} \times (-\infty, T)$. This completes the proof. □

Moreover, we have the following subsolution of (1.2).

Lemma 3.3 *For any* $(\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$*, the function* $\underline{u}(x,t)$ *defined by*

$$
\underline{u}(x,t) := \max\left\{\Phi^+_{\omega_1}(x,t), \Phi^-_{\omega_2}(x,t)\right\}
$$

is a subsolution of (1.2) *on* $\mathbb{R} \times (-\infty, +\infty)$.

Proof The proof is obvious. We omit it here. □

4 Proof of the main result

Based on the construction of sub- and supersolutions of (1.2), we first prove the assertions of Theorem 1.1 for the case $(\theta_1, \theta_2) = (\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$. Then, we improve the results to any $(\theta_1, \theta_2) \in \mathbb{R}^2$.

4.1 Entire solutions for $(\theta_1, \theta_2) = (\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$

As mentioned in the Introduction, we always assume that (G) and (B1) hold and (1.2) has a traveling wave front $U(x + ct)$ with speed $c > 0$.

4.1.1 Existence of entire solutions

Theorem 4.1 *For any* $(\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$*, there exists an entire solution* $\Phi_{\omega_1,\omega_2}(x,t)$ *of* (1.2) *satisfying the following statements.*

(1) $\partial_t \Phi_{\omega_1,\omega_2}(x,t) > 0$ *and* $0 < \Phi_{\omega_1,\omega_2}(x,t) < K$ *for all* $(x,t) \in \mathbb{R}^2$.
(2) *For any* $(x,t) \in \mathbb{R}^2$, $\Phi_{\omega_1,\omega_2}(x,t)$ *is increasing with respect to* (ω_1, ω_2).
(3) $\lim_{t\to-\infty} \{\sup_{x\le 0} |\Phi_{\omega_1,\omega_2}(x,t) - \Phi^-_{\omega_1}(x,t)| + \sup_{x\ge 0} |\Phi_{\omega_1,\omega_2}(x,t) - \Phi^+_{\omega_2}(x,t)|\} = 0$,
$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |\Phi_{\omega_1,\omega_2}(x,t) - K| = 0$, $\lim_{t\to-\infty} \sup_{|x|\le N_0} \Phi_{\omega_1,\omega_2}(x,t) = 0$, $\forall\, N_0 \in \mathbb{R}$,
$\lim_{|x|\to+\infty} \sup_{t\ge t_0} |\Phi_{\omega_1,\omega_2}(x,t) - K| = 0$, $\forall\, t_0 \in \mathbb{R}$.
(4) *For any* $\omega_1^*, \omega_2^* \in (-\infty, \bar{\omega}]$*, there exists* $(x_0, t_0) \in \mathbb{R}^2$ *depending on* $\omega_1, \omega_2, \omega_1^*, \omega_2^*$ *such that* $\Phi_{\omega_1^*,\omega_2^*}(\cdot,\cdot) = \Phi_{\omega_1,\omega_2}(\cdot + x_0, \cdot + t_0)$ *on* $\mathbb{R}^2$.

Proof For any $(\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$ and $n \in (-T, +\infty) \cap \mathbb{N}$, let $\Phi^n(x,t)$ be the unique solution of the following initial value problem:

$$\begin{cases} \Phi^n_t = D[J * \Phi^n - \Phi^n] - d\Phi^n + (G \star b(\Phi^n))(x,t), & \text{for } x \in \mathbb{R},\ t > -n; \\ \Phi^n(x,s) = \underline{u}(x,s), & \text{for } x \in \mathbb{R},\ s \in [-n-\tau, -n]. \end{cases} \tag{4.1}$$

By Lemmas 2.2, 3.2 and 3.3, we have

$$\begin{array}{ll} & \underline{u}(x,t) \le \Phi^n(x,t) \le \Phi^{n+1}(x,t) \le \overline{u}(x,t), \quad \text{for } x \in \mathbb{R},\ -n \le t < T, \\ \text{and} & \underline{u}(x,t) \le \Phi^n(x,t) \le K, \quad \text{for } x \in \mathbb{R},\ t > -n. \end{array} \tag{4.2}$$

Then, there exists a function $\Phi(x,t)$ such that $\lim_{n\to\infty} \Phi^n(x,t) = \Phi(x,t)$ for any $(x,t) \in \mathbb{R}^2$. Moreover, for any given $t_0 \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that $t_0 > -n$ and $\Phi^n(x,t)$ satisfies

$$\Phi^n(t)(x) = T(t-t_0)\Phi^n(t_0)(x) + \int_{t_0}^t T(t-r)F\big[(\Phi^n)^r\big](x)dr,$$

where $T(t)$ and $F[\cdot]$ are defined as in Sect. 2.1. By Lebesgue's dominated convergence theorem, we obtain

$$\Phi(t)(x) = T(t-t_0)\Phi(t_0)(x) + \int_{t_0}^t T(t-r)F\big[(\Phi)^r\big](x)dr.$$

This implies that $\Phi(x,t)$ is continuous and differentiable with respect to t. In addition, one can show that

$$\Phi_t = D\big[J * \Phi - \Phi\big] - d\Phi(x,t) + \big(G \star b(\Phi)\big)(x,t).$$

Therefore, $\Phi_{\omega_1,\omega_2}(x,t) := \Phi(x,t)$ is an entire solution of (1.2). Now we prove the assertions of (1)–(4) in the sequel.

