

# A Permutation Related to Non-compact Global Attractors for Slowly Non-dissipative Systems

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**Abstract** We consider scalar reaction-diffusion equations with non-dissipative nonlinearities generating global semiflows which exhibit blow-up in infinite time. This type of equations was only recently approached and the corresponding dynamical systems are known as slowly non-dissipative systems. The existence of unbounded solutions, referred to as growth solutions, requires the introduction of some objects interpreted as equilibria at infinity. By extending known results, we are able to obtain a complete decomposition of the associated non-compact global attractor. The connecting orbit structure is determined based on the Sturm permutation method, which yields a simple criterion for the existence of heteroclinic connections.

**Keywords** Global attractors · Slowly non-dissipative equations · Heteroclinic orbits · Blow-up solutions

## 1 Introduction

We consider the following scalar reaction-diffusion equation

$$\begin{cases} u_t = u_{xx} + f(x, u, u_x), & x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0. \end{cases} \quad (1)$$

where  $f(x, u, u_x) = bu + g(x, u, u_x)$  and  $g : [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is a bounded  $C^2$  function. This implies, from standard theory (see, for instance, [1, 16, 19]), that we are provided with a local solution semigroup defined by  $u(t, \cdot) = S(t)u_0$  for  $t \geq 0$  and initial condition  $u_0$  in

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an adequate phase space. By the comparison principle if  $b < 0$  this semigroup is dissipative, therefore possessing a compact global attractor, i.e., a nonempty compact invariant set attracting every bounded subset of the state space, see for instance [2, 16]. Also, due to the existence of a Lyapunov function [22, 30], the semigroup  $S(t)$  has a gradient-like behavior and, consequently, the attractor is composed of the equilibria and the heteroclinic orbits connecting them. When  $b > 0$  the semigroup is no longer dissipative but we still obtain a non-compact global attractor defined as the nonempty minimal set attracting all bounded sets of the state space. In [6, 7], Ben-Gal has considered this problem for the case  $f = f(u)$  and described the non-compact global attractor in terms of bounded equilibria, equilibria at infinity, and their heteroclinic connecting orbits. In addition, the heteroclinic connectivity was described in terms of previously known blocking principles involving the Morse indices of equilibria and the zero numbers of the differences between equilibrium solutions, see for example [3, 4, 12]. In this paper we consider the general case  $f = f(x, u, u_x)$  and describe the heteroclinic connectivity by means of a permutation introduced in [11] and used extensively for the characterization and equivalence of global attractors, as we can see for instance in [12–14, 29].

Our main result, contained in Theorem 2, describes a simple criterion for the existence of heteroclinic connections on the non-compact global attractor. A permutation of all the equilibria, both bounded and at infinity, allows for the description of the connecting orbit structure in terms of an adjacency notion introduced in [29].

This paper is organized as follows. In Sect. 2 we introduce the class of slowly non-dissipative equations. We also present preliminary results that lead to the existence of grow-up solutions, and provide the definition of non-compact global attractor. In Sect. 3 we obtain the asymptotic behavior of grow-up solutions and introduce the concept of equilibria at infinity. We then prove that, under nondegeneracy conditions, the set of equilibria is non-empty.

In Sect. 4 the heteroclinic connections with the equilibria at infinity are considered. The theory of inertial manifolds is used here to obtain the exact limits of the grow-up solutions. Also, the connecting orbit structure within infinity is described.

The main goal of Sect. 5 is to obtain a permutation associated with Eq. (1) characterizing the non-compact global attractor. We define the suspension of any given meander permutation and consider suspensions of the usual permutation related to (1), obtained from the ordering of the equilibrium points according to their values at  $x = 0$  and  $\pi$ . The permutation obtained from the suspensions is then proved to be realizable by a dissipative equation and is then associated with the non-dissipative Eq. (1).

In Sect. 6 we prove our main result. First, the Morse indices and zero numbers of the equilibria of the dissipative equation are calculated. Then, a correspondence between the equilibria of the non-dissipative Eq. (1) and the equilibria of the dissipative equation is made in such a way that the nodal properties are preserved. We prove that the permutation derived from the suspensions determines the heteroclinic connections in the non-compact global attractor, which is then characterized in terms of the adjacency notion.

## 2 Set-Up and Preliminary Results

We consider the Hilbert space  $X = L^2([0, \pi])$  with norm  $\|\cdot\|$ . Moreover, we denote by  $A$  the sectorial operator  $-\partial_{xx} - bI$ . We know that  $-A$  generates the analytic semigroup  $e^{-tA}$ , [19, 25]. For  $A_1 = A + bI$ , the fractional power spaces

$$X^\alpha := D(A_1^\alpha),$$

for each  $\alpha \geq 0$ , are well defined with the graph norm  $\|x\|_\alpha := \|A_1^\alpha x\|$ ,  $x \in X^\alpha$ .

We assume that  $g = g(x, u, p) : [0, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is a bounded  $C^2$  function, globally Lipschitz in  $(u, p)$ , i.e., there exist positive constants  $L_1$  and  $L_2$  such that

$$|g(x, u^1, p^1) - g(x, u^2, p^2)| \leq L_1|u^1 - u^2| + L_2|p^1 - p^2|. \tag{2}$$

Moreover, to control the decay behavior of the number of zeros of solutions at infinity needed in Sect. 4, we assume that  $L_2 < 1$ . On  $X^\alpha$ , with  $\alpha > \frac{3}{4}$ , the Nemitskii operator of  $g$  defined by

$$G(u)(x) = g(x, u, u_x),$$

takes values in  $X$  and is globally bounded and Lipschitz in  $u \in X^\alpha$ . We take  $\alpha > \frac{3}{4}$  so that  $X^\alpha \subset C^1$ . The solution semigroup  $S(t)$  is then defined in the underlying space  $X^\alpha$  for  $t \geq 0$ ,

$$S(t) : X^\alpha \longrightarrow X^\alpha.$$

The dynamical system generated by Eq. (1) is dissipative if there is a fixed large ball in the state space which attracts all the solutions. A dynamical system is fast non-dissipative if for at least one initial condition the corresponding solution exhibits a blow-up in finite time. A recently introduced class of systems called *slowly non-dissipative* consists of all systems for which global existence and uniqueness is guaranteed for every initial condition, but some solutions blow-up in infinite time. See [5].

Since  $g$  is bounded, the dynamical system generated by (1) is globally defined. Indeed,  $\|S(t)u_0\|_\alpha$  is bounded for each  $0 < t < \infty$  and  $\alpha \in (3/4, 1)$ . If  $t(u_0)$  denotes the maximal time of existence of the solution through  $u_0$ , then it follows from [19] that  $t(u_0) = +\infty$  for all  $u_0 \in X^\alpha$ . This implies that a finite-time blow-up does not take place and the dynamical system obtained is either dissipative or slowly non-dissipative.

We consider an orthonormal basis  $\{\varphi_j(x)\}_{j \in \mathbb{N}_0}$  in  $L^2([0, \pi])$  comprised of eigenfunctions of the operator  $A$  with Neumann boundary conditions, i.e.,  $\varphi_j(x) = \sqrt{\frac{2}{\pi}} \cos jx$  for  $j = 1, 2, \dots$  and  $\varphi_0(x) = \sqrt{\frac{1}{\pi}}$ . We further denote by  $\lambda_j$  the corresponding eigenvalues, which are given by  $\lambda_j = j^2 - b$  for each  $j \in \mathbb{N}_0$ .

**Lemma 1** *If  $b > 0$  then the dynamical system generated by Eq. (1) is slowly non-dissipative.*

*Proof* We consider the eigenspaces  $E_j$  of  $A$  associated with each eigenvalue  $\lambda_j$ . Using the  $L^2$ -eigenprojections any solution  $u(t, x)$  of (1) is represented by

$$u(t, x) = \sum_{j=0}^{\infty} \hat{u}_j(t) \varphi_j(x),$$

where  $\hat{u}_j(t) = \langle u(t, \cdot), \varphi_j(\cdot) \rangle_{L^2}$  with the inner product in  $L^2([0, \pi])$ . Applying the  $E_j$ -projection to Eq. (1) we obtain

$$\frac{d}{dt} \hat{u}_j(t) = \langle u_t(t, \cdot), \varphi_j(\cdot) \rangle_{L^2} = \langle u_{xx} + bu, \varphi_j(\cdot) \rangle_{L^2} + \langle g(x, u, u_x), \varphi_j(\cdot) \rangle_{L^2},$$

which, by selfadjointness of  $A$ , implies

$$\frac{d}{dt} \hat{u}_j(t) = -\lambda_j \hat{u}_j(t) + \langle g(x, u, u_x), \varphi_j(x) \rangle_{L^2}. \tag{3}$$

