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Stability of Concatenated Traveling Waves

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Abstract We consider a reaction–diffusion equation in one space dimension whose initial condition is approximately a sequence of widely separated traveling waves with increasing velocity, each of which is individually asymptotically stable. We show that the sequence of traveling waves is itself asymptotically stable: as $t \to \infty$, the solution approaches the concatenated wave pattern, with different shifts of each wave allowed. Essentially the same result was previously proved by Wright (J Dyn Differ Equ 21:315–328, 2009) and Selle (Decomposition and stability of multifronts and multipulses, 2009), who regarded the concatenated wave pattern as a sum of traveling waves. In contrast to their work, we regard the pattern as a sequence of traveling waves restricted to subintervals of $\mathbb{R}$ and separated at any finite time by small jump discontinuities. Our proof uses spatial dynamics and Laplace transform.

Keywords Interaction of waves · Reaction–diffusion equation · Spatial dynamics · Laplace transform · Exponential dichotomy in trace space

1 Introduction

Consider the system of reaction–diffusion equations in one space dimension

$$u_t = u_{xx} + f(u), \tag{1.1}$$

where $f \in C^2(\mathbb{R}^n)$. Throughout this paper we assume that the solutions of (1.1) are in $H^{2,1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$ and both sides of (1.1) are in $L^2_{loc}(\mathbb{R} \times \mathbb{R}^+)$. Notice that $H^{2,1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$ is continuously imbedded in $C_{loc}(\mathbb{R} \times \mathbb{R}^+)$ therefore $f(u) \in L^2_{loc}(\mathbb{R} \times \mathbb{R}^+)$. We choose the diffusion terms u_{xx} to simplify the illustration of our method. More general systems, such as $u_t = Au_{xx} + f(u)$ in [24,26], where A is positive definite, can be treated by our method.

Dedicated to Professor John Mallet-Parets 60th birthday.

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We assume that (1.1) has m traveling wave solutions, with widely separated centers, that connect $m+1$ spatially constant, time-independent solutions. These spatially constant solutions correspond to $m+1$ equilibria $e_0, \dots, e_m$ of the ordinary differential equation $u_t = f(u)$. The jth traveling wave, which has speed c_j, is $q_j(\xi_j)$, $\xi_j = x - y_j - c_j t$. It connects $q_j(-\infty) = e_{j-1}$ to $q_j(\infty) = e_j$.

We write ξ instead of ξ_j if it is clear which y_j and c_j are used. In the coordinates (ξ, t), $q_j(\xi)$ is a stationary solution of

$$u_t = u_{\xi\xi} + c_j u_\xi + f(u), \quad \xi = \xi_j = x - y_j - c_j t. \tag{1.2}$$

The traveling wave $q_j(\xi)$ satisfies the ODE

$$q_j'' + c_j q_j' + f(q_j) = 0, \quad 1 \le j \le m.$$

The function $(u(\xi), v(\xi)) = (q_j(\xi), q_j'(\xi))$ is a heteroclinic orbit of the associated first-order system

$$u_\xi = v, \quad v_\xi = -c_j v - f(q_j) \tag{1.3}$$

that connects the equilibria $(e_{j-1}, 0)$ and $(e_j, 0)$.

After a phase shift, we may assume for definiteness that $|q_j'(0)| = \max\{|q_j'(\xi)| : \xi \in \mathbb{R}\}$. Then $q_j(0)$, which we regard as the center of the wave q_j, travels on the characteristic line $\xi = 0$, which corresponds to $x = y_j + c_j t$. We assume the waves are widely separated, i.e., $y_1 << y_2 << \cdots << y_m$, and we assume $c_1 < c_2 < \cdots < c_m$.

We define a concatenated wave pattern by dividing the domain $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ into m sectors and placing one traveling wave in each sector. More precisely, for $1 \le j \le m-1$, let $\bar{c}_j = (c_j + c_{j+1})/2$ be the average speed of the waves q_j and q_{j+1}, and let $x_j = (y_j + y_{j+1})/2$, . For convenience let $x_0 = -\infty$ and $x_m = \infty$. Define

$$\begin{aligned} M_j &= \{(x,t) : x = y_j + c_j t, \ t \ge 0\}, \\ \Gamma_j &= \{(x,t) : x = x_j + \bar{c}_j t, \ t \ge 0\}, \\ \Omega_j &= \{(x,t) : x_{j-1} + \bar{c}_{j-1} t < x < x_j + \bar{c}_j t, \ t \ge 0\}, \end{aligned}$$

so that M_j is inside Ω_j, and Γ_j separates Ω_j and Ω_{j+1}. Define the concatenated wave pattern to be

$$u^{\text{con}}(x,t) = q_j(x - y_j - c_j t) \text{ for } (x,t) \in \Omega_j, \ 1 \le j \le m.$$

The center of the wave q_j in Ω_j moves on the line M_j, and the lines $M_1, \dots, M_m$ spread apart as $t \to \infty$. The concatenated pattern satisfies (1.1) in each Ω_j but is not continuous across the Γ_j (Fig. 1).

For $\eta > 0$ and $\pi/2 < \theta < \pi$, define the sector

$$\Sigma(-\eta, \theta) = \{s \in \mathbb{C} : |\arg(s + \eta)| \le \theta\}.$$

$\Sigma(-\eta, \theta)$ has vertex at $s = -\eta$ and opens to the right with opening angle 2θ. It contains the half plane $\Re(\lambda) \ge -\eta$.

For $1 \le j \le m$, the linearization of (1.2) at the traveling wave $q_j(\xi)$ is

$$u_t = u_{\xi\xi} + c_j u_\xi + Df(q_j(\xi))u, \quad \xi = \xi_j = x - y_j - c_j t. \tag{1.4}$$

Define the linear operator L_j on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ by

$$L_j u = u_{\xi\xi} + c_j u_\xi + Df(q_j(\xi))u.$$

Throughout this paper we make the following standard assumptions.

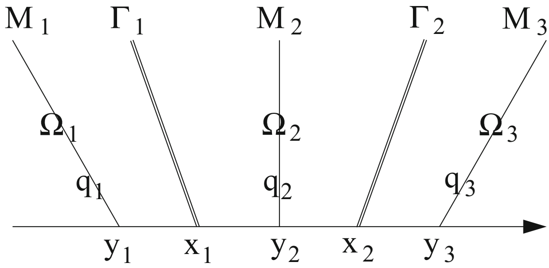

Fig. 1 For the case $m = 3$, the concatenated pattern consists three waves separated by *two lines* Γ_1 and Γ_2

H1 For $0 \le j \le m$, $\Re\sigma(Df(e_j)) < 0$.
H2 For $1 \le j \le m$, the operator L_j on $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$, has the simple eigenvalue $\lambda = 0$, with one-dimensional eigenspace spanned by q_j'.

From H1, for $0 \le j \le m$, the linear first-order system $u_\xi = v,\ v_\xi = -cv - Df(e_j)u$ has, counting multiplicity, n eigenvalues with negative real part and n eigenvalues with positive real part. Together with H2, we can show that there are numbers $\eta > 0$ and θ, with $\pi/2 < \theta < \pi$, such that

(A1) for $0 \le j \le m$, the spectrum of the operator $u \to u_{\xi\xi} + c_j u_\xi + Df(e_j)u$ on L^2 is contained in the complement of $\Sigma(-\eta, \theta)$;
(A2) for $1 \le j \le m$, the spectrum of the operator L_j on L^2, is contained in the complement of $\Sigma(-\eta, \theta)$ (essential spectrum), plus the simple eigenvalue 0.

Let L_j^* be the adjoint operator for L_j on $L^2(\mathbb{R})$, with domain $H^2(\mathbb{R})$:

$$L_j^* z = z_{\xi\xi} - c_j z_\xi + Df(q_j(\xi))^* z. \tag{1.5}$$

Hypothesis (H2) implies that the adjoint equation $L_j^* z = 0$ has a unique (up to constant multiples) bounded solution z_j. Moreover, since q_j' is not in the range of L_j, $\int_{-\infty}^{\infty} < z_j, q_j' > d\xi \neq 0$. Assume that

$$\int_{-\infty}^{\infty} < z_j, q_j' > d\xi = 1, \quad 1 \le j \le m. \tag{1.6}$$

Let $H^{2,1}(\Omega_j, \gamma),\ \gamma \le 0$, be the space of functions u on Ω_j such that $e^{-\gamma t}u(x,t) \in H^{2,1}(\Omega_j)$. Let I_j be the interval (x_{j-1}, x_j).

For $u \in H^{2,1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$, the function $t \to u(\cdot, t)$ is continuous in $H^1_{loc}(\mathbb{R})$. So it is natural to consider the initial condition $u(x,0) = u_0(x) \in H^1_{loc}(\mathbb{R})$. We assume further that on the first and last intervals, $u_0(\cdot) - q(\cdot - y_j) \in H^1(I_j),\ \ j = 1, m$. If $v \in H^{2,1}(\Omega_j)$, and if Γ is a line in the closure of Ω_j, then by the trace theory, (v, v_x) has well-defined limit in $H^{0.75}(\Gamma) \times H^{0.25}(\Gamma) \overset{def}{=} H^{0.75\times 0.25}(\Gamma)$, denoted by $(v(\Gamma), v_x(\Gamma))$. In particular, let

$$W_j = (q_j, q_j'),\ J_{j0} = W_j(\Gamma_j) - W_{j+1}(\Gamma_j),\ \text{then } J_{j0} \in H^{0.75\times 0.25}(\Gamma_j).$$

Consider

$$\begin{aligned} q_j(\Gamma_j) =& q_j(x_j - y_j + (\bar{c}_j - c_j)t), \\ q_j(\Gamma_{j-1}) =& q_j(x_{j-1} - y_j + (\bar{c}_{j-1} - c_j)t). \end{aligned}$$

From $x_j - y_j = y_j - x_{j-1} \geq \inf\{y_{j+1} - y_j\}/2$, there exist $\bar{C} > 0$, $\mu < 0$, $-\eta < 0$ such that for all $1 \leq j \leq m$,

$$\begin{aligned} |q_j(\Gamma_j) - e_j| + |q_{j\xi}(\Gamma_j)| &\leq \bar{C} e^{-\eta \inf\{y_{j+1}-y_j\}/2}, \quad t = 0, \\ |q_{j+1}(\Gamma_j) - e_j| + |q_{j+1,\xi}(\Gamma_j)| &\leq \bar{C} e^{-\eta \inf\{y_{j+1}-y_j\}/2}, \quad t = 0, \\ |q_j(\Gamma_j) - e_j| + |\partial_t q_j(\Gamma_j)| &\leq \bar{C} e^{\mu t}, \quad t \geq 0, \\ |q_{j+1}(\Gamma_j) - e_j| + |\partial_t q_{j+1}(\Gamma_j)| &\leq \bar{C} e^{\mu t}, \quad t \geq 0, \end{aligned} \tag{1.7}$$

Definition 1.1 The concatenated wave pattern $u^{\text{con}}(x, t)$ is *exponentially stable with the rate* $e^{\gamma t}$, provided there exist $\gamma < 0$ and $\delta_0 > 0$ for which the following is true.

(1) The set $S_{init} := \{u_0 \in H^1_{loc}(\mathbb{R}) : \max_j\{|u_0(x) - q_j(x - y_j)|_{H^1(I_j)} < \delta_0\}$ is nonempty.

(2) For any $u_0 \in S_{init}$, there exist a unique solution $u(x, t) \in H^{2,1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$ to (1.1) and a sequence of numbers $r_1, \ldots, r_m$ such that $u(x, 0) = u_0(x)$. Moreover, if $\rho := \max_j\{|u_0(x) - q_j(x - y_j)|_{H^1(I_j)}$, and if in each Ω_j, $u(x, t) = q_j(x - y_j - c_j t + r_j) + u_j(x, t)$, then

$$\partial_t u_j \in L^2(\Omega_j, \gamma) \text{ and } |u_j(\cdot, t)|_{H^1(x)} < C\rho e^{\gamma t}, \quad t \geq 0.$$

Intuitively, on each Ω_j, $u(x, t)$ exponentially approaches a shifted concatenated wave as $t \to \infty$. Different shifts are allowed in different Ω_j. Note if $\max\{|u_0(x) - q_j(x - y_j|)_{H^1(I_j)}\} = \rho$, then $\max\{|J_{j0}|\} \leq C\rho$. Given a concatenated patter u^{con}, if δ_0 is too small, then S_{init} is an empty set.

We now state the main result of this paper.

Theorem 1.1 *Assume that the conditions (H1) and (H2) hold. Let $-\eta$ and μ be the constants in (A1), (A2) and* (1.7) *and let γ satisfies* $\max\{-\eta, \mu\} < \gamma < 0$. *Then there exists a sufficiently large $\ell > 0$, a small $\delta_0 > 0$ and a constant $C_1 > 0$ such that if*

$$\inf\{y_{j+1} - y_j\} \geq \ell, \text{ and } \bar{C} e^{-\eta \inf\{y_{j+1}-y_j\}/2} < \delta_0,$$

then the concatenated wave $u^{\text{con}}(x, t)$ is stable with the rate $e^{\gamma t}$. Moreover, $u_0 \in H^1_{loc}(\mathbb{R})$ is in S_{init} if

$$\bar{C} e^{-\eta \inf\{y_{j+1}-y_j\}/2} < \max\{|u_0(x) - q_j(x - y_j)|_{H^1(I_j)}\} < \delta_0.$$

Remark 1.1 First, ℓ must be sufficiently large so that the existence of exponential dichotomies and related contraction rates conditions, as will be introduced later, are satisfied. We may need to choose $\inf\{y_{j+1} - y_j\}$ even greater so the set S_{init} is nonempty.

The "spatial dynamics" used in this paper were developed by Kirchgassner [7], Renardy [20], Mielke [18], Sandstede, Scheel, and collaborators [1,19], and others. This approach treats the space variable as "time", and evolve functions of t which is natural to handle the concatenated waves that are placed side by side with jumps along common boundaries. In [11,12], the interaction of stable, standing waves for a boundary value problem of parabolic systems in finite domain was considered by the method similar to that used in this paper. However, $\lambda = 0$ was not an eigenvalue and wave speed was not an issue in those papers. The new contribution of this paper is to treat the eigenvalue $\lambda = 0$ and the variation of wave speeds and shifts related to $\lambda = 0$. To simplify the notation, the equation considered in this paper is similar to that of [11]. Using the ideas of this paper, but changing the trace spaces to the more general ones used in [12], we should be able to handle interactions of traveling waves of general higher order parabolic systems as in [12]: $u_t + (-1)^m D_x^{2m} u = f(u, u_x, \ldots, (D_x)^{2m-1} u), \quad u \in \mathbb{R}^n$.

To illustrate our method, consider the simple case of two traveling waves $q_j(x - c_j t)$, $j = 1, 2$ of (1.1) moving in opposite direction: $c_1 < 0 < c_2$. Define the concatenated wave $u^{con}(x, t)$ separated by $\Gamma = \{x = 0, t \geq 0\}$ as follows

$$u^{con}(x, t) = q_1(x + N - c_1 t) \text{ if } x < 0, \quad u^{con}(x, t) = q_2(x - N - c_2 t) \text{ if } x > 0. \quad (1.8)$$

Assume $N > 0$ is a large constant so that the jumps along Γ, $[u^{con}, u_x^{con}](\Gamma)$, are small and decay to zero as functions of time t.

Consider the perturbation of the initial data around $u^{con}(x, 0)$. Notice that u^{con} is not a solution of (1.1). Let the exact solution be

$$u(x, t) = u^{con} + u_1(x, t) \text{ for } x < 0, \quad u(x, t) = u^{con} + u_2(x, t) \text{ for } x > 0. \quad (1.9)$$

The corrections $u_1(x, t)$ and $u_2(x, t)$ will be solved as initial-boundary value problems of PDEs in $x \leq 0$ and $x \geq 0$ respectively, cf. (2.3). The boundary values are determined by two conditions: (1) The boundary values for u_1, u_2 at Γ must compensate the jumps of u^{con} at Γ as follows

$$\begin{aligned} u_2(0, t) - u_1(0, t) &= -(u^{con}(0+, t) - u^{con}(0-, t)), \\ u_{2x}(0, t) - u_{1x}(0, t) &= -(u_x^{con}(0+, t) - u_x^{con}(0-, t)). \end{aligned}$$

(2) The boundary conditions for u_1 at $x = 0-$ (or for u_2 at $x = 0+$) must belong to the unstable subspace (or stable subspace) of the dichotomies of the "spatial dynamics" of the system (such dichotomies exist at least near each equilibirum point). So with the help of the variations of wave speeds as parameters, the solution $u_1(x, t)$ (or $u_2(x, t)$) can pass the center of q_1 (or q_2) where the left half and right half of exponential dichotomies do not match, and still decay to zero as $x \to -\infty$ (or $x \to \infty$). The condition (2) may sound complicated but it is based on how Lions and Magenes treated the boundary values of PDEs in the popular text book [15].