(1) By (4.2), it's obvious that

$$\begin{array}{ll} & \underline{u}(x,t) \le \Phi_{\omega_1,\omega_2}(x,t) \le \overline{u}(x,t), \quad \text{for all } x \in \mathbb{R},\ t < T, \\ \text{and} & \underline{u}(x,t) \le \Phi_{\omega_1,\omega_2}(x,t) \le K, \quad \text{for all } (x,t) \in \mathbb{R}^2. \end{array} \tag{4.3}$$

Clearly, $\Phi_{\omega_1,\omega_2}(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2$. Since

$$\Phi^n(x,t) \ge \underline{u}(x,t) \ge \underline{u}(x,s) = \Phi^n(x,s)$$

for $(x,t) \in \mathbb{R} \times [-n, +\infty)$ and $s \in [-n-\tau, -n]$, by Lemma 2.2, we have $\partial_t \Phi^n(x,t) \ge 0$ for $(x,t) \in \mathbb{R} \times (-n, +\infty)$. This yields to $\partial_t \Phi_{\omega_1,\omega_2}(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^2$.

Moreover, from (4.1), we have

$$\begin{aligned}\partial_{tt}\Phi_{\omega_1,\omega_2} =&D\big[J*\partial_t\Phi_{\omega_1,\omega_2}-\partial_t\Phi_{\omega_1,\omega_2}\big]-d\partial_t\Phi_{\omega_1,\omega_2}\\ &+\int_0^\tau\int_{-\infty}^{+\infty}G(x-y,s)b'\big(\Phi_{\omega_1,\omega_2}(y,t-s)\big)\partial_t\Phi_{\omega_1,\omega_2}(y,t-s)dyds\\ \geq&-(D+d)\partial_t\Phi_{\omega_1,\omega_2},\quad \text{for } (x,t)\in\mathbb{R}^2,\end{aligned}$$

which implies

$$\partial_t\Phi_{\omega_1,\omega_2}(x,t)\geq\partial_t\Phi_{\omega_1,\omega_2}(x,s)e^{-(D+d)(t-s)},\ \ \forall s<t. \tag{4.4}$$

Suppose that the first part of (1) is false, then there exists a $(x_0,t_0)\in\mathbb{R}^2$ such that $\partial_t\Phi_{\omega_1,\omega_2}(x_0,t_0)=0$ and it follows from (4.4) that $\partial_t\Phi_{\omega_1,\omega_2}(x_0,t)=0$ for all $t\leq t_0$. Hence $\Phi_{\omega_1,\omega_2}(x_0,t)=\Phi_{\omega_1,\omega_2}(x_0,t_0)$ for all $t\leq t_0$, which implies that

$$\lim_{t\to-\infty}\Phi_{\omega_1,\omega_2}(x_0,t)=\Phi_{\omega_1,\omega_2}(x_0,t_0).$$

On the other hand, from (4.3), we have

$$\lim_{t\to-\infty}\Phi_{\omega_1,\omega_2}(x_0,t)=0\ \text{ and } \Phi_{\omega_1,\omega_2}(x_0,t_0)>0.$$

This contradiction yields that $\partial_t\Phi_{\omega_1,\omega_2}(x,t)>0$ for all $(x,t)\in\mathbb{R}^2$. Moreover, we can show that $\Phi_{\omega_1,\omega_2}(x,t)<K$ for all $(x,t)\in\mathbb{R}^2$.

(2) Noting that $U'(z)>0$ and $0<U(z)<1$ for $z\in\mathbb{R}$, then it follows that $\Phi_{\omega_1,\omega_2}(x,t)$ is increasing with respect to (ω_1,ω_2).

(3) & (4) Using (4.3), the proofs of these parts are straightforward and thus omitted. The proof is complete. ◻

4.1.2 Uniqueness and stability of entire solutions

In order to prove the uniqueness, stability and continuous dependence on the shift parameters ω_1,ω_2 of the entire solution $\Phi_{\omega_1,\omega_2}(x,t)$, we construct different sub-supersolution pairs of (1.2) to trap the entire solution.

Lemma 4.2 *There exist $\delta_0\in(0,K)$, $\rho_0>0$ and $\sigma_0>0$ such that for any $\gamma\in\mathbb{R}$, $\delta\in(0,\delta_0]$ and $\sigma\geq\sigma_0$, the functions $u^\pm(x,t)$ defined by*

$$u^\pm(x,t)=\Phi_{\omega_1,\omega_2}\left(x,t+\gamma\pm\sigma\delta\big(1-e^{-\rho_0 t}\big)\right)\pm\delta e^{-\rho_0 t}$$

constitute a pair of super- and subsolution of (1.2) on $[0,+\infty)$.