If we write  $\langle g(x, u, u_x), \varphi_j(x) \rangle_{L^2} = \langle G(u)(x), \varphi_j(x) \rangle_{L^2}$ , where  $G(u)$  is the Nemitskii operator for  $g(u)$ , and denote

$$\hat{g}_j(t) := \langle G(u(t, x))(x), \varphi_j(x) \rangle_{L^2},$$

we finally obtain from (3),

$$\frac{d}{dt} \hat{u}_j(t) = -\lambda_j \hat{u}_j(t) + \hat{g}_j(t). \tag{4}$$

We notice that if  $b > 0$  then  $\lambda_j = j^2 - b < 0$  at least for  $j = 0$ . Then the general solution of the linear non-homogeneous ODE (4) has the form  $\hat{u}_j(t) = \hat{u}_j^p(t) + \hat{u}_j^h(t)$ , where  $\hat{u}_j^p(t)$  is a particular solution of (4) and  $\hat{u}_j^h(t)$  is a solution of the corresponding homogeneous equation. As a particular solution we take  $\hat{u}_j^p(t) = \int_{-\infty}^t e^{-\lambda_j(t-s)} \hat{g}_j(s) ds$  and the solution with initial condition  $\hat{u}_j(0) = \int_{-\infty}^0 e^{\lambda_j s} \hat{g}_j(s) ds + \hat{u}_j^h(0)$ , is given by

$$\hat{u}_j(t) = \hat{u}_j^h(0) e^{-\lambda_j t} + \int_{-\infty}^t e^{-\lambda_j(t-s)} \hat{g}_j(s) ds. \tag{5}$$

Since  $G(u)$  is bounded in  $L^2$  we have that  $\hat{g}_j(t)$  is uniformly bounded for all  $j \in \mathbb{N}_0$ . Consequently, the particular solution  $\hat{u}_j^p(t)$  is uniformly bounded for all  $j \in \mathbb{N}_0$  such that  $\lambda_j < 0$ . On the other hand, whenever  $\lambda_j$  is strictly negative and  $\hat{u}_j^h(0) \neq 0$ , then  $\hat{u}_j^h(t) = \hat{u}_j^h(0) e^{-\lambda_j t}$  grows exponentially as  $t$  goes to infinity.

From the Fourier decomposition using the eigenprojections we conclude that the  $L^2$ -norm of a solution  $u(t, x)$  of (4) is  $\|u(t, \cdot)\|^2 = \sum_{i=0}^{\infty} (\hat{u}_i(t))^2$ . Since we have  $\lambda_j < 0$  at least for  $j = 0$ , if we take an initial condition  $u_0$  such that  $\hat{u}_0^h(0) \neq 0$ , then  $\hat{u}_0(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Thus, the corresponding solution  $u(t, x)$  for such an initial condition exhibits infinite-time blow-up. □

This proof also shows that if  $u(t, \cdot)$  is a grow-up solution then at least one mode  $\hat{u}_j(t)$  for  $j \leq \sqrt{b}$  will grow-up to infinity with  $t$ , since all the remaining modes  $\hat{u}_l(t)$  for  $l > \sqrt{b}$  must remain bounded. In fact, in this case  $\lambda_l = l^2 - b > 0$  and, writing the general solution of (4) as

$$\hat{u}_l(t) = \hat{u}_l(0) e^{-\lambda_l t} + \int_0^t e^{-\lambda_l(t-s)} \hat{g}_l(s) ds, \tag{6}$$

where  $\hat{u}_l(0) = \langle u_0(\cdot), \varphi_l(\cdot) \rangle_{L^2}$ , we obtain

$$|\hat{u}_l(t)| \leq |\hat{u}_l(0)| + \frac{\Gamma}{\lambda_l},$$

using the uniform bound  $\Gamma$  for  $\hat{g}_l$ .

Despite the non-dissipativity of Eq. (1), we can still guarantee the existence of a Lyapunov functional given in the form

$$V(u) = \int_0^\pi H(x, u, u_x) dx \tag{7}$$

where  $H(x, u, p)$  is a smooth function on  $[0, \pi] \times \mathbb{R}^2$  with  $H_{pp} > 0$  (see [22,30]). The functional  $V$  satisfies

$$\frac{d}{dt} V(u) = - \int_0^\pi H_{pp}(x, u, u_x) u_t^2 dx \leq 0.$$

Solutions  $u(t, \cdot) \in X^\alpha$  which are unbounded in time are called *grow-up solutions*. In order to begin the discussion regarding the asymptotic behavior of these unbounded solutions of Eq. (1), we recall the next lemma derived in [6].

**Lemma 2** Consider the Eq. (1) with  $b > 0$ . Then the solution  $u(t, \cdot)$  either converges to some bounded equilibrium as  $t$  goes to infinity or  $u(t, \cdot)$  is a grow-up solution.

The proof follows from an application of LaSalle’s invariance principle [15] to the Lyapunov functional (7). One immediate consequence of these two lemmas is that there does not exist a global attractor for the semigroup  $S(t)$ . We then present the following definition of non-compact global attractor introduced in [6].

**Definition 1** A non-compact global attractor for the Eq. (1) is a non-empty minimal set in  $X^\alpha$  attracting all bounded sets of  $X^\alpha$ .

### 3 Non-compact Global Attractor

Next we consider a grow-up solution and its normalized trajectory in order to obtain a description of the explicit behavior of grow-up solutions. We recall the following lemma obtained in [6].

**Lemma 3** Let  $b > 0$ ,  $b \neq n^2$  for  $n \in \mathbb{N}$ . Consider a grow-up solution  $u(t, \cdot)$  of (1) and its normalized trajectory  $\frac{u(t, \cdot)}{\|u(t, \cdot)\|}$ . Then, there is  $m < \sqrt{b}$  such that  $\frac{u(t, \cdot)}{\|u(t, \cdot)\|}$  converges to  $\varphi_m^l$  in  $L^2$ , where  $\varphi_m^l = \iota\varphi_m$  and  $\iota \in \{+1, -1\}$ . Moreover, a necessary and sufficient condition for the rescaled trajectory to converge to  $\varphi_m^l$  in  $L^2$  is that

$$\lim_{t \rightarrow \infty} \frac{\hat{u}_m^2(t)}{\sum_{j=0}^\infty \hat{u}_j^2(t)} = 1, \tag{8}$$

and the sign of  $\varphi_m^l(0)$  should be the same as  $u(t, 0)$  for all  $t \in (T, \infty)$ , for some  $T > 0$ .

*Proof* Again we let  $u(t, x) = \sum_{j=0}^\infty \hat{u}_j(t)\varphi_j(x)$ , and we remark that  $|\hat{u}_j(t)| \leq c/j^2$  for all  $j > \sqrt{b}$ , since these modes decay. By Lemma 1 at least one mode  $\hat{u}_m(t)$  for some  $m < \sqrt{b}$  will grow-up to infinity with  $t$ . If  $\hat{u}_m$  is the only infinitely growing mode, then

$$\lim_{t \rightarrow \infty} \frac{\hat{u}_l(t)}{(\sum_{j=0}^\infty \hat{u}_j^2(t))^{\frac{1}{2}}} = 0$$

for all  $l \neq m$ . If, on the other hand,  $u(t, \cdot)$  has more than one infinitely growing mode, let  $\hat{u}_m$  denote the one with the lowest subscript. Reasoning as in (5) we see that the growth to infinity of a mode  $\hat{u}_i$  is determined by the term  $\hat{u}_i^h(0)e^{-\lambda_i t}$ . Hence the mode  $\hat{u}_m(t)$  grows exponentially faster than any other growing mode  $\hat{u}_i(t)$ ,  $m < i < \sqrt{b}$ , and we still have

$$\lim_{t \rightarrow \infty} \frac{\hat{u}_i(t)}{(\sum_{j=0}^\infty \hat{u}_j^2(t))^{\frac{1}{2}}} = 0.$$

Then we obtain (8) and, from a zero number stabilization argument, we conclude that the rescaled solution  $\frac{u(t, \cdot)}{\|u(t, \cdot)\|}$  converges to either  $\varphi_m$  or  $-\varphi_m$ . The sign condition on  $u(t, 0)$  for all  $t \in (T, \infty)$  then implies that the limit is  $\varphi_m^l = \iota\varphi_m$ .