Now consider the concatenation of m traveling waves. After linearization, the correction term u_j defined in Ω_j, $j = 1, \ldots, m$, should satisfy the initial-boundary value problems with prescribed jump $J_j(\Gamma)$ along Γ_j, as in (2.6):

$$\begin{aligned} &u_{jt} = u_{j,xx} + Df(q_j) u_j + h_j(x, t), \quad u_j(x, 0) = u_{j0}(x), \\ &([\{u_j\}], [\{u_{jx}\}])(\Gamma_j) = J_j(\Gamma_j). \end{aligned}$$

If the linear system can be solved then the exact $\{u_j\}_1^m$ is obtained by the contraction mapping argument. Compared to the "inverse systems" used in other papers to treat wave interactions, this system is simpler, and highly localized such that the coefficient of the jth equation only depends on q_j. It can easily be adapted to many nonstandard cases where q_j is not a saddle to saddle connection, or some e_j is non-hyperbolic and q_j connects to its center manifold, or weighted norm must be used to ensure the stability of each individual wave, etc. See discussions of the generalized Fisher/KPP equation in Sect. 6.

Essentially the same result was proved by Wright [26] and Selle [24], who regarded the concatenated wave pattern as a sum of traveling waves. Besides being easier to treat some less standard systems as mentioned above, the other advantage of our approach is that it directly links the wave speeds and phase shifts to the perturbations of initial conditions and the jumps between adjacent waves, cf. (4.6), (4.22) and (5.4) where $\beta_j(t) q_j'(\xi)$ is in the eigenspace associated to $\lambda = 0$. This information can be useful in some practical applications where we are not only interested in the existence of the the concatenated traveling wave, but also in how each wave component is changed by the interaction with other waves.

Here is a brief outline of the paper. In Sect. 2 we outline the proof. The structure of the proof is based on the approach of Sattinger [23], in which the linear variational system is obtained around the original traveling wave, not around an undetermined shift of the wave (here shifts of the waves). When linear variational systems are considered, the unknown shifts appear as multiples of q_j'. The nonlinear system is considered in the last section where we solve for the entire solution and asymptotic shift (here shifts) simultaneously by a contraction mapping principle. We remark that Rottmann-Matthes has developed a method parallel to Sattinger's approach [21,22].

In Sect. 3 we give some background about exponential dichotomies and Laplace transform. In Sect. 3.1, we discuss exponential dichotomies in frequency domain where the equation can be treated pointwise in s. In Sect. 3.2, we discuss exponential dichotomies where the equation in frequency domain cannot be treated pointwise in s. In Sect. 3.3, we discuss the roughness of exponential dichotomies for general abstract equations in Banach spaces. In Sect. 3.4 we discuss exponential dichotomies for linear variational system near the equilibrium e_j. In Sect. 3.5 we discuss exponential dichotomies for linear variational system near the traveling wave solution q_j. In Sect. 4 we study the linear non-homogeneous system obtained by linearizing (1.1) at the discontinuous concatenated wave solution $u^{\text{con}}(x,t)$. A solution to the non-homogeneous system, ignoring jump discontinuities along the Γ_j, is obtained in Sect. 4.1, and a solution to the homogeneous system with prescribed jumps is obtained in Sect. 4.2. In Sect. 5 we complete the proof of Theorem 1.1 by solving the nonlinear initial value problem using our solution of the linearized problem and a contraction mapping argument. In Sect. 6, we discuss the wave interaction of the generalized Fisher/KPP equation where an important proposition used in [26] is not satisfied, but may still be treated by our method.

2 Outline of Proof

Let $\Omega = I \times \mathbb{R}^+$ or an open subset of $\mathbb{R} \times \mathbb{R}^+$, always thought of as xt-space. Define the following Banach spaces:

$$\begin{aligned}
H^k(\mathbb{R}^+) &= W^{k,2}(\mathbb{R}^+, \mathbb{R}^n), \;\; k \geq 0, \;\text{the usual Sobolev space.}\\
H^{k_1\times k_2}(\mathbb{R}^+) &= H^{k_1}(\mathbb{R}^+) \times H^{k_2}(\mathbb{R}^+), \;\; k_1 \geq 0, \; k_2 \geq 0.\\
H^{2,1}(\Omega) &= \{u : \Omega \to \mathbb{R}^n \mid u, u_{xx} \text{ and } u_t \in L^2(\Omega; \mathbb{R}^n\}.\\
|u|_{H^{2,1}(\Omega)} &= |u|_{L^2} + |u_{xx}|_{L^2} + |u_t|_{L^2}.
\end{aligned}$$

As usual, $H^0 = L^2$ and $H_0^k(\mathbb{R}^+) \subseteq H^k(\mathbb{R}^+)$ consists of functions that are 0 at $t = 0$. We say $u(x,t) \in H^{2,1}_{loc}(\Omega)$ if it is in $H^{2,1}$ when restricted to a bounded subset of Ω.

For a constant $\gamma < 0$, define:

$$\begin{aligned}
H^k(\mathbb{R}^+, \gamma) &= \{u : \mathbb{R}^+ \to \mathbb{R}^n \mid e^{-\gamma t}u \in H^k(\mathbb{R}^+\}; \;\; |u|_{H^k(\mathbb{R}^+,\gamma)} = |e^{-\gamma t}u|_{H^k(\mathbb{R}^+)}.\\
H^{k_1\times k_2}(\mathbb{R}^+, \gamma) &= H^{k_1}(\mathbb{R}^+, \gamma) \times H^{k_2}(\mathbb{R}^+, \gamma).\\
L^2(\Omega, \gamma) &= \{u : \Omega \to \mathbb{R}^n \mid e^{-\gamma t}u \in L^2(\Omega)\}; \;\; |u|_{L^2(\Omega,\gamma)} = |e^{-\gamma t}u|_{L^2(\Omega)}.\\
H^{2,1}(\Omega, \gamma) &= \{u : \Omega \to \mathbb{R}^n \mid e^{-\gamma t}u \in H^{2,1}(\Omega)\}; \;\; |u|_{H^{2,1}(\Omega,\gamma)} = |e^{-\gamma t}u|_{H^{2,1}(\Omega)}.
\end{aligned}$$

Let $X^1(\mathbb{R}^+, \gamma)$, or $X^1(\gamma)$ for short, be the space of functions $\beta(t)$ such that $\dot{\beta} \in L^2(\gamma),\ \gamma < 0$. Define the norm in $X^1(\gamma)$ as

$$|\beta|_{X^1(\gamma)} := |\beta(0)| + |\dot{\beta}|_{L^2(\gamma)}.$$

For $\beta \in X^1(\gamma)$, the limit $\beta(\infty)$ exists. By the Cauchy-Schwarz inequality, have

$$|\beta(t) - \beta(\infty)| = |\int_t^\infty e^{\gamma t}(e^{-\gamma t}\dot{\beta}(t))dt| \le Ce^{\gamma t}|\dot{\beta}|_{L^2(\gamma)}. \tag{2.1}$$

The change of coordinates $\xi = x - y - ct$ converts Ω to a subset $\tilde{\Omega}$ of $\mathbb{R} \times \mathbb{R}^+$, with coordinates (ξ, t), and converts a function $u(x, t)$ on Ω to a function $\tilde{u}(\xi, t) = u(\xi + y + ct, t)$ on $\tilde{\Omega}$.

Lemma 2.1 *The map $u \to \tilde{u}$ is a linear isomorphism of $H^{2,1}(\Omega, \gamma)$ to $H^{2,1}(\tilde{\Omega}, \gamma)$. The map $u \to \tilde{u}$ and its inverse $\tilde{u} \to u$ both have norm at most* $1 + |c|$.

Proof Let $u \in H_0^{2,1}(\Omega, \gamma)$. Then $\tilde{u}_\xi = u_x,\ \tilde{u}_{\xi\xi} = u_{xx},\ \tilde{u}_t = u_t + c_j u_x$. Thus

$$|\tilde{u}| + |\tilde{u}_\xi| + |\tilde{u}_{\xi\xi}| + |\tilde{u}_t| \le |u| + |u_x| + |u_{xx}| + |u_t| + |c||u_x|.$$

Here all the norms are in $L^2(\Omega, \gamma)$. The lemma follows easily. □

Let $\Gamma(x_0, c) = \{(x, t) : x = x_0 - ct,\ t \ge 0\}$.

Lemma 2.2 *If $\Gamma(x_0, c) \subset \Omega$, then the mapping $u \to u|_{\Gamma(x_0,c)}$ is bounded linear from $H^{2,1}(\Omega, \gamma)$ to $H^{0.75\times0.25}(\mathbb{R}^+, \gamma)$. Moreover, there is a number $K > 0$, independent of x_0 and c, such the norm of the linear map is at most* $K(1 + |c|)$.

Proof For $c = 0$, see [15,25]. For $c \ne 0$, use Lemma 2.1 followed by letting $c = 0$. □

Now assume that $v \in H_{\text{loc}}^{2,1}(\mathbb{R} \times \mathbb{R}^+)$, and $\Gamma(x_0, c)$ is the line as in Lemma 2.2. It follows from a localization process and Lemmas 2.1 and 2.2, the restriction of v to Γ, as a function of t, belongs to $H_{\text{loc}}^{0.75\times0.25}(\mathbb{R}^+)$.

Lemma 2.3 *Let $x = 0$ be the line that divides $\{x \in \mathbb{R}, t \ge 0\}$ into two regions: $\Omega^- = \{(x, t) : x < 0, t \ge 0\}$ and $\Omega^- = \{(x, t) : x > 0, t \ge 0\}$. Let $v^- \in H^{2,1}(\Omega^-)$ and $v^+ \in H^{2,1}(\Omega^+)$. Assume the traces $v^-|_{x=0} = v^+|_{x=0}$ in the space $H^{0.75\times0.25}(\mathbb{R}^+)$. Then the function v that equals v^- on Ω^- and v^+ on Ω^+ is in the space* $H^{2,1}(\mathbb{R} \times \mathbb{R}^+)$.

Proof The proof is a simple excise of the trace theory, and is outlined below. Let w be defined on $\{x \in \mathbb{R}, t \ge 0\}$, and equals to $D_x v^-$ (or $D_x v^+$) on Ω^- (or Ω^+). Using integration by parts, it is easy to show that $D_x v = w$ in $\mathbb{R} \times \mathbb{R}^+$. Thus $v_x \in L^2(\mathbb{R} \times \mathbb{R}^+)$. Same proof shows that $v_{xx}, v_t \in L^2(\mathbb{R} \times \mathbb{R}^+)$. Therefore $v \in H^{2,1}(\mathbb{R} \times \mathbb{R}^+)$. □

Remark 2.1 Suppose $v \in L_{\text{loc}}^2(\mathbb{R} \times \mathbb{R}^+)$, and its restrictions to the left and right of Γ, v^- and v^+, are locally $H^{2,1}$ functions. Using the cut-off functions and change of coordinates as in Lemma 2.1, it is easy to see that $v \in H_{\text{loc}}^{2,1}(\mathbb{R} \times \mathbb{R}^+)$ if and only if the traces of v^- and v^+ on Γ are equal.

2.1 Deriving the Linear Variational System

Write the exact solution of (1.1), with the initial condition $u^{ex}(x,0) = u_0^{ex}(x)$, as $u^{ex}(x,t) = u^{ap}(x,t) + u^{cor}(x,t)$, $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, with $u^{ap}(x,t) = q_j(x - y_j - c_j t + r_j)$ in Ω_j. For the rest of the paper, denote $u^{cor}(x,t)$ by $u(x,t)$, and its restriction to Ω_j by u_j.

Let $\{r_j\}_{j=1}^m = (r_1, \dots, r_m)$, often with the range of j omitted, same notation for a sequence of functions $\{u_j\}$. The $\{r_j\}$ are parameters to be determined so that $u_j(x,t)$ will lie in the appropriate space. The equation for u_j, the perturbation to $q_j(x - y_j - c_j t + r_j)$, is

$$u_t = u_{xx} + Df(q_j(x - y_j - c_j t + r_j))u + \mathcal{O}(u^2), \quad (x,t) \in \Omega_j$$

The initial value for $u_j(x,t)$ in Ω_j is

$$u_{j0}(x) = u_j(x,0) = u_0^{ex}(x) - q_j(x - y_j + r_j), \quad x \in I_j = (x_{j-1}, x_j).$$

Recall that $\Gamma_j = \{(x,t) : x = x_j + \bar{c}_j t,\ t \geq 0\}$ separates Ω_j, Ω_{j+1}. The traces $u_j(\Gamma_j)$ and $u_{j+1}(\Gamma_j)$ exist. Define the jump of $\{u_j\}$ across Γ_j, a function of t, by

$$[\{u_j\}](\Gamma_j) = u_{j+1}(\Gamma_j) - u_j(\Gamma_j).$$

We will use this notation for any sequence of functions defined on $\{\Omega_j\}$, such as $\{q_j(\xi + y_j + c_j t)\}$, and even after the change of variables to ξ_j in Ω_j. With this notation, the jump conditions along Γ_j are

$$[\{u_j\}, \{u_{jx}\}](\Gamma_j) = -[u^{ap}, u_x^{ap}](\Gamma_j), \quad 1 \leq j \leq m-1. \tag{2.2}$$

The jump conditions depend on the parameters $\{r_j\}$ since u^{ap} does.

Notice the compatibility between the jump conditions at $t = 0$ and the jumps of the initial condition at $x = x_j$:

$$[\{u_j\}, \{u_{jx}\}](\Gamma_j)|_{t=0} = [\{u_j 0\}, \{\dot{u}_{j0}\}](x_j), \quad 1 \leq j \leq m-1.$$

The unknown $\{r_j\}$ appears in the argument of $u^{ap}(x,t)$. To avoid having an undetermined r_j in $Df(u^{ap})$, we shall follow the idea of Sattinger [23] to linearize around $q_j(x - y_j - c_j t)$. With the moving coordinate $\xi_j = x - y_j - c_j t$, denoted by ξ when there should be no ambiguity, the exact solution in Ω_j becomes $\tilde{u}^{ex}(\xi,t) = u^{ex}(\xi + y_j + c_j t, t)$. However, both the approximate solution and the perturbation depend on the parameter r_j. To show this dependence, in Ω_j we write

$$\tilde{u}^{ex}(\xi,t) = q(\xi + r_j) + \tilde{u}(\xi,t;r_j), \quad \tilde{u}(\xi,t;r_j) = u^{ex}(\xi + y_j + c_j t, t) - q_j(\xi + r_j).$$

We find that $\tilde{u}(\xi,t;r_j)$ is a solution of the following differential equation:

$$u_t = u_{\xi\xi} + c_i u_\xi + Df(q_j(\xi))u + B_j(r_j)u + R_j(u, r_j), \quad (\xi,t) \in \Omega_j, \tag{2.3}$$

where $B_j(r_j)u + R_j(u,r_j) = f(q_j(\xi + r_j) + u) - f(q_j(\xi + r_j)) - Df(q_j(\xi))u$, and

$$B_j(r_j) = Df(q_j(\xi + r_j)) - Df(q_j(\xi)) = r_j \int_0^1 D^2 f(q_j(\xi + s r_j)) q_j'(\xi + s r_j) ds = \mathcal{O}(r_j),$$

$$R_j(u,r_j) = f(q_j(\xi + r_j) + u) - f(q_j(\xi + r_j)) - Df(q_j(\xi + r_j))u = \mathcal{O}(|u|^2).$$

Since $B(r_j)u = \mathcal{O}(r_j|u|)$, so the terms $B(r_j)u$, $R(u,r_j)$ are of second order in (u, r_j).

When $r_j = 0$, the initial condition for the perturbation $\tilde{u}$ is

$$\tilde{u}(\xi, 0; 0) = u^{ex}(\xi + y_j, 0) - q_j(\xi) \stackrel{def}{=} \bar{u}_{j0}(\xi).$$

For general r_j, the initial condition for the perturbation is

$$\tilde{u}(\xi, 0; r_j) = u^{ex}(\xi + y_j, 0) - q_j(\xi + r_j) = \bar{u}_{j0}(\xi) + q_j(\xi) - q_j(\xi + r_j).$$

Let $g_j(\xi, r_j) = q_j(\xi) - q_j(\xi + r_j) + r_j q_j'(\xi) = \mathcal{O}(r_j^2)$. We have the initial conditions for the correction term

$$u_j(\xi, 0) = \bar{u}_{j0}(\xi) - r_j q_j'(\xi) + g_j(\xi, r_j), \quad x \in I_j. \tag{2.4}$$

Let $W_j = (q_j, q_j'), \ \{r_j\} = (r_1, \ldots, r_m)$. We rewrite the jump conditions (2.2) to emphasize the dependence on $\{r_j\}$

$$\begin{aligned} -[u^{ap}, u^{ap}_\xi](\Gamma_j) \stackrel{def}{=} J_j(\Gamma_j, \{r_j\}) &= W_j(\Gamma_j + r_j) - W_{j+1}(\Gamma_j + r_{j+1}) \\ &= W_j(\Gamma_j) - W_{j+1}(\Gamma_j) + G_j(\{r_j\}), \end{aligned}$$

where

$$\begin{aligned} G_j(\{r\}) := W_{j+1}(\Gamma_j) &- W_{j+1}(\Gamma_j + r_{j+1}) \\ &+ W_j(\Gamma_j + r_j) - W_j(\Gamma_j). \end{aligned}$$

Recall that $J_{j0} = W_j(\Gamma_j) - W_{j+1}(\Gamma_j)$, which is the jump condition when $\{r_j\} = 0$. Now (2.2) can be written as

$$[\{u_j\}, \{u_{jx}\}](\Gamma_j) = J_j(\Gamma_j, \{r_j\}) = J_{j0}(\Gamma_j) + G_j(\{r_j\}), \quad 1 \le j \le m - 1. \tag{2.5}$$

As shown in Lemma 2.3 and the remark that follows, to have $u^{con} \in H^{2,1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$, the jumps across each Γ_j must satisfy

$$\begin{aligned} u_{j+1}(\Gamma_j) - u_j(\Gamma_j) &= q_j(\Gamma_j + r_j) - q_{j+1}(\Gamma_j + r_{j+1}) \in H^{0.75}(\Gamma_j), \\ u_{j+1,x}(\Gamma_j) - u_{j,x}(\Gamma_j) &= q_{j,x}(\Gamma_j + r_j) - q_{j+1,x}(\Gamma_j + r_{j+1}) \in H^{0.25}(\Gamma_j). \end{aligned}$$

In order to solve the nonlinear system (2.3), (2.4), (2.5), we shall first consider the following nonhomogeneous linear system:

$$u_{jt} = u_{j,xx} + Df(q_j)u_j + h_j(x, t), \quad u_j(x, 0) = u_{j0}(x), \tag{2.6a}$$

$$([\{u_j\}], [\{u_{jx}\}])(\Gamma_j) = J_j(\Gamma_j), \tag{2.6b}$$

$$[\{u_{j0}\}, \{\dot{u}_{j0}\}](x_j) = J_j(\Gamma_j)|_{t=0}. \tag{2.6c}$$

In these equations, for $j = 1, \ldots m$, $h_j \in L^2(\Omega_j, \gamma)$, $\gamma < 0$, $u_j(x, 0) \in H^1(I_j)$; and for $j = 1, \ldots m - 1$, $J_j(\Gamma_j) \in H^{0.75 \times 0.25}(\gamma)$. Temporarily they do not depend on $(\{r_j\}, \{u_j\})$. The last one is the compatibility between the initial conditions and the jump conditions. We look for $u_j \in H^{2,1}(\Omega_j, \gamma) + span\{\beta(t)q_j'(\xi), \beta(t) \in X^1(\gamma)\}$, as will be specified in Sect. 4.

