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Multi-type Entire Solutions in a Nonlocal Dispersal Epidemic Model

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Abstract This paper deals with entire solutions of a nonlocal dispersal epidemic model. Unlike local (random) dispersal problems, a nonlocal dispersal operator is not compact and the solutions of nonlocal dispersal system studied here lack regularity in suitable spaces, which affects the uniform convergence of the solution sequences and the technique details in constructing the entire solutions. In the monostable case, some new types of entire solutions are constructed by combining leftward and rightward traveling fronts with different speeds and a spatially independent solution. In the bistable case, the existence of many different entire solutions with merging fronts are proved by constructing different sub- and super-solutions. Various qualitative features of the entire solutions are also investigated. A key idea is to characterize the asymptotic behaviors of the traveling wave solutions at infinite in terms of appropriate sub- and super-solutions. Finally, we also obtain the smoothness of the entire solutions in space, i.e., the solutions established in our paper are global Lipschitz continuous in space.

Keywords Entire solutions · Nonlocal dispersal · Epidemic model · Traveling wave solutions · Asymptotic behavior

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1 Introduction and Main Results

The spatial spread of epidemics is an important subject in mathematical epidemiology. In order to model the cholera epidemic which spread in the European Mediterranean regions in

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1973, Capasso and Paveri-Fontana [9] proposed a system of two ordinary differential equations. Furthermore, Capasso and Maddalena [7] considered the spatial mobile and assumed that the bacteria disperse randomly while the small mobility of the infective human population is neglected, they obtained the following reaction-diffusion system

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = d\frac{\partial^2 u}{\partial x^2}(x,t) - a_{11}u(x,t) + a_{12}v(x,t), \\ \frac{\partial v}{\partial t}(x,t) = -a_{22}v(x,t) + g(u(x,t)), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, d, a_{11}, a_{12} and a_{22} are all positive constants. The variables $u(x,t)$ and $v(x,t)$ respectively stand for the spatial densities of the infectious agent and the infective human population at location x and time t, the parameter d denotes the diffusion coefficient of the agent, $1/a_{11}$ is the mean lifetime of the agent in the environment, $1/a_{22}$ is the infectious period of the infective human, a_{12} is the multiplicative factor of the infectious agent due to the human populations, and $g(u)$ is the infection rate of human under the assumption that total susceptible human population is constant during the evolution of the epidemic.

Traveling wave solutions of the system (1.1) have been widely studied. For example, Zhao and Wang [51] established the existence of monotone traveling waves and the minimal wave speed of (1.1) with *monostable* nonlinearity. Xu and Zhao [47] proved the existence, uniqueness and global exponential stability of traveling waves of (1.1) with *bistable* nonlinearity. For more related works, we refer to [5–8].

It is well known that traveling wave solutions are only special examples of the so-called entire solutions which are defined in the whole space and for all time $t \in \mathbb{R}$. The study on entire solutions is crucial and significant in the following sense: (i) From the dynamical system point of view, entire solutions can help us for the mathematical understanding of transient dynamics, and has the implication that dynamics of two solutions can have distinct histories in the configuration, though their asymptotic profiles as $t \to +\infty$ coincide [32]. Moreover, it can help us fully understand the structures of the global attractors which consist of entire solutions. (ii) From the viewpoint of biology, the entire solutions provide some new spread and invasion ways of the epidemic and species, see [29,32]. In the recent years, there are many works devoted to the entire solutions of scalar reaction-diffusion equations with and without delays [11,13,14,18,20,22,23,27,32,40,48], lattice differential equations [41], nonlocal dispersal equations [26,38], reaction-advection-diffusion equations in cylinders [28,30], and reaction-diffusion systems [21,29,33,39,44–46]. More recently, Wu [44] and Wu et al [46] studied the entire solutions of system (1.1) with bistable and monostable nonlinearity, respectively.

Note that the Laplacian operator which is used to describe the diffusion of the infectious agent in (1.1) only depicts a local and short range diffusion process. However, in reality, the migration or diffusion of the individuals are not just limited in a local or short range, see e.g. Lee et al. [25] and Murray [34]. So it is not enough or very accurate to formulate the diffusion of individuals in a long range by Laplacian operator. One method in overcoming the shortcoming of the Laplacian operator is to describe these models concerning with the spatial migration by the following nonlocal operator

$$(\mathcal{D}u)(x,t) = (J * u)(x,t) - u(x,t) = \int_{\mathbb{R}} J(x-y)[u(u,t) - u(x,t)]dy.$$

Taking this fact into account,we propose the following nonlocal dispersal epidemic system:

$$\begin{cases} u_t(x,t) = d(J * u(x,t) - u(x,t)) - a_{11}u(x,t) + a_{12}v(x,t), \\ v_t(x,t) = -a_{22}v(x,t) + g(u(x,t)), \end{cases} \tag{1.2}$$

The meaning of parameters here are same as in system (1.1), and $(J * u - u)$ is nonlocal dispersal operator which is used to describe the diffusion of the infectious agent.

In view of the great significance of the entire solutions, it is a very interesting and important problem to model the spread process of the epidemic. The dynamics of the process in mathematically characterized by traveling wave solutions or entire solutions. So the first purpose of this paper is to provide many different spread ways of the epidemic. That is to say, we shall establish some different types of entire solutions of (1.2) with monostable and bistable nonlinearity, respectively. The second purpose is to obtain a smooth property of the entire solutions since a lack of regularizing effect occurs in nonlocal dispersal system (1.2).

Mathematically, it suffices to study the following rescaled system

$$\begin{cases} u_t(x,t) = d(J * u(x,t) - u(x,t)) - u(x,t) + \alpha v(x,t), \\ v_t(x,t) = -\beta v(x,t) + g(u(x,t)), \end{cases} \tag{1.3}$$

where $\alpha = a_{12}/a_{11}^2$ and $\beta = a_{22}/a_{11}$.

We first list some assumptions on the functions J and g which are needed throughout this paper.

- **(J1)** $J \in C^1(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(x)dx = 1$, and $\int_{\mathbb{R}} J(x)e^{-\lambda x}dx < +\infty$, $\forall \lambda > 0$.
- **(J2)** J is compactly supported and $M := \sup\big\{ \mid y \mid : y \in supp(J)\big\} > 0$.
- **(GM)** $g \in C^2(\mathbb{R}, \mathbb{R})$ and there exists a constant $K_1 > 0$ such that $g(0) = \alpha g(K_1) - \beta K_1 = 0$. Moreover, $g'(0) > \frac{\beta}{\alpha} > g'(K_1)$, $g'(u) > 0$ for $u \geq 0$, $\frac{\beta}{\alpha}u < g(u)$ for $u \in (0, K_1)$, and $g'(u) \leq g'(0)$ for $u \in [0, +\infty)$ (*Monostable*).
- **(GB)** $g \in C^2(\mathbb{R}, \mathbb{R})$, $g(0) = 0$, $g'(0) \geq 0$, $g'(u) > 0$ for $u > 0$, $\lim_{u\to+\infty} g(u) = 1$, and there is a $u_0 > 0$ such that $g''(u) > 0$ for $u \in (0, u_0)$ and $g''(u) < 0$ for $u > u_0$. Furthermore, $g'(0) < \frac{\beta}{\alpha} < \gamma$, where γ is a positive constant such that the equation $g(u) = \gamma u$ has one and only one positive root (*Bistable*).

If g satisfies (GM), we obtain a monostable case, then the diffusion-free system of (1.3) admits only an unstable equilibrium $E^- = \mathbf{0} = (0, 0)$ and a stable equilibrium $E^+ = \mathbf{K} = (K_1, K_2)$, where $K_1 = \alpha K_2$. If g satisfies (GB), we obtain a bistable case, and the diffusion-free system of (1.3) has three equilibria $E_0 = (0, 0)$, $E_1 = (u_1^*, v_1^*)$ and $E_2 = (u_2^*, v_2^*)$, where $g(u_i^*) = (\beta/\alpha)u_i^*$ and $u_i^* = \alpha v_i^*$, $i = 1, 2$. E_1 is a saddle point, E_0 and E_2 are stable nodes.

Hereafter, a solution $w(x,t) := (u(x,t), v(x,t))$ of (1.3) is called a traveling wave solution connecting E_i and $E_j (i \neq j)$ with speed c, if $(u(x,t), v(x,t)) = (\phi_c(\xi), \psi_c(\xi))$, $\xi = x + ct$ for some function $(\phi_c, \psi_c) \in C^1(\mathbb{R}, \mathbb{R}^2)$ which satisfies

$$\begin{cases} c\phi_c'(\xi) = d(J * \phi_c(\xi) - \phi_c(\xi)) - \phi_c(\xi) + \alpha\psi_c(\xi), \\ c\psi_c'(\xi) = -\beta\psi_c(\xi) + g(\phi_c(\xi)), \end{cases} \tag{1.4}$$

and

$$\lim_{\xi\to-\infty} (\phi_c(\xi), \psi_c(\xi)) = E_i, \quad \lim_{\xi\to+\infty} (\phi_c(\xi), \psi_c(\xi)) = E_j. \tag{1.5}$$

Moreover, we say (ϕ_c, ψ_c) is a traveling (wave) front if (ϕ_c, ψ_c) is monotone.

Since system (1.4) can be decoupled by solving the second equation and transformed into the following scalar integro-differential equation

$$c\phi_c'(\xi) = d(J * \phi_c(\xi) - \phi_c(\xi)) - \phi_c(\xi) + \frac{\alpha}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi - s)} g(\phi_c(s))ds. \tag{1.6}$$

In order to consider the traveling fronts of (1.3) satisfies (1.5), it suffices to consider the monotone solutions of (1.6) subject to (1.5).

In the recent years, there are many works devoted to the traveling wave solutions for variaties of nonlocal dispersal equations with monostable and bistable nonlinearity. For monostable case, our Eq. (1.6) satisfies the conditions of Pan [35] under the assumptions (J1) and (GM), so it is easy to prove that there exists a number $c^* > 0$ such that (1.3) has a traveling front $\Phi_c(\cdot) = (\phi_c(\cdot), \psi_c(\cdot))$ connecting E^- and E^+ for any $c \geq c^*$, also see [49,50]. For bistable case, from the abstract theory established by Bates et al [3] and Chen [12], see also Fang and Zhao [16], we know that under the assumptions (J1)–(J2) and (GB), there exists a unique constant $c \in \mathbb{R}$ such that (1.3) has a traveling front $\Phi(\cdot) = (\phi(\cdot), \psi(\cdot))$ connecting E_0 and E_2 with speed c. Moreover, if we restrict $g(u)$ in the interval $[0, u_1^*]$ or $[u_1^*, u_2^*]$, then system (1.3) can be regarded as a monostable system in $[0, u_1^*] \times [0, v_1^*]$ or $[u_1^*, u_2^*] \times [v_1^*, v_2^*]$. Assume further the following

(GBS) $g(u) \leq g(u_1^*) + g'(u_1^*)(u - u_1^*)$ for $u \in [u_1^*, u_2^*]$ and $g(u) \geq g(u_1^*) + g'(u_1^*)(u - u_1^*)$ for $u \in [0, u_1^*]$.

Thus there exists $c_1^* < 0$ and $c_2^* > 0$ such that (1.3) has two families of traveling fronts $\Phi_1(\cdot) = (\phi_1(\cdot), \psi_1(\cdot))$ connecting $(0, 0)$ with (u_1^*, v_1^*) for any $c_1 \leq c_1^*$ and $\Phi_2(\cdot) = (\phi_2(\cdot), \psi_2(\cdot))$ connecting (u_1^*, v_1^*) and (u_2^*, v_2^*) for any $c_2 \geq c_2^*$, respectively.

In the *monostable* case, since (1.3) allows a continuous family of traveling fronts $\Phi_c(x+ct)$ with different speeds, we construct a family of entire solutions of (1.3) by a combination of traveling fronts with different speeds and spatially independent solutions. In the *bistable* case, we explore some new types of entire solutions to (1.3). The first type is characterized by two monostable fronts $\Phi_1(x + c_1 t)$ and $\Phi_2(x + c_2 t)$ emerge from the left axis and right axis respectively. The second type is constructed from a monostable front $\Phi_1(-x + c_1 t)$ merging with a bistable front $\Phi(x + ct)$ under the assumption $c > -c_1$. The last type behaves like two traveling fronts $\Phi(-x + ct)$ and $\Phi(x + ct)$ propagating from both sides of the x-axis and annihilating at a finite time.

The basic idea is to use traveling fronts to build different sub and supersolutions of (1.3) and then deduce the existence of entire solutions trapped between these sub and supersoutions. Although the basic idea is similar to the works [21,29,30,32,33,44,46], the technique details are different. For example, in our system (1.3), since a lack of regularizing effect occurs in u-equation due to the nonlocal dispersal and in v-equation due to the zero diffusion coefficient, the solution sequences $\{w_n(x, t)\}$ of the Cauchy problem for (1.3) are not smooth enough with respect to x, hence its uniform convergence is not ensured. In order to obtain the continuous entire solutions with respect to t and x, we have to make $\{w_n(x, t)\}$ possess a property which is similar to a global Lipschitz condition with respect to x (see Lemma 6.1). A similar method was successfully applied in the work [46] to partially degenerate reaction-diffusion systems.

Now we state our main results as follows.

Theorem 1.1 *Assume that* (J1) *and* (GM) *hold. Let* $\Phi_{c_i} = (\phi_{c_i}, \psi_{c_i})$ *be the traveling fronts of* (1.3) *connecting* $\mathbf{0}$ *and* $\mathbf{K}$ *with* $c_i \geq c^*$, $i = 1, 2$. *Then for any* $\theta_1, \theta_2 \in \mathbb{R}$, *system* (1.3) *possesses an entire solution* $W_{c_1,c_2,\theta_1,\theta_2}(x, t) = (u(x, t), v(x, t)) : \mathbb{R}^2 \to [0, K_1] \times [0, K_2]$ *such that*

(i) $\frac{\partial}{\partial t} W_{c_1,c_2,\theta_1,\theta_2}(x, t) > 0$ *for any* $(x, t) \in \mathbb{R}^2$.

(ii)

$$\lim_{t\to-\infty}\sup_{x\ge 0}\| W_{c_1,c_2,\theta_1,\theta_2}(x,t)-\Phi_{c_1}(x+c_1t+\theta_1)\|=0,$$
$$\lim_{t\to-\infty}\sup_{x\le 0}\| W_{c_1,c_2,\theta_1,\theta_2}(x,t)-\Phi_{c_2}(-x+c_2t+\theta_2)\|=0.$$

(iii) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\|W_{c_1,c_2,\theta_1,\theta_2}(x,t)-\mathbf{K}\|=0.$

(iv) *For any* $x_1 < x_2$, $\lim_{t\to-\infty}\sup_{x\in[x_1,x_2]}\|W_{c_1,c_2,\theta_1,\theta_2}(x,t)\|=0.$

(v) *For any* $t_0\in\mathbb{R}$, $\lim_{|x|\to+\infty}\sup_{t\in[t_0,+\infty)}\|W_{c_1,c_2,\theta_1,\theta_2}(x,t)-\mathbf{K}\|=0.$

We have construct some new entire solutions connecting two traveling fronts in Theorem 1.1. Next, we consider any combination of traveling fronts and the spatially independent solutions to construct some new entire solutions. The existence of the spatially independent solution of (1.3) follows from Wu [46, Theorem 2].

Proposition 1.2 *Assume* (GM) *holds. Then system* (1.3) *has a spatially independent solution* $\Gamma(t)=(\Gamma_1(t),\Gamma_2(t))$ *which satisfies*

$$\Gamma(-\infty)=\mathbf{0},\quad \Gamma(+\infty)=\mathbf{K},\quad \Gamma'(t)\gg 0,\quad \lim_{t\to-\infty}\Gamma(t)e^{-\lambda^*t}=(1,b^*),\quad \Gamma(t)\le(1,b^*)e^{\lambda^*t},$$

for all $t\in\mathbb{R}$*, where* $b^*=g'(0)/(\beta+\lambda^*)$ *and* λ^* *is the positive real root of the equation* $\lambda^2+(\beta+1)\lambda+\beta-\alpha g'(0)=0.$

For convenience, we define

$$\max\{w_1,w_2\}=(\max\{u_1,u_2\},\max\{v_1,v_2\}),\ \min\{w_1,w_2\}=(\min\{u_1,u_2\},\min\{v_1,v_2\}),$$

for $w_1=(u_1,v_1)$ and $w_2=(u_2,v_2)$.

Theorem 1.3 *Assume* (J1) *and* (GM) *hold. Let* $\Phi_{c_i}=(\phi_{c_i},\psi_{c_i})$ *be the traveling fronts of* (1.3) *connecting* $\mathbf{0}$ *and* $\mathbf{K}$ *with* $c_i\ge c^*$, $i=1,2$, *and* $\Gamma(t)=(\Gamma_1(t),\Gamma_2(t))$ *be the spatially independent solution of* (1.3) *described as in Proposition 1.2. Then for any given* $c_1,c_2\ge c^*$, $\theta_1,\theta_2,\theta_3\in\mathbb{R}$ *and* $\chi_1,\chi_2\in\{0,1\}$ *with* $\chi_1+\chi_2\ge 1$, *system* (1.3) *possesses an entire solution* $W_\Gamma(x,t)=(u(x,t),v(x,t)):\mathbb{R}^2\to[0,K_1]\times[0,K_2]$ *such that*

$$\max\{\chi_1\Phi_{c_1}(x+c_1t+\theta_1),\chi_2\Phi_{c_2}(-x+c_2t+\theta_2),\Gamma(t+\theta_3)\}$$
$$\le W_\Gamma(x,t)\le\min\left\{\chi_1\Phi_{c_1}(x+p_1(t))+\chi_2\Phi_{c_2}(-x+p_2(t))+(1,b^*)e^{\lambda^*(t+\theta_3)},\mathbf{K}\right\}$$

for $(x,t)\in\mathbb{R}\times(-\infty,0]$*, where* $p_i(t)(i=1,2)$ *with* $0<p_i(t)-c_it-\theta_i\le R_0e^{\nu t}$ *are monotone increasing functions on* $(-\infty,T]$, $T\le 0$, $R_0>0$, $\nu>0$ *are constants. Moreover, the assertions (i) and (iii)–(v) in Theorem 1.1 still hold for* $W_\Gamma(x,t)$ *as for* $W_{c_1,c_2,\theta_1,\theta_2}(x,t)$.

Remark 1 Notice that the entire solutions established in Theorem 1.1 are completely different from those of Theorem 1.3 since $W_{c_1,c_2,\theta_1,\theta_2}(x,t)$ and $W_\Gamma(x,t)$ have different decay rate when $t\to-\infty$ due to $\lambda^*<c\lambda_1(c)$ for any $c\ge c^*$ (see Lemma 3.2).

For bistable case, we obtain the following several different types of entire solutions by considering a combination of the traveling fronts Φ, Φ_1 and Φ_2.