Conversely, from

$$\begin{aligned} \lim_{t \rightarrow \infty} \left\| \frac{u(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_m^l(\cdot) \right\|^2 &= 2 - 2 \lim_{t \rightarrow \infty} \frac{\langle u(t, \cdot), \varphi_m^l(\cdot) \rangle}{\|u(t, \cdot)\|} \\ &= 2 - \lim_{t \rightarrow \infty} 2t \frac{\hat{u}_m(t)}{(\sum_{j=0}^\infty \hat{u}_j^2(t))^{\frac{1}{2}}} = 0 \end{aligned}$$

we conclude that

$$\lim_{t \rightarrow \infty} \frac{\hat{u}_m(t)}{(\sum_{j=0}^{\infty} \hat{u}_j^2(t))^{\frac{1}{2}}} = \iota,$$

which implies (8). The  $L^2$  convergence of the rescaled trajectory to  $\varphi'_m$  then implies that  $u(t, 0)$  has the same sign of  $\varphi'_m(0)$  for all  $t \in (T, \infty)$ . □

As we are interested in the limiting objects at infinity, we are particularly concerned with the modes with  $j \leq \sqrt{b}$ . In [18] Hell used Poincaré projections in Hilbert space to study the grow-up solutions interpreted as heteroclinic orbits to points at infinity. Using this approach we show that the projections to infinite norm of the equilibria of the linear equation  $u_t = u_{xx} + bu$  play the role of equilibria at infinity for the given nonlinear Eq. (1). Following [18] we consider the Poincaré projection of the space  $X^\alpha \times \{1\}$  onto the infinite dimensional upper hemisphere

$$\mathcal{H} = \{(\chi, z) \in X^\alpha \times \mathbb{R} : \langle \chi, \chi \rangle_\alpha + z^2 = 1, z \geq 0\},$$

where  $\langle u, v \rangle_\alpha = \langle A_1^\alpha u, A_1^\alpha v \rangle_{L^2}$  is the inner product in  $X^\alpha$ . The hyperplane  $X^\alpha \times \{1\}$  is tangent to the unit sphere in  $X^\alpha \times \mathbb{R}$ . The Poincaré projection is defined in the following way: for a given point  $M$  on the hyperplane, the straight line through  $M$  and the center of the unit sphere  $(0, 0)$  intersects the sphere at two antipodal points, one on the upper hemisphere and one in the lower. The Poincaré projection of  $M$ , which will be denoted by  $\mathcal{P}(M)$ , is defined as the intersection point on the upper hemisphere. As  $M$  goes to infinity on  $X^\alpha \times \{1\}$ ,  $\mathcal{P}(M)$  goes to the boundary of  $\mathcal{H}$ , that is, the equator of the unit sphere  $\mathcal{H}_e := \{(\chi, 0) \in X^\alpha \times \mathbb{R} : \langle \chi, \chi \rangle_\alpha = 1\}$ . By examining the Eq. (1) on  $X^\alpha$  transformed by the projection  $\mathcal{P}$ , Hell has obtained the equilibrium points on the equator

$$\Phi_j^\pm = \{(\chi, z) : \chi_j = \pm 1, z = 0 \text{ and } \chi_n = 0 \forall n \neq j\}, \tag{9}$$

for all  $j \in \mathbb{N}_0$ .

Because the infinity of  $X^\alpha$  is projected onto the equator  $\mathcal{H}_e$  and  $\Phi_j^\pm$  are equilibrium points on  $\mathcal{H}_e$ , we define objects  $\Phi_j^{\infty, \pm}$  at infinity as

$$\mathcal{P}(\Phi_j^{\infty, \pm}) = \Phi_j^\pm,$$

and refer to these as equilibria at infinity. However, we have only a finite number of equilibria at infinity. This follows from Lemma 3 and from the fact that the eigenmodes with  $j > \sqrt{b}$  remain bounded for any grow-up solution. The set of equilibria at infinity is given precisely by

$$E_f^\infty = \{\Phi_j^{\infty, \pm} : j = 0, \dots, [\sqrt{b}]\},$$

where  $[\cdot]$  denotes the integer part.

We decompose the global attractor as

$$\mathcal{A}_f = \mathcal{A}_f^c \cup \mathcal{A}_f^\infty,$$

where  $\mathcal{A}_f^c$  is the maximal compact invariant subset contained in some sufficiently large ball  $B \subset X^\alpha$  and  $\mathcal{A}_f^\infty$  is an unbounded subset. The bounded subset is composed of the set of solutions in  $\mathcal{A}_f$  which remain bounded in the state space  $X^\alpha$  for  $t \geq 0$ . Therefore, we may apply the standard theory for dissipative equations and conclude that the bounded subset  $\mathcal{A}_f^c$  is composed of the set of bounded equilibria  $E_f^c$  and their connecting heteroclinic orbits (see [16]). The unbounded subset  $\mathcal{A}_f^\infty$  contains the set of equilibria at infinity  $E_f^\infty$  and the subset of grow-up solutions.

In the sequel we want to prove that  $E_f^c$  is non-empty. The bounded equilibria  $v \in E_f^c$  of Eq. (1) are the solutions of the ODE boundary value problem

$$\begin{aligned} u_{xx} + bu + g(x, u, u_x) &= 0, \quad x \in (0, \pi) \\ u_x &= 0, \quad x = 0, \pi. \end{aligned} \tag{10}$$

We then associate with (10) the initial value problem

$$\begin{aligned} u_x &= v \\ v_x &= -bu - g(x, u, v), \\ u(0) &= u_0, \quad v(0) = 0. \end{aligned} \tag{11}$$

We remark that the set of solutions  $u = u(x, u_0), v = v(x, u_0)$  of (11) defines the two-dimensional manifold in  $[0, \pi] \times \mathbb{R}^2$

$$L = \{(x, u, v) : u = u(x, u_0), v = v(x, u_0) \text{ solves (11), } u_0 \in \mathbb{R}, 0 \leq x \leq \pi\}.$$

Under the setting of Eq. (1), all trajectories of the initial value problem (11) are guaranteed to exist for all  $x \in [0, \pi]$ . Also let the section curve of  $L$  at  $x = \pi$  be denoted by  $\gamma$ , i.e.,

$$\gamma := \{(x, u(x, u_0), v(x, u_0)) : u_0 \in \mathbb{R}, x = \pi\}. \tag{12}$$

The intersection points of  $\gamma$  with the plane  $v = 0$  correspond to the equilibria of (1) (see [11, 28]).

We next show that under appropriate conditions Eq. (1) has at least one stationary solution. The assertion is already known if (1) is dissipative, as it follows from a series of results derived in [17, 20, 30]. In which concerns non-dissipative equations in the form (1), according to Matano, no satisfactory result was obtained except for the case  $f(x, u, u_x) = f(u)$ , see [21, Remark 4.6]. Also, this state of affairs does not seem to have changed much in the last 30 years. Then, in the next lemma we address this matter and consider non-dissipative equations in the general form (1), under the nondegeneracy condition  $b \neq n^2$ .

**Lemma 4** *If the dynamical system generated by Eq. (1) is*

- (i) *dissipative (i.e., with  $b < 0$ ), or*
- (ii) *slowly non-dissipative (i.e., with  $b > 0$ ) and  $b \neq n^2$  for  $n \in \mathbb{N}$ ,*

*then the set of equilibria  $E_f^c$  is not empty.*

*Proof* To verify the non-emptiness of  $E_f^c$  we prove that  $\gamma$  intersects the plane  $v = 0$ . We address the slowly non-dissipative case (ii). We rewrite (11) in the form

$$\begin{aligned} u_x &= \sqrt{b}w, \\ w_x &= -\sqrt{b}u - \frac{1}{\sqrt{b}}g(x, u, \sqrt{b}w), \\ u(0) &= u_0, \quad w(0) = 0, \end{aligned} \tag{13}$$

and introduce the change to polar coordinates

$$u = \rho \cos \theta, \quad w = -\rho \sin \theta, \tag{14}$$

with  $\rho(0, u_0) = u_0$  and  $\theta(0, u_0) = 0$ . This corresponds to the well-known Prüfer transformation normally used in Sturm-Liouville eigenvalue problems to prove Sturm comparison theorem. From (13) and (14), we obtain

$$\begin{aligned}\rho_x \cos \theta - \rho \theta_x \sin \theta &= -\sqrt{b} \rho \sin \theta, \\ -\rho_x \sin \theta - \rho \theta_x \cos \theta &= -\sqrt{b} \rho \cos \theta - \frac{1}{\sqrt{b}} g(x, \rho \cos \theta, -\sqrt{b} \rho \sin \theta).\end{aligned}$$

This implies (for  $\rho \neq 0$ )

$$\rho_x = \frac{1}{\sqrt{b}} g(x, \rho \cos \theta, -\sqrt{b} \rho \sin \theta) \sin \theta \quad (15)$$

$$\theta_x = \sqrt{b} + \frac{1}{\sqrt{b} \rho} g(x, \rho \cos \theta, -\sqrt{b} \rho \sin \theta) \cos \theta. \quad (16)$$

Since  $g$  is uniformly bounded, given  $\delta > 0$  there exists  $\hat{R} > 0$  large enough such that  $|\rho(x, u_0)| > \hat{R}$  implies

$$\left| \frac{1}{\sqrt{b} \rho} g(x, \rho \cos \theta, -\sqrt{b} \rho \sin \theta) \cos \theta \right| < \delta.$$

Moreover, we have

$$\begin{aligned}|\rho(x, u_0) - \rho(0, u_0)| &= \left| \int_0^x \rho_x(x, u_0) dx \right| \\ &\leq \int_0^x \left| \frac{1}{\sqrt{b}} g(x, \rho \cos \theta, -\sqrt{b} \rho \sin \theta) \sin \theta \right| dx \leq C, \quad (17)\end{aligned}$$

for all  $x \in [0, \pi]$  and some positive constant  $C$ . From (17) we conclude that, for a given  $\hat{R} > 0$ , if  $|\rho(0, u_0)|$  is sufficiently large, then  $|\rho(x, u_0)| > \hat{R}$  uniformly on  $[0, \pi]$ . Therefore, if  $|\rho(0, u_0)|$  is sufficiently large, we have

$$\left| \frac{1}{\sqrt{b} \rho} g(x, \rho \cos \theta, -\sqrt{b} \rho \sin \theta) \cos \theta \right| < \delta, \text{ for all } x \in [0, \pi].$$