For the nonlinear systems (2.3–2.5), the forcing terms, initial and jump conditions depend on the parameters $\{r_j\}$, since u^{ap} does. In Sect. 5, the correction term $u(x, t)$, together with shifts $\{r_j\}$ will be solved by letting

$$\begin{aligned} h_j &= B_j(r_j)u + R_j(r_j, u_j), \\ u_{j0} &= \bar{u}_{j0}(\xi) - r_j q_j'(x - y_j) + g_j(\xi, r_j) \text{ for } x \in I_j, \\ J_j(\Gamma_j) &= J_{j0}(\Gamma_j) + G_j(\{r_j\}). \end{aligned} \tag{2.7}$$

We look for $(\{u_j\}, \{r_j\})$ with $u_j \in H^{2,1}(\Omega_j, \gamma)$ by using a contraction mapping argument adapted from [23].

3 Function Spaces and Exponential Dichotomies

The following definitions come from [11]. A function $f(s)$ is in the Hardy-Lebesgue class $\mathcal{H}(\gamma)$, $\gamma \in \mathbb{R}$, if

(1) $f(s)$ is analytic in $\Re(s) > \gamma$;
(2) $\{\sup_{\sigma>\gamma}(\int_{-\infty}^{\infty} |f(\sigma + i\omega)|^2 d\omega)^{1/2}\} < \infty$.

$\mathcal{H}(\gamma)$ is a Banach space with norm defined by the left side of (2).

According to the Paley–Wiener Theorem [28], $u(t) \in L^2(\mathbb{R}^+, \gamma)$ if and only if its Laplace transform $\hat{u}(s) \in \mathcal{H}(\gamma)$, and the mapping $u \to \hat{u}$ is a Banach space isomorphism.

For $k, k_1, k_2 \geq 0$ and $\gamma \in \mathbb{R}$, let

$$\begin{aligned}\mathcal{H}^k(\gamma) &= \{u(s) \mid u(s) \text{ and } (s-\gamma)^k u(s) \in \mathcal{H}(\gamma)\},\\ |u|_{\mathcal{H}^k(\gamma)} &= |u|_{\mathcal{H}^k(\gamma)} + |(s-\gamma)^k u|_{\mathcal{H}^k(\gamma)},\\ \mathcal{H}^{k_1\times k_2}(\gamma) &= \mathcal{H}^{k_1}(\gamma) \times \mathcal{H}^{k_2}(\gamma).\end{aligned}$$

An equivalent norm on $\mathcal{H}^k(\gamma)$ is

$$|u|_{\mathcal{H}^k(\gamma)} = \left(\sup_{\sigma>\gamma} \int_{-\infty}^{\infty} |u(\sigma + i\omega)|^2 (1 + |\sigma + i\omega|^{2k}) d\omega\right)^{1/2}.$$

It can be shown that $u(t) \in H_0^k(\mathbb{R}^+, \gamma)$ if and only if $\hat{u}(s) \in \mathcal{H}^k(\gamma)$, and the mapping $u \to \hat{u}$ is a Banach space isomorphism. Clearly $(u, v) \in H_0^{k_1\times k_2}(\mathbb{R}^+, \gamma)$, $k_1, k_2 \geq 0$, if and only if $(\hat{u}, \hat{v}) \in \mathcal{H}^{k_1\times k_2}(\gamma)$, and the mapping $(u, v) \to (\hat{u}, \hat{v})$ is a Banach space isomorphism.

To treat Laplace transformed linear systems that depend on the parameter s, following [11,12], we introduce the following family of norms on $u \in \mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^n$:

Definition 3.1 For $Re(s) > \gamma$ and $k_1 \geq 0$, let $E^{k_1}(s)$ denote $\mathbb{C}^n$ with the weighted norm

$$|u|_{E^{k_1}(s)} = (1 + |s|^{k_1})|u|,$$

and let $E^{k_1\times k_2}(s)$ denote $\mathbb{C}^n \times \mathbb{C}^n$ with the weighted norm

$$\left|(u, v)^\tau\right|_{E^{k_1\times k_2}(s)} = (1 + |s|^{k_1})|u| + (1 + |s|^{k_2})|v|,$$

where $|u|$ and $|v|$ are the usual norms on $\mathbb{C}^n$.

Using $E^{k_1}(s)$ and $E^{k_1\times k_2}(s)$, we define some equivalent norms for $u \in \mathcal{H}^{k_1}(\gamma)$ and $(u, v)^\tau \in \mathcal{H}^{k_1\times k_2}(\gamma)$:

$$\left(\sup_{\sigma>\gamma} \int_{-\infty}^{\infty} |u|^2_{E^{k_1}(\sigma+i\omega)} d\omega\right)^{1/2}, \quad \left(\sup_{\sigma>\gamma} \int_{-\infty}^{\infty} |(u, v)^\tau|^2_{E^{k_1\times k_2}(\sigma+i\omega)} d\omega\right)^{1/2}.$$

Consider the second order linear equation and its Laplace transform

$$u_t = u_{\xi\xi} + cu_\xi + A(\xi, t)u, \quad u(\xi, 0) = 0, \tag{3.1}$$

$$\hat{u}_{\xi\xi} = s\hat{u} - c\hat{u}_\xi - \hat{A}(\xi, s) \overset{s}{*} \hat{u}. \tag{3.2}$$

Here $A(\xi, t)$ is C^1 in $t \in \mathbb{R}^+$ for each fixed ξ, and is piecewise continuous in ξ in the $C^1(\mathbb{R}^+)$ norm. Examples are $A(\xi, t) = Df(e_j)$, $A(\xi, t) = Df(q_j(\xi))$, and $A(\xi, t) =$

$Df(q_j(\xi+kt)), t \geq 0, \xi \in \mathbb{R}$. The convolution represents the operator $\mathcal{L}(A(\xi,t)\mathcal{L}^{-1}\hat{u}(\xi,s))$ and is performed along the vertical axis in $\mathbb{C}$ where both $\hat{b}$ and $\hat{u}$ are defined:

$$\hat{A}(\xi,s) \overset{s}{*} \hat{u}(\xi,s) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{A}(\xi,p)\hat{u}(\xi,s-p)dp.$$

Convert (3.1), (3.2) to the equivalent first order system and its Laplace transform

$$u_\xi = v, \quad v_\xi = u_t - cv - A(\xi,t)u, \quad u(\xi,0)=0, \tag{3.3}$$

$$\hat{u}_\xi = \hat{v}, \quad \hat{v}_\xi = s\hat{u} - c\hat{v} - \hat{A}(\xi,s) \overset{s}{*} \hat{u}. \tag{3.4}$$

3.1 Exponential Dichotomies if $A(\xi)$ is Independent of t

If $A(\xi,t) = A(\xi)$ is independent of time t, then (3.4) becomes

$$\hat{u}_\xi = \hat{v}, \quad \hat{v}_\xi = s\hat{u} - c\hat{v} - A(\xi)\hat{u}. \tag{3.5}$$

This equation is defined point-wise in s and can be solved one s at a time.

Let $T(\xi,\zeta;s)$ be the principal matrix solution for (3.5), s be a parameter in $\mathcal{S} \subset \mathbb{C}$, $I \subset \mathbb{R}$ be an interval, and Id be an identity matrix.

Definition 3.2 We say that (3.5) has an s-dependent exponential dichotomy for $s \in \mathcal{S}$ and $\xi \in I$ if there exist projections $P_s(\xi,s)+P_u(\xi,s) = Id$ on $\mathbb{C}^{2n}$, analytic in s and continuous in ξ, such that, with the s-dependent constants $K(s), \beta(s) > 0$, the following properties hold:

$$\begin{aligned} T(\xi,\zeta;s)P_s(\zeta,s) &= P_s(\xi,s)T(\xi,\zeta;s), \quad \xi \geq \zeta, \\ |T(\xi,\zeta;s)P_s(\zeta,s)|_{R^{2n}} &\leq K(s)e^{-\beta(s)|\xi-\zeta|}, \quad \xi \geq \zeta, \\ |T(\xi,\zeta;s)P_u(\zeta,s)|_{R^{2n}} &\leq K(s)e^{-\beta(s)|\xi-\zeta|}, \quad \xi \leq \zeta. \end{aligned} \tag{3.6}$$

We say that (3.5) has a uniform exponential dichotomy on the spaces $E^{(k+0.5)\times k}(s)$, $k \geq 0$ for $s \in \mathcal{S}$ and $\xi \in I$ if it has an s-dependent exponential dichotomy, and in addition there are constants $K, \alpha > 0$, independent of s and ξ, such that

(1) $|P_s(\xi,s)| \leq K$ and $|P_u(\xi,s)| \leq K$ for all $s \in \mathcal{S}$ and $\xi \in I$,
(2) each $K(s) \leq K$, and
(3) $\beta(s) = \alpha(1+|s|^{0.5})$.

Here $|P_s(\xi,s)|$ and $|P_s(\xi,s)|$ are calculated using the norms on $E^{(k+0.5)\times k}(s)$. The s dependent stable and unstable subspaces for the dichotomy shall be denoted by

$$E_s(\xi,s) = RP_s(\xi,s), \quad E_u(\xi,s) = RP_u(\xi,s).$$

Given $\xi_0 \in \mathbb{R}$, if $I = (-\infty,\xi_0]$, then the unstable subspace $E_u(\xi,s), \xi \in I$ is unique, although the exponential dichotomy in I is not unique. Similarly, If $I = [\xi_0,\infty)$, then $E_s(\xi,s)$ is unique, although the exponential dichotomy in I is not unique.

3.2 Exponential Dichotomies if $A(\xi,t)$ Depends on t

In general (3.4) involves a global operator $\hat{u}(\xi,s) \to \hat{A}(\xi,s) \overset{s}{*} \hat{u}(\xi,s)$ so the exponential dichotomy cannot be considered by fixing one s at a time. We find the following lemma useful.

Lemma 3.1 *Let $B(\xi, t)$ be a C^1 bounded function in t for each ξ and is piecewise continuous in ξ in the norm of $|B|_{C^1(t)}$. Then*

$$B(\xi, t)u(\xi, t)|_{H_0^{0.75}(\mathbb{R}^+)} \leq |B(\xi, t)|_{C^1(t)}|u|_{H_0^{0.75}(\mathbb{R}^+)}.$$

Moreover, after the Laplace transform, we have

$$|\hat{B}(\xi, s) \overset{s}{*} \hat{u}(\xi, s)|_{\mathcal{H}^{0.75}(0)} \leq |B(\xi, t)|_{C^1(t)}|\hat{u}(\xi, s)|_{\mathcal{H}^{0.75}(0)}.$$

Proof It is straightforward to show that $u(\xi, t) \to B(\xi, t)u(\xi, t)$ is bounded in the spaces $H_0^k(\mathbb{R}^+)$, for $k = 0, 1$:

$$|B(\xi, t)u(\xi, t)|_{H_0^k(\mathbb{R}^+)} \leq |B|_{C^1}|u|_{H_0^k(\mathbb{R}^+)}, \ k = 0, 1.$$

Expressed as the interpolation of two spaces $H_0^{0.75} = [L^2, H_0^1]_{0.75}$, the first estimate of the lemma can be obtained by the theory of interpolations [15,16]. The second estimate can be obtained by applying the Laplace transform to Bu. □

Consider the abstract differential equation $U_\xi = L(\xi)U, \ \xi \in I$ in the Banach space E. Here I is a bounded or unbounded interval in $\mathbb{R}$, $L(\xi) : E \to E$ is a linear (possibly unbounded) operator for each $\xi \in I$.

Definition 3.3 We say $U_\xi = L(\xi)U$ has an exponential dichotomy on E defined for $\xi \in I$, if there exist projections $P_s(\xi) + P_u(\xi) = Id$ in E, continuous in $\xi \in I$, and a solution operator $T(\xi, \zeta)$ that is defined and invariant on subspaces of E as in (3.7a), (3.7b). Moreover there exist constants $K, \alpha > 0$ such that the last inequalities (3.7c), (3.7d) are satisfied.

$$T(\xi, \zeta) : RP_s(\zeta) \to RP_s(\xi) \text{ is defined and continuous for } \xi \geq \zeta; \quad (3.7a)$$

$$T(\xi, \zeta) : RP_u(\zeta) \to RP_u(\xi) \text{ is defined and continuous for } \xi \leq \zeta; \quad (3.7b)$$

$$|T(\xi, \zeta)P_s(\zeta)|_E \leq Ke^{-\alpha|\xi-\zeta|}, \quad \xi \geq \zeta; \quad (3.7c)$$

$$|T(\xi, \zeta)P_u(\zeta)|_E \leq Ke^{-\alpha|\xi-\zeta|}, \quad \xi \leq \zeta. \quad (3.7d)$$

We assume that $T(\xi, \zeta)u$ is a solution of the differential equation $u_\xi = L(\xi)u$ if

(1) $u \in RP_s(\zeta)$ and $\xi \geq \zeta$, or
(2) $u \in RP_u(\zeta)$ and $\xi \leq \zeta$.

For each initial data $(\hat{u}, \hat{v}) \in \mathcal{H}^{0.75\times0.25}(\gamma)$ at $\zeta \in I$, there may not exist a solution of (3.4) in $\mathcal{H}^{0.75\times0.25}(\gamma)$ for $\xi \leq \zeta$ or $\xi \geq \zeta$. Assume that $\hat{u}(\xi, s) \to \hat{A}(\xi, s) \overset{s}{*} \hat{u}(\xi, s)$ is a bounded operator in $\mathcal{H}^{0.25}(\gamma)$, (3.4) can be written as $U_\xi = L(\xi)U$ in the Banach space $E = \mathcal{H}^{0.75\times0.25}(\gamma)$ where $U = (\hat{u}, \hat{v})$.

Definition 3.4 We say that (3.4) has an exponential dichotomy in $E = \mathcal{H}^{0.75\times0.25}(\gamma)$ for $\xi \in I$ if there exist projections $P_s(\xi) + P_u(\xi) = Id$, partially defined solution operator $T(\xi, \zeta)$ and constants $K, \alpha > 0$ as in Definition 3.3.

Suppose that $u \in H_0^{2,1}(\mathbb{R} \times \mathbb{R}^+)$ is a solution of (3.1). Then for any $\xi_0 \in \mathbb{R}$ the trace $(u, u_\xi)(\xi_0)$ can be defined and is a continuous function $\mathbb{R} \to H_0^{0.75\times0.25}(\mathbb{R}^+)$, cf. [15]. However, to each $(u_0, v_0) \in H_0^{0.75\times0.25}(\mathbb{R}^+)$, there may not exist a solution for (3.1) in $H_0^{2,1}(I \times \mathbb{R}^+)$ such that the trace at ξ_0 is (u_0, v_0).

To be more specific, consider (3.3) as a first order system with the independent variable ξ, we look for $(u, v) \in H_0^{0.75\times0.25}(\gamma)$ in the space of functions in t. The function $A(\xi, t)$ should be smooth enough such that the mapping $u \to A(\xi, t)u$ is continuous from $H_0^{0.75}(\gamma) \to H_0^{0.25}(\gamma)$.

Definition 3.5 We say that (3.3) has an exponential dichotomy in $E = H_0^{0.75\times 0.25}(\gamma)$ for $\xi \in I$ if if there exist projections $\check{P}_s(\xi) + \check{P}_u(\xi) = Id$, partially defined solution operator $\check{T}(\xi, \zeta)$ and constants $K, \alpha > 0$ as in Definition 3.3.