Theorem 1.4 *Assume that* (J1)–(J2) *and* (GB)–(GBS) *hold. Let* $\Phi_1(\cdot)$ *be the traveling front connecting* E_0 *and* E_1 *with speed* $c_1 \le c_1^*$, $\Phi_2(\cdot)$ *be the traveling front connecting* E_1 *and* E_2 *with speed* $c_2 \ge c_2^*$, *and* $\Phi(\cdot)$ *be the traveling front connecting* E_0 *and* E_2 *with speed* $c > 0$. *Then* (1.3) *has an entire solution* $W_1(x,t) = (U_1(x,t), V_1(x,t)) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ *such that*

$$\lim_{t\to-\infty}\left\{\sup_{x\le\frac{(c_1+c_2)t}{2}} \| W_1(x,t) - \Phi_1(x+c_1t) \| + \sup_{x\ge\frac{(c_1+c_2)t}{2}} \| W_1(x,t) - \Phi_2(x+c_2t) \|\right\} = 0.$$

Theorem 1.5 *Let all the assumptions of Theorem 1.4 be satisfied. Then* (1.3) *possesses an entire solution* $W_2(x,t) = (U_2(x,t), V_2(x,t)) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ *such that*

$$\lim_{t\to-\infty}\left\{\sup_{x\le\frac{(c_1+c_2)t}{2}} \| W_2(x,t) - \Phi_2(-x+c_2t) \| + \sup_{x\ge\frac{(c_1+c_2)t}{2}} \| W_2(x,t) - \Phi_1(-x+c_1t) \|\right\} = 0,$$

Remark 2 Note that for this nonlocal dispersal system (1.3), there is no related results like Theorem 3.1 of [17] and Theorem 3.5 of [36] which is similar to a stability property of the traveling fronts. So we can not obtain the convergence of the entire solutions established in Theorems 1.4 and 1.5 to the bistable front $\Phi(x+ct)$ as $t \to +\infty$.

Theorem 1.6 *Let all the assumptions of Theorem 1.4 be satisfied. If* $c > -c_1$, *then* (1.3) *admits an entire solution* $W_3(x,t) = (U_3(x,t), V_3(x,t)) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ *satisfying*

$$\lim_{t\to-\infty}\left\{\sup_{x\le\frac{(c_1+c)t}{2}} \| W_3(x,t) - \Phi_1(-x+c_1t) \| + \sup_{x\ge\frac{(c_1+c)t}{2}} \| W_3(x,t) - \Phi(x+ct) \|\right\} = 0.$$

Moreover, for any given $a > 0$,

$$\lim_{t\to+\infty}\inf_{x\in\mathbb{R}} \|W_3(x,t) - E_1\| = 0 \ \textit{and} \ \lim_{t\to+\infty}\sup_{x\in[-a,+\infty)} \|W_3(x,t) - E_2\| = 0.$$

Theorem 1.7 *Assume that* (J1)–(J2) *and* (GB) *hold. Let* $\Phi(\cdot)$ *be the traveling front of* (1.3) *connecting* E_0 *and* E_2 *with* $c > 0$. *Then* (1.3) *admits an entire solution* $W_4(x,t) = (U_4(x,t), V_4(x,t)) : \mathbb{R}^2 \to [0, u_2^*] \times [0, v_2^*]$ *satisfying*

$$\lim_{t\to-\infty}\left\{\sup_{x\le 0} \| W_4(x,t) - \Phi(-x+ct) \| + \sup_{x\ge 0} \| W_4(x,t) - \Phi(x+ct) \|\right\} = 0.$$

Moreover, the following properties hold:

(i) $\frac{\partial W_4}{\partial t}(x,t) > 0$ *for all* $(x,t) \in \mathbb{R}^2$.
(ii) $\lim_{t\to-\infty} \| W_4(x,t) \| = 0$ *locally in* $x \in \mathbb{R}$.
(iii) $\lim_{t\to+\infty} \|W_4(x,t) - E_2\| = 0$ *for all* $(x,t) \in \mathbb{R}^2$.

We would like to point out that when the speed of the traveling fronts $\Phi(x+ct)$ connecting E_0 and E_2 is negative, i.e. $c < 0$, we can obtain similar results on entire solutions as in Theorems 1.4–1.7. We leave the details to the readers. However, we can not deal with the case $c = 0$ in this paper.

Remark 3 From the viewpoint of diseases transmission, the entire solutions established in Theorems 1.1–1.7 represent some different spread ways of the epidemic. For example, the entire solution in Theorem 1.1 can be viewed as the infectious agent spread from the both sides of the living areas of human population as $t \to -\infty$, and then tends to the positive stable state as $t \to +\infty$. That is to say, the disease spread from the both sides of the living areas successfully. In addition, the entire solution established in Theorem 1.6 indicates that the infectious agent and the infective human spread from the both sides of the x-axis in the same directions and finally the faster one might catch the slower one.

Note that the entire solutions of (1.3) established in Theorems 1.1–1.7 are differentiable with respect to t, but it is not smooth enough with respect to x since a lack of regularizing effect occurs in nonlocal dispersal system (1.3). Thus we further prove a smooth property of the entire solutions $w(x,t) = (u(x,t), v(x,t))$ established in Theorems 1.1–1.7 which is similar to global Lipschitz continuous with respect to x under the following assumption.

(H) $\sup_{u\in[0,K_1]} g'(u) < \frac{\beta}{\alpha}(1+d)$ if (GM) holds, and $\sup_{u\in[0,u_2^*]} g'(u) < \frac{\beta}{\alpha}(1+d)$ if (GB) holds.

Theorem 1.8 *Assume that* (J1)–(J2) *and* (H) *hold. Let* $w(x,t) = (u(x,t), v(x,t))$ *be the entire solutions of* (1.3) *established in Theorems 1.1–1.7. Then there exist positive constants* D_1 *and* D_2 *such that for any* $(x,t) \in \mathbb{R}^2$ *and* $\eta > 0$,

$$\|w(x+\eta,t) - w(x,t)\| \le D_1\eta \ \ \text{and} \ \ \left\| \frac{\partial w}{\partial t}(x+\eta,t) - \frac{\partial w}{\partial t}(x,t) \right\| \le D_2\eta. \tag{1.7}$$

We remark that a similar result is firstly established by Li et al. [26] for scalar nonlocal dispersal equations with monostable nonlinearity. See also [38] for the bistable nonlinearity. We extend this result to nonlocal dispersal systems successfully.

The remainder of this paper is organized as follows. In Sect. 2, we make some preparations which are important and necessary in what follows. In Sect. 3, we study the asymptotic behaviors of traveling fronts at infinity since they are essential in the proofs of the main Theorems. Sections 4 and 5 focus on the existence of the desired entire solutions of system (1.3) in monostable and bistable cases respectively by constructing appropriate super and subsolutions. In Sect. 6, we prove Theorem 1.8 with the help of an ordinary differential equation. At last, we finish this article by providing some interesting discussions.

2 Preliminaries

In this section, we will make some preparations for getting our main results latter.

In what follows, we use the usual notations for the standard ordering in $\mathbb{R}^2$. That is, for $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$, we denote $w_1 \le w_2$ if $u_1 \le u_2$ and $v_1 \le v_2$, $w_1 < w_2$ if $w_1 \le w_2$ and $w_1 \ne w_2$, and $w_1 \ll w_2$ if $u_1 < u_2$ and $v_1 < v_2$. If $w_1 < w_2$, we denote $(w_1, w_2) = \{w \in \mathbb{R}^2 : w_1 < w < w_2\}$, $(w_1, w_2] = \{w \in \mathbb{R}^2 : w_1 < w \le w_2\}$, $[w_1, w_2) = \{w \in \mathbb{R}^2 : w_1 \le w < w_2\}$, and $[w_1, w_2] = \{w \in \mathbb{R}^2 : w_1 \le w \le w_2\}$. Let $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$.

Let $X = BUC(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$ with the supermum norm $\|\cdot\|_X$. Let $X^+ = \{w = (u,v) \in X : u(x) \ge 0, v(x) \ge 0, x \in \mathbb{R}\}$. It is easy to see that X^+ is a closed cone of X. For any $w_1, w_2 \in X$, we write $w_1 \le_X w_2$ if $w_2 - w_1 \in X^+$, $w_1 <_X w_2$ if $w_2 - w_1 \in X^+ \backslash \{0\}$, and $w_1 \ll_X w_2$ if $w_2 - w_1 \in Int(X^+)$. For $w_1, w_2 \in X$ with $w_1 \le_X w_2$, we denote $[w_1, w_2]_X = \{w \in X : w_1 \le_X w \le_X w_2\}$.

Now, we consider the following Cauchy problem of (1.3):

$$\begin{cases} u_t(x,t) = d(J*u(x,t) - u(x,t)) - u(x,t) + \alpha v(x,t), & (x,t) \in \mathbb{R} \times (0,+\infty), \\ v_t(x,t) = -\beta v(x,t) + g(u(x,t)), & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

For $w = (u, v) \in X$, we define $T_1(t)u = e^{-(d+1)t}u$, $T_2(t)v = e^{-\beta t}v$. Clearly, $T(t) = (T_1(t), T_2(t))$ is a linear semigroup on X. Moreover, it is clear that (2.1) is equivalent to the following integral equation

$$w(t) = T(t)w_0 + \int_0^t T(t-s)B(w(s))ds,$$

where

$$w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad T(t) = \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix}, \quad w_0(x) = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix},$$

and

$$B(w) = \begin{pmatrix} B_1(w) \\ B_2(w) \end{pmatrix} = \begin{pmatrix} d(J*u) + \alpha v \\ g(u) \end{pmatrix}.$$

Definition 2.1 A continuous function $w = (u, v) : [\tau, T) \to X$, $\tau < T$, is called a supersolution (subsolution) of (1.3) on $[\tau, T)$ if

$$w(t) \geq (\leq) T(t-s)w(s) + \int_s^t T(t-r)B(w(r))dr$$

for any $\tau \leq s < t < T$. A function $w : (-\infty, T) \to X$ is called a supersolution (subsolution) of (1.3) on $(-\infty, T)$, if for any $\tau < T$, w is a supersolution (subsolution) of (1.3) on $[\tau, T)$.

Fortunately, the function $B(w)$ here satisfies the quasi-monotonicity in [31], so by [31, Corollary 5] (taking delay as zero) and [31, Theorem 1], we obtain the following lemma. For convenience, we denote $\mathbf{0} := (0, 0)$ and

$$\tilde{\mathbf{K}} = \begin{cases} (K_1, K_2) & \text{if (GM) holds,} \\ (u_2^*, v_2^*) & \text{if (GB) holds.} \end{cases}$$

Lemma 2.2 *Assume* (J1) *and* (GM) *or* (GB).

(i) *For any* $w_0 \in [\mathbf{0}, \tilde{\mathbf{K}}]_X$, (1.3) *has a unique classical solution* $w(x, t; w_0)$ *on* $(x, t) \in \mathbb{R} \times [0, \infty)$ *with* $w(x, 0; w_0) = w_0(x)$ *and* $\mathbf{0} \leq w(x, t; w_0) \leq \tilde{\mathbf{K}}$ *for* $x \in \mathbb{R}, t \geq 0$.
(ii) *For any pair of supersolution* $w^+(x, t)$ *and subsolution* $w^-(x, t)$ *of* (1.3) *on* $[0, \infty)$ *with* $w^+(x, 0) \geq w^-(x, 0)$ *and* $\mathbf{0} \leq w^-(x, t), w^+(x, t) \leq \tilde{\mathbf{K}}$ *for* $(x, t) \in \mathbb{R} \times [0, \infty)$, *there holds* $\mathbf{0} \leq w^-(x, t) \leq w^+(x, t) \leq \tilde{\mathbf{K}}$ *for all* $(x, t) \in \mathbb{R} \times [0, \infty)$.

3 Asymptotic Behavior of Traveling Fronts

In this section, we will use the method developed by Carr and Chamj [4] and Ikahara's Theorem to obtain the asymptotic behavior of traveling fronts of (1.3). We always assume that J satisfies (J1).

3.1 Monostable Case

In this subsection, we assume that g satisfies (GM). For $c > 0$ and $\lambda \in \mathbb{C}\backslash\{-\beta/c\}$, we define two characteristic functions:

$$\Delta_1(\lambda, c) = d\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}dy - c\lambda - (d+1) + \frac{\alpha g'(0)}{\beta + c\lambda},$$
$$\Delta_2(\lambda, c) = d\int_{-\infty}^{+\infty} J(y)e^{-\lambda y}dy - c\lambda - (d+1) + \frac{\alpha g'(K_1)}{\beta + c\lambda}.$$

By some simple computations, we obtain the following lemma.

Lemma 3.1 (i) *There exist $c^*, \lambda_* > 0$ such that $\Delta_1(\lambda_*, c^*) = 0$ and $\frac{\partial}{\partial\lambda}\Delta_1(\lambda, c)|_{\lambda=\lambda_*} = 0$. Moreover, the equation $\Delta_1(\lambda, c) = 0$ has only two distinct positive real roots $\lambda_1 < \lambda_2$ for $c > c^*$ and has no real root for $c < c^*$.*

(ii) *For any $c > 0$, $\Delta_2(\lambda, c) = 0$ has two distinct real roots $\lambda_3 \in \left(-\frac{\beta}{c}, 0\right)$ and $\lambda_4 \in (0, +\infty)$.*

Lemma 3.2 *For any $c \geq c^*$, there holds $c\lambda_1(c) > \lambda^*$, where $c^*, \lambda_1(c)$ and λ^* are defined as in Lemma 3.1 and Proposition 1.2.*

Proof Note that

$$c\lambda_1(c) - \lambda^* = d\left(\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}dy - 1\right) - 1 + \frac{\alpha g'(0)}{\beta + c\lambda_1(c)} - \lambda^* > -1 + \frac{\alpha g'(0)}{\beta + c\lambda_1(c)} - \lambda^*.$$

If there exists $c_0 \geq c^*$ such that $c_0\lambda_1(c_0) \leq \lambda^*$, then

$$0 \geq c_0\lambda_1(c_0) - \lambda^* > -1 + \frac{\alpha g'(0)}{\beta + c_0\lambda_1(c_0)} - \lambda^* \geq -1 + \frac{\alpha g'(0)}{\beta + \lambda^*} - \lambda^* = 0,$$

which is a contradiction. The proof is complete. □

Next, we provide a technical lemma which is important to obtain the asymptotic behavior of traveling fronts.

Lemma 3.3 (Ikehara [4]) *Let $u(\xi)$ be a positive decreasing function in $\mathbb{R}$ and $F(\lambda) = \int_0^{+\infty} e^{-\lambda\xi}u(\xi)d\xi$, if F can be written as $F(\lambda) = H(\lambda)/(\lambda + \lambda_0)^{k+1}$ for some constants $k > -1$, $\lambda_0 > 0$, and some analytic function H in the strip $-\lambda_0 \leq Re\lambda < 0$, then*

$$\lim_{\xi\to+\infty} \frac{u(\xi)}{\xi^k e^{-\lambda_0\xi}} = \frac{H(-\lambda_0)}{\Gamma(\lambda_0 + 1)}.$$

Theorem 3.4 *Assume that* (J1) *and* (GM) *hold. Let $\Phi_c(\xi) = (\phi_c(\xi), \psi_c(\xi))$ be a monotone increasing traveling wave solution of* (1.3) *connecting* **0** *and* **K** *with speed $c \geq c^*$, then the following statements hold:*

(i) *For $c > c^*$,*

$$\lim_{\xi\to-\infty} \Phi_c(\xi)e^{-\lambda_1\xi} = (1, A_0)a_0, \quad \lim_{\xi\to-\infty} \Phi_c'(\xi)e^{-\lambda_1\xi} = (1, A_0)a_0\lambda_1.$$

For $c = c^*$,

$$\lim_{\xi\to-\infty} \Phi_c(\xi)\xi^{-1}e^{-\lambda_*\xi} = -(1, A_0)a_0, \quad \lim_{\xi\to-\infty} \Phi_c'(\xi)\xi^{-1}e^{-\lambda_*\xi} = -(1, A_0)a_0\lambda_*.$$

(ii) *For* $c \geq c^*$,

$$\lim_{\xi \to +\infty} (\mathbf{K} - \Phi_c(\xi))e^{-\lambda_3 \xi} = (1, A_1)a_1, \quad \lim_{\xi \to +\infty} \Phi_c'(\xi)e^{-\lambda_3 \xi} = -(1, A_1)a_1\lambda_3,$$

where a_0, a_1 *are positive constants which determined by* c, $A_0 = \frac{g'(0)}{\beta + c\lambda_1}$ *for* $c > c^*$, $A_0 = \frac{g'(0)}{\beta + c\lambda_*}$ *for* $c = c^*$, *and* $A_1 = \frac{g'(K_1)}{\beta + c\lambda_3}$.

Proof We only prove the assertion (i), since the assertion (ii) can be shown similarly. The proof is divided into three steps.

Step 1. We prove that there exists $\xi' \in \mathbb{R}$ such that $\phi_c(\xi)$ is integrable on $(-\infty, \xi']$, that is $\int_{-\infty}^{\xi'} \phi_c(\xi)d\xi < +\infty$.

For convenience, we define $f(\phi_c(\xi)) = -\phi_c(\xi) + \frac{\alpha}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi - s)} g(\phi_c(s))ds$, then it follows from (1.6) that $\phi_c(\xi)$ satisfies

$$c\phi_c'(\xi) = d(J * \phi_c(\xi) - \phi_c(\xi)) + f(\phi_c(\xi)). \tag{3.1}$$

Note that $f'(0) = -1 + \frac{\alpha}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi - s)} g'(0)ds = \frac{\alpha}{\beta}g'(0) - 1 > 0$ and $\lim_{\xi \to -\infty} \phi_c(\xi) = 0$, then there exists $\xi' < 0$ small enough such that for any $\xi \leq \xi'$, $\frac{1}{2}f'(0)\phi_c(\xi) \geq K_0\phi_c^2(\xi)$, where $K_0 := \frac{1}{2}\max_{\phi \in [0, K_1]} |f''(\phi)|$. Then according to Taylor's expansion, for any $\xi \leq \xi'$,

$$f(\phi_c(\xi)) = f'(0)\phi_c(\xi) + \frac{f''(s)}{2}\phi_c^2(\xi) \geq f'(0)\phi_c(\xi) - K_0\phi_c^2(\xi) \geq \frac{1}{2}f'(0)\phi_c(\xi),$$

for some $s \in [0, K_1]$. Then for $\xi \leq \xi'$, we conclude from (3.1) that

$$c\phi_c'(\xi) \geq d(J * \phi_c(\xi) - \phi_c(\xi)) + \frac{1}{2}f'(0)\phi_c(\xi). \tag{3.2}$$

Integrating (3.2) from η to ξ with $\eta < \xi \leq \xi'$, we get

$$c(\phi_c(\xi) - \phi_c(\eta)) \geq d\int_{\eta}^{\xi} (J * \phi_c(s) - \phi_c(s))ds + \frac{1}{2}f'(0)\int_{\eta}^{\xi} \phi_c(s)ds. \tag{3.3}$$

Note $\phi_c(-\infty) = 0$, then by Fubini's theorem and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{\eta \to -\infty} \int_{\eta}^{\xi} (J * \phi_c(s) - \phi_c(s))ds$$
$$= - \lim_{\eta \to -\infty} \int_{\eta}^{\xi} \int_{-\infty}^{+\infty} J(y)y \int_0^1 \phi_c'(s - \theta y)d\theta dy ds = -\int_{-\infty}^{+\infty} J(y)y \int_0^1 \phi_c(\xi - \theta y)d\theta dy.$$

Letting $\eta \to -\infty$ in (3.3), we have

$$c\phi_c(\xi) + d\int_{-\infty}^{+\infty} J(y)y \int_0^1 \phi_c(\xi - \theta y)d\theta dy \geq \frac{1}{2}f'(0)\int_{-\infty}^{\xi} \phi_c(s)ds, \tag{3.4}$$

which shows that $\phi_c(\xi)$ is integrable on $(-\infty, \xi']$.