Thus, from (16) we have

$$\left| \int_0^\pi (\theta_x - \sqrt{b}) dx \right| \leq \int_0^\pi \left| \frac{1}{\sqrt{b} \rho} g(x, \rho \cos \theta, -\sqrt{b} \rho \sin \theta) \cos \theta \right| dx < \delta \pi,$$

and, since  $\theta(0, u_0) = 0$ , we obtain  $|\theta(\pi, u_0) - \sqrt{b}\pi| < \delta\pi$ . Letting  $\varepsilon = \delta\pi$ , we finally obtain that for a given  $\varepsilon > 0$  there exists  $R > 0$  such that  $|\rho(0, u_0)| > R$  implies

$$|\theta(\pi, u_0) - \sqrt{b}\pi| < \varepsilon. \quad (18)$$

As  $|\rho(\pi, u_0)|$  grows with  $|\rho(0, u_0)| = |u_0|$  and  $b \neq n^2$  for  $n \in \mathbb{N}$ , we conclude from (18) that for both  $u_0 > R$  and  $u_0 < -R$  the curve  $\gamma$  is asymptotic to a straight line with inclination angle  $\sqrt{b}\pi$ , therefore going from one (upper or lower) semiplane of  $\{(x, u, v) : x = \pi\}$  to the other. Hence, the continuous curve  $\gamma$  must intersect the plane  $v = 0$ , which implies that  $E_f^c$  is not empty.

For the proof of (i) we refer the reader to [17, 20, 30].  $\square$

#### 4 Heteroclinics to Infinity

In the sequel we let  $b > 0$  and  $b \neq n^2$  for  $n \in \mathbb{N}$ . From the proof of Lemma 4 this implies that all the solutions of (10) have their values contained in a bounded subset of the phase plane  $(u, u_x)$ . In addition, we also make the generic assumption that all bounded equilibria



$v \in E_f^c$  are *hyperbolic*. Hyperbolicity of  $v \in E_f^c$  means that  $\lambda = 0$  is not an eigenvalue of the linearization of (10) at  $v$

$$\lambda u = u_{xx} + bu + g_u(x, v, v_x)u + g_p(x, v, v_x)u_x \tag{19}$$

with Neumann boundary conditions. Hence, from here on we deal with only a finite set  $E_f^c = \{v_1, \dots, v_n\}$ .

As usual, the *Morse index*  $i(v)$  of the hyperbolic  $v \in E_f^c$  denotes the number of strictly positive eigenvalues of (19), alias, the dimension of the unstable manifold of  $v$ ,  $W^u(v)$ .

The nondegeneracy condition  $b \neq n^2$  in Lemma 4 corresponds to the hyperbolicity of the equilibria at infinity, as we can see in [18] and [7]. This concept is related to the projected equation obtained on the equator  $\mathcal{H}_e$  from which we find the equilibria at infinity.

We intend to determine the bounded equilibria connecting to infinity as well as the exact limit of the grow-up solutions. In order to do that, it is necessary to work in higher norms than the  $L^2$ -norm. This is due to the fact that the  $L^2$ -norm is insufficient for determining the exact limiting object at infinity since the  $L^2$ -norm alone does not prevent the zero number to drop at  $t = \infty$ . As it was observed in [5], despite the fact that the influence of the bounded nonlinearity  $g$  must decrease as the norm of  $u(t, \cdot)$  grows to infinity, we are unable to study the limit in the  $X^\alpha$  norm. To obtain the limits in  $C^1$ , we appeal to the theory of inertial manifolds. We have the following result from Miklavčič [24] for the existence of an inertial manifold for Eq. (1).

**Lemma 5** *Consider the Eq. (1) with  $g$  satisfying the inequality (2) for some positive constants  $L_1 < +\infty$  and  $L_2 < 1$ . Then there exists an inertial manifold  $\mathcal{M}$  for Eq. (1) and  $\mathcal{M}$  contains the non-compact global attractor  $\mathcal{A}_f$ . Moreover,  $\mathcal{M}$  is Lipschitz continuous with values in  $C^1$ .*

In the dissipative case and for  $f = f(u)$  Brunovský [8] used the inertial manifold to show that the global attractor is the graph of a  $C^1$  smooth function over a subset of a finite dimensional linear space of dimension equal to the maximal Morse index of the equilibria in the attractor. Later Brunovský and Tereščák [9] extended the existence result of the inertial manifold to the case of  $f = f(x, u, u_x)$  with a small Lipschitz constant  $L_2 < \varepsilon$ . Finally, Miklavčič [24] obtained this existence result for the case  $L_2 < 1$ . Using these results, Matano and Nakamura [23, Theorem E] extended the smooth graph result of Brunovský to the case of  $f = f(u, u_x)$  with  $L_2 < 1$  (under periodic boundary conditions).

The next lemma determines the limiting object at infinity by just requiring information on the zero number of the grow-up solution.

**Lemma 6** *Let  $v$  be a hyperbolic equilibrium for Eq. (1) and  $u(t, \cdot)$  a grow-up solution in the unstable manifold of  $v$ ,  $W^u(v)$ , with*

$$z(u(t, \cdot) - v(\cdot)) = m$$

for all  $t \in (T, \infty)$  for some  $T > 0$ . Then

$$\lim_{t \rightarrow \infty} \left\| \frac{u(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_m^t \right\|_{C^1} = 0,$$

where  $\varphi_m^t = \iota\varphi_m$  and  $\iota = \text{sign}(u(t, 0) - v(0))$  for  $t \in (T, \infty)$ .

*Proof* From the existence of the inertial manifold  $\mathcal{M}$  we have that  $\mathcal{M} = \text{graph}[\Psi]$  where

$$\Psi : P_N D(A) \longrightarrow Q_N D(A)$$

is a Lipschitz mapping with values in  $C^1$ , and we decompose  $u(t, \cdot)$  as  $u(t, \cdot) = p(t, \cdot) + q(t, \cdot)$  where

$$p(t, \cdot) = \sum_{j=0}^N \langle u(t, \cdot), \varphi_j(\cdot) \rangle_{L^2} \varphi_j(\cdot) \in P_N D(A),$$

$$q(t, \cdot) = \Psi(p(t, \cdot)) \in Q_N D(A).$$

We know that all modes  $\hat{u}_j(t)$  with  $j > \sqrt{b}$  remain bounded and also, from Lemma 3, it follows that

$$\lim_{t \rightarrow \infty} \left\| \frac{u(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_i^t \right\| = 0$$

for some integer  $i$  with  $i < \sqrt{b}$ .

The mapping  $\Psi$  is uniformly bounded in  $C^1$ , thus  $q(t, \cdot)$  remains bounded as  $p(t, \cdot)$  grows to infinity along with  $u(t, \cdot)$ . As a result, we have that

$$\lim_{t \rightarrow \infty} \left\| \frac{p(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_i^t \right\| = 0$$

and

$$\lim_{t \rightarrow \infty} \left\| \frac{u(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_i^t \right\|_{C^1} = \lim_{t \rightarrow \infty} \left\| \frac{p(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_i^t \right\|_{C^1}.$$

Since  $p(t, \cdot)$  and  $\varphi_i^t(\cdot)$  are both in the finite-dimensional subspace  $P_N D(A)$ , due to norm equivalence in finite dimension it follows that

$$\lim_{t \rightarrow \infty} \left\| \frac{u(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_i^t \right\|_{C^1} \leq \lim_{t \rightarrow \infty} C \left\| \frac{p(t, \cdot)}{\|u(t, \cdot)\|} - \varphi_i^t \right\| = 0.$$

Finally, since  $\|u\|$  grows up to infinity as  $t \rightarrow \infty$ , we must have

$$z(u - v) = z\left(\frac{u - v}{\|u\|}\right) = z(\varphi_i^t)$$

for all  $t > T$ . This implies that  $i = m$  and  $\iota = \text{sign}(u(t, 0) - v(0))$  for all  $t \in (T, \infty)$ .  $\square$

This lemma asserts that the zero number  $z(u(t, \cdot) - v(\cdot))$  of a grow-up solution  $u \in W^u(v)$  can only drop at finite times  $t < T$  (thus excluding the disturbing possibility  $t = \infty$ ), hence explicitly identifying the equilibrium at infinity to which the grow-up solution connects to.

For dissipative equations with Dirichlet boundary conditions, Brunovský and Fiedler have constructed the  $y$ -map which is an important tool for describing the heteroclinic connections on the global attractor, see [3]. For each initial condition  $u_0$  in the unstable manifold of a hyperbolic equilibrium  $v$ ,  $y(u_0)$  determines the zero number  $z(u(t, \cdot) - v(\cdot))$  of the corresponding shifted solution and the sign of  $u(t, 0) - v(0)$ . An extended form of the  $y$ -map was designed by Ben-Gal [5, 6] to deal with slowly non-dissipative equations with Neumann boundary conditions. For any hyperbolic equilibrium  $v$  and any positive integer  $m < i(v)$ , the surjectiveness of the  $y$ -map (see [3, Lemma 2.2]) implies the existence of a solution  $u(t, \cdot)$  in  $W^u(v)$  with  $z(u(t, \cdot) - v(\cdot)) = m$  for all  $t > 0$ .