Lemma 3.2 (1) *Assume that* (3.4) *or* (3.5) *has an exponential dichotomy in* $\mathcal{H}^{0.75\times 0.25}(\gamma)$ *for* $\xi \in \mathbb{R}$. *Then* (3.3) *has an exponential dichotomy in* $H_0^{0.75\times 0.25}(\gamma)$ *with the projections* $\check{P}_j(\xi) = \mathcal{L}^{-1} P_j(\xi)\mathcal{L}$ *where* $j = s, u$ *and the partially defined solution operator* $\check{T}(\xi, \zeta) = \mathcal{L}^{-1} T(\xi, \zeta)\mathcal{L}$ *for* $\xi, \zeta \in I$.

(2) *Assume that* (3.5) *has an exponential dichotomy in* $E^{(k+0.5)\times k}(s)$ *for* $k \geq 0$, $Re(s) \geq \gamma, \xi \in \mathbb{R}$. *Then* (3.5) *has an exponential dichotomy in* $\mathcal{H}^{(k+0.5)\times k}(\gamma)$ *with the same projections derived from those in* $E^{(k+0.5)\times k}(s)$*, and the same constants* K, α.

Proof (1) Observe that $(u_0, v_0) \to \mathcal{L}(u_0, v_0)$, $H_0^{0.75\times 0.25}(\gamma) \to \mathcal{H}^{0.75\times 0.25}(\gamma)$ is a Banach spaces isomorphism.

(2) The proof of part (2) follows from that of Lemma 3.1 in [11].

□

3.3 Roughness of Exponential Dichotomies

Consider the abstract differential equation $u_\xi = A(\xi)u$, $\xi \in I$ in the Banach space E. The following result gives the basic facts about persistence of exponential dichotomies under perturbation in a Banach space E.

Theorem 3.3 (Roughness of Exponential Dichotomies) *Assume that* I *is a bounded or unbounded interval in* $\mathbb{R}$, $A(\xi) : E \to E$ *is a bounded operator for each* $\xi \in I$ *and is in* $L^\infty(I)$ *in the norm of bounded operators in* E*, and the linear differential equation in* E, $u_\xi = A(\xi)u$*, has an exponential dichotomy on* I *with projections* $P^0(\xi) + Q^0(\xi) = Id$ *and constants* K_0, $\alpha_0 > 0$. *Assume that* $B(\xi) : E \to E$ *is another bounded linear operator in* $L^\infty(I)$ *with* $\delta = \sup\{|B(\xi)|, \xi \in I\} < \infty$.

Consider the perturbed linear equation

$$u_\xi = (A(\xi) + B(\xi))u. \tag{3.8}$$

Let $0 < \tilde{\alpha} < \alpha_0$*, and assume that* δ *is sufficiently small so that*

$$C_1\delta < 1 \text{ and } C_2\delta < 1, \text{ where } C_1 = \frac{2K_0}{\alpha_0 - \tilde{\alpha}},\ C_2 = \frac{2K_0^2}{(\alpha_0 - \tilde{\alpha})(1 - C_1\delta)}. \tag{3.9}$$

Then (3.8) *also has an exponential dichotomy on* I *with projections* $\tilde{P}(\xi) + \tilde{Q}(\xi) = Id$ *and the exponent* $\tilde{\alpha}$. *The multiplicative constant is* $\tilde{K} = K_0(1 - C_1\delta)^{-1}(1 - C_2\delta)^{-1}$ *and the following inequalities hold for* ξ, $\zeta \in I$*: There exists a partially defined and invariant solution operator* $T_B(\xi, \zeta)$ *for the linear system* (3.8) *that satisfies* (3.7a), (3.7b) *with* T *replaced by* T_B. *And*

$$|T_B(\xi, \zeta)\tilde{P}(\zeta)| \leq \tilde{K} e^{-\tilde{\alpha}(\xi - \zeta)}, \quad \zeta \leq \xi;$$
$$|T_B(\xi, \zeta)\tilde{Q}(\zeta)| \leq \tilde{K} e^{-\tilde{\alpha}(\zeta - \xi)}, \quad \xi \leq \zeta;$$
$$|\tilde{P}(\xi) - P^0(\xi)| \leq \frac{C_2\delta}{1 - C_2\delta}.$$

If E is finite dimensional, then the proof of Theorem 3.3 is well known, [3]. If E is an infinite dimensional Banach space, we cannot write the solution operator backwards in time, the proof is quite different, [5,10]. For a shorter proof with almost identical notations, see [13] (simply replace the rate function $a(x)$ by e^x and the decay rate $(a(x)/a(y))^{-\alpha}$ be $e^{-\alpha(x-y)}$).

3.4 Exponential Dichotomies for Linear Variational Systems Around e_j

We study the linear variational system around e_j and a small perturbation that depends on time t. Consider the linear variational system around e_j and its Laplace transform:

$$u_\xi = v, \quad v_\xi = u_t - \bar{c}_j v - Df(e_j)u, \quad u(\xi, 0) = 0, \tag{3.10}$$

$$\hat{u}_\xi = \hat{v}, \quad \hat{v}_\xi = (sI - Df(e_j))\hat{u} - \bar{c}_j \hat{v}. \tag{3.11}$$

Let $b(\xi, t)$ be piecewise continuous in ξ and is C^1 in $t \in \mathbb{R}^+$. Consider the following perturbed system and its Laplace transform:

$$u_\xi = v, \quad v_\xi = u_t - \bar{c}_j v - Df(e_j)u - b(\xi, t)u, \quad u(\xi, 0) = 0, \tag{3.12}$$

$$\hat{u}_\xi = \hat{v}, \quad \hat{v}_\xi = (sI - Df(e_j))\hat{u} - \hat{b} \overset{s}{*} \hat{u} - \bar{c}_j \hat{v}. \tag{3.13}$$

Lemma 3.4 (1) *If (H1) is satisfied, then system* (3.11) *has an exponential dichotomy in the function space* $E^{(k+0.5)\times k}(s)$ *for* $k \geq 0$, $Re(s) \geq \gamma$, $\xi \in \mathbb{R}$.

(2) *Assume that the linear operator* $\hat{u}(\xi, s) \to \hat{b} \overset{s}{*} \hat{u}$ *is piecewise continuous in* $\xi \in \mathbb{R}$, *uniformly bounded from* $\mathcal{H}^{0.75}(\gamma) \to \mathcal{H}^{0.25}(\gamma)$, *and satisfies*

$$|\hat{b} \overset{s}{*} \hat{u}|_{\mathcal{H}^{0.25}(\gamma)} \leq \delta |\hat{u}(\xi, s)|_{\mathcal{H}^{0.75}(\gamma)}, \quad \xi \in \mathbb{R}.$$

Then if $\delta > 0$ *is sufficiently small, the first order system* (3.13) *also has an exponential dichotomy in* $\mathcal{H}^{0.75\times 0.25}(\gamma)$ *for* $\xi \in \mathbb{R}$.

Proof (1) Simply use the spectral projections of (3.11) as the projections of the dichotomy.
(2) Result from part (1) implies that (3.11) has an exponential dichotomy in $\mathcal{H}^{0.75\times 0.25}(\gamma)$. If δ is small, we can treat (3.13) as a small perturbation of (3.11), then apply Theorem 3.3. □

Lemma 3.5 *Assume the conditions of Lemma* 3.4 *are satisfied. Let* $E_s(\xi)$, $E_u(\xi)$ *be the stable and unstable subspaces of the dichotomy for* (3.13) *in* $\mathcal{H}^{0.75\times 0.25}(\gamma)$.

(1) *Let* $\phi \in E_s(a)$. *For* $\xi \geq a$, *define* $(u, v)^\tau(\xi, t) = \mathcal{L}^{-1}(T(\xi, a)P_s(a)\phi)$. *Then* $u \in H_0^{2,1}([a, \infty) \times \mathbb{R}^+, \gamma)$ *and is a solution to* (3.1). *Moreover*

$$|u|_{H^{2,1}(\gamma)} \leq C|\phi|_{\mathcal{H}^{0.75\times 0.25}(\gamma)}. \tag{3.14}$$

(2) *Let* $\phi \in E_u(a)$. *For* $\xi \leq a$, *define* $(u, v)^\tau(\xi, t) = \mathcal{L}^{-1}(T(\xi, a)P_u(a)\phi)$. *Then* $u \in H_0^{2,1}((-\infty, a] \times \mathbb{R}^+, \gamma)$ *and is a solution to* (3.1). *Estimate* (3.14) *is also satisfied.*

Proof We shall prove (1) only. By the definition of (u, v), we have $\hat{u} \in \mathcal{H}^{0.75}(\gamma)$.

From Lemma 3.1, $\hat{b} \overset{s}{*} \hat{u} \in \mathcal{H}^{0.75}(\gamma) \subset \mathcal{H}^0(\gamma)$. Thus $g = (0, \hat{b} \overset{s}{*} \hat{u})^\tau \in \mathcal{H}^{0.5\times 0}(\gamma)$. Rewrite (3.13) as a first order system

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}_\xi = \begin{pmatrix} 0 & I \\ sI - Df(e_j) & -\bar{c}_j \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} + g. \tag{3.15}$$

Let the projections of the dichotomy for (3.10), in $E^{0.75\times 0.25}(s)$ be $P_s(\xi, s)$ and $P_u(\xi, s)$. Using the solution mapping $T(\xi, \zeta; s)$, the solution of (3.15) in $\xi \geq a$ can be expressed as

$$(\hat{u}, \hat{v})^\tau = T(\xi, a; s)P_s(a, s)\phi(s) + \int_a^\xi T(\xi, \zeta; s)P_s(\zeta, s)g(\zeta)d\zeta$$
$$+ \int_\infty^\xi T(\xi, \zeta; s)P_u(\zeta, s)g(\zeta)d\zeta.$$

Therefore, $(u, v)^\tau) = \mathcal{L}^{-1}(\hat{u}, \hat{v})^\tau$ can be expressed as $(u^{(1)}, v^{(1)})^\tau + (u^{(2)}, v^{(2)})^\tau$, where $(u^{(1)}, v^{(1)})^\tau$ and $(u^{(2)}, v^{(2)})^\tau$ are the inverse L-transform of the first term and the two integral terms respectively. From Lemma 3.1 in [11], $u^{(1)} \in H_0^{2,1}(\xi \geq a, \gamma)$ and is bounded by $|\phi|_{\mathcal{H}^{0.75\times 0.25}(\gamma)}$. From Lemma 3.8 in [11], $u^{(2)} \in H_0^{2,1}(\xi \geq a, \gamma)$ and is bounded by $|g|_{\mathcal{H}^{0.5\times 0}(\gamma)} \leq C|\phi|_{\mathcal{H}^{0.75\times 0.25}(\gamma)}$. The proof of part (1) has been completed. □

3.5 Exponential Dichotomies for Linear Variational Systems Around q_j

We remark that if $b(\xi, t) = b(\xi)$ is independent of t, then

$$\hat{b}(\xi) \overset{s}{*} \hat{u}(\xi, s)) = b(\xi)\hat{u}(\xi, s).$$

This is the case considered in this subsection where $b(\xi) = Df(q_j(\xi)) - Df(e_k), k = j-1$ or j. The principle matrix solution $T(\xi, \eta, s)$ with a parameter s of the linear system

$$U_\xi = V, \quad V_\xi = (sI - Df(q_j(\xi)))U - c_jV, \quad \xi \in \mathbb{R}. \tag{3.16}$$

can be viewed as a linear flow in the Banach space $E^{0.75\times 0.25}(s)$. We now consider the existence of exponential dichotomies for the linear system (3.16).

Lemma 3.6 *Let $(q_j(\xi), q_j'(\xi))$ be the heteroclinic solution connecting $(u, v) = (e_{j-1}, 0)$ to $(e_j, 0)$. Assume that (H1) and (H2) are satisfied. Then in the space $E^{0.75\times 0.25}(s)$, system (3.16), with $s \in \Sigma(-\eta, \theta)\backslash\{0\}$, has an exponential dichotomy on $\mathbb{R}$. The projections of the dichotomy are analytic in s. For any $\epsilon > 0$, if $|s| \geq \epsilon$ then the Projections $P_s(\xi, s)$ and $P_u(\xi, s)$ are uniformly bounded by $K(\epsilon) > 0$. The exponent is $\alpha(1 + |s|^{0.5})$ for some $\alpha > 0$.*

Moreover, let $P_s(e_{j-1}, s)$ and $P_u(e_j, s)$ be the spectral projections at the two limiting points $(e_{j-1}, 0)$ or $(e_j, 0)$. There is a large constant $M > 0$ such that depending on $|s| \geq M$ or $|s| < M$, we have

$$\begin{aligned} |P_s(\xi, s) - P_s(e_{j-1}, s)| &\leq \frac{16K^2(\epsilon)\delta_k}{\alpha(1 + |s|^{0.5})}, \quad \xi \leq -N, \\ |P_u(\xi, s) - P_u(e_j, s)| &\leq \frac{16K^2(\epsilon)\delta_k}{\alpha(1 + |s|^{0.5})}, \quad \xi \geq N, \end{aligned} \tag{3.17}$$

where the constant δ_k are as follows. If $|s| \geq M$ then $k = 1$ and if $\epsilon \leq |s| \leq M$ then $k = 2$, with $\delta_1 = \sup_\xi |Df(q_j(\xi))|$ and δ_2 as in (3.20).

Proof The proof is adapted from that of [11], see also [14].

Step 1: Exponential dichotomy for $|s| \geq M$. Let $M > 0$ be a sufficiently large constant. In the region $\{|s| \geq M\} \cap \Sigma(-\eta, \theta)$, we treat (3.16) as perturbations to the system

$$U_\xi = V, \quad V_\xi = sU + c_jV. \tag{3.18}$$

From [11], the system above has an exponential dichotomy in $E^{0.75\times 0.25}(s)$ with the constant K_0 and the exponent $\alpha_0 = \alpha(1 + |s|^{0.5})$.

Although $\delta_1 = \sup_\xi |Df(q_j(\xi))|$ is not small, but the conditions $C_1\delta_1 < 1$ and $C_2\delta_1 < 1$ in Theorem 3.3 can be satisfied if we choose $\tilde{\alpha} = \alpha(1 + |s|^{0.5})/2$. Then from $\alpha_0 = \alpha(1 + |s|^{0.5})$, $\alpha_0 - \tilde{\alpha} = \alpha(1 + |s|^{0.5})/2$ can be large from the condition $|s| \geq M$ for a large constant M. If M is sufficiently large then (3.9) in Theorem 3.3 is satisfied and system (3.16) has exponential dichotomies in $E^{0.75\times 0.25}(s)$ with the constant $\tilde{K}$ independent of s. The exponent of the dichotomy is $\tilde{\alpha} = \frac{\alpha}{2}(1 + |s|^{0.5})$. The projections satisfy (3.17) with $k = 1$.

Observe that the stable and unstable subspaces of (3.18) are analytic in s. Since the perturbed equaion is analytic in s, and the contraction mapping principle is used to find the stable and unstalbe subspace of (3.16), Thus the projections $P_s(\xi, s)$ and $P_u(\xi, s)$ are analitic in s for $\{|s| > M\} \cap \Sigma(-\eta, \theta)$.

Step 2: Exponential dichotomies on $\mathbb{R}$ for $0 < |s| \le M$. After $M > 0$ has been determined, for any $0 < \epsilon < M$, we consider the spectral equation in the compact set $\{\epsilon \le |s| \le M\} \cap \Sigma(-\eta, \theta)$.

Assume that $N > 0$ is a sufficiently large constant so that on $I_- = (-\infty, -N]$ or $I_+ = [N, \infty)$, $q_j(\xi)$ is close to e_{j-1} or e_j respectively. Consider the following system with constant coefficient, where s as a parameter:

$$U_\xi = V, \quad V_\xi = (sI - Df(e_k))U - c_j V, \quad k = j-1, j. \tag{3.19}$$

From (H1), the eigenvalues for the constant system has n eigenvalues with positive real parts and n eigenvalues with negative real parts. Thus, (3.19) has exponential dichotomies with the common exponent $\alpha_0(s) > 0$, and the projections depend analytically on s. Also in $\Sigma(-\eta, \theta)$, the constant K is uniformly valid with respect to s.

For such $N > 0$, let

$$\delta_2 = \max\{\sup\{|Df(q_j(\xi)) - Df(e_{j-1})| : \xi \le -N\}, \sup\{|Df(q_j(\xi)) - Df(e_j)| : \xi \ge N\}\}. \tag{3.20}$$

If δ_2 as in (3.20) is sufficiently small, then system (3.16) has nonunique exponential dichotomies in I_- and I_+. The unstable subspace $E_u(\xi, s)$, $\xi \le -N$ and the stable subspace $E_s(\xi, s)$, $\xi \ge N$ are unique. Since they are constructed by contraction mapping principle, both spaces depend analytically on $s \in \Sigma(-\eta, \theta)$. We shall use them to construct the unified dichotomy on $\mathbb{R}$. The stable subspace $E_s(\xi, s)$, $\xi \le -N$ and unstable subspace $E_u(\xi, s)$, $\xi \ge N$ are not unique, and shall be modified as follows.