Step 2. Next we will show that there exists a constant $\gamma > 0$ such that $\phi_c(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$. Define $U(\xi) = \int_{-\infty}^{\xi} \phi_c(s)ds$, it is easy to see that $U(\xi)$ is a well-defined non-decreasing smooth function with $U(-\infty) = 0$. First we prove $U(\xi)$ is integrable on $(-\infty, \xi']$. Integrating (3.2) from $-\infty$ to ξ, we get

$$c\phi_c(\xi) \geq d(J * U(\xi) - U(\xi)) + \frac{1}{2}f'(0)U(\xi). \tag{3.5}$$

Then integrating (3.5) from $-\infty$ to ξ again, there is

$$cU(\xi) \geq d \int_{-\infty}^{\xi} (J * U(s) - U(s))ds + \frac{1}{2} f'(0) \int_{-\infty}^{\xi} U(s)ds.$$

Since $U(\xi)$ is increasing and $U(-\infty) = 0$, for $\xi \leq \xi'$ we get

$$\begin{aligned}\int_{-\infty}^{\xi} (J * U(s) - U(s))ds &= \int_{-\infty}^{\xi} \int_{\mathbb{R}} J(y)[U(s-y) - U(s)]dyds \\ &= \int_{-\infty}^{\xi} \int_{0}^{+\infty} J(y)[(U(s+y) - U(s)) - (U(s) - U(s-y))]dyds \\ &= \int_{0}^{+\infty} J(y) \left[\int_{\xi}^{\xi+y} U(s)ds - \int_{\xi-y}^{\xi} U(s)ds \right] dy \geq 0.\end{aligned}$$

So

$$cU(\xi) \geq \frac{1}{2} f'(0) \int_{-\infty}^{\xi} U(s)ds \quad \text{for } \xi \leq \xi'. \tag{3.6}$$

Thus, $U(\xi)$ is integrable on $(-\infty, \xi']$. In view of U is non-negative and increasing, then for any $r > 0$ and $\xi \leq \xi'$,

$$cU(\xi) \geq \frac{1}{2} f'(0) \int_{-\infty}^{\xi} U(s)ds \geq \frac{1}{2} f'(0) \int_{\xi-r}^{\xi} U(s)ds \geq \frac{1}{2} f'(0) r U(\xi - r).$$

Choose $r_0 > 0$ sufficiently large such that $\theta_0 := \frac{2c}{f'(0)r_0} \in (0, 1)$, then $U(\xi - r_0) \leq \theta_0 U(\xi)$, $\xi \leq \xi'$. Define $\tilde{U}(\xi) = U(\xi)e^{-\gamma\xi}$, where $\gamma = \frac{1}{r_0} \ln \frac{1}{\theta_0}$, then for any $\xi \leq \xi'$,

$$\tilde{U}(\xi - r_0) = U(\xi - r_0)e^{-\gamma(\xi - r_0)} = \frac{1}{\theta_0} U(\xi - r_0)e^{-\gamma\xi} \leq U(\xi)e^{-\gamma\xi} = \tilde{U}(\xi).$$

Therefore,

$$0 \leq \tilde{U}(\xi) \leq K' := \max \left\{ \tilde{U}(s) | s \in [\xi' - r_0, \xi'] \right\} \quad \text{for } \xi \leq \xi',$$

which implies that $U(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$.

In view of $g(\phi_c) \leq g'(0)\phi_c$ for $\phi_c \in (0, K_1)$ and $\phi_c(\cdot)$ is nondecreasing, we get

$$c\phi_c'(\xi) \leq d(J * \phi_c(\xi) - \phi_c(\xi)) - \phi_c(\xi) + \frac{\alpha}{\beta} g'(0)\phi_c(\xi). \tag{3.7}$$

Integrating (3.7) from $-\infty$ to ξ, $\xi \leq \xi'$, one has

$$c\phi_c(\xi) \leq d(J * U(\xi) - U(\xi)) - U(\xi) + \frac{\alpha}{\beta} g'(0)U(\xi), \tag{3.8}$$

According to (J1) and $U(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$, we have

$$J * U(\xi) = \int_{\mathbb{R}} J(y)U(\xi - y)dy = O(e^{\gamma\xi}) \quad \text{as} \quad \xi \to -\infty.$$

Thus (3.8) implies that $\phi_c(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$.

Step 3. In the following, we prove the main results of this theorem. Based on the discussions above, for $\lambda \in \mathbb{C}$ with $0 < \text{Re}\lambda < \gamma$, we can define a two-sided Laplace transform of ϕ_c by

$$\mathcal{L}(\lambda) = \int_{-\infty}^{+\infty} \phi_c(\xi)e^{-\lambda\xi} d\xi.$$

Rewrite equation (1.6) as

$$d\left(J*\phi_c(\xi)-\phi_c(\xi)\right)-c\phi_c'(\xi)-\phi_c(\xi)+\frac{\alpha g'(0)}{c}\int_{-\infty}^{\xi}e^{-\frac{\beta}{c}(\xi-s)}\phi_c(s)ds$$
$$=\frac{\alpha}{c}\int_{-\infty}^{\xi}e^{-\frac{\beta}{c}(\xi-s)}\left[g'(0)\phi_c(s)-g(\phi_c(s))\right]ds. \tag{3.9}$$

Note that

$$\int_{\mathbb{R}}e^{-\lambda\xi}(J*\phi_c(\xi))d\xi=\int_{\mathbb{R}}e^{-\lambda y}J(y)\int_{\mathbb{R}}\phi_c(\xi-y)e^{-\lambda(\xi-y)}d\xi dy=\mathcal{L}(\lambda)\int_{\mathbb{R}}J(y)e^{-\lambda y}dy,$$

and

$$\frac{\alpha g'(0)}{c}\int_{-\infty}^{+\infty}e^{-\lambda\xi}\int_{-\infty}^{\xi}e^{-\frac{\beta}{c}(\xi-s)}\phi_c(s)dsd\xi=\frac{\alpha g'(0)}{\beta+c\lambda}\mathcal{L}(\lambda).$$

Multiply both sides of (3.9) by $e^{-\lambda\xi}$ and integrating along ξ on $\mathbb{R}$, we get

$$\mathcal{L}(\lambda)\Delta_1(\lambda,c)=\frac{\alpha}{c}\int_{\mathbb{R}}e^{-\lambda\xi}\left(\int_{-\infty}^{\xi}e^{-\frac{\beta}{c}(\xi-s)}[g'(0)\phi_c(s)-g(\phi_c(s))]ds\right)d\xi. \tag{3.10}$$

Let $V(\xi)=\phi_c(-\xi)$ and $\Lambda=-\lambda$ in (3.10), then $V(+\infty)=0$ and $V(\cdot)$ is decreasing and

$$\mathcal{L}_1(\Lambda)\Delta_1(-\Lambda,c)=\frac{\alpha}{c}\int_{-\infty}^{+\infty}e^{-\Lambda\xi}h(\xi)d\xi, \tag{3.11}$$

where

$$\mathcal{L}_1(\Lambda)=\int_{-\infty}^{+\infty}e^{-\Lambda\xi}V(\xi)d\xi \ \text{ and } \ h(\xi)=\int_{\xi}^{+\infty}e^{\frac{\beta}{c}(\xi-s)}[g'(0)V(s)-g(V(s))]ds.$$

From $g\in C^2(\mathbb{R})$, $g(0)=0$, $V(\xi)=O(e^{-\gamma\xi})$ as $\xi\to+\infty$, and Taylor's expansion, one has

$$g'(0)V(s)-g(V(s))=O(V^2(s))=O(e^{-2\gamma s})\text{ as }s\to+\infty.$$

Therefore, the right side of (3.11) is well defined for $-2\gamma<Re\Lambda<0$. We now use a property of Laplace transform (Widder [43]). According to $V(\xi)>0$, there exists a constant μ such that $\mathcal{L}_1(\Lambda)$ is analytic for $\mu<Re\Lambda<0$ and $\mathcal{L}_1(\Lambda)$ has a singularity at $\Lambda=\mu$. Hence $\mathcal{L}_1(\Lambda)$ is well defined until Λ is a zero of $\Delta_1(-\Lambda,c)=0$, it follows from Lemma 3.1 that $\mathcal{L}_1(\Lambda)$ is well defined for $-\lambda_1<Re\Lambda<0$ since $0<\lambda_1<\lambda_2$.

From (3.11), we can define

$$F(\Lambda):=\int_0^{+\infty}V(\xi)e^{-\Lambda\xi}d\xi=\frac{\frac{\alpha}{c}\int_{-\infty}^{+\infty}e^{-\Lambda\xi}h(\xi)d\xi}{\Delta_1(-\Lambda,c)}-\int_{-\infty}^{0}V(\xi)e^{-\Lambda\xi}d\xi.$$

In order to apply Lemma 3.3, we define

$$H(\Lambda):=\frac{\frac{\alpha}{c}\int_{-\infty}^{+\infty}e^{-\Lambda\xi}h(\xi)d\xi}{\Delta_1(-\Lambda,c)/(\Lambda+\lambda_1)^{k+1}}-(\Lambda+\lambda_1)^{k+1}\int_{-\infty}^{0}V(\xi)e^{-\Lambda\xi}d\xi,$$

where $k=0$ for $c>c^*$ and $k=1$ for $c=c^*$ since $\Delta_1(-\Lambda,c)$ has a simple root λ_1 when $c>c^*$ and a double root λ_1 when $c=c^*$. Note that if $c=c^*$, then $\lambda_1=\lambda_*$. Clearly, $F(\Lambda)=H(\Lambda)/(\Lambda+\lambda_1)^{k+1}$.

Now we claim that $H(\Lambda)$ is analytic in the strip $S := \{\Lambda \in \mathbb{C} | -\lambda_1 \le \mathrm{Re}\Lambda < 0\}$. Clearly, it suffices to show that the function

$$J(\Lambda) := \frac{\frac{\alpha}{c}\int_{-\infty}^{+\infty} e^{-\Lambda\xi} h(\xi) d\xi}{\Delta_1(-\Lambda, c)/(\Lambda + \lambda_1)^{k+1}}$$

is analytic in S. Since $J(\Lambda) = \mathcal{L}_1(\Lambda)(\Lambda + \lambda_1)^{k+1}$, and $\mathcal{L}_1(\Lambda)$ is well defined for $-\lambda_1 < \mathrm{Re}\Lambda < 0$, we know that $J(\Lambda)$ is analytic for $-\lambda_1 < \mathrm{Re}\Lambda < 0$. Next we just show $J(\Lambda)$ is analytic for $\mathrm{Re}\Lambda = -\lambda_1$. we claim that $\Delta_1(-\Lambda, c) = 0$ does not have any zeros with $\mathrm{Re}\Lambda = -\lambda_1$ other than $\Lambda = -\lambda_1$. Actually, let $\Lambda = -\lambda_1 + \omega i$, then follows from $\Delta_1(-\Lambda, c) = 0$ and $\Delta_1(\lambda_1, c) = 0$, we have

$$d \int_{\mathbb{R}} J(y) e^{-\lambda_1 y} \sin^2\left(\frac{\omega y}{2}\right) dy + \frac{\alpha g'(0) c^2 \omega^2}{[(\beta + c\lambda_1)^2 + c^2\omega^2](\beta + c\lambda_1)} = 0,$$

and

$$d \int_{\mathbb{R}} J(y) e^{-\lambda_1 y} \sin(\omega y) dy + \frac{\alpha g'(0) c\omega}{(\beta + c\lambda_1)^2 + c^2\omega^2} + c\omega = 0,$$

which implies that $\omega = 0$. Thus $J(\Lambda)$ is analytic for $\mathrm{Re}\Lambda = -\lambda_1$, and $H(\Lambda)$ is analytic in S. Then by Lemma 3.3 we get that $\lim_{\xi\to-\infty} \phi_c(\xi) e^{-\lambda_1\xi} = \lim_{\xi\to+\infty} V(\xi) e^{\lambda_1\xi}$ exists for $c > c^*$, and

$$\lim_{\xi\to-\infty} \phi_c(\xi)\xi^{-1} e^{-\lambda_1\xi} = -\lim_{\xi\to+\infty} V(\xi)\xi^{-1} e^{\lambda_1\xi} \text{ exists for } c = c^*.$$

Take $a_0 = a_0(c) := \lim_{\xi\to-\infty} \phi_c(\xi) e^{-\lambda_1\xi}$ and $a_0 = a_0(c^*) := -\lim_{\xi\to-\infty} \phi_c(\xi)\xi^{-1} e^{-\lambda_1\xi}$.

Moreover, by using Lebesgue's dominated convergence theorem, it is easy to show that $\lim_{\xi\to-\infty} e^{-\lambda_1\xi}\phi_c'(\xi) = a_0\lambda_1$. Similarly, we can prove for $c = c^*$, $\lim_{\xi\to-\infty} \phi_c'(\xi)\xi^{-1} e^{-\lambda_1\xi} = -a_0\lambda_1$. Noting that $\psi_c(\xi) = \frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi - s)} g(\phi_c(s)) ds$, we have $\lim_{\xi\to-\infty} e^{-\lambda_1\xi}\psi_c(\xi) = \frac{g'(0)a_0}{\beta + c\lambda_1}$. The other conclusions can be obtained similarly. The proof is complete. □

Remark 4 From Lemma 3.1, we know that for any $c \ge c^*$, $\Delta_2(\lambda, c) = 0$ has only one simple root $\lambda_3 < 0$, so in the proof of (ii) of Theorem 3.4, we just choose $k = 0$ for $c \ge c^*$.

3.2 Bistable Case

In this subsection, we assume that g satisfies (GB). Define the following characteristic functions:

$$\Delta_3(\lambda, c) = d \int_{-\infty}^{+\infty} J(y) e^{-\lambda y} dy - c\lambda - (d+1) + \frac{\alpha g'(0)}{\beta + c\lambda},$$
$$\Delta_4(\lambda, c) = d \int_{-\infty}^{+\infty} J(y) e^{-\lambda y} dy - c\lambda - (d+1) + \frac{\alpha g'(u_2^*)}{\beta + c\lambda},$$

where $\lambda \in \mathbb{C} \backslash \{-\beta/c\}$. Then by a similar argument as Lemma 3.1, we obtain

Lemma 3.5 *For any $c > 0$, $\Delta_3(\lambda, c) = 0$ has two real roots $\lambda_5 \in (-\beta/c, 0)$ and $\lambda_6 \in (0, +\infty)$, and $\Delta_4(\lambda, c) = 0$ also has two real roots $\lambda_7 \in (-\beta/c, 0)$ and $\lambda_8 \in (0, +\infty)$.*

Theorem 3.6 *Assume* (J1) *and* (GB) *hold. Let $\Phi(\xi) = (\phi(\xi), \psi(\xi))$ be an increasing traveling wave solution of* (1.3) *satisfying $\Phi(-\infty) = E_0$ and $\Phi(+\infty) = E_2$ with speed $c \neq 0$. Then the following statements hold:*

(i) $\lim_{\xi\to-\infty}\Phi(\xi)e^{-\lambda_6\xi}=(1,B_0)b_0$, $\lim_{\xi\to-\infty}\Phi'(\xi)e^{-\lambda_6\xi}=(1,B_0)b_0\lambda_6$,
(ii) $\lim_{\xi\to+\infty}(E_2-\Phi(\xi))e^{-\lambda_7\xi}=(1,B_1)b_1$, $\lim_{\xi\to+\infty}\Phi'(\xi)e^{-\lambda_7\xi}=-(1,B_1)b_1\lambda_7$,

where b_0 and b_1 are some positive constants, $B_0=\frac{g'(0)}{\beta+c\lambda_6}>0$ and $B_1=\frac{g'(u_2^*)}{\beta+c\lambda_7}>0$.

Proof This lemma can be proved by making a modification of Theorem 2.5 of Wu [46], so we omit the details. □

Remark 5 The readers must notice that $\Delta_1(\lambda,c)$ and $\Delta_3(\lambda,c)$ are different functions since $g'(0)$ in them are different.

4 Entire Solutions in Monostable Case

In this section, we will establish the existence of entire solutions of (1.3) by using super-sub-solutions method and comparison principle.

Before the proof of Theorem 1.1 we first give some useful lemmas. According to Theorem 3.4, we obtain the following estimates directly.

Lemma 4.1 *Let $\Phi_c(\cdot)=(\phi_c(\cdot),\psi_c(\cdot))$ be an increasing traveling wave front of (1.3) connecting $(0,0)$ and (K_1,K_2) with speed $c\geq c^*$. Then there exist positive constants $k(c), K(c), m(c), M(c)$ and $\delta(c)$ such that for $c\geq c^*$ and $x\geq 0$,*

$$\begin{aligned}
k(c)e^{\lambda_3(c)x}&\leq K_1-\phi_c(x)\leq K(c)e^{\lambda_3(c)x},\\
\delta(c)k(c)e^{\lambda_3(c)x}&\leq \delta(c)(K_1-\phi_c(x))\leq \phi_c'(x),\\
m(c)(K_1-\phi_c(x))&\leq K_2-\psi_c(x)\leq M(c)(K_1-\phi_c(x)),\\
m(c)\delta(c)k(c)e^{\lambda_3(c)x}&\leq \delta(c)(K_2-\psi_c(x))\leq \psi_c'(x).
\end{aligned}$$

For $c>c^*$, $x\leq 0$,

$$\begin{aligned}
k(c)e^{\lambda_1(c)x}&\leq \phi_c(x)\leq K(c)e^{\lambda_1(c)x},\quad \delta(c)\phi_c(x)\leq\phi_c'(x),\\
m(c)\phi_c(x)&\leq\psi_c(x)\leq M(c)\phi_c(x),\quad \delta(c)\psi_c(x)\leq\psi_c'(x),
\end{aligned}$$

and for $c=c^*$, $x\leq 0$, let $\varepsilon\in(0,\lambda_*)$, there exists $K_\varepsilon>0$ such that

$$\begin{aligned}
&\phi_{c^*}(x)\leq K_\varepsilon e^{(\lambda_*-\varepsilon)x},\ \ \delta(c^*)\phi_{c^*}(x)\leq\phi_{c^*}'(x),\ \ \delta(c^*)\psi_{c^*}(x)\leq\psi_{c^*}'(x),\\
&m(c^*)\phi_{c^*}(x)\leq\psi_{c^*}(x)\leq M(c^*)\phi_{c^*}(x).
\end{aligned}\tag{4.1}$$

Next we consider the following coupled system of ordinary differential equations:

$$\begin{cases}
p_1'(t)=c_1+Ne^{\mu p_1(t)}, & t<0,\\
p_2'(t)=c_2+Ne^{\mu p_1(t)}, & t<0,\\
p_1(0)\leq 0,\quad p_2(0)\leq 0.
\end{cases}\tag{4.2}$$

where c_1, c_2, N and μ are positive constants and $c_2\geq c_1\geq c^*$. Solving (4.2) explicitly, we obtain

$$p_i(t)=p_i(0)+c_it-\frac{1}{\mu}\ln\left\{1+\frac{N}{c_1}e^{\mu p_1(0)}(1-e^{c_1\mu t})\right\},\ i=1,2.\tag{4.3}$$

Obviously, $p_i(t)$ is increasing, $i=1,2$. Let

$$\omega_1=p_1(0)-\frac{1}{\mu}\ln\left\{1+\frac{N}{c_1}e^{\mu p_1(0)}\right\},\quad \omega_2=p_2(0)-\frac{1}{\mu}\ln\left\{1+\frac{N}{c_1}e^{\mu p_1(0)}\right\}.\tag{4.4}$$

Then according to the identity $p_i(t) - c_i t - \omega_i = -\frac{1}{\mu}\ln\left\{1 - re^{c_1\mu t}/(1+r)\right\}$, $i = 1, 2$, where $r = Ne^{\mu p_1(0)}/c_1$, we have

$$0 < p_1(t) - c_1 t - \omega_1 = p_2(t) - c_2 t - \omega_2 \le R_0 e^{c_1\mu t}, \text{ for } t \le 0, \tag{4.5}$$

where R_0 is some positive constant. Since $p_2' - p_1' = c_2 - c_1 \ge 0$, we obtain $p_2(t) \le p_1(t) \le 0 (t \le 0)$ if $p_2(0) \le p_1(0) \le 0$.