Given an equilibrium  $v \in E_f^c$ , we say that  $v$  has a heteroclinic connection to the object at infinity  $\Phi^\infty \in \{\Phi_j^{\infty, \pm} : j = 0, \dots, [\sqrt{b}]\}$  if there exists a grow-up solution  $u(t, \cdot)$  satisfying

$$\lim_{t \rightarrow -\infty} u(t, \cdot) = v \text{ and } \lim_{t \rightarrow \infty} \left\| \frac{u(t, \cdot)}{\|u(t, \cdot)\|} - \varphi(\cdot) \right\|_{C^1} = 0,$$

where  $\varphi \in \{\varphi_j^\pm\}_{j \in \mathbb{N}}$  and  $\varphi$  corresponds to the equilibrium  $\Phi \in \{\Phi_j^\pm\}_{j \in \mathbb{N}}$  on the sphere at infinity which is the projection of  $\Phi^\infty$ .

We define the zero number of  $\Phi_j^{\infty, \pm}$  as

$$z(\Phi_j^{\infty, \pm}) := z(\varphi_j^\pm) = j \tag{20}$$

and

$$z(\Phi_j^{\infty, \pm} - v) := z(\Phi_j^{\infty, \pm}), \text{ for all } v \in E_f^c. \tag{21}$$

Also, we let

$$\text{sign}(\Phi_j^{\infty, \pm}(0)) := \text{sign}(\varphi_j^\pm(0)) = \pm 1 \tag{22}$$

and

$$\text{sign}(\Phi_j^{\infty, \pm}(0) - v(0)) := \text{sign}(\Phi_j^{\infty, \pm}(0)), \text{ for all } v \in E_f^c. \tag{23}$$

Under the above setting, we postulate

$$\Phi_j^{\infty, -}(0) < v(0) < \Phi_j^{\infty, +}(0), \text{ for all } v \in E_f^c. \tag{24}$$

We notice that the zero numbers of the objects at infinity  $\{\Phi_j^\pm\}_{j \in \mathbb{N}}$  are consistent in the sense that the zero numbers of the corresponding heteroclinic orbits do not drop at  $t = \infty$  and the eigenfunctions  $\{\pm\varphi_j\}_{j \in \mathbb{N}}$  only have simple zeros.

The next definition adapts to the equilibria at infinity the notion of adjacency first introduced in [29].

**Definition 2** Let  $v \in E_f^c$  and let  $\Phi^\infty \in \{\Phi_j^{\infty, \pm} : j = 0, \dots, [\sqrt{b}]\}$  denote an equilibrium at infinity with  $z(\Phi^\infty - v) = m$ , for some  $m \in \mathbb{N}$ . We say that  $v$  and  $\Phi^\infty$  are  $m$ -adjacent if there does not exist  $w \in E_f^c$  with  $z(\Phi^\infty - w) = z(v - w) = m$  and

- (i)  $v(0) < w(0) < \Phi^\infty(0)$ , if  $v(0) < \Phi^\infty(0)$ ,
- (ii)  $\Phi^\infty(0) < w(0) < v(0)$ , if  $\Phi^\infty(0) < v(0)$ .

With this adjacency definition we address the heteroclinic connectivity of finite equilibria to objects at infinity via grow-up solutions.

**Lemma 7 (Infinite Blocking Lemma)** Let  $v \in E_f^c$  be a hyperbolic equilibrium and let  $\Phi^\infty \in \{\Phi_j^{\infty, \pm} : j = 0, \dots, [\sqrt{b}]\}$  denote an equilibrium at infinity with  $z(\Phi^\infty - v) = m$ . If  $v$  and  $\Phi^\infty$  are not  $m$ -adjacent, then  $v$  does not have any heteroclinic connection to the object  $\Phi^\infty$  at infinity.

*Proof* First notice that (21) implies that  $m \leq [\sqrt{b}]$ . Assume that there exists a grow-up solution  $u(t, \cdot)$  connecting  $v$  to  $\Phi^\infty$ . Since  $v$  and  $\Phi^\infty$  are not  $m$ -adjacent, there exists  $w \in E_f^c$  such that

$$z(\Phi^\infty - w) = z(v - w) = m \tag{25}$$

and

$$\Phi^\infty(0) < w(0) < v(0) \text{ or } v(0) < w(0) < \Phi^\infty(0). \tag{26}$$

Then  $\tilde{u} = u - w$  is a trajectory from  $v - w$  to  $\Phi^\infty - w$  satisfying  $\tilde{u}_t = \tilde{u}_{xx} + b\tilde{u} + \tilde{g}(x, \tilde{u}, \tilde{u}_x)$  with  $\tilde{g}(x, \tilde{u}, \tilde{u}_x) := g(x, w + \tilde{u}, w_x + \tilde{u}_x) - g(x, w, w_x)$ . By Lemma 6 we conclude that there exists  $T > 0$  such that  $z(u(t, \cdot) - w(\cdot)) = m$  for all  $t > T$ . Since (26) holds, the value of  $w(0)$  lies between  $u(t, 0)$  and  $v(0)$ , increasing the value of  $t$  if necessary. Thus

$$u(t, 0) - w(0) < 0 < v(0) - w(0) \text{ or } v(0) - w(0) < 0 < u(t, 0) - w(0).$$

Moreover, by the nonincreasing property of  $z(\tilde{u}(t, \cdot))$  we have that  $z(v - w) \geq z(u(t, \cdot) - w)$ . On the other hand, by (25)

$$z(v - w) = m = z(u(t, \cdot) - w).$$

However, since  $v(0) - w(0)$  and  $u(t, \cdot) - w(0)$  have opposite signs and  $\tilde{u}(t, 0) \neq 0$  for all non-dropping times, we must have some dropping time before  $T$  and  $z(v - w) > z(u(t, \cdot) - w)$  which is a contradiction.  $\square$

The next lemma ensures the existence of connections whenever they are not blocked.

**Lemma 8** (*Infinite Liberalism Lemma*) *Let  $v \in E_f^c$  be a hyperbolic equilibrium, let  $m$  denote an integer such that  $0 \leq m \leq i(v) - 1$ , and let  $\Phi^\infty \in \{\Phi_j^{\infty, \pm} : j = 0, \dots, [\sqrt{b}]\}$  denote an equilibrium at infinity with  $z(\Phi^\infty - v) = m$ . If  $v$  and  $\Phi^\infty$  are  $m$ -adjacent, then  $v$  has a heteroclinic connection to the object  $\Phi^\infty$  at infinity.*

*Proof* Using the  $y$ -map we obtain an initial condition  $u_0 \in W^u(v)$  such that the corresponding solution  $u(t, \cdot)$  satisfies

$$z(u(t, \cdot) - v(\cdot)) = m$$

for all  $0 \leq t < \infty$ , with  $\text{sign}(u(t, 0) - v(0)) = \iota := \text{sign}(\Phi^\infty(0))$ . For more details see [5, 6, 26]. We show first that there does not exist any bounded equilibrium  $w$  of (1) such that  $u(t, \cdot)$  converges to  $w$  as  $t$  goes to infinity.

Let us assume that  $\lim_{t \rightarrow \infty} u(t, \cdot) = w$  for some equilibrium  $w \in E_f^c$ . Then, as the zero number of the shifted solution  $u - v$  is nonincreasing

$$\lim_{t \rightarrow \infty} z(u(t, \cdot) - v(\cdot)) = z(w - v)$$

must be less than or equal to  $m$ . Suppose  $z(w - v) < m$ . This would imply that the zero number of the shifted solution drops at infinity and we show why this is not possible. Let

$$\lim_{t \rightarrow \infty} (u(t, \cdot) - v(\cdot)) = w - v =: \tilde{w}.$$

Since  $\tilde{w} \neq 0$ , it follows that  $\tilde{w}$  has only simple zeros, as it solves the ODE

$$0 = \tilde{u}_{xx} + b(\tilde{u}) + \tilde{g}(x, \tilde{u}, \tilde{u}_x).$$

Therefore, any solution in a sufficiently small neighborhood of  $\tilde{w}$  must also have only simple zeros. It implies that  $z(u(t, \cdot) - v(\cdot))$  is constant over some small neighborhood of  $t = \infty$  and, thus,

$$z(u(t, \cdot) - v(\cdot)) = z(\tilde{w}(\cdot)) = z(w - v) < m$$

for all very large  $t$ . However, in this case  $z(u(t, \cdot) - v(\cdot))$  would have to drop at some finite time as  $u(t, \cdot)$  converges to  $w$ , which is a contradiction. Therefore, if  $v$  connects to  $w$  via  $u(t, \cdot)$  then  $z(w - v) = m$ . Since  $\text{sign}(u(t, 0) - v(0)) = \iota$  for all  $0 \leq t < \infty$ , the sign of  $u(t, 0) - v(0)$  remains always positive or negative in forward time. Then,  $\text{sign}(w(0) - v(0)) = \iota$ . However, there does not exist any bounded equilibrium  $w$  fulfilling these conditions, i.e.,

$$z(w - v) = m \text{ and } \text{sign}(w(0) - v(0)) = \iota = \text{sign}(\Phi^\infty(0) - v(0)),$$

since  $v$  and  $\Phi^\infty$  are  $m$ -adjacent.

