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A Shilnikov Phenomenon Due to State-Dependent Delay, by Means of the Fixed Point Index

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Abstract The first part of this paper is a general approach towards chaotic dynamics for a continuous map $f : X \supset M \to X$ which employs the fixed point index and continuation. The second part deals with the differential equation

$$x'(t) = -\alpha\, x(t - d_\Delta(x_t)).$$

with state-dependent delay. For a suitable parameter α close to $5\pi/2$ we construct a delay functional d_Δ, constant near the origin, so that the previous equation has a homoclinic solution, $h(t) \to 0$ as $t \to \pm\infty$, with certain regularity properties of the linearization of the semiflow along the flowline $t \mapsto h_t$. The third part applies the method from the beginning to a return map which describes solution behaviour close to the homoclinic loop, and yields the existence of chaotic motion.

Keywords State-dependent delay · Symbolic dynamics · Fixed point index

Mathematics Subject Classification 34K23 · 37B10

1 Overview

The present paper consists of three different parts. The first part in Sect. 2 below is a general approach towards chaotic dynamics for a continuous map $f : X \supset M \to X$ which employs the fixed point index and continuation.

The second and third parts deal with the differential equation

$$x'(t) = -\alpha\, x\, (t - d_\Delta(x_t)) \tag{1.1}$$

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with state-dependent delay which for small solutions coincides with the basic linear differential equation

$$x'(t) = -\alpha\, x(t-1)$$

modelling negative feedback with a *constant* time lag. The underlying motivation is to understand better what a variable, state-dependent delay can do to the dynamics in an otherwise simple system. This may be seen in contrast to, say, ordinary differential equations, where solutions follow the vectorfield, or to delay differential equations like

$$x'(t) = -\mu + f\,(x(t-1))$$

with a constant time lag. For the latter results obtained since the 1950ies provide some insight into how the shape of the real function f and the parameter $\mu > 0$ are related to the behaviour of solution curves $t \mapsto x_t$ in the space of initial data $[-1, 0] \to \mathbb{R}$.

In Sects. 3–9, which constitute the second part of the paper, we construct a delay functional d_Δ, of constant value 1 near the origin, so that Eq. (1.1) has a homoclinic solution, $h(t) \to 0$ as $t \to \pm\infty$, with certain regularity properties of the linearization of the semiflow along the flowline $t \mapsto h_t$. Section 3 contains a detailed introduction into this part of the paper. The main result of Sects. 4–9 is stated in Theorem 9.2.

The third part in Sects. 10–15 applies the method from Sect. 2 to a map which describes the behaviour of solutions close to the homoclinic loop, and yields the existence of chaotic motion. This final result is stated as Theorem 15.3.

Notation For $r > 0$ and $t \in \mathbb{R}$ the segment $x_t : [t-r, t] \to M$ of a map $x : \mathbb{R} \supset J \to M$ with $[t-r, t] \subset J$ is defined by $x_t(s) = x(t+s)$.

For given maps f, m and for x in the domain of m, $m(x)$ in the domain of f, we write $f(m(x))$ as $f \circ m(x)$ also in cases where the full image of m is not contained in the domain of f.

The j-th component of $(x_1, \ldots, x_n) \in M_1 \times \cdots \times M_n$ is written x_j.

The closure, the interior, and the boundary of a subset M of a topological space are denoted by $\overline{M}$, $\mathrm{int}(M)$, and ∂M, respectively. The norm on a Banach space B is written $|\cdot|$, except for the norms $|\cdot|_{0,n}$ and $|\cdot|_{1,n}$, $|\cdot|_1$ introduced in Sect. 3 below; $U_r(x)$ is the open ball of radius r and center x in B, and $B_r := U_r(0)$. The Lipschitz constant of a map $m : M \to E$, $M \subset B$, B and E Banach spaces, is defined by

$$\mathrm{Lip}(m) = \sup_{x \neq y} \frac{|m(y) - m(x)|}{|y - x|} \quad (\leq \infty).$$

The support of a map $\phi : B \supset U \to \mathbb{R}$ is the set $\mathrm{supp}\,(\phi) = \overline{\phi^{-1}(0)}$.

A curve is a continuous map from an interval $I \subset \mathbb{R}$ into a Banach space. The tangent cone $T_x M$ of a subset $M \subset B$ of a Banach space B, at $x \in M$, is the set of all tangent vectors $v = c'(0)$ of differentiable curves $c : I \to B$ with $0 \in I$, $c(I) \subset M$, $c(0) = x$.

The Banach space of linear continuous operators from B into another Banach E is denoted by $L_c(B, E)$.

On products $B_1 \times \cdots \times B_n$ of normed spaces we use the norm given by $|(b_1, \ldots, b_n)| = \max_{j=1,\ldots,n} |b_j|$ unless stated otherwise.

The canonical unit vectors of $\mathbb{R}^n$ are denoted by $e_1, \ldots, e_n$. The unit sphere in $\mathbb{R}^{n+1}$ is denoted by S^n.

On Euclidean spaces we always use the Euclidean norm.

Derivatives and partial derivatives as continuous linear maps are written $Df(x)$ and $D_j f(x, y)$, $j \in \{1, 2\}$. For derivatives of maps x on domains $J \subset \mathbb{R}$ as elements of the target space, at $t \in J$, we have $x'(t) = Dx(t)1$.

In the sequel the prefix C^1- and formulations like C^1-*smooth* or *of class* C^1 mean that maps or submanifolds are continuously differentiable.

2 A Framework for the Detection of Symbolic Dynamics

We describe a very simple general approach to the description of the dynamics of a map f, restricted to some invariant subset of its domain, by the index shift on a space of symbol sequences. The main tool we use is the Leray–Schauder fixed point index in the following context: If U is an open subset of the Banach space E and $f : U \to E$ is continuous and compact, and the fixed point set $\text{Fix}(f)$ is compact, then the index $\text{ind}(f, U)$ is defined. (See [3], §12, in particular, Sect. 3, p. 311, or [22], Chapter 12, pp. 527–529. In the latter reference, it is assumed in addition that U is bounded and f is defined on the closure $\overline{U}$, with no fixed points on the boundary ∂U.) If $M \subset E$ is closed and such that $M = \overline{\text{int}(M)}$ and f has no fixed points on the boundary ∂M, then we use the notation $\text{ind}(f, M)$ with the same meaning as $\text{ind}(f, \text{int}(M))$, if the latter index is defined.

The method described here is much inspired by [23], but different in the following aspects:

(1) Our conditions on homotopies which leave the relevant fixed point indices invariant are free of assumptions related to the computation of the fixed point index, and are therefore simpler. The actual calculation of fixed point indices (for the map on the 'simpler' end of the homotopy) remains as a specific task in each application.
(2) We do not assume finite dimension, as it is for example the case in [15,23] or [2], and also in the paper [21] on delay equations.

Definition 2.1 Let a topological space X and a closed subset $M \subset X$ be given.

(1) A continuous map $f : M \to X$ is called M-admissible if

$$\forall m \in \mathbb{N} : \ \text{Fix}\left(f^m\right) \cap \partial M = \emptyset. \tag{2.1}$$

(2) Two continuous maps $f_0, \ f_1 : M \to X$ are called $M - homotopic$ (to each other) if there exists a homotopy $f : [0, 1] \times M \to X, \ (\lambda, x) \mapsto f_\lambda(x)$ (which is then called an M-homotopy) such that all maps f_λ are M-admissible, i.e.,

$$\forall m \in \mathbb{N} \quad \forall \lambda \in [0, 1] : \ \text{Fix}\left(f_\lambda^m\right) \cap \partial M = \emptyset. \tag{2.2}$$

We provide a simple criterion for maps to be M-admissible, respectively M-homotopic.

Proposition 2.2 *Let X be a topological space and $M \subset X$ closed.*

(1) *If $g : M \to X$ is continuous and*

$$\partial M \cap g(M) \cap g^{-1}(M) = \emptyset \tag{2.3}$$

then g is M-admissible.

(2) *This is true, in particular, if $\partial M = \partial_1 M \cup \partial_2 M$ and these two subsets satisfy*

$$g(\partial_1 M) \cap M = \emptyset = \partial_2 M \cap g(M). \tag{2.4}$$

(3) *If* $f : [0,1] \times M \to X, (\lambda, x) \mapsto f_\lambda(x)$ *is continuous, and* ∂M *is the union of two subsets* $\partial_1 M, \partial_2 M$ *of* ∂M *such that condition* (2.4) *holds for all* $\lambda \in [0,1]$, *then* f *is an* M*-homotopy.*

Proof Obviously, for $m \in \mathbb{N}$ one has $\operatorname{Fix}(g^m) \cap \partial M \subset g(M) \cap g^{-1}(M) \cap \partial M$, so condition (2.3) implies (2.1) for g.

If (2.4) holds then $\partial_1 M \cap g^{-1}(M) = \emptyset$ and

$$\begin{aligned} \partial M \cap g(M) \cap g^{-1}(M) &= \{ \underbrace{[\partial_1 M \cap g^{-1}(M)]}_{=\emptyset} \cap M \} \\ &\quad \cup \{ \underbrace{[\partial_2 M) \cap g(M)]}_{=\emptyset} \cap g^{-1}(M) \} \\ &= \emptyset, \end{aligned}$$

so (2.3) is satisfied. Assertion (3) is clear.

Remark Condition (2.1) (which demands that f has no periodic points on the boundary of M) is, of course, satisfied if the invariant set of f within M (i.e., the set $\left\{ x \in M \,\middle|\, \exists (x_n)_{n\in\mathbb{Z}} \in M^{\mathbb{Z}} : x_n = f(x_{n-1}) \ (n \in \mathbb{Z}),\ x_0 = x \right\}$) does not intersect ∂M.

We shall use the homotopy invariance of the fixed point index in the following version:

Assume that E is a Banach space, $\Omega \subset [0,1] \times E$ is open, and $f : \Omega \to E,\ (\lambda, x) \mapsto f_\lambda(x)$ is continuous, the set $\Sigma := \left\{ (\lambda, x) \in \Omega \,\middle|\, x = f_\lambda(x) \right\}$ is compact, and f is compact on an open neighbourhood Γ of Σ. Setting $\Omega_\lambda := \left\{ x \in E \,\middle|\, (\lambda, x) \in \Omega \right\}$ for $\lambda \in [0,1]$, the fixed point index $\operatorname{ind}(f_\lambda, \Omega_\lambda)$ is then defined for all $\lambda \in [0,1]$ and independent of λ.

(See [14], noting that $\operatorname{ind}(f, M) = \deg(\operatorname{id} - f, M)$, where deg denotes the Leray–Schauder degree; see also [9], p. 198, Theorem 2.2., part iii). The version from [14] is more general than the one from [9], but easy to obtain from the latter by restricting f to a bounded open neighbourhood of Σ. A slightly weaker formulation than ours, assuming that Ω is bounded and that f is compact on all of Ω, is called 'generalized homotopy invariance' in [22], Chapter 13, p. 572.)

The following statement is a version of Theorem 2.2 from [23], suitable for our context.

Lemma 2.3 *Let* $m \in \mathbb{N}$ *and let* $M_0, \ldots, M_m$ *be closed subsets of a Banach space E with non-empty interior, and such that with* $M := M_0 \cup \ldots \cup M_m$ *one has* $\partial M = \bigcup_{j=0}^m \partial M_j$. *Assume that* $f : [0,1] \times M \to E$ *is an* M*-homotopy, and compact (i.e., the closure* $\overline{f([0,1] \times M)}$ *of the image of* f *is compact). Define* $\Omega_\lambda := \bigcap_{j=0}^m f_\lambda^{-j}(\operatorname{int}(M_j))$ *for* $\lambda \in [0,1]$. *Then the fixed point index* $\operatorname{ind}(f_\lambda^m, \Omega_\lambda)$ *is defined for all* $\lambda \in [0,1]$, *and independent of* λ.

Proof Set $\Omega := \bigcup_{\lambda\in[0,1]} \{\lambda\} \times \Omega_\lambda$. If $(\lambda, x) \in \Omega$ then $f_\lambda^j(x) \in \operatorname{int}(M_j)$ for $j = 0, \ldots, m$. Continuity of f implies existence of $\delta > 0$ such that for $(\mu, y) \in [0,1] \times E$ with $|\mu - \lambda| < \delta$ and $|y - x| < \delta$, one has $f_\mu^j(y) \in \operatorname{int}(M_j)$, $j = 0, \ldots, m$, so $((\lambda - \delta, \lambda + \delta) \cap [0,1]) \times U_\delta(x) \subset \Omega$. Hence Ω is open in $[0,1] \times E$, and the assertion of the lemma follows from compactness of f and from the homotopy invariance of the fixed point index, if we prove the following property:

$$F := \left\{ (\lambda, x) \in \Omega \,\middle|\, f_\lambda^m(x) = x \right\} \text{ is compact.} \tag{2.5}$$

Note that

$$F = \left\{ (\lambda, x) \in [0, 1] \times M_0 \mid x \in \bigcap_{j=0}^{m} f_\lambda^{-j}(\operatorname{int}(M_j)),\ f_\lambda^m(x) = x \right\}. \tag{2.6}$$

Now the set $\tilde{F} := \left\{ (\lambda, x) \in [0, 1] \times M_0 \mid x \in \bigcap_{j=0}^{m} f_\lambda^{-j}(M_j),\ f_\lambda^m(x) = x \right\}$ is compact, since it is closed and contained in the compact set $[0, 1] \times \overline{f([0, 1] \times M_{m-1})}$. Clearly $F \subset \tilde{F}$, so to prove (2.5) it suffices to show

$$\tilde{F} \setminus F = \emptyset. \tag{2.7}$$

We have

$$\begin{aligned}
\tilde{F} \setminus F = \big\{ (\lambda, x) \in [0, 1] \times M_0 \,\big|\, & x = f_\lambda^m(x), \\
& x \in \bigcap_{j=0}^{m} f_\lambda^{-j}(M_j) \setminus \bigcap_{j=0}^{m} f_\lambda^{-j}(\operatorname{int}(M_j)) \big\} \\
= \big\{ (\lambda, x) \in [0, 1] \times M_0 \,\big|\, & x = f_\lambda^m(x),\ x \in \bigcap_{j=0}^{m} f_\lambda^{-j}(M_j), \\
& \exists\, l \in \{0, \dots, m\} : x \notin f_\lambda^{-l}(\operatorname{int}(M_l)) \big\} \\
= \big\{ (\lambda, x) \in [0, 1] \times M_0 \big\} \,\big|\, & x = f_\lambda^m(x),\ x \in \bigcap_{j=0}^{m} f_\lambda^{-j}(M_j), \\
& \exists\, l \in \{0, \dots, m\} : f_\lambda^l(x) \in \partial M_l \big\}.
\end{aligned}$$

Thus, if $(\lambda, x) \in \tilde{F} \setminus F$ then there exists $l \in \{0, \dots, m\}$ such that $f_\lambda^l(x) \in \partial M_l \subset \partial M$, which contradicts the fact that f is an M-homotopy. Hence (2.7) is proved, which implies (2.5) and concludes the proof.

We turn towards symbolic dynamics now, and we restrict considerations to the simplest case of two symbols. For a map f and a subset M of its domain, we define

$$\operatorname{traj}(f, M) := \left\{ (x_j)_{j \in \mathbb{Z}} \in M^{\mathbb{Z}} \mid \forall\, j \in \mathbb{Z} : x_j = f(x_{j-1}) \right\}.$$

Let N_0, N_1 be disjoint, closed, nonempty subsets of a Banach space E with $N_j = \overline{\operatorname{int}(N_j)}$, $j = 0, 1$, and set $N := N_0 \cup N_1$. (Then $\operatorname{int}(N) = \operatorname{int}(N_0) \cup \operatorname{int}(N_1)$, from which one sees that automatically $\partial N = \partial N_0 \cup \partial N_1$.) For $\mathbf{s} = (s_0, s_1, \dots, s_m) \in \{0, 1\}^{m+1}$ and a map $f : N \to E$ we use the notation

$$\begin{aligned}
N_{\mathbf{s}, f} &:= \bigcap_{j=0}^{m} f^{-j}(\operatorname{int}(N_{s_j})) \\
&= \left\{ x \in \operatorname{int}(N_{s_0}) \mid f^j(x) \in \operatorname{int}(N_{s_j}),\ j = 1, \dots, m \right\}.
\end{aligned}$$

If f is continuous, compact and $\operatorname{Fix}(f^j) \cap \partial N = \emptyset$ for all $j \in \mathbb{N}$ then Lemma 2.3 (applied to the special case of a homotopy independent of λ) shows that $\operatorname{ind}(f^m, N_{\mathbf{s}, f})$ is defined for all $m \in \mathbb{N}$.

Corollary 2.4 *Let N_0, N_1 and $N = N_0 \cup N_1$ be as above, and assume that $f : [0,1] \times N \to E$ is compact and an N-homotopy. Further, assume that for all $m \in \mathbb{N}$ and all $\mathbf{s} = (s_0, \ldots, s_m) \in \{0,1\}^{m+1}$ with $s_0 = s_m$, one has*

$$ind\left(f_1^m, N_{\mathbf{s},f_1}\right) \neq 0. \tag{2.8}$$

Then f_0 has symbolic dynamics in the following sense: With the 'position map' $p : N \to \{0,1\}$, $p = 0$ on N_0 and $p = 1$ on N_1, the map

$$\sigma : traj\ (f_0, N) \ni (x_j)_{j\in\mathbb{Z}} \mapsto \left(p(x_j)\right)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$$

is surjective. For a periodic sequence $\mathbf{s} \in \{0,1\}^{\mathbb{Z}}$, there exists a periodic orbit $(x_j)_{j\in\mathbb{Z}} \in traj\ (f_0, N)$ with $\sigma((x_j)) = \mathbf{s}$, with the same minimal period.

Proof The set $\overline{f(N)} \cap N$ is compact, so $(\overline{f(N)} \cap N)^{\mathbb{Z}}$ is compact with the product topology. Now

$$\text{traj}\ (f, N) = \bigcap_{k\in\mathbb{Z}} \left\{(x_j) \in (\overline{f(N)} \cap N)^{\mathbb{Z}} \mid x_k = f(x_{k-1})\right\}$$

is a closed subset of $(\overline{f(N)} \cap N)^{\mathbb{Z}}$ in this topology (as follows from continuity of f and of the evaluation maps $(x_j) \mapsto x_k$), and hence traj (f, N) is also compact. The map σ is continuous with respect to the product topologies on traj (f, N) and on $\{0,1\}^{\mathbb{Z}}$, since N_0 and N_1 are closed and disjoint (the position map p is locally constant). It follows that the image of σ is compact, and hence closed in $\{0,1\}^{\mathbb{Z}}$. Since f is an N-homotopy, Lemma 2.3 shows that property (2.8) also holds with f_0 instead of f_1. We conclude from the existence property of the fixed point index that for every $m \in \mathbb{N}$ and every m-periodic sequence $(s_j) \in \{0,1\}^{\mathbb{Z}}$, there exists an m-periodic point $x \in N$ with $f^j(x) \in N_{s_j}$ $(j \in \mathbb{N})$. (The assertion on periodic orbits is proved.) It follows that the image of σ contains all periodic sequences (of all periods) in $\{0,1\}^{\mathbb{Z}}$. Since these are dense in $\{0,1\}^{\mathbb{Z}}$ with the product topology, and the image of σ is closed, it must be all of $\{0,1\}^{\mathbb{Z}}$.

Remark The idea of employing the fixed point index to obtain periodic orbits obeying periodic symbol sequences, and then to use a density argument to conclude that for every symbol sequence there exists a corresponding trajectory, is well-known. It was used, e.g., in [15], see Remark 1, p. 71 there.

The last part of this section is less general than the results so far, but more specific for our application later, namely for the computation of the fixed point index for the map on the 'simpler' end of an M-homotopy.

Proposition 2.5 *Let $n \in \mathbb{N}$ and let $B_1 \subset \mathbb{R}^n$ be homeomorphic to the closed unit ball in $\mathbb{R}^n$ (w.r. to some norm $||\ ||$), and assume $g : B_1 \to g(B_1) \subset \mathbb{R}^n$ is a homeomorphism such that*

$$B_1 \subset int(g(B_1)).$$

Then the fixed point index $ind(g,\ int(B_1))$ is defined and equals $+1$ or -1.

Proof Note first the following consequence of the open mapping theorem ([22], Theorem 16C, p. 705):

$$\begin{aligned}&\text{A homeomorphism between two closed subsets } A_1, A_2 \text{ of } \mathbb{R}^n\\ &\text{maps int}(A_1) \text{ to int}(A_2) \text{ and } \partial A_1 \text{ to } \partial A_2.\end{aligned} \tag{2.9}$$

Set $B_2 := g(B_1)$, so both sets B_1 and B_2 are homeomorphic to the closed unit ball $K_1 := \overline{U_1(0)}$. We have $g(\partial B_1) = \partial B_2$, and, since $B_1 \subset \operatorname{int}(B_2)$, the map g has no fixed points on ∂B_1, and $\operatorname{ind}(g, B_1)$ is defined.

Choose now a homeomorphism $\varphi : K_1 \to B_2$. We set $\tilde{B}_1 := \varphi^{-1}(B_1)$ and $\tilde{g} := \varphi^{-1} \circ g \circ \varphi|_{\tilde{B}_1}$. The commutativity property of the fixed point index ([22], formula (36), p. 573) together with (2.9) implies that

$$\begin{aligned} \operatorname{ind}(g,\ \operatorname{int}(B_1)) &= \operatorname{ind}[\varphi \circ (\varphi^{-1} \circ g),\ \operatorname{int}(B_1)] = \\ &= \operatorname{ind}[(\varphi^{-1} \circ g) \circ \varphi|_{\tilde{B}_1}, \varphi^{-1}(\operatorname{int}(B_1))] \\ &= \operatorname{ind}(\tilde{g},\ \operatorname{int}(\tilde{B}_1)). \end{aligned}$$

Under $\tilde{g}$, the set $\tilde{B}_1$ is mapped homeomorphically to the unit ball K_1, and $\tilde{B}_1 \subset \operatorname{int}(K_1)$, so $|x| < 1$ for $x \in \tilde{B}_1$, in particular, for $x \in \partial\tilde{B}_1$. With $h(t, x) := (1 - t)x - \tilde{g}(x)$ for $x \in \tilde{B}_1$ and $t \in [0, 1]$, we thus have

$$\forall\, x \in \partial\tilde{B}_1 :\ |h(t, x)| \geq \underbrace{|\tilde{g}(x)|}_{=1} - |x| > 0.$$

It follows (writing 'deg' for the Brouwer or Leray-Schauder degree) that

$$\begin{aligned} \operatorname{ind}\left(\tilde{g},\ \operatorname{int}(\tilde{B}_1)\right) &= \deg\left(\operatorname{id} - \tilde{g},\ \operatorname{int}(\tilde{B}_1), 0\right) = \deg\left(h(0, \cdot),\ \operatorname{int}(\tilde{B}_1), 0\right) \\ &= \deg\left(h(1, \cdot),\ \operatorname{int}(\tilde{B}_1), 0\right) = \deg\left(-\tilde{g},\ \operatorname{int}(\tilde{B}_1), 0\right). \end{aligned}$$

Now since $\tilde{g}$ is a homeomorphism (and, clearly, assumes the value 0 in $\tilde{B}_1$), the degree $\deg(-\tilde{g},\ \operatorname{int}(\tilde{B}_1), 0)$ equals $+1$ or -1 (see [22], Chapter 13, property (HD), p. 578).

Lemma 2.6 *Let $n \in \mathbb{N}$ and let N_0, N_1 be disjoint sets, each homeomorphic to the closed unit ball in $\mathbb{R}^n$. Let $f : N_0 \cup N_1 \to \mathbb{R}^n$ map each N_j homeomorphically to its image and such that*

$$\textit{int}(f(N_0)) \supset N_0 \cup N_1 \ \textit{and}\ \ \textit{int}(f(N_1)) \supset N_0 \cup N_1 \ \text{(Fig. 1)}. \tag{2.10}$$

Then, for every $m \in \mathbb{N}$ and every $\mathbf{s} = (s_0, \ldots, s_m) \in \{0, 1\}^{m+1}$ with $s_0 = s_m$, the index $ind(f^m, N_{\mathbf{s},f})$ is defined and equals $+1$ or -1.

Proof In the proof, we use the expressions *closed ball* and *open ball* (in italics) for sets which are homeomorphic to the closed respectively open unit ball in $\mathbb{R}^n$. Further, we write $A \underset{f}{\simeq} B$, if f maps the set A homeomorphically to B. Recall also property (2.9) from the proof of Proposition 2.5.

Claim 1 For $m \in \mathbb{N}_0$ and $\mathbf{s} = (s_0, \ldots, s_m) \in \{0, 1\}^{m+1}$ (not necessarily with $s_0 = s_m$), the following is true:

(a) $N_{\mathbf{s},f} = \bigcap_{j=0}^{m} f^{-j}(\operatorname{int}(N_{s_j}))$ is an *open ball*, and $N_{\mathbf{s},f} \underset{f^m}{\simeq} \operatorname{int}(N_{s_m})$.

(b) $\overline{N_{\mathbf{s},f}}$ is a *closed ball* with $N_{\mathbf{s},f} = \operatorname{int}(\overline{N_{\mathbf{s},f}})$. $\overline{N_{\mathbf{s},f}} = \bigcap_{j=0}^{m} f^{-j}(N_{s_j})$, and $\overline{N_{\mathbf{s},f}} \underset{f^m}{\simeq} N_{s_m}$.

(c) In case $m \geq 1$, one has $\overline{N_{\mathbf{s},f}} \subset \operatorname{int}(N_{s_0})$.

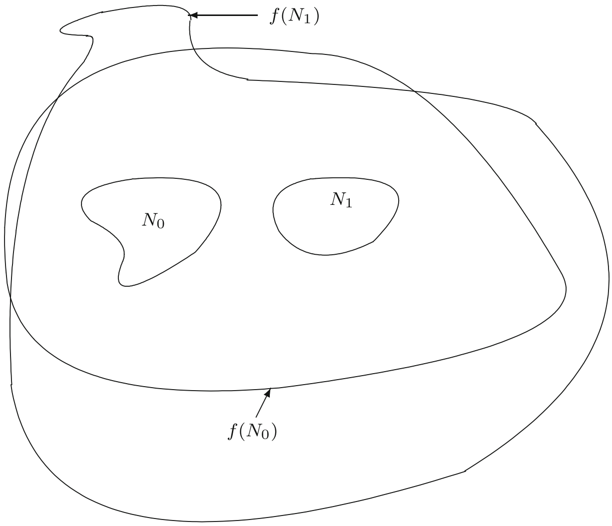


Fig. 1 The sets N_0, N_1, and their images

Proof (Induction on m.)
$m = 0$: If $\mathbf{s} = (s_0)$ then $N_{\mathbf{s},f} = f^0(\operatorname{int}(N_{s_0})) = \operatorname{int}(N_{s_0})$ is an *open ball*, and $\overline{N_{\mathbf{s},f}} = \overline{\operatorname{int}(N_{s_0})} = N_{s_0}$, as follows from (2.9), since N_{s_0} is a *closed ball*.

The remaining assertions of the claim are trivial in case $m = 0$.
$m \to m+1$: Assume $\mathbf{s} = (s_0, \ldots, s_{m+1})$, and set $\tilde{\mathbf{s}} := (s_1, \ldots, s_{m+1})$. We have

$$\begin{aligned} N_{\mathbf{s},f} &= \bigcap_{j=0}^{m+1} f^{-j}\left(\operatorname{int}(N_{s_j})\right) = \operatorname{int}(N_{s_0}) \cap \bigcap_{j=1}^{m+1} f^{-j}\left(\operatorname{int}(N_{s_j})\right) \\ &= \operatorname{int}(N_{s_0}) \cap f^{-1}\left[\bigcap_{j=1}^{m+1} f^{-(j-1)}(\operatorname{int}(N_{s_j}))\right] \\ &= \operatorname{int}(N_{s_0}) \cap f^{-1}\left[\bigcap_{j=0}^{m} f^{-j}(\operatorname{int}(N_{s_{j+1}}))\right] \\ &= \operatorname{int}(N_{s_0}) \cap f^{-1}\left(N_{\tilde{\mathbf{s}},f}\right). \end{aligned} \tag{2.11}$$

From the induction hypothesis, $N_{\tilde{\mathbf{s}},f}$ is an open ball, which by definition is contained in $\operatorname{int}(N_{s_1})$). From (2.10) and (2.9), we have

$$N_{\tilde{\mathbf{s}},f} \subset N_{s_1} \subset \operatorname{int}(f(N_{s_0})) = f(\operatorname{int}(N_{s_0})).$$

Now since $f|_{N_{s_0}}$ is homeomorphic onto its image, the same is true for $f|_{\operatorname{int}(N_{s_0})}$, and we conclude that the set

$$\operatorname{int}(N_{s_0}) \cap f^{-1}(N_{\tilde{\mathbf{s}},f}) = [f|_{\operatorname{int}(N_{s_0})}]^{-1}(N_{\tilde{\mathbf{s}},f})$$

is an *open ball*, so in view of (2.11) the same is true for $N_{\mathbf{s},f}$. Further, $f\big|_{N_{s_0}}$ maps $N_{\mathbf{s},f}$ homeomorphically to $N_{\tilde{\mathbf{s}},f}$, and, from the induction hypothesis, $N_{\tilde{\mathbf{s}},f} \underset{f^m}{\simeq} \operatorname{int}(N_{s_{m+1}})$. Together, we have

$$N_{\mathbf{s},f} \underset{f}{\simeq} N_{\tilde{\mathbf{s}},f} \underset{f^m}{\simeq} \operatorname{int}(N_{s_{m+1}}),$$

and it follows that $N_{\mathbf{s},f} \underset{f^{m+1}}{\simeq} \operatorname{int}(N_{s_{m+1}})$. (The assertions of (a) are proved.)