It is clear that if $(\underline{u}_1(x,t), \underline{v}_1(x,t))$ and $(\underline{u}_2(x,t), \underline{v}_2(x,t))$ are two subsolutions of (1.3) on $t \in (T_1, T_2)$, then the pairing of $(\underline{u}, \underline{v}) := (\max_{x\in\mathbb{R}}\{\underline{u}_1, \underline{u}_2\}, \max_{x\in\mathbb{R}}\{\underline{v}_1, \underline{v}_2\})$ is a subsolution of (1.3) on $t \in (T_1, T_2)$. Similarly, if $(\bar{u}_1(x,t), \bar{v}_1(x,t))$ and $(\bar{u}_2(x,t), \bar{v}_2(x,t))$ are supersolutions of (1.3) on $t \in (T_1, T_2)$, then the pairing of $(\bar{u}, \bar{v}) := (\min_{x\in\mathbb{R}}\{\bar{u}_1, \bar{u}_2\}, \min_{x\in\mathbb{R}}\{\bar{v}_1, \bar{v}_2\})$ is a supersolution of (1.3) on $t \in (T_1, T_2)$. Thus we have the following lemma.

Lemma 4.2 *The function $\underline{w}(x,t) = (\underline{u}(x,t), \underline{v}(x,t))$ defined by*

$$\underline{w}(x,t) = \max\{\Phi_{c_1}(x + c_1 t + \omega_1), \Phi_{c_2}(-x + c_2 t + \omega_2)\}$$

is a subsolution of (1.3) on $(-\infty, +\infty)$, where ω_i is defined in (4.4).

Lemma 4.3 *Assume* (J1) *and* (GM) *hold. Given c_1 and c_2 such that $c_2 \ge c_1 \ge c^*$, let $L = \max_{u\in[0,K_1]}|g''(u)|$, N and μ of (4.2) satisfy*

(i) *if $c_2 = c_1 = c^*$, let $\mu = \lambda_* - \varepsilon$ and $N \ge \frac{LK_\varepsilon}{\delta(c^*)m(c^*)}$ for some $\varepsilon \in (0, \lambda_*)$.*
(ii) *if $c_2 > c_1 = c^*$, let $\mu = \lambda_1(c_2)$ and*

$$N \ge \max\left\{\frac{LK_\varepsilon}{\delta(c_2)m(c_2)}, \frac{LK(c_2)}{\delta(c^*)m(c^*)}, \frac{LK_\varepsilon}{\delta(c^*)m(c^*)}, \frac{LK(c_2)}{\delta(c_2)m(c_2)}\right\},$$

for some $\varepsilon \in (0, \lambda_* - \lambda_1(c_2))$.
(iii) *if $c_2 \ge c_1 > c^*$, let $\mu = \lambda_1(c_2)$ and*

$$N \ge \max\left\{\frac{LK(c_1)}{\delta(c_2)m(c_2)}, \frac{LK(c_2)}{\delta(c_1)m(c_1)}, \frac{LK(c_1)}{\delta(c_1)m(c_1)}, \frac{LK(c_2)}{\delta(c_2)m(c_2)}\right\}.$$

Then for the solution $(p_1(t), p_2(t))$ of (4.2) with $p_2(0) \le p_1(0) \le 0$, the function $\bar{w}(x,t) = (\bar{u}(x,t), \bar{v}(x,t))$ defined by

$$\bar{w}(x,t) = \Phi_{c_1}(x + p_1(t)) + \Phi_{c_2}(-x + p_2(t)),$$

is a supersolution of (1.3) on $t \in (-\infty, 0]$.

Proof For convenience, we denote $E[\bar{w}](x,t) = (E_1[\bar{w}](x,t), E_2[\bar{w}](x,t))$, where

$$E_1[\bar{w}] := \bar{u}_t - d(J * \bar{u} - \bar{u}) + \bar{u} - \alpha\bar{v},$$
$$E_2[\bar{w}] := \bar{v}_t + \beta\bar{v} - g(\bar{u}).$$

Then we just need to prove $E_1[\bar{w}](x,t) \ge 0$ and $E_2[\bar{w}](x,t) \ge 0$ for all $(x,t) \in \mathbb{R}\times(-\infty, 0]$.

Direct computations show that

$$E_1[\bar{w}] = (p_1' - c_1)\phi_{c_1}' + (p_2' - c_2)\phi_{c_2}' = Ne^{\mu p_1}(\phi_{c_1}' + \phi_{c_2}') \ge 0.$$

Next, we show that $E_2[\bar{w}](x,t) \ge 0$ for $(x,t) \in \mathbb{R}\times(-\infty, 0]$. Similarly we get

$$\begin{aligned} E_2[\bar{w}] &= \psi_{c_1}'(p_1' - c_1) + \psi_{c_2}'(p_2' - c_2) + g(\phi_{c_1}) + g(\phi_{c_2}) - g(\phi_{c_1} + \phi_{c_2}) \\ &= (\psi_{c_1}' + \psi_{c_2}')\left[Ne^{\mu p_1} - H(x,t)\right], \end{aligned} \tag{4.6}$$

where

$$H(x,t)=\frac{G(x,t)}{\psi'_{c_1}(x+p_1(t))+\psi'_{c_2}(-x+p_2(t))}, \tag{4.7}$$

and

$$G(x,t)=g(\phi_{c_1}(x+p_1(t))+\phi_{c_2}(-x+p_2(t)))-g(\phi_{c_1}(x+p_1(t)))-g(\phi_{c_2}(-x+p_2(t))). \tag{4.8}$$

For $u_1,u_2\in[0,K_1]$, recalling that $g(0)=0$ and $g'(u)\le g'(0)$ for $u\in[0,2K_1]$, we obtain

$$g(u_1+u_2)-g(u_1)-g(u_2)\le Lu_i^2,\ i=1,2.$$

Thus we get $G(x,t)\le L\phi_{c_i}^2((-1)^{i-1}x+p_i(t)),\ i=1,2$. Similar to the proof of [46, Lemma 18], we can show that $\bar{w}(x,t)$ is a supersolution of (1.3) on $(-\infty,0]$. This completes the proof. □

Proof of Theorem 1.1 For $n\in\mathbb{N}$, we denote

$$\varphi^n(x):=(\varphi_1^n(x),\varphi_2^n(x))=\max\{\Phi_{c_1}(x-c_1n+\omega_1),\Phi_{c_2}(-x-c_2n+\omega_2)\},\quad x\in\mathbb{R}.$$

Consider the following initial value problem of (1.3):

$$\begin{cases}(u_n)_t(x,t)=d(J*u_n(x,t)-u_n(x,t))-u_n(x,t)+\alpha v_n(x,t), & x\in\mathbb{R},t>-n,\\ (v_n)_t(x,t)=-\beta v_n(x,t)+g(u_n(x,t)), & x\in\mathbb{R},t>-n,\\ (u_n(x,-n),v_n(x,-n))=(u_{n,0}(x),v_{n,0}(x))=\varphi^n(x), & x\in\mathbb{R}.\end{cases} \tag{4.9}$$

From Lemma 2.2, we know that system (4.9) has a unique solution $w_n(x,t;\varphi^n)=(u_n(x,t;\varphi^n),v_n(x,t;\varphi^n))$ which satisfies $\mathbf{0}\le w_n(x,t)\le\mathbf{K}$ for $(x,t)\in\mathbb{R}\times[-n,+\infty)$ and $w_n(x,-n)=\underline{w}(x,-n)\le w_{n+1}(x,-n)\le\mathbf{K}$, then by comparison principle, we get $\mathbf{0}\le\underline{w}(x,t)\le w_n(x,t)\le w_{n+1}(x,t)\le\min\{\mathbf{K},\bar{w}(x,t)\}$. That is to say, $\{w_n(x,t)\}_{n=1}^{\infty}$ is bounded and non-decreasing about n for any $(x,t)\in\mathbb{R}\times(-n,+\infty)$. Then there exists a function $w(x,t)=(u(x,t),v(x,t))$ satisfying $\mathbf{0}\le(u(x,t),v(x,t))\le\mathbf{K}$ such that for any $(x,t)\in\mathbb{R}^2$, there is

$$\lim_{n\to\infty}(u_n(x,t),v_n(x,t))=(u(x,t),v(x,t)).$$

For any given $t_0\in\mathbb{R}$, there exists $n\in\mathbb{N}$ such that $t_0>-n$ and $w_n=(u_n,v_n)$ satisfies

$$w_n(x,t)=T(t-t_0)w_n(x,t_0)+\int_{t_0}^{t}T(t-s)B(w_n(x,s))ds,$$

where $T(t)$ and B are defined as in Sect. 2. Then by Lebesgue dominated convergence theorem, we get

$$w(x,t)=T(t-t_0)w(x,t_0)+\int_{t_0}^{t}T(t-s)B(w(x,s))ds.$$

It is easy to see that $w(x,t)$ is continuous and differentiable about t. Thus we obtain that

$$\begin{aligned} u_t(x,t) &= -(d+1)e^{-(d+1)(t-t_0)}u(x,t_0) - (d+1)\int_{t_0}^{t} e^{-(d+1)(t-s)}B_1(u(x,s),v(x,s))ds \\ &\quad + B_1(u(x,t),v(x,t)) \\ &= -(d+1)u(x,t) + B_1(u(x,t),v(x,t)) \\ &= d(J*u(x,t)-u(x,t)) - u(x,t) + \alpha v(x,t). \\ v_t(x,t) &= -\beta e^{-\beta(t-t_0)}v(x,t_0) - \beta\int_{t_0}^{t} e^{-\beta(t-s)}B_2(u(x,s),v(x,s))ds \\ &\quad + B_2(u(x,t),v(x,t)) \\ &= -\beta v(x,t) + B_2(u(x,t),v(x,t)) \\ &= -\beta v(x,t) + g(u(x,t)). \end{aligned}$$

Therefore, $w(x,t) = (u(x,t), v(x,t))$ is an entire solution of (1.3) and satisfies

$$\begin{aligned} &\underline{w}(x,t) \le w(x,t) \le \bar{w}(x,t), \text{ on } (x,t) \in \mathbb{R}\times(-\infty,0], \\ &\underline{w}(x,t) \le w(x,t) \le \mathbf{K}, \text{ on } (x,t) \in \mathbb{R}^2. \end{aligned} \tag{4.10}$$

Furthermore, since $\lim_{t\to-\infty}\sup_{x\in\mathbb{R}} \| \bar{w}(x,t) - \underline{w}(x,t) \| = 0$, we get the entire solution $w(x,t)$ of (1.3) satisfying the following asymptotic behaviors:

$$\begin{aligned} &\lim_{t\to-\infty}\sup_{x\ge0} \| w(x,t) - \Phi_{c_1}(x+c_1t+\omega_1) \| = 0, \\ &\lim_{t\to-\infty}\sup_{x\le0} \| w(x,t) - \Phi_{c_2}(-x+c_2t+\omega_2) \| = 0. \end{aligned}$$

Moreover, by (4.10), it is easy to see that $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}} \| w(x,t) - \mathbf{K} \| = 0$.

Now we prove the assertion (i). Since $w_n(x,t) \ge \underline{w}(x,t) \ge \underline{w}(x,-n) = w_n(x,-n)$ for $(x,t) \in \mathbb{R}\times(-n,+\infty)$. Let $\varepsilon > 0$, following $w_n(x,\varepsilon-n) \ge w_n(x,-n)$ we have $w_n(x,t+\varepsilon) \ge w_n(x,t)$ for any $t > -n$ and $x \in \mathbb{R}$. This implies that $\frac{\partial}{\partial t}w_n(x,t) \ge 0$ for $(x,t) \in \mathbb{R}\times(-n,+\infty)$ which yields $\frac{\partial}{\partial t}w(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^2$. Next, we show that $\frac{\partial}{\partial t}w(x,t) \gg 0$ for all $(x,t) \in \mathbb{R}^2$. Note that

$$u_{tt} = d(J*u_t - u_t) - u_t + \alpha v_t \ge -(d+1)u_t,$$

then for any $x \in \mathbb{R}$ and $\tau < t$, we have

$$u_t(x,t) \ge u_t(x,\tau)e^{-(d+1)(t-\tau)} \ge 0.$$

Suppose for the contrary that there exists a point $(x_0,t_0) \in \mathbb{R}^2$ such that $u_t(x_0,t_0) = 0$, then $u_t(x_0,\tau) = 0$ for all $\tau \le t_0$. Hence, $\lim_{t\to-\infty} u(x_0,t) = u(x_0,t_0)$. But (4.10) shows that $\lim_{t\to-\infty} u(x_0,t) = 0$ and $u(x_0,t_0) > 0$. This contradiction yields that $u_t(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2$. Similarly, we can show that $v_t(x,t) > 0$ for all $(x,t) \in \mathbb{R}^2$. The proofs of (iii)–(v) are straightforward, so we omit them. Take $W_{c_1,c_2,\omega_1,\omega_2}(x,t) = w(x,t)$, then Theorem 1.1 holds for $\theta_i = \omega_i$, $i = 1, 2$.

For any $\theta_1, \theta_2 \in \mathbb{R}$, define $W_{c_1,c_2,\theta_1,\theta_2}(\cdot,\cdot) = W_{c_1,c_2,\omega_1,\omega_2}(\cdot+\xi,\cdot+\eta)$ with

$$\xi = \frac{c_2(\theta_1-\omega_1) - c_1(\theta_2-\omega_2)}{c_1+c_2} \text{ and } \eta = \frac{\theta_1+\theta_2-\omega_1-\omega_2}{c_1+c_2}.$$

Thus, $W_{c_1,c_2,\theta_1,\theta_2}(x,t)$ is also an entire solution of (1.3). The proof is complete. □

Proof of Theorem 1.3 We just constructing a pair of super and subsolutions of (1.3) since the other discussions are similar to that of Theorem 1.1, and we omit them. □

Let $\Gamma(t)$ be an increasing solution of (1.3) described as in Proposition 1.2.

Lemma 4.4 *Suppose that* (J1) *and* (GM) *hold. Then*

$$\underline{w}(x,t) = \max\left\{\chi_1\Phi_{c_1}(x+c_1t+\omega_1),\, \chi_2\Phi_{c_2}(-x+c_2t+\omega_2),\, \Gamma(t+\omega_3)\right\}$$

is a subsolution of (1.3) *on* $\mathbb{R}\times(-\infty,+\infty)$*, where* ω_1 *and* ω_2 *are defined by* (4.4)*,* $\omega_3 \in \mathbb{R}$ *and* $\chi_1, \chi_2 \in \{0,1\}$ *with* $\chi_1+\chi_2 \geq 1$.

Proof The proof is similar to that of Lemma 4.2, see also [27, Lemma 3.6]. So we omit it here. □

Lemma 4.5 *Assume that* (J1) *and* (GM) *hold. Then there exists* $T \leq 0$ *such that*

$$\bar{w}(x,t) = \min\left\{\chi_1\Phi_{c_1}(x+p_1(t)) + \chi_2\Phi_{c_2}(-x+p_2(t)) + (1,b^*)e^{\lambda^*(t+\theta_3)},\, \mathbf{K}\right\}$$

is a supersolution of (1.3) *on* $\mathbb{R}\times(-\infty,T)$*, where* $\chi_1,\chi_2\in\{0,1\}$ *with* $\chi_1+\chi_2\geq 1$*,* $\theta_3\in\mathbb{R}$*, and* N *and* μ *in* (4.2) *are defined as in Lemma 4.3.*

Proof We only consider the case $\chi_1=\chi_2=1$ since the other cases can be proved similarly. Denote $\rho(t)=(\rho_1(t),\rho_2(t))=(1,b^*)e^{\lambda^*(t+\theta_3)}$, then $\rho(t)$ satisfies

$$\begin{cases}\rho_1'(t) = -\rho_1(t)+\alpha\rho_2(t),\\ \rho_2'(t) = -\beta\rho_2(t)+g'(0)\rho_1(t).\end{cases}$$

Define

$$\begin{aligned}
S_1 &= \{(x,t)\in\mathbb{R}\times(-\infty,0] \,|\, \phi_{c_1}(x+p_1(t))+\phi_{c_2}(-x+p_2(t))+\rho_1(t) > K_1\},\\
S_2 &= \{(x,t)\in\mathbb{R}\times(-\infty,0] \,|\, \phi_{c_1}(x+p_1(t))+\phi_{c_2}(-x+p_2(t))+\rho_1(t) < K_1\},\\
S_3 &= \{(x,t)\in\mathbb{R}\times(-\infty,0] \,|\, \psi_{c_1}(x+p_1(t))+\psi_{c_2}(-x+p_2(t))+\rho_2(t) > K_2\},\\
S_4 &= \{(x,t)\in\mathbb{R}\times(-\infty,0] \,|\, \psi_{c_1}(x+p_1(t))+\psi_{c_2}(-x+p_2(t))+\rho_2(t) < K_2\}.
\end{aligned}$$

We divide the reminder of the proof into three steps.

Step 1. We first verify that $E_1[\bar{w}](x,t)\geq 0$ for $(x,t)\in S_1\cup S_2$.

(i) If $(x,t)\in S_1$, then $\bar{u}(x,t)=K_1$, $\bar{v}(x,t)\leq K_2$, and $E_1[\bar{w}](x,t)\geq K_1-\alpha K_2=0$.
(ii) If $(x,t)\in S_2$, then $\bar{u}(x,t)=\phi_{c_1}(x+p_1(t))+\phi_{c_2}(-x+p_2(t))+\rho_1(t)$ and $\bar{v}(x,t) \leq \psi_{c_1}(x+p_1(t))+\psi_{c_2}(-x+p_2(t))+\rho_2(t)$. Thus,

$$\begin{aligned}
&E_1[\bar{w}](x,t)\\
&= \phi_{c_1}'p_1'+\phi_{c_2}'p_2'+\rho_1'-d(J*\phi_{c_1}-\phi_{c_1})-d(J*\phi_{c_2}-\phi_{c_2})+\phi_{c_1}+\phi_{c_2}+\rho_1-\alpha\bar{v}\\
&= \phi_{c_1}'(p_1'-c_1)+\phi_{c_2}'(p_2'-c_2)+\rho_1'+\rho_1-\alpha\bar{v}+\alpha\psi_{c_1}+\alpha\psi_{c_2}\\
&= (\phi_{c_1}'+\phi_{c_2}')Ne^{\mu p_1(t)}-\alpha[\bar{v}-(\psi_{c_1}+\psi_{c_2}+\rho_2)]\geq 0.
\end{aligned}$$

Step 2. Now we prove that $E_2[\bar{w}](x,t)\geq 0$ for $(x,t)\in S_3\cup S_4$.

(i) For $(x,t)\in S_3$, $\bar{v}(x,t)=K_2$, $\bar{u}(x,t)\leq K_1$, and $E_2[\bar{w}](x,t)\geq \beta K_2-g(K_1)=0$.

(ii) For $(x,t) \in S_4$, $\bar{v}(x,t) = \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) + \rho_2(t)$ and $\bar{u}(x,t) \le \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) + \rho_1(t)$. In view of $g'(u) \le g'(0)$ for $u \in [0, +\infty)$, we have

$$\begin{aligned} E_2[\bar{w}](x,t) &= \psi'_{c_1}(p'_1 - c_1) + \psi'_{c_2}(p'_2 - c_2) + g(\phi_{c_1}) + g(\phi_{c_2}) - g(\bar{u}) + g'(0)\rho_1 \\ &\ge Ne^{\mu p_1(t)}(\psi_{c'_1} + \psi'_{c_2}) + g(\phi_{c_1}) + g(\phi_{c_2}) - g(\phi_{c_1} + \phi_{c_2}) \\ &\quad + g'(0)\rho_1 - [g(\phi_{c_1} + \phi_{c_2} + \rho_1) - g(\phi_{c_1} + \phi_{c_2})] \\ &\ge (\psi_{c'_1} + \psi'_{c_2})\left[Ne^{\mu p_1(t)} - H(x,t)\right], \end{aligned}$$

where $H(x,t)$ is given by (4.7). Then by using a similar argument as in the proof of Lemma 4.3, we get $E_2[\bar{w}](x,t) \ge 0$ for $(x,t) \in S_4$.