Proof We only prove that $u^+(x,t)$ is a supersolution of (1.2) on $[0,+\infty)$. Following the same arguments, we can also show that $u^-(x,t)$ is a subsolution. Since

$$\lim_{(\rho,\bar\omega)\to(0,b'(0))}\big[-\rho+d-e^{\rho\tau}\bar\omega\big]=d-b'(0)>0,$$
and
$$\lim_{(\rho,\bar\omega)\to(0,b'(K))}\big[-\rho+d-e^{\rho\tau}\bar\omega\big]=d-b'(K)>0,$$

we can fix $\rho_0>0$ and $0<\delta_1\ll K$ such that

$$\begin{aligned}&-\rho_0+d-e^{\rho_0\tau}\bar\omega>0,\quad \text{for } \bar\omega\in\big[b'(0)-\delta_1,b'(0)+\delta_1\big],\\ \text{and}\quad&-\rho_0+d-e^{\rho_0\tau}\bar\omega>0,\quad \text{for } \bar\omega\in\big[b'(K)-\delta_1,b'(K)+\delta_1\big].\end{aligned} \tag{4.5}$$

Now we choose $\delta_0 \in (0, \delta_1)$ and $\nu \in (0, K)$ such that $\delta_0 e^{\rho_0 \tau} L_2 \le \delta_1/4$,

$$\text{and} \quad \begin{array}{ll} b'(u) \in \left[b'(0) - \delta_1/2, b'(0) + \delta_1/2\right], & \text{for } u \in [0, \nu], \\ b'(u) \in \left[b'(K) - \delta_1/2, b'(K) + \delta_1/2\right], & \text{for } u \in [K - \nu, K + \nu]. \end{array} \tag{4.6}$$

By assumption (G), there exists an $M > 0$ such that

$$L_1 \left\{ \int_0^\tau \int_{-\infty}^{-M} + \int_0^\tau \int_M^{+\infty} \right\} G(y, s) dyds \in (0, \delta_1/4), \tag{4.7}$$

$$\int_0^\tau \int_{-M}^{M} G(y, s) dyds \ge \left(b'(K) - \frac{3}{4}\delta_1\right) \Big/ \left(b'(K) - \frac{1}{2}\delta_1\right), \quad \text{if } b'(K) > 0, \tag{4.8}$$

$$\int_0^\tau \int_{-M}^{M} G(y, s) dyds \ge \left(b'(0) - \frac{3}{4}\delta_1\right) \Big/ \left(b'(0) - \frac{1}{2}\delta_1\right), \quad \text{if } b'(0) > 0. \tag{4.9}$$

Take $X > 0$ such that

$$\text{and} \quad \begin{array}{ll} U(x) \in (0, \nu/4), & \text{for } x \le -X + M \\ U(x) \in (K - \nu/4, K + \nu/4), & \text{for } x \ge X - M - c\tau. \end{array} \tag{4.10}$$

Since

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| \int_0^\tau \int_{-\infty}^{+\infty} G(y, s) b'\big(\Phi_{\omega_1, \omega_2}(x - y, t - s)\big) dyds - b'(K) \right| = 0,$$

there exists a $T_1 > \tau$ such that

$$\int_0^\tau \int_{-\infty}^{+\infty} G(y, s) b'\big(\Phi_{\omega_1, \omega_2}(x - y, t - s)\big) dyds \in \left[b'(K) - \delta_1/2, b'(K) + \delta_1/2\right], \tag{4.11}$$

for any $t > T_1$ and $x \in \mathbb{R}$. In view of $\lim_{t \to -\infty} [p_i(t) - ct - \omega_i] = 0$, $i = 1, 2$, we can take $T_2 \le T$, where $T < 0$ is defined in Lemma 3.2, such that

$$2 \max_{i=1,2} |p_i(t) - ct - \omega_i| \max_{x \in \mathbb{R}} U'(x) \in (0, \nu/4), \quad \text{for } t \le T_2 - \tau. \tag{4.12}$$

Letting $\kappa_1 := \min_{|x| \le X} U'(x) > 0$, then there exists a $\sigma_1 > 0$ such that

$$\frac{1}{2}\kappa_1 \sigma_1 \rho_0 - \rho_0 + d - e^{\rho_0 \tau} L_1 > 0. \tag{4.13}$$

Set $\Psi(x, t) := \Phi^+_{\omega_1}(x, t) + \Phi^-_{\omega_2}(x, t)$. One can easily show that

$$\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} \left\| \Phi_{\omega_1, \omega_2}(x, \cdot) - \Psi(x, \cdot) \right\|_{C^0((-\infty, t])} = 0.$$

Since $0 < \Phi_{\omega_1, \omega_2}(x, t) < K$, $\forall (x, t) \in \mathbb{R}^2$, one can verify that

$$0 < \partial_t \Phi_{\omega_1, \omega_2}(x, t) \le L_3 := (D + d)K \text{ and } |\partial_{tt} \Phi_{\omega_1, \omega_2}(x, t)| \le L_4 := [2D + d + L_1] L_3,$$

for $(x, t) \in \mathbb{R}^2$. Similarly, we have $|\Psi_{tt}(x, t)| \le 2L_4/c^2$ for $(x, t) \in \mathbb{R}^2$. Then, by the interpolation $\| \cdot \|_{C^1} \le 2\sqrt{\| \cdot \|_{C^0} \| \cdot \|_{C^2}}$, we obtain

$$\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} \left\| \Phi_{\omega_1, \omega_2}(x, \cdot) - \Psi(x, \cdot) \right\|_{C^1((-\infty, t])} = 0.$$

Thus, there exists a $T_3 \le T_2$ such that

$$\sup_{x\in\mathbb{R}} \left\| \Phi_{\omega_1,\omega_2}(x,\cdot) - \Psi(x,\cdot) \right\|_{C^1((-\infty,t])} < \kappa_1/2, \quad \text{for any } t \le T_3. \tag{4.14}$$