Since $\overline{N_{\tilde{\mathbf{s}},f}} \subset \overline{N_{s_1}} = N_{s_1}$, and since N_{s_1} is contained the set $\operatorname{int}(f(N_{s_0}))$, which (compare (2.9)) equals $f(\operatorname{int}(N_{s_0}))$, we have that

$$B := N_{s_0} \cap f^{-1}(\overline{N_{\tilde{\mathbf{s}},f}}) = [f\big|_{\operatorname{int}(N_{s_0})}]^{-1}(\overline{N_{\tilde{\mathbf{s}},f}}),$$

in particular,

$$B \subset \operatorname{int}(N_{s_0}). \tag{2.12}$$

From the induction hypothesis, $\overline{N_{\tilde{\mathbf{s}},f}}$ is a *closed ball*, so the set B is also a *closed ball* (since $f\big|_{\operatorname{int}(N_{s_0})}$ is homeomorphic onto its image). Using the property $\operatorname{int}(\overline{N_{\tilde{\mathbf{s}},f}}) = N_{\tilde{\mathbf{s}},f}$ from the induction hypothesis and the definition of $N_{\mathbf{s},f}$, we see that the interior of this closed ball equals

$$\operatorname{int}(B) = [f\big|_{\operatorname{int}(N_{s_0})}]^{-1}(\operatorname{int}(\overline{N_{\tilde{\mathbf{s}},f}})) = [f\big|_{\operatorname{int}(N_{s_0})}]^{-1}(N_{\tilde{\mathbf{s}},f}) = N_{\mathbf{s},f}$$

(see (2.11)). It follows that $\overline{N_{\mathbf{s},f}} = \overline{\operatorname{int}(B)} = B$ (here we used (2.9), hence $\operatorname{int}(\overline{N_{\mathbf{s},f}}) = \operatorname{int}(B) = N_{\mathbf{s},f}$. Further, the induction hypothesis gives $\overline{N_{\tilde{\mathbf{s}},f}} = \bigcap_{j=0}^{m} f^{-j}(N_{s_{j+1}})$, so with the definition of B we conclude

$$\overline{N_{\mathbf{s},f}} = B = N_{s_0} \cap f^{-1}\left(\bigcap_{j=0}^{m} f^{-j}(N_{s_{j+1}})\right) = \bigcap_{j=0}^{m+1} f^{-j}\left(N_{s_j}\right).$$

Finally, $f\big|_{N_{s_0}}$ maps $\overline{N_{\mathbf{s},f}} = B$ homeomorphically to $\overline{N_{\tilde{\mathbf{s}},f}}$, and (from the induction hypothesis) $\overline{N_{\tilde{\mathbf{s}},f}} \underset{f^m}{\simeq} N_{s_{m+1}}$, so we have $\overline{N_{\mathbf{s},f}} \underset{f^{m+1}}{\simeq} N_{s_{m+1}}$. The assertions of (b) are also proved, and assertion (c) follows from $\overline{N_{\mathbf{s},f}} = B$ and (2.12). (The claim is proved.)

Let now $m \in \mathbb{N}$ and $\mathbf{s}$ as in the lemma with $s_0 = s_m$ be given. From the above claim we know that $\overline{N_{\mathbf{s},f}} \underset{f^m}{\simeq} N_{s_m} = N_{s_0}$, both sets are *closed balls*, and since $m \geq 1$, have $\overline{N_{\mathbf{s},f}} \subset \operatorname{int}(N_{s_0})$. The statement on the fixed point index thus follows directly from Proposition 2.5, applied with $g := f^m\big|_{\overline{N_{\mathbf{s},f}}}$.

3 Introduction to the Construction of a Delay Functional

The linear equation

$$x'(t) = -\alpha\, x(t-1) \tag{3.1}$$

with parameter $\alpha > 0$ defines a strongly continuous semigroup T_α of bounded linear operators $T_\alpha(t)$ on the Banach space $C = C([-2, 0], \mathbb{R})$ of continuous functions $[-2, 0] \to \mathbb{R}$,

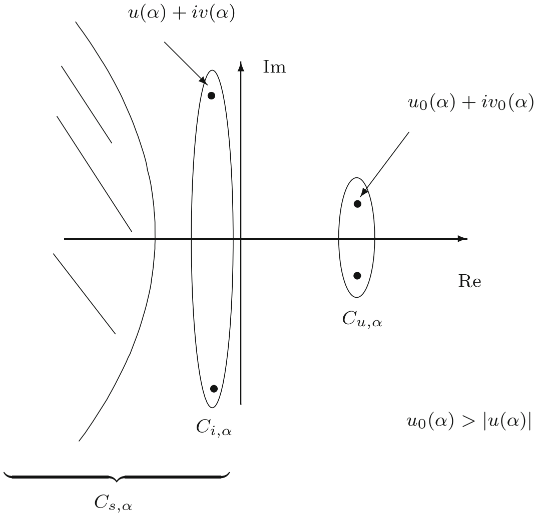


Fig. 2 The spectrum of the (complexified) infinitesimal generator G_α, with the subspaces $C_{u,\alpha}$, $C_{i,\alpha}$ and $C_{s,\alpha}$ indicated at the corresponding subsets of the spectrum

with the norm given by $|\phi| = \max_{-2\le t\le 0} |\phi(t)|$. This is easily seen as in the more familiar case of the space $C([-1,0],\mathbb{R})$. For $\frac{\pi}{2} < \alpha < \frac{5\pi}{2}$ the semigroup is hyperbolic with 2-dimensional unstable space $C_{u,\alpha} \subset C$. There is a complex conjugate pair $\lambda_0(\alpha), \overline{\lambda_0(\alpha)}$ of simple eigenvalues of the generator G_α of T_α in the open right half-plane, with $\mathrm{Re}(\lambda_0(\alpha)) = u_0(\alpha) > 0$ and $\frac{\pi}{2} < \mathrm{Im}(\lambda_0(\alpha)) = v_0(\alpha) < \pi$, and there is a leading complex conjugate pair $\lambda(\alpha), \overline{\lambda(\alpha)}$ of simple eigenvalues with maximal real part in the open left half-plane, with $\mathrm{Re}(\lambda(\alpha)) = u(\alpha) < 0$ and $2\pi < \mathrm{Im}(\lambda(\alpha)) = v(\alpha) < \frac{5\pi}{2}$; all other eigenvalues have real parts strictly less than $u(\alpha)$. The leading pair in the left half-plane defines a 2-dimensional *leading stable space* $C_{i,\alpha} \subset C_{s,\alpha}$ of the stable subspace $C_{s,\alpha} \subset C$ of the semigroup (Fig. 2).

In [18] we obtained a continuously differentiable *delay functional* $d_U : C \supset U \to (0,2)$, U open, with $d_U(\phi) = 1$ on a neighbourhood of $0 \in U$, so that the equation

$$x'(t) = -\alpha\, x\,(t - d_U(x_t)) \tag{3.2}$$

with state-dependent delay has a twice continuously differentiable solution $h : \mathbb{R} \to \mathbb{R}$ which is homoclinic to the zero solution,

$$h_t \neq 0 \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad h(t) \to 0 \quad \text{as} \quad |t| \to \infty.$$

Here and in the sequel we use the notation x_t for the solution segment in C given by $x_t(s) = x(t+s)$. The construction in [18] was done for $\alpha \in (\frac{\pi}{2}, \frac{5\pi}{2})$ sufficiently close to $\frac{5\pi}{2}$, in which case we also have

$$u_0(\alpha) + u(\alpha) > 0. \tag{3.3}$$

A major part of this construction concerns a regularity property of d_U, which is that along the homoclinic curve $t \mapsto h_t$ the intersection of the stable and unstable manifolds at the stationary point 0 is one-dimensional, thus minimal. In order to make the preceding statement precise we need to recall basic facts about well-posedness for initial value problems of the form

$$x'(t) = f(x_t) \quad \text{for} \quad t \ge 0, \tag{3.4}$$

$$x_0 = \phi, \tag{3.5}$$

which apply to differential equations with state-dependent delay. Proofs are found in [16,17], also see [5]. For $r > 0$ and $n \in \mathbb{N}$ let C_n denote the Banach space of continuous functions $[-r, 0] \to \mathbb{R}^n$, with the norm given by $|\phi|_{n,0} = \max_{-r \le t \le 0} |\phi(t)|$, so $C = C_1$ and $|\phi| = |\phi|_{1,0}$ for $\phi \in C$. Similarly let C_n^1 denote the Banach space of continuously differentiable functions $[-r, 0] \to \mathbb{R}^n$, with the norm given by $|\phi|_{n,1} = |\phi|_{n,0} + |\phi'|_{n,0}$, and abbreviate $C^1 = C_1^1, |\cdot|_1 = |\cdot|_{1,1}$. Let a continuously differentiable map $f : C_n^1 \supset U_1 \to \mathbb{R}^n, U_1 \subset C_n^1$ open, be given. Assume in addition that

(e) each derivative $Df(\phi) : C_n^1 \to \mathbb{R}^n$, $\phi \in U_1$, has a linear extension $D_e f(\phi) : C_n \to \mathbb{R}^n$, and the map

$$U_1 \times C_n \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

Then the set

$$X = X_f = \left\{\phi \in U_1 : \phi'(0) = f(\phi)\right\},$$

if non-empty, is a continuously differentiable submanifold of C_n^1, with codimension n, and every $\phi \in X$ determines a maximal continuously differentiable map $x^\phi : [-r, t_e(\phi)) \to \mathbb{R}^n$, $0 < t_e(\phi) \le \infty$, which satisfies the initial value problem (3.4)–(3.5) and is unique in the sense that any other continuously differentiable solution $x : [-r, s) \to \mathbb{R}^n$, $0 < s$, of the same initial value problem is a restriction of x^ϕ. These maximal solutions define a continuous semiflow $F = F_f$ on X, given by $F(t, \phi) = x_t^\phi$ for arguments in the domain $\Omega = \Omega_f = \{(t, \phi) \in [0, \infty) \times X : t < t_e(\phi)\}$. All solution operators F_t, $t \ge 0$, with nonempty domain $\Omega_t = \{\phi \in X : t < t_e(\phi)\}$ and $F_t(\phi) = F(t, \phi)$ are continuously differentiable. For $t \ge 0$, $\phi \in \Omega_t$, and $\chi \in T_\phi X$ we have

$$DF_t(\phi)\chi = v_t^{\phi,\chi}$$

with the continuously differentiable map $v^{\phi,\chi} : [-r, t_e(\phi)) \to \mathbb{R}^n$ satisfying

$$\begin{aligned} v'(t) &= Df\,(F(t, \phi))\, v_t \quad \text{for} \quad t \ge 0,\\ v_0 &= \chi. \end{aligned}$$

Moreover the restriction of F to the set $\{(t, \phi) \in \Omega : r < t\}$ is continuously differentiable, with

$$D_1 F\,(t, \phi)\, 1 = \left(x_t^\phi\right)' = \left((x^\phi)'\right)_t \in C_n^1.$$

It follows that for every continuously differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ which satisfies Eq. (3.4) for all $t \in \mathbb{R}$ the flowline $\xi : \mathbb{R} \ni t \mapsto x_t \in C_n^1$ is continuously differentiable with $D\xi(t)1 = (x')_t = (x_t)' \in C_n^1$ for all $t \in \mathbb{R}$.

At a stationary point $\phi_0 \in X$ the linearization of F, namely, the strongly continuous semigroup of the operators

$$D_2 F(t, \phi_0) : T_{\phi_0} X \to T_{\phi_0} X, \quad t \ge 0,$$

is given by restricting the semigroup $(S(t))_{t \ge 0}$ on $C_n \supset C_n^1 \supset T_{\phi_0} X$ which is defined by the solutions $v = v^\chi$ of the initial value problems

$$\begin{aligned} v'(t) &= D_e f(\phi_0) v_t,\\ v_0 &= \chi \in C_n. \end{aligned}$$

(These solutions $v : [-r, \infty) \to \mathbb{R}^n$ are continuous, $v|[0, \infty)$ is differentiable and satisfies the differential equation, and $S(t)\chi = v_t^\chi$ [1,4].) The spectra of the generators of both semigroups coincide, and for each pair of complex conjugate eigenvalues the associated realified generalized eigenspaces are the same (so belong to $T_{\phi_0}X$).

We return to Eq. (3.2) with the delay functional d_U from [18]. Recall that the evaluation map $ev : C \times [-2, 0] \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R}$ is continuous (but not locally Lipschitz), and that the restricted map $ev_1 : C^1 \times (-2, 0) \ni (\phi, t) \mapsto ev(\phi, t) \in \mathbb{R}$ is continuously differentiable with

$$Dev_1(\phi, t)(\eta, s) = D_1 ev_1(\phi, t)\eta + D_2 ev_1(\phi, t)s = \eta(t) + s\,\phi'(t).$$

It follows that the map $f : C^1 \supset U_1 \to \mathbb{R}$ given by $U_1 = U \cap C^1$ and

$$f(\phi) = -\alpha\,\phi(-d_U(\phi)) = -\alpha\, ev_1\,(\phi, d_U(\phi))$$

is continuously differentiable with

$$\begin{aligned} Df(\phi)\eta &= -\alpha\left\{\eta(-d_U(\phi)) - \phi'(-d_U(\phi))D(d_U|U_1)(\phi)\eta\right\} \\ &= -\alpha\left\{\eta(-d_U(\phi)) - \phi'(-d_U(\phi))Dd_U(\phi)\eta\right\} \end{aligned}$$

for all $\phi \in U_1$ and $\eta \in C^1$. We easily deduce that condition (e) is satisfied, and obtain a semiflow F on the manifold

$$X = \left\{\phi \in C^1 : \phi'(0) = -\alpha\,\phi(-d_U(\phi))\right\}$$

as described above. The segments $\phi \in X$ in a neighbourhood of $0 \in X$ belong to the closed subspace

$$Y = \left\{\phi \in C^1 : \phi'(0) = -\alpha\,\phi(-1)\right\} = T_0 X,$$

and the local stable and unstable manifolds of the stationary point $0 \in X$ of the semiflow F are simply open neighbourhoods of 0 in $Y_{s,\alpha} = Y \cap C_{s,\alpha}$ and in $Y_{u,\alpha} = C_{u,\alpha} \subset Y$, with tangent spaces $Y_{s,\alpha}$ and $C_{u,\alpha}$, respectively.

We drop the index and argument α from now on whenever convenient.

The precise statement of the minimal intersection property mentioned above is that for $\tau < 0$ with $h_\tau \in Y$ and $t > 0$, $-\tau$ and t sufficiently large, we have

$$(D_2 F(t - \tau, h_\tau)C_u) \cap Y_s = \mathbb{R}\,h_t'; \tag{3.6}$$

$h_t' \in T_{h_t}X \subset C^1$ is tangent to the flowline $H_1 : \mathbb{R} \ni \tilde{t} \mapsto h_{\tilde{t}} \in C^1$ at $\tilde{t} = t$.

What has been described so far is an infinite-dimensional analogue of Shilnikov's vector fields on $\mathbb{R}^4$ with a flowline homoclinic to 0, with complex conjugate pairs of eigenvalues of the linearized vector field in each open half-plane, at unequal distances from the imaginary axis, and with minimal intersection of stable and unstable manifolds along the homoclinic curve. Shilnikov's well-known result is that under these conditions there are infinitely many periodic orbits close to the homoclinic loop [11], compare also [6,13]. What can be said about the flowlines of F close to the homoclinic loop $H_1(\mathbb{R}) \cup \{0\} \subset X$? A difference between our scenario and Shilnikov's in addition to dimensionality is, of course, that the solution operators F_t, $t > 0$, are not diffeomorphisms, and their derivatives not isomorphisms.

A natural question at this point is perhaps whether there also exist a parameter α and a delay functional d_U so that Eq. (3.2), with the linearization of the semiflow at zero given by Eq. (3.1), generates a homoclinic solution as in Shilnikov's earlier result [10] on complicated dynamics for a smooth vectorfield v on $\mathbb{R}^3$, with one positive eigenvalue of $Dv(0)$ and the others complex conjugate with negative real part. Let us briefly explain why this is not the case. The desired spectral properties require for the linearization at zero Eq. (3.1) with $\alpha < 0$ (which models positive feedback); for suitable $\alpha < 0$ there is one positive eigenvalue of the associated generator while all others form complex conjugate pairs with negative real parts. The one-dimensional unstable eigenspace of the positive eigenvalue sits in the wedge of data without sign change, and the complementary stable space intersects with the wedge only at the origin. Notice that the wedge is positively invariant under *any* equation of the form (3.2) with $\alpha < 0$! Knowing this it is not hard to exclude for the latter the possibility of solutions homoclinic to zero.

Another question which may be asked is whether a homoclinic solution of Eq. (3.2), with the linearized semiflow given by Eq. (3.1), can be achieved by a delay functional of the simple form

$$d_U(\phi) = \delta(\phi(0))$$

with a function $\delta : \mathbb{R} \to (0, 2)$. Again, this is not the case: From $d_U(\phi) = 1$ for small ϕ we would have $\delta(\xi) = 1$ in some interval $(-\epsilon, \epsilon) \neq \emptyset$. The elements $\phi \neq 0$ of the unstable space C_u have at most one sign change, and one can show that each element of the stable space C_s has at least 2 zeros spaced at a distance less than 1. It follows that any homoclinic solution of Eq. (3.2) would have zeros $z < z' \leq z + 1$ with $h(t) \neq 0$ for $z - 1 \leq t < z$. In case $h(t) > 0$ on $[z - 1, z)$ this yields

$$h'(t) = -\alpha\, h\big(t - \delta(h(t))\big) = -\alpha\, h(t - 1) < 0$$

for all $t \in [z, z + 1)$ with $-\epsilon < h(t) \leq 0$, which in turn yields a contradiction to $h(z') = 0$. The argument in case $h(t) < 0$ on $[z - 1, z)$ is analogous.

In [19] we obtained a set of flowlines $\mathbb{R} \ni t \mapsto x_t \in C^1$ of F close to the homoclinic loop which have complicated histories in the sense that their behaviour for $t \leq 0$ is encoded by the backward symbol sequences $-\mathbb{N}_0 \ni j \mapsto s_j \in \{-, +\}$; there is a pair of disjoint sets $H_\pm$ so that $x_{t_j} \in H_{s_j}$ for all integers $j \leq 0$, and $t_j \searrow -\infty$ as $j \to -\infty$. Also,

$$0 \neq p_u x_{t_j} \to 0 \quad \text{as} \quad j \to -\infty$$

for the projection $p_u : Y \to Y$, $Y = Y_s \oplus C_u$, along Y_s onto C_u; none of these flowlines is periodic.

It is perhaps interesting that the proof in [19] does not make use of property (3.3).

In any case, a proof that close to the homoclinic loop a set of flowlines exists whose behaviour is encoded by the *entire* symbol sequences $\mathbb{Z} \to \{-, +\}$ seems to require further properties of F. In the present paper we keep the parameter α as chosen in Section 2 of [18] and consider the function h and the delay function $d : \mathbb{R} \to \mathbb{R}$ found in Sections 3 and 4 of [18], so that

$$h'(t) = -\alpha\, h\big(t - d(t)\big) \tag{3.7}$$

for all $t \in \mathbb{R}$. Starting from α, d, and h we construct a new delay functional $d_\Delta : C \supset \Delta \to (0, 2)$, Δ open, with $d_\Delta(\phi) = 1$ on a neighbourhood of $0 \in \Delta$ and $d_\Delta(h_t) = d(t)$ for all $t \in \mathbb{R}$, so that h solves the equation

$$x'(t) = -\alpha\, x\big(t - d_\Delta(x_t)\big) \tag{3.8}$$

for all $t \in \mathbb{R}$ and has the minimal intersection property (3.6), and in addition the semiflow F on

$$X = \big\{\phi \in \Delta \cap C^1 : \phi'(0) = -\alpha\, \phi(-d_\Delta(\phi))\big\}$$

given by Eq. (3.8) satisfies

$$D_2F(t-\tau, h_\tau)(C_i \oplus C_u) = C_i \oplus C_u \tag{3.9}$$

for $-\tau > 0$ and $t > 0$ sufficiently large. In other words, for such $\tau < 0$ and $t > 0$, with h_τ and h_t close to 0, the linearization $DF_{t-\tau}(h_\tau)$ defines an automorphism of the leading 4-dimensional invariant subspace of the semigroup T, which also is the leading invariant subspace for the linearization of F at $0 \in X$. Equation (3.9) in combination with (3.3) and the minimal intersection property (3.6) will enable us to obtain the desired result on symbolic dynamics close to the homoclinic loop.

We shall obtain the delay functional d_Δ as a special case of a more general construction whose result is stated as Theorem 9.2 below. Loosely speaking it says that for every integer $k \geq 2$ there exist continuously differentiable delay functionals d_{Δ_k} on open subsets of the space C, with $d_{\Delta_k}(\phi) = 1$ close to 0, so that the equation

$$x'(t) = -\alpha\, x\big(t - d_{\Delta_k}(x_t)\big)$$

has a solution homoclinic to 0 and the associated solution operators have linearizations along the homoclinic orbit *with prescribed behaviour on certain spaces of dimension* $k+1$.

4 Preliminaries: A Delay Function

Consider $a > 0$ and $\alpha \in (\frac{\pi}{2}, \frac{5\pi}{2})$ chosen in Section 2 of [18]. It will be convenient to write a_h instead of a in the sequel. Recall the solution

$$w : \mathbb{R} \ni t \mapsto e^{u_0 t} \sin(v_0 t) \in \mathbb{R}$$

of Eq. (3.1), which has all segments w_t in C_u, and the solution

$$y : \mathbb{R} \ni t \mapsto e^{ut} \sin(vt) \in \mathbb{R}$$

of Eq. (3.1), which has all segments y_t in C_i. The segments w_t' and y_t' also belong to C_u and C_i, respectively. The largest negative zero of w is at $t = -\frac{\pi}{v_0}$, and Eq. (3.1) implies that the largest negative extremum of w is $m = -\frac{\pi}{v_0} + 1$. Set $\beta = \frac{5\pi}{2}$ as in Section 2 of [18]. As $\alpha < \beta$ we have $v_0 = v_0(\alpha) < v_0(\beta)$, see for example [20]. Hence

$$\begin{aligned} m &< -\frac{\pi}{v_0(\beta)} + 1 \; (= m_\beta) \\ &< z < 0, \end{aligned} \tag{4.1}$$

by the choice of z in Section 2 of [18]. Using $v_0 > \frac{\pi}{2}$ we also get

$$-1 < m. \tag{4.2}$$

We turn to the strictly increasing sequences of zeros z_j, $j \in \mathbb{Z}$, and local extrema $m_j = z_{j-3} + 1$ of y, with $z_0 = 0$. We have

$$z_0 < m_1 < z_1 < m_2 < z_2 < m_3 = 1. \tag{4.3}$$

The construction of the delay function $d : \mathbb{R} \to \mathbb{R}$ begins in Section 3 of [18] with the choice of $d|(-\infty, t_*]$ where $t_* > 0$ had been fixed earlier with

$$0 < t_* < \frac{1}{\beta} = \frac{2}{5\pi} < m_1,$$

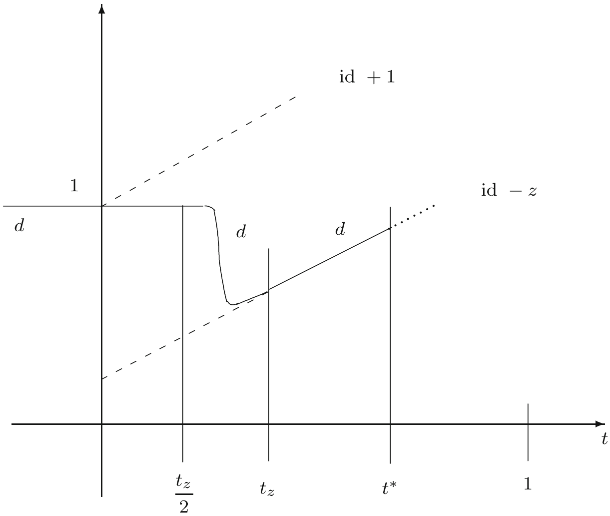


Fig. 3 d for $t \le t_*$

see (2.6) in [18]. The only restrictions on the C^1-function $d|(-\infty, t_*]$ are that for a number $t_z \in (0, t_*)$ chosen in Section 3 of [18] we have

$$d(t) = 1 \quad \text{on} \quad \left(-\infty, \frac{t_z}{2}\right], \tag{4.4}$$

$$-1 < t - d(t) < z \quad \text{on} \quad \left(\frac{t_z}{2}, t_z\right), \tag{4.5}$$

$$t - d(t) = z \quad \text{on} \quad [t_z, t_*]. \tag{4.6}$$

A look at Fig. 3 (which is a reproduction of Figure 6 in [18]) reveals that in addition we may assume

$$d'(t) < 1 \quad \text{on} \quad [0, t_z). \tag{4.7}$$

Now consider $d : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ as constructed in Sections 3 and 4 of [18] with the additional property that (4.7) holds. It is convenient to list further properties of d and h which are stated in Sections 3 and 4 of [18]:

$$h(t) = w(t) \quad \text{on} \quad \left(-\infty, \frac{t_z}{2}\right], \tag{4.8}$$

$$h(t) = a_h y(t) \quad \text{on} \quad [t''_y, \infty) \tag{4.9}$$

$$\text{with} \quad z_1 < t''_y < m_2,$$

$$h'(t) > 0 \quad \text{on} \quad [0, m_1), h'(t) < 0 \quad \text{on} \quad (m_1, m_2). \tag{4.10}$$

There are $\epsilon > 0$ and $\delta \in (0, \frac{m_2 - m_1}{2})$ with

$$d(t) = 1 \quad \text{on} \quad \left(-\infty, \frac{t_z}{2}\right] \cup [m_1 + 1 - \epsilon, m_1 + 1 + \delta] \cup [m_2 + 1 - \delta, \infty). \tag{4.11}$$

We have

$$t - d(t) \le 0 \quad \text{on} \quad [0, m_1] \quad \text{and} \quad z \le t - d(t) < m_1 \quad \text{for} \quad t_z < t < m_1 + 1 \tag{4.12}$$

and

$$m_1 < t - d(t) < m_2 \quad \text{for} \quad m_1 + 1 < t < m_2 + 1. \tag{4.13}$$

Proposition 4.1 *There is a unique zero $\tilde{t}$ of the function*

$$\mathbb{R} \ni t \mapsto t - d(t) - m \in \mathbb{R}$$

in $[0, t_z]$, and $0 < \tilde{t} < t_z$. The zeros of the function

$$\mathbb{R} \ni t \mapsto h'(t - d(t)) \in \mathbb{R}$$

in $(0, \infty)$ are $\tilde{t}$ and the numbers $m_j + 1$, $j \in \mathbb{N}$.

Proof 1. By (4.1) and (4.2), $-1 < m < z < 0$. Due to (4.7) the function $[0, t_z] \ni t \mapsto t - d(t) - m \in \mathbb{R}$ is strictly increasing with values $-1 - m < 0$ at $t = 0$ and $z - m > 0$ at $t = t_z$. Therefore it has a unique zero $\tilde{t}$ in $[0, t_z]$, and $0 < \tilde{t} < t_z$.

2. On $[0, t_z]$ we have $-1 \leq t - d(t) \leq z$, see (4.4) and (4.5), and m is the only zero of w' in $[-1, z]$. Using (4.8) we obtain that $\tilde{t}$ is the only zero of $\mathbb{R} \ni t \mapsto h'(t - d(t)) \in \mathbb{R}$ in $[0, t_z]$. Using (4.12), (4.8), and (4.10) we see that $h'(t - d(t)) > 0$ on $(t_z, m_1 + 1)$. From (4.11) and (4.9), (4.10) we infer

$$h'(m_j + 1 - d(m_j + 1)) = h'(m_j) = 0 \quad \text{for} \quad j \in \{1, 2\}.$$

From (4.13) and (4.10) combined we get $h'(t - d(t)) < 0$ in $(m_1 + 1, m_2 + 1)$. For $t > m_2 + 1$ we use (4.9) and (4.11) and find $h'(t - d(t)) = a_h y'(t - 1)$, hence $h'(t - d(t)) = 0$ and $t > m_2 + 1$ if and only if $t - 1 = m_j$ with $3 \leq j \in \mathbb{N}$.