Step 3. Finally, we prove that there exists $T \le 0$ such that $\bar{w}(x,t)$ is a supersolution of (1.3) on $(-\infty, T)$. This proof is completely similar to that of Lemma 18 of [46], so we omit it. Thus the proof is complete. □

5 Entire Solutions in Bistable Case

In the bistable case, our main results are Theorems 1.4–1.7. Since the proofs of Theorems 1.4 and 1.5 are similar, we only prove Theorems 1.4, 1.6 and 1.7. Before to prove the main Theorems, we first give some preliminaries.

5.1 Preliminaries

In this subsection, we give some main estimates which are essential in our proofs by using the results about the asymptotic behaviors of traveling fronts given in Sect. 3.

Proposition 5.1 *There exist some positive numbers $C_0, C_1, C_2, \eta_1, \eta_2$ and ρ such that for $\xi \le M$,*

$$|\phi'(\xi)|, |\psi'(\xi)|, |\phi'_1(\xi)|, |\psi'_1(\xi)|, |\phi'_2(\xi)|, |\psi'_2(\xi)| \le C_0 e^{\eta_1 \xi}, \tag{5.1}$$

$$C_1 e^{\eta_1 \xi} \le |\phi(\xi)|, |\psi(\xi)|, |\phi_1(\xi)|, |\psi_1(\xi)|, |\phi_2(\xi) - u_1^*|, |\psi_2(\xi) - v_1^*| \le C_2 e^{\eta_1 \xi}, \tag{5.2}$$

$$\frac{|\phi'(\xi)|}{|\phi(\xi)|}, \frac{|\psi'(\xi)|}{|\psi(\xi)|}, \frac{|\phi'_1(\xi)|}{|\phi_1(\xi)|}, \frac{|\psi'_1(\xi)|}{|\psi_1(\xi)|}, \frac{|\phi'_2(\xi)|}{|\phi_2(\xi) - u_1^*|}, \frac{|\psi'_2(\xi)|}{|\psi_2(\xi) - v_1^*|} \ge \rho, \tag{5.3}$$

and for $\xi \ge -M$,

$$|\phi'(\xi)|, |\psi'(\xi)|, |\phi'_1(\xi)|, |\psi'_1(\xi)|, |\phi'_2(\xi)|, |\psi'_2(\xi)| \le C_0 e^{-\eta_2 \xi}, \tag{5.4}$$

$$\begin{aligned} C_1 e^{-\eta_2 \xi} &\le |u_2^* - \phi(\xi)|, |v_2^* - \psi(\xi)|, |u_1^* - \phi_1(\xi)|, |v_1^* - \psi_1(\xi)|, |u_2^* - \phi_2(\xi)|, \\ &\quad |v_2^* - \psi_2(\xi)| \le C_2 e^{-\eta_2 \xi}, \end{aligned} \tag{5.5}$$

$$\frac{|\phi'(\xi)|}{|\phi(\xi) - u_2^*|}, \frac{|\psi'(\xi)|}{|\psi(\xi) - v_2^*|}, \frac{|\phi'_1(\xi)|}{|\phi_1(\xi) - u_1^*|}, \frac{|\psi'_1(\xi)|}{|\psi_1(\xi) - v_1^*|}, \frac{|\phi'_2(\xi)|}{|\phi_2(\xi) - u_2^*|}, \frac{|\psi'_2(\xi)|}{|\psi_2(\xi) - v_2^*|} \ge \rho, \tag{5.6}$$

where M is defined in (J2).

Now we consider the following two ordinary differential equations [32]:

$$p_1'(t) = c - Ne^{\sigma p_1(t)}, \quad t \le 0, \tag{5.7}$$

$$p_2'(t) = c + Ne^{\sigma p_2(t)}, \quad t \le 0, \tag{5.8}$$

where c, N, σ are positive constants, the initial value $p_1(0) \le p_2(0) < 0$. In particular, if we assume $c - Ne^{\sigma p_1(0)} > 0$, then $p_1(0) < \min\{\frac{1}{\sigma}\ln\frac{c}{N}, 0\}$. We notice that (5.7) and (5.8) plays an elementary role in constructing of the sub and supersolutions. We can solve (5.7) and (5.8) explicitly as

$$p_1(t) = p_1(0) + ct - \frac{1}{\sigma}\ln\left\{1 - \frac{N}{c}e^{\sigma p_1(0)}(1 - e^{c\sigma t})\right\},$$
$$p_2(t) = p_2(0) + ct - \frac{1}{\sigma}\ln\left\{1 + \frac{N}{c}e^{\sigma p_2(0)}(1 - e^{c\sigma t})\right\}.$$

If we define

$$\omega_1 := p_1(0) - \frac{1}{\sigma}\ln\left\{1 - \frac{N}{c}e^{\sigma p_1(0)}\right\}, \quad \omega_2 = p_2(0) - \frac{1}{\sigma}\ln\left\{1 + \frac{N}{c}e^{\sigma p_2(0)}\right\},$$

then

$$p_1(t) - ct - \omega_1 = -\frac{1}{\sigma}\ln\left\{\left(1 - \frac{r_1}{1+r_1}\right)e^{c\sigma t}\right\}, \quad r_1 = -\frac{N}{c}e^{\sigma p_1(0)},$$
$$p_2(t) - ct - \omega_2 = -\frac{1}{\sigma}\ln\left\{\left(1 - \frac{r_2}{1+r_2}\right)e^{c\sigma t}\right\}, \quad r_2 = \frac{N}{c}e^{\sigma p_2(0)}.$$

Thus we have

$$0 < p_2(t) - p_1(t) \le R_0 e^{c\sigma t}, \quad t \le 0, \tag{5.9}$$

for some finite positive constant R_0.

5.2 Proof of Theorem 1.4

In this subsection, we prove Theorem 1.4 by constructing appropriate sub- and supersolutions.

Firstly, we transform system (1.3) into the following system by a transformation $(u(x,t), v(x,t)) = (U(z,t), V(z,t))$, $z = x + \bar{c}t$, where $\bar{c}$ is an any given constant.

$$\begin{cases} U_t(z,t) = d(J * U - U)(z,t) - \bar{c}U_z(z,t) + f_1(U(z,t), V(z,t)), \\ V_t(z,t) = -\bar{c}V_z(z,t) + f_2(U(z,t), V(z,t)), \end{cases} \tag{5.10}$$

where $(z,t) \in \mathbb{R}^2$, $f_1(U,V) = -U + \alpha V$ and $f_2(U,V) = -\beta V + g(U)$. It is easy to see that $(u(x,t), v(x,t))$ is a solution of (1.3) if and only if $(U(z,t), V(z,t))$ is a solution of (5.10). Thus we just consider the entire solutions of (5.10).

The definition of supersolution and subsolution of system (5.10) is similar to that of (1.3), see Definition 2.1.

Let $(\phi_1(x + c_1t), \psi_1(x + c_1t))$ and $(\phi_2(x + c_2t), \psi_2(x + c_2t))$ be the traveling fronts of (1.3), then $(\phi_1(z - c_0t), \psi_1(z - c_0t))$ and $(\phi_2(z + c_0t), \psi_2(z + c_0t))$ are two traveling fronts of (5.10) with $\bar{c} = (c_1 + c_2)/2$, and $c_0 = (c_2 - c_1)/2$. Motivated by Morita [32], we define the following auxiliary functions:

$$Q_1(x,y)=\frac{(u_2^*-u_1^*)xy}{x(y-u_1^*)+u_1^*(u_2^*-y)},\quad (x,y)\in D_1:=\{[0,u_1^*]\times[u_1^*,u_2^*]\}\setminus\{(0,u_2^*)\},\tag{5.11}$$

$$Q_2(x,y)=\frac{(v_2^*-v_1^*)xy}{x(y-v_1^*)+v_1^*(v_2^*-y)},\quad (x,y)\in D_2:=\{[0,v_1^*]\times[v_1^*,v_2^*]\}\setminus\{(0,v_2^*)\}.\tag{5.12}$$

Denote

$$Q_{ix}=\frac{\partial Q_i}{\partial x},\quad Q_{iy}=\frac{\partial Q_i}{\partial y},\quad Q_{ixx}=\frac{\partial^2 Q_i}{\partial x^2},\quad Q_{ixy}=\frac{\partial^2 Q_i}{\partial x\partial y},\quad Q_{iyy}=\frac{\partial^2 Q_i}{\partial y^2},\quad i=1,2.$$

Since the functions Q_1 and Q_2 satisfy

$$\begin{aligned}Q_1(x,y)&=x+x(y-u_1^*)\left\{\frac{u_2^*-x}{x(y-u_1^*)+u_1^*(u_2^*-y)}\right\}\\&=y+(x-u_1^*)(y-u_2^*)\left\{\frac{-y}{x(y-u_1^*)+u_1^*(u_2^*-y)}\right\},\text{ for }(x,y)\in D_1.\\Q_2(x,y)&=x+x(y-v_1^*)\left\{\frac{v_2^*-x}{x(y-v_1^*)+v_1^*(v_2^*-y)}\right\}\\&=y+(x-v_1^*)(y-v_2^*)\left\{\frac{-y}{x(y-v_1^*)+v_1^*(v_2^*-y)}\right\},\text{ for }(x,y)\in D_2.\end{aligned}$$

It follows from Morita and Ninomiya [32] that $Q_i(i=1,2)$ possess the following properties.

Lemma 5.2 *The functions $Q_i(i=1,2)$ defined by (5.11) and (5.12) satisfy*

$$\begin{aligned}&Q_{1x}(x,u_1^*)=Q_{1y}(u_1^*,y)=1,\quad Q_{1x}(x,u_2^*)=Q_{1y}(0,y)=0,\qquad (x,y)\in D_1.\\&Q_{2x}(x,v_1^*)=Q_{2y}(v_1^*,y)=1,\quad Q_{2x}(x,v_2^*)=Q_{2y}(0,y)=0,\qquad (x,y)\in D_2.\end{aligned}$$

and

$$\begin{aligned}&Q_{1xx}(x,u_1^*)=Q_{1xx}(x,u_2^*)=Q_{1yy}(0,y)=Q_{1yy}(u_1^*,y)=0,\qquad (x,y)\in D_1.\\&Q_{2xx}(x,v_1^*)=Q_{2xx}(x,v_2^*)=Q_{2yy}(0,y)=Q_{2yy}(v_1^*,y)=0,\qquad (x,y)\in D_2.\end{aligned}$$

Moreover, there exist functions $\widetilde{Q}_{111j},\widetilde{Q}_{122j}\in C^1(D_1)$ and $\widetilde{Q}_{211j},\widetilde{Q}_{222j}\in C^1(D_2)$, $j=1,2$ satisfying

$$\begin{aligned}&Q_{1xx}(x,y)=(y-u_1^*)\tilde{Q}_{1111}(x,y)=(y-u_2^*)\tilde{Q}_{1112}(x,y),\\&Q_{1yy}(x,y)=x\tilde{Q}_{1221}(x,y)=(x-u_1^*)\tilde{Q}_{1222}(x,y),\qquad (x,y)\in D_1,\\&Q_{2xx}(x,y)=(y-v_1^*)\tilde{Q}_{2111}(x,y)=(y-v_2^*)\tilde{Q}_{2112}(x,y),\\&Q_{2yy}(x,y)=x\tilde{Q}_{2221}(x,y)=(x-v_1^*)\tilde{Q}_{2222}(x,y),\qquad (x,y)\in D_2.\end{aligned}$$

In what follows, we construct a pair of super and subsolutions to prove Theorem 1.4.

Lemma 5.3 *Let all the assumptions of Theorem 1.4 be satisfied. Set $\bar{c}=(c_1+c_2)/2$ and $c_0=(c_2-c_1)/2$. Let $(p_1(t),c_0)$ and $(p_2(t),c_0)$ be the solutions of (5.7) and (5.8) respectively. Then the functions defined by*

$$\begin{cases}\overline{U}(z,t):=Q_1(\phi_1(z-p_1(t)),\phi_2(z+p_2(t))),\\\overline{V}(z,t):=Q_2(\psi_1(z-p_1(t)),\psi_2(z+p_2(t))),\end{cases}$$

and

$$\begin{cases} \underline{U}(z,t) := Q_1(\phi_1(z-p_2(t)), \phi_2(z+p_1(t))), \\ \underline{V}(z,t) := Q_2(\psi_1(z-p_2(t)), \psi_2(z+p_1(t))), \end{cases}$$

are a pair of super and subsolutions of (5.10) *for* $t \le 0$. *Moreover, there are*

$$\underline{U}(z,t) \le \overline{U}(z,t), \quad \sup_{z\in\mathbb{R}}(\overline{U}(z,t) - \underline{U}(z,t)) \le Ce^{c_0\sigma t}, \quad t \le 0, \tag{5.13}$$

$$\underline{V}(z,t) \le \overline{V}(z,t), \quad \sup_{z\in\mathbb{R}}(\overline{V}(z,t) - \underline{V}(z,t)) \le Ce^{c_0\sigma t}, \quad t \le 0, \tag{5.14}$$

for some positive constant C, *and* σ *as in* (5.7).

Proof From $c_1 < 0 < c_2$, we have $c_0 > 0$. For convinence, we denote

$$\begin{aligned} \mathcal{F}_1(U,V) &= U_t - d(J*U - U) + \bar{c}U_z - f_1(U,V), \\ \mathcal{F}_2(U,V) &= V_t + \bar{c}V_z - f_2(U,V). \end{aligned} \tag{5.15}$$

To prove this lemma, it suffices to show that

$$\mathcal{F}_i(\overline{U},\overline{V}) \ge 0 \text{ and } \mathcal{F}_i(\underline{U},\underline{V}) \le 0, \quad i = 1,2$$

for $(z,t) \in \mathbb{R}\times(-\infty,0]$. By using the above prepared results, direct calculations give that

$$\begin{aligned} &\mathcal{F}_1(\overline{U},\overline{V}) \\ &= Q_{1x}\phi_1'(-p_1' + \bar{c}) + Q_{1y}\phi_2'(p_2' + \bar{c}) - f_1(Q_1,Q_2) - d[J*Q_1 - Q_1] \\ &= Q_{1x}\phi_1'(-p_1' + \bar{c} - c_1) + Q_{1y}\phi_2'(p_2' + \bar{c} - c_2) + Q_{1x}f_1(\phi_1,\psi_1) + Q_{1y}f_1(\phi_2,\psi_2) \\ &\quad - f_1(Q_1,Q_2) + d[Q_{1x}(J*\phi_1 - \phi_1) + Q_{1y}(J*\phi_2 - \phi_2) - (J*Q_1 - Q_1)] \\ &= Q_{1x}\phi_1' Ne^{\sigma p_1(t)} + Q_{1y}\phi_2' Ne^{\sigma p_2(t)} - F_1(\phi_1,\phi_2,\psi_1,\psi_2) - H_1(\phi_1,\phi_2), \end{aligned} \tag{5.16}$$

where $Q_1 = Q_1(\phi_1,\phi_2)$, $Q_2 = Q_2(\psi_1,\psi_2)$ and

$$\begin{aligned} F_1(\phi_1,\phi_2,\psi_1,\psi_2) &= f_1(Q_1,Q_2) - Q_{1x}f_1(\phi_1,\psi_1) - Q_{1y}f_1(\phi_2,\psi_2), \\ H_1(\phi_1,\phi_2) &= d[(J*Q_1 - Q_1) - Q_{1x}(J*\phi_1 - \phi_1) - Q_{1y}(J*\phi_2 - \phi_2)]. \end{aligned}$$

By virtue of (5.9), we know that $e^{\sigma p_2(t)} \ge e^{\sigma p_1(t)}$ for $t \le 0$, then it follows from (5.16) that

$$\mathcal{F}_1(\overline{U},\overline{V}) \ge A_1(\phi_1,\phi_2)\left[Ne^{\sigma p_1(t)} - G_1(\phi_1,\phi_2,\psi_1,\psi_2)\right], \tag{5.17}$$

where

$$A_1(\phi_1,\phi_2) := Q_{1x}\phi_1' + Q_{1y}\phi_2',$$

$$G_1(\phi_1,\phi_2,\psi_1,\psi_2) := \frac{F_1(\phi_1,\phi_2,\psi_1,\psi_2) + H_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)}.$$

Indeed, following from (5.11) and (5.12), we have

$$Q_{1x}(x,y) = \frac{u_1^*(u_2^* - u_1^*)y(u_2^* - y)}{[x(y-u_1^*) + u_1^*(u_2^* - y)]^2}, \quad Q_{1y}(x,y) = \frac{u_1^*(u_2^* - u_1^*)x(u_2^* - x)}{[x(y-u_1^*) + u_1^*(u_2^* - y)]^2},$$

$$Q_{2x}(x,y) = \frac{v_1^*(v_2^* - v_1^*)y(v_2^* - y)}{[x(y-v_1^*) + v_1^*(v_2^* - y)]^2}, \quad Q_{2y}(x,y) = \frac{v_1^*(v_2^* - v_1^*)x(v_2^* - x)}{[x(y-v_1^*) + v_1^*(v_2^* - y)]^2}.$$

By virtue of the facts $0 < \phi_1 < u_1^*$, $0 < \psi_1 < v_1^*$, $u_1^* < \phi_2 < u_2^*$, $v_1^* < \phi_2 < v_2^*$, and $\phi_i' > 0$ $(i = 1,2)$ for all $(z,t) \in \mathbb{R}^2$, we have $A_1(\phi_1,\phi_2) > 0$ for all $(z,t) \in \mathbb{R}\times(-\infty,0]$.

Now we verify that $\mathcal{F}_1\left(\overline{U}(z,t), \overline{V}(z,t)\right) \geq 0$ for $(z,t) \in \mathbb{R} \times (-\infty, 0]$. The remainder of the proof is divided into three steps.