Since

$$\lim_{|x|\to+\infty} \max_{t\in[T_3,T_1]} \left| \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) b'\big(\Phi_{\omega_1,\omega_2}(x-y,t-s)\big) dyds - b'(K) \right| = 0,$$

we can take $X_1 > 0$ such that (4.11) holds for any $|x| > X_1$ and $t \in [T_3, T_1]$. In addition, let $\kappa_2 := \min_{|x|\le X_1, t\in[T_3,T_1]} \partial_t \Phi_{\omega_1,\omega_2}(x,t) > 0$ and take $\sigma_0 > \sigma_1$ such that

$$\kappa_2 \sigma_0 \rho_0 - \rho_0 + d - e^{\rho_0 \tau} L_1 > 0. \tag{4.15}$$

Then, for $\gamma \in \mathbb{R}$, $\delta \in (0, \delta_0]$ and $\sigma \ge \sigma_0$, we denote $\xi(t) := t + \gamma + \sigma\delta(1 - e^{-\rho_0 t})$. Clearly, $\xi(t-s) \le \xi(t) - s$ for $t \ge 0$ and $s \in [0, \tau]$. Since $\partial_t \Phi_{\omega_1,\omega_2}(x,t) > 0$ for $(x,t) \in \mathbb{R}^2$ and $|b'(u) - b'(v)| \le L_2|u - v|$ for $u, v \in [0, 2K]$, direct computations show that

$$\begin{aligned}
&\mathcal{F}(u^+)(x,t)\\
&\quad := u_t^+ - D\big[(J * u^+)(x,t) - u^+(x,t)\big] + du^+ - \big(G \star b(u^+)\big)(x,t)\\
&\quad = \partial_t \Phi_{\omega_1,\omega_2}(x,\xi(t))\big(1 + \sigma\delta\rho_0 e^{-\rho_0 t}\big) - \rho_0 \delta e^{-\rho_0 t}\\
&\qquad - D\Big[\big(J * \Phi_{\omega_1,\omega_2}\big)(x,\xi(t)) - \Phi_{\omega_1,\omega_2}(x,\xi(t))\Big]\\
&\qquad + d\Phi_{\omega_1,\omega_2}(x,\xi(t)) + d\delta e^{-\rho_0 t} - G \star b\big(u^+\big)(x,t)\\
&\quad = \delta e^{-\rho_0 t}\big[\sigma\rho_0 \partial_t \Phi_{\omega_1,\omega_2}(x,\xi(t)) - \rho_0 + d\big] + \big(G \star b(\Phi_{\omega_1,\omega_2})\big)(x,\xi(t))\\
&\qquad - \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) b\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t-s)) + \delta e^{-\rho_0(t-s)}\big) dyds\\
&\quad \ge \delta e^{-\rho_0 t}\Big[\sigma\rho_0 \partial_t \Phi_{\omega_1,\omega_2}(x,\xi(t)) - \rho_0 + d\Big]\\
&\qquad + \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) b\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s)\big) dyds\\
&\qquad - \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) b\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s) + \delta e^{-\rho_0(t-s)}\big) dyds\\
&\quad \ge \delta e^{-\rho_0 t}\Big[\sigma\rho_0 \partial_t \Phi_{\omega_1,\omega_2}(x,\xi(t)) - \rho_0 + d\\
&\qquad\quad - e^{\rho_0\tau} \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) b'\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s) + \theta_1 \delta e^{-\rho_0(t-s)}\big) dyds\Big] \qquad (4.16)\\
&\quad \ge \delta e^{-\rho_0 t}\Big\{\sigma\rho_0 \partial_t \Phi_{\omega_1,\omega_2}(x,\xi(t)) - \rho_0 + d - e^{\rho_0\tau}\Big[\delta e^{\rho_0\tau} L_2\\
&\qquad\quad + \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) b'\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s)\big) dyds\Big]\Big\}\\
&\quad \ge \delta e^{-\rho_0 t}\left\{-\rho_0 + d - e^{\rho_0\tau}\left[\frac{\delta_1}{4} + \int_0^\tau \int_{-\infty}^{+\infty} G(y,s) b'\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s)\big) dyds\right]\right\},
\end{aligned} \tag{4.17}$$

where $\theta_1 \in (0, 1)$. Moreover, for $\xi(t) \le T_3$, Lemmas 3.2 and 3.3 imply that

$$\begin{aligned}
&\max\big\{U(x-y+c\xi(t)-cs+\omega_1),\, U(-x+y+c\xi(t)-cs+\omega_2)\big\}\\
&\quad \le \Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s) \le U(x-y+p_1(\xi(t)-s)) + U(-x+y+p_2(\xi(t)-s))
\end{aligned}$$

$$\leq 2 \max_{i=1,2} \left| p_i(\xi(t)-s) - c(\xi(t)-s) - \omega_i \right| \max_{x\in\mathbb{R}} U'(x)$$
$$+ U(x-y+c\xi(t)-cs+\omega_1) + U(-x+y+c\xi(t)-cs+\omega_2). \tag{4.18}$$

Now, we consider the following seven cases.