In view of (4.11) and $0 < \tilde{t} < t_z \leq t_* < m_1$ we choose $\rho > 0$ with $\rho < \min\{\epsilon, \delta\}$ such that

$$d(t) = 1 \quad \text{on} \quad (-\infty, \rho] \cup [m_1 + 1 - \rho, m_1 + 1 + \rho] \cup [m_2 + 1 - \rho, \infty) \tag{4.14}$$

and

$$\rho < \tilde{t} - \rho \quad \text{and} \quad \tilde{t} + \rho < m_1 - \delta. \tag{4.15}$$

From $\rho < \delta$ we have

$$m_1 + \rho < m_2 - \rho.$$

5 Nonautonomous Differential Equations with Parametrized Variable Delay and an Associated Autonomous System

Let $n \in \mathbb{N}$, $n \geq 2$, be given. The construction of the desired delay functional relies on solutions to a n-parameter-family of nonautonomous differential equations with variable delay. For each parameter we shall consider the solution of the corresponding initial value problem at $t_0 = 0$ for a particular initial function, which also depends on the parameter. All of these solutions extend to the whole real line. Segments of the extensions will form a set on which we shall later begin with the definition of the delay functional. The present section provides facts about nonautonomous equations and initial values of the form we need.

Let C^1-functions $d_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, be given so that for every $j \in \{1, \ldots, n\}$ the function $d_* = d_j$ satisfies

$$d_*(t) = 0 \quad \text{on} \quad (-\infty, 0] \cup [m_2 + 1, \infty). \tag{5.1}$$

Using (4.10), (5.1), continuity, and compactness of $[0, m_2 + 1]$ we infer that the set

$$V_n := \left\{ c \in \mathbb{R}^n : 0 < d(t) + \sum_1^n c_j d_j(t) < 2 \quad \text{for all} \quad t \in \mathbb{R} \right\}$$

is open. Notice $0 \in V_n$. The n-parameter family of differential equations with variable delay addressed above are the equations

$$x'(t) = -\alpha\, x\left(t - \left[d(t) + \sum_1^n c_j d_j(t)\right]\right) \tag{5.2}$$

with parameter $c \in V_n$. It is easy to see by integrations on successive intervals of length

$$\min\left\{d(t) + \sum_1^n c_j d_j(t) : t \in \mathbb{R}\right\}$$

that each initial function $\phi \in C^1$ with $\phi'(0) = -\alpha\,\phi(-1)$ uniquely determines a C^1-function $x = x^\phi$, $x : [-2, \infty) \to \mathbb{R}$, which satisfies Eq. (5.2) for all $t \geq 0$ and $x_0 = \phi$.

In addition to the functions d_j let C^1-solutions $w_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, of Eq. (3.1) be given and set

$$\phi_j := w_{j,0} \in C^1 \quad \text{for} \quad j \in \{1, \ldots, n\}.$$

The particular initial functions mentioned above are given by

$$\phi_c = h_0 + \sum_1^n c_j \phi_j$$

for $c \in V_n$. It is convenient to introduce the restricted affine linear map

$$E : V_n \ni c \mapsto \phi_c \in C^1.$$

Because of (4.10), (5.1), $h(t) = w(t)$ on $(-\infty, 0]$, and $\phi_j = w_{j,0}$ we obtain that the continuously differentiable functions $x^c : \mathbb{R} \to \mathbb{R}$ given by $x^c(t) = x^{E(c)}(t)$ for $t \geq -2$ and $x^c(t) = h(t) + \sum_1^n c_j w_j(t)$ for $t < -2$ solve Eq. (5.2) for all $t \in \mathbb{R}$. Notice that

$$x^0(t) = h(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{5.3}$$

The remainder of this section prepares a proof that the map

$$I : \mathbb{R} \times V_n \ni (t, c) \mapsto x_t^c \in C^1$$

is C^1-smooth, and the computation of DI. This will be done by means of a natural auxiliary system

$$x'(t) = g(x_t) \in \mathbb{R}^{n+2} \tag{5.4}$$

of autonomous differential equations with state-dependent delay. We now introduce the functional g. Consider the spaces C_{n+2} and C^1_{n+2}. The set

$$U_{n+2} := \left\{\phi \in C^1_{n+2} : (\phi_2(0), \ldots, \phi_{n+1}(0)) \in V_n\right\}$$

is open, and the delay functional

$$\hat{d} : C^1_{n+2} \supset U_{n+2} \to (0, 2)$$

given by

$$\hat{d}(\phi) = d\big(\phi_{n+2}(0)\big) + \sum_{j=2}^{n+1} \phi_j(0) d_{j-1}\big(\phi_{n+2}(0)\big)$$

is C^1-smooth with

$$D\hat{d}(\phi)\eta = d'\big(\phi_{n+2}(0)\big)\eta_{n+2}(0) + \sum_{j=2}^{n+1} \big\{\eta_j(0) d_{j-1}(\phi_{n+2}(0)) + \phi_j(0) d'_{j-1}(\phi_{n+2}(0))\eta_{n+2}(0)\big\}.$$

Consider the functional $g : C^1_{n+2} \supset U_{n+2} \to \mathbb{R}^{n+2}$ given by

$$g_1(\phi) = -\alpha\,\phi_1\big(-\hat{d}(\phi)\big),$$
$$g_j(\phi) = 0 \quad \text{for} \quad j \in \{2, \ldots, n+1\},$$
$$g_{n+2}(\phi) = 1.$$

The next result is obvious.

Corollary 5.1 *For every $c \in V_n$ the map $x^{c,n+2} : \mathbb{R} \to \mathbb{R}^{n+2}$ given by*

$$x_1^{c,n+2}(t) = x^c(t),$$
$$x_j^{c,n+2}(t) = c_{j-1} \quad \textit{for} \quad j \in \{2, \ldots, n+1\},$$
$$x_{n+2}^{c,n+2}(t) = t$$

is C^1-smooth, $x := x^{c,n+2}$ satisfies Eq. (5.4) for all $t \in \mathbb{R}$, and

$$x(t) = \begin{pmatrix} h(t) + \sum_1^n c_j\phi_j(t) \\ c_1 \\ \vdots \\ c_n \\ t \end{pmatrix} \quad \textit{on} \quad [-2, 0].$$

We need smoothness properties of g. The components g_j, $j \in \{2, \ldots, n+2\}$, are C^1-smooth with all derivatives $Dg_j(\phi) : C^1_{n+2} \to \mathbb{R}$, $\phi \in U_{n+2}$, zero. For the first component we have

$$g_1(\phi) = -\alpha\, ev_1\left(\phi_1, -\hat{d}(\phi)\right).$$

As in Sect. 3 we obtain that g_1 is C^1-smooth with

$$\begin{aligned} Dg_1(\phi)\eta &= -\alpha\big\{\eta_1(-\hat{d}(\phi)) - \phi_1'(-\hat{d}(\phi)) D\hat{d}(\phi)\eta\big\} \\ &= -\alpha\Big\{\eta_1(-\hat{d}(\phi)) - \phi_1'(-\hat{d}(\phi))\big[d'(\phi_{n+2}(0))\eta_{n+2}(0) \\ &\quad + \sum_{j=2}^{n+1}\{\eta_j(0) d_{j-1}(\phi_{n+2}(0)) + \phi_j(0) d'_{j-1}(\phi_{n+2}(0))\eta_{n+2}(0)\}\big]\Big\}. \end{aligned} \tag{5.5}$$

The preceding expression does not contain derivatives of η and can be used to extend $Dg_1(\phi)$ to a linear map $D_e g_1(\phi) : C_{n+2} \to \mathbb{R}$. Using the continuity of $ev : C \times [-2, 0] \to \mathbb{R}$ we easily obtain that the map

$$U_{n+2} \times C \ni (\phi, \eta) \mapsto D_e g_1(\phi)\eta \in \mathbb{R}$$

is continuous. It follows that the functional g has the extension property (e) from Sect. 3. Consequently the maximal C^1-solutions $x^\phi : [-2, t_e(\phi)) \to \mathbb{R}^{n+2}$ of the initial value problem given by Eq. (5.4) for $t \geq 0$ and $x_0 = \phi$ in the C^1-submanifold

$$X_g := \{\phi \in U_{n+2} : \phi'(0) = g(\phi)\}$$

define a continuous semiflow $G : \Omega_g \to X_g$ on X_g, by

$$\Omega_g = \{(t, \phi) \in [0, \infty) \times X_g : t < t_e(\phi)\} \quad \text{and} \quad G(t, \phi) = x_t^\phi.$$

For the C^1-maps $DG_t : \Omega_{g,t} \to X_g$, $t \geq 0$, with nonempty domain

$$\Omega_{g,t} := \{\phi \in X_g : t < t_e(\phi)\}$$

we have

$$DG_t(\phi)\eta = v_t^{\phi,\eta}$$

with the C^1-solution $v = v^{\phi,\eta}$, $v : [-2, t_e(\phi)) \to \mathbb{R}^{n+2}$, of the initial value problem

$$v'(t) = Dg\big(G(t, \phi)\big)v_t \quad \text{for} \quad t \geq 0,$$
$$v_0 = \eta \in T_\phi X_g.$$

The restriction of G to the set $\{(t, \phi) \in \Omega_g : t > 2\}$ is C^1-smooth, with

$$D_1 G(t, \phi)1 = \big((x^\phi)'\big)_t = \big(x_t^\phi\big)'.$$

We return to the solutions $x^c : \mathbb{R} \to \mathbb{R}$, $c \in V_n$, of Eq. (5.2). It is convenient to introduce the restricted affine linear map $\hat{E} : V_n \to C^1_{n+2}$ given by

$$\hat{E}_1 = E,$$
$$\hat{E}_j(c)(t) = c_{j-1} \quad \text{for all} \quad j \in \{2, \ldots, n+1\} \quad \text{and} \quad t \in [-2, 0],$$
$$\hat{E}_{n+2}(c)(t) = t \quad \text{for all} \quad t \in [-2, 0].$$

Then

$$\hat{E}(c) = x_0^{c,n+2} \quad \text{for all} \quad c \in V_n,$$

see Corollary 5.1. In particular,

$$\hat{E}(0)(t) = \begin{pmatrix} h(t) \\ 0 \\ \vdots \\ 0 \\ t \end{pmatrix} \quad \text{on} \quad [-2, 0].$$

Equation (5.4) at $t = 0$ yields

$$\hat{E}(V_n) \subset X_g.$$

Observe that Corollary 5.1 also yields

$$\hat{E}(V_n) \subset \Omega_{g,t} \quad \text{for every} \quad t \geq 0,$$

and

$$I(t,c) = x_t^c = \mathrm{pr}_1 G_t\big(\hat{E}(c)\big) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad c \in V_n,$$

with the projection

$$\mathrm{pr}_1 : C^1_{n+2} \ni \phi \mapsto \phi_1 \in C^1.$$

Corollary 5.2 *Let $j \in \{1, \ldots, n\}$ and $d_* = d_j$. For every $t \geq 0$ we have*

$$D\big(pr_1 \circ G_t \circ \hat{E}\big)(0)e_j = pr_1 v_t^{\hat{E}(0), D\hat{E}(0)e_j},$$

and $b = (v^{\hat{E}(0), D\hat{E}(0)e_j})_1$ satisfies

$$b'(t) = -\alpha\big\{b(t-d(t)) - h'(t-d(t))d_*(t)\big\} \quad \textit{for all} \quad t \geq 0, \tag{5.6}$$

$$b_0 = \phi_j. \tag{5.7}$$

Proof We have

$$D\left(\mathrm{pr}_1 \circ G_t \circ \hat{E}\right)(0)e_j = \mathrm{pr}_1 DG_t\left(\hat{E}(0)\right) D\hat{E}(0)e_j = \mathrm{pr}_1 v_t^{\hat{E}(0), D\hat{E}(0)e_j}$$

for all $t \geq 0$ and

$$D\hat{E}(0)e_j = \begin{pmatrix} \phi_j \\ 0 \\ \vdots \\ 0 \\ \underline{1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in C^1_{n+2}$$

with $\underline{1} : [-2, 0] \ni t \mapsto 1 \in \mathbb{R}$ as the $(j+1)$-th component. As

$$\hat{h} : \mathbb{R} \ni t \mapsto \begin{pmatrix} h(t) \\ 0 \\ \vdots \\ 0 \\ t \end{pmatrix} \in \mathbb{R}^{n+2}$$

is a continuously differentiable solution of Eq. (5.4) (see Corollary 5.1) and $\hat{E}(0) = \hat{h}_0$ we obtain that $v = v^{\hat{E}(0), D\hat{E}(0)e_j}$ satisfies

$$v'(t) = Dg\big(G(t, \hat{E}(0))\big)v_t = Dg\big(\hat{h}_t\big)v_t \quad \text{for all} \quad t \geq 0.$$

According to (5.5),

$$\begin{aligned}\big(Dg(\hat{h}_t)v_t\big)_1 = -\alpha\big\{&v_1(t-d(t)) - h'(t-d(t))[d'(t)v_{n+2}(t) \\ &+ \sum_{k=2}^{n+1}\{v_k(t)d_{k-1}(t) + 0 \cdot d'_{k-1}(t)v_{n+2}(t)\}]\big\},\end{aligned}$$

and $(Dg(\hat{h}_t)v_t)_j = 0$ for all $j \in \{2,\dots,n+2\}$. Using the initial condition $v_0 = D\hat{E}(0)e_j$ and the preceding equations we find $v_{j+1}(t) = 1$ for all $t \geq -2$ and $v_k(t) = 0$ for all $k \in \{2,\dots,n+2\} \setminus \{j+1\}$ and all $t \geq -2$. Consequently,

$$\begin{aligned} b'(t) = v_1'(t) &= -\alpha\left\{v_1(t-d(t)) - h'(t-d(t))[d'(t)\cdot 0 + 1\cdot d_j(t)]\right\} \\ &= -\alpha\left\{b(t-d(t)) - h'(t-d(t))d_j(t)\right\} \quad \text{for all} \quad t \geq 0. \end{aligned}$$

Also, $b_0 = v_{1,0} = (D\hat{E}(0)e_j)_1 = \phi_j$.

Proposition 5.3 (Uniqueness) *For every $j \in \{1,\dots,n\}$ there is at most one C^1-function $b : [-2,\infty) \to \mathbb{R}$ satisfying (5.6) for all $t \geq 0$ and (5.7).*

Proof Let $j \in \{1,\dots,n\}$ and suppose $b : [-2,\infty) \to \mathbb{R}$ and $B : [-2,\infty) \to \mathbb{R}$ are C^1-smooth and satisfy Eq. (5.6) for all $t \geq 0$, and $b_0 = B_0$, and $b(t) \neq B(t)$ for some $t > 0$. For $t_0 = \inf\{t > 0 : b(t) \neq B(t)\}$ we get $t_0 \geq 0$ and $b(t) = B(t)$ on $[-2, t_0]$. Using $d(t_0) > 0$ we find $\epsilon' > 0$ with $t - d(t) < t_0$ for $t_0 \leq t < t_0 + \epsilon'$. Then Eq. (5.6) yields $b'(t) = B'(t)$ on $[t_0, t_0+\epsilon']$. It follows that $b(t) = B(t)$ on $[-2,t_0] \cup [t_0, t_0+\epsilon']$, hence $t_0 = \inf\{t > 0 : b(t) \neq B(t)\} \geq t_0 + \epsilon'$, which contradicts $\epsilon' > 0$.

6 Prescribed Solution Behaviour

The first result of this section shows that we can obtain solutions $b : \mathbb{R} \to \mathbb{R}$ of Eq. (5.6) with prescribed ends $b|(-\infty, 0]$ and $b|[m_2+1,\infty)$ by a suitable choice of the delay function $d_* : \mathbb{R} \to \mathbb{R}$.

Proposition 6.1 *For each pair of C^1-solutions $w_* : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ of Eq. (3.1) there exist C^1-functions $b : \mathbb{R} \to \mathbb{R}$ and $d_* : \mathbb{R} \to \mathbb{R}$ with the following properties: Eq. (5.6) is satisfied for all $t \in \mathbb{R}$, (5.1) holds, and*

$$\begin{aligned} b(t) &= w_*(t) \quad \textit{on} \quad (-\infty, 0], \\ b(t) &= q(t) \quad \textit{on} \quad [m_2, \infty). \end{aligned}$$

Proof 1. The functions w_* and q have derivatives of arbitrary order. By (4.15), $[\tilde{t}-\rho, \tilde{t}+\rho] \subset [0, m_1]$, hence $t - d(t) \leq 0$ on $[\tilde{t}-\rho, \tilde{t}+\rho]$ because of (4.12). From $m_2 < m_1 + 1$ we infer

$$[m_2 - \rho, \infty) \supset [m_1 + 1 - \rho, m_1 + 1 + \rho].$$

In particular,

$$t + 1 \in [m_2 - \rho, \infty) \quad \text{for all} \quad t \in [m_1 - \rho, m_1 + \rho].$$

There exists a twice continuously differentiable function $b : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} b(t) &= w_*(t) \quad \text{on} \quad (-\infty, 0], \\ b'(t) &= -\alpha\, w_*(t - d(t)) \quad \text{on} \quad [\tilde{t}-\rho, \tilde{t}+\rho], \\ b(t) &= -\frac{q'(t+1)}{\alpha} \quad \text{on} \quad [m_1 - \rho, m_1 + \rho], \\ b(t) &= q(t) \quad \text{on} \quad [m_2 - \rho, \infty). \end{aligned}$$

We define $d_* : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} d_*(t) &= 0 \quad \text{on} \quad (-\infty, \rho] \cup [m_2 + 1 - \rho, \infty), \\ d_*(\tilde{t}) &= 0, \\ d_*(m_1 + 1) &= 0, \\ d_*(t) &= \frac{b'(t) + \alpha\, b(t - d(t))}{\alpha\, h'(t - d(t))} \quad \text{on} \quad (\rho, m_2 + 1 - \rho) \setminus \{\tilde{t}, m_1 + 1\}. \end{aligned}$$

2. Proof that d_* is C^1-smooth. The restriction of d_* to the open set $\mathbb{R} \setminus \{\rho, \tilde{t}, m_1 + 1, m_2 + 1 - \rho\}$ is C^1-smooth. The C^1-function

$$\tilde{d} : (0, m_2 + 1) \setminus \{\tilde{t}, m_1 + 1\} \ni t \mapsto \frac{b'(t) + \alpha\, b(t - d(t))}{\alpha\, h'(t - d(t))} \in \mathbb{R}$$

satisfies $\tilde{d} = 0$ on $(0, \rho]$, because of $d(t) = 1$ and $b(t) = w_*(t)$ on $[0, \rho]$ and Eq. (3.1) for w_*. Hence $d_*(t) = 0 = \tilde{d}(t)$ on $[0, \rho]$. It follows that d_* and $\tilde{d}$ coincide on $[0, \tilde{t})$, which yields that $d_*|(-\infty, \tilde{t})$ is C^1-smooth.

On $(\tilde{t} - \rho, \tilde{t} + \rho) \setminus \{\tilde{t}\}$ we have $d_*(t) = 0$, because of

$$b'(t) = -\alpha\, w_*(t - d(t)) = -\alpha\, b(t - d(t)) \quad (\text{since} \quad t - d(t) \le 0).$$

As $d_*(\tilde{t}) = 0$ we see that $d_*|(\tilde{t} - \rho, \tilde{t} + \rho)$ is C^1-smooth.

On

$$[m_1 + 1 - \rho, m_1 + 1 + \rho] \setminus \{m_1 + 1\} \subset (\rho, m_2 + 1 - \rho) \setminus \{\tilde{t}, m_1 + 1\}$$

we have

$$\begin{aligned} -\alpha\, b(t - d(t)) &= -\alpha\, b(t - 1) \qquad (\text{see } 4.14) \\ &= q'(t) = b'(t) \quad (\text{since} \quad m_2 - \rho < m_1 + 1 - \rho) \end{aligned}$$

and consequently $d_*(t) = 0$. As $d_*(m_1 + 1) = 0$ we see that $d_*|(m_1 + 1 - \rho, m_1 + 1 + \rho)$ is C^1-smooth.

Finally, consider $(m_1 + 1, m_2 + 1) \ni m_2 + 1 - \rho$. On the subinterval

$$(m_1 + 1, m_2 + 1 - \rho) \subset (\rho, m_2 + 1 - \rho) \setminus \{\tilde{t}, m_1 + 1\}$$

we have $d_*(t) = \tilde{d}(t)$. On the subinterval $[m_2 + 1 - \rho, m_2 + 1)$ we have $d(t) = 1$ and $b(t) = q(t)$, hence

$$\begin{aligned} b'(t) = q'(t) &= -\alpha\, q(t - 1) = -\alpha\, b(t - 1) \quad (\text{since} \quad t - 1 \ge m_2 - \rho) \\ &= -\,\alpha\, b(t - d(t)), \end{aligned}$$

and thereby $\tilde{d}(t) = 0 = d_*(t)$. So $\tilde{d}$ and d_* coincide on $(m_1 + 1, m_2 + 1)$, which shows that $d_*|(m_1 + 1, m_2 + 1)$ is C^1-smooth. Now the assertion is obvious.

3. Verification of Eq. (5.6). The definition of d_* shows that b satisfies Eq. (5.6) on

$$(\rho, m_2 + 1 - \delta) \setminus \{\tilde{t}, m_1 + 1\}.$$

At $t = \tilde{t}$ we have $d_*(\tilde{t}) = 0$ and

$$\begin{aligned} b'(\tilde{t}) &= -\alpha\, w_*(\tilde{t} - d(\tilde{t})) = -\alpha\, b(\tilde{t} - d(\tilde{t})) \quad (\text{since} \quad \tilde{t} - d(\tilde{t}) = 0) \\ &= -\alpha\{b(\tilde{t} - d(\tilde{t})) - h'(\tilde{t} - d(\tilde{t}))d_*(\tilde{t})\}. \end{aligned}$$

At $t = m_1 + 1$ we have $d_*(m_1 + 1) = 0$ and $d(m_1 + 1) = 1$ and

$$b(m_1) = -\frac{q'(m_1+1)}{\alpha} = -\frac{b'(m_1+1)}{\alpha}$$

(since $m_1 + 1 > m_2$), hence

$$\begin{aligned} b'(m_1+1) &= -\alpha\, b(m_1) \\ &= -\alpha\big\{b(m_1+1-d(m_1+1)) \\ &\quad - h'(m_1+1-d(m_1+1))d_*(m_1+1)\big\}. \end{aligned}$$

On $(-\infty, \rho]$ we have $d(t) = 1$ and $d_*(t) = 0$ and $t - 1 < 0$, hence

$$\begin{aligned} b'(t) = w_*'(t) &= -\alpha\, w_*(t-1) = -\alpha\, b(t-d(t)) \\ &= -\alpha\big\{b(t-d(t)) - h'(t-d(t))d_*(t)\big\}. \end{aligned}$$

On $[m_2 + 1 - \rho, \infty)$ we have $d(t) = 1$ and $d_*(t) = 0$ and $t - 1 \geq m_2 - \rho$, hence

$$\begin{aligned} b'(t) = q'(t) &= -\alpha\, q(t-1) = -\alpha\, b(t-d(t)) \\ &= -\alpha\big\{b(t-d(t)) - h'(t-d(t))d_*(t)\big\}. \end{aligned}$$

Proposition 6.2 *Let $n \in \mathbb{N}$ and let analytic solutions $w_j : \mathbb{R} \to \mathbb{R}$ and $q_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, of Eq. (3.1) be given with $w_0', w_{1,0}, \ldots, w_{n,0}$ linearly independent and $a\, y_{m_2+2}', q_{1,m_2+2}, \ldots, q_{n,m_2+2}$ linearly independent. For every $j \in \{1, \ldots, n\}$ let a C^1-function $d_j : \mathbb{R} \to \mathbb{R}$ and a C^1-solution $b_j : \mathbb{R} \to \mathbb{R}$ of Eq. (5.6) with $d_* = d_j$ be given as in Proposition 6.1, with $b_j(t) = w_j(t)$ on $(-\infty, 0]$ and $b_j(t) = q_j(t)$ on $[m_2, \infty)$. Then the segments $h_t', b_{1,t}, \ldots, b_{n,t}$ are linearly independent for each $t \in \mathbb{R}$.*

Proof Analyticity and the hypothesis on linear independence combined imply that for every open interval $J \subset \mathbb{R}$ the restrictions of $w', w_1, \ldots, w_n$ to J are linearly independent, as well as the restrictions of $a\, y', q_1, \ldots, q_n$ to J. This implies the assertion for all $t < 2$ since for such t the interval $[t-2, t]$ contains an open subinterval J on which $h'(t) = w'(t)$ and $b_j(t) = w_j(t)$ for all $j \in \{1, \ldots, n\}$. Analogously we have for $t \geq 2 > m_2$ that $[t-2, t]$ contains an open subinterval J on which $h'(t) = a_h y'(t)$ and $b_j(t) = q_j(t)$ for all $j \in \{1, \ldots, n\}$.

7 Delay Functionals on Finite-Dimensional Manifolds

Let analytic solutions $w_j : \mathbb{R} \to \mathbb{R}$ and $q_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, of Eq. (3.1) be given as in the hypothesis of Proposition 6.2, and C^1-functions $d_j : \mathbb{R} \to \mathbb{R}$ and $b_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, as guaranteed by Proposition 6.1, so that for each $j \in \{1, \ldots, n\}$ we have

$$\begin{aligned} & d_j(t) = 0 \quad \text{on} \quad (-\infty, 0] \cup [m_2+1, \infty), \\ & b_j'(t) = -\alpha\{b_j(t-d(t)) - h'(t-d(t))d_j(t)\} \quad \text{for all} \quad t \in \mathbb{R}, \\ & b_j(t) = w_j(t) \quad \text{on} \quad (-\infty, 0], \\ & b_j(t) = q_j(t) \quad \text{on} \quad [m_2, \infty). \end{aligned}$$

All of these functions will be kept fixed from here on until Proposition 9.1 and its proof. Set $\phi_j := w_{j,0} \in C^1$ for $j \in \{1, \ldots, n\}$. Notice that all results from Sect. 5 apply. We proceed accordingly and obtain the map

$$I : \mathbb{R} \times V_n \ni (t, c) \mapsto x_t^c \in C^1$$

Recall $0 \in V_n$. From (5.3) we have

$$I(t, 0) = h_t \quad \text{for all} \quad t \in \mathbb{R}. \tag{7.1}$$

Proposition 7.1 *The map I is C^1-smooth with*

$$D_1 I(t, 0)1 = h'_t$$

and

$$D_{j+1} I(t, 0)1 = b_{j,t} \quad \textit{for all} \quad j \in \{1, \ldots, n\} \quad \textit{and} \quad t \in \mathbb{R}.$$

Proof 1. (Smoothness) According to Corollary 5.1 each map $x = x^{c,n+2}$, $c \in V_n$, is C^1-smooth and satisfies Eq. (5.4) for all $t \in \mathbb{R}$, and $x_0^{c,n+2} = \hat{E}(c)$. Hence

$$I(t, c) = x_t^c = \text{pr}_1 x_t^{c,n+2} \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad c \in V_n.$$

For $t \geq 0$ and $c \in V_n$ this yields

$$I(t, c) = \text{pr}_1 G\big(t, \hat{E}(c)\big). \tag{7.2}$$

It follows that the restriction of I to $(2, \infty) \times V_n$ is C^1-smooth.

Next, let $t_0 \leq 2$ and $c_0 \in V_n$ be given. Choose $t_1 < t_0 - 3$. For every $(t, c) \in (t_0 - 1, t_0 + 1) \times V_n$ we then have $t = s + t_1$ with

$$s = t - t_1 \in (t_0 - t_1 - 1, t_0 - t_1 + 1) \subset (2, \infty).$$

Also, $x_t^{c,n+2} = G(t - t_1, x_{t_1}^{c,n+2})$, hence

$$I(t, c) = x_t^c = \text{pr}_1 x_t^{c,n+2} = \text{pr}_1 G\big(t - t_1, x_{t_1}^{c,n+2}\big).$$

In view of the chain rule and $t - t_1 > 2$ we obtain that $I|(t_0 - 1, t_0 + 1) \times V_n$ is C^1-smooth provided the map

$$V_n \ni c \mapsto x_{t_1}^{c,n+2} \in C_{n+2}^1$$

is C^1-smooth, which is obvious from

$$x_{t_1}^{c,n+2} = \begin{pmatrix} w_{t_1} + \sum_1^n c_j w_{j,t_1} \\ c_1 \cdot \underline{1} \\ \vdots \\ c_n \cdot \underline{1} \\ id_{t_1} \end{pmatrix}$$

for all $c \in V_n$.