Since  $u(t, \cdot)$  cannot converge to any bounded equilibrium, by Lemma 2 we conclude that  $\lim_{t \rightarrow \infty} \|u(t, \cdot)\| = \infty$  and thus  $u(t, \cdot)$  is a grow-up solution. Finally, it was proven

in Lemma 6 that in this case the normalized trajectory  $\frac{u(t, \cdot)}{\|u(t, \cdot)\|}$  converges in the  $C^1$ -norm to the eigenfunction  $\varphi_m^l$ . This means that  $v$  has a heteroclinic connection to the object  $\Phi_m^{\infty, \iota}$  at infinity. Since  $z(\Phi^\infty) = z(\Phi^\infty - v) = m$  and  $\text{sign}(\Phi^\infty(0)) = \iota$ , it follows from (20) and (22) that  $\Phi^\infty = \Phi_m^{\infty, \iota}$ .  $\square$

The connecting orbit structure within infinity was obtained in [18]. The intra-infinite heteroclinics are namely the following:

$$\Phi_j^{\infty, \iota} \text{ connects to } \Phi_i^{\infty, \pm}, \text{ for each } \iota \in \{-, +\} \text{ and all } i \leq j - 1, \tag{27}$$

with  $i, j \in \{1, \dots, [\sqrt{b}]\}$ .

### 5 Suspension

In this section we obtain a permutation related to Eq. (1) in the sense of [11]. We begin by recalling some facts which hold true in the dissipative case.

Let  $h$  be a function such that the dynamical system induced by

$$\begin{cases} u_t = u_{xx} + h(x, u, u_x), & x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0 \end{cases} \tag{28}$$

is dissipative. Assume also that all the equilibria are hyperbolic. We denote by  $E_h = \{w_1, \dots, w_N\}$  the set of equilibria of Eq. (28) ordered by their values at  $x = 0$ ,

$$w_1(0) < w_2(0) < \dots < w_N(0).$$

Let  $\sigma_h \in S(N)$  denote the permutation defined by the reordering at  $x = \pi$ ,

$$w_{\sigma_h(1)}(\pi) < w_{\sigma_h(2)}(\pi) < \dots < w_{\sigma_h(N)}(\pi). \tag{29}$$

This permutation corresponds to the ordering of the points of intersection of the plane  $v = 0$  with the curve  $\gamma_h$  defined for the stationary equation for (28) as in (12). Permutations related in this way to a Jordan curve like  $\gamma_h$  (in the Poincaré sphere) are called *meander permutations*.

We say that a permutation  $\sigma \in S(N)$  is a *dissipative permutation* if  $\sigma(1) = 1$  and  $\sigma(N) = N$ . Given any  $\sigma \in S(N)$  we also define the vector  $(i_j(\sigma))_{1 \leq j \leq N}$  by

$$\begin{aligned} i_1(\sigma) &= 0 \\ i_{j+1}(\sigma) &= i_j(\sigma) + (-1)^{j+1} \text{sign}(\sigma^{-1}(j+1) - \sigma^{-1}(j)) \end{aligned} \tag{30}$$

for  $j = 1, \dots, N - 1$ . Then  $\sigma$  is called a *Morse permutation* if  $i_j(\sigma) \geq 0$  for all  $1 \leq j \leq N$ . We say that the permutation  $\sigma$  is realizable by a dissipative Eq. (28) if  $\sigma = \sigma_h$ , with  $\sigma_h$  defined in (29). A permutation that is realizable by a dissipative equation is referred to as a *Sturm permutation*. The motivation for these definitions is the realization result obtained in [13] and recalled in Theorem 1 further below in this section.

For each equilibrium  $w_j \in E_h$ , the Morse index  $i(w_j)$  coincides with  $i_j(\sigma_h)$ . That is to say that

$$i(w_j) = i_j(\sigma_h), \text{ for all } j = 1, \dots, N. \tag{31}$$

This is shown in [27]. The indices  $i_j(\sigma_h)$  can alternatively be written as

$$i_j(\sigma_h) = \sum_{m=1}^{j-1} (-1)^{m+1} \text{sign}(\sigma_h^{-1}(m+1) - \sigma_h^{-1}(m)), \tag{32}$$

with empty sums denoting zero (see [12]). For dissipative equations with only hyperbolic equilibria, the number of equilibria  $N$  is odd and the permutation defined in (29) is a dissipative Morse meander permutation. Moreover, a permutation  $\sigma$  associated to dissipative systems satisfies (31) and we have  $i(w_1) = i(w_N) = 0$ .

As a consequence of (1) being non-dissipative we see in the next lemma that the Morse indices  $i(v_1)$  and  $i(v_n)$  are both different from zero.

The equilibria in  $E_f^c = \{v_1, \dots, v_n\}$  are ordered by their values at  $x = 0$ . We denote by  $\bar{\theta} = \bar{\theta}(x, u_0)$  the solution of the differential equation

$$\bar{\theta}_x = \sqrt{b} + \frac{1}{\sqrt{b}}q_0(x, u_0) \cos^2 \bar{\theta} - q_1(x, u_0) \cos \bar{\theta} \sin \bar{\theta}, \quad \bar{\theta}(0, u_0) = 0 \tag{33}$$

where

$$q_0(x, u_0) = g_u(x, u(x, u_0), v(x, u_0)) \text{ and } q_1(x, u_0) = g_p(x, u(x, u_0), v(x, u_0)).$$

This  $\bar{\theta} = \bar{\theta}(\cdot, u_0)$  corresponds to the angle swept by the unit vector tangent to the  $x$ -section of  $L$  along the solution  $(u(\cdot, u_0), v(\cdot, u_0))$ , as obtained from the linearization of (28), with  $u_0$  corresponding to the initial value of problem (11). We also denote by  $s(u_0) := (\pi, u(\pi, u_0), v(\pi, u_0))$  the point in the section curve  $\gamma$  corresponding to the initial condition  $u_0$ . Then  $\bar{\theta}(\pi, u_0)$  can also be read as the angle swept clockwise by the unit vector tangent to  $\gamma$  at  $s(u_0)$  as  $s(u_0)$  describes  $\gamma$ , with  $u_0$  going from  $-\infty$  to  $\infty$  (see [27,28]). Finally, let  $u_0^j$  be the initial value satisfying  $u_0^j = v_j(0)$  for the equilibrium  $v_j$ . Then, the Morse index of  $v_j$  is given by  $i(v_j) = 1 + [\bar{\theta}(\pi, u_0^j)/\pi]$ . See [27,28].

**Lemma 9** *Assume that all equilibria of (1) are hyperbolic. Then the Morse index of the minimal and maximal equilibria,  $v_1$  and  $v_n$ , are given by*

$$i(v_1) = i(v_n) = 1 + [\sqrt{b}]. \tag{34}$$

*Proof* As already mentioned, the Morse index of an equilibrium  $v_j$  is given by  $i(v_j) = 1 + [\bar{\theta}(\pi, u_0^j)/\pi]$  for all  $j = 1, \dots, n$ , where  $u_0^j = v_j(0)$ . The equilibria  $v_1$  and  $v_n$  being hyperbolic implies that  $\bar{\theta}(\pi, u_0^1), \bar{\theta}(\pi, u_0^n) \neq m\pi$ , for all  $m \in \mathbb{N}$  (see [27]). In the following we show that  $[\bar{\theta}(\pi, u_0^j)/\pi] = [\sqrt{b}]$  for  $j = n$ , the case  $j = 0$  being similar.

Let  $u = u(x, u_0), v = v(x, u_0)$  denote the solutions of (11) and let  $\theta(\pi, u_0)$  be the angle swept clockwise by the position vector  $(u(x, u_0), v(x, u_0))$  as  $x$  runs from  $x = 0$  to  $x = \pi$ . We have seen in the proof of Lemma 4 that  $\theta(x, u_0)$  satisfies (16) and  $\lim_{|u_0| \rightarrow \infty} \theta(\pi, u_0) = \sqrt{b}\pi$  (see (18)). This limit does not depend on the bounded nonlinearity  $g$ , hence it does not change if  $g$  is modified by an adequate cut-off outside a large subset of  $[0, \pi] \times \mathbb{R}^2$ . Let  $C_1 > 0$  denote a bound on the solutions  $u = u(x, u_0), v = v(x, u_0)$  of (11) with  $u_0 \in [u_0^1, u_0^n]$ ,

$$\max \{|u(x, u_0)|, |v(x, u_0)| : x \in [0, \pi], u_0 \in [u_0^1, u_0^n]\} < C_1.$$

Also, let  $C_2 > 0$  denote a bound for the solutions  $u \in \mathcal{A}_f^c$  on the compact global attractor of (1), which as we recall is in  $X^\alpha \subset C^1$ ,

$$\sup \{|u(x)|, |u_x(x)| : x \in [0, \pi], u \in \mathcal{A}_f^c\} < C_2.$$

Then, for an open neighborhood  $D \subset \mathbb{R}^2$  of the set

$$\{(u, v) \in \mathbb{R}^2 : |u|, |v| \leq \max\{C_1, C_2\}\} \subset D, \tag{35}$$

we define a  $C^2$  function  $\tilde{g}$  compactly supported in  $[0, \pi] \times D$  such that

$$\tilde{g}(x, u, v) = g(x, u, v) \tag{36}$$

for all  $|u|, |v| \leq \max\{C_1, C_2\}$  and  $x \in [0, \pi]$ . The dynamical system generated by (1) with  $g$  replaced by  $\tilde{g}$  is non-dissipative and it possesses a noncompact global attractor

$$\mathcal{A}_{\tilde{f}} = \mathcal{A}_{\tilde{f}}^c \cup \mathcal{A}_{\tilde{f}}^\infty,$$

where  $\tilde{f}(x, u, p) = bu + \tilde{g}(x, u, p)$  and  $\mathcal{A}_{\tilde{f}}^c$  is the maximal compact invariant set. Then, in view of (36) we have that  $\mathcal{A}_f^c \subset \mathcal{A}_{\tilde{f}}^c$  and the replacement of  $g$  by  $\tilde{g}$ , assumed from here on, does not affect the Morse indices of any equilibria. In fact, this replacement does not change the section curve  $\gamma$  for  $u_0 \in [u_0^1, u_0^n]$ .