(1) $\xi(t) > T_1$. Following (4.5), (4.11) and (4.17), we have $\mathcal{F}(u^+)(x,t) \geq 0$.
(2) $\xi(t) \leq T_3$ and $x + c\xi(t) + \omega_1 \geq X$. Then $-x < -X$. It follows from (4.7) and (4.17) that

$$\mathcal{F}(u^+)(x,t) \geq \delta e^{-\rho_0 t}\Big\{ -\rho_0 + d - e^{\rho_0\tau}\Big[\delta_1/2 + \int_0^\tau \int_{-M}^M G(y,s) b'\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s)\big)dyds\Big]\Big\}. \tag{4.19}$$

Moreover, from (4.10), (4.12) and (4.18), we have

$$\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s) \in (K-\nu, K+\nu), \text{ for } y \in [-M,M] \text{ and } s \in [0,\tau].$$

If $b'(K) > 0$, it then follows from (4.6) and (4.8) that

$$\int_0^\tau \int_{-M}^M G(y,s) b'\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s)\big)dyds \in \big[b'(K) - 3\delta_1/4, b'(K) + \delta_1/2\big].$$

Moreover, if $b'(K) = 0$, then

$$\int_0^\tau \int_{-M}^M G(y,s) b'\big(\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s)\big)dyds \in \big[b'(K), b'(K) + \delta_1/2\big].$$

By (4.5) and (4.19), we conclude that $\mathcal{F}(u^+)(x,t) \geq 0$.
(3) $\xi(t) \leq T_3$ and $-x+c\xi(t)+\omega_2 \geq X$. Similar to case (2), we can prove that $\mathcal{F}(u^+)(x,t) \geq 0$.
(4) $\xi(t) \leq T_3$, $x + c\xi(t) + \omega_1 \leq -X$ and $-x + c\xi(t) + \omega_2 \leq -X$. Using (4.10), (4.12) and (4.18), we have

$$\Phi_{\omega_1,\omega_2}(x-y,\xi(t)-s) \in (0,\nu), \quad \text{for } y \in [-M,M] \text{ and } s \in [0,\tau].$$

Similar to case (2), we can show that $\mathcal{F}(u^+)(x,t) \geq 0$ by using (4.5), (4.6), (4.9) and (4.19).
(5) $\xi(t) \leq T_3$ and $x + c\xi(t) + \omega_1 \in [-X, X]$ or $-x + c\xi(t) + \omega_2 \in [-X, X]$. According to (4.14), we have

$$\partial_t \Phi_{\omega_1,\omega_2}(x,\xi(t)) \geq \partial_t \Psi(x,\xi(t)) - \frac{\kappa_1}{2}$$
$$= U'(x + c\xi(t) + \omega_1) + U'(-x + c\xi(t) + \omega_2) - \frac{\kappa_1}{2} \geq \frac{\kappa_1}{2}.$$

Then, by (4.13) and (4.16), we obtain $\mathcal{F}(u^+)(x,t) \geq 0$.
(6) $T_3 \leq \xi(t) \leq T_1$ and $|x| > X_1$. Noting that (4.11) holds for any $|x| > X_1$ and $t \in [T_3, T_1]$, it follows from (4.5), (4.11) and (4.17) that $\mathcal{F}(u^+)(x,t) \geq 0$.
(7) $T_3 \leq \xi(t) \leq T_1$ and $|x| \leq X_1$. Following (4.15) and (4.16), it must be $\mathcal{F}(u^+)(x,t) \geq 0$.

Summing up the above seven cases, we see that $\mathcal{F}(u^+)(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times [0,+\infty)$, i.e. $u^+(x,t)$ is a supersolution of (1.2) on $[0,+\infty)$. The proof is complete. □

Lemma 4.3 *There exist $\delta_* > 0$, $\rho_* > 0$ and $\sigma_* > 0$ such that for any $\gamma \in \mathbb{R}$, $\delta \in (0, \delta_*]$ and $\sigma \geq \sigma_*$, the functions $V^{\pm}(x,t)$ defined by*

$$V^{\pm}(x,t) := U\Big(-x + ct + \gamma \pm \sigma\delta\big(1 - e^{-\rho_* t}\big)\Big) \pm \delta e^{-\rho_* t}$$

constitute a pair of super- and subsolutions of (1.2) *on* $[0, +\infty)$.

Proof The proof is similar to that of Lemma 4.2, we omit it. □

Let σ_0, ρ_0, δ_0 and σ_*, ρ_*, δ_* be the positive constants given in Lemmas 4.2 and 4.3, respectively. We have the following results.

Theorem 4.4 *Let $\Phi_{\omega_1,\omega_2}(x,t)$ be the entire solution of* (1.2) *decided in* Theorem 4.1*, then the following statements hold.*

(1) *If $\widetilde{\Phi}(x,t)$ is an entire solution of* (1.2) *satisfying the first property of* (3) *of* Theorem 4.1*, then $\widetilde{\Phi}(x,t) = \Phi_{\omega_1,\omega_2}(x,t)$.*
(2) *For any $(x,t) \in \mathbb{R}^2$, $\Phi_{\omega_1,\omega_2}(x,t)$ converges to $\begin{cases} \Phi^+_{\omega_2}(x,t) \text{ as } \omega_1 \to -\infty; \\ \Phi^-_{\omega_1}(x,t) \text{ as } \omega_2 \to -\infty. \end{cases}$*
(3) *$\Phi_{\omega_1,\omega_2}(x,t)$ depends continuously on $(\omega_1,\omega_2) \in (-\infty, \bar{\omega}]^2$.*
(4) *$\Phi_{\omega_1,\omega_2}(x,t)$ is Liapunov stable in the sense of part* (7) *of* Theorem 1.1.