2. (Computation of derivatives) Using (7.1) and the fact that h is twice continuously differentiable we get

$$D_1 I(t, 0)1 = \frac{d}{ds}(\mathbb{R} \ni s \mapsto h_s \in C^1)(t)1 = h'_t \quad \text{for all} \quad t \in \mathbb{R}.$$

Then let $j \in \{1, \ldots, n\}$ be given. For each $t < 0$ and $c \in V_n$ we have

$$I(t, c) = w_t + \sum_1^n c_j w_{j,t},$$

hence $D_{j+1}I(t,c) = w_{j,t} = b_{j,t}$. For every $t \ge 0$ and $c \in V_n$ we obtain from (7.2) the equation

$$I(t,c) = \big(\mathrm{pr}_1 \circ G_t \circ \hat{E}\big)(c),$$

and thereby

$$D_{j+1}I(t,0)1 = D_j\big(\mathrm{pr}_1 \circ G_t \circ \hat{E}\big)(0)1 = D\big(\mathrm{pr}_1 \circ G_t \circ \hat{E}\big)(0)e_j.$$

Corollary 5.2 yields $D(\mathrm{pr}_1 \circ G_t \circ \hat{E})(0)e_j = b_t$ with a C^1-function $b : [-2,\infty) \to \mathbb{R}$ satisfying Eq. (5.6) for all $t \ge 0$ and (5.7). As $b_j|[-2,\infty)$ satisfies the same initial value problem we obtain from Proposition 5.3 (uniqueness) that

$$D_{j+1}I(t,0)1 = b_t = b_{j,t}.$$

Corollary 7.2 *Let $J \subset \mathbb{R}$ be a compact interval. Then there exists $s = s_J > 0$ with $(-s,s)^n \subset V_n$ so that the restriction $I|J \times (-s,s)^n$ itself and all its derivatives $DI(t,c)$, $(t,c) \in J \times (-s,s)^n$, are injective.*

Proof 1. Let $J \subset \mathbb{R}$ be a compact interval. As $V_n \ni 0$ is open there exists $s_0 > 0$ with $(-s_0,s_0)^n \subset V_n$. Suppose the assertion concerning I is false. Then there are sequences of reals $t_j \in J \ni \hat{t}_j$ and $c_j \in (-s_0,s_0)^n \ni \hat{c}_j$, $j \in \mathbb{N}$, with $c_j \to 0$ and $\hat{c}_j \to 0$ for $j \to \infty$, and for all $j \in \mathbb{N}$, $(t_j,c_j) \ne (\hat{t}_j,\hat{c}_j)$ and $I(t_j,c_j) = I(\hat{t}_j,\hat{c}_j)$. Passing to subsequences we may assume $t_j \to t \in J$ and $\hat{t}_j \to \hat{t} \in J$ as $j \to \infty$. In case $t \ne \hat{t}$ we get $h_t = I(t,0) = I(\hat{t},0) = h_{\hat{t}}$, which contradicts injectivity of the flowline $t \mapsto h_t$ (Proposition 3.2 of [18]).

In case $t = \hat{t}$ the mean value theorem yields

$$\begin{aligned} 0 &= I\big(\hat{t}_j,\hat{c}_j\big) - I(t_j,c_j) \\ &= \int_0^1 DI\big((t_j,c_j) + \theta[(\hat{t}_j,\hat{c}_j) - (t_j,c_j)]\big)[\ldots]d\theta \\ &= (\hat{t}_j - t_j)\int_0^1 D_1 I(\ldots)1d\theta + \sum_{k=1}^{n}\big(\hat{c}_{j,k} - c_{j,k}\big)\int_0^1 D_{k+1}I(\ldots)1d\theta \end{aligned}$$

for every $j \in \mathbb{N}$. Setting $r_j = |(\hat{t}_j,\hat{c}_j) - (t_j,c_j)|$ $(\ne 0)$ for $j \in \mathbb{N}$ we have

$$\left|\frac{1}{r_j}\big((\hat{t}_j,\hat{c}_j) - (t_j,c_j)\big)\right| = 1$$

for all $j \in \mathbb{N}$. Passing to subsequences we may assume

$$\frac{1}{r_j}\big((\hat{t}_j,\hat{c}_j) - (t_j,c_j)\big) \to (\bar{t},c) \in S^n \subset \mathbb{R}^{n+1} \quad \text{for} \quad j \to \infty.$$

As

$$\int_0^1 D_1 I(\ldots)1d\theta \to D_1 I(t,0)1 = h_t'$$

and

$$\int_0^1 D_{k+1}I(\ldots)1d\theta \to D_{k+1}I(t,0)1 = b_{k,t}$$

for $j \to \infty$ we arrive at

$$0 = \bar{t}\, h'_t + \sum_{k=2}^{n+1} c_{k-1} b_{k-1,t}$$

which is a contradiction to linear independence (Proposition 6.2).

It follows that for some $\hat{s}_J \in (0, s_0)$ the restriction $I\big|_{J \times (-\hat{s}_J, \hat{s}_J)^n}$ is injective.

2. Suppose the assertion concerning DI is false. Then there are sequences of reals $t_j \in J$ and $c_j \in (-s_0, s_0)^n$, $j \in \mathbb{N}$, with $c_j \to 0$ and $DI(t_j, c_j)$ not injective. It follows that for each $j \in \mathbb{N}$ the vectors

$$D_k I(t_j, c_j)1 = DI(t_j, c_j)e_k, \quad k \in \{1, \dots, n+1\},$$

are linearly dependent, and there exist $r_j \in S^n \subset \mathbb{R}^{n+1}$ with

$$0 = \sum_{k=1}^{n+1} r_{j,k} D_k I(t_j, c_j)1 \quad \text{for all} \quad j \in \mathbb{N}.$$

Passing to subsequences we may assume $r_j \to r_0 \in S^n$ and $t_j \to t \in J$ for $j \to \infty$. Passing to limits we arrive at

$$0 = \sum_{k=1}^{n+1} r_{0,k} D_k I(t, 0)1 = r_{0,1} h'_t + \sum_{k=2}^{n+1} r_{0,k} b_{k-1,t}$$

which is a contradiction as in part 1 of the proof.

It follows that for some $s_J \in (0, \hat{s}_J)$ all derivatives $DI(t, c)$, $(t, c) \in J \times (-s_J, s_J)^n$, are injective.

We fix $t_1 < 0$ and $t_2 > m_2 + 2$, set $J := [t_1, t_2]$, and choose $s = s_J$ according to Corollary 7.2.

Corollary 7.3 *The set $M := I((t_1, t_2) \times (-s, s)^n) \subset C^1 \subset C$ is an $(n+1)$-dimensional C^1-submanifold of the space C, and the map $I_C : (t_1, t_2) \times (-s, s)^n \to M$ given by $I_C(t, c) = I(t, c)$ is a C^1-diffeomorphism.*

Proof Use Corollary 7.2, employ the inclusion map $C^1 \to C$, and apply Proposition 10.5 from [18].

The C^1-map

$$\overline{d} : \mathbb{R} \times \mathbb{R}^n \ni (t, c) \mapsto d(t) + \sum_{1}^{n} c_j d_j(t) \in \mathbb{R}$$

satisfies

$$\begin{aligned} &\overline{d}(\mathbb{R} \times V_n) \subset (0, 2), \\ &\quad \overline{d}(t, c) = d(t) \quad \text{on} \quad \big((-\infty, 0] \cup [m_2 + 1, \infty)\big) \times \mathbb{R}^n, \\ &\quad \overline{d}(t, 0) = d(t) \quad \text{on} \quad \mathbb{R}. \end{aligned}$$

It follows that the delay functional $d_M : C \supset M \to (0, 2)$ given by

$$d_M(\phi) = \overline{d}\big(I_C^{-1}(\phi)\big)$$

is C^1-smooth. For each $(t, c) \in (t_1, t_2) \times (-s, s)^n$ we have

$$d_M(x_t^c) = d_M\big(I_C(t, c)\big) = \overline{d}(t, c) = d(t) + \sum_1^n c_j d_j(t). \tag{7.3}$$

Using this and Eq. (5.2) we obtain that for each $c \in (-s, s)^n$ the function $x = x^c$ satisfies the autonomous equation

$$x'(t) = -\alpha\, x\big(t - d_M(x_t)\big) \tag{7.4}$$

with state-dependent delay for all $t \in (t_1, t_2)$. In particular,

$$h'(t) = -\alpha\, h\big(t - d_M(h_t)\big) \quad \text{on} \quad (t_1, t_2),$$

because of (5.3). Notice that for $t \in (t_1, 0) \cup (m_2 + 2, t_2)$ and $c \in (-s, s)^n$ we have

$$d_M(x_t^c) = d(t).$$

8 Delay Functionals on Neighbourhoods of the Homoclinic Loop

This section follows almost verbatim Sections 7 and 8 from [18]. In the first part, which corresponds to Section 7 from [18], we extend a restriction of d_M to a compact neighbourhood of the orbit piece $\{h_t : 0 \le t \le m_2 + 2\}$ in M to an open neighbourhood of M in the ambient space C.

Fix $t_{10} \in (t_1, 0)$ and $t_{20} \in (m_2 + 2, t_2)$. For every $t \in [t_{10}, t_{20}]$ there are an open neighbourhood U_t of $h_t \in M$ in C, a radius $r(t) > 0$, a closed subspace Q_t of codimension $n + 1$ in C, and a C^1-diffeomorphism R_t from U_t onto $\mathbb{R}^{n+1}_{r(t)} \times Q_{r(t)}$, with

$$R_t(U_t \cap M) = \mathbb{R}^{n+1}_{r(t)} \times \{0\}.$$

As $H : \mathbb{R} \ni t \mapsto h_t \in C$ is injective (Proposition 3.2 from [18]) we can choose the neigbourhoods U_t in such a way that

$$h_{t_{10}} \notin \overline{U_t} \quad \text{for all } t \in (t_{10}, t_{20}] \text{ and } h_{t_{20}} \notin \overline{U_t} \text{ for all } t \in [t_{10}, t_{20}). \tag{8.1}$$

By compactness of the orbit piece $\{h_t : t_{10} \le t \le t_{20}\}$ there exist $s_1 < \ldots < s_m$ in $[t_{10}, t_{20}]$ so that the sets $U_\mu = U_{s_\mu}$, $\mu \in \{1, \ldots, m\}$, cover the orbit piece $H([t_{10}, t_{20}])$. Observe that (8.1) implies $s_1 = t_{10}$ and $s_m = t_{20}$.

Using compactness once again we find $r \in (0, s_J)$ so that

$$K := I_C\big([t_{10}, t_{20}] \times [-r, r]^n\big) \subset \cup_{\mu=1}^m U_\mu.$$

For the open covering $(U_\mu)_{\mu=1}^m$ of the compact subset K of the manifold M there exists a subordinate continuously differentiable partition of unity $(\eta_\iota)_{\iota=1}^j$, that is, each $\eta_\iota : M \to [0, 1]$ is continuously differentiable and has compact support, for every $\iota \in \{1, \ldots, j\}$ there exists $\mu \in \{1, \ldots, m\}$ with $\operatorname{supp}(\eta_\iota) \subset U_\mu \cap M$, and for every $\phi \in K$,

$$\sum_{\iota=1}^j \eta_\iota(\phi) = 1.$$

There exists a map $\{1, \ldots, j\} \ni \iota \mapsto \mu(\iota) \in \{1, \ldots, m\}$ with

$$\begin{aligned} &\text{supp }(\eta_\iota) \subset U_{\mu(\iota)}, \\ &\quad \mu(\iota) = 1 \quad \text{for all} \quad \iota \in \\ &\qquad J_1 = \left\{\iota' \in \{1, \ldots, j\} : \text{ supp }(\eta_{\iota'}) \subset U_1\right\}, \\ &\quad \mu(\iota) = m \quad \text{for all} \quad \iota \in \\ &\qquad J_m = \left\{\iota' \in \{1, \ldots, j\} : \text{ supp }(\eta_{\iota'}) \subset U_m\right\}. \end{aligned}$$

As in the first part of the proof of Proposition 8.1 of [18] we get

$$J_1 \neq \emptyset \neq J_m.$$

Now let $\iota \in \{1, \ldots, j\}$ be given. The next objective is the construction of a C^1-function

$$\overline{d}_\iota : \Delta_\iota \to \mathbb{R}, \quad \Delta_\iota \subset C \quad \text{open},$$

with $M \subset \Delta_\iota$ and

$$\overline{d}_\iota(\phi) = \eta_\iota(\phi) d_M(\phi) \quad \text{for all} \quad \phi \in M.$$

We abbreviate

$$U_* := U_{\mu(\iota)}, \, R_* := R_{s_{\mu(\iota)}}, \, r_* := r(s_{\mu(\iota)}), \, Q_* := Q_{s_{\mu(\iota)}}.$$

Then

$$R_*(U_* \cap M) = \mathbb{R}^{n+1}_{r_*} \times \{0\} \subset \mathbb{R}^{n+1}_{r_*} \times Q_{r_*/4}.$$

Set

$$V_{\mu(\iota)} := R^{-1}_{r_*}(\mathbb{R}^{n+1}_{r_*} \times Q_{r_*/4}) \supset U_* \cap M.$$

Obviously,

$$V_{\mu(\iota)} \subset U_*, \text{ supp }(\eta_\iota) \subset U_* \cap M \subset V_{\mu(\iota)},$$

and

$$R_*(\text{ supp }(\eta_\iota)) = \text{pr}_1 R_*(\text{ supp }(\eta_\iota)) \times \{0\},$$

with the projection

$$\text{pr}_1 : \mathbb{R}^{n+1} \times Q_* \to \mathbb{R}^{n+1}$$

onto the first factor. The map $\hat{d} = \overline{d}_\iota$, $\hat{d} : V_{\mu(\iota)} \to \mathbb{R}$, given by

$$\hat{d}(\phi) = \eta_\iota\big(R^{-1}_*(\text{pr}_1 R_*(\phi), 0)\big) d_M\big(R^{-1}_*(\text{pr}_1 R_*(\phi), 0)\big)$$

is C^1-smooth (Fig. 4).

Proposition 8.1 *Let $\iota \in \{1, \ldots, j\}$ be given. Every $\phi \in M \setminus$ supp (η_ι) has an open neighbourhood $V_{\phi,\iota}$ in C with*

$$V_{\phi,\iota} \cap R^{-1}_*\big(pr_1 R_*(\text{ supp }(\eta_\iota)) \times \overline{Q_{r_*/2}}\big) = \emptyset.$$

In particular, $V_{\phi,\iota} \cap$ supp $(\eta_\iota) = \emptyset$.

Proof See the proof of Proposition 7.1 in [18].

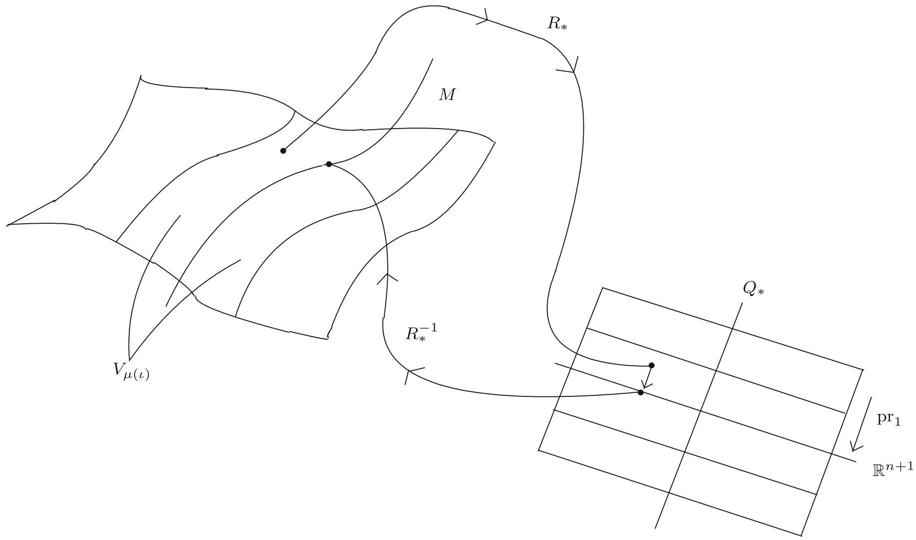


Fig. 4 The argument of η_ι and d_M in the formula defining $\hat{d}(\phi)$, $\phi \in V_{\mu(\iota)}$

For $\iota \in \{1, \ldots, j\}$ given we continue as in Section 7 of [18], choose neighbourhoods $V_{\phi,\iota}$ according to Proposition 8.1, and consider the set

$$\hat{V}_\iota := \cup_{\phi \in M \setminus \text{ supp } (\eta_\iota)} V_{\phi,\iota},$$

which is open in C. We have

$$\hat{V}_\iota \cap R_*^{-1}\big(\mathrm{pr}_1 R_*(\text{ supp } (\eta_\iota)) \times \overline{Q_{r_*/2}}\big) = \emptyset,$$

and the open set

$$\Delta_\iota := \hat{V}_\iota \cup V_{\mu(\iota)}$$

contains

$$\big(M \setminus \text{ supp } (\eta_\iota)\big) \cup \text{ supp } (\eta_\iota) = M.$$

Proposition 8.2 *Let $\iota \in \{1, \ldots, j\}$ be given. For every $\psi \in \hat{V}_\iota \cap V_{\mu(\iota)}$ we have $\overline{d}_\iota(\psi) = 0$.*

Proof See the proof of Proposition 7.2 in [18].

For each $\iota \in \{1, \ldots, j\}$ we extend $\overline{d}_\iota : V_{\mu(\iota)} \to \mathbb{R}$ to a map on Δ_ι by $\overline{d}_\iota(\psi) = 0$ on $\hat{V}_\iota$. The extended map $\overline{d}_\iota : \Delta_\iota \to \mathbb{R}$ is C^1-smooth.

Corollary 8.3 *Let $\iota \in \{1, \ldots, j\}$ be given. For all $\psi \in M$ we have $\overline{d}_\iota(\psi) = \eta_\iota(\psi) d_M(\psi)$.*

Proof See the proof of Corollary 7.3 in [18].

The set $\Delta^* := \cap_{\iota=1}^{j} \Delta_\iota \quad (\supset M)$ is open in C, and the map

$$d^* : \Delta^* \to \mathbb{R}$$

given by $d^*(\phi) = \sum_{\iota=1}^{j} \overline{d}_\iota(\phi)$ is C^1-smooth.

Corollary 8.4 *For every $\phi \in K \subset M$ we have $d^*(\phi) = d_M(\phi)$.*

Proof Use Corollary 8.3 and

$$d^*(\phi) = \sum_{\iota=1}^{j} \overline{d}_\iota(\phi) = \sum_{\iota=1}^{j} \eta_\iota(\phi) d_M(\phi) = d_M(\phi)$$

for $\phi \in K \subset M$.

The construction of the desired delay functional on a neighbourhood of the homoclinic loop $H(\mathbb{R}) \cup \{0\} \subset C$ requires a modification of d^*. This is done as in Section 8 of [18].

The next intermediate step is to find $t_{11} \in (t_{10}, 0)$ and an open neighbourhood $V_{11} \subset \Delta^*$ of $h_{t_{11}}$ in C so that

$$d^*(\phi) = 1 \quad \text{on} \quad V_{11}.$$

Observe that for all $\iota \in J_1$ and $\phi \in V_1$ we have

$$\overline{d}_\iota(\phi) = \eta_\iota(R_1^{-1}(\mathrm{pr}_1 R_1(\phi), 0)) d_M(R_1^{-1}(\mathrm{pr}_1 R_1(\phi), 0)). \tag{8.2}$$

Proposition 8.5 *For every $\iota \in J_1' = \{1, \ldots, j\} \setminus J_1$ we have $\mu(\iota) \in \{2, \ldots, j\}$, and for all $\phi \in (U_1 \setminus \cup_{\mu=2}^m \overline{U_\mu}) \cap \Delta_\iota$ we have*

$$\overline{d}_\iota(\phi) = 0.$$

Proof See the proof of Proposition 8.1 (ii) in [18].

By (8.1) the open set $U_1 \setminus \cup_{\mu=2}^m \overline{U_\mu}$ contains $h_{t_{10}}$. As H is continuous there exists $t_{11} \in (t_{10}, 0)$ with

$$h_{t_{11}} \in U_1 \setminus \cup_{\mu=2}^m \overline{U_\mu}.$$

Recall $U_1 = U_{s_1}$. Then

$$R_{s_1}(h_{t_{11}}) \in \mathbb{R}^{n+1}_{r(s_1)} \times \{0\}.$$

As I_C is a C^1-diffeomorphism the set $I_C((t_{10}, 0) \times (-r, r)^n)$ is an open subset of M which contains $h_{t_{11}}$. By continuity there exists $\rho_1 \in (0, \frac{r(s_1)}{4})$ so that

$$\mathbb{R}^{n+1}_{\rho_1} + \mathrm{pr}_1 R_{s_1}(h_{t_{11}}) \subset \mathbb{R}^{n+1}_{r(s_1)}, \tag{8.3}$$

$$R_{s_1}^{-1}\big((\mathbb{R}^{n+1}_{\rho_1} + \mathrm{pr}_1 R_{s_1}(h_{t_{11}})) \times Q_{1,\rho_1}\big) \subset U_1 \setminus \cup_{\mu=2}^m \overline{U_\mu},$$

and

$$R_{s_1}^{-1}\big((\mathbb{R}^{n+1}_{\rho_1} + \mathrm{pr}_1 R_{s_1}(h_{t_{11}})) \times \{0\}\big) \subset I_C\big((t_{10}, 0) \times (-r, r)^n\big) \subset K. \tag{8.4}$$

For every $\phi \in R_{s_1}^{-1}((\mathbb{R}^{n+1}_{\rho_1} + \mathrm{pr}_1 R_{s_1}(h_{t_{11}})) \times \{0\})$ we infer from (8.4) that

$$\begin{aligned} d_M(\phi) &= d_M\big(I_C(t, c)\big) \quad \big(\text{with} \quad t_{10} < t < 0, c \in (-r, r)^n\big) \\ &= d(t) + \sum_1^n c_\nu d_\nu(t) \quad (\text{see } 7.3) \\ &= d(t) \quad (\text{since} \quad t < 0) \\ &= 1 \qquad (\text{since} \quad t < 0). \end{aligned}$$

The set

$$V_{11} := R_{s_1}^{-1}\big((\mathbb{R}^{n+1}_{\rho_1} + \mathrm{pr}_1 R_{s_1}(h_{t_{11}})\big) \times Q_{1,\rho_1}\big) \cap \Delta^*$$

is open in C and contains $h_{t_{11}}$. Using $\rho_1 < \frac{r(s_1)}{4}$ and (8.3) we get

$$V_{11} \subset V_1.$$

Proposition 8.6 *For every $\phi \in V_{11}$, $d^*(\phi) = 1$.*

Proof See the proof of Proposition 8.2 in [18].

In the same way as above we find $t_{21} \in (m_2+2, t_{20})$ and an open neighbourhood $V_{21} \subset \Delta^*$ of $h_{t_{21}}$ in C so that

$$d^*(\phi) = 1 \quad \text{on} \quad V_{21}.$$

Now we can complete the construction of the delay functional on a neighbourhood of $H(\mathbb{R}) \cup \{0\}$ in C. We choose $t'_{11} \in (t_{10}, t_{11})$ and $t''_{11} \in (t_{11}, 0)$ so that

$$H\big([t'_{11}, t''_{11}]\big) \subset V_{11}$$

and similarly $t'_{21} \in (m_2 + 2, t_{21})$ and $t''_{21} \in (t_{21}, t_{20})$ so that

$$H\big([t'_{21}, t''_{21}]\big) \subset V_{21}.$$

The sets $\{0\} \cup H((-\infty, t'_{11}]) \cup H([t''_{21}, \infty))$ and $H([t''_{11}, t'_{21}]) \subset M \subset \Delta^*$ are compact and disjoint since H is injective, see Proposition 3.2 in [18]. Consequently there are disjoint open neighbourhoods N_0 of $\{0\} \cup H((-\infty, t'_{11}]) \cup H([t''_{21}, \infty))$ in C and N of $H([t''_{11}, t'_{21}])$ in C. We may assume $N \subset \Delta^*$. Since $d_M(M) \subset (0, 2)$ and $d^*(\phi) = d_M(\phi)$ on $K \supset H([t''_{11}, t'_{21}])$ (see Corollary 8.4) we may also assume $d^*(\phi) \in (0, 2)$ on N. The open subset

$$\Delta := N_0 \cup V_{11} \cup N \cup V_{21}$$

of C contains $H(\mathbb{R}) \cup \{0\}$. On $N \cap (V_{11} \cup V_{21})$ we have $d_*(\phi) = 1$. It follows that the equations

$$d_\Delta(\phi) = 1 \quad \text{for} \quad \phi \in N_0 \cup V_{11} \cup V_{21},$$
$$d_\Delta(\phi) = d^*(\phi) \quad \text{for} \quad \phi \in N,$$

define a C^1-map $d_\Delta : \Delta \to (0, 2)$. The continuity of I_C and the compactness of $H([t''_{11}, t'_{21}]) \subset N$ imply that there exists $r_\Delta \in (0, r)$ so that

$$K_\Delta := I_C\big([t''_{11}, t'_{21}] \times (-r_\Delta, r_\Delta)^n\big)$$

is contained in N.

Proposition 8.7 *For every $t \in \mathbb{R}$ we have $d_\Delta(h_t) = d(t)$, and for all $t \in [t''_{11}, t'_{21}]$ and $c \in (-r_\Delta, r_\Delta)^n$,*

$$I_C(t, c) \in \Delta \quad \textit{and} \quad d_\Delta\big(I_C(t, c)\big) = d(t) + \sum_1^n c_\nu d_\nu(t).$$

Proof (Compare the proof of Proposition 8.3 in [18]) For $t \le t''_{11}$ we have $h_t \in N_0 \cup V_{11}$, hence $d_\Delta(h_t) = 1$. As $t < 0$ we also have $d(t) = 1$. Analogously one finds $d_\Delta(h_t) = 1 = d(t)$ for $t \ge t'_{21}$.

For $t''_{11} \le t \le t'_{21}$ and $c \in (-r_\Delta, r_\Delta)^n$ we have $I_C(t,c) \in K_\Delta \subset N$ $(\subset \Delta)$, hence $d_\Delta(I_C(t,c)) = d^*(I_C(t,c))$.

As $t_{10} < t_{11} < t''_{11} < 0$ and $m_2 + 2 < t'_{21} < t_{21} < t''_{21} < t_{20}$ and $c \in (-r_\Delta, r_\Delta)^n$ we also have $I_C(t,c) \in K$. Hence

$$\begin{aligned} d_\Delta\big(I_C(t,c)\big) &= d^*\big(I_C(t,c)\big) = d_M\big(I_C(t,c)\big) \quad \text{(see Corollary 8.4)} \\ &= d(t) + \sum_1^n c_\nu d_\nu(t) \quad \text{(see 7.3)}. \end{aligned}$$

For $c = 0$, obviously

$$d_\Delta(h_t) = d_\Delta\big(I(t,0)\big) = d(t)$$

also for $t''_{11} \le t \le t''_{21}$.

It follows that the solution $x = h$ of Eq. (3.7) also satisfies Eq. (3.8),

$$x'(t) = -\alpha\, x\big(t - d_\Delta(x_t)\big)$$

for all $t \in \mathbb{R}$, and that the solutions x^c of Eq. (5.2), $c \in (-r_\Delta, r_\Delta)^n$, satisfy Eq. (3.8) for all $t \in [t''_{11}, t'_{21}]$.

For the next section we also need the following result.

Corollary 8.8 *Let reals $t_- \le t_+$ be given. There exists $\bar r \in (0, r_\Delta)$ with $I_C([t_-, t_+] \times (-\bar r, \bar r)^n) \in \Delta$ and*

$$d_\Delta\big(I_C(t,c)\big) = d(t) + \sum_1^n c_\nu d_\nu(t) \quad \textit{on} \quad [t_-, t_+] \times (-\bar r, \bar r)^n.$$

Proof (See the proof of Corollary 8.4 in [18]) In case $t_- < t''_{11}$ we have $H([t_-, t''_{11}]) \subset N_0 \cup V_{11}$. Using compactness and continuity we find $\bar r \in (0, r_\Delta)$ with

$$I_C\big([t_-, t''_{11}] \times (-\bar r, \bar r)^n\big) \subset N_0 \cup V_{11}.$$

On $[t_-, t''_{11}] \times [-\bar r, \bar r]^n$ we get

$$\begin{aligned} d_\Delta\big(I_C(t,c)\big) &= 1 = d(t) + 0 \quad (\text{since} \quad t \le t''_{11} < 0) \\ &= d(t) + \sum_1^n c_\nu d_\nu(t) \quad (\text{since} \quad d_\nu(t) = 0 \quad \text{on} \quad (-\infty, 0]). \end{aligned}$$

Proposition 8.7 contains the desired equation on $[t''_{11}, t'_{21}] \times [-\bar r, \bar r]^n$. Now it becomes obvious how to complete the proof using Proposition 8.7 and $d(t) = 1$ for $t \ge t'_{21}$ and $d_\Delta(\phi) = 1$ on $N_0 \cup V_{21}$.