Using the unique subspaces $E_u(-N, s)$, $E_s(N, s)$, we extend them by

$$E_u(\xi, s) = T(\xi, -N, s)E_u(-N, s), \quad \text{for } -N \le \xi \le \infty,$$
$$E_s(\xi, s) = T(\xi, N, s)E_s(N, s) \quad \text{for } -\infty \le \xi \le N.$$

From (H1) and (H2), if $s \ne 0$, $T(N, -N, s)E_u(-N, s)$ intersects with $E_s(N, s)$ transversely, or equivalently $T(-N, N, s)E_s(N, s)$ intersects with $E_u(-N, s)$ transversely. The dichotomy has been extended to $\xi \in \mathbb{R}$, and is analytic for $s \in \Sigma(-\eta, \theta)\backslash\{0\}$ and $|s| \le M$. The exponent of the dichotomy is $\alpha_1(1 + |s|^{0.5})$ where α_1 is independent of s.

In the compact set $\{\epsilon \le |s| \le M\} \cap \Sigma(-\eta, \theta)$, the angle between $E_u(\pm N, s)$ and $E_s(\pm N, s)$ are bounded below by a constant that depends on ϵ. Thus, the constant $K(\epsilon)$ depends on ϵ.

Final Step: If we combine the two cases and select any $0 < \alpha < \min\{\tilde{\alpha}/2, \alpha_1\}$, then (3.16) has an exponential dichotomy in $E^{0.75\times 0.25}(s)$ for $\xi \in \mathbb{R}$ and $s \in \Sigma(-\eta, \theta)\backslash\{0\}$. The exponent is $\alpha(1 + |s|^{0.5})$. This completes the proof of the lemma □

In the next lemma we discuss exponential dichotomies of (3.16) for $s \approx 0$ which is treated as a perturbation of $s = 0$.

Lemma 3.7 *Let $a = -N$ or N where $N > 0$ is the constant as in* (3.20).

(1) *For a small $\epsilon > 0$ and $|s| \le \epsilon$, let $E_u(\xi, s)$, $\xi \le a$ be the unstable subspace and $E_s(\xi, s)$, $\xi \ge a$ be the stable subspace for* (3.16)*. Then the angle between $E_u(a_-, s)$ and $E_s(a_+, s)$ are bounded below by $C|s|$, $C > 0$.*

(2) *For a small* $\epsilon > 0$*, if* $|s| \leq \epsilon$*, then* (3.16) *has two separate dichotomies on* $\xi \in (-\infty, a]$ *and* $[a, \infty)$ *respectively. The two separate dichotomies are not unique. However they can be constructed such that the projections, denoted by*

$$P_s^-(\xi, s) + P_u^-(\xi, s) = Id, \ \xi \leq a; \quad P_s^+(\xi, s) + P_u^+(\xi, s) = Id, \ \xi \geq a,$$

are analytic in s and satisfy the property

$$|P_s^{\pm}(\xi, s)| + |P_u^{\pm}(\xi, s)| \leq K, \ \textit{for all } |s| \leq \epsilon.$$

Proof We prove part (1) first. For each vector $\phi \in E_s(\xi, s = 0)$ there exists a unique $\tilde{\phi} \in E_s(\xi, s)$ such that $\tilde{\phi} - \phi \in E_u(\xi, s = 0)$ and $|\tilde{\phi} - \phi| = \mathcal{O}(|s|)$. Similarly for each vector $\phi \in E_s(\xi, s = 0)$ there exists a unique $\tilde{\phi} \in E_u(\xi, s)$ such that $\tilde{\phi} - \phi \in E_s(\xi, s = 0)$ and $|\tilde{\phi} - \phi| = \mathcal{O}(|s|)$. The perturbation argument used in the proof also shows that the spaces $E_s(\xi, s)$ and $E_u(\xi, s)$ are analytic in s. When $s = 0$, the intersection of $E_u(a_-, s = 0)$ and $E_s(a_+, s = 0)$ is one dimensional, spanned by $(q_j'(a), q_j''(a))$. Melnikov's method can be used to show that the 1D intersection breaks if $s \neq 0$ and small. And the angle is of $\mathcal{O}(|s|)$. See Lemma 3.9 of [9].

To prove part (2), let us consider $a = -N$ only for the case $a = N$ is similar. In $(-\infty, -N]$, define $E_s(-N, s)$ to be a subspace that is orthogonal to $E_u(-N, s = 0)$. Then use the flow $T(\xi, -N, s)$ to define $E_s(\xi, s)$ for $\xi \leq -N$. In $[-N, \infty)$, let the stable subspace be the extension of $E_s(N, s)$ by the flow. Define $E_u(-N, s)$ to be the subspace that is orthogonal to $E_s(-N, s = 0)$, then extend it to $\xi \geq -N$ by the flow. Once the subspaces $E_s(\xi, s)$ and $E_u(\xi, s)$ are defined for $\xi \leq -N$ and $\xi \geq -N$ respectively , the exponential dichotomies on the two separate intervals are determined.

The validity of extension of dichotomies used above has been proved in Lemmas 2.3 and 2.4 in [8]. ◻

The definition of angles between two subspaces and its relation to the norms of P_u, P_s can be found in Lemma 3.9 of [9]. In Lemma 3.10 of that paper, perturbation of a linear system from $\epsilon = 0$ to $\epsilon \neq 0$ but small is discussed, see also [4]. The result can apply to our case by changing ϵ to s. The perturbation argument used in the proof also shows that the dichotomies near $s = 0$ are analytic in s. In particular, based on part (1) of Lemma 3.7, we have the following corollary.

Corollary 3.8 *For the unified dichotomy as in Lemma 3.6, the projections satisfy the property* $|P_s(0, s)| + |P_u(0, s)| \leq C/|s|$.

4 Solution of the Nonhomogeneous Linear System (2.6)

In Sect. 4.1, we solve the initial value problem (2.6a), ignoring the jump condition (2.6b). Then in Sect. 4.2, we solve the full linear system (2.6) with $h_j = 0$ and $u_{j0} = 0$. These results can be combined to solve (2.6).

4.1 Solve the Nonhomogeneous System with Initial Conditions

In this subsection we look for a solution u_j of the nonhomogenous system with the initial condition in each Ω_j, $1 \leq j \leq m$.

$$u_{jt} = u_{jxx} + Df(q_j)u_j + h_j, \quad u_j(x, 0) = u_{j0}(x), \tag{4.1}$$

where $u_{j0}(x)$ is the restriction of $u_0(x)$ to (x_{j-1}, x_j). We will ignore the jump conditions and leave them for the next subsection.

Assume that $u_{j0} \in H^1(x_{j-1}, x_j)$ and $h_j \in L^2(\Omega_j, \gamma), \gamma < 0$. We extend the initial data and the forcing term to the whole space $u_{j0}(x) \to \tilde{u}_{j0}(x),\ h_j(x,t) \to \tilde{h}_j(x,t)$ so the fundamental solution can be used to solve (4.1). In particular, we make the zero extension of h_j to $\tilde{h}_j$ outside Ω_j. We extend $u_{j0} \in H^1(x_{j-1}, x_j)$ to $\tilde{u}_{j0} \in H^1(\mathbb{R})$ by a bounded extension operator $H^1(x_{j-1}, x_j) \to H^1(\mathbb{R})$. Then consider the initial value problem in $\mathbb{R} \times \mathbb{R}^+$,

$$\tilde{u}_{jt} = \tilde{u}_{jxx} + Df(q_j)\tilde{u}_j + \tilde{h}_j(x,t), \quad \tilde{u}_j(x,0) = \tilde{u}_{j0}(x).$$

In the moving coordinates $\xi = x - y_j - c_j t$, we have

$$\tilde{u}_{jt} = \tilde{u}_{j\xi\xi} + c_j\tilde{u}_{j\xi} + Df(q_j)\tilde{u}_j + \tilde{h}_j(\xi,t), \quad \tilde{u}_j(\xi,0) = \tilde{u}_{j0}(\xi). \tag{4.2}$$

Recall that $\lambda = 0$ is always an eigenvalue for the associated homogeneous equation to (4.2). We want to show that the solution u_j will have a term $\beta_j(t)q_j'$ in the eigenspace associated to $\lambda = 0$, and the remaining part approaches zero exponentially.

Consider the general linear equation

$$U_t = U_{\xi\xi} + c_jU_\xi + Df(q_j)U, \quad 1 \le j \le m.$$

Recall that $q_j'(\xi) \in \ker(L_{cj})$, $z_j \in \ker(L_{cj}^*)$ and $\int_{-\infty}^{\infty} < z_j, q_j > dx = 1,\ 1 \le j \le m$. The spectral projection to the eigenspace corresponding to $\lambda = 0$ is

$$P_jU(x) = \left(\int_{-\infty}^{\infty} < z_j(x), U(x) > dx\right) q_j'(x).$$

The complementary projection is $Q_j := Id - P_j$. Define

$$\mathcal{X}_j := \{Y : \int_{-\infty}^{\infty} < z_j, Y > dx = 0\}. \tag{4.3}$$

Then $RQ_j = \mathcal{X}_j$, which is an invariant subspace with all the spectrum points in the complement of $\Sigma(-\eta, \theta)$. See the condition (A2) following H1 and H2.

By the spectral decomposition, $u_j(\xi,t) = Y_j(\xi,t) + \beta_j(t)q_j'$ where $Y_j \in \mathcal{X}_j$. The operator $L_jY = Y_{xx} + c_jY_x + Df(q_j)Y$ defined on $\mathcal{X}_j$ is sectorial and generates an analytic semigroup $e^{L_{cj}t}$. For $\tilde{u}_{j0} \in H^1(\mathbb{R})$ and $\tilde{h}_j \in L^2(\gamma)$, we have

$$Y_j = e^{L_jt}Q_j\tilde{u}_{j0} + \int_0^t e^{L_j(t-\tau)}Q_j\tilde{h}_j(\tau)d\tau. \tag{4.4}$$

From Lemma 3.11 of [11], it is easy to show if $-\eta < \gamma < 0$, then $Y_j \in H^{2,1}(\mathbb{R} \times \mathbb{R}^+, \gamma)$ and satisfies

$$|Y_j|_{H^{2,1}(\gamma)} \le C(|\tilde{u}_{j0}| + |\tilde{h}_j|) \le C(|u_{j0}|_{H^1(\mathbb{R})} + |h_j|_{L^2(\gamma)}). \tag{4.5}$$

We then consider the equation in RP_j:

$$\dot{\beta}_j(t) = \int_{-\infty}^{\infty} < z_j(\xi), \tilde{h}_j(\xi,t) > d\xi, \quad \beta_j(0) = \int_{-\infty}^{\infty} < z_j(\xi), \tilde{u}_{j0}(\xi) > d\xi. \tag{4.6}$$

Since $\tilde{h}_j \in L^2(\gamma)$, we have $\dot{\beta}_j \in L^2(\gamma), \gamma < 0$. By solving this ODE we obtain the solution $\beta_j(t)q_j'(x)$ in the space RP_j. Using also (2.1), we have

$$\begin{aligned} &|\beta_j(0)| \le C|\tilde{u}_{j0}| \le C|u_{j0}|_{H^1(\mathbb{R})}, \quad |\dot{\beta}_j|_{L^2(\gamma)} \le C|h_j|_{L^2(\gamma)}, \\ &|\beta_j(\infty) - \beta_j(t)| \le Ce^{\gamma t}|h_j|_{L^2(\gamma)}, \quad |\beta_j|_{X^1(\gamma)} \le C(|u_{j0}|_{H^1(\mathbb{R})} + |h_j|_{L^2(\gamma)}). \end{aligned} \tag{4.7}$$

By restricting Y_j to Ω_j, we have the following theorem.

Theorem 4.1 *In each* $\Omega_j, 1 \le j \le m$, *the initial value problem* (4.1) *has a solution* $u_j(\xi,t) = Y_j(\xi,t) + \beta_j(t)q_j'$ *where* $Y_j \in H^{2,1}(\Omega_j,\gamma)$ *satisfies* (4.5) *and* $\beta_j \in X^1(\gamma)$ *satisfies* (4.7). *The solution is the restriction of the solution of* (4.2) *in* $\mathbb{R}\times\mathbb{R}^+$ *to* Ω_j, *and is unique once the extension operators are fixed.*

4.2 System of Equations with Jumps Along Γ_j

Let $\{u_j^{(1)}(x,t)\}$ be the solution of (4.1) obtained in Sect. 4.1 and

$$\tilde{J}_j(\Gamma_j) := -([\{u_j^{ap} + u_j^{(1)}\}, \{u_{jx}^{ap} + u_{jx}^{(1)}\}])(\Gamma_j) = J_j(\Gamma_j) - ([\{u_j^{(1)}\}, \{u_{jx}^{(1)}\}])(\Gamma_j).$$

In this subsection, we consider the linear system for u_j defined on Ω_j with nonzero jump conditions along Γ_j:

$$u_{jt} = u_{j,xx} + Df(q_j)u_j,\ (x,t)\in\Omega_j,\quad u_j(x,0)=0,\quad ([\{u_j\}\{u_{jx}\}])(\Gamma_j) = \tilde{J}_j(\Gamma_i), \tag{4.8}$$

The main results of this subsection are summarized in the following theorem:

Theorem 4.2 *Given* $\{\tilde{J}_j(\Gamma_j)\} \in \prod_1^{m-1} H_0^{0.75\times0.25}(\gamma)$, *under (H1)–(H2), if* $y_{j+1}-y_j$ *is sufficiently large, then the linear system* (4.8) *has a unique solution* $\{u_j(x,t)\}$ *that can be expressed as* $u_j = \beta_j(t)q_j' + Y_j$, *where* $Y_j \in H_0^{2,1}(\Omega_j,\gamma)$, $\beta_j(0)=0$, $\dot{\beta}_j(t)\in L^2(\gamma)$. *The solution mapping, expressed as* $\{\tilde{J}_j(\Gamma_j)\}_{j=1}^{m-1} \to (\{(Y_j,\beta_j)\}_{j=1}^m$ *is a bounded operator*

$$\prod_1^{m-1} H_0^{0.75\times0.25}(\gamma) \to \prod_1^m (H_0^{2,1}(\Omega_j,\gamma)\times X^1(\gamma)).$$

Proof Let N be the fixed large constant defined in Lemma 3.6 and let

$$\begin{aligned} y_j^- &= y_j - N,\quad M_j^- = \{(x,t): x = y_j^- + c_jt,\ t\ge 0\},\\ y_j^+ &= y_j + N,\quad M_j^+ = \{(x,t): x = y_j^+ + c_jt,\ t\ge 0\}. \end{aligned} \tag{4.9}$$

See Fig. 2. The proof of the theorem is based on an iteration process by repeating Part A and Part B described below. First, we use Pat A to achieve the prescribed jumps along $\Gamma_j, 1\le j\le m-1$. In doing so we introduced some jump error along the line $M_j^{\pm}, 1\le j\le m$. Then we use Pat B to eliminate the jumps along $M_j^{\pm}$ which in turn introduces some jump errors back to Γ_j. However the jump errors along Γ_j are exponentially smaller than the prescribed jumps along Γ_j. We can repeat procedures in Part A and Part B to treat the jump errors along Γ_j, each time reduce the errors by an exponentially small factor. The iteration process converges to the exact solution with the prescribed jump conditions along Γ_j.

Due to the lack of a unified exponential dichotomy when looking for a solution with the prescribed jump along $M_j^{\pm}$, we introduce a term $\beta_j(t)q_j'$ in the solution of the linear system. This is done each time the iteration is performed so the term $\beta_j(t)q_j'$ is the sum of an infinite series that converges at the rate of a geometric series. Details will be given at the end of this section. □

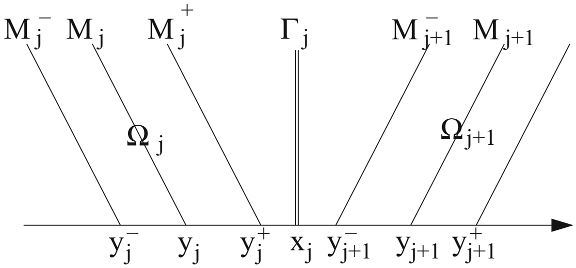

Fig. 2 Illustration of the lines $M_j^{\pm}$ defined in (4.9)

Part A: We look for a piecewise smooth $u(x,t)$ that is defined between M_j^+ and M_{j+1}^-, with the jump δ_j along Γ_j, and satisfies the equations:

$$u_t = u_{xx} + Df(q_j)u, \quad \text{if } (x,t) \text{ is between } M_j^+ \text{ and } \Gamma_j, \tag{4.10}$$

$$u_t = u_{xx} + Df(q_{j+1})u, \quad \text{if } (x,t) \text{ is between } \Gamma_j \text{ and } M_{j+1}^-. \tag{4.11}$$

$$u(x,0) = 0 \quad [(u,u_x)](\Gamma_j) = \delta_j. \tag{4.12}$$

We are interested in solutions that decay exponentially as (x,t) moves away from Γ_j. The solution between M_j^+ and M_{j+1}^- is non-unique, depends on the modification of the vector fields to the left of M_j^+ and the right of M_{j+1}^-, as will be specified in the proof.