Proof (1) Suppose that $\widetilde{\Phi}(x,t)$ is an entire solution of (1.2) satisfying the first property of (3) of Theorem 4.1. Given any $t_1 < 0$, we define

$$\eta := \sup_{x\in\mathbb{R}} \big\|\widetilde{\Phi}(x,\cdot + t_1) - \Phi_{\omega_1,\omega_2}(x,\cdot + t_1)\big\|_{L^\infty[-\tau,0]}.$$

It suffices to show that $\eta = 0$. By our assumptions, for any $\delta \in (0, \delta_0]$, there exist a $t_2 < t_1 - \tau$ such that $\sup_{x\in\mathbb{R}} \|\widetilde{\Phi}(x,\cdot + t_2) - \Phi_{\omega_1,\omega_2}(x,\cdot + t_2)\|_{L^\infty[-\tau,0]} < \delta$. Hence,

$$\begin{aligned}&\Phi_{\omega_1,\omega_2}\big(x, s + t_2 - \sigma_0\delta\left(e^{\rho_0\tau} - e^{-\rho_0 s}\right)\big) - \delta e^{-\rho_0 s}\\ &\quad\leq \widetilde{\Phi}(x, s + t_2) \leq \Phi_{\omega_1,\omega_2}\big(x, s + t_2 + \sigma_0\delta\left(e^{\rho_0\tau} - e^{-\rho_0 s}\right)\big) + \delta e^{-\rho_0 s},\end{aligned}$$

for $x \in \mathbb{R}$, $s \in [-\tau, 0]$. In addition, by Lemmas 2.2 and 4.2, we have

$$\begin{aligned}&\Phi_{\omega_1,\omega_2}\big(x, t + t_2 - \sigma_0\delta\left(e^{\rho_0\tau} - e^{-\rho_0 t}\right)\big) - \delta e^{-\rho_0 t}\\ &\quad\leq \widetilde{\Phi}(x, t + t_2) \leq \Phi_{\omega_1,\omega_2}\big(x, t + t_2 + \sigma_0\delta\left(e^{\rho_0\tau} - e^{-\rho_0 t}\right)\big) + \delta e^{-\rho_0 t}\end{aligned}$$

for $x \in \mathbb{R}$ and $t \geq 0$. Noting that $s + t_1 - t_2 > 0$, then we obtain

$$\begin{aligned}&\Phi_{\omega_1,\omega_2}\big(x, s + t_1 - \sigma_0\delta(e^{\rho_0\tau} - e^{\rho_0(s+t_1-t_2)})\big) - \delta\\ &\quad\leq \widetilde{\Phi}(x, s + t_1) \leq \Phi_{\omega_1,\omega_2}\big(x, s + t_1 + \sigma_0\delta(e^{\rho_0\tau} - e^{\rho_0(s+t_1-t_2)})\big) + \delta.\end{aligned}$$

Therefore, for all $x \in \mathbb{R}$, it follows that

$$\Phi_{\omega_1,\omega_2}\big(x, s + t_1 - \sigma_0\delta e^{\rho_0\tau}\big) - \delta \leq \widetilde{\Phi}(x, s + t_1) \leq \Phi_{\omega_1,\omega_2}\big(x, s + t_1 + \sigma_0\delta e^{\rho_0\tau}\big) + \delta. \tag{4.20}$$

On the other hand, since $0 < \partial_t\Phi_{\omega_1,\omega_2}(x,t) \leq M_3 := (D + d)K$ for any $(x,t) \in \mathbb{R}^2$, then (4.20) implies that

$$\sup_{x\in\mathbb{R}} \big\|\widetilde{\Phi}(x, s + t_1) - \Phi_{\omega_1,\omega_2}(x, s + t_1)\big\|_{L^\infty[-\tau,0]} \leq \big(1 + M_3\sigma_0 e^{\rho_0\tau}\big)\,\delta.$$

Thus we have $\eta \leq (1 + M_3\sigma_0 e^{\rho_0\tau})\delta$. By the arbitrariness of δ, we see that $\eta = 0$.

(2) Let $\{(\omega_1, \omega_2^k)\}_{k\in\mathbb{N}}$ be a sequence satisfying $(\omega_1, \omega_2^k) \in (-\infty, \bar{\omega}]^2$, $\omega_2^{k+1} < \omega_2^k < \omega_1$ and $\omega_2^k \to -\infty$ as $k \to -\infty$. According to Theorem 4.1, for each $k \in \mathbb{N}$, there exist an entire solution $\Phi_{\omega_1,\omega_2^k}(x,t)$ of (1.2) such that for any $x \in \mathbb{R}$ and $t < T$, there holds

$$\begin{aligned}\Phi^-_{\omega_1}(x,t) &\le \max\left\{\Phi^-_{\omega_1}(x,t), \Phi^+_{\omega_2^k}(x,t)\right\} \le \Phi_{\omega_1,\omega_2^{k+1}}(x,t) \le \Phi_{\omega_1,\omega_2^k}(x,t)\\ &\le U\left(-x + p_1\big(t;\omega_1,\omega_2^k\big)\right) + U\left(x + p_2\big(t;\omega_1,\omega_2^k\big)\right)\\ &= U\left(-x + \widetilde{p}_1(t;\omega_1)\right) + U\left(x + \widetilde{p}_2\big(t;\omega_1,\omega_2^k\big)\right). \end{aligned} \tag{4.21}$$

By the monotonicity of $\Phi_{\omega_1,\omega_2^k}(x,t)$ on k, there exists a function $\Psi(x,t)$ such that $\lim_{k\to+\infty} \Phi_{\omega_1,\omega_2^k}(x,t) = \Psi(x,t)$. It then follows from (4.21) that

$$\Phi^-_{\omega_1}(x,t) \le \Psi(x,t) \le U\big(-x + \widetilde{p}_1(t;\omega_1)\big), \quad \text{for any } x \in \mathbb{R} \text{ and } t < T. \tag{4.22}$$