9 Linearization Along the Homoclinic Curve

As in Sect. 3 we obtain from C^1-smoothness of the map $d_\Delta : C \supset \Delta \to (0, 2)$ that the maximal C^1-solutions $x = x^\phi$, $x : [-r, t_e(\phi)) \to \mathbb{R}^n$, $0 < t_e(\phi) \le \infty$, of the initial value problem given by Eq. (3.8) and the initial condition

$$x_0 = \phi \in \Big\{\psi \in \Delta \cap C^1 : \psi'(0) = -\alpha\,\psi\big(-d_\Delta(\phi)\big)\Big\} =: X_\Delta$$

define a continuous semiflow $F : \Omega \to X$ on the C^1-submanifold $X := X_\Delta$ of C^1, with domain $\Omega := \{(t,\phi) \in [0,\infty) \times X : t < t_e(\phi)\}$ and $F(t,\phi) = x_t^\phi$. Let

$$f : \Delta \cap C^1 \to \mathbb{R}$$

be given by $f(\phi) = -\alpha\,\phi(-d_\Delta(\phi))$. The C^1-maps F_t, $t \geq 0$, with nonempty domain $\Omega_t := \{\phi \in X : t < t_e(\phi)\}$ and $F_t(\phi) = F(t,\phi)$, satisfy

$$DF_t(\phi)\chi = v_t^{\phi,\chi}$$

with the C^1-solution $v = v^{\phi,\chi}$, $v : [-r, t_e(\phi)) \to \mathbb{R}^n$, of the initial value problem

$$\begin{aligned} v'(t) &= Df\big(F(t,\phi)\big)v_t \quad \text{for} \quad t \geq 0,\\ v_0 &= \chi \in T_\phi X. \end{aligned}$$

The restriction of F to the set $\{(t,\phi) \in \Omega : 2 < t\}$ is C^1-smooth, with

$$D_1F(t,\phi)1 = \big(x_t^\phi\big)' = \big((x^\phi)'\big)_t \in C^1.$$

From Eq. (3.8) for $x = h$ we infer

$$F(t-s, h_s) = h_t \quad \text{for all} \quad t \geq s.$$

It follows that

$$D_2F(t-s,h_s)h_s' = h_t' \quad \text{for all} \quad t \geq s. \tag{9.1}$$

Proposition 9.1 *For every $j \in \{1,\ldots,n\}$, for all reals $s \leq t_{11}'$ and for all reals $t \geq t_{21}''$ we have*

$$D_2F(t-s,h_s)w_{j,s} = q_{j,t}.$$

Proof Let $j \in \{1,\ldots,n\}$ be given, and let $d_{\Delta,1}$ denote the C^1-map $C^1 \supset \Delta \cap C^1 \ni \phi \mapsto d_\Delta(\phi) \in (0,2)$. For each $t \in \mathbb{R}$ we get

$$\begin{aligned} Dd_{\Delta,1}(h_t)b_{j,t} &= Dd_{\Delta,1}\big(I_C(t,0)\big)D_{j+1}I(t,0)1 \qquad \text{(Proposition 7.1)}\\ &= D_{j+1}\big(d_{\Delta,1} \circ I\big)(t,0)1 \qquad \text{(the chain rule)}\\ &= D_{j+1}\Big((s,c) \mapsto d(s) + \sum_1^n c_j d_j(s)\Big)(t,0)1 \qquad \text{(Corollary 8.8)}\\ &= d_j(t). \end{aligned}$$

A computation as in Sect. 3 shows that for every $\phi \in \Delta \cap C^1$ and for all $\chi \in C^1$ we have

$$Df(\phi)\chi = -\alpha\big\{\chi(-d_\Delta(\phi)) - \phi'(-d_\Delta(\phi))Dd_{\Delta,1}(\phi)\chi\big\}.$$

It follows that for every $t \in \mathbb{R}$,

$$\begin{aligned} Df(h_t)b_{j,t} &= -\alpha\big\{b_j(t - d_{\Delta,1}(h_t)) - h'(t - d_{\Delta,1}(h_t))Dd_{\Delta,1}(h_t)b_{j,t}\big\}\\ &= -\alpha\big\{b_j(t - d_{\Delta,1}(h_t)) - h'(t - d_{\Delta,1}(h_t))d_j(t)\big\}\\ &\qquad \text{(by the computation above)}\\ &= (b_j)'(t) \quad \text{(by the choice of } b_j \text{ in Sect. 5)} \end{aligned}$$

The preceding equation implies that for all reals s and $\tau \geq 0$ we have

$$D_2 F(\tau, h_s) b_{j,s} = b_{j,\tau+s}.$$

Finally, use $b_j(t) = w_j(t)$ on $(-\infty, 0]$ and $b_j(t) = q_j(t)$ on $[m_2, \infty)$.

Before we state what has been achieved in a theorem it may be convenient to recall that for $\alpha \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ we defined $w : \mathbb{R} \to \mathbb{R}$ by $w(t) = e^{u_0 t} \sin(v_0 t)$ and $y : \mathbb{R} \to \mathbb{R}$ by $y(t) = e^{ut} \sin(vt)$, with $\lambda_0 = u_0 + i v_0$ the eigenvalue of the generator of the semigroup T_α in $(0, \infty) + i(0, \infty)$ and $\lambda = u + iv$ the eigenvalue in $(-\infty, 0) + i(0, \infty)$ with largest real part.

Theorem 9.2 *There exist $\alpha_0 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ so that for every $\alpha \in \left(\alpha_0, \frac{5\pi}{2}\right)$ there is a real $a_h > 0$ with the following properties. For every $n \in \mathbb{N}$, and for all families of analytic solutions $w_j : \mathbb{R} \to \mathbb{R}$ and $q_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, of Eq. (3.1) with $w_0', w_{1,0}, \ldots, w_{n,0}$ linearly independent and $y_{m_2+2}', q_{1,m_2+2}, \ldots, q_{n,m_2+2}$ linearly independent there are an open neighbourhood Δ of 0 in C and a C^1-functional $d_\Delta : C \supset \Delta \to (0, 2)$ so that*

(i) *$d_\Delta(\phi) = 1$ on a neigbourhood of 0 in C,*
(ii) *Eq. (3.8),*

$$x'(t) = -\alpha\, x\,(t - d_\Delta(x_t))\,,$$

has a C^1-solution $h : \mathbb{R} \to \mathbb{R}$ with $h(t) = w(t)$ on $(-\infty, 0]$ and $h(t) = a_h y(t)$ on $[1, \infty)$; in particular, $h(t) \to 0$ for $|t| \to \infty$,
(iii) The maximal C^1-solutions $[-2, t_e) \to \mathbb{R}$ of Eq. (3.8) define a semiflow F on the C^1-submanifold

$$X := \{\phi \in \Delta \cap C^1 : \phi'(0) = -\alpha\, \phi(-d_\Delta(\phi))\}.$$

There exist $s_0 \leq 0$ and $t_0 \geq 3$ so that for all $s \leq s_0$ and all $t \geq t_0$ we have

$$D_2 F(t - s, h_s) h_s' = h_t',$$

and for every $j \in \{1, \ldots, n\}$,

$$w_{j,s} \in T_{h_s} X, \quad q_{j,t} \in T_{h_t} X, \quad \text{and} \quad D_2 F(t - s, h_s) w_{j,s} = q_{j,t}.$$

Corollary 9.3 *There exist $\alpha_0 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ so that for every $\alpha \in \left(\alpha_0, \frac{5\pi}{2}\right)$ there is a real $a_h > 0$ with the following properties. There are an open neighbourhood Δ of 0 in C and a C^1-functional $d_\Delta : C \supset \Delta \to (0, 2)$ so that*

(i) $d_\Delta(\phi) = 1$ on a neigbourhood of 0 in C,
(ii) and Eq. (3.8), namely,

$$x'(t) = -\alpha\, x\,(t - d_\Delta(x_t))$$

has a C^1-solution $h : \mathbb{R} \to \mathbb{R}$ with $h(t) = w(t)$ on $(-\infty, 0]$ and $h(t) = a_h y(t)$ on $[1, \infty)$; in particular, $h(t) \to 0$ for $|t| \to \infty$,
(iii) The maximal C^1-solutions $[-2, t_e) \to \mathbb{R}$ of Eq. (3.8) define a semiflow F on the C^1-submanifold

$$X := \left\{\phi \in \Delta \cap C^1 : \phi'(0) = -\alpha\, \phi(-d_\Delta(\phi))\right\}.$$

There exist $s_0 \le 0$ *and* $t_0 \ge 3$ *so that for all* $s \le s_0$ *and all* $t \ge t_0$*, with*

$$Y := T_0 X = \{\chi \in C^1 : \chi'(0) = -\alpha\chi(-1)\} \quad \textit{and} \quad Y_s := C_s \cap Y \supset C_i,$$

we have

$$T_{h_s} X = T_{h_t} X = Y = Y_s \oplus C_u,$$

and

$$\begin{aligned} h'_s &\in C_u, \\ h'_t &\in C_i, \\ D_2 F(t-s, h_s) h'_s &= h'_t, \\ D_2 F(t-s, h_s)(C_i \oplus C_u) &= (C_i \oplus C_u) \quad (\textit{this is 3.9}), \\ (D_2 F(t-s, h_s) C_u) \cap Y_s &= \mathbb{R} h'_t \qquad (\textit{this is 3.6}). \end{aligned}$$

Proof Recall $0 \ne w'_t \in C_u$ and $0 \ne y'_t \in C_i$ for all $t \in \mathbb{R}$. There are analytic solutions $w_j : \mathbb{R} \to \mathbb{R}$ and $q_j : \mathbb{R} \to \mathbb{R}$ of Eq. (3.1), $j \in \{1, 2, 3\}$, so that for all $t \in \mathbb{R}$ $w'_t, w_{1,t}$ form a basis of C_u and $w_{2,t}, w_{3,t}$ form a basis of C_i, $y'_t, q_{1,t}$ form a basis of C_i, and $q_{2,t}, q_{3,t}$ form a basis of C_u. Theorem 9.2 with $n = 3$ yields that for $s \le s_0$ and $t \ge t_0$ the derivative $D_2 F(t-s, h_s) : T_{h_s} X \to T_{h_t} X$ maps a basis of $C_i \oplus C_u$ onto a basis of the same space.

In particular we can arrange that $D_2 F(t-s, h_s) w_{1,s} = q_{2,t} \quad (\in C_u)$ which yields the minimal intersection property

$$\big(D_2 F(t-s, h_s) C_u\big) \cap Y_s = \mathbb{R} h'_t$$

for all $s \le s_0$ and $t \ge t_0$.

10 The Inner Map

From here on we consider the delay functional $d_\Delta : C \supset \Delta \to (0, 2)$ from Corollary 9.3. Then there exists $\theta > m_2 + 2$ so that for all $s \le -\theta$ and for all $t \ge \theta$ we have (3.9) and the minimal intersection property (3.6).

Let $W \subset \Delta \subset C$ denote a neighbourhood of $0 \in C$ on which $d(\phi) = 1$. Then

$$X \cap W = \{\phi \in W \cap C^1 : \phi'(0) = -\alpha\,\phi(-1)\} = Y \cap W$$

and for every $t \ge 0$ and $\phi \in X \cap W$ with $F([0, t] \times \{\phi\}) \subset W$,

$$F(t, \phi) = T(t)\phi.$$

In the sequel we introduce hypersurfaces H_i and H_u which will be solid tori in $Y \cap W$ with *central circles* $S_i \subset C_i$ and $S_u \subset C_u$, respectively. Upon that we define the inner map as the shift along phase curves from $H_i \setminus Y_s$ to $H_u \setminus S_u = H_u \setminus C_u$. This requires some preparation concerning the semigroups T on C and $(D_2 F(t, 0))_{t \ge 0}$ on Y. Recall that $D_2 F(t, 0)\chi = T(t)\chi$ for all $\chi \in Y$.

Recall $\lambda_0 = u_0 + i v_0$, $\lambda = u + i v$ from Sects. 3 and 4 and let $C_< \subset C$ denote the realified generalized eigenspace associated with the subset of the spectrum of the generator of the semigroup T given by $\mathrm{Re}(\zeta) < u < 0$. From the invariant decomposition $C = C_< \oplus C_i \oplus C_u$ we obtain the decomposition

$$Y = Y_< \oplus C_i \oplus C_u \tag{10.1}$$

with $Y_< = C_< \cap Y$ which is positively invariant under the operators $D_2F(t,0) : Y \to Y$, $t \geq 0$. The projections $Y \to Y$ onto $Y_<$, C_i, C_u which are given by the decomposition (10.1) are denoted by $P_<$, P_i, P_u, respectively.

For the exponential decay of phase curves $T(\cdot)\chi$ in $Y_<$ we have the estimate

$$|T(t)y_<|_1 \leq c_< e^{-\eta_< t}|y_<|_1 \quad \text{for all} \quad y_< \in Y_<, \quad t \geq 0, \tag{10.2}$$

with constants $c_< \geq 1$ and $-\eta_< < u < 0$.

We turn to the action of T on $C_i \oplus C_u$. The complex-valued functions $e^{\lambda_0 \cdot} : [-2,0] \to \mathbb{C}$ and $e^{\lambda \cdot} : [-2,0] \to \mathbb{C}$ are eigenvectors associated with the eigenvalues $\lambda_0 = u_0 + iv_0$ and $\lambda = u + iv$ of the generator of T. The functions $c_u : [-2,0] \to \mathbb{R}$ and $s_u : [-2,0] \to \mathbb{R}$ given by

$$c_u(t) = e^{u_0 t}\cos(v_0 t) = \text{ Re}\left(e^{\lambda_0 t}\right), \; s_u(t) = e^{u_0 t}\sin(v_0 t) = \text{ Im}\left(e^{\lambda_0 t}\right),$$

form a basis of C_u, and the functions $c_i : [-2,0] \to \mathbb{R}$ and $s_i : [-2,0] \to \mathbb{R}\}$ given by

$$c_i(t) = e^{ut}\cos(vt) = \text{ Re}\left(e^{\lambda t}\right), \; s_i(t) = e^{ut}\sin(vt) = \text{ Im}\left(e^{\lambda t}\right)$$

form a basis of C_i. For reals a, b and $t \geq 0$ and $z = a + ib \in \mathbb{C}$, $z = |z|e^{i\phi}$ with $\phi \in \mathbb{R}$, we use the extension of the semigroup to complex-valued data $[-2,0] \to \mathbb{C}$ and obtain

$$T(t)z \cdot e^{\lambda \cdot} = ze^{\lambda t}e^{\lambda \cdot},$$

hence

$$\begin{aligned} T(t)(a \cdot c_i - b \cdot s_i) &= T(t) \text{ Re}\left(z \cdot e^{\lambda \cdot}\right) \\ &= \text{ Re}\left(T(t)z \cdot e^{\lambda \cdot}\right) \\ &= \text{ Re}\left(ze^{\lambda t}e^{\lambda \cdot}\right) \\ &= \text{ Re}\left(|z|e^{i\phi + ut + ivt}(c_i + is_i)\right) \\ &= |z|e^{ut}\left(\cos(\phi + vt) \cdot c_i - \sin(\phi + vt) \cdot s_i\right). \end{aligned} \tag{10.3}$$

Analogously,

$$T(t)(a \cdot c_u - b \cdot s_u) = |z|e^{u_0 t}(\cos(\phi + v_0 t) \cdot c_u - \sin(\phi + v_0 t) \cdot s_u). \tag{10.4}$$

It will be convenient to introduce the isomorphism

$$\begin{aligned} K : Y_< \times \mathbb{C} \times \mathbb{C} \to Y, K(y_<, z, z_0) = y_< + &\text{ Re}(z) \cdot c_i - \text{ Im}(z) \cdot s_i \\ &+ \text{Re}(z_0) \cdot c_u - \text{ Im}(z_0) \cdot s_u, \end{aligned}$$

with $\mathbb{C}$ considered as a vector space over $\mathbb{R}$. A first consequence is the formula

$$K^{-1}T(t)K(y_<, z, z_0) = T(t)y_< + |z|e^{ut}e^{i(\phi + vt)} + |z_0|e^{u_0 t}e^{i(\psi + v_0 t)} \tag{10.5}$$

for $y_< \in Y_<$, $z = |z|e^{i\phi} \in \mathbb{C}$, and $|z_0|e^{i\psi} \in \mathbb{C}$, with reals ϕ, ψ.

Now choose $\epsilon_0 > 0$ so that

$$W_0 := K\left(Y_{<,\epsilon_0} \times \mathbb{C}_{\epsilon_0} \times \mathbb{C}_{\epsilon_0}\right)$$

is contained in W. Then choose positive reals $r < R_i < R$ with

$$R_i < R\, e^{-u_0\theta}, \quad R < \min\left\{\epsilon_0, e^{-u_0\theta}\right\} \;\; (< 1), \quad c_< r < \epsilon_0, \quad r < A\, e^{u\theta}$$

and such that for all $y_< \in Y_<$ and for all positive reals $q \le R_i$ we have

$$\left| T\left(\frac{1}{u_0}\log\left(\frac{R}{q}\right)\right) y_< \right|_1 \le |y_<|_1. \tag{10.6}$$

Consider the hypersurfaces

$$H_i := \{K(y_<, z, z_0) : |y_<|_1 \le r, |z| = r, |z_0| \le R_i\},$$
$$H_u := \{K(y_<, z, z_0) : |y_<|_1 \le r, |z| \le r, |z_0| = R\}$$

in $Y \cap W = X \cap W$.

The *central circles* in these solid tori are the sets

$$S_i := \{K(0, z, 0) : |z| = r\}$$
and
$$S_u := \{K(0, 0, z_0) : |z_0| = R\},$$

respectively (Fig. 5).

For every $t \le 0$ the homoclinic solution h satisfies $h_t \in C_u$, and for all $a \in [-2, 0]$,

$$h_t(a) = e^{u_0(t+a)} \sin\big(v_0(t+a)\big) = e^{u_0 t}\big(\sin(v_0 t)c_0(a) + \cos(v_0 t)s_0(a)\big),$$

hence

$$h_t = K(0, 0, e^{u_0 t}\big(\sin(v_0 t) - i\ \cos(v_0 t)\big),$$

and thereby,

$$|K^{-1}h_t| = e^{u_0 t}$$

for all $t \le 0$. Analogously we have for all $t \ge m_2 + 2$ that $h_t \in C_i$ and

$$|K^{-1}h_t| = a_h\, e^{ut}.$$

The choice of $R < e^{-u_0\theta}$ and $r < A\, e^{u\theta}$ above implies that there exist $t_u \le -\theta$ and $t_i \ge \theta$ with $h_{t_u} \in H_u$ and $h_{t_i} \in H_i$.

Using (10.5) we see that a phase curve $[0, \infty) \ni t \mapsto T(t)\chi \in C^1$ of the semigroup T which starts from $\chi = K(y_<, z, z_0) \in H_i \setminus Y_s$, that is, with $0 < |z_0| \le R_i < R$, reaches the level set

$$\left\{\tilde{\chi} \in Y : |K^{-1}P_u\tilde{\chi}| = R\right\}$$

at

$$t = \frac{1}{u_0}\log\left(\frac{R}{|z_0|}\right).$$

Let $\sigma_0 : H_i \setminus Y_s \to (0, \infty)$ be the *stopping time map* given by

$$\sigma_0(\chi) = \frac{1}{u_0}\log\left(\frac{R}{|z_0|}\right)$$

for $\chi = K(y_<, z, z_0) \in H_i \setminus Y_s$. It will be convenient to introduce also the map

$$\tau : (0, \infty) \to \mathbb{R},\ \ \tau(q) = \frac{1}{u_0}\log\left(\frac{R}{q}\right),$$

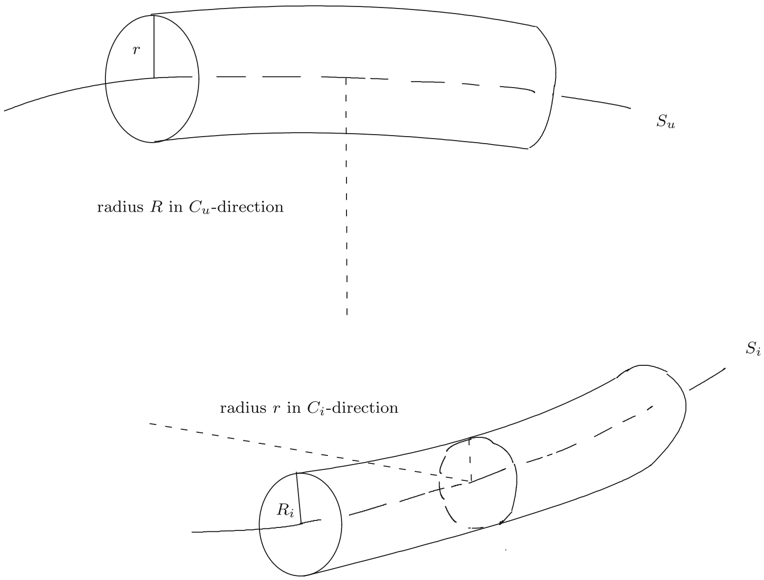


Fig. 5 The sets H_i and H_u with central circles S_i and S_u. The $Y_<$-components are omitted

which permits us to write

$$\sigma_0(\chi) = \tau(|z_0|)$$

for $\chi = K(y_<, z, z_0) \in H_i \setminus Y_s$.

The estimate (10.2), the choice $c_< r < \epsilon_0$, and the representations (10.3) and (10.4) of the semigroup on C_i and on C_u combined show that all $T(t)\chi$ with $0 \le t \le \sigma_0(\chi)$, $\chi \in H_i \setminus Y_s$, belong to a bounded set $W_b \subset W$, hence $T(t)\chi = F(t, \chi)$ for these t and χ. Using this fact and (10.5) we see that the inner map

$$\Sigma_0 : H_i \setminus Y_s \ni \chi \mapsto F(\sigma_0(\chi), \chi) \in X$$

is given as follows (Fig. 6).

For $\chi = K(y_<, z, z_0)$, $|y_<|_1 \le r$, $z = re^{i\phi}$, $z_0 = |z_0|e^{i\psi}$ with $0 < |z_0| \le R_i < R$ and reals ϕ, ψ, we have $\Sigma_0(\chi) = K(\hat{y}_<, \hat{z}, \hat{z_0})$ with

$$\hat{y}_< = T\Big(\tau(|z_0|)\Big)y_< \in Y_<, \tag{10.7}$$

$$\hat{z} = r\left(\frac{R}{|z_0|}\right)^{\frac{u}{u_0}} e^{i\left(\phi + v\tau(|z_0|)\right)} \in \mathbb{C}, \tag{10.8}$$

$$\hat{z_0} = R\, e^{i\left(\psi + v_0\tau(|z_0|)\right)} \in \mathbb{C}. \tag{10.9}$$

Using (10.6)–(10.9) we infer

$$\Sigma_0(H_i \setminus Y_s) \subset H_u \setminus C_u.$$

Proposition 10.1 *$\Sigma_0(H_i \setminus Y_s)$ has compact closure in Y.*

Proof The inequality

$$|z_0| \le R_i \le R\, e^{-u_0\theta} \le R\, e^{-2u_0}$$

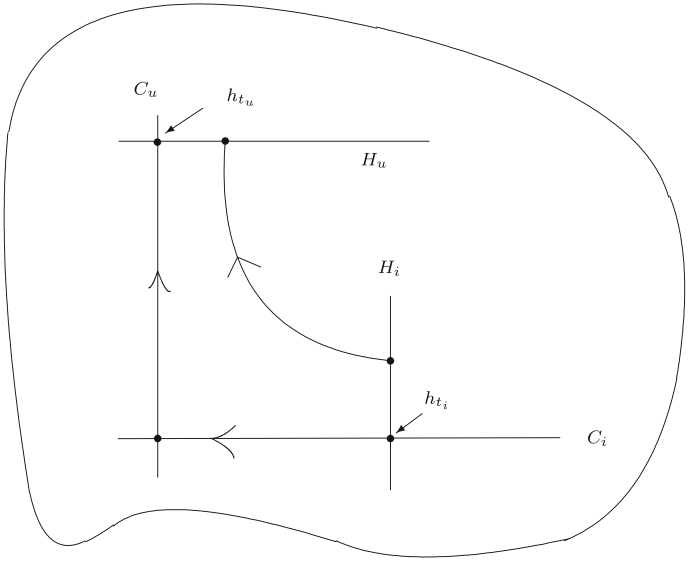


Fig. 6 The inner map, with $Y_<$-components and one dimension in each of C_i and C_u omitted

yields $\sigma_0(\chi) \geq 2$ on $H_i \setminus Y_s$. The fact that the set of all $T(t)\chi, 0 \leq t \leq \sigma_0(\chi)$ and $\chi \in H_i \setminus Y_s$, is bounded means that the solutions $y^\chi : [-2, \infty) \to \mathbb{R}$ of the initial value problem

$$y'(t) = -\alpha\, y(t-1), \quad y_0 = \chi \in H_i \setminus Y_s,$$

and their derivatives are uniformly bounded on $[-2, \sigma_0(\chi)]$. It follows that there is a constant $L \geq 0$ such that

$$\text{Lip}\big(y^\chi|[-2, \sigma_0(\chi)]\big) \leq L \quad \text{for all} \quad \chi \in H_i \setminus Y_s.$$

Using the preceding equation we infer that $\text{Lip}((y^\chi)'|[0, \sigma_0(\chi)]) \leq \alpha\, L$ for all $\chi \in H_i \setminus Y_s$. As $2 \leq \sigma_0(\chi)$ this yields $\text{Lip}((y^\chi_{\sigma_0(\chi)})') \leq \alpha L$ for all $\chi \in H_i \setminus Y_s$. Altogether,

$$\sup_{\chi \in H_i \setminus Y_s} |\Sigma_0(\chi)|_1 + \sup_{\chi \in H_i \setminus Y_s} \text{Lip}\big(\Sigma_0(\chi)\big) + \sup_{\chi \in H_i \setminus Y_s} \text{Lip}\big((\Sigma_0(\chi))'\big) \quad < \quad \infty.$$

Now a twofold application of the Arzelà–Ascoli theorem leads to the assertion.

11 The Outer Map

In this section we define an outer map following phase curves from a neighbourhood of h_{t_u} in H_u to their intersection with H_i. The first step towards the outer map prepares the existence of a suitable stopping time map.