Lemma 4.3 *For each $\delta_j \in H_0^{0.75\times 0.25}(\gamma)$ defined on Γ_j, there exists a piecewise smooth solution u defined on $\mathbb{R}\times\mathbb{R}^+$ that satisfies equations (4.10), (4.11) and jump condition (4.12). The support of u is between M_j^+ and M_{j+1}^-. Moreover the solutions satisfy the following estimates*

$$|u(x,t)|_{M_j^+}| + |u(x,t)|_{M_{j+1}^-}| \le C(e^{-\alpha(x_j-y_j-N)} + e^{-\alpha(y_{j+1}-x_j-N)})|\delta_j|. \tag{4.13}$$

where all the norms are in $H_0^{0.75\times 0.25}(\gamma)$.

Proof Using the moving coordinate $\xi = x - x_j - \bar{c}_j t$, the line Γ_j becomes $\xi = 0$. Equations (4.10), (4.11) become

$$u_t = u_{\xi\xi} + \bar{c}_j u_\xi + Df(q_k)u, \ k = j, j+1$$

respectively.

From the definitions of $y_j^{\pm}$ and $M_j^{\pm}$ in (4.9) and $\xi = x - x_j - \bar{c}_j t$, we have

$$M_j^+ = \{(\xi,t) : \xi = y_j - x_j + N + (c_j - \bar{c}_j)t, \ t \ge 0\},$$
$$M_{j+1}^- = \{(\xi,t) : \xi = y_{j+1} - x_j - N + (c_{j+1} - \bar{c}_j)t, \ t \ge 0\}.$$

To the left and right of Γ_j, let $u^{ap}(x,t) = q_k(x - y_k - c_k t)$, $k = j, j+1$. If (ξ,t) is between M_j^+ and M_{j+1}^-, let $A(\xi,t) = Df(u^{ap}(\xi + x_j + \bar{c}_j t, t))$. Using a smooth cut-off function, we can extend $A(\xi,t)$ to all $\mathbb{R}\times\mathbb{R}^+$, so that to the left of M_j^+,

$$A(\xi,t) = Df(q_j(N), \quad \xi = y_j - x_j + N + (c_j - \bar{c}_j)t,$$
$$A(\xi,t) = Df(e_j), \quad \xi < y_j - x_j + N - 1 + (c_j - \bar{c}_j)t.$$

Similarly to the right of M_{j+1}^-,

$$A(\xi,t) = Df(q_{j+1}(-N)), \quad \xi = y_{j+1} - x_j - N + (c_{j+1} - \bar{c}_j)t,$$
$$A(\xi,t) = Df(e_j), \quad \xi \geq y_{j+1} - x_j - N + 1 + (c_{j+1} - \bar{c}_j)t.$$

Also $B(\xi,t) := A(\xi,t) - Df(e_j) = \mathcal{O}(e^{\gamma t})$ is piecewise continuous in ξ and C^1 in t. It is uniformly small for all $\xi \in \mathbb{R}$ if N is sufficiently large. Since the system $u_\xi = v,\ v_\xi = u_t - \bar{c}_j v - Df(e_j)u$ has an exponential dichotomy for $\xi \in \mathbb{R}$, by the roughness of exponential dichotomies, the linear system

$$u_\xi = v, \;\; v_\xi = u_t - \bar{c}_j v - A(\xi,t)u$$

has an exponential dichotomy for all $\xi \in \mathbb{R}$ in $H_0^{0.75\times 0.25}(\gamma)$. Applying the exponential weight function to Bu and u, from Lemma 3.1, we have $|Bu|_{H_0^{0.75}(\gamma)} \leq |B|_{C^1}|u|_{H_0^{0.75}(\gamma)}$. Since $\delta = |\hat{B}(\xi,s)|_{C^1}$ can be arbitrarily small if N is sufficiently large, the existence of the exponential dichotomy follows from Theorem 3.3.

Let the projections of this dichotomy be denoted $\check{P}_u(0-) + \check{P}_s(0+) = Id$ at $\xi = 0$. For the given $\delta_j \in H_0^{0.75\times 0.25}(\gamma)$, let

$$u_-^1(\xi) = -T(\xi,0)\check{P}_u(0-)\delta_j, \quad u_+^1(\xi) = T(\xi,0)\check{P}_s(0+)\delta_j.$$

Then

$$|u_-^1(y_j^+)| \leq Ce^{-\alpha(x_j-y_j-N)}|\delta_j|, \quad |u_+^1(y_{j+1}^-)| \leq Ce^{-\alpha(y_{j+1}-x_j-N)}|\delta_j|.$$

To the left of $\xi = y_j^+$, or to the right of $\xi = y_{j+1}^-$, we have

$$|u_-^1|_{H^{2,1}(\gamma)} \leq Ce^{-\alpha(x_j-y_j-N)}|\delta_j|, \quad |u_+^1|_{H^{2,1}(\gamma)} \leq Ce^{-\alpha(y_{j+1}-x_j-N)}|\delta_j|,$$

Therefore, the traces of u_-^1 and u_+^1 on M_j^+ and M_{j+1}^- are exponentially small. This proves (4.13). Finally we truncate $u_\pm^1$ so that to the left of M_j^+ and to the right of M_{j+1}^-, $u_\pm^1 = 0$. □

For the truncated $u_\pm^1$, the jump condition (4.12) along Γ_j is satisfied, but the function $u_\pm^1$ has jump discontinuities along M_j^+ and M_{j+1}^-. Notice that

$$x_j - y_j - N = y_{j+1} - x_j - N = (y_{j+1} - y_j)/2 - N, \quad 1 \leq j \leq m-1.$$

From (4.13), the jumps are exponentially small in $H_0^{0.75\times 0.25}(\gamma)$ if $y_{j+1} - y_j,\ 1 \leq j \leq m-1$ are sufficiently large.

Part B: We consider a linear variational PDE around $q_j(\xi_j)$ in the domain $\mathbb{R}\times\mathbb{R}^+$ with the zero initial condition and two prescribed jumps along $M_j^\pm$:

$$u_t = u_{xx} + Df(q_j)u, \quad u(x,0) = 0, \;\; [(u,u_x)](M_j^\pm) = \phi_j^\pm.$$

We can treat one jump at a time. To combine the two cases, let $a = -N$ or N, where $N > 0$ is the fixed large constant in Lemma 3.6 and let $M_a := \{\xi = a\} = \{x = a + y_j + c_j t\}$. In the moving coordinates, the equations before and after the Laplace transform are:

$$u_t = u_{\xi\xi} + c_j u_\xi + Df(q_j(\xi))u, \quad u(\xi,0) = 0, \quad [(u,u_\xi)](M_a) = \phi_a, \tag{4.14}$$
$$0 = \hat{u}_{\xi\xi} + c_j\hat{u}_\xi - s\hat{u} + Df(q_j(\xi))\hat{u}, \quad [(\hat{u},\hat{u}_\xi)](M_a) = \hat{\phi}_a. \tag{4.15}$$

Converting to the first order system

$$\hat{u}_\xi = \hat{v}, \;\; \hat{v}_\xi = (sI - Df(q_j(\xi)))\hat{u} - c_j\hat{v}, \quad [(\hat{u},\hat{v})](M_a) = \hat{\phi}_a. \tag{4.16}$$

The specified jump $\phi_a(t)$ is a function in $H_0^{0.75\times 0.25}(\gamma)$, and $\hat{\phi}_a(s)$ is in $\mathcal{H}^{0.75\times 0.25}(\gamma)$. We look for solutions that decay to zero as ξ moves away from M_a.

Lemma 4.4 *For $s \in \Sigma(-\eta,\theta)\backslash\{0\}$, system (4.15) has a unique solution that decays exponentially as $\xi \to \pm\infty$. If $0 < \epsilon \le |s|$, then the solution satisfies*

$$|\hat{u}(\cdot,s)|_{L^2(\xi\le a)} + |\hat{u}(\cdot,s)|_{L^2(\xi\ge a)} \le \frac{C(\epsilon)}{1+|s|}|\hat{\phi}_a|_{E^{0.75\times 0.25}(s)}, \tag{4.17}$$

$$|(\hat{u},\hat{v})(\xi,s)|_{E^{0.72\times 0.25}(s)} \le C(\epsilon)e^{-\alpha|\xi-a|}|\hat{\phi}_a|_{E^{0.72\times 0.25}(s)}. \tag{4.18}$$

The constant $C(\epsilon) = \mathcal{O}(1/\epsilon)$ as $\epsilon \to 0$.

Proof Using the unified exponential dichotomy which is analytic in $s \in \Sigma(-\eta,\theta)\backslash\{0\}$, we can express the solution of (4.15) as follows:

$$\begin{aligned}(\hat{u}(\xi,s),\hat{v}(\xi,s))^\tau &= -T(\xi,a,s)P_u(a,s)\hat{\phi}_a(s), \quad \xi \le a,\\ (\hat{u}(\xi,s),\hat{v}(\xi,s))^\tau &= T(\xi,a,s)P_s(a,s)\hat{\phi}_a(s), \quad \xi \ge a.\end{aligned} \tag{4.19}$$

The analytic functions $(\hat{u},\hat{v})$ may have a simple pole at $s = 0$.

The proof of (4.17), (4.18) follows from the existence of an exponential dichotomy for (4.15) and part (2) of Lemma 3.1 in [11]. □

Our next step is to treat (4.15) at $s \approx 0$. To this end, we write $u(\xi,t) = Y(\xi,t)+\beta(t)q'(\xi)$ where $Y(\cdot,t) \in \mathcal{X}_j$ is defined in (4.3). The initial conditions are $Y(\xi,0) = 0$ and $\beta(0) = 0$. Then before and after the Laplace transform, we have

$$\begin{aligned}&Y_t = Y_{\xi\xi} + c_jY_\xi + Df(q_j)Y - \dot{\beta}(t)q'(\xi), \quad [(Y,Y_\xi)](M_a) = \hat{\phi}_a,\\ &\hat{Y}_{\xi\xi} + c_j\hat{Y}_\xi + Df(q_j)\hat{Y} - s\hat{Y} = s\beta(s)q'(\xi), \quad [(\hat{Y},\hat{Y}_\xi)](a) = \hat{\phi}_a.\end{aligned} \tag{4.20}$$

Multiplying by z_j and integrating by parts, we obtain a necessary condition for (4.20) to be solvable in the domain $\hat{Y} \in \mathcal{X}_j$, $s \in \Sigma(-\eta,\theta)$:

$$\int_{-\infty}^{\infty} < z_j(\xi), s\beta(s)q'(\xi) > d\xi + < (c_jz_j - z_j', z_j)(a), \hat{\phi}_a >= 0. \tag{4.21}$$

From (4.21) and $\int < z_j(\xi), q_j'(\xi) > d\xi = 1$, we have:

$$s\hat{\beta}(s) = - < (c_jz_j - z_j', z_j)(a), \hat{\phi}_a > . \tag{4.22}$$

From $\hat{\phi}_a \in \mathcal{H}^{0.75\times 0.25}(\gamma)$, $s\beta(s) \in \mathcal{H}^{0.25}(\gamma)$. Thus, for $t \ge 0$, $\dot{\beta}(t) = \mathcal{L}^{-1}(s\beta(s))$ is determined and $|\dot{\beta}(t)|_{L^2(\gamma)} \le C|\phi(a)|_{L^2(\gamma)}$, $\gamma < 0$. This together with $\beta(0) = 0$ determines $\beta(t)$ for all $t \ge 0$. We also find that $|\beta(t) - \beta(\infty)| \le Ce^{\gamma t}|\phi(a)|_{L^2(\gamma)}$. From (4.22), the function $s\beta(s)$ is analytic for $s \in \Sigma(-\eta,\theta)$, $\beta(s)$ has an isolated pole at $s = 0$. Denote $\hat{h} = (0, s\beta(s)q')^\tau$. Clearly for each ξ, $\hat{h}(\xi,\cdot) \in \mathcal{H}^{0.75\times 0.25}(\gamma)$ and hence as a function of s, $|\hat{h}|_{L^2(\mathbb{R})} \in \mathcal{H}^{0.75\times 0.25}(\gamma)$.

Express (4.20) as a first order system:

$$(\hat{Y},\hat{Z})_\xi^\tau = (\hat{Z}, (sI - Df(q_j))\hat{Y} - c_j\hat{Z})^\tau + \hat{h}(\xi,s), \quad [(\hat{Y},\hat{Y}_\xi)](a) = \hat{\phi}_a. \tag{4.23}$$

In Lemma 4.4, we have obtained (u,u_ξ) for $s \in \Sigma(-\eta,\theta)\backslash\{0\}$. Converting the results to $(\hat{Y},\hat{Z})$, we find that $(\hat{Y},\hat{Z})$ are analytic for $s \in \Sigma(-\eta,\theta)\backslash\{0\}$ and satisfy

$$\begin{aligned}&|\hat{Y}(\cdot,s)|_{L^2(\xi\le a)} + |\hat{Y}(\cdot,s)|_{L^2(\xi\ge a)} \le \frac{C(\epsilon)}{1+|s|}|\hat{\phi}_a|_{E^{0.75\times 0.25}(s)},\\ &|(\hat{Y},\hat{Z})(\xi,s)|_{E^{0.72\times 0.25}(s)} \le C(\epsilon)e^{-\alpha|\xi-a|}|\hat{\phi}_a|_{E^{0.72\times 0.25}(s)}.\end{aligned} \tag{4.24}$$

Lemma 4.5 *If for the dichotomies to the left and right of* $\xi = a$,

$$E_s(a_+, s=0) \cap E_u(a_-, s=0) = span\{(q_j'(a), q_j''(a))\},$$

and if $\hat{\beta}(s)$ *satisfies* (4.22)*, then in a neighborhood of* $s = 0$*, the functions* $(\hat{Y}, \hat{Z})$ *are holomorphically extendable over* $s = 0$*. Moreover, if* $|s| \le \epsilon$,

$$|\hat{Y}(\cdot, s)|_{L^2(\xi \le a)} + |\hat{Y}(\cdot, s)|_{L^2(\xi \ge a)} \le C|\hat{\phi}_a|_{E^{0.75\times 0.25}(s)}, \tag{4.25}$$

$$|(\hat{Y}, \hat{Z})(\xi, s)|_{E^{0.72\times 0.25}(s)} \le Ce^{-\alpha|\xi - a|}|\hat{\phi}_a|_{E^{0.72\times 0.25}(s)}. \tag{4.26}$$

Proof For each $|s| \le \epsilon$, there exist two separate dichotomies for (4.23), one for $\xi \in (-\infty, a]$ the other for $\xi \in [a, \infty)$. The projections are denoted by $P_s^- + P_u^- = Id$ for $\xi \le a$ and $P_s^+ + P_u^+ = Id$ for $\xi \ge a$. Observe that unlike the unified dichotomy defined for all $\xi \in \mathbb{R}$, The two separate dichotomies satisfy the property

$$|P_s^{\pm}(\xi, s)| + |P_u^{\pm}(\xi, s)| \le K, \text{ for all } |s| \le \epsilon.$$

We can express the solution of (4.23) as follows:

$$\begin{aligned}
(\hat{Y}(\xi, s), \hat{Z}(\xi, s))^\tau &= T(\xi, a, s)P_u^-(a, s)(\hat{Y}(a, s), \hat{Z}(a, s)) \\
&\quad + \int_{-\infty}^{\xi} T(\xi, \zeta, s)P_s^-(\zeta, s)(\hat{h}(\zeta, s))d\zeta + \int_a^{\xi} T(\xi, \zeta, s)P_u^-(\zeta, s)(\hat{h}(\zeta, s))d\zeta, \text{ for } \xi \le a, \\
(\hat{Y}(\xi, s), \hat{Z}(\xi, s))^\tau &= T(\xi, a, s)P_s^+(a, s)(\hat{Y}(a, s), \hat{Z}(a, s)) \\
&\quad + \int_a^{\xi} T(\xi, \zeta, s)P_s^+(\zeta, s)(\hat{h}(\zeta, s))d\zeta + \int_{\infty}^{\xi} T(\xi, \zeta, s)P_u^+(\zeta, s)(\hat{h}(\zeta, s))d\zeta, \text{ for } a \le \xi.
\end{aligned} \tag{4.27}$$

The solution is determined by a pair of vectors:

$$\mu_u(a_-, s) := P_u^-(a, s)(\hat{Y}(a, s), \hat{Z}(a, s)), \quad \mu_s(a_+, s) := P_s^+(a, s)(\hat{Y}(a, s), \hat{Z}(a, s)).$$

To satisfy the jump condition at $\xi = a$, we need

$$\begin{aligned}
&\mu_s(a_+, s) - \mu_u(a_-, s) = \\
&\quad \hat{\phi}_a + \int_{-\infty}^{a} T(a, \zeta, s)P_s^-(\zeta, s)(\hat{h}(\zeta, s))d\zeta + \int_a^{\infty} T(a, \zeta, s)P_u^+(\zeta, s)(\hat{h}(\zeta, s))d\zeta.
\end{aligned} \tag{4.28}$$

Let the right hand side of (4.28) be $d(a, s)$ which is analytic for $|s| \le \epsilon$ and satisfies

$$|d(a, s)|_{E^{0.75\times 0.25}(s)} \le C|\phi_a|_{E^{0.75\times 0.25}(s)}.$$

We wish to solve the equation $\mu_s(a_+, s) - \mu_u(a_-, s) = d(a, s)$ in $E^{0.75\times 0.25}(s)$ such that $\mu_s(a_+, s) \in RP_s^+(a_+, s)$ and $\mu_u(a_-, s) \in RP_u^-(a_-, s)$. Notice that

$$\begin{aligned}
&RP_s^+(a_+, s) = E_s(a_+, s), \quad RP_u^-(a_-, s) = E_u(a_-, s), \\
&E_s^+(a_+, s) \oplus E_u^-(a_-, s) = \mathbb{R}^{2n}, \quad \text{if } s \in \Sigma(-\eta, \theta)\backslash\{0\}.
\end{aligned}$$

For such s, the unique pair of solutions $(\mu_s(a_+, s), \mu_u(a_-, s))$ can be expressed by the unified dichotomy that is defined on all $\mathbb{R}$:

$$\mu_s(a_+, s) = P_s(a, s)d(a, s), \quad \mu_u(a_-, s) = -P_u(a, s)d(a, s). \tag{4.29}$$

The analytic functions $(\mu_s(a_+, s), \mu_u(a_-, s))$ may have a simple pole at $s = 0$. We now show that the pole is removable, that is, in $E^{0.75\times 0.25}(s)$,

$$|\mu_s(a_+, s)| + |\mu_u(a_-, s)| \leq C|\hat{\phi}_a|, \text{ if } 0 < |s| \leq \epsilon. \tag{4.30}$$

In the above, as well as in the rest of this section, any unmarked norms are $E^{0.75\times 0.25}(s)$ norms.