Step 1. We give some estimates on the functions $Q_1(\phi_1(z-p_1(t)), \phi_2(z+p_2(t)))$ and $Q_2(\psi_1(z-p_1(t)), \psi_2(z+p_2(t)))$. If $p_2(0) \ll -1$, then $p_2(t)$ can small enough, it follows from (5.1) and (5.3) of Proposition 5.1 that

$$0 < \phi_2(z+p_2(t)) - u_1^* \leq \frac{C_0}{\rho} e^{\eta_1(z+p_2)} \leq \frac{C_0}{\rho} e^{\eta_1 p_2} \leq \frac{u_2^* - u_1^*}{2}, \text{ for } z \leq 0,\ t \leq 0. \quad (5.18)$$

Thus, there exists a positive constant $\mu_1 > 0$ such that

$$Q_{1x}(\phi_1, \phi_2) = \frac{u_1^*(u_2^* - u_1^*)\phi_2(u_2^* - \phi_2)}{[\phi_1(\phi_2 - u_1^*) + u_1^*(u_2^* - \phi_2)]^2} \geq \frac{(u_1^*)^2(u_2^* - u_1^*)(u_2^* - \phi_2)}{[2u_1^*(u_2^* - u_1^*)]^2} \geq \mu_1 \quad (5.19)$$

for $z \leq 0,\ t \leq 0$. By a similar argument, if $p_1(0) \ll -1$, we have

$$0 < u_1^* - \phi_1(z - p_1(t)) \leq \frac{C_0}{\rho} e^{-\eta_2(z - p_1(t))} \leq \frac{C_0}{\rho} e^{\eta_2 p_1(t)} \leq \frac{u_1^*}{2}, \text{ for } z \geq 0,\ t \leq 0. \quad (5.20)$$

Therefore, there exists $\mu_2 > 0$ such that

$$Q_{1y}(\phi_1, \phi_2) = \frac{u_1^*(u_2^* - u_1^*)\phi_1(u_2^* - \phi_1)}{[\phi_1(\phi_2 - u_1^*) + u_1^*(u_2^* - \phi_2)]^2} \geq \frac{u_1^*(u_2^* - u_1^*)^2\phi_1}{[2u_1^*(u_2^* - u_1^*)]^2} \geq \mu_2, \quad (5.21)$$

for $z \geq 0,\ t \leq 0$. Moreover, we have the following estimates about Q_1.

$$Q_{1xx}(\phi_1, \phi_2) = (\phi_2 - u_1^*)(\phi_2 - u_2^*) \frac{2u_1^*(u_2^* - u_1^*)\phi_2}{[\phi_1(\phi_2 - u_1^*) + u_1^*(u_2^* - \phi_2)]^3}, \quad (5.22)$$

$$Q_{1xy}(\phi_1, \phi_2) = u_1^*(u_2^* - u_1^*) \frac{(2u_1^* - u_2^*)\phi_1\phi_2 + u_1^* u_2^*(u_2^* - \phi_1 - \phi_2)}{[\phi_1(\phi_2 - u_1^*) + u_1^*(u_2^* - \phi_2)]^3}, \quad (5.23)$$

$$Q_{1yy}(\phi_1, \phi_2) = \phi_1(\phi_1 - u_1^*) \frac{2u_1^*(u_2^* - u_1^*)(\phi_1 - u_2^*)}{[\phi_1(\phi_2 - u_1^*) + u_1^*(u_2^* - \phi_2)]^3}. \quad (5.24)$$

From (5.20) we have $\phi_1(z - p_1(t)) \geq u_1^*/2$ for $z \geq 0$ and $t \leq 0$, then

$$\begin{aligned}\phi_1(\phi_2 - u_1^*) + u_1^*(u_2^* - \phi_2) &\geq \frac{u_1^*}{2}[\phi_2(z+p_2(t)) - u_1^*] + u_1^*[u_2^* - \phi_2(z+p_2(t))] \\ &= \frac{u_1^*}{2}[2u_2^* - u_1^* - \phi_2(z+p_2(t))] \geq \frac{u_1^*(u_2^* - u_1^*)}{2},\end{aligned}$$

for $z \geq 0,\ t \leq 0$. Similarly, from (5.18) we get

$$\phi_1(\phi_2 - u_1^*) + u_1^*(u_2^* - \phi_2) \geq \frac{u_1^*(u_2^* - u_1^*)}{2}, \text{ for } z \leq 0,\ t \leq 0.$$

Thus, there exists a constant C' such that

$$\begin{aligned}&|Q_{1xx}(\phi_1(z-p_1(t)), \phi_2(z+p_2(t)))|, \quad \left|Q_{1xy}(\phi_1(z-p_1(t)), \phi_2(z+p_2(t)))\right|, \\ &\left|Q_{1yy}(\phi_1(z-p_1(t)), \phi_2(z+p_2(t)))\right| \leq C', \text{ uniformly in } (z,t) \in \mathbb{R} \times (-\infty, 0]. \quad (5.25)\end{aligned}$$

Step 2. We now estimate

$$\frac{F_1(\phi_1, \phi_2, \psi_1, \psi_2)}{A_1(\phi_1, \phi_2)} \leq L_1 e^{\eta_1 p_2(t)},\ z \leq 0 \text{ and } \frac{F_1(\phi_1, \phi_2, \psi_1, \psi_2)}{A_1(\phi_1, \phi_2)} \leq L_1 e^{\eta_2 p_1(t)},\ z \geq 0. \quad (5.26)$$

for some constant $L_1 > 0$. Let $x = \phi_1(z - p_1(t))$, $y = \phi_2(z + p_2(t))$ in Q_1 and $x = \psi_1(z - p_1(t))$, $y = \psi_2(z + p_2(t))$ in Q_2, then F_1 satisfies

$$\begin{aligned} F_1(\phi_1, \phi_2, \psi_1, \psi_2) &= f_1(Q_1, Q_2) - Q_{1x} f_1(\phi_1, \psi_1) - Q_{1y} f_1(\phi_2, \psi_2) \\ &= -Q_1 + \alpha Q_2 - Q_{1x}(-\phi_1 + \alpha\psi_1) - Q_{1y}(-\phi_2 + \alpha\psi_2). \end{aligned}$$

Then by Lemma 5.2, we obtain

$$F_1(\phi_1, u_1^*, \psi_1, v_1^*) = -\phi_1 + \alpha\psi_1 - (-\phi_1 + \alpha\psi_1) - Q_{1y}(\phi_1, u_1^*)(-u_1^* + \alpha v_1^*) = 0.$$

Similarly, we have

$$F_1(\phi_1, u_1^*, \psi_1, v_1^*) = F_1(\phi_1, u_2^*, \psi_1, v_2^*) = F_1(0, \phi_2, 0, \psi_2) = F_1(u_1^*, \phi_2, v_1^*, \psi_2) = 0.$$

Thus, there exist functions $F_{11}, F_{12}, F_{13} \in C(D_1 \times D_2)$ such that for $z \le p_1(t)$, we have the expression

$$F_1(\phi_1, \phi_2, \psi_1, \psi_2) = (\phi_1 + \psi_1)[(\phi_2 - u_1^*) + (\psi_2 - v_1^*)]F_{11}(\phi_1, \phi_2, \psi_1, \psi_2).$$

Similarly, we have

$$F_1(\phi_1, \phi_2, \psi_1, \psi_2) = [(\phi_1 - u_1^*) + (\psi_1 - v_1^*)][(\phi_2 - u_2^*) + (\psi_2 - v_2^*)]F_{12}(\phi_1, \phi_2, \psi_1, \psi_2)$$

for $z \ge -p_2(t)$, and

$$F_1(\phi_1, \phi_2, \psi_1, \psi_2) = [(\phi_1 - u_1^*) + (\psi_1 - v_1^*)][(\phi_2 - u_1^*) + (\psi_2 - v_1^*)]F_{13}(\phi_1, \phi_2, \psi_1, \psi_2)$$

for $p_1(t) \le z \le -p_2(t)$, where $\phi_1 = \phi_1(z - p_1(t))$, $\phi_2 = \phi_2(z + p_2(t))$, $\psi_1 = \psi_1(z - p_1(t))$, $\psi_2 = \psi_2(z + p_2(t))$. It is easy to see that there exists a positive constant C_3 such that $|(F_{11}, F_{12}, F_{13})(\phi_1, \phi_2, \psi_1, \psi_2)| \le C_3$.

Next we consider two cases $z \in (-\infty, p_1(t)] \cup [-p_2(t), +\infty)$ and $z \in [p_1(t), -p_2(t)]$, respectively.

Case I. $z \in (-\infty, p_1(t)] \cup [-p_2(t), +\infty)$, then by using Proposition 5.1, (5.19) and the above prepared results, for $z \le p_1(t)$ and $t \le 0$ we have

$$\begin{aligned} \frac{F_1(\phi_1, \phi_2, \psi_1, \psi_2)}{A_1(\phi_1, \phi_2)} &= \frac{(\phi_1 + \psi_1)[(\phi_2 - u_1^*) + (\psi_2 - v_1^*)]F_{11}(\phi_1, \phi_2, \psi_1, \psi_2)}{Q_{1x}\phi_1' + Q_{1y}\phi_2'} \\ &\le \frac{(1 + \psi_1/\phi_1)(|\phi_2 - u_1^*| + |\psi_2 - v_1^*|)|F_{11}(\phi_1, \phi_2, \psi_1, \psi_2)|}{Q_{1x}\phi_1'/\phi_1} \\ &\le \frac{(1 + C_2/C_1)(|\phi_2'|/\rho + |\psi_2'|/\rho)C_3}{\mu_1 \rho} \\ &\le \frac{C_4}{\mu_1 \rho^2} e^{\eta_1(z + p_2(t))} \le L_2 e^{\eta_1 p_2(t)}, \end{aligned} \tag{5.27}$$

for some constant $L_2 > 0$. Similarly, there exists some constant $L_3 > 0$ such that

$$\begin{aligned} \frac{F_1(\phi_1, \phi_2, \psi_1, \psi_2)}{A_1(\phi_1, \phi_2)} &= \frac{[(\phi_1 - u_1^*) + (\psi_1 - v_1^*)][(\phi_2 - u_2^*) + (\psi_2 - v_2^*)]F_{12}(\phi_1, \phi_2, \psi_1, \psi_2)}{Q_{1x}\phi_1' + Q_{1y}\phi_2'} \\ &\le \frac{[1 + (v_2^* - \psi_2)/(u_2^* - \phi_2)](|\phi_1 - u_1^*| + |\psi_1 - v_1^*|)|F_{12}|}{Q_{1y}\phi_2'/(u_2^* - \phi_2)} \\ &\le L_3 e^{\eta_2 p_1(t)}, \ \text{for } z \ge -p_2(t), \ t \le 0. \end{aligned} \tag{5.28}$$

Case II. $z \in [p_1(t), -p_2(t)]$. Firstly, for $p_1(t) \le z \le 0$ and $t \le 0$, there is

$$\frac{F_1(\phi_1,\phi_2,\psi_1,\psi_2)}{A_1(\phi_1,\phi_2)} = \frac{[(\phi_1-u_1^*)+(\psi_1-v_1^*)][(\phi_2-u_1^*)+(\psi_2-v_1^*)]F_{13}(\phi_1,\phi_2,\psi_1,\psi_2)}{Q_{1x}\phi_1'+Q_{1y}\phi_2'} \le \frac{[1+(v_1^*-\psi_1)/(u_1^*-\phi_1)]2C_3C_2e^{\eta_1(z+p_2(t))}}{Q_{1x}\phi_1'/(u_1^*-\phi_1)} \le L_4e^{\eta_1 p_2(t)}. \quad (5.29)$$

For $0 \le z \le -p_2(t)$ and $t \le 0$, we also have

$$\frac{F_1(\phi_1,\phi_2,\psi_1,\psi_2)}{A_1(\phi_1,\phi_2)} = \frac{[(\phi_1-u_1^*)+(\psi_1-v_1^*)][(\phi_2-u_1^*)+(\psi_2-v_1^*)]F_{13}(\phi_1,\phi_2,\psi_1,\psi_2)}{Q_{1x}\phi_1'+Q_{1y}\phi_2'} \le \frac{[1+(\psi_2-v_1^*)/(\phi_2-u_1^*)]2C_3C_2e^{-\eta_2(z-p_1(t))}}{Q_{1y}\phi_2'/(\phi_2-u_1^*)} \le L_5e^{\eta_2 p_1(t)}. \quad (5.30)$$

Then take $L_1 = \max\{L_i, i = 2,3,4,5\}$ and combing (5.27)–(5.30), we conclude that (5.26) hold.

Step 3. Next we estimate the following inequalities:

$$\frac{H_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_1'e^{\eta_1 p_2(t)},\ z \le 0 \text{ and } \frac{H_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_1'e^{\eta_2 p_1(t)},\ z \ge 0, \quad (5.31)$$

for some constant $L_1' > 0$. For simplicity, let's denote

$$\hat\phi_1(\theta) = \phi_1(z-p_1(t)-\theta r) \text{ and } \hat\phi_2(\theta) = \phi_2(z+p_2(t)-\theta r),\ \theta \in [0,1], r \in \mathbb{R}.$$

Note that

$$\begin{aligned}
&H_1(\phi_1,\phi_2)\\
&= d\int_{\mathbb{R}} J(r)\big[Q_1\big(\hat\phi_1(1),\hat\phi_2(1)\big) - Q_1\big(\hat\phi_1(0),\hat\phi_2(0)\big)\big]dr\\
&\quad - dQ_{1x}\int_{\mathbb{R}} J(r)[\hat\phi_1(1)-\hat\phi_1(0)]dr - dQ_{1y}\int_{\mathbb{R}} J(r)[\hat\phi_2(1)-\hat\phi_2(0)]dr\\
&= d\int_{\mathbb{R}} J(r)Q_{1x}\big(\theta_1\hat\phi_1(1)+(1-\theta_1)\hat\phi_1(0),\hat\phi_2(1)\big)\big[\hat\phi_1(1)-\hat\phi_1(0)\big]dr\\
&\quad + d\int_{\mathbb{R}} J(r)Q_{1y}\big(\hat\phi_1(0),\theta_2\hat\phi_2(1)+(1-\theta_2)\hat\phi_2(0)\big)\big[\hat\phi_2(1)-\hat\phi_2(0)\big]dr\\
&\quad - d\int_{\mathbb{R}} J(r)Q_{1x}\big(\hat\phi_1(0),\hat\phi_2(0)\big)[\hat\phi_1(1)-\hat\phi_1(0)]dr\\
&\quad - d\int_{\mathbb{R}} J(r)Q_{1y}\big(\hat\phi_1(0),\hat\phi_2(0)\big)[\hat\phi_2(1)-\hat\phi_2(0)]dr\\
&= d\int_{\mathbb{R}} J(r)\Big\{\big[Q_{1x}\big(\theta_1\hat\phi_1(1)+(1-\theta_1)\hat\phi_1(0),\hat\phi_2(1)\big)-Q_{1x}\big(\hat\phi_1(0),\hat\phi_2(0)\big)\big]\big[\hat\phi_1(1)-\hat\phi_1(0)\big]\\
&\quad + \big[Q_{1y}\big(\hat\phi_1(0),\theta_2\hat\phi_2(1)+(1-\theta_2)\hat\phi_2(0)\big)-Q_{1y}\big(\hat\phi_1(0),\hat\phi_2(0)\big)\big]\big[\hat\phi_2(1)-\hat\phi_2(0)\big]\Big\}dr\\
&= d\int_{\mathbb{R}} J(r)\Big\{Q_{1xx}\big(\theta_3\hat\phi_1(1)+(1-\theta_3)\hat\phi_1(0),\hat\phi_2(1)\big)\theta_1\big[\hat\phi_1(1)-\hat\phi_1(0)\big]^2\\
&\quad + Q_{1xy}\big(\hat\phi_1(0),\theta_4\hat\phi_2(1)+(1-\theta_4)\hat\phi_2(0)\big)\big[\hat\phi_1(1)-\hat\phi_1(0)\big]\big[\hat\phi_2(1)-\hat\phi_2(0)\big]\\
&\quad + Q_{1yy}\big(\hat\phi_1(0),\theta_5\hat\phi_2(1)+(1-\theta_5)\hat\phi_2(0)\big)\theta_2\big[\hat\phi_2(1)-\hat\phi_2(0)\big]^2\Big\}dr,
\end{aligned}$$

where $\theta_i \in (0,1)(i = 1, \cdots, 5)$. Note that from (5.22)–(5.25), there exists a positive constant $\widetilde{C}'$ such that

$$\left|Q_{1xx}(\theta_3\hat{\phi}_1(1) + (1-\theta_3)\hat{\phi}_1(0), \hat{\phi}_2(1))\right| \le \widetilde{C}'[\hat{\phi}_2(1) - u_1^*],$$
$$\left|Q_{1yy}(\hat{\phi}_1(0), \theta_5\hat{\phi}_2(1) + (1-\theta_5)\hat{\phi}_2(0))\right| \le \widetilde{C}'\hat{\phi}_1(0),$$
$$\left|Q_{1xy}(\hat{\phi}_1(0), \theta_4\hat{\phi}_2(1) + (1-\theta_4)\hat{\phi}_2(0))\right| \le \widetilde{C}'.$$

Therefore, we have

$$\frac{H_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le d\widetilde{C}' \int_{\mathbb{R}} J(r) \left\{ \frac{[\hat{\phi}_1(1) - \hat{\phi}_1(0)]^2[\hat{\phi}_2(1) - u_1^*]}{Q_{1x}\phi_1'(z-p_1) + Q_{1y}\phi_2'(z+p_2)} \right.$$
$$\left. + \frac{[\hat{\phi}_1(1) - \hat{\phi}_1(0)][\hat{\phi}_2(1) - \hat{\phi}_2(0)] + [\hat{\phi}_2(1) - \hat{\phi}_2(0)]^2\hat{\phi}_1(0)}{Q_{1x}\phi_1'(z-p_1) + Q_{1y}\phi_2'(z+p_2)} \right\} dr.$$

Let

$$B_1(\phi_1,\phi_2) = r^2[\phi_1'(z - p_1 - \theta_6 r)]^2[\phi_2(z + p_2 - r) - u_1^*],$$
$$C_1(\phi_1,\phi_2) = r^2\phi_1'(z - p_1 - \theta_7 r)\phi_2'(z + p_2 - \theta_8 r),$$
$$D_1(\phi_1,\phi_2) = r^2[\phi_2'(z + p_2 - \theta_9 r)]^2\phi_1(z - p_1),$$

where $\theta_i \in (0,1)(i = 6, \cdots, 9)$ and $r \in [-M, M]$, M is defined in (J2).

For $z \le p_1(t) < 0$, we have $z - p_1(t) - \theta_6 r \le M$ and $z + p_2(t) - r \le M$, then by (5.1), (5.3), (5.19) and (5.2) we get

$$\frac{B_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le \frac{r^2[\phi_1'(z - p_1 - \theta_6 r)]^2[\phi_2(z + p_2 - r) - u_1^*]}{Q_{1x}\phi_1'(z-p_1)}$$
$$\le \frac{M^2C_0^2e^{2\eta_1(z-p_1-\theta_6 r)}}{\mu_1\rho\phi_1(z-p_1)}[\phi_2(z + p_2 - r) - u_1^*]$$
$$\le \frac{M^2C_0^2e^{2\eta_1(z-p_1-\theta_6 r)}}{\mu_1\rho C_1 e^{\eta_1(z-p_1)}} C_2 e^{\eta_1(z+p_2-r)} \le L_2' e^{\eta_1 p_2(t)},$$

for some constant $L_2' > 0$. Similarly, for $p_1(t) \le z \le 0$, there also exists a constant $L_3' > 0$ such that

$$\frac{B_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_3' e^{\eta_1 p_2(t)}.$$

By a similar argument as above, we obtain

$$\frac{B_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_4' e^{\eta_2 p_1(t)},\quad \frac{C_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_6' e^{\eta_2 p_1(t)},\quad \frac{D_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_8' e^{\eta_2 p_1(t)} \text{ for } z \ge 0,$$
$$\frac{C_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_5' e^{\eta_1 p_2(t)},\quad \frac{D_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_7' e^{\eta_1 p_2(t)} \text{ for } z \le 0,$$

for some constants $L_i' > 0$, $(i = 4, \cdots, 8)$. Then taking $L_1' = d\widetilde{C}'\Sigma_{i=2}^8 L_i'$, we get

$$\frac{H_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le d\widetilde{C}' \int_{\mathbb{R}} J(r) \left\{ \frac{B_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} + \frac{C_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} + \frac{D_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \right\} dr$$
$$\le L_1' e^{\eta_1 p_2(t)}, \text{ for } z \le 0, t \le 0,$$

and

$$\frac{H_1(\phi_1,\phi_2)}{A_1(\phi_1,\phi_2)} \le L_1' e^{\eta_2 p_1(t)}, \text{ for } z \ge 0, t \le 0.$$

Choose $0 < \eta_3 < \min\{\eta_1, (p_2(0)\eta_1)/p_1(0)\}$. It is easy to show that $\eta_1 p_2(t) \le \eta_3 p_1(t) < 0$. Finally, applying (5.26) and (5.31), letting $N \ge L_1 + L_1'$ and $\sigma \le \min\{\eta_2, \eta_3\}$, we have

$$\begin{aligned}\mathcal{F}_1(\overline{U},\overline{V}) &\ge A_1(\phi_1,\phi_2)\left[Ne^{\sigma p_1(t)} - (L_1+L_1')e^{\eta_1 p_2(t)}\right]\\ &\ge A_1(\phi_1,\phi_2)\left[Ne^{\sigma p_1(t)} - (L_1+L_1')e^{\eta_3 p_1(t)}\right] \ge 0,\end{aligned}$$

uniformly in $(z,t) \in (-\infty,0]\times(-\infty,0]$. And

$$\mathcal{F}_1(\overline{U},\overline{V}) \ge A_1(\phi_1,\phi_2)\left[Ne^{\sigma p_1(t)} - (L_1+L_1')e^{\eta_2 p_1(t)}\right] \ge 0,$$

uniformly in $(z,t) \in [0,+\infty)\times(-\infty,0]$. Thus, $\mathcal{F}_1(\overline{U},\overline{V}) \ge 0$ for all $(z,t) \in \mathbb{R}\times(-\infty,0]$.