For every tangent vector $z \in T_{h_{t_i}} H_i$ there is a differentiable curve ζ in $H_i \subset Y_< + S_i^1 + C_u$ with $\zeta(0) = h_{t_i}$ and $z = \zeta'(0)$. The function $c_r \circ \zeta$, with $c_r : Y \ni \chi \mapsto |K_i^{-1} P_i \chi| \in \mathbb{R}$, is constant. This implies

$$Dc_r(h_{t_i})z = D(c_r \circ \zeta)(0) = 0.$$

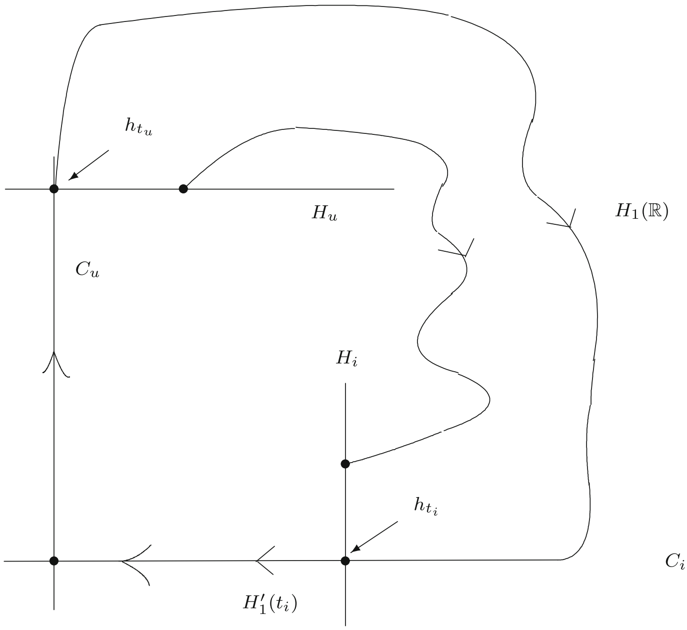


Fig. 7 The outer map, with $Y_<$-components and one dimension in each of C_i and C_u omitted

For the phase curve $H_1 : \mathbb{R} \ni t \mapsto h_t \in C^1$ with range in X and for $t \geq m_2 + 2$ we obtain $c_r(H_1(t)) = |K^{-1} P_i(H_1(t))| = A\, e^{ut}$, hence

$$Dc_r(h_{t_i})H_1'(t_i) \neq 0,$$

which yields

$$h_{t_i}' = H_1'(t_i) \notin T_{H_1(t_i)} H_i. \tag{11.1}$$

See [18] for the equation. The transversality condition (11.1), the fact that the semiflow F is continuously differentiable on the part of its domain given by $t > 2$, and the inequality $t_i - t_u > 2$ combined yield a continuously differentiable stopping time map

$$\sigma_1 : V_{\sigma_1} \to (2, \infty)$$

on an open neighbourhood $V_{\sigma_1} \subset W_0$ of h_{t_u} in Y, with

$$\sigma_1(h_{t_u}) = t_i - t_u \quad \text{and} \quad |K^{-1} P_i F(\sigma_1(\chi), \chi)| = r \quad \text{for all} \quad \chi \in V_{\sigma_1}.$$

As $h_{t_i} = F(\sigma_1(h_{t_u}), h_{t_u})$ is in C_i the components of h_{t_i} in $Y_<$ and in C_u vanish. It follows that there is an open neighbourhood $V \subset V_{\sigma_1}$ of h_{t_u} in Y so that each $F(\sigma_1(\chi), \chi) \in H_i$, $\chi \in V$, belongs to the C^1-submanifold

$$\overset{\circ}{H_i} := \{K(y_<, z, z_0) : |y_<|_1 < r, |z| = r, |z_0| < R_i\} \subset H_i$$

of the space Y, and we obtain the continuously differentiable outer map

$$\Sigma_1 : V \ni \chi \mapsto F(\sigma_1(\chi), \chi) \in \overset{\circ}{H_i}$$

with

$$\Sigma_1(h_{t_u}) = h_{t_i} \text{ (see Fig.7).}$$

Recall that for any $\chi \in Y$,

$$D\Sigma_1(h_{t_u})\chi = P_h D_2 F\left(t_i - t_u, h_{t_u}\right)\chi$$

with the projection $P_h : Y \to Y$ along $\mathbb{R}h'_{t_i}$ onto $T_{h_{t_i}} H_i$, because of the relations

$$\begin{aligned} T_{h_{t_i}} H_i \ni D\Sigma_1(h_{t_u})\chi &= D_1 F\left(t_i - t_u, h_{t_u}\right) D\sigma_1(h_{t_u})\chi + D_2 F\left(t_i - t_u, h_{t_u}\right)\chi \\ &= D\sigma_1(h_{t_u})\chi \cdot h'_{t_i} + P_h D_2 F\left(t_i - t_u, h_{t_u}\right)\chi \\ &\quad + (id_Y - P_h)\, D_2 F\left(t_i - t_u, h_{t_u}\right)\chi. \end{aligned}$$

We have

$$T_{h_{t_u}} H_u = Y_< + C_i + \mathbb{R}\tau_u$$

with $\tau_u = \omega'(0) \neq 0$ for the curve

$$\omega : \mathbb{R} \to S_u^1 \subset H_u \cap C_u, \;\; \omega(\psi) = K\left(0, 0, R\, e^{i(\psi+\psi_u)}\right)$$

where $\psi_u \in [-\pi, \pi)$ and

$$h_{t_u} = K\left(0, 0, R\, e^{i\psi_u}\right).$$

Similarly,

$$T_{h_{t_i}} H_i = Y_< + \mathbb{R}\tau_i + C_u$$

with $\tau_i = \rho'(0) \neq 0$ for the curve

$$\rho : \mathbb{R} \to S_i^1 \subset H_i \cap C_i, \;\; \rho(\phi) = K\left(0, r\, e^{i(\phi+\phi_i)}, 0\right)$$

where $\phi_i \in [-\pi, \pi)$ and

$$h_{t_i} = K\left(0, r\, e^{i\phi_i}, 0\right).$$

Because of (11.1) the vectors $\tau_i \in C_i$ and $h'_{t_i} \in C_i$ are linearly independent, and because of the relation

$$h'_{t_u} \notin T_{h_{t_u}} H_u$$

analogous to (11.1) the vectors $\tau_u \in C_u$ and $h'_{t_u} \in C_u$ are linearly independent. For all $y_< \in Y_<$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\chi_u \in C_u$ we have

$$P_h\left(y_< + a\tau_i + bh'_{t_i} + \chi_u\right) = y_< + a\tau_i + \chi_u. \tag{11.2}$$

It is convenient to recall here that

$$D_2 F\left(t_i - t_u, h_{t_u}\right)(C_i \oplus C_u) = C_i \oplus C_u. \tag{11.3}$$

Proposition 11.1

$$D\Sigma_1\left(h_{t_u}\right)(C_i \oplus \mathbb{R}\tau_u) = \mathbb{R}\tau_i \oplus C_u.$$

Proof Using (11.3) and (11.2) we infer

$$D\Sigma_1(h_{t_u})(C_i \oplus \mathbb{R}\tau_u) \subset P_h(C_i \oplus C_u) = \mathbb{R}\tau_i \oplus C_u.$$

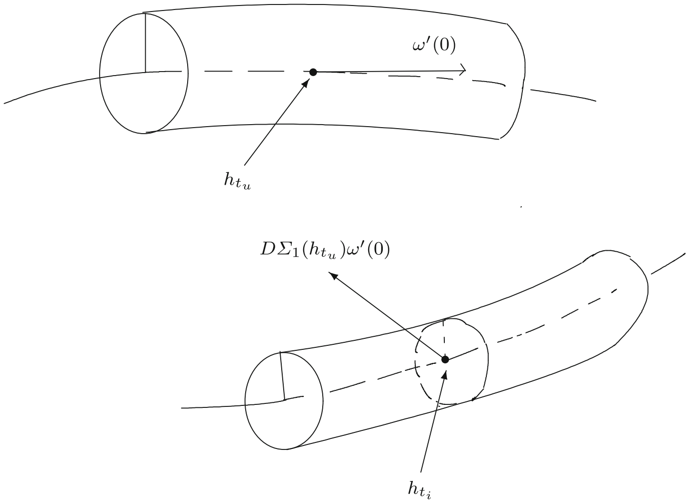


Fig. 8 The transversality condition

It remains to show that the restriction of $D\Sigma_1(h_{t_u})$ to $C_i \oplus \mathbb{R}\tau_u$ is injective. So let $\chi \in C_i \oplus \mathbb{R}\tau_u$ with $0 = D\Sigma_1(h_{t_u})\chi = P_h D_2 F(t_i - t_u, h_{t_u})\chi$ be given. Then $D_2 F(t_i - t_u, h_{t_u})\chi \in \mathbb{R}h'_{t_i}$. Using $D_2 F(t_i - t_u, h_{t_u})h'_{t_u} = h'_{t_i}$ (see Theorem 9.2), $h'_{t_u} \in C_u$ and (11.3) we obtain

$$\chi \in \mathbb{R}h'_{t_u},$$

and it follows that $\chi \in \mathbb{R}h'_{t_u} \cap (C_i \oplus \mathbb{R}\tau_u) = \{0\}$.

We proceed to a transversality condition for the outer map.

Proposition 11.2

$$P_u D\Sigma_1(h_{t_u})\tau_u \neq 0 \text{ (see Fig. 8).}$$

Proof 1. From (11.3) we get $D_2 F(t_i - t_u, h_{t_u})\tau_u \in C_i \oplus C_u$. Suppose $D_2 F(t_i - t_u, h_{t_u})\tau_u \in C_i$. As τ_u and h'_{t_u} form a basis of C_u and

$$D_2 F\left(t_i - t_u, h_{t_u}\right) h'_{t_u} = h'_{t_i} \in C_i$$

we obtain $D_2 F(t_i - t_u, h_{t_u})C_u \subset C_i \subset Y_s$ which in view of (11.3) yields

$$\dim\left(D_2 F(t_i - t_u, h_{t_u})C_u\right) \cap Y_s = 2,$$

in contradiction to the minimal intersection property (3.6) with $t_u \leq -\theta$, $t_i \geq \theta$.

2. We just showed $D_2 F(t_i - t_u, h_{t_u})\tau_u \in (C_i \oplus C_u) \setminus C_i$. The decompositions

$$Y = Y_< \oplus \mathbb{R}\tau_i \oplus \mathbb{R}h'_{t_i} \oplus C_u$$

and

$$T_{h_{t_i}} H_i = Y_< \oplus \mathbb{R}\tau_i \oplus C_u$$

in combination with

$$D_2 F(t_i - t_u, h_{t_u})\tau_u = a\tau_i + bh'_{t_i} + \chi_u$$

for some a, b in $\mathbb{R}$ and $0 \neq \chi_u \in C_u$, the latter because of part 1, yield

$$\begin{aligned} P_u D\Sigma_1(h_{t_u})\tau_u &= P_u P_h D_2 F(t_i - t_u, h_{t_u})\tau_u \\ &= P_u P_h(a\tau_i + bh'_{t_i} + \chi_u) = P_u(a\tau_i + \chi_u) \quad \text{(see 11.2)} \\ &= \chi_u \neq 0. \end{aligned}$$

For later use we translate the previous results into statements about global coordinates on H_u and H_i, respectively. Consider the injective maps

$$\mathbf{C}_u : \overline{Y_{<,r}} \times \overline{\mathbb{C}_r} \times [-\pi, \pi) \to Y,\ \mathbf{C}_u(y_<, z, \psi) = K\left(y_<, z, R\, e^{i(\psi+\psi_u)}\right)$$

and

$$\mathbf{C}_i : \overline{Y_{<,r}} \times [-\pi, \pi) \times \overline{\mathbb{C}_r} \to Y,\ \mathbf{C}_i(y_<, \phi, z_0) = K\left(y_<, r\, e^{i(\phi+\phi_i)}, z_0\right).$$

We have

$$\mathbf{C}_u\left(\overline{Y_{<,r}} \times \overline{\mathbb{C}_r} \times [-\pi, \pi)\right) = H_u \quad \text{and} \quad \mathbf{C}_i\left(\overline{Y_{<,r}} \times [-\pi, \pi) \times \overline{\mathbb{C}_r}\right) = H_i.$$

The map $\mathbf{C}_u$ defines a C^1-diffeomorphism from $Y_{<,r} \times \mathbb{C}_r \times (-\pi, \pi)$ into the C^1-submanifold

$$\overset{\circ}{H}_u := \{K(y_<, z, z_0) : |y_<|_1 < r, |z| < r, |z_0| = R\} \subset H_u$$

of the space Y, with

$$\mathbf{C}_u\left(\{0_<\} \times \mathbb{C}_r \times (-\pi, \pi)\right) \subset \{0_<\} + C_i + S_u,$$

and the map $\mathbf{C}_i$ defines a C^1-diffeomorphism from $Y_{<,r} \times (-\pi, \pi) \times \mathbb{C}_r$ into the C^1-submanifold $\overset{\circ}{H}_i \subset H_i$ of the space Y, with

$$\mathbf{C}_i(\{0_<\} \times (-\pi, \pi) \times \mathbb{C}_r) \subset \{0_<\} + S_i + C_u.$$

Let us distinguish the null elements of the spaces $Y_<, \mathbb{C}, \mathbb{R}$ by writing $0_<, 0_{\mathbb{C}}, 0_{\mathbb{R}}$, respectively, and define

$$0_u := \left(0_<, 0_{\mathbb{C}}, 0_{\mathbb{R}}\right) \in \overline{Y_{<,r}} \times \overline{\mathbb{C}_r} \times [-\pi, \pi),\ 0_i := \left(0_<, 0_{\mathbb{R}}, 0_{\mathbb{C}}\right) \in \overline{Y_{<,r}} \times [-\pi, \pi) \times \overline{\mathbb{C}_r}.$$

Then

$$\mathbf{C}_u\left(0_u\right) = h_{t_u}, \tag{11.4}$$

$$\begin{aligned} D\mathbf{C}_u(0_u)\left(\{0_<\} \times \mathbb{C} \times \mathbb{R}\right) &= \{0_<\} + C_i + T_{h_{t_u}} S_u \\ &= \{0_<\} + C_i + \mathbb{R}\tau_u, \end{aligned} \tag{11.5}$$

$$D\mathbf{C}_u(0_u)\left(0_<, 0_{\mathbb{C}}, 1\right) = \tau_u, \tag{11.6}$$

$$\mathbf{C}_i(0_i) = h_{t_i}, \tag{11.7}$$

$$\begin{aligned} D\mathbf{C}_i(0_i)\left(\{0_<\} \times \mathbb{R} \times \mathbb{C}\right) &= \{0_<\} + T_{h_{t_i}} S_i + C_u \\ &= \{0_<\} + \mathbb{R}\tau_i + C_u, \end{aligned} \tag{11.8}$$

$$D\mathbf{C}_i(0_i)\left(0_<, 1, 0_{\mathbb{C}}\right) = \tau_i. \tag{11.9}$$

Now consider the *outer map Σ_1 in terms of coordinates*, namely, the map

$$P_1 : (\mathbf{C}_u)^{-1}(V) \to Y_< \times \mathbb{R} \times \mathbb{C}$$

given by

$$P_1(\eta, z, \psi) = (\mathbf{C}_i)^{-1}\left(\Sigma_1\left(\mathbf{C}_u(\eta, z, \psi)\right)\right).$$

The map P_1 is defined on a neighbourhood of the origin in $Y_< \times \mathbb{C} \times \mathbb{R}$, has its range in $\overline{Y_{<,r}} \times [-\pi, \pi) \times \overline{\mathbb{C}_r}$, satisfies

$$P_1(0_u) = 0_i,$$

and is continuously differentiable on

$$(\mathbf{C}_u)^{-1}(V) \cap (Y_{<,r} \times \mathbb{C}_r \times (-\pi, \pi)).$$

Proposition 11.1 in combination with (11.4)–(11.19) yields

$$D P_1(0_u)(\{0_<\} \times \mathbb{C} \times \mathbb{R}) = \{0_<\} \times \mathbb{R} \times \mathbb{C}.$$

It follows that

(T1) the induced map $D_1 : \{0_<\} \times \mathbb{C} \times \mathbb{R} \to \{0_<\} \times \mathbb{R} \times \mathbb{C}$ is an isomorphism

(of three-dimensional vector spaces over $\mathbb{R}$). Observe that the inverse of the derivative of the C^1-diffeomorphism

$$Y_{<,r} \times (-\pi, \pi) \times \mathbb{C}_r \stackrel{\mathbf{C}_i}{\to} \mathbf{C}_i(Y_{<,r} \times (-\pi, \pi) \times \mathbb{C}_r) \subset \overset{\circ}{H_i}$$

at h_{t_i} is the linear map $[D\mathbf{C}_i(0_i)]^{-1}$. Using this we infer from Proposition 11.2 that the vector

$$\begin{aligned}\xi &:= D P_1\big(0_u\big)\big(0_<, 0_{\mathbb{C}}, 1\big) = D_1\big(0_<, 0_{\mathbb{C}}, 1\big)\\ &= \big[D\mathbf{C}_i(0_i)\big]^{-1} D\Sigma_1(h_{t_u}) D\mathbf{C}_u(0_u)\big(0_<, 0_{\mathbb{C}}, 1\big)\\ &= \big[D\mathbf{C}_i(0_i)\big]^{-1} D\Sigma_1(h_{t_u})\tau_u\end{aligned}$$

and the projection

$$\mathrm{pr}_2 : \{0_<\} \times \mathbb{R} \times \mathbb{C} \to \{0_<\} \times \mathbb{R} \times \mathbb{C},\ \ \mathrm{pr}_2(0_<, \phi, z_0) = \big(0_<, 0_{\mathbb{R}}, z_0\big),$$

satisfy

(T2) $\mathrm{pr}_2 \xi \neq (0_<, 0_{\mathbb{R}}, 0_{\mathbb{C}})$.

Clearly the nullspace of pr_2 is

$$\{0_<\} \times \mathbb{R} \times \{0_{\mathbb{C}}\} = \mathbb{R}\, e_\phi \ \text{ with } e_\phi := \big(0_<, 1, 0_{\mathbb{C}}\big).$$

We end this section with further technical preparations concerning the isomorphism D_1. As a consequence of (T1), the vector $\xi = D_1(0_<, 0_{\mathbb{C}}, 1) \in D_1(\{0_<\} \times \{0_{\mathbb{C}}\} \times \mathbb{R})$ does not belong to the two-dimensional space (Fig. 9)

$$U_1 := D_1\big(\{0_<\} \times \mathbb{C} \times \{0_{\mathbb{R}}\}\big).$$

Therefore the range of D_1 satisfies

$$D_1(\{0_<\} \times \mathbb{C} \times \mathbb{R}) = \{0_<\} \times \mathbb{R} \times \mathbb{C} = U_1 \oplus \mathbb{R}\xi. \tag{11.10}$$

Notice that (T2) yields

$$\xi \notin \mathbb{R}\, e_\phi. \tag{11.11}$$

From (11.10) and (11.11) we see that there are uniquely determined $\mu \in \mathbb{R}$ and $f_1 \in U_1 \setminus \{(0_i\}$ such that

$$e_\phi = f_1 + \mu\, \xi. \tag{11.12}$$

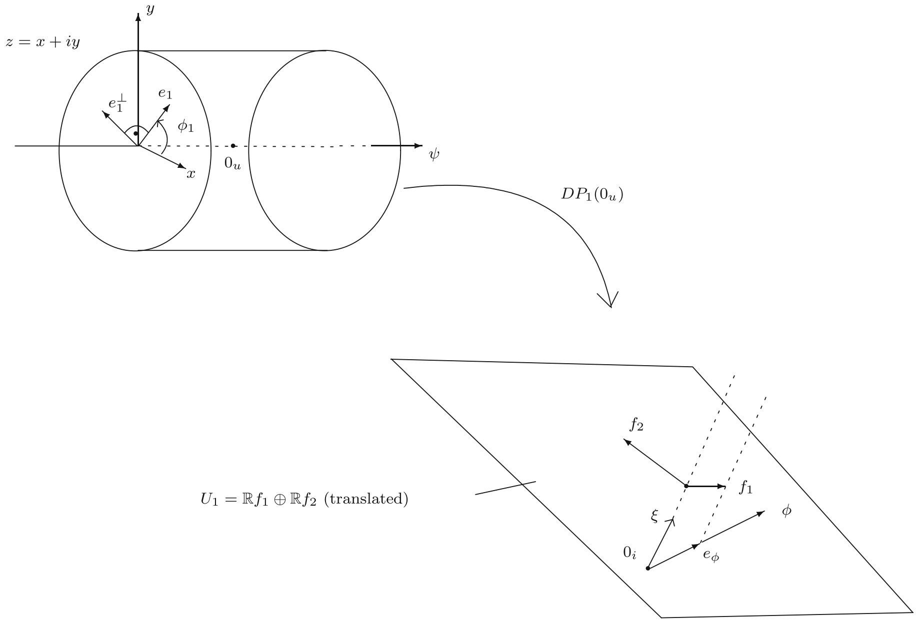


Fig. 9 The vectors f_1, f_2, ξ, e_1, and $e_1^\perp$. (The direction of f_1 is geometrically obtained by intersecting the plane spanned by ξ and e_ϕ with the space U_1.)

Set $e_1 := D_1^{-1} f_1 \in \{0_<\} \times \mathbb{C} \times \{0_\mathbb{R}\}$. Then $e_1 = (0_<, p_1 e^{i\phi_1}, 0_\mathbb{R})$ with $p_1 > 0$ and $0 \leq \phi_1 < 2\pi$ uniquely determined. Define

$$e_1^\perp := (0_<, p_1 e^{i(\phi_1 + \frac{\pi}{2})}, 0_\mathbb{R}).$$

Then

$$\{0_<\} \times \mathbb{C} \times \{0_\mathbb{R}\} = \mathbb{R}\, e_1 \oplus \mathbb{R}\, e_1^\perp.$$

Setting $f_2 := D_1 e_1^\perp$ we arrive at

$$U_1 = D_1\big(\mathbb{R}\, e_1 \oplus \mathbb{R}\, e_1^\perp\big) = \mathbb{R}\, f_1 \oplus \mathbb{R}\, f_2, \tag{11.13}$$

which in combination with (11.10) yields

$$\{0_<\} \times \mathbb{R} \times \mathbb{C} = \mathbb{R}\, f_1 \oplus \mathbb{R}\, f_2 \oplus \mathbb{R}\, \xi \tag{11.14}$$

for the range of D_1.

Next we consider the plane $H := \mathbb{R}\, f_2 \oplus \mathbb{R}\, \xi \subset \{0_<\} \times \mathbb{R} \times \mathbb{C}$. Using (11.12) and (11.14) we see that the vector e_ϕ spanning the nullspace of pr_2 does not belong to H. Consequently the restriction $\text{pr}_2\big|_H$ defines an isomorphism onto the space $\{0_<\} \times \{0_\mathbb{R}\} \times \mathbb{C}$. Therefore the vectors $\text{pr}_2\xi$ and $\text{pr}_2 f_2$ form a basis of the space $\{0_<\} \times \{0_\mathbb{R}\} \times \mathbb{C}$, which in turn guarantees a constant $\gamma_2 > 0$ such that for all reals a, b we have

$$\big|\text{pr}_2(a\, f_2 + b\, \xi)\big| \geq \gamma_2\big(|a| + |b|\big). \tag{11.15}$$

In Sect. 13 we will approximate the map P by a map with values in the space $H \oplus \mathbb{R} \cdot e_\phi$, and then consider a simplifying homotopy which eliminates the components in e_ϕ-direction, and replaces the values in H by their projection to $\{0_<\} \times \{0_{\mathbb{R}}\} \times \mathbb{C}$ (there property (11.15) is important). The geometric idea of finding disjoint subsets N_0, N_1 in the domain of P, to which the methods from Sect. 2 can be applied, is to define subsets which (ignoring the $Y_<$-part) get mapped to 'different sides' of the plane H. This means that the components of $P(x)$ in e_ϕ-direction will be positive for $x \in N_0$ and negative for $x \in N_1$. In order to control these values, we need to control the values of $P_0(x)$ in the direction of e_1 and $e_1^\perp$, and we prepare this now.

Let $< \cdot, \cdot >: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ denote the euclidean scalar product, i.e.,

$$\langle a + bi, c + di \rangle = ac + bd \quad \text{for all } a, b, c, d, \in \mathbb{R}.$$

Obviously, $e^{i\phi_1}$ and $e^{i(\phi_1 + \frac{\pi}{2})}$ are orthogonal unit vectors with respect to $< \cdot, \cdot >$. From the definitions of e_1 and $e_1^\perp$ we obtain for every $z \in \mathbb{C}$

$$\left(0_<, z, 0_{\mathbb{R}}\right) = L(z)e_1 + L^\perp(z)e_1^\perp, \tag{11.16}$$

with the $\mathbb{R}-$linear functionals $L : \mathbb{C} \to \mathbb{R}$ and $L^\perp : \mathbb{C} \to \mathbb{R}$ given by

$$L(z) = \frac{1}{p_1} < z, e^{i\phi_1} >, \quad L^\perp(z) = \frac{1}{p_1} < z, e^{i(\phi_1 + \pi/2)} > . \tag{11.17}$$

For $0_{\mathbb{C}} \neq z = |z| \cdot e^{i\phi}$ we get

$$L(z) = \frac{|z|}{p_1} \cos(\phi - \phi_1), \quad L^\perp(z) = \frac{|z|}{p_1} \sin(\phi - \phi_1). \tag{11.18}$$

In view of (11.8), we can find $d_1 \in (0, \pi/2)$ (close to $\pi/2$) and $\varepsilon_1 > 0$ such that

$$0 < d_1 - \varepsilon_1 < d_1 + \varepsilon_1 < \pi/2, \tag{11.19}$$

and such that if $0_{\mathbb{C}} \neq z = |z|e^{i\phi}$ then with μ from (11.12) one has the implication

$$|\phi - \phi_1| \in [d_1 - \varepsilon_1, d_1 + \varepsilon_1] + \mathbb{Z}\pi \implies |L^\perp(z)| \geq 2|\mu||L(z)|, \quad |L^\perp(z)| \geq \frac{1}{2}\frac{|z|}{p_1}. \tag{11.20}$$

12 Composition

This section begins with neighbourhoods of the point h_{t_u} in the domain V of the outer map which are given by small components in $Y_<$ and in C_i and small arcs on $S_u \ni h_{t_u}$. We find preimages of these neighbourhoods under the inner map on which the composition of the inner and outer maps is defined.

Recall that V is a neighbourhood of $h_{t_u} = K(0, 0, R\, e^{i\psi_u})$ in Y. There exist $\gamma_V \in (0, \pi)$ and $r_V \in (0, r]$ with

$$R \left(\frac{r_V}{r}\right)^{-\frac{u_0}{u}} \leq R_i \tag{12.1}$$

such that for every $\gamma \in (0, \gamma_V]$, $\tilde{r} \in (0, r_V]$, and $\hat{r} \in (0, r_V]$ the closed set

$$V(\gamma, \tilde{r}, \hat{r}) := \{K(y_<, z, z_0) \in Y : |y_<|_1 \leq \hat{r}, \ |z| \leq \tilde{r}, \ z_0 = R\, e^{i\psi} \text{ with } \psi_u - \gamma \leq \psi \leq \psi_u + \gamma\}$$

is a subset of V which contains h_{t_u} (Fig. 10).

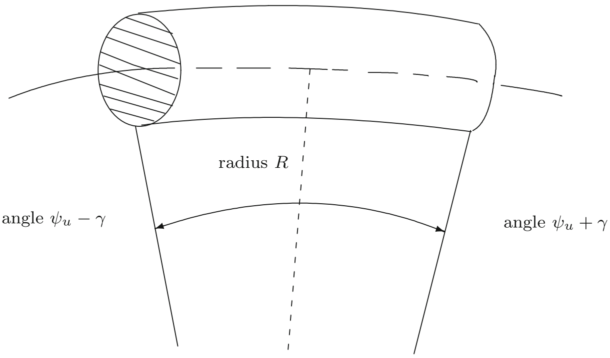
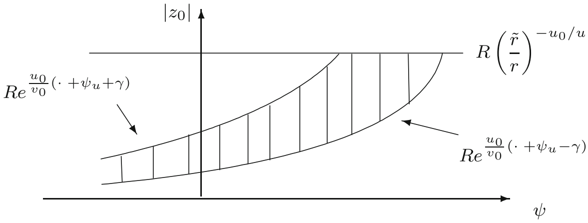


Fig. 10 The set $V(\gamma, \tilde{r}, \hat{r})$, with $Y_<$-components omitted



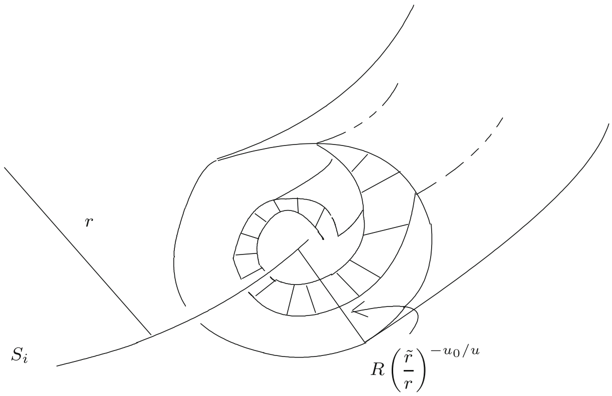


Fig. 11 The inequalities (12.2) and (12.3), and the set $H_i(\gamma, \tilde{r}, \hat{r})$, with $Y_<$-components omitted

For the same $\gamma, \tilde{r}, \hat{r}$ define (Fig. 11)

$$H_i(\gamma, \tilde{r}, \hat{r}) := \Big\{\chi = K(y_<, z, z_0) \in Y : |y_<|_1 \le r,\ \ |z| = r,$$
$$z_0 = |z_0|\, e^{i\psi} \text{ with } \psi \in \mathbb{R} \text{ satisfies}$$
$$0 < |z_0| \le R\left(\frac{\tilde{r}}{r}\right)^{-\frac{u_0}{u}} \tag{12.2}$$

and

$$R\, e^{\frac{u_0}{v_0}(\psi - \psi_u - \gamma)} \le |z_0| \le R\, e^{\frac{u_0}{v_0}(\psi - \psi_u + \gamma)},$$
$$\text{and}$$
$$|P_< \Sigma_0(\chi)|_1 \le \hat{r}\Big\}. \tag{12.3}$$

Then

$$H_i(\gamma,\tilde r,\hat r)\subset H_i\setminus Y_s$$

(and $h_{t_i}\notin H_i(\gamma,\tilde r,\hat r)$).

Proposition 12.1 *For every $\gamma\in(0,\gamma_V]$, $\tilde r\in(0,r_V]$, and $\hat r\in(0,r_V]$ we have*

$$\Sigma_0\big(H_i(\gamma,\tilde r,\hat r)\big)\subset V\big(\gamma,\tilde r,\hat r\big).$$

Proof Let $\chi=K(y_<,z,z_0)\in H_i(\gamma,\tilde r,\hat r)\subset H_i\setminus Y_s$ be given, with $|y_<|_1\le r$, $|z|=r$, $z_0=|z_0|e^{i\psi}$ with $\psi\in\mathbb{R}$ satisfying (12.2) and (12.3), and

$$|P_<\Sigma_0(\chi)|_1\le\hat r.$$

Using (10.7)–(10.9) we obtain $\Sigma_0(\chi)=K(\hat y_<,\hat z,\hat{z_0})$ with

$$\begin{aligned}|\hat y_<|_1&=|T(\tau(|z_0|))y_<|_1=|T(\sigma_0(\chi))y_<|_1=|T(\sigma_0(\chi))P_<\chi|_1\\&=|P_<T(\sigma_0(\chi))\chi|_1=|P_<\Sigma_0(\chi)|_1\le\hat r\end{aligned}$$

and

$$|\hat z|=r\left(\frac{R}{|z_0|}\right)^{\frac{u}{u_0}}$$

which is not larger than $\tilde r$ because of (12.2). Finally,

$$\hat{z_0}=R\,e^{i\hat\psi}$$

with

$$\hat\psi=\psi+v_0\tau(|z_0|)=\psi+\frac{v_0}{u_0}\log\left(\frac{R}{|z_0|}\right),$$

and (12.3) yields $\psi_u-\gamma\le\hat\psi\le\psi_u+\gamma$. Altogether,

$$\Sigma_0(\chi)=K(\hat y_<,\hat z,\hat{z_0})\in V(\gamma,\tilde r,\hat r).$$

Remark It is not hard to show that we actually have

$$\Sigma_0(H_i(\gamma,\tilde r,\hat r))=V(\gamma,\tilde r,\hat r),$$

see Proposition 4.1 in [19]. Notice that the sets $H_i(\gamma,\tilde r,\hat r)$ are not closed as $S_i\subset\overline{H_i(\gamma,\tilde r,\hat r)}\setminus H_i(\gamma,\tilde r,\hat r)$.