If  $|u_0|$  is sufficiently large, then  $(u(x, u_0), v(x, u_0)) \notin D$  and we have that  $\tilde{g}(x, u(x, u_0), v(x, u_0)) = 0$  for all  $x \in [0, \pi]$ . In this case, since  $\bar{\theta} = \bar{\theta}(x, u_0)$  satisfies (33) and  $q_0(x, u_0) \equiv q_1(x, u_0) \equiv 0$  we conclude that  $\bar{\theta}_x(x, u_0) = \sqrt{b}$ . Therefore  $\bar{\theta}(\pi, u_0) = \sqrt{b}\pi$  for large values of  $|u_0|$  and

$$\bar{\theta}(\pi, r)/\pi = \sqrt{b} \tag{37}$$

for  $r > u_0^n$  large. Then, let  $\gamma_n$  denote the arc of the section curve  $\gamma$  between the points  $(u_0^n, 0)$  and  $(u(\pi, r), v(\pi, r))$ ,

$$\gamma_n = \{(\pi, u(\pi, u_0), v(\pi, u_0)) : u_0^n \leq u_0 \leq r\}.$$

Also, let  $\bar{\gamma}_n$  denote the horizontal line segment between the points  $(u_0^n, 0)$  and  $(u(\pi, r), 0)$ ,

$$\bar{\gamma}_n = \{(\pi, u, 0) : u_0^n < u < u(\pi, r)\},$$

and finally let  $\bar{\bar{\gamma}}_n$  denote the vertical line segment between  $(u(\pi, r), 0)$  and  $(u(\pi, r), v(\pi, r))$ ,

$$\bar{\bar{\gamma}}_n = \{(\pi, u(\pi, r), p) : 0 \leq p < v(\pi, r)\}.$$

We then define the closed curve  $\Gamma_n = \gamma_n \cup \bar{\gamma}_n \cup \bar{\bar{\gamma}}_n$ . Since  $v_n$  is the equilibrium with maximal initial value  $u_0$ , the curve  $\gamma_n$  intersects the horizontal axis only at  $(u(\pi, u_0), 0)$ , hence  $\gamma_n \cap \bar{\gamma}_n = \emptyset$  is empty. Moreover, for  $r$  sufficiently large we also have  $\gamma_n \cap \bar{\bar{\gamma}}_n = \emptyset$  by (18). We therefore conclude that  $\Gamma_n$  is a Jordan curve, i.e., a piecewise differentiable simple and closed curve. It then follows from (37) and the turning tangent theorem ([10]) applied to  $\Gamma_n$  that

$$|\bar{\theta}(\pi, u_0^n) - \bar{\theta}(\pi, r)| < \pi.$$

This implies

$$\left[ \frac{\bar{\theta}(\pi, u_0^n) - \bar{\theta}(\pi, r)}{\pi} \right] = 0,$$

and using (37) we have  $[\bar{\theta}(\pi, u_0^n)/\pi] = [\sqrt{b}]$  as required. □

As a consequence of this lemma, the dynamical system generated by Eq. (1) is non-dissipative for  $b > 0$  and  $i(v_1) = i(v_n) = k$  with  $k := 1 + [\sqrt{b}]$  strictly positive. Then, if dissipative properties are not verified, we do not expect the Morse indices of the equilibria to satisfy (32). In spite of this fact, we want to obtain the Morse indices of the equilibria  $v_j \in E_f^c$  in terms of an associated permutation defined in an analogous manner as for dissipative equations.

We similarly define for Eq. (1) a permutation  $\sigma_f \in S(n)$  by ordering the equilibria in  $E_f^c$  firstly at  $x = 0$  and then according to their values at  $x = \pi$ . The permutation  $\sigma_f$  can also be defined from the corresponding section curve  $\gamma = \gamma_f$  obtained for (1), as we can see in [28]. Therefore  $\sigma_f$  is a meander permutation.

For this permutation  $\sigma = \sigma_f \in S(n)$  we define the vector  $(i_j(\sigma))_{1 \leq j \leq n}$  by

$$\begin{aligned} i_1(\sigma) &= k \\ i_{j+1}(\sigma) &= i_j(\sigma) + (-1)^{j+1+k} \text{sign}(\sigma^{-1}(j+1) - \sigma^{-1}(j)), \end{aligned} \tag{38}$$

for  $j = 1, \dots, n - 1$ . We next verify that the dimension of the unstable manifold of the equilibrium  $v_j$ , which we are denoting by  $i(v_j)$ , coincides with  $i_j(\sigma_f)$  for all  $1 \leq j \leq n$ .

**Lemma 10** *Let  $k = 1 + \lceil \sqrt{b} \rceil$ . If  $\sigma = \sigma_f \in S(n)$  denotes the meander permutation corresponding to the section curve  $\gamma = \gamma_f$  of (1) and  $(i_j(\sigma_f))_{1 \leq j \leq n}$  is the vector defined by (38), then*

$$i(v_j) = i_j(\sigma_f), \quad 1 \leq j \leq n. \tag{39}$$

*Proof* It was established in [27] that the Morse indices  $i(v_j)$  of the equilibria  $v_j \in E_f^c$  are determined in terms of the curve  $\gamma$  defined in (12). As already mentioned, the relation is given by

$$i(v_j) = 1 + \lceil \bar{\theta}(\pi, v_j(0)) / \pi \rceil \tag{40}$$

where  $\bar{\theta}$  is the solution of (33). We now want to write (40) explicitly in terms of  $\sigma_f$ .

Since  $\bar{\theta}(\pi, v_j(0))$  can also be read as the angle swept clockwise by the unit vector tangent to  $\gamma$  at  $s(u_0) = (\pi, u(\pi, u_0), p(\pi, u_0))$ , with  $p = u_x$ , as  $u_0$  goes from  $-\infty$  to  $v_j(0)$ , we obtain from (40) that

$$i(v_{j+1}) = i(v_j) + c_j \tag{41}$$

with  $c_j \in \{-1, +1\}$ . From the alternative definition of  $\bar{\theta}(\pi, v_j(0))$  in terms of  $\gamma_f$  and from (40), one obtains that

$$c_j = \text{sign}(v_{j+1}(\pi) - v_j(\pi)) \text{sign}(p_{u_0}(\pi, v_j(0))),$$

where  $p_{u_0}$  denotes the derivative of  $p$  with respect to  $u_0$ . It follows from the definition of  $\sigma_f$  that

$$\text{sign}(v_{j+1}(\pi) - v_j(\pi)) = \text{sign}(\sigma_f^{-1}(j+1) - \sigma_f^{-1}(j)). \tag{42}$$

We also have the relation

$$\text{sign}(p_{u_0}(\pi, v_l(0))) = -\text{sign}(p_{u_0}(\pi, v_{l+1}(0))), \text{ for all } 1 \leq l \leq n. \tag{43}$$

Moreover, the following inequalities hold

$$\begin{aligned} p_{u_0}(\pi, v_1(0)) &> 0 \text{ if } k \text{ is even,} \\ p_{u_0}(\pi, v_1(0)) &< 0 \text{ if } k \text{ is odd.} \end{aligned}$$

As a result, the alternation rule

$$(-1)^{j+1} (-1)^k p_{u_0}(\pi, v_j(0)) > 0$$

follows from (43). Equivalently we have

$$\text{sign}(p_{u_0}(\pi, v_j(0))) = (-1)^{j+1+k}. \tag{44}$$

From (42) and (44) we therefore conclude that

$$c_j = (-1)^{j+1+k} \text{sign}(\sigma_f^{-1}(j+1) - \sigma_f^{-1}(j)),$$



that is,

$$i(v_{j+1}) = i(v_j) + (-1)^{j+1+k} \text{sign}(\sigma_f^{-1}(j+1) - \sigma_f^{-1}(j)), \tag{45}$$

for all  $1 \leq j \leq n$ , which compares with (38) as we wanted to prove. □

Remark that by (34) and (39) we have  $i_1(\sigma_f) = i_n(\sigma_f) = k$ . In the sequel, we present the definition of suspension of a permutation  $\sigma \in S(n)$ .