Next, we show that $\mathcal{F}_2(\overline{U},\overline{V}) \ge 0$ for $(z,t) \in \mathbb{R}\times(-\infty,0]$.

$$\begin{aligned}\mathcal{F}_2(\overline{U},\overline{V}) &= -Q_{2x}\psi_1' p_1' + Q_{2y}\psi_2' p_2' + \bar{c}Q_{2x}\psi_1' + \bar{c}Q_{2y}\psi_2' + \beta Q_2 - g(Q_1)\\ &= Q_{2x}\psi_1'(-p_1' + \bar{c} - c_1) + Q_{2y}\psi_2'(p_2' + \bar{c} - c_2) - [\beta Q_{2x}\psi_1 + \beta Q_{2y}\psi_2 - \beta Q_2]\\ &\quad - [g(Q_1) - Q_{2x}g(\phi_1) - Q_{2y}g(\phi_2)]\\ &= Q_{2x}\psi_1' Ne^{\sigma p_1(t)} + Q_{2y}\psi_2' Ne^{\sigma p_2(t)} - H_2(\phi_1,\phi_2,\psi_1,\psi_2),\end{aligned}$$

where

$$H_2(\phi_1,\phi_2,\psi_1,\psi_2) = [g(Q_1) - Q_{2x}g(\phi_1) - Q_{2y}g(\phi_2)] + [\beta Q_{2x}\psi_1 + \beta Q_{2y}\psi_2 - \beta Q_2].$$

By virtue of (5.9), we have

$$\mathcal{F}_2(\overline{U},\overline{V}) \ge A_2(\psi_1,\psi_2)\left[Ne^{\sigma p_1(t)} - \frac{H_2(\phi_1,\phi_2,\psi_1,\psi_2)}{A_2(\psi_1,\psi_2)}\right],$$

where $A_2(\psi_1,\psi_2) = Q_{2x}\psi_1' + Q_{2y}\psi_2'$. Similar to those argument about $A_1(\phi_1,\phi_2)$, we get $A_2(\psi_1,\psi_2) > 0$ for all $(z,t) \in \mathbb{R}\times(-\infty,0]$. Now we show that

$$\frac{H_2(\phi_1,\phi_2,\psi_1,\psi_2)}{A_2(\psi_1,\psi_2)} \le N_1 e^{\eta_1 p_2(t)}, \qquad z \le 0, \tag{5.32}$$

$$\frac{H_2(\phi_1,\phi_2,\psi_1,\psi_2)}{A_2(\psi_1,\psi_2)} \le N_1 e^{\eta_2 p_1(t)}, \qquad z \ge 0, \tag{5.33}$$

for some constant $N_1 > 0$. With the aid of Lemma 5.2 we obtain that

$$\begin{aligned}H_2(\phi_1,u_1^*,\psi_1,v_1^*) &= g(Q_1(\phi_1,u_1^*)) - g(\phi_1) - Q_{2y}g(u_1^*) - \beta Q_2(\psi_1,v_1^*) + \beta\psi_1 + \beta Q_{2y}v_1^*\\ &= g(\phi_1) - g(\phi_1) - Q_{2y}g(u_1^*) - \beta\psi_1 + \beta\psi_1 + \beta Q_{2y}v_1^* = 0.\end{aligned}$$

Similarly,

$$H_2(\phi_1,u_1^*,\psi_1,v_1^*) = H_2(\phi_1,u_2^*,\psi_1,v_2^*) = H_2(0,\phi_2,0,\psi_2) = H_2(u_1^*,\phi_2,v_1^*,\psi_2) = 0.$$

Thus we have the following expressions

$H_2(\phi_1, \phi_2, \psi_1, \psi_2) = (\phi_1 + \psi_1)[(\phi_2 - u_1^*) + (\psi_2 - v_1^*)]H_{21}(\phi_1, \phi_2, \psi_1, \psi_2)$, for $z \le p_1(t)$,
$H_2(\phi_1, \phi_2, \psi_1, \psi_2) = [(\phi_1 - u_1^*) + (\psi_1 - v_1^*)][(\phi_2 - u_2^*) + (\psi_2 - v_2^*)]H_{22}(\phi_1, \phi_2, \psi_1, \psi_2)$, for $z \ge -p_2(t)$,
$H_2(\phi_1, \phi_2, \psi_1, \psi_2) = [(\phi_1 - u_1^*) + (\psi_1 - v_1^*)][(\phi_2 - u_1^*) + (\psi_2 - v_1^*)]H_{23}(\phi_1, \phi_2, \psi_1, \psi_2)$, for $p_1(t) \le z \le -p_2(t)$.

Then by a similar argument as $\mathcal{F}_1(\overline{U}, \overline{V})$, we can prove that

$$\mathcal{F}_2(\overline{U}, \overline{V}) \ge 0, \text{ for all } (z, t) \in \mathbb{R} \times (-\infty, 0].$$

The proofs of $\mathcal{F}_i(\underline{U}, \underline{V}) \le 0, i = 1, 2$ are similar to that of $\mathcal{F}_i(\overline{U}, \overline{V}) \ge 0, i = 1, 2$, so we omit the details.

Finally, we show (5.13) and (5.14), we only prove (5.13) since (5.14) can be proved similarly. In fact, it is easy to show that $Q_{ix} \ge 0$ and $Q_{iy} \ge 0$ on D_i, $i = 1, 2$, and

$$\begin{aligned} &\overline{U}(z, t) - \underline{U}(z, t) \\ &= Q_{1x}(\theta_1\hat{\phi}_1(0) + (1 - \theta_1)\phi_1(z - p_2), \hat{\phi}_2(0))[\hat{\phi}_1(0) - \phi_1(z - p_2)] \\ &\quad + Q_{1y}(\phi_1(z - p_2), \theta_2\hat{\phi}_2(0) + (1 - \theta_2)\phi_2(z + p_1))[\hat{\phi}_2(0) - \phi_2(z + p_1)]. \end{aligned}$$

From (5.9) and $\phi_i' > 0 (i = 1, 2)$, we know that $\hat{\phi}_1(0) - \phi_1(z - p_2) \ge 0$ and $\hat{\phi}_2(0) - \phi_2(z + p_1) \ge 0$. Consequently, we have $\overline{U}(z, t) \ge \underline{U}(z, t)$ and

$$\sup_{x \in \mathbb{R}} \left(\overline{U}(z, t) - \underline{U}(z, t)\right) \le |Q_{1x}||\phi_1'|(p_2(t) - p_1(t)) + |Q_{1y}||\phi_2'|(p_2(t) - p_1(t)) \le Ce^{c_0\sigma t}.$$

This complete the proof. □

From the equivalent of system (1.3) and (5.10), it is easy to verify that

$$\begin{cases} \bar{u}(x, t) := Q_1(\phi_1(x + \bar{c}t - p_1(t)), \phi_2(x + \bar{c}t + p_2(t))), \\ \bar{v}(x, t) := Q_2(\psi_1(x + \bar{c}t - p_1(t)), \psi_2(x + \bar{c}t + p_2(t))), \end{cases}$$

and

$$\begin{cases} \underline{u}(x, t) := Q_1(\phi_1(x + \bar{c}t - p_2(t)), \phi_2(x + \bar{c}t + p_1(t))), \\ \underline{v}(x, t) := Q_2(\psi_1(x + \bar{c}t - p_2(t)), \psi_2(x + \bar{c}t + p_1(t))), \end{cases}$$

is a pair of super and subsolutions of (1.3) for $x \in \mathbb{R}$ and $t \le 0$.

Proof of Theorem 1.4 ire solutions of (1.3) described as Theorem 1.4. Consider the following Cauchy problem

$$\begin{cases} (u_n)_t(x, t) = d(J * u_n - u_n)(x, t) - u_n(x, t) + \alpha v_n(x, t), \quad x \in \mathbb{R}, t > -n, \\ (v_n)_t(x, t) = -\beta v_n(x, t) + g(u_n(x, t)), \quad x \in \mathbb{R}, t > -n, \\ u_n(x, -n) := \underline{u}(x, -n) = Q_1(\phi_1(x - \bar{c}n - p_2(-n)), \phi_2(x - \bar{c}n + p_1(-n))), \quad x \in \mathbb{R}, \\ v_n(x, -n) := \underline{v}(x, -n) = Q_2(\psi_1(x - \bar{c}n - p_2(-n)), \psi_2(x - \bar{c}n + p_1(-n))), \quad x \in \mathbb{R}. \end{cases}$$

Then the remainder of the proof is almost same as that of Theorem 1.1, so we omit it. □

5.3 Proofs of Theorems 1.6 and 1.7

We first define the following auxiliary functions:

$$Q_1^*(x,y)=\frac{u_1^*u_2^*(x+y)-(u_1^*+u_2^*)xy}{u_1^*u_2^*-xy},\quad (x,y)\in D_1^*, \tag{5.34}$$

$$Q_2^*(x,y)=\frac{v_1^*v_2^*(x+y)-(v_1^*+v_2^*)xy}{v_1^*v_2^*-xy},\quad (x,y)\in D_2^*, \tag{5.35}$$

where $D_1^*:=\{[0,u_1^*]\times[0,u_2^*]\}\setminus\{(u_1^*,u_2^*)\}$ and $D_2^*:=\{[0,v_1^*]\times[0,v_2^*]\}\setminus\{(v_1^*,v_2^*)\}$. Then the functions $Q_1^*(x,y)$ satisfies

$$Q_1^*(x,y)=x+y(x-u_1^*)\left\{\frac{x-u_2^*}{u_1^*u_2^*-xy}\right\}=y+x(y-u_2^*)\left\{\frac{y-u_1^*}{u_1^*u_2^*-xy}\right\},$$

$$Q_{1x}^*(x,y)=\frac{u_1^*u_2^*(u_1^*-y)(u_2^*-y)}{(u_1^*u_2^*-xy)^2},\quad Q_{1y}^*(x,y)=\frac{u_1^*u_2^*(u_1^*-x)(u_2^*-x)}{(u_1^*u_2^*-xy)^2},$$

$$Q_{1xy}^*(x,y)=\frac{-u_1^*u_2^*\left\{u_2^*(x-u_1^*)(y-u_1^*)+u_1^*(x-u_2^*)(y-u_2^*)\right\}}{(u_1^*u_2^*-xy)^3},$$

$$Q_{1xx}^*(x,y)=y(y-u_2^*)\left\{\frac{2u_1^*u_2^*(y-u_1^*)}{(u_1^*u_2^*-xy)^3}\right\},$$

$$Q_{1yy}^*(x,y)=x(x-u_2^*)\left\{\frac{2u_1^*u_2^*(x-u_1^*)}{(u_1^*u_2^*-xy)^3}\right\},$$

for $(x,y)\in D_1^*$, $Q_2^*(x,t)$ also has the similar properties as $Q_1^*(x,y)$. We define

$$\begin{cases}\overline{U}^*(x,t):=Q_1^*(\phi_1(-z-p_2(t)),\phi(z+p_2(t))),\\ \overline{V}^*(x,t):=Q_2^*(\psi_1(-z-p_2(t)),\psi(z+p_2(t))),\end{cases}$$

and

$$\begin{cases}\underline{U}^*(x,t):=Q_1^*(\phi_1(-z-p_1(t)),\phi(z+p_1(t))),\\ \underline{V}^*(x,t):=Q_2^*(\psi_1(-z-p_1(t)),\psi(z+p_1(t))),\end{cases}$$

for $(z,t)\in\mathbb{R}\times(-\infty,0]$, where Q_1^* and Q_2^* are defined by (5.34) and (5.35) respectively. Then by a similar argument as Lemma 5.3, we can obtain the following lemmas.

Lemma 5.4 *Let all the assumptions of Theorem 1.4 be satisfied. Let $\bar{c}=(c-c_1)/2$, $c_0=(c+c_1)/2$, and $(p_i(t),c_0)(i=1,2)$ be the solutions of (5.7) and (5.8). If $c>-c_1$, then the functions defined by*

$$\begin{cases}\bar{u}^*(x,t):=Q_1^*(\phi_1(-x-\bar{c}t-p_2(t)),\phi(x+\bar{c}t+p_2(t))),\\ \bar{v}^*(x,t):=Q_2^*(\psi_1(-x-\bar{c}t-p_2(t)),\psi(x+\bar{c}t+p_2(t))),\end{cases}$$

and

$$\begin{cases}\underline{u}^*(x,t):=Q_1^*(\phi_1(-x-\bar{c}t-p_1(t)),\phi(x+\bar{c}t+p_1(t))),\\ \underline{v}^*(x,t):=Q_2^*(\psi_1(-x-\bar{c}t-p_1(t)),\psi(x+\bar{c}t+p_1(t))),\end{cases}$$

are a pair of super and subsolutions of (1.3) for $(x,t)\in\mathbb{R}\times(-\infty,0]$. Moreover, (5.13) and (5.14) hold for $(\bar{u}^*(x,t),\bar{v}^*(x,t))$ and $(\underline{u}^*(x,t),\underline{v}^*(x,t))$.

Lemma 5.5 *Assume* (J1)–(J2) *and* (GB) *hold. Let* $\Phi(\cdot)$ *be the traveling front of* (1.3) *connecting* E_0 *and* E_2 *with* $c > 0$ *and* $(p_i(t), c)(i = 1, 2)$ *be the solutions of* (5.7) *and* (5.8). *Then the functions defined by*

$$\begin{cases} \bar{u}(x,t) := Q_1^*(\phi(x - p_2(t)), \phi(-x + p_2(t))), \\ \bar{v}(x,t) := Q_2^*(\psi(x - p_2(t)), \psi(-x + p_2(t))), \end{cases}$$

and

$$\begin{cases} \underline{u}(x,t) := Q_1^*(\phi(x - p_1(t)), \phi(-x + p_1(t))), \\ \underline{v}(x,t) := Q_2^*(\psi(x - p_1(t)), \psi(-x + p_1(t))), \end{cases}$$

are a pair of super and subsolutions of (1.3) *for* $(x,t) \in \mathbb{R} \times (-\infty, 0]$. *Moreover,* (5.13) *and* (5.14) *hold for* $(\bar{u}(x,t), \bar{v}(x,t))$ *and* $(\underline{u}(x,t), \underline{v}(x,t))$.

The proofs of Theorems 1.6 and 1.7 are completely similar to that of Theorem 1.4, so we omit them.

6 Smooth Properties of Entire Solutions

In this section, we prove Theorem 1.8. We only prove that the entire solutions $w(x,t) = (u(x,t), v(x,t))$ established in Theorem 1.1 satisfy (1.7) since the entire solutions established in Theorems 1.3–1.7 can be proved similarly. We first give a continuous lemma for our nonlocal problem (2.1) which plays an important role in the proof.

Lemma 6.1 *Assume* (J1)–(J2) *and* (GM) *and* (H) *hold. Let* $w(x,t) = (u(x,t), v(x,t))$ *be a solution of* (2.1) *with initial value* $w_0(x,0) = (u_0(x), v_0(x)) \in [\mathbf{0}, \mathbf{K}]_X$*, then there exists a positive constant* $M' > 0$*, independent of* w_0*, such that for any* $x \in \mathbb{R}$ *and* $t > 0$,

$$|u_t(x,t)|,\ |u_{tt}(x,t)|,\ |v_t(x,t)|,\ |v_{tt}(x,t)| \le M'.$$

In addition, if there exists $L_0 > 0$ *such that for any* $\eta > 0$,

$$\sup_{x \in \mathbb{R}} |u_0(x+\eta) - u_0(x)| \le L_0 \eta, \quad \sup_{x \in \mathbb{R}} |v_0(x+\eta) - v_0(x)| \le L_0 \eta,$$

then for any $\eta > 0$, $x \in \mathbb{R}$ *and* $t > 0$, *we have*

$$\|w(x+\eta,t) - w(x,t)\| \le M''\eta, \quad \left\| \frac{\partial w}{\partial t}(x+\eta,t) - \frac{\partial w}{\partial t}(x,t) \right\| \le M''\eta, \tag{6.1}$$

where $M'' > 0$ *is some constant which is independent of* w_0 *and* η.