Corollary 12.2 *$\overline{\Sigma_1\circ\Sigma_0(H_i(\gamma,\tilde r,\hat r))}$ is compact and contained in the set $\Sigma_1(V(\gamma,\tilde r,\hat r))$.*

Proof As $V(\gamma,\tilde r,\hat r)$ is closed we have $\overline{\Sigma_0(H_i(\gamma,\tilde r,\hat r))}\subset V(\gamma,\tilde r,\hat r)$. Proposition 10.1 yields that $\overline{\Sigma_0(H_i(\gamma,\tilde r,\hat r))}\subset\overline{\Sigma_0(H_i\setminus Y_s)}$ is compact. It follows that $\overline{\Sigma_1\circ\Sigma_0(H_i(\gamma,\tilde r,\hat r))}\subset\Sigma_1(\overline{\Sigma_0(H_i(\gamma,\tilde r,\hat r))})$ is compact and contained in $\Sigma_1(V(\gamma,\tilde r,\hat r))$.

We express the *return map*

$$H_i(\gamma,\tilde r,\hat r)\ni\chi\mapsto\Sigma_1(\Sigma_0(\chi))\in H_i$$

in terms of coordinates as follows. The *inner map in terms of coordinates*, namely, the map

$$\begin{aligned}P_0:\mathbf{C}_i^{-1}\big(H_i(\gamma,\tilde r,\hat r)\big)&\to\overline{Y_{<,r}}\times\overline{\mathbb{C}_r}\times[-\pi,\pi),\\P_0(y_<,\phi,z_0)&=\mathbf{C}_u^{-1}\big(\Sigma_0(\mathbf{C}_i(y_<,\phi,z_0))\big),\end{aligned}$$

has its values in $\mathbf{C}_u^{-1}(V(\gamma, \tilde{r}, \hat{r})) \subset \mathbf{C}_u^{-1}(V)$, which is the domain of P_1, the outer map in terms of coordinates, and

$$P : \mathbf{C}_i^{-1}\big(H_i(\gamma, \tilde{r}, \hat{r})\big) \to Y_< \times \mathbb{R} \times \mathbb{C}$$

given by $P(x) = P_1(P_0(x))$ is the *return map in terms of coordinates*.

Using the definitions of the maps $\mathbf{C}_i, \mathbf{C}_u$ and (10.7)–(10.9) we infer $P_0(y_<, \phi, z_0) = (\tilde{y}_<, \tilde{z}, \tilde{\psi})$ with

$$\tilde{y}_< = T(\tau(|z_0|))y_<, \tag{12.4}$$

$$\tilde{z} = r\left(\frac{R}{|z_0|}\right)^{\frac{u}{u_0}} e^{i\left(\phi + v\tau(|z_0|) + \phi_i\right)}, \tag{12.5}$$

$$\tilde{\psi} = \psi + v_0\tau\big(|z_0|\big) - \psi_u. \tag{12.6}$$

Corollary 12.2 implies that P maps its domain into a compact subset of $Y_< \times \mathbb{R} \times \mathbb{C}$ which is contained in the domain $\overline{Y_{<,r}} \times [-\pi, \pi) \times \overline{\mathbb{C}_r}$ of $\mathbf{C}_i$.

13 Definition of Suitable Subsets N_0, N_1

In this section we define disjoint closed subsets N_0, N_1 of the domain of $P_1 \circ P_0$ for which we can prove that $P = P_1 \circ P_0$ has symbolic dynamics in the sense of Corollary 2.4.

Choose first $\bar{\delta}_2 \in (0, \min\{\gamma_V, r_V\}]$ such that P_1 is defined on the set $Y_{<,\bar{\delta}_2} \times \mathbb{C}_{\bar{\delta}_2} \times (-\bar{\delta}_2, \bar{\delta}_2)$ and that with constants $L_1, c > 0$, with ξ from (T2), and γ_2 from (11.15), the following estimates hold for y and $\tilde{y}$ in $Y_{<,\bar{\delta}_2} \times \mathbb{C}_{\bar{\delta}_2} \times (-\bar{\delta}_2, \bar{\delta}_2)$:

$$|P_1(y) - P_1(\tilde{y})| \le L_1|y - \tilde{y}| \tag{13.1}$$

$$P_1(y) = \underbrace{P_1(0_u)}_{=0_i} + DP_1(0_u)(y - 0_u) + \nu(y), \text{ where} \tag{13.2}$$

$$|\nu(y)| \le c|y - 0_u| \text{ and } c \le \min\left\{\frac{|\mathrm{pr}_2\xi|}{16}, \frac{\gamma_2}{16p_1}, \gamma_2\right\}. \tag{13.3}$$

Choose $\delta_1 \in (0, 1]$ such that with ε_1 from (11.19) and p_1 from the definition of e_1 in Sect. 11, one has

$$2\delta_1 L_1 \le \min\left\{\frac{\gamma_2}{16p_1}, c\right\}, \quad \delta_1 < \varepsilon_1/2. \tag{13.4}$$

We set $r_< := r/c_<$ (see (10.2)), so that for $t \ge 0$ one has $r_< c_< \exp(-\eta_< t) \le r$. Next we choose $\delta_2 \in (0, \bar{\delta}_2]$ satisfying the following conditions (with $I_2 := [-\delta_2, \delta_2]$; recall also that $d_1 < \pi/2$, and the eigenvalues $u + iv$ and $u_0 + iv_0$):

$$\frac{v}{v_0}\delta_2 < \varepsilon_1/2, \tag{13.5}$$

$$L_1\delta_2 < \min\{r_<, \delta_1\}, \tag{13.6}$$

$$|u|\left(\frac{I_2}{v_0} + \frac{[-d_1, d_1]}{v}\right) \subset \left[-\frac{|u|\pi}{v}, \frac{|u|\pi}{v}\right]. \tag{13.7}$$

For $\psi < \psi_u - \delta_2$ we define the interval

$$\mathcal{R}(\psi) := R \cdot \exp\left[\frac{u_0}{v_0}(I_2 + \psi - \psi_u)\right] \text{ (compare formula (12.3))} \tag{13.8}$$

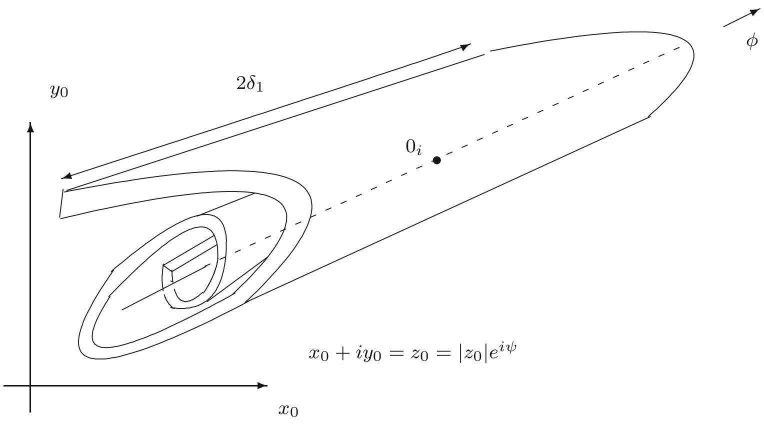


Fig. 12 A part of the set D_{ϑ^*,δ_1}, without components in $Y_<$

which is contained in $(0, R)$, and for $\vartheta > 0$ we define the following subset of $Y_< \times \mathbb{R} \times \mathbb{C}$:

$$D_{\vartheta,\delta_1} := \Big\{ \big(y_<, \phi, |z_0|e^{i\psi}\big) \mid |y_<|_1 \le r_<,\ |\phi| \le \delta_1,\ -\infty < \psi \le -\delta_2 - |\psi_u| - \vartheta,\ |z_0| \in \mathcal{R}(\psi) \Big\}.$$

Note that $\max \mathcal{R}(\psi) \to 0$ and $\min \Big\{ \tau(|z_0|) \mid |z_0| \in \mathcal{R}(\psi) \Big\} \to \infty$ as $\psi \to -\infty$. It is clear from Proposition 12.1 and the definition of the sets $H_i(\ldots)$ that there exists $\bar{\vartheta} > \delta_2$ such that for $\vartheta \ge \bar{\vartheta}$ one has $D_{\vartheta,\delta_1} \subset \mathbf{C}_i^{-1}(H_i(\delta_2, \delta_2, \delta_2))$, which implies that

$$\begin{aligned} &\text{for all } \vartheta \ge \bar{\vartheta}, \text{ the maps } P_0 \text{ and } P_1 \circ P_0 \text{ are defined on } D_{\vartheta,\delta_1}, \text{ and} \\ &P_0\big(D_{\vartheta,\delta_1}\big) \subset \mathbf{C}_u^{-1}\big(V(\delta_2, \delta_2, \delta_2)\big) = \overline{Y_{<,\delta_2}} \times \overline{\mathbb{C}_{\delta_2}} \times I_2. \end{aligned} \tag{13.9}$$

Recall that $-\eta_< < u < 0$ and that $u_0 > |u|$ (see (3.3)), and set $q := \exp[3\pi|u|/v]$. Choose $\vartheta^* > \bar{\vartheta}$ such that for $x = (y_<, \phi, z_0) \in D_{\vartheta^*,\delta_1}$ one has

$$|z_0| \le \frac{|\mathrm{pr}_2\xi|\delta_2}{8}, \tag{13.10}$$

$$e^{-\eta_<\tau(|z_0|)} \le \delta_1 e^{u\tau\left(|z_0|\right)}, \tag{13.11}$$

$$\frac{R}{r} q \exp\big[-(u + u_0)\vartheta^*/v_0\big] < \frac{1}{16p_1}\min\{\gamma_2, 1\}, \tag{13.12}$$

and consider the set D_{ϑ^*,δ_1} from now on (Fig. 12).

The projection of D_{ϑ^*,δ_1} to the z_0-plane is the area bounded by the two logarithmic spirals given by $|z_0| = \max \mathcal{R}(\psi)$ and $|z_0| = \min \mathcal{R}(\psi)$, $\psi \in (-\infty, -\delta_2 - |\psi_u| - \vartheta^*]$.

The relative positions of D_{ϑ^*,δ_1} and its image under P are qualitatively as shown in Fig. 13. This is not obvious at this point, but will be shown in Sects. 13 and 14. In particular, the fact that $P(D_{\vartheta^*,\delta_1})$ extends further in the directions of ξ and f_2 than D_{ϑ^*,δ_1} is contained in the proof of Lemma 14.1.

Note that for $\vartheta, \vartheta' \in (-\infty, -\delta_2 - |\psi_u| - \vartheta^*]$ one has the implication

$$\vartheta' = \vartheta - 2k\pi \text{ for some } k \in \mathbb{N} \Longrightarrow \max \mathcal{R}(\vartheta') < \min \mathcal{R}(\vartheta), \tag{13.13}$$

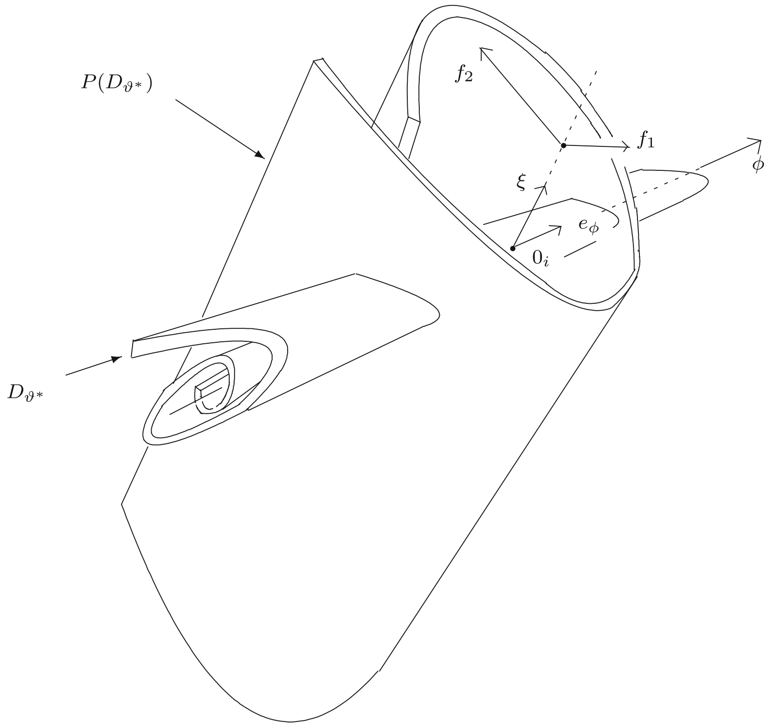


Fig. 13 The set D_{ϑ^*} and its image under P (qualitatively)

since $2\delta_2 < 2k\pi$. Thus, for $(y_<, \phi, |z_0|e^{i\psi}) \in D_{\vartheta^*,\delta_1}$, the number $\psi \in (-\infty, -\delta_2 - \psi_u - \vartheta^*]$ is uniquely determined by $|z_0|$ (not only modulo 2π). Recall the numbers ϕ_1 and d_1 from Sect. 11. We now choose $k^* \in \mathbb{N}$ such that $\psi_u + \frac{v_0}{v}(\phi_i - \phi_1 - 2k^*\pi + d_1) < -\delta_2 - |\psi_u| - \vartheta^*$, and such that with

$$\begin{aligned} r_{\min} &:= r \exp\left[\frac{|u|}{v}(\phi_i - \phi_1 - \pi - 2k^*\pi)\right] \exp\left[-\frac{|u|}{v}\pi\right], \\ r_{\max} &:= r \exp\left[\frac{|u|}{v}(\phi_i - \phi_1 - 2k^*\pi)\right] \exp\left[\frac{|u|}{v}\pi\right] \end{aligned} \tag{13.14}$$

one has

$$\frac{\mu r_{\max}}{p_1} \le \delta_2/2, \quad \frac{r_{\max}}{p_1} \le \delta_2 |\mathrm{pr}_2 \xi| \min\left\{1, \frac{1}{8|\mathrm{pr}_2 f_2|}\right\}, \quad 2r_{\max} \le \delta_2. \tag{13.15}$$

Then the intervals

$$\begin{aligned} J_0 &:= \psi_u + \frac{v_0}{v}\big(\phi_i - \phi_1 - 2k^*\pi + [-d_1, d_1]\big), \\ J_1 &:= \psi_u + \frac{v_0}{v}\big(\phi_i - (\phi_1 + \pi) - 2k^*\pi + [-d_1, d_1]\big) \end{aligned}$$

satisfy $\max J_1 < \min J_0 < \max J_0 < -\delta_2 - |\psi_u| - \vartheta^*$ (for the first inequality, recall that $d_1 < \pi/2$).

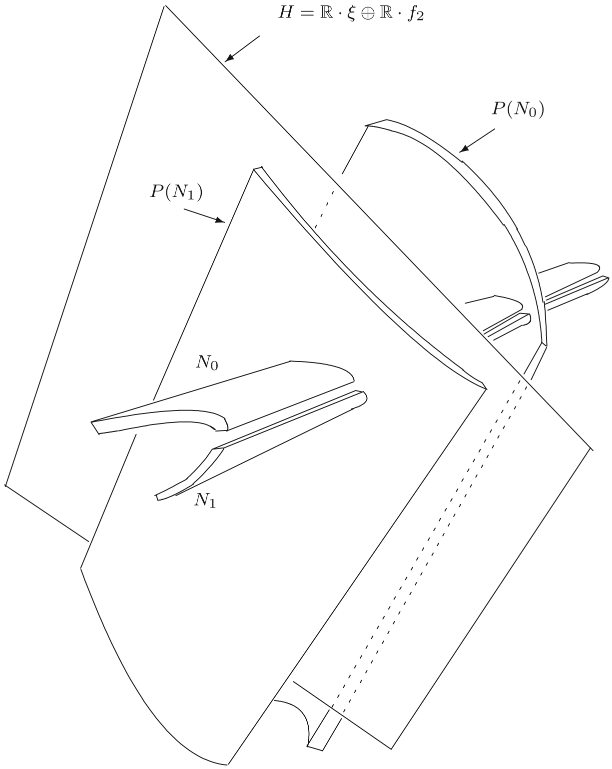


Fig. 14 The sets N_0 and N_1, their images under P, and the hyperplane $0_i + H$ (qualitatively, with only the three-dimensional part shown)

Finally we define

$$N_0 := \Big\{ \big(y_<, \phi, |z_0|e^{i\psi} \big) \;\Big|\; |y_<|_1 \le r_<,\ |\phi| \le \delta_1, \\ \psi \in J_0,\ |z_0| \in \mathcal{R}(\psi) \Big\},$$

$$N_1 := \Big\{ \big(y_<, \phi, |z_0|e^{i\psi} \big) \;\Big|\; |y_<|_1 \le r_<,\ |\phi| \le \delta_1, \\ \psi \in J_1,\ |z_0| \in \mathcal{R}(\psi) \Big\},$$

and $N := N_0 \cup N_1$.

These sets are closed subsets of D_{ϑ^*,δ_1}, and disjointness of J_0 and J_1 together with property (13.13) imply that $N_0 \cap N_1 = \emptyset$. Note also that $q = r_{\max}/r_{\min}$ (independently of the choice of k^*).

The intersection properties of N_0, N_1 and their images under P are as indicated in Fig. 14. This is proved partially in Proposition 13.1 (in particular, how the boundaries of N_0 and N_1 are mapped under P), and partially in the proof of Lemma 14.1, where we a construct a homotopy to a simpler model map. Parts (c) and (d) of Proposition 13.1 describe, in geometric interpretation, that N_0 and N_1 get mapped to different sides of the plane H.

Proposition 13.1 *Assume* $x = (y_<, \phi, |z_0|e^{i\psi}) \in N$ *(with* $\psi \in J_0 \cup J_1$*, and* $\phi \in [-\delta_1, \delta_1]$*), and set*

$$\tau := \tau(|z_0|),\ \tilde{y}_< := T(\tau)y_<,\ r' := re^{u\tau},\ \tilde{\phi} := \phi_i + \phi + v\tau,\ \tilde{\psi} := \psi + v_0\tau - \psi_u.$$

Then

$$P_0(x) = \big(\tilde{y}_<, r'e^{i\tilde{\phi}}, \tilde{\psi}\big). \tag{13.16}$$

The following properties (in particular, 'boundary correspondences') hold:

(a) $\tau \geq \vartheta^*/v_0$.
(b) $\tilde{\psi} \in [-\delta_2, \delta_2]$*, and*

$$|z_0| = \min \mathcal{R}(\psi) \implies \tilde{\psi} = \delta_2, \qquad |z_0| = \max \mathcal{R}(\psi) \implies \tilde{\psi} = -\delta_2.$$

(c)

$$\begin{aligned}\psi \in J_0 &\implies \tilde{\phi} \in \phi_1 + [-d_1 - \varepsilon_1, d_1 + \varepsilon_1] + 2k^*\pi, \text{ and}\\ \psi = \min J_0 &\implies \tilde{\phi} - \phi_1 \in d_1 + [-\varepsilon_1, \varepsilon_1] + 2k^*\pi,\\ \psi = \max J_0 &\implies \tilde{\phi} - \phi_1 \in -d_1 + [-\varepsilon_1, \varepsilon_1] + 2k^*\pi.\end{aligned}$$

(d)

$$\begin{aligned}\psi \in J_1 &\implies \tilde{\phi} \in \phi_1 + \pi + [-d_1 - \varepsilon_1, d_1 + \varepsilon_1] + 2k^*\pi, \text{ and}\\ \psi = \min J_1 &\implies \tilde{\phi} - (\phi_1 + \pi) \in d_1 + [-\varepsilon_1, \varepsilon_1] + 2k^*\pi,\\ \psi = \max J_1 &\implies \tilde{\phi} - (\phi_1 + \pi) \in -d_1 + [-\varepsilon_1, \varepsilon_1] + 2k^*\pi.\end{aligned}$$

(e) $r' \in [r_{\min}, r_{\max}]$.
(f) $|z_0| \leq \dfrac{r_{\min}}{16p_1}\min\{\gamma_2, 1\}$.

Proof Equality (13.16) is clear from (12.4)–(12.6).

Ad (a) and (b): From the definition of $\mathcal{R}(\psi)$,

$$\begin{aligned}\tau(\mathcal{R}(\psi)) &= \frac{1}{u_0}\log(\frac{R}{\mathcal{R}(\psi)}) = \frac{1}{u_0}\log\left(\exp\left[-\frac{u_0}{v_0}(I_2 + \psi - \psi_u)\right]\right)\\ &= -\frac{(I_2 + \psi - \psi_u)}{v_0},\end{aligned}$$

which shows that $\tilde{\psi} = \psi + v_0\tau - \psi_u \in \psi - (I_2 + \psi - \psi_u) - \psi_u = -I_2 = I_2 = [-\delta_2, \delta_2]$, and also the boundary relations in (b). (The inclusion $\tilde{\psi} \in I_2$ can also be seen from (13.9)). Further, $\psi \leq -\delta_2 - |\psi_u| - \vartheta^*$ implies $\tau \geq (-\delta_2 + \delta_2 + |\psi_u| + \vartheta^* + \psi_u)/v_0 \geq \vartheta^*/v_0$, which proves (a).

Ad (c): $\tilde{\phi} = \phi_i + \phi + v\tau \in \phi_i + \phi - \dfrac{v}{v_0}(I_2 + \psi - \psi_u)$, so $\psi \in J_0$ implies

$$\begin{aligned}\tilde{\phi} &\in \phi_i + \phi - \frac{v}{v_0}\big(I_2 + J_0 - \psi_u\big)\\ &= \phi_i + \phi - \frac{v}{v_0}\Big[I_2 + \frac{v_0}{v}(\phi_i - \phi_1 - 2k^*\pi + [-d_1, d_1])\Big]\\ &\subset \phi_i + [-\delta_1, \delta_1] + \frac{v}{v_0}I_2 - \phi_i + \phi_1 + 2k^*\pi - [-d_1, d_1]\\ &= [-\delta_1, \delta_1] + \frac{v}{v_0}I_2 + \phi_1 + 2k^*\pi - [-d_1, d_1].\end{aligned}$$

Using (13.4) and (13.5), we obtain

$$\begin{aligned}\tilde{\phi} &\in [-\varepsilon_1/2, \varepsilon_1/2] + [-\varepsilon_1/2, \varepsilon_1/2] + \phi_1 + 2k^*\pi - [-d_1, d_1] \\ &= \phi_1 + [-d_1 - \varepsilon_1, d_1 + \varepsilon_1] + 2k^*\pi.\end{aligned}$$

If $\psi = \min J_0 = \psi_u + \frac{v_0}{v}(\phi_i - \phi_1 - 2k^*\pi - d_1)$ then $\tilde{\phi} \in [-\varepsilon_1, \varepsilon_1] + \phi_1 + 2k^*\pi + d_1$, and if $\psi = \max J_0$, the same is true with d_1 replaced by $-d_1$.

Ad (d): The proof is analogous to the proof of b), with ϕ_1 replaced by $\phi_1 + \pi$ (compare the definitions of J_0 and J_1).

Ad (e): If $x \in N_0$ then (recall that $|u| = -u$, and formula 12.5)

$$\begin{aligned} r' &= r\left(\frac{R}{|z_0|}\right)^{u/u_0} \in r\left[\frac{1}{\exp[\frac{u_0}{v_0}(I_2 + \psi - \psi_u)]}\right]^{u/u_0} = r\exp\left[\frac{|u|}{v_0}(I_2 + \psi - \psi_u)\right] \\ &\in r\exp\left[\frac{|u|}{v_0}(I_2 + J_0 - \psi_u)\right] \\ &= r\exp\left[\frac{|u|}{v_0}I_2\right]\cdot\exp\left\{\frac{|u|}{v_0}\cdot\frac{v_0}{v}[\phi_i - \phi_1 - 2k^*\pi + [-d_1, d_1]]\right\} \\ &= r\exp\left[\frac{|u|}{v_0}I_2\right]\cdot\exp\left\{\frac{|u|}{v}[\phi_i - \phi_1 - 2k^*\pi]\right\}\cdot\exp\left\{\frac{|u|}{v}[-d_1, d_1]\right\} \\ &= r\exp\left[\frac{|u|}{v}(\phi_i - \phi_1 - 2k^*\pi)\right]\cdot\exp\left[|u|\left(\frac{I_2}{v_0} + \frac{[-d_1, d_1]}{v}\right)\right]. \end{aligned}$$

Using (13.7), we see that this set is contained in $r\exp[\frac{|u|}{v}(\phi_i - \phi_1 - 2k^*\pi)]\cdot\exp([-\frac{|u|}{v}\pi, \frac{|u|}{v}\pi])$. A similar estimate, with J_0 replaced by J_1 and $(\phi_1 + \pi)$ in place of ϕ_1 shows that if $x \in N_1$ then $r' \in r\exp[\frac{|u|}{v}(\phi_i - \phi_1 - \pi - 2k^*\pi)]\cdot\exp[-\frac{|u|}{v}\pi, \frac{|u|}{v}\pi]$. Together with the definitions of $r_{\min}$ and $r_{\max}$ one sees that $r_{\min} \le r' \le r_{\max}$.

Ad (f): Recall that $r_{\max}/r_{\min} = \exp[3\pi|u|/v] = q$. We have $|z_0| = Re^{-u_0\tau}$ and

$$q\, r_{\min} = r_{\max} \ge r' = re^{u\tau} = \frac{r}{R}\underbrace{Re^{-u_0\tau}}_{=|z_0|}e^{(u_0+u)\tau},$$

so $|z_0| \le \frac{R}{r}q\, r_{\min}e^{-(u_0+u)\tau}$. Using part a) and (13.12), we conclude

$$|z_0| \le \frac{R}{r}q\, r_{\min}e^{-(u_0+u)\vartheta^*/v_0} \le \frac{r_{\min}}{16p_1}\min\{\gamma_2, 1\}.$$

Recall the functionals L and $L^\perp$ from Sect. 11. We use the notation of Proposition 13.1, and the abbreviations

$$a := L\left(e^{i(\phi_i + v\tau)}\right), \quad b := L^\perp\left(e^{i(\phi_i + v\tau)}\right).$$

(Note that, compared to the formula for $\tilde{\phi}$ in Proposition 13.1, the variable ϕ does not appear in the definitions of a and b.)

Proposition 13.2 *For $x \in N$, we have*

$$\begin{aligned} P(x) &= r'\big[af_1 + bf_2\big] + \tilde{\psi}\xi + R_1 + R_2 \\ &= \big[\tilde{\psi} - \mu r'a\big]\xi + r'bf_2 + r'ae_\phi + R_1 + R_2, \end{aligned} \tag{13.17}$$

where

$$|R_1| \le 2L_1 r' \delta_1, \tag{13.18}$$
$$|R_2| \le c[r' + |\tilde{\psi}|], \tag{13.19}$$
$$|R_1| + |R_2| \le \frac{\delta_2 |pr_2 \xi|}{8}. \tag{13.20}$$

Proof We use the notation of (13.16). For $x \in N$,

$$P(x) = P_1\big(\tilde{y}_<, r' e^{i(\phi_i + \phi + \upsilon\tau)}, \tilde{\psi}\big) = P_1\big(0_<, r' e^{i(\phi_i + \upsilon\tau)}, \tilde{\psi}\big) + R_1, \tag{13.21}$$

where (according to (13.1) and the definition of $r_< = \frac{r}{c_<}$)

$$\begin{aligned} |R_1| &\le L_1[r'|\phi| + |\tilde{y}_<|_1] \le L_1[r'\delta_1 + c_< e^{-\eta_< \tau} r_<] \\ &= L_1[r'\delta_1 + r e^{-\eta_< \tau}] \qquad \text{(see 13.11)} \\ &\le L_1(r'\delta_1 + \delta_1 \underbrace{r e^{u\tau}}_{=r'}] = 2L_1 r' \delta_1. \end{aligned}$$

Further,

$$\begin{aligned} P_1\big(0_<, r' e^{i(\phi_i + \upsilon\tau)}, \tilde{\psi}\big) &= P_1\big(0_<, r' e^{i(\phi_i + \upsilon\tau)}, \tilde{\psi}\big) - P_1 \underbrace{\big(0_<, 0_{\mathbb{C}}, 0_{\mathbb{R}}\big)}_{=0_u} \\ &= DP_1(0_u)\big[0_<, r' e^{i(\phi_i + \upsilon\tau)}, \tilde{\psi}\big] + R_2, \end{aligned} \tag{13.22}$$

where according to (13.3) one has $|R_2| \le c(r' + |\tilde{\psi}|)$.

We see that properties (13.18)–(13.19) hold (but (13.17) is still to be proved). Recall from Sect. 11 that the projection of $DP_1(0_u)[0_<, r' e^{i(\phi_i + \upsilon\tau)}, \tilde{\psi}]$ onto $Y_< \times \{0\} \times \{0_{\mathbb{C}}\}$ is zero in our situation. From the definitions of D_1, f_1, f_2 and ξ we see that

$$\begin{aligned} DP_1(0_u)\big[0_<, r' e^{i(\phi_i + \upsilon\tau)}, \tilde{\psi}\big] &= D_1\big[r'a \cdot e_1 + r'b \cdot e_1^{\perp} + \tilde{\psi} \cdot (0_<, 0_{\mathbb{C}}, 1)\big] \\ &= r'\big[af_1 + bf_2\big] + \tilde{\psi}\xi. \end{aligned} \tag{13.23}$$

Combination of (13.21)–(13.23) proves the first equation in (13.17), and the second is obtained from (11.12), replacing f_1 by $e_\phi - \mu\xi$.