The idea of the proof follows from that of Lemma 3.10 of [11], which also shows that the dichotomy has a pole at $s = 0$. Recall that z_j satisfies the adjoint equation

$$z_j'' - c_j z_j + Df(q_j)^* z_j = 0.$$

Converting this second order equation into a first order system, we can show that the adjoint equation of (4.16) has a bounded solution $\Psi = (\psi_1, \psi_2) := (c_j z_j - z_{j\xi}, z_j)$, which satisfies

$$\Psi(a) \perp E_u(a, s = 0) + E_s(a, s = 0).$$

From (4.21), we have $< \Psi(a), d(a, s = 0) >= 0$. Therefore $| < \Psi(a), d(a, s) > | \leq C|s|$. Apply the orthogonal projections to $d(a, s)$ so that $d(a, s) = d^T(a, s) + d^\perp(a, s)$ where $d^T(a, s) \in E_s(a_+, s = 0) + E_u(a_-, s = 0)$ and $d^\perp(a, s) \in \text{span}\{\Psi(a)\}$. From $< \Psi(a), d(a, s = 0) >= 0$, we have $d^\perp(a, 0) = 0$, thus $|d^\perp(a, s)| \leq C|s|$. From Lemma 3.7, the unified projections satisfy $|P_s(a, s)| + |P_u(a, s)| \leq C/|s|$. Therefore

$$|P_s(a, s)d^\perp(a, s)| + |P_u(a, s)d^\perp(a, s)| \leq C|\phi_a|, \quad \text{if } 0 < |s| \leq \epsilon. \tag{4.31}$$

We now prove that if $0 < |s| \leq \epsilon$ for a small $\epsilon > 0$, then

$$|P_s(a, s)d^T(a, s)| + |P_u(a, s)d^T(a, s)| \leq C|\phi_a|. \tag{4.32}$$

We can write $d^T(a, s) = d_1(a, s) + d_2(a, s)$ where $d_1(a, s) \in E_s(a_+, s = 0)$ and $d_2(a, s) \in E_u(a_-, s = 0)$. We further require that $d_2(a, s) \perp \text{span}\{(q_j'(a), q_j''(a))\}$ so the decomposition is unique and satisfies:

$$|d_1(a, s)| + |d_2(a, s)| \leq C|d^T(a, s)|.$$

We now consider the perturbations of $d_1(a, s)$ and $d_2(a, s)$. First, a perturbation theorem to the stable subspace, see Lemma 3.5 of [10], shows that for $0 < |s| \leq \epsilon$, there exists a unique $\tilde{d}_1(a, s)$ such that $\tilde{d}_1(a, s) \in E_s(a_+, s)$ and $\tilde{d}_1(a, s) - d_1(a, s) \in E_u(a_+, s = 0)$. Moreover $d_1(a, s) - \tilde{d}_1(a, s) = \mathcal{O}(s)$. A simpler proof for finite dimensional spaces can be find in Lemma 2.3 of [13] (simply change the algebraic decay rate to exponential decay rate). Similarly, there exists a unique $\tilde{d}_2(a, s)$ such that $\tilde{d}_2(a, s) \in E_u(a_-, s)$ and $d_2(a, s) - \tilde{d}_2(a, s) \in E_a(a_-, s = 0)$. Moreover $d_2(a, s) - \tilde{d}_2(a, s) = \mathcal{O}(s)$.

We can easily check the following:

$$P_s(a, s)d^T(a, s) = P_s(a, s)(d_1 + d_2) = \tilde{d}_1 + P_s\left((d_1 - \tilde{d}_1) + (d_2 - \tilde{d}_2)\right),$$
$$|P_s(a, s)d^T(a, s)| \leq |\tilde{d}_1(a, s)| + |P_s|(|\tilde{d}_1 - d_1| + |\tilde{d}_2 - d_2|).$$

Using $|\tilde{d}_1(a, s)| \leq C|d_1(a, s)| \leq C|d^T(a, s)|$, $|P_s| = \mathcal{O}(1/s)$ and $|d_j(a, s) - \tilde{d}_j(a, s)| = \mathcal{O}(s)$, $j = 1.2$, we have $|P_s(a, s)d^T(a, s)| \leq C|d^T(a, s)|$ for $0 < |s| \leq \epsilon$.

Similarly we can prove that $|P_u(a, s)d^T(a, s)| \leq C|d^T(a, s)|$ for $0 < |s| \leq \epsilon$. Combining (4.29), (4.31) and (4.32), we have shown that $(\mu_s(a_+, s), \mu_u(a_-, s))$ are holomorphically extendable over $s = 0$, and satisfies (4.30).

If $s = 0$ were not singular for the projections P_s, P_u, then from (4.27) we could prove that $\hat{Y} \in H_0^{2,1}(\xi \leq a)$ and $\hat{Y} \in H_0^{2,1}(\xi \geq a)$ just like [11]. The idea of the proof still works under the restriction $|s| \le \epsilon$ for a small $\epsilon > 0$. In particular, part (2) of Lemma 3.1 in [11] implies that

$$|T(\xi, a, s)P_u^-(a, s)(\hat{Y}(a, s), \hat{Z}(a, s))|_{L^2(\xi \le a)} \le C|\hat{\phi}_a|_{E^{0.75\times 0.25}(s)}.$$

The proof of Lemma 3.8 in [11] implies that the $L^2(\xi \le a)$ norms of the two terms

$$\int_{-\infty}^{\xi} T(\xi, \zeta, s)P_s^-(\zeta, s)(\hat{h}(\zeta, s))d\zeta \text{ and } \int_a^{\xi} T(\xi, \zeta, s)P_u^-(\zeta, s)(\hat{h}(\zeta, s))d\zeta$$

are also bounded by $C|\hat{\phi}_a|_{E^{0.75\times 0.25}(s)}$. Therefore

$$|\hat{Y}|_{L^2(\xi \le a)} \le C|\hat{\phi}_a|_{E^{0.75\times 0.25}(s)}, \quad |s| \le \epsilon.$$

Similar estimates can be obtained for $\xi \ge a$ from the second half of (4.27). This proves (4.25).

Now consider the three terms in (4.27) for $\xi \le a, \ |s| \le \epsilon$ again. By (4.30), we have

$$|(\hat{Y}(a_-, s), \hat{Z}(a_-, s))|_{E^{0.72\times 0.25}(\gamma)} \le C|\hat{\phi}_a|_{E^{0.72\times 0.25}(\gamma)},$$
$$|T(\xi, a, s)P_u^-(a, s)(\hat{Y}(a_-, s), \hat{Z}(a_-, s))^\tau|_{E^{0.72\times 0.25}(\gamma)} \le Ce^{-\alpha|\xi - a|}|\hat{\phi}_a|_{E^{0.72\times 0.25}(\gamma)}.$$

Using the fact $0 < \alpha < \alpha_1$, it is easy to check that for $|s| \le \epsilon$,

$$|\hat{h}(\zeta, s)|_{E^{0.72\times 0.25}(s)} \le Ce^{-\alpha_1|\zeta - a|}|\hat{\phi}_a|_{E^{0.72\times 0.25}(s)},$$
$$\left|\int_{-\infty}^{\xi} T(\xi, \zeta, s)P_s^-(\zeta, s)(\hat{h}(\zeta, s))d\zeta + \int_a^{\xi} T(\xi, \zeta, s)P_u^-(\zeta, s)(\hat{h}(\zeta, s))d\zeta\right|_{E^{0.72\times 0.25}(s)}$$
$$\le Ce^{-\alpha|\xi - a|}|\hat{\phi}_a|_{E^{0.72\times 0.25}(s)}.$$

This proves (4.26) for $\xi \le a$. The proof for $\xi \ge a$ is similar. □

Proof (The proof of Theorem 4.2 continued.) For $|s| = \epsilon$, the functions $(\hat{Y}, \hat{Z})$ have been constructed two times – converted from $(\hat{u}, \hat{v})$ obtained in Lemma 4.4, and directly from Lemma 4.5. However, the solution $(\hat{Y}, \hat{Z})$ is unique for any given $s \in \Sigma(-\eta, \theta)$. This proves that $(\hat{Y}, \hat{Z})$ is analytic in the entire region $s \in \Sigma(-\eta, \theta)$.

Combining (4.24) and (4.25), we have for $\xi \le a$,

$$|\hat{Y}|_{L^2(\xi \le a)} \le \frac{C}{(1 + |s|)}|\hat{\phi}_a|_{E^{0.75\times 0.25}(s)}, \quad \text{if } s \in \Sigma(-\eta, \theta).$$
$$\int_{-\infty}^{\infty} (1 + |s|)^2 |\hat{Y}|^2_{L^2(\xi \le a)} d\omega \le C \int_{-\infty}^{\infty} |\hat{\phi}_a|^2_{E^{0.75\times 0.25}(s)} d\omega \le C|\hat{\phi}_a|^2_{\mathcal{H}^{0.75\times 0.25}(\gamma)}.$$

Similar results can be obtained for $\xi \ge a$. The inverse Laplace transform shows that both for $\xi \ge a$ and $\xi \le a$, $Y \in H_0^{2,1}(\gamma)$ for some $-\eta < \gamma < 0$.

By combining (4.24) and (4.26), and using the inverse Laplace transform, we have

$$|(\hat{Y}, \hat{Z})|_{E^{0.75\times 0.25}(s)} \le Ce^{-\alpha|\xi - a|}|\hat{\phi}_a|_{E^{0.75\times 0.25}}(s), \quad s \in \Sigma(-\eta, \theta),$$
$$|(Y, Z)|_{H_0^{0.75\times 0.25}(\gamma)} \le Ce^{-\alpha|\xi - a|}|\phi_a|_{H_0^{0.75\times 0.25}}(\gamma).$$

The distance of Γ_j to $M_j^{\pm}$ is greater than $x_j - y_j - N$. So (Y, Y_ξ) at Γ_j is bounded by $Ce^{-\alpha|x_j - y_j - N|}|\phi_a|$. Recall that $(u_j, u_{j\xi}) = (Y, Z) + (\beta(t)q_j', \beta(t)q_j'')$, which is also bounded by $Ce^{-\alpha|x_j - y_j - N|}|\phi_a|$ at Γ_j.

Using the result of Part B, we can eliminate the jump errors along $M_j^{\pm}$, $1 \le j \le m$. The process will induce exponentially small errors along Γ_j again. Repeating the process, the jump error along Γ_j and $M_j^{\pm}$ can be eliminated. We introduce a function $\beta_j(t)$ in each iteration, which is added up to form the final $\beta_j(t)$ for each u_j. □

5 Solution of the Nonlinear System

In Sect. 4, we solved the nonhomogeneous linear system (2.6), rewritten here for the reader's convenience:

$$\begin{aligned} &u_{jt} = u_{j,xx} + Df(q_j)u_j + h_j,\ \ uj(x,0) = u_{j0}(x),\ \text{ for } (x,t) \in \Omega_j,\\ &[(u_j, u_{jx})](\Gamma_j) = J_j(\Gamma_j),\\ &[u_{j0}, u_{j0,x}] = J_j(\Gamma_j)|_{t=0}. \end{aligned} \tag{5.1}$$

where $h_j \in L^2(\Gamma_j, \gamma)$, $u_{j0}(x) \in H^1(x_{j-1}, x_j)$, and $J_j(\Gamma_j) \in H^{0.75\times 0.25}(\gamma)$ are temporarily given functions, independent of $(\{u_j\}, \{r_j\})$.

We obtained the solution in the form $u_j(x,t) = u_j^{(1)}(x,t) + u_j^{(2)}(x,t)$ where $u_j^{(1)}$ is the solution of a nonhomogeneous initial value problem (4.1) without jump conditions, and $u_j^{(2)}$ is the solution of (4.8) with nonzero jump conditions along Γ_j.

We now prove the main result of the paper – Theorem 1.1. To obtain the solution to the nonlinear problem, as in (2.7), we set $h_j = B(r_j)u_j + R(r_j, u_j)$, $u_{j0}(x) = \bar{u}_{j0}(x) - r_j q_j'(\xi) + g_j(\xi, r_j)$ and $J_j(\Gamma_j) = J_j(\Gamma_j, \{r_j\}) = J_{j0} + G_j(\{r_j\})$ in (5.1). In the resulting nonlinear system, we look for $\{r_j\}$, so that (5.1) has a solution $u_j \in H^{2,1}(\Omega_j, \gamma)$, $1 \le j \le m$.

The solution $u_j^{(1)}(x,t)$ in Ω_j can be expressed as $U = \beta_j^{(1)}(t) q_j'(\xi_j) + Y_j^{(1)}(\xi, t)$. To simplify the notation, the inner product for L^2 functions $\int < a(\xi), b(\xi) > d\xi$ will be denoted by $< a, b >$. Before restricting to Ω_j, the equations for $\beta_j^{(1)}$ and $Y_j^{(1)}$ are

$$\begin{aligned} Y_{jt}^{(1)} &= L_j Y_j^{(1)} + Q_j \tilde{h}_j,\\ \dot{\beta}_j^{(1)}(t) &= \langle z_j, \tilde{h}_j \rangle,\\ \beta_j^{(1)}(0) &= \langle z_j, \bar{u}_{j0} \rangle - r_j + \langle z_j, g_j(\cdot, r_j) \rangle. \end{aligned}$$

Let $\mathcal{K}_j$ be the integral operator as in (4.4) and $(\cdot)_{\Omega_j}$ be the restriction of a function to Ω_j. Then in Ω_j,

$$\begin{aligned} \beta_j^{(1)}(t) &= \beta_j^{(1)}(\infty) + \int_{\infty}^{t} \langle z_j, \tilde{h}_j \rangle dt,\\ Y_j^{(1)} &= \Big(\mathcal{K}_j Q_j(-\tilde{h}_j) + e^{tL_j} Q_j(\bar{u}_{j0} + g_j(\xi, r_j)) \Big)_{\Omega_j},\\ r_j &= \langle z_j, \bar{u}_{j0} \rangle + \langle z_j, g_j(\cdot, r_j) \rangle - \beta_j^{(1)}(0). \end{aligned}$$

We have $\dot{\beta}_j^{(1)} \in L^2(\gamma)$, $\beta_j^{(1)} \in X^1(\gamma)$.