Proof From lemma 2.2, we see that $(0,0) \le (u(x,t), v(x,t)) \le (K_1, K_2)$ for $(x,t) \in \mathbb{R} \times [0, +\infty)$. By (2.1), we obtain that for $x \in \mathbb{R}$, $t \ge 0$,

$$\begin{aligned} &|u_t| \le d|J * u| + (d+1)|u| + \alpha|v| \le (2d+1)K_1 + \alpha K_2 := M_1, \\ &|v_t| \le \beta|v| + |g(u)| \le \beta K_2 + g(K_1) := M_2, \\ &|u_{tt}| = |d(J * u_t) - (d+1)u_t + \alpha v_t| \le (2d+1)M_1 + \alpha M_2 := M_3, \\ &|v_{tt}| = |-\beta v_t + g'(u)u_t| \le \beta M_2 + M_1 \max_{u \in [0, K_1]} g'(u) := M_4. \end{aligned}$$

Take $M' = \max\{M_i, i = 1, 2, 3, 4\}$, then the first statement of this lemma follows. Now we prove (6.1). Note that

$$v(x,t) = e^{-\beta t} v_0(x) + \int_0^t e^{-\beta(t-s)} g(u(x,s))ds, \quad \forall x \in \mathbb{R}, t > 0.$$

Then

$$u_t(x,t) = d(J * u(x,t) - u(x,t)) - u(x,t) + \alpha \left(e^{-\beta t} v_0(x) + \int_0^t e^{-\beta(t-s)} g(u(x,s))ds \right).$$

For any $\eta > 0$, let $(\delta u)(x,t) = u(x+\eta,t) - u(x,t)$, without loss of generality, we assume that $(\delta u)(x,t) \geq 0$. Then

$$\begin{cases} (\delta u)_t \leq d \int_{\mathbb{R}} |J(x+\eta-y) - J(x-y)| u(y,t)dy - (d+1)(\delta u) + \alpha e^{-\beta t} L_0 \eta \\ \qquad\qquad + \alpha \int_0^t e^{-\beta(t-s)} [g(u(x+\eta,s)) - g(u(x,s))]ds, \\ (\delta u)(x,0) = u_0(x+\eta) - u_0(x) \leq L_0 \eta. \end{cases}$$

Since $J' \in L^1(\mathbb{R})$ by (J1) and (J2), there exists $L' > 0$ such that

$$\int_{\mathbb{R}} |J(x+\eta-y) - J(x-y)|dy = \eta \int_{\mathbb{R}} \int_0^1 |J'(x-y+\theta\eta)| d\theta dy \leq L'\eta, \quad \forall \eta > 0.$$

and

$$\alpha \int_0^t e^{-\beta(t-s)} [g(u(x+\eta,s)) - g(u(x,s))]ds \leq \alpha m \int_0^t e^{-\beta(t-s)} (\delta u)(x,s)ds,$$

where $m := \sup_{u \in [0,K_1]} g'(u) > \frac{\beta}{\alpha}$. Thus we get

$$(\delta u)_t \leq dK_1 L' \eta - (d+1)(\delta u) + \alpha L_0 \eta + \alpha m \int_0^t e^{-\beta(t-s)} (\delta u)(x,s)ds.$$

Now we consider the following ordinary equation

$$z'(t) = a_1 \eta - a_2 z(t) + a_3 \int_0^t e^{-\beta(t-s)} z(s)ds, \tag{6.2}$$

where $a_1 = dK_1 L' + \alpha L_0$, $a_2 = d + 1$, $a_3 = \alpha m$. Differential (6.2) about t, we obtain

$$z''(t) = -a_2 z'(t) + a_3 z(t) - \beta a_3 \int_0^t e^{-\beta(t-s)} z(s)ds. \tag{6.3}$$

Combing (6.2) with (6.3), we have

$$\begin{cases} z''(t) + (a_2 + \beta) z'(t) + (a_2 \beta - a_3) z(t) - \beta a_1 \eta = 0, \\ z(0) = L_0 \eta, \quad z'(0) = (a_1 - a_2 L_0)\eta. \end{cases} \tag{6.4}$$

By the linear ordinary differential equations theory, we set $z(t) = c_1(t) e^{\lambda_1 t} + c_2(t) e^{\lambda_2 t}$ is the solution of (6.4), where $\lambda_1 < \lambda_2 < 0$ are the eigenvalues of the following characteristic equation

$$\lambda^2 + (a_2 + \beta)\lambda + (a_2 \beta - a_3) = 0,$$

since (H) implies that $a_2\beta - a_3 = (d+1)\beta - \alpha m > 0$. Then z(t) satisfies

$$\begin{cases} e^{\lambda_1 t}c_1'(t) + e^{\lambda_2 t}c_2'(t) = 0, \\ \lambda_1 e^{\lambda_1 t}c_1'(t) + \lambda_2 e^{\lambda_2 t}c_2'(t) = \beta a_1\eta. \end{cases} \tag{6.5}$$

By (6.5) we get

$$\begin{cases} c_1(t) = \frac{\beta a_1\eta}{\lambda_1(\lambda_2-\lambda_1)}e^{-\lambda_1 t} + k_1, \\ c_2(t) = \frac{\beta a_1\eta}{\lambda_2(\lambda_1-\lambda_2)}e^{-\lambda_2 t} + k_2. \end{cases}$$

Recalling that $z(0) = L_0\eta$ and $z'(0) = (a_1 - a_2L_0)\eta$, we further have

$$\begin{cases} c_1(0) + c_2(0) = L_0\eta, \\ c_1'(0) + c_1(0)\lambda_1 + c_2'(0) + c_2(0)\lambda_2 = (a_1 - a_2L_0)\eta. \end{cases}$$

Then

$$\begin{cases} k_1 + k_2 = L_0\eta - \frac{\beta a_1\eta}{\lambda_1\lambda_2}, \\ k_1\lambda_1 + k_2\lambda_2 = (a_1 - a_2L_0)\eta, \end{cases}$$

that is

$$\begin{aligned} k_1 &= \frac{a_1\lambda_1\eta + \beta a_1\eta - a_2L_0\lambda_1\eta - L_0\lambda_1\lambda_2\eta}{\lambda_1(\lambda_1-\lambda_2)}, \\ k_2 &= \frac{a_1\lambda_2\eta + \beta a_1\eta - a_2L_0\lambda_2\eta - L_0\lambda_1\lambda_2\eta}{\lambda_2(\lambda_2-\lambda_1)}. \end{aligned}$$

Therefore, $z(t) = k_1e^{\lambda_1 t} + k_2e^{\lambda_2 t} + \frac{\beta a_1\eta}{\lambda_1\lambda_2}$. Note that $\lambda_1 < \lambda_2 < 0$ and

$$\begin{aligned} |k_1| &\le \frac{|\beta a_1 + a_1\lambda_1 - a_2L_0\lambda_1 - L_0\lambda_1\lambda_2|}{\lambda_1(\lambda_1-\lambda_2)}\eta \le M_5\eta, \\ |k_2| &\le \frac{|\beta a_1 + a_1\lambda_2 - a_2L_0\lambda_2 - L_0\lambda_1\lambda_2|}{\lambda_2(\lambda_1-\lambda_2)}\eta \le M_6\eta. \end{aligned}$$

Thus $z(t) \le M_5\eta + M_6\eta + \frac{\beta a_1}{\lambda_1\lambda_2}\eta \le M_7\eta$.

Note that δu satisfies

$$\begin{cases} (\delta u)_t \le a_1\eta - a_2(\delta u) + a_3\int_0^t e^{-\beta(t-s)}(\delta u)(x,s)ds, \\ (\delta u)(x,0) \le L_0\eta. \end{cases}$$

Then by the comparison of the ordinary differential equation, we get that for any $x \in \mathbb{R}$ and $t > 0$,

$$|(\delta u)(x,t)| \le z(t) \le M_7\eta,$$

and

$$(\delta v)(x,t) = e^{-\beta t}(v_0(x+\eta) - v_0(x)) + \int_0^t e^{-\beta(t-s)}(g(u(x+\eta,s)) - g(u(x,s)))ds.$$

Therefore,

$$|\delta v| \le L_0\eta + m\int_0^t e^{-\beta(t-s)}|\delta u(x,s)|ds \le L_0\eta + \frac{mM_7\eta}{\beta}(1-e^{-\beta t}) \le M_8\eta.$$

Moreover, for any $x \in \mathbb{R}$ and $t > 0$, we have

$$\begin{aligned}
&\left|\frac{\partial u}{\partial t}(x+\eta,t) - \frac{\partial u}{\partial t}(x,t)\right| \\
&\quad = \left|d\int_{\mathbb{R}} (J(x+\eta-y) - J(x-y))u(y,t)dy - (d+1)(\delta u)(x,t) + \alpha(\delta v)(x,t)\right| \\
&\quad \le dK_1L'\eta + (d+1)M_7\eta + \alpha M_8\eta \le M_9\eta,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial v}{\partial t}(x+\eta,t) - \frac{\partial v}{\partial t}(x,t)\right| \\
&\quad \le \beta|(\delta v)(x,t)| + |g(u(x+\eta,t)) - g(u(x,t))| \\
&\quad \le \beta M_8\eta + mM_7\eta \le M_{10}\eta.
\end{aligned}$$

Then take $M'' = \max\{M_i, i = 7, 8, 9, 10\}$, we obtain (6.1). The proof is complete. □

Proof of Theorem 1.8 Now we consider the initial value problem (4.9). Since $\Phi_i(\cdot)$ and $\Phi_i'(\cdot)$ are uniformly bounded on $\mathbb{R}$, it is easy to show that there exists $L_0 > 0$, such that for any $x \in \mathbb{R}$ and $\eta > 0$,

$$\sup_{x\in\mathbb{R}} \|w_{n,0}(x+\eta) - w_{n,0}(x)\| \le L_0\eta. \tag{6.6}$$

Then the conclusions of Lemma 6.1 are valid for the solution $(u_n(x,t), v_n(x,t)$ of (4.9). Consequently, by Arzela-Ascoli Theorem and a diagonal extraction process, there exists a function $(u_*(x,t), v_*(x,t))$ and a subsequence $(u_{n_i}(x,t), v_{n_i}(x,t))$ of $(u_n(x,t), v_n(x,t))$, such that

$$u_{n_i}(x,t), \quad v_{n_i}(x,t), \quad \frac{\partial}{\partial t}u_{n_i}(x,t), \quad \frac{\partial}{\partial t}v_{n_i}(x,t),$$

converge uniformly in any compact set $S \subset \mathbb{R}^2$ to

$$u_*(x,t), \quad v_*(x,t), \quad \frac{\partial}{\partial t}u_*(x,t), \quad \frac{\partial}{\partial t}v_*(x,t).$$

Then combining the proof of Theorem 1.1 with the uniqueness of the limit, we have $(u_*(x,t), v_*(x,t)) = (u(x,t), v(x,t))$. Let $S \subset \mathbb{R}^2$ be a compact subset with $(x,t), (x+\eta,t) \in S$, then there exists $I_0 \in \mathbb{N}$ such that for any $i > I_0$,

$$|u(y,t) - u_{n_i}(y,t)| \le \eta \quad \text{for any} \quad (y,t) \in S.$$

Let $D_1 = 2 + M''$, where M'' is defined in Lemma 6.1. Therefore, we have

$$\begin{aligned}
&|u(x+\eta,t) - u(x,t)| \\
&\quad \le |u(x+\eta,t) - u_{n_i}(x+\eta,t)| + |u_{n_i}(x+\eta,t) - u_{n_i}(x,t)| + |u_{n_i}(x,t) - u(x,t)| \\
&\quad \le D_1\eta.
\end{aligned}$$

The other inequalities in Theorem 1.8 can be proved similarly. Thus we have proved that Theorem 1.8 is valid for the entire solutions obtained in Theorem 1.1. The proof is complete. □

7 Discussions

We would like to point out that our main results can be extended to the following partially degenerate nonlocal dispersal system

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = d(J * u(x,t) - u(x,t)) + f(u(x,t), v(x,t)), \\ \frac{\partial v}{\partial t}(x,t) = -a_{22}v(x,t) + g(u(x,t)). \end{cases} \tag{7.1}$$

About the local diffusion, we can see [44,46]. An important example is

$$f(u,v) = f_1(u) - a_{11}u + a_{12}v \text{ and } g(u) = ku.$$

We note that in the bistable case, we need the condition (J2) which is used to construct the sub- and supersolutions (see Lemma 5.3). We guess that it is possible to weaken the condition (J2) by changing sub- and supersolutions in bistable case, while it seems very difficult in mathematics. We leave it as a further investigation.

In addition, the condition (J2) is also needed to prove the Lipschitz continuous of the entire solutions established in the current arguments. Though we hope that the results of Theorem 1.8 can be extended to a general kernel function $J \in L^1$, it is difficult to mathematically prove it. We also leave it as a further investigation.

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References

1. Andreu-Vaillo, F., Mazón, J.M., Rossi, J.D., Toledo-Melero, J.: Nonlocal Diffusion Problems, Mathematical Surveys and Monographs. AMS, Providence (2010)
2. Bates, P.W.: On some nonlocal evolution equations arising in materials science. In: Brunner, H., Zhao, X.Q., Zou, X. (eds.) Nonlinear Dynamics and Evolution Equations. Fields Institute Communications, vol. 48, pp. 13–52. AMS, Providence (2006)
3. Bates, P., Fife, P., Ren, X., Wang, X.: Traveling waves in a convolution model for phase transitions. Arch. Ration. Mech. Anal. **138**, 105–136 (1997)
4. Carr, J., Chmaj, A.: Uniqueness of travelling waves for nonlocal monostable equations. Proc. Am. Math. Soc. **132**, 2433–2439 (2004)
5. Capasso, V.: Mathematical Structures of Epidemic Systems, Lecture Notes in Biomath, vol. 97. Springer-Verlag, Heidelberg (1993)
6. Capasso, V., Kunisch, K.: A reaction-diffusion system arising in modelling man-environment diseases. Q. Appl. Math. **46**, 431–450 (1988)
7. Capasso, V., Maddalena, L.: Convergence to equilibrium states for a reaction-diffusion system modeling the spatial spread of a class of bacterial and viral diseases. J. Math. Biol. **13**, 173–184 (1981)
8. Capasso, V., Maddalena, L.: Saddle point behavior for a reaction-diffusion system: application to a class of epidemic models. Math. Comput. Simul. **24**, 540–547 (1982)
9. Capasso, V., Paveri-Fontana, S.: A mathematical model for the 1973 cholera epidemic in the European Mediterranean region. Revue d'Epidemical. et de Santé Publique. **27**, 121–132 (1979)
10. Coville, J., Dupaigne, L.: On a nonlocal reaction-diffusion eqution arising in population dynamics. Proc. R. Soc. Edinburgh **137A**, 727–755 (2007)
11. Crooks, E.C.M., Tsai, J.C.: Front-like entire solutions for equations with convection. J. Differ. Equ. **253**, 1206–1249 (2012)
12. Chen, X.: Existence, uniqueness and asymptotical stability of travelling fronts in non-local evolution equations. Adv. Differ. Equ. **2**, 125–160 (1997)
13. Chen, X., Guo, J.S.: Existence and uniqueness of entire solutions for a reaction-diffusion equation. J. Differ. Equ. **212**, 62–84 (2005)
14. Chen, X., Guo, J.S., Ninomiya, H.: Entire solutions of reaction-diffusion equations with balanced bistable nonlinearity. Proc. R. Soc. Edinburgh **136A**, 1207–1237 (2006)

15. Ermentrout, B., Mcleod, J.: Existence and uniqueness of traveling waves for a neural network. Proc. R. Soc. Edinburgh **123A**, 461–478 (1994)
16. Fang, J., Zhao, X.Q.: Bistable traveling waves for monotone semiflows with applications. J. Eur. Math. Soc. **7**, 173–213 (2005)
17. Fife, P.C., McLeod, J.B.: The approach of solutions of nonlinear diffusion equations to travelling front solutions. Arch. Ration. Mech. Anal. **65**, 335–361 (1977)
18. Fukao, Y., Morita, Y., Ninomiya, H.: Some entire solutions of the Allen–Cahn equation. Taiwan. J. Math. **8**, 15–32 (2004)
19. Gourley, S.A., Wu, J.: Delayed nonlocal diffusive systems in biological invasion and disease spread. Fields Inst. Commun. **48**, 137–200 (2006)
20. Guo, J.S., Morita, Y.: Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations. Discret. Contin. Dyn. Syst. **12**, 193–212 (2005)
21. Guo, J.S., Wu, C.H.: Entire solutions for a two-component competition system in a lattice. Tohoku Math. J. **62**, 17–28 (2010)
22. Hamel, F., Nadirashvili, N.: Entire solution of the KPP eqution. Commun. Pure Appl. Math. **52**, 1255–1276 (1999)
23. Hamel, F., Nadirashvili, N.: Travelling fronts and entire solutions of the Fisher-KPP equation in R^N. Arch. Ration. Mech. Anal. **157**, 91–163 (2001)
24. Kao, C.Y., Lou, Y., Shen, W.: Random dispersal vs non-local dispersal. Discret. Contin. Dyn. Syst. **26**, 551–596 (2010)
25. Lee, C.T., et al.: Non-local concepts in models in biology. J. Theor. Biol. **210**, 201–219 (2001)
26. Li, W.T., Sun, Y.J., Wang, Z.C.: Entire solutions in the Fisher-KPP equation with nonlocal dispersal. Nonlinear Anal. Real World Appl. **11**, 2302–2313 (2010)
27. Li, W.T., Wang, Z.C., Wu, J.: Entire solutions in monostable reaction-diffusion eqautions with delayed nonlinearity. J. Differ. Equ. **245**, 102–129 (2008)
28. Li, W.T., Liu, N.W., Wang, Z.C.: Entire solutions in reaction-advection-diffusion equations in cylinders. J. Math. Pures Appl. **90**, 492–504 (2008)
29. Li, W.T., Zhang, L., Zhang, G.B.: Invasion entire solutions in a competition system with nonlocal dispersal. Discrete Contin. Dyn. Syst. **35**, 1531–1560 (2015)
30. Liu, N.W., Li, W.T., Wang, Z.C.: Entire solutions of reaction-advection-diffusion equations with bistable nonlinearity in cylinders. J. Differ. Equ. **246**, 4249–4267 (2009)
31. Martin, R.H., Smith, H.L.: Abstract functional differential equations and reaction-diffusion equations. Trans. Am. Math. Soc. **321**, 1–44 (1990)
32. Morita, Y., Ninomiya, H.: Entire solutions with merging fronts to reaction-diffusion equations. J. Dyn. Diff. Eqns. **18**, 841–861 (2006)
33. Morita, Y., Tachibana, K.: An entire solution to the Lotka–Volterra competition-diffusion equations. SIAM J. Math. Anal. **40**, 2217–2240 (2009)
34. Murray, J.: Mathematical Biology, 3rd edn. Springer, Berlin, Heidelberg, New York (1993)
35. Pan, S., Li, W.T., Lin, G.: Travelling wave fronts in nonlocal reaction-diffusion systems and applications. Z. Angew. Math. Phys. **60**, 377–392 (2009)
36. Roquejoffre, J.M.: Eventual monotonicity and convergence to travelling fronts for the solutions of parabolic equations in cylinders. Ann. Inst. H. Poincaré Anal. Non Linéaire **14**, 499–552 (1997)
37. Schumacher, K.: Traveling-front solutions for integro-differential equations. I. J. Reine Angew. Math. **316**, 54–70 (1980)
38. Sun, Y.J., Li, W.T., Wang, Z.C.: Entire solutions in nonlocal dispersal equations with bistable nonlinearity. J. Differ. Equ. **251**, 551–581 (2011)
39. Wang, M., Lv, G.: Entire solutions of a diffusion and competitive Lotka–Volterra type system with nonlocal delayed. Nonlinearity **23**, 1609–1630 (2010)
40. Wang, Z.C., Li, W.T., Ruan, S.: Entire solutions in bistable reaction-diffusion equations with nonlocal delayed nonlinearity. Trans. Am. Math. Soc. **361**, 2047–2084 (2009)
41. Wang, Z.C., Li, W.T., Wu, J.: Entire solutions in delayed lattice differential equations with monostable nonlinearity. SIAM J. Math. Anal. **40**, 2392–2420 (2009)
42. Weinberger, H.F.: Long-time behavior of a class of biological models. SIAM J. Math. Anal. **13**, 353–396 (1982)
43. Widder, D.V.: The Laplace Transform. Princeton University Press, Princeton (1941)
44. Wu, S.L.: Entire solutions in a bistable reaction-diffusion system modeling man-environment-man epidemics. Nonlinear Anal. Real World Appl. **13**, 1991–2005 (2012)
45. Wu, S.L., Wang, H.Y.: Front-like entire solutions for monostable reaction-diffusion systems. J. Dyn. Differ. Equ. **25**, 505–533 (2013)

46. Wu, S.L., Sun, Y.J., Liu, S.Y.: Traveling fronts and entire solutions in partially degenerate reaction-diffusion systems with monostable nonlinearity. Discret. Contin. Dyn. Syst. **33**, 921–946 (2013)
47. Xu, D., Zhao, X.Q.: Erratum to Bistable waves in an epidemic model. J. Dyn. Differ. Equ. **17**, 219–247 (2005)
48. Yagisita, H.: Back and global solutions characterizing annihilation dynamics of traveling fronts. Publ. Res. Inst. Math. Sci. **39**, 117–164 (2003)
49. Yu, Z., Yuan, R.: Existence of traveling wave solutions in nonlocal reaction-diffusion systems with delays and applications. ANZIAM. J. **51**, 49–66 (2009)
50. Yu, Z., Yuan, R.: Existence and asymptotics of traveling waves for nonlocal diffusion systems. Chaos Solitons Fractals **45**, 1361–1367 (2012)
51. Zhao, X.Q., Wang, W.: Fisher waves in an epidemic model. Discret. Contin. Dyn. Syst. **4B**, 1117–1128 (2004)