Proof of (13.20):

$$\begin{aligned} |R_1| + |R_2| &\le r'[2L_1\delta_1 + c] + c|\tilde{\psi}| \qquad \text{(see Proposition 13.1, (e) and (b))} \\ &\le r_{\max}[2L_1\delta_1 + c] + c\delta_2 \qquad \text{(see 13.4)} \\ &\le r_{\max} 2c + c\delta_2 \qquad \text{(see 13.3)} \\ &\le \frac{|\mathrm{pr}_2\xi|}{16}[2r_{\max} + \delta_2] \qquad \text{(see 13.15)} \\ &\le \frac{|\mathrm{pr}_2\xi|}{16}[\delta_2 + \delta_2] = \frac{|\mathrm{pr}_2\xi|}{8}\delta_2. \end{aligned} \tag{13.24}$$

14 Homotopy to a Simpler Map

Motivated by (13.17), we introduce a simplified model map $Q : N \to Y_< \times \mathbb{R} \times \mathbb{C}$ for $P|N$ by

$$Q(x) := \mathrm{pr}_2[\tilde{\psi} \cdot \xi + r_{\max} L^{\perp}(e^{i(\phi_i + \upsilon\tau)}) f_2] \quad (x \in N = N_0 \cup N_1). \tag{14.1}$$

(Here, as above, $\tau = \tau(|z_0|)$, $\tilde{\psi} = \psi + v_0\tau - \psi_u$, if $x = (y_<, \phi, |z_0|e^{i\psi})$, $\psi \in J_0 \cup J_1$, $z_0 \in \mathcal{R}(\psi)$). The homotopy from $P_{|N}$ to Q in the lemma below is the main step in the proof of the symbolic dynamics result. Comparing (14.1) and (13.17), we see that it achieves the following simplifications:

(1) The dependence of the mapping P on the coordinates $y_<$ and ϕ is eliminated, and the dimension of the image is reduced to two;
(2) The component of $Q(x)$ in the direction of ξ depends only on $\tilde{\psi}$;
(3) In the component in f_2-direction, the x-dependent value of r' is replaced by the constant $r_{\max}$.
(4) The remainder terms R_1, R_2 are omitted.

Recall the notion 'N-homotopic' from Sect. 2.

Lemma 14.1 *$P_{|N}$ and Q are N-homotopic, with a compact homotopy.*

Proof We define $f : [0,1] \times N \to Y_< \times \mathbb{R} \times \mathbb{C}$, $(\lambda, x) \mapsto f_\lambda(x)$ by $f_\lambda(x) := (1-\lambda)P(x) + \lambda Q(x)$. Clearly, f is continuous and compact (since P is compact, and Q is finite-dimensional).
Using (13.17) and (14.1), and writing again τ for $\tau(|z_0|)$ and a, b instead of $L(e^{i(\phi_i+v\tau)})$ and $L^\perp(e^{i(\phi_i+v\tau)})$, we see that for $x = (y_<, \phi, z_0) \in N$

$$f_\lambda(x) = (1-\lambda)\left\{[\tilde{\psi} - \mu r' a]\xi + r' b f_2 + r' a e_\phi + R_1 + R_2\right\} + \lambda \mathrm{pr}_2\left[\tilde{\psi}\xi + r_{\max} b f_2\right]. \quad (14.2)$$

Note that with $\tilde{\phi} := \phi_i + v\tau$

$$\max\{|a|, |b|\} = \max\left\{|L(e^{i\tilde{\phi}})|, |L^\perp(e^{i\tilde{\phi}})|\right\} \le \left|e^{i\tilde{\phi}}\right|/p_1 = 1/p_1. \quad (14.3)$$

With the projection $\mathrm{pr}_3 : Y_< \times \mathbb{R} \times \mathbb{C} \to \{0_<\} \times \mathbb{R} \times \mathbb{C}$ defined by $\mathrm{pr}_3(y_<, \phi, z_0) := (0_<, \phi, z_0)$ and $\mathrm{pr}_2 e_\phi = 0$, we have

$$\begin{aligned} \mathrm{pr}_2\mathrm{pr}_3 f_\lambda(x) = \left[\tilde{\psi} - (1-\lambda)\mu r' a\right]\mathrm{pr}_2\xi + \left[(1-\lambda)r' + \lambda r_{\max}\right] \cdot b \cdot \mathrm{pr}_2 f_2 \\ + (1-\lambda)\mathrm{pr}_2\mathrm{pr}_3(R_1 + R_2). \end{aligned} \quad (14.4)$$

In order to prove that f is an N-homotopy, we use part (3) of Proposition 2.2. For $j \in \{0, 1\}$ we define

$$\partial_1 N_j := \left\{(y_<, \phi, |z_0|e^{i\psi}) \in N_j \;\middle|\; |z_0| \in \{\min \mathcal{R}(\psi), \max \mathcal{R}(\psi)\} \text{ or } \psi \in \{\min J_j, \max J_j\}\right\},$$

and

$$\partial_2 N_j := \left\{(y_<, \phi, z_0) \in N_j \;\middle|\; |\phi| = \delta_1 \text{ or } |y_<|_1 = r_<\right\}.$$

Then $\partial N_j = \partial_1 N_j \cup \partial_2 N_j$, and the assertion of the lemma is proved if we show

$$\forall \lambda \in [0,1]: \quad f_\lambda(\partial_1 N_j) \cap N = \emptyset = \partial_2 N_j \cap f_\lambda(N), \quad j = 0, 1, \quad (14.5)$$

since then part (3) of Proposition 2.2 applies with $\partial_k N := \partial_k N_0 \cup \partial_k N_1$, $k = 1, 2$. Let now $j \in \{0, 1\}$, $\lambda \in [0, 1]$, and $x = (y_<, \phi, |z_0|e^{i\psi}) \in N_j$ (with $\psi \in J_j$) be given.

1. Assume first $x \in \partial_1 N_j$. Then

(i) $|z_0| \in \{\min \mathcal{R}(\psi), \max \mathcal{R}(\psi)\}$ or
(ii) $\psi \in \{\min J_j, \max J_j\}$.

In case (i), we see from Proposition 13.1, (b) that $|\tilde{\psi}| = \delta_2$. From (14.4) we conclude, using that $r' \leq r_{\max}$ (see Proposition 13.1, e) and (14.3), that

$$|\mathrm{pr}_2\mathrm{pr}_3 f_\lambda(x)| \geq |\tilde{\psi} - (1-\lambda)\mu r' a| \cdot |\mathrm{pr}_2\xi| - r_{\max}|b| \cdot |\mathrm{pr}_2 f_2| - (|R_1| + |R_2|)$$
$$\geq \big(\delta_2 - \frac{\mu r_{\max}}{p_1}\big)|\mathrm{pr}_2\xi| - \frac{r_{\max}}{p_1}|\mathrm{pr}_2 f_2| - (|R_1| + |R_2|).$$

Using also (13.15) and (13.20) we get

$$|\mathrm{pr}_2\mathrm{pr}_3 f_\lambda(x)| \geq \big(\delta_2 - \frac{\delta_2}{2}\big)|\mathrm{pr}_2\xi| - \frac{\delta_2|\mathrm{pr}_2\xi|}{8} - \frac{\delta_2|\mathrm{pr}_2\xi|}{8} = \frac{\delta_2|\mathrm{pr}_2\xi|}{4}. \tag{14.6}$$

On the other hand, for $\hat{x} = (\hat{y}_<, \hat{\phi}, w_0) \in N$, we have from Proposition 13.1, (f) and from (13.15)

$$|\mathrm{pr}_2\mathrm{pr}_3\hat{x}| = |w_0| \leq \frac{r_{\min}}{16p_1} \leq \frac{r_{\max}}{16p_1} \leq \frac{\delta_2|\mathrm{pr}_2\xi|}{16}.$$

Thus we see that in case (i) $f_\lambda(x) \notin N$.

In case (ii), we apply Proposition 13.1 with $\phi = 0$ and obtain from parts c) and d) that $(\tilde{\phi} - \phi_1) \in \{\pm d_1\} + [-\varepsilon_1, \varepsilon_1] + \mathbb{Z}\pi$. Then (11.20) shows that $|\mu||a| \leq |b|/2$ and $|b| \geq 1/(2p_1)$. From (11.15) and (14.4) we now derive, using also (13.18) and (13.19), that

$$|\mathrm{pr}_2\mathrm{pr}_3 f_\lambda(x)| \geq \gamma_2\{|\tilde{\psi} - (1-\lambda)\mu r' a| + \underbrace{[(1-\lambda)r' + \lambda r_{\max}]}_{\geq r'}|b|\}$$
$$- (|R_1| + |R_2|)$$
$$\geq \gamma_2\{|\tilde{\psi}| - |\mu| r' |a| + r'|b|\} - (|R_1| + |R_2|)$$
$$\geq \gamma_2 r'(|b| - |\mu||a|) + \gamma_2|\tilde{\psi}| - (|R_1| + |R_2|)$$
$$\geq \gamma_2 r' \frac{|b|}{2} + \gamma_2|\tilde{\psi}| - 2L_1 r'\delta_1 - cr' - c|\tilde{\psi}|$$
$$= \big(\frac{\gamma_2|b|}{2} - 2L_1\delta_1 - c\big)r' + (\gamma_2 - c)|\tilde{\psi}|.$$

In view of (13.3) and (13.4) we obtain (since $\gamma_2 \geq c$ and $|b| \geq 1/(2p_1)$)

$$|\mathrm{pr}_2\mathrm{pr}_3 f_\lambda(x)| \geq \left(\frac{\gamma_2}{4p_1} - \frac{\gamma_2}{16p_1} - \frac{\gamma_2}{16p_1}\right)r' = \frac{\gamma_2}{8p_1}r' \geq \frac{\gamma_2}{8p_1}r_{\min}. \tag{14.7}$$

But, for $\hat{x} = (\hat{y}_<, \hat{\phi}, w_0) \in N$, we have from Proposition 13.1, (f):

$$|\mathrm{pr}_2\mathrm{pr}_3\hat{x}| = |w_0| \leq \frac{\gamma_2}{16p_1}r_{\min}.$$

Hence, also in case (ii) $f_\lambda(x) \notin N$. Together, we have shown

$$f_\lambda(\partial_1 N_j) \cap N = \emptyset. \tag{14.8}$$

2. Now we assume that $x = (y_<, \phi, |z_0|e^{i\psi}) \in \partial_2 N_j$, which means that

(i) $|\phi| = \delta_1$ or (ii) $|y_<|_1 = r_<$.

Consider $\tilde{x} = (\tilde{y}_<, \tilde{\phi}, w_0) \in N$, and define $\hat{y}_< \in Y_<$, $\hat{\phi} \in \mathbb{R}$, and $\hat{z}_0 \in \mathbb{C}$ by $f_\lambda(\tilde{x}) = (\hat{y}_<, \hat{\phi}, \hat{z}_0)$. With the projection $\text{pr}_< : Y_< \times \mathbb{R} \times \mathbb{C} \to Y_<$ we have $\text{pr}_< Q(\tilde{x}) = 0$ and

$$\begin{aligned} |\hat{y}_<|_1 &= |\text{pr}_< f_\lambda(\tilde{x})|_1 = |(1-\lambda)\text{pr}_< P_1 P_0(\tilde{x})|_1 \\ &= |(1-\lambda)\text{pr}_<[P_1 P_0(\tilde{x}) - P_1(0_u)]|_1 \\ &\le |P_1 P_0(\tilde{x}) - P_1(0_u)| \quad \text{(see (13.1))} \\ &\le L_1 |P_0(\tilde{x}) - 0_u| \le L_1 \delta_2 \quad \text{(see (13.9))} \\ &< r_< \qquad \text{(see (13.6))}. \end{aligned}$$

It follows that $\hat{y}_< \neq y_<$ in case (ii), so $x \notin f_\lambda(N)$ in case (ii). Further, with the projection $\text{pr}_1 : \{0_<\} \times \mathbb{R} \times \mathbb{C} \to \mathbb{R}$, we have $\text{pr}_1 \text{pr}_3 Q(x) = \text{pr}_1 \text{pr}_3 0_i = 0$, and thus an argument similar to the one above shows

$$\begin{aligned} |\hat{\phi}| &= |\text{pr}_1 \text{pr}_3 f_\lambda(\tilde{x})| = |(1-\lambda)\text{pr}_1 \text{pr}_3[(P_1 \circ P_0)(\tilde{x}) - 0_i]| \\ &\le |(P_1 \circ P_0)(\tilde{x}) - P_1(0_u)| \le L_1 \delta_2 < \delta_1 \text{ (see (13.6))}. \end{aligned}$$

We see that also in case (i), where $|\text{pr}_1 \text{pr}_3 x| = |\phi| = \delta_1$, one has $x \notin f_\lambda(N)$, and thus

$$f_\lambda(N) \cap \partial_2 N_j = \emptyset. \tag{14.9}$$

Now (14.9) and (14.8) together give (14.5), which proves the lemma.

15 Computation of the Fixed Point Index and Symbolic Dynamics Theorem

In order to apply Corollary 2.4 to the N-homotopy from Lemma 14.1, it is necessary to show that

$$\text{for all } m \in \mathbb{N} \text{ and all } \mathbf{s} = (s_0, \dots s_m) \in \{0,1\}^{m+1} \text{ with } s_0 = s_m \text{ we have } \text{ind}(Q^m, N_{\mathbf{s},Q}) \neq 0. \tag{15.1}$$

From the definition of Q in (14.1) it is obvious that Q (and hence also Q^m for $m \in \mathbb{N}$) maps into the plane $E := \{0_<\} \times \{0_\mathbb{R}\} \times \mathbb{C}$. We write ${}_{E|}Q^m$ for the restriction of Q^m in the image space. The map

$$\iota : \mathbb{C} \to E, \ \iota(z_0) := (0_<, 0_\mathbb{R}, z_0) \in E$$

is a homeomorphism. We set

$$\tilde{N}_j := \left\{ |z_0| e^{i\psi} \in \mathbb{C} \ \middle|\ \psi \in J_j, \ |z_0| \in \mathcal{R}(\psi) \right\} \quad (j = 0, 1),$$

and $\tilde{N} := \tilde{N}_0 \cup \tilde{N}_1$. Further, we define $\tilde{Q} : \tilde{N} \to \mathbb{C}$ by

$$Q(0_<, 0_\mathbb{R}, z_0) = (0_<, 0_\mathbb{R}, \tilde{Q}(z_0)).$$

For $\tilde{\xi}, \tilde{f}_2 \in \mathbb{C}$ defined by $\text{pr}_2 \xi = (0_<, 0_\mathbb{R}, \tilde{\xi})$, $\text{pr}_2 f_2 = (0_<, 0_\mathbb{R}, \tilde{f}_2)$, we see from (11.15) that $\tilde{\xi}$ and $\tilde{f}_2$ are $\mathbb{R}-$linearly independent, and the definitions of Q and $\tilde{Q}$ show that for $z_0 = |z_0| e^{i\psi} \in \tilde{N}$ ($\psi \in J_0 \cup J_1$) we have

$$\tilde{Q}(z_0) = [\psi + v_0 \tau(|z_0|) - \psi_u] \cdot \tilde{\xi} + r_{\max} L^\perp (e^{i(\phi_i + v\tau(|z_0|))}) \cdot \tilde{f}_2. \tag{15.2}$$

Proposition 15.1 *For m and $\mathbf{s}$ as in (15.1), one has*

$$ind(Q^m, N_{\mathbf{s},Q}) = ind(\tilde{Q}^m, \tilde{N}_{\mathbf{s},\tilde{Q}}). \tag{15.3}$$

Proof We first show that

$$\text{for all } m \in \mathbb{N} \text{ and all } x \in \tilde{N} \text{ in the domain of } \tilde{Q}^m \text{ we have} \quad \tilde{Q}^m(x) = \iota^{-1} \circ {}_{E|}Q^m \circ \iota(x). \tag{15.4}$$

For $z_0 \in \tilde{N}$, we have $\iota(\tilde{Q}(z_0)) = (0_<, 0_{\mathbb{R}}, \tilde{Q}(z_0))$, and from the definitions of Q and $\tilde{Q}$,

$$Q(\iota(z_0)) = (0_<, 0_{\mathbb{R}}, \tilde{Q}(z_0)) = \iota(\tilde{Q}(z_0)).$$

We have shown ${}_{E|}Q \circ \iota = \iota \circ \tilde{Q}$ on $\tilde{N}$, from which (15.4) follows. Using the reduction or contraction property of the fixed point index (see [3], §12, p. 316, property VIII), and the fact that Q^m maps into E, we obtain

$$\text{ind}(Q^m, N_{\mathbf{s},Q}) = \text{ind}({}_{E|}Q^m{}_{|E}, N_{\mathbf{s},Q} \cap E). \tag{15.5}$$

From the commutativity property of the fixed point index (see [3], §12, p. 308, property VII), or alternatively from the invariance of the Leray–Schauder-degree under homeomorphisms (see [22], §13.7, p. 578, formula (41)), we see that the last index equals $\text{ind}(\iota^{-1} \circ {}_{E|}Q^m \circ \iota,\ \iota^{-1}(N_{\mathbf{s},Q} \cap E))$, which in view of (15.4) equals $\text{ind}(\tilde{Q}^m, \iota^{-1}(N_{\mathbf{s},Q} \cap E))$, so we have

$$\text{ind}(Q^m, N_{\mathbf{s},Q}) = \text{ind}(\tilde{Q}^m, \iota^{-1}(N_{\mathbf{s},Q} \cap E)). \tag{15.6}$$

Now

$$\begin{aligned} N_{\mathbf{s},Q} \cap E &= (N_{s_0} \cap E) \cap \bigcap_{j=1}^{m} Q^{-j}(N_{s_j}) = \text{ (since } Q \text{ maps into } E) \\ &= (N_{s_0} \cap E) \cap \bigcap_{j=1}^{m} Q^{-j}(N_{s_j} \cap E). \end{aligned}$$

Since $N_j \cap E = \iota(\tilde{N}_j),\ j = 0, 1$, we obtain $N_{\mathbf{s},Q} \cap E = \bigcap_{j=0}^{m} Q^{-j}(\iota(\tilde{N}_{s_j}))$. It follows from (15.4) that

$$\iota^{-1}(N_{\mathbf{s},Q} \cap E) = \bigcap_{j=0}^{m} \iota^{-1}(Q^{-j}(\iota(\tilde{N}_{s_j}))) = \bigcap_{j=0}^{m} \tilde{Q}^{-j}(\tilde{N}_{s_j}) = \tilde{N}_{\mathbf{s},\tilde{Q}}. \tag{15.7}$$

Now (15.3) is obtained by inserting (15.7) into (15.6).

Proposition 15.2 *For* $j = 0, 1$, *the function* $\tilde{Q}_{|\tilde{N}_j}$ *maps* $\tilde{N}_j$ *homeomorphically to its image, and* $\tilde{N}_0 \cup \tilde{N}_1 \subset$ *int*$(\tilde{Q}(\tilde{N}_j))$.

Proof Claim 1. $\tilde{Q}_{|\tilde{N}_j}$ is injective for $j = 0, 1$.

Proof Assume $z_0 = |z_0|e^{i\psi}$ and $\tilde{z}_0 = |\tilde{z}_0|e^{i\tilde{\psi}} \in \tilde{N}_0$ first, with $\{\psi, \tilde{\psi}\} \subset J_0$. Then Proposition 13.1, (c) (applied with $\phi := 0$) shows

$$\phi_i + \{v\tau(|z_0|), v\tau(|\tilde{z}_0|)\} \subset \phi_1 + 2k^*\pi + \big[-d_1 - \varepsilon_1, d_1 + \varepsilon_1\big]. \tag{15.8}$$

From (11.19) we know $[d_1 - \varepsilon_1, d_1 + \varepsilon_1] \subset (0, \pi/2)$, and for $s \in [-d_1 - \varepsilon_1, d_1 + \varepsilon_1] \subset (-\pi/2, \pi/2)$ we see from (11.18) that

$$L^\perp\big(e^{i(\phi_1+2k^*\pi+s)}\big) = L^\perp\big(e^{i(\phi_1+s)}\big) = \frac{1}{p_1}\sin(s).$$

Hence,

$$\text{the map } [-d_1 - \varepsilon_1, d_1 + \varepsilon_1] \ni s \mapsto L^\perp\big(e^{i(\phi_1+2k^*\pi+s)}\big) \in \mathbb{R} \text{ is injective.} \tag{15.9}$$

Now assume $\tilde{Q}(z_0) = \tilde{Q}(\tilde{z}_0)$. Then linear independence of $\tilde{\xi}$ and $\tilde{f}_2$ in formula (15.2) for $\tilde{Q}$ gives

$$\begin{aligned} L^\perp\big(e^{i(\phi_i+v\tau(|z_0|))}\big) &= L^\perp\big(e^{i(\phi_i+v\tau(|\tilde{z}_0|))}\big), \text{ and} \\ \psi + v_0\tau(|z_0|) - \psi_u &= \tilde{\psi} + v_0\tau(|\tilde{z}_0|) - \psi_u. \end{aligned} \tag{15.10}$$

It follows from (15.8), (15.9) and the first equality in (15.10) that $\tau(|z_0|) = \tau(|\tilde{z}_0|)$, and hence $|z_0| = |\tilde{z}_0|$. The second equality in (15.10) then shows $\psi = \tilde{\psi}$, so $z_0 = \tilde{z}_0$.

The proof for the case $z_0, \tilde{z}_0 \in \tilde{N}_1$ is analogous.

Since $\tilde{N}_j$ is compact, we obtain from Claim 1 that $\tilde{Q}\big|_{\tilde{N}_j} : \tilde{N}_j \to \tilde{Q}(\tilde{N}_j)$ is a homeomorphism ($j = 0, 1$), which is the first part of the proposition.

Claim 2 $\tilde{N}_0 \cup \tilde{N}_1 \subset \operatorname{int}(\tilde{Q}(\tilde{N}_j))$.

Proof We set $R_0 := \frac{r_{\min}}{16p_1}\min\{\gamma_2, 1\}$; then Proposition 13.1, (f) and (13.15) show

$$\tilde{N}_0 \cup \tilde{N}_1 \subset \overline{U_{R_0}(0)}, \text{ and } R_0 \le \frac{r_{\max}}{16p_1} \le \frac{\delta_2|\mathrm{pr}_2\xi|}{16}. \tag{15.11}$$

Further, we set $R_1 := \min\{\frac{\gamma_2}{8p_1}r_{\min}, \frac{\delta_2|\mathrm{pr}_2\xi|}{4}\}$, so $R_1 > R_0$.

Now if $z_0 \in \partial\tilde{N}_j$ (the boundary of $\tilde{N}_j$ in $\mathbb{C}$) for $j = 0$ or $j = 1$, then $(0_<, 0_\mathbb{R}, z_0) \in \partial_1 N_j$, with $\partial_1 N_j$ as in the proof of Lemma 14.1. We then see from (14.6) and (14.7) (for the special case $\lambda = 1$) that

$$|\tilde{Q}(z_0)| \ge R_1, \tag{15.12}$$

which shows that $\tilde{Q}(\partial\tilde{N}_j) \cap U_{R_1}(0) = \emptyset$, and from (2.9) we know that $\tilde{Q}(\partial\tilde{N}_j) = \partial(\tilde{Q}(\tilde{N}_j))$, so we obtain $\partial(\tilde{Q}(\tilde{N}_j)) \cap B(0; R_1) = \emptyset$ ($j = 0, 1$), and hence, in order to prove

$$\tilde{Q}(\tilde{N}_j) \supset B(0; R_1) \supset \overline{B(0; R_0)} \supset \tilde{N}_0 \cup \tilde{N}_1, \tag{15.13}$$

it suffices to show

$$0 \in \tilde{Q}(\tilde{N}_j), \ \ j = 0, 1. \tag{15.14}$$

Proof of (15.14) *for* $j = 0$. The number $\bar{\psi} := \psi_u + \frac{v_0}{v}(\phi_i - \phi_1 - 2k^*\pi)$ lies in J_0, and the number $\bar{r}_2 := R\exp[\frac{u_0}{v_0}(\bar{\psi} - \psi_u)]$ lies in $\mathcal{R}(\bar{\psi})$ (see 13.8), so the complex number

$\bar{z}_0 := \bar{r}_2 e^{i\bar{\psi}}$ lies in $\tilde{N}_0$. One has

$$\tau(|\bar{z}_0|) = \frac{1}{u_0}\log(R/\bar{r}_2) = \frac{1}{u_0}\frac{u_0}{v_0}(\psi_u - \bar{\psi}) = \frac{1}{v}(\phi_1 + 2k^*\pi - \phi_i),$$

so $\phi_i + v\tau(|\bar{z}_0|) = \phi_1 + 2k^*\pi$, and hence (compare 11.18)

$$L^{\perp}\left(e^{i(\phi_i + v\tau(|\bar{z}_0|))}\right) = L^{\perp}\left(e^{i(\phi_1 + 2k^*\pi)}\right) = L^{\perp}\left(e^{i\phi_1}\right) = 0.$$

Further, $\bar{\psi} + v_0\tau(|z_0|) - \psi_u = \bar{\psi} + \psi_u - \bar{\psi} - \psi_u = 0$, so formula (15.2) shows $\tilde{Q}(\bar{z}_0) = 0$.

The proof of (15.14) for the case $j = 1$ is analogous.

Now (15.13), and hence Claim 2 (the remaining part of the proposition) are proved.

We are now ready to prove a symbolic dynamics result for the map P, with the obvious consequences for the dynamics of the map $\Sigma_1 \circ \Sigma_0$, and thus for the state-dependent delay equation (3.8) from Theorem 9.2.

Theorem 15.3 *(a) The map $P = P_1 \circ P_0$ has symbolic dynamics w.r. to the two sets N_0, N_1 in the sense of Corollary 2.4.*

(b) The same is true for the map $\Sigma_1 \circ \Sigma_0$ and the sets $\mathbf{C}_i(N_0), \mathbf{C}_i(N_1)$.

(c) In particular, to every periodic symbol sequence in $\{0,1\}^{\mathbb{Z}}$ there exists a corresponding periodic solution of equation (3.8) (see Corollary 9.3) with phase curve orbitally close to the image of the homoclinic phase curve (i.e., to $\left\{h_t \;\middle|\; t \in \mathbb{R}\right\}$), and passing through $\mathbf{C}_i(N_0), \mathbf{C}_i(N_1)$ according to the periodic pattern.

Proof Ad (a): Clearly, $\tilde{N}_j$ is homeomorphic to a closed two-dimensional ball, $j = 0, 1$. From Proposition 15.2 and Lemma 2.6 we obtain that for m and $\mathbf{s}$ as above, $\operatorname{ind}(\tilde{Q}^m, \tilde{N}_{\mathbf{s},\tilde{Q}}) = \pm 1$. Using Proposition 15.1, we obtain property (15.1). Now Corollary 2.4 and Lemma 14.1 show the symbolic dynamics result for the map P.

Parts (b) and (c) are obvious from the relation between P_0 and Σ_0, respectively P_1 and Σ_1, and from the constructions of Σ_1 and Σ_0 via stopping times and the semiflow F generated by equation (3.8) in Sects. 10 and 11.

Remark (a) One sees from the construction of the sets N_0 and N_1, in particular from the choice of the number $k^* \in \mathbb{N}$, that a whole sequence of such sets N_0^k, N_1^k can be found, corresponding to all $k \geq k^*$. Thus, in the homoclinic situation, a countable sequence of such subsets containing symbolic dynamics as described in the above theorem exists. One could then also study trajectories of P moving between different $N_j^k,\ j = 0, 1,\ k \geq k^*$, analogous to considerations in [12]. We do not pursue this.

(b) It is essentially clear that nearby equations will give rise to nearby return maps $\tilde{P}$ (at least C^0−close to P). Thus, given particular sets N_0, N_1 as above, it follows from robustness of the fixed point index that $\tilde{P}$ will also have symbolic dynamics on $N_0 \cup N_1$. Note, however, that the perturbation arguments for Poincaré maps as given in [8] in a C^1-setting do not apply to the case of state-dependent delay equations.

(c) It would probably be possible to replace the use of the topological method for the construction of a semi-conjugacy to a symbol shift by purely analytical techniques - but at the expense of considerable technical effort. We also feel that the topological approach captures the essential reasons for the presence of the chaotic motion more clearly. For similar reasons, a mixed topological-analytical technique was chosen in [7], in a situation analogous to the classical Shilnikov result in dimension three. (Intermediate value theorem and implicit function theorem for forward symbol sequences,then compactness arguments for backward symbol sequences.) The use of the intermediate value theorem was possible because the unstable direction was one-dimensional. In the situation of the present paper, the gain of proof economy by the topological method is more significant, due to the higher dimension (two) of the unstable manifold.

It is true that analytical methods may yield a complete description of the whole invariant set of P in suitable subsets of its domain, which cannot be achieved via fixed-point index methods.

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