Moreover, given any $t_3 < T$, we define

$$\bar{\eta} := \sup_{x\in\mathbb{R}} \left\|\Psi(x, t_3 + \cdot) - U(-x + c(t_3 + \cdot) + \omega_1)\right\|_{L^\infty[-\tau,0]}.$$

For any $\delta \in (0, \delta_*]$, since $\widetilde{p}_1(t;\omega_1) - ct - \omega_1 \to 0$ as $t \to -\infty$, it follows from (4.22) that there exists $t_4 < t_3 - \tau$ such that for any $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, there holds

$$\Phi^-_{\omega_1}(x, s+t_4) \le \Psi(x, s+t_4) \le U\left(-x + c(s+t_4) + \omega_1 + \sigma\delta\big(e^{\rho_*\tau} - e^{-\rho_* s}\big)\right) + \delta e^{-\rho_* s}.$$

By comparison principle and Lemma 4.3, we have

$$\Phi^-_{\omega_1}(x,t) \le \Psi(x,t) \le U\left(-x + ct + \omega_1 + \sigma\delta\left(e^{\rho_*\tau} - e^{-\rho_*(t-t_4)}\right)\right) + \delta e^{-\rho_*(t-t_4)},$$

for any $x \in \mathbb{R}$ and $t > t_4$. Then it follows that

$$\begin{aligned}\Phi^-_{\omega_1}(x, t_3+s) \le &\Psi(x, t_3+s)\\ \le &U\left(-x + c(t_3+s) + \omega_1 + \sigma\delta\left(e^{\rho_*\tau} - e^{-\rho_*(t_3+s-t_4)}\right)\right) + \delta e^{-\rho_*(t_3+s-t_4)}\\ \le &U\left(-x + c(t_3+s) + \omega_1 + \sigma\delta e^{\rho_*\tau}\right) + \delta, \quad \text{for } x \in \mathbb{R},\end{aligned}$$

which implies that

$$\sup_{x\in\mathbb{R}} \left\|\Psi(x, (t_3+\cdot)) - \Phi^-_{\omega_1}(x, t_3+\cdot)\right\|_{L^\infty[-\tau,0]} \le \delta + \sigma\delta e^{\rho_*\tau} \max_{z\in\mathbb{R}} U'(z).$$

According to the arbitrariness of δ, we obtain $\bar{\eta} = 0$. Thus, $\Psi(x,t) = \Phi^-_{\omega_1}(x,t)$ for any $(x,t) \in \mathbb{R}^2$. Since $\Phi_{\omega_1,\omega_2}(x,t)$ is increasing with respect to ω_2, we obtain

$$\lim_{\omega_2\to-\infty} \Phi_{\omega_1,\omega_2}(x,t) = \Phi^-_{\omega_1}(x,t), \; for\ any\ (x,t) \in \mathbb{R}^2.$$

Similarly, we can show the other assertion of this part.

(3) Given any $(\omega_1^0, \omega_2^0) \in (-\infty, \bar{\omega}]^2$, we choose two sequences $\{(\omega^k_{\pm,1}, \omega^k_{\pm,2})\}$ with $(\omega^k_{\pm,1}, \omega^k_{\pm,2}) \in \mathbb{R}^2$ such that $\lim_{k\to+\infty} (\omega^k_{\pm,1}, \omega^k_{\pm,2}) \to (\omega_1^0, \omega_2^0)$ and

$$\left(\omega^k_{-,1}, \omega^k_{-,2}\right) \le \left(\omega^{k+1}_{-,1}, \omega^{k+1}_{-,2}\right) < \left(\omega^0_1, \omega^0_2\right) < \left(\omega^{k+1}_{+,1}, \omega^{k+1}_{+,2}\right) \le \left(\omega^k_{+,1}, \omega^k_{+,2}\right), \quad \forall k \in \mathbb{N}.$$

From Theorem 4.1, there exist entire solutions $\Phi_{\omega_1^0,\omega_2^0}(x,t)$ and $\Phi_{\omega_{\pm,1}^k,\omega_{\pm,2}^k}(x,t)$ of (1.2) for $k \in \mathbb{N}$ which satisfy

$$0 \le \Phi_{\omega_{-,1}^{k+1},\omega_{-,2}^{k+1}} \le \Phi_{\omega_{-,1}^{k},\omega_{-,2}^{k}} \le \Phi_{\omega_1^0,\omega_2^0} \le \Phi_{\omega_{+,1}^{k},\omega_{+,2}^{k}} \le \Phi_{\omega_{+,1}^{k+1},\omega_{+,2}^{k+1}} \le K$$

for $(x,t) \in \mathbb{R}^2$. Then, there exist $\Phi_\pm(x,t)$ such that $\lim_{k\to\infty} \Phi_{\omega_{\pm,1}^k,\omega_{\pm,2}^k}(x,t) = \Phi_\pm(x,t)$ and $\Phi_\pm(x,t)$ are entire solutions of (1.2). Since $0 < U'(z) \le M_3/c$ for $z \in \mathbb{R}$, $0 < p_i(t;\omega_{+,1}^k,\omega_{+,2}^k) - c_i t - \omega_{+,i}^k \le R_0 e^{c\lambda_{12}t}$ for $t \le 0$ and