The solution $u^{(2)}(x,t) = \left(u_1^{(2)}, \ldots, u_m^{(2)}\right)$ can be expressed by operators

$$F_j : \{J_j(\Gamma_j, \{r_j\}) - ([u^{(1)}], [u_x^{(1)}])(\Gamma_j)\}_{j=1}^{m-1} \to u_j^{(2)},\ \ 1 \le j \le m.$$

In process in Sect. 4.2 yields the solution of the jump problem in the form $u_j^{(2)} = \beta_j^{(2)}(t)q_j' + Y_j^{(2)}(x,t)$. Define $F_j^\sharp(\{J_j\}) := \beta_j^{(2)}(t)q_j'$ and $F_j^\flat(\{J_j\}) := Y_j^{(2)}(x,t)$. Then

$$u_j^{(2)} = F_j(\{J_j\}) = F_j^\sharp(\{J_j\}) + F_j^\flat(\{J_j\}), \quad J_j := J_j(\Gamma_j, \{r_j\}) - ([u^{(1)}], [u_x^{(1)}])(\Gamma_j)$$

We look for a solution of (5.1): $u_j = \beta_j(t)q_j' + Y_j$, $1 \le j \le m$, with $\beta_j(\infty) = 0$. Notice that $\beta_j^{(2)}(0) = 0$. From $\beta_j(t) = \beta_j^{(1)}(t) + \beta_j^{(2)}(t)$, we obtain $\beta_j^{(1)}(\infty) = -\beta_j^{(2)}(\infty)$ and $\beta_j^{(1)}(0) = -\beta_j^{(2)}(\infty) + \int_\infty^0 \langle z_j, \tilde{h}_j\rangle dt$. Thus

$$\beta_j(t) = \int_\infty^t \langle z_j, \tilde{h}_j\rangle dt - \beta_j^{(2)}(\infty) + \beta_j^{(2)}(t).$$

It is easy to check that $\beta_j^{(2)} = \langle z_j, F_j^\sharp(\{J_j\})\rangle \in X^1(\gamma)$ with $J_j = J_{j0} + G_j(\{r_j\}) - ([u^{(1)}], [u_x^{(1)}])(\Gamma_j)$. Together, we consider a system for r_j and $u_j = \beta_j(t)q_j' + Y_j$:

$$\begin{aligned} \beta_j(t) &= \int_\infty^t \langle z_j, \tilde{h}_j\rangle dt + \beta_j^{(2)}(t) - \beta_j^{(2)}(\infty). \\ Y_j &= \Big(\mathcal{K}_j Q_j \tilde{h}_j + e^{tL} Q_j(\bar{u}_{j0} + g_j(\xi, r))\Big)_{\Omega_j} + F_j^\flat(\{J_j\}), \\ r_j &= \langle z_j, \bar{u}_{j0}\rangle + \langle z_j, g_j(\cdot, r_j)\rangle + \int_0^\infty \langle z_j, \tilde{h}_j\rangle dt + \beta_j^{(2)}(\infty). \end{aligned} \tag{5.2}$$

System (5.2) can be expressed as

$$\{\beta_j, Y_j, r_j\} = \Phi(\{\bar{u}_{j0}\}, \{J_{j0}\}, \{\beta_j\}, \{Y_j\}, \{r_j\}), \quad 1 \le j \le m. \tag{5.3}$$

We shall solve this as a fixed point problem by the contraction mapping principle on the unknown variables (β_j, Y_j, r_j). Let $B(\epsilon)$ be an ϵ-ball in

$$\prod_1^m X^1(\gamma) \times \prod_1^m H^{2,1}(\Omega_j, \gamma) \times \mathbb{R}^m.$$

$$B(\epsilon) := \left\{ (\{\beta_j\}, \{Y_j\}, \{r_j\}) : \sum_{1 \le j \le m} (|\beta_j| + |Y_j| + |r_j|) \le \epsilon \right\}.$$

To ensure that Φ is a contraction mapping on $B(\epsilon)$, it suffices to have

(1) $|\Phi(\{\bar{u}_{j0}\}, \{J_{j0}\}, 0, 0, 0)| \le \epsilon/2$, and
(2) the Lipschitz numbers of h_j, g_j and $J_j(r_j)$ with respect to (β_j, Y_j, r_j) are sufficiently small on $B(\epsilon)$ so that

$$|\Phi(\{\bar{u}_{j0}\}, \{J_{j0}\}, \{\beta_j\}, \{Y_j\}, \{r_j\}) - \Phi(\{\bar{u}_{j0}\}, \{J_{j0}\}, 0, 0, 0)| \le \epsilon/2.$$

Condition (1) is satisfied if $\min\{|y_{j+1} - y_j| : 1 \le j \le m-1\}$ is sufficiently large and if for $1 \le j \le m$, $|\bar{u}_{j0}| < \rho$, $|\{J_{j0}\}| < C\rho$ are sufficiently small, that is, if $\min\{|y_{j+1} - y_j| : 1 \le j \le m-1\}$ and ρ satisfies conditions specified in Theorem 1.1.

To ensure that (2) is satisfied, and Φ is a contraction mapping on (β, Y_j, r_j), recall

$$\begin{aligned} h_j &= B(r_j)u_j + R(r_j, u_j), \quad g_j = q_j(\xi_j) - q_j(\xi_j + r_j) + r_j q_j'(\xi_j), \\ G_j(\{r_j\}) &= W_{j+1}(\Gamma_j) - W_{j+1}(\Gamma_j + r_{j+1}) + W_j(\Gamma_j + r_j) - W_j(\Gamma_j). \end{aligned}$$

It is straightforward to check that h_j, g_j are all small terms in the sense that:

(1) $|h_j| + |g_j| \le C(|u_j|^2 + |r_j|^2)$,
(2) the derivatives of h_j and g_j with respect to r_j, u_j are small.

It remains to verify that under the conditions of Theorem 1.1, the the jumps G_j and dG_j/dr_k are small for all $1 \le j \le m-1$, $1 \le k \le m$.

Along the line Γ_j, $G_j(\{r_j\}) = \mathcal{O}(|\{W_{jx}\}||\{r_j\}|)$. Therefore,

$$|G_j(\{r_j\})| \le C \max\{e^{-\eta|x_j - y_j|} : 1 \le j \le m-1\}|\{r_j\}| << \epsilon$$

For $k \ne j, j+1$, $dJ_j/dr_k = 0$, while for $k = j$ or $j+1$,

$$|dG_j/dr_k| \le C \max\{e^{-\eta|x_j - y_j|} : 1 \le j \le m-1\} << 1.$$

Therefore the Lipschitz number of $\{G_j(\{r_j\})\}$ with respect to $\{r_k\}$ is exponentially small if $\min\{|y_{j+1} - y_j| : 1 \le j \le m-1\} > \ell$ is sufficiently large. The constant ℓ is independent of δ_0 as in Theorem 1.1 and Remark 1.1.

We have proved that system (5.2) can be solved by the contraction mapping principle on $\{r_j\}, \{Y_j\}$ and $\{\beta_j\}$, and $u_j \in H^{2,1}(\Omega_j, \gamma)$. The solution u_j is a continuous function $t \in \mathbb{R}^+ \to H^1(\Omega_j)$. To check estimates (2) of Definition (1.1), let the solution of the linear problem be

$$(\{\beta_j\}, \{Y_j\}, \{r_j\})^{(0)} = \Phi(\{\bar{u}_{j0}\}, \{J_{j0}\}, 0, 0, 0). \tag{5.4}$$

From Sect. 4, $|(\{Y_j\}, \{\beta_j\}, \{r_j\})^{(0)}| \le C\rho$. If the rate of the contraction map Φ is $0 < k < 1$, from (5.3) and (5.4), we have

$$|(\{Y_j\}, \{\beta_j\}, \{r_j\}) - (\{Y_j\}, \{\beta_j\}, \{r_j\})^{(0)}| \le k|(\{Y_j\}, \{\beta_j\}, \{r_j\})|,$$
$$|(\{Y_j\}, \{\beta_j\}, \{r_j\})| \le (1-k)^{-1}|(\{Y_j\}, \{\beta_j\}, \{r_j\})^{(0)}| \le C\rho.$$

Thus, $|Y_j|_{H^{2,1}(\Omega_j,\gamma)} + |\beta_j|_{X^1(\gamma)} \le C\rho$. So u_j is a continuous function $t \in \mathbb{R}^+ \to H^1(\Omega_j)$ and $|e^{-\gamma t} u_j|_{H^1(\Omega_j)} \le C(|Y_j| + |\beta_j|) \le C\rho$. This proves estimate (2) of Definition (1.1).

6 Generalized Fisher/KPP Equations and Final Remarks

In this section we briefly consider the concatenation of two traveling waves of the generalized Fisher/KPP equation where our assumptions **H1** and **H2** are not satisfied. We hope to show that concatenation of waves and spatial dynamics can be useful in dealing with such nonstandard case.

The Fisher-KPP equation $u_t = u_{xx} + 2u(1-u)$ has a traveling wave solution $u(x-3t)$ connecting $u = 1$ to $u = 0$. The change of variable $u \to 1-u$ yields $u_t = u_{xx} - 2u(1-u)$, which has has a traveling wave $u(x-3t)$ connecting $u = 0$ to $u = 1$. We now consider the generalized Fisher-KPP equation and the associated first order system satisfied by the traveling wave $u(\xi) = u(x-3t)$:

$$\begin{aligned} &u_t = u_{xx} - 2u^n(1-u), \quad n \in \mathbb{N}, \\ &u' = v, \quad v' = -3v + 2u^n(1-u). \end{aligned} \tag{6.1}$$

Denote the traveling wave $u(x-3t)$ by $q_2(x - c_2 t)$. Let $q_1(x - c_1 t)$ be a traveling wave that moves to the left with the speed $c_1 < 0$. (One such example is to flip the axis $x \to -x$ so that $q_1(x - c_1 t) = q_2(-x - c_2 t)$ and $c_1 = -c_2$.) For each fixed t, as x increases from $-\infty$ to ∞, $q_1(x - c_1 t)$ (or $q_2(x - c_2 t)$) connects $u = 1$ to $u = 0$ (or $u = 0$ to $u = 1$).

Define the concatenated wave $u(x,t)^{con}$ separated by $\Gamma = \{x = 0, t \ge 0\}$ as in (1.8). Let $u(\xi, t) = q_j(\xi) + u_j(\xi, t)$ be the exact solution near u^{con}, where $j = 1$ for $x < 0$ and

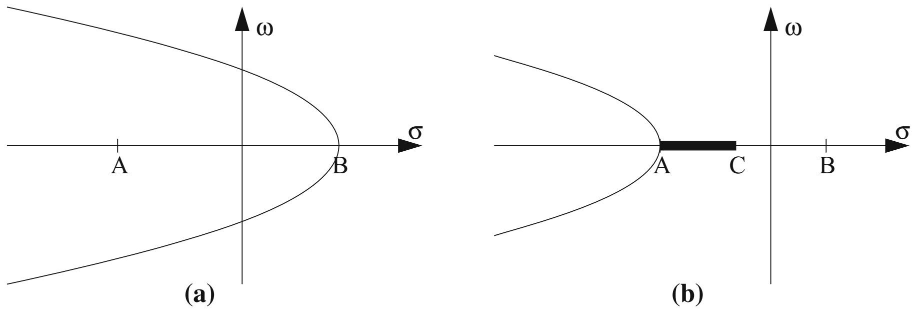

Fig. 3 The spectrum for the wave q_2 when $n = 1$. **a** Without the weight function the spectrum is bounded to the right by a parabola with the vertex at $B = Df(1) > 0$. **b** With the weight function the spectrum is bounded to the right by a parabola with the vertex at $A = Df(0)$, plus the line segment $\overline{AC}$, where $C = Df(1) - c_2^2/4 < 0$

$j = 2$ for $x > 0$. For $n \in \mathbb{N}$, each single wave q_1 (or q_2) is stable only under some weighted norms applied to large ξ and/or large $-\xi$. Let $w_j(\xi) \geq 1$ be a suitable weight function. The weighted norms are designed to limit the allowed perturbations to q_1 and q_2 by requiring that $\|u_j\|_{H^k_w} := \|w_j u_j\|_{H^k} < \infty$.

For $n = 1$, the traveling wave q_1 is a node to saddle connection and q_2 is a saddle to node connection. As in [23], we can choose $w_1(\xi) = e^{c_1\xi/2}$ for $\xi < 0$ and $w_1(\xi) = 1$ for $\xi \geq 0$; and choose $w_2(\xi)$ similarly. See Fig. 3 for the spectrum before and after adding the weighted norms. Observe that $w_1(\xi) \to \infty$ as $\xi \to -\infty$, and $w_2(\xi) \to \infty$ as $\xi \to \infty$, those are the left end and right end of the concatenated wave. Therefore the same weights can be applied to $u_j(x, t)$, $j = 1, 2$, to put restriction on perturbations of the initial data of the concatenated wave to ensure its stability.

For $n \geq 2$, $(u, v) = (0, 0)$ is non-hyperbolic with eigenvalues $\lambda_1 = 0, \lambda_2 \neq 0$. The traveling wave q_1 connects $(1, 0)$ to the center manifold of $(0, 0)$ and q_2 connects the center manifold of $(0, 0)$ to $(1, 0)$. The system looks similar to that studied by Wu, Xing and Ye [27], but is not the same. For the initial data of $u_1(x, t), x < 0$ (or $u_2(x, t), x > 0$), we can use the same exponential weight functions $w_j(\xi)$ as when $n = 1$. However, if $n \geq 2$, it is known that the linear variational system around $q_1(\xi), \xi \geq 0$ (or $q_2(\xi), \xi \leq 0$) has an algebraic dichotomy (rather than an exponential dichotomy), see [27]. To restrict the perturbations of each traveling wave, for $u_1(x, t), x > 0$ (or $u_2(x, t), x < 0$), we may use $w_j(\xi) = c_j(1 + |\xi|)^\gamma,\ \gamma > 0$ for $\xi > 0,\ \ j = 1$ (or $\xi < 0,\ \ j = 2$), as in [13,27]. However, for the concatenated wave, since u_1 exists only for $x < 0$ and u_2 exists only for $x > 0$ so the boundedness of the weighted norms does not put any restriction on the initial values of u_1 for large x (or initial values of u_2 for large $-x$).

In comparison, our method of eliminating jumps between the waves does not depend on evolution operators in time. It depends on evolution operators in space x, so it is more flexible to deal with weights or jumps in x direction. As in [14], we might be able to replace the weighted norms by some boundary conditions to the left and right of Γ, which also helps to restrict the allowed perturbations of q_1 and q_2.

To summarize, the main ideas of our method, as outlined in Sect. 1, should work both for bistable, and generalized Fisher/KPP type traveling waves. In our future work, we hope to find suitable function spaces so that the linear variational system may have exponential dichotomies and the stability of the concatenated wave may be proved by method similar to that used in this paper.

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References

1. Beck, M., Sandstede, B., Zumbrun, K.: Nonlinear stability of time-periodic viscous shocks. Arch. Ration. Mech. Anal. **196**, 1011–1076 (2010)
2. Beyn, W.J., Thummler, V.: Freezing solutions of equivariant evolution equations. SIAM J. Appl. Dyn. Syst. **3**, 85–116 (2004)
3. Coppel, W.A.: Dichotomies in Stability Theory, Lecture Notes in Mathematics, vol. 629. Springer, New York (1978)
4. Gardner, R.A., Jones, C.K.R.T.: Traveling waves of a perturbed diffusion equation arising in a phase field model. Indiana Univ. Math. J. **38**, 1197–1222 (1989)
5. Hale, J.K., Lin, X.-B.: Heteroclinic orbits for retarded functional differential equations. J. Differ. Equ. **65**, 175–202 (1986)
6. Henry, D.: Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics No. 840. Springer, New York (1980)
7. Kirchgassner, K.: Homoclinic bifurcation of perturbed reversible systems. In: Knobloch, W., Smith, K. (eds.) Lecture Notes in Mathematics 1017, pp. 328–363. Springer, New York (1982)
8. Lin, X.-B.: Exponential dichotomies and homoclinic orbits in functional differential equations. J. Differ. Equ. **63**, 227–254 (1986)
9. Lin, X.-B.: Shadowing lemma and singularly perturbed boundary value problems. SIAM J. Appl. Math. **49**(1), 26–54 (1989)
10. Lin, X.-B.: Exponential dichotomies in intermediate spaces with applications to a diffusively perturbed predator-prey model. J. Differ. Equ. **108**, 36–63 (1994)
11. Lin, X.-B.: Shadowing matching errors for wave-front-like solutions. J. Differ. Equ. **129**, 403–457 (1996)
12. Lin, X.-B.: Local and global existence of multiple waves near formal approximations. Progr. Nonlinear Differ. Equ. Appl. **19**, 385–404 (1996)
13. Lin, X.-B.: Algebraic dichotomies with an application to the stability of Riemann solutions of conservation laws. J. Differ. Equ. **11**, 2924–2965 (2009)
14. Lin, X.-B.: Stability of standing waves for monostable reaction-convection equations in a large bounded domain with boundary conditions. J. Differ. Equ. **255**, 58–84 (2013)
15. Lions, J.L., Magenes, E.: Non-homogeneous Boundary Value Problems and Applications. Springer, Berlin (1972)
16. Lunardi, A.: Interpolation Theory, (Publications of the Scuola Normale Superiore/Lecture Notes (Scuola Normale Superiore)) (2009)
17. Mallet-Paret, J.: Generic properties of retarded functional differential equations. Bull. Am. Math. Soc. **81**, 750–752 (1975)
18. Mielke, A.: A reduction principle for nonautonomous systems in infinite-dimensional spaces. J. Differ. Equ. **65**, 68–88 (1986)
19. Peterhof, D., Sandstede, B., Scheel, A.: Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders. J. Differ. Equ. **140**, 266–308 (1997)
20. Renardy, M.: Bifurcation of singular solutions in reversible systems and applications to reaction–diffusion equations. Adv. Math. **3**, 324–406 (1982)
21. Rottmann-Matthes, J.: Stability of parabolic-hyperbolic traveling waves. Dyn. PDE **9**, 29–62 (2012)
22. Rottmann-Matthes, J.: Stability and freezing of waves in non-linear hyperbolic-parabolic systems. IMA J. Appl. Math. **77**, 420–429 (2012)
23. Sattinger, D.H.: On the stability of waves of nonlinear parabolic systems. Adv. Math. **22**, 312–355 (1976)
24. Selle, S.: Decomposition and Stability of Multifronts and Multipulses, Ph.D. thesis, University of Bielefeld (2009)
25. Traves, F.: Basic Linear Partial Differential Equations. Academic Press, New York (1975)
26. Wright, J.D.: Separating dissipative pulses: the exit manifold. J. Dyn. Differ. Equ. **21**, 315–328 (2009)
27. Yaping, W., Xiuxia, X.: Stability of travelling waves with algebraic decay for n-degree Fisher-type equations. Discret. Contin. Dyn. Syst. **16**(1), 4766 (2006)
28. Yosida, K.: Functional Analysis, Grundlehren der mathematischen Wissenschaften, vol. 123. Springer, New York (1980)