$$\begin{aligned}\max\left\{\Phi^-_{\omega_1^0}(x,t), \Phi^+_{\omega_2^0}(x,t)\right\} \le& \max\left\{\Phi^-_{\omega_{+,1}^k}(x,t), \Phi^+_{\omega_{+,2}^k}(x,t)\right\} \le \Phi_{\omega_{+,1}^k,\omega_{+,2}^k}(x,t)\\ \le& U\big(-x + p_1\big(t;\omega_{+,1}^k,\omega_{+,2}^k\big)\big) + U\big(x + p_2\big(t;\omega_{+,1}^k,\omega_{+,2}^k\big)\big)\end{aligned}$$

for any $t < T$, $x \in \mathbb{R}$ and $k \in \mathbb{N}$, we can easily show that

$$\lim_{t\to-\infty}\left\{\sup_{x\le 0}|\Phi_+(x,t) - \Phi^-_{\omega_1^0}(x,t)| + \sup_{x\ge 0}|\Phi_+(x,t) - \Phi^+_{\omega_2^0}(x,t)|\right\} = 0.$$

By the uniqueness of entire solutions, we have $\Phi_+(x,t) = \Phi_{\omega_1^0,\omega_2^0}(x,t)$. Similarly, one can prove that $\Phi_-(x,t) = \Phi_{\omega_1^0,\omega_2^0}(x,t)$. Hence, we can easily show that $\Phi_{\omega_1,\omega_2}(x,t)$ depends continuously on (ω_1,ω_2).

(4) Given any $\epsilon > 0$, let us define $\tilde{\delta} := \tilde{\delta}(\epsilon) = \epsilon/(2M_3) > 0$. Then, for all $|z| \le \tilde{\delta}$, it follows that

$$\sup_{x,t\in\mathbb{R}}\left|\Phi_{\omega_1,\omega_2}(x,t) - \Phi_{\omega_1,\omega_2}(x,t+z)\right| \le \sup_{x,t\in\mathbb{R}}\left|\partial_t\Phi_{\omega_1,\omega_2}(x,t)\right||z| \le M_3\tilde{\delta} \le \epsilon/2. \tag{4.23}$$

Let $\bar{\delta} := \min\{\epsilon/2, \tilde{\delta}/(\sigma_0 e^{\rho_0\tau}), \delta_0\}$. For any $\varphi \in \mathcal{C}_{[0,K]}$ satisfying

$$\sup_{x\in\mathbb{R}}\left\|\varphi(x,\cdot) - \Phi_{\omega_1,\omega_2}(x+x_0,\cdot+t_0)\right\|_{L^\infty[-\tau,0]} < \bar{\delta},$$

we have

$$\begin{aligned}&\Phi_{\omega_1,\omega_2}\big(x+x_0, s+t_0-\sigma_0\bar{\delta}\left(e^{\rho_0\tau}-e^{-\rho_0 s}\right)\big) - \bar{\delta}e^{-\rho_0 s}\\ &\quad\le \varphi(x,s) \le \Phi_{\omega_1,\omega_2}\big(x+x_0, s+t_0+\sigma_0\bar{\delta}\left(e^{\rho_0\tau}-e^{-\rho_0 s}\right)\big) + \bar{\delta}e^{-\rho_0 s}\end{aligned}$$

for $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. By comparison principle and Lemma 4.2, we obtain

$$\begin{aligned}&\Phi_{\omega_1,\omega_2}\big(x+x_0, t+t_0-\sigma_0\bar{\delta}\left(e^{\rho_0\tau}-e^{-\rho_0 t}\right)\big) - \bar{\delta}e^{-\rho_0 t}\\ &\quad\le u(x,t;\varphi) \le \Phi_{\omega_1,\omega_2}\big(x+x_0, t+t_0+\sigma_0\bar{\delta}\left(e^{\rho_0\tau}-e^{-\rho_0 t}\right)\big) + \bar{\delta}e^{-\rho_0 t}\end{aligned} \tag{4.24}$$

for $x \in \mathbb{R}$ and $t \ge 0$. It then follows form (4.23) and (4.24) that

$$\left|u(x,t;\varphi) - \Phi_{\omega_1,\omega_2}(x+x_0,t+t_0)\right| \le M_3\sigma_0\bar{\delta}e^{\rho_0\tau} + \bar{\delta} \le \epsilon, \tag{4.25}$$

for all $x \in \mathbb{R}$ and $t \ge 0$. The proof is complete. □

Based on the results of the previous subsection, we are ready to prove Theorem 1.1.

4.2 Proof of Theorem 1.1

For any $\theta_1, \theta_2 \in \mathbb{R}$, there exists a $T_0 < 0$ such that $cT_0 + \theta_1 < \bar{\omega}$ and $cT_0 + \theta_2 < \bar{\omega}$. Take

$$\omega_1 := cT_0 + \theta_1 \text{ and } \omega_2 := cT_0 + \theta_2.$$

Clearly, $(\omega_1, \omega_2) \in (-\infty, \bar{\omega}]^2$. Therefore, there exists an entire solution $\Phi_{\omega_1,\omega_2}(x,t)$ satisfying the assertions of Theorems 4.1 and 4.4. Let

$$\Phi_{\theta_1,\theta_2}(x,t) := \Phi_{\omega_1,\omega_2}(x, t - T_0),$$

then $\Phi_{\theta_1,\theta_2}(x,t)$ is also an entire solution of (1.2) which satisfies the assertions of Theorem 1.1.

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