**Definition 3** Let  $\sigma \in S(n)$  denote a meander permutation and  $k$  a positive integer. We define the suspension  $\hat{\sigma}^k$  of the permutation  $\sigma$  as the permutation  $\hat{\sigma}^k \in S(n+2)$  which satisfies:

- (i)  $\hat{\sigma}^k(j) = \sigma(j-1) + 1$ , for  $j \in \{2, \dots, n+1\}$ ; and
- (ii) if  $k$  is odd

$$\hat{\sigma}^k(1) = 1 \text{ and } \hat{\sigma}^k(n+2) = n+2,$$

and if  $k$  is even

$$\hat{\sigma}^k(1) = n+2 \text{ and } \hat{\sigma}^k(n+2) = 1.$$

It is clear that  $\hat{\sigma}^k$  is a meander permutation. For this permutation we define the vector  $(i_j(\hat{\sigma}^k))_{1 \leq j \leq n+2}$  by (38) with  $k$  replaced by  $k-1$ . That is,

$$i_1(\hat{\sigma}^k) = k-1, \tag{46}$$

and

$$i_{j+1}(\hat{\sigma}^k) = i_j(\hat{\sigma}^k) + (-1)^{j+1+k-1} \text{sign}((\hat{\sigma}^k)^{-1}(j+1) - (\hat{\sigma}^k)^{-1}(j)), \tag{47}$$

for  $j = 1, \dots, n+1$ . For an illustration of a permutation suspension see Fig. 1 further below in Sect. 6.

From (46) we conclude that after  $k$  suspensions of  $\sigma_f$  one obtains a meander permutation  $\hat{\sigma}_f^1 \in S(n+2k)$  with a vector  $(i_j(\hat{\sigma}_f^1))_{1 \leq j \leq n+2k}$  satisfying

$$i_1(\hat{\sigma}_f^1) = i_{n+2k}(\hat{\sigma}_f^1) = 0. \tag{48}$$

Indeed, since  $\hat{\sigma}_f^j$  is the suspension of  $\hat{\sigma}_f^{j+1}$  and  $i_1(\hat{\sigma}_f^j) = j-1$ , for  $1 \leq j \leq k$ , it follows that  $i_1(\hat{\sigma}_f^1) = 0$ . In order to prove that  $i_{n+2k}(\hat{\sigma}_f^1) = 0$  we show that  $i_{n+2}(\hat{\sigma}^k) = k-1$  which implies the former equality by reverse induction on  $k$ . By (47) we have

$$i_{n+2}(\hat{\sigma}^k) = i_{n+1}(\hat{\sigma}^k) + (-1)^{n+2+k-1} \text{sign}((\hat{\sigma}^k)^{-1}(n+2) - (\hat{\sigma}^k)^{-1}(n+1)).$$

Hence, it follows that

$$\begin{aligned} i_{n+2}(\hat{\sigma}^k) &= i_{n+1}(\hat{\sigma}^k) + \begin{cases} -\text{sign}(n+2 - (\hat{\sigma}^k)^{-1}(n+1)), & \text{if } k \text{ is odd} \\ \text{sign}(1 - (\hat{\sigma}^k)^{-1}(n+1)), & \text{if } k \text{ is even} \end{cases} \\ &= i_{n+1}(\hat{\sigma}^k) - 1, \end{aligned} \tag{49}$$

since  $1 < (\hat{\sigma}^k)^{-1}(n+1) < n+2$ . We next prove that  $i_{n+1}(\hat{\sigma}^k) = k$ . For that we need to verify the following

$$i_{j+1}(\hat{\sigma}^k) = i_j(\sigma), \text{ for all } 1 \leq j \leq n. \tag{50}$$

If  $j = 1$ , we have from (47)

$$\begin{aligned} i_2(\hat{\sigma}^k) &= i_1(\hat{\sigma}^k) + (-1)^{2+k-1} \text{sign}((\hat{\sigma}^k)^{-1}(2) - (\hat{\sigma}^k)^{-1}(1)) \\ &= k - 1 + \begin{cases} \text{sign}((\hat{\sigma}^k)^{-1}(2) - 1), & \text{if } k \text{ is odd} \\ -\text{sign}((\hat{\sigma}^k)^{-1}(2) - (n + 2)), & \text{if } k \text{ is even} \end{cases} \\ &= k - 1 + 1 = k = i_1(\sigma). \end{aligned}$$

Suppose now that  $i_j(\hat{\sigma}^k) = i_{j-1}(\sigma)$  for  $j \leq n - 1$ . Then

$$\begin{aligned} i_{j+1}(\hat{\sigma}^k) &= i_j(\hat{\sigma}^k) + (-1)^{j+1+k-1} \text{sign}((\hat{\sigma}^k)^{-1}(j + 1) - (\hat{\sigma}^k)^{-1}(j)) \\ &= i_{j-1}(\sigma) + (-1)^{j+k} \text{sign}(\sigma^{-1}(j) - \sigma^{-1}(j - 1)) \\ &= i_j(\sigma). \end{aligned}$$

We thus conclude inductively that (50) holds and, consequently,  $i_{n+1}(\hat{\sigma}^k) = i_n(\sigma) = k$ . By (49) this implies  $i_{n+2}(\hat{\sigma}^k) = k - 1$  and completes (48).

It is clear that  $\hat{\sigma}_f^1 \in S(N)$  with  $N = n + 2k$ . Moreover, for all  $j \in \{1, \dots, k\} \cup \{n + k + 1, \dots, n + 2k\}$ , the permutation  $\hat{\sigma}_f^1$  satisfies

$$\hat{\sigma}_f^1(j) = \begin{cases} j, & \text{for } j \text{ odd} \\ n + 2k - (j - 1), & \text{for } j \text{ even.} \end{cases} \tag{51}$$

Furthermore, by (i) of Definition 3,  $\hat{\sigma}_f^1$  and  $\sigma_f$  are related by

$$\hat{\sigma}_f^1(k + l) = k + \sigma_f(l), \text{ for } l = 1, \dots, n. \tag{52}$$

For an illustration of a sequence of  $k$  suspensions see Fig. 2 of Sect. 6.

The next step is to verify the existence of a function  $h$  realizing the permutation  $\hat{\sigma}_f^1$ . To do that we recall the following result

**Theorem 1** [13] *There exists a  $C^2$  function  $h$  realizing the permutation  $\sigma \in S(N)$  if, and only if,  $N$  is odd and  $\sigma$  is a dissipative Morse meander permutation.*

We know that  $\hat{\sigma}_f^1 \in S(N)$  is a meander permutation which is dissipative since  $N = n + 2k$  is odd and  $\hat{\sigma}_f^1(1) = 1, \hat{\sigma}_f^1(N) = N$  by (51). Moreover, the vector  $(i_j(\hat{\sigma}_f^1))_{1 \leq j \leq N}$  defined in (30) satisfies

$$i_j(\hat{\sigma}_f^1) \geq 0, \tag{53}$$

for all  $j = 1, \dots, N$ . Indeed, it follows from (48) that (53) holds for  $j = 1$  and  $j = N$ . If  $j \in \{k + 1, \dots, k + n\}$ , we have from (50) that

$$i_{l+k}(\hat{\sigma}_f^1) = i_{l+k-1}(\hat{\sigma}_f^2) = \dots = i_{l+1}(\hat{\sigma}_f^k) = i_l(\sigma) \geq 0, \tag{54}$$

for  $l = 1, \dots, n$ . Lastly, if  $j \in \{2, \dots, k\} \cup \{n + k + 1, \dots, n + 2k - 1\}$  we invoke (30) and use (51). This yields

$$\begin{aligned} i_{j+1}(\hat{\sigma}_f^1) &= i_j(\hat{\sigma}_f^1) + (-1)^{j+1} \text{sign}((\hat{\sigma}_f^1)^{-1}(j + 1) - (\hat{\sigma}_f^1)^{-1}(j)) \\ &= i_j(\hat{\sigma}_f^1) + \begin{cases} -\text{sign}(j + 1 - (n + 2k - (j - 1))), & \text{for } j \text{ even} \\ \text{sign}(n + 2k - j - j), & \text{for } j \text{ odd} \end{cases} \end{aligned}$$

if  $j \in \{1, \dots, k - 1\} \cup \{n + k + 1, \dots, n + 2k - 1\}$ , which is equivalent to

$$i_{j+1}(\hat{\sigma}_f^1) = i_j(\hat{\sigma}_f^1) + 1, \text{ for } j \in \{1, \dots, k - 1\}$$

and to

$$i_{j+1}(\hat{\sigma}_f^1) = i_j(\hat{\sigma}_f^1) - 1, \text{ for } j \in \{n + k + 1, \dots, n + 2k - 1\}.$$

Then, by recalling that  $i_1(\hat{\sigma}_f^1) = i_{n+2k}(\hat{\sigma}_f^1) = 0$ , it follows that

$$i_j(\hat{\sigma}_f^1) = j - 1 \geq 0 \text{ and } i_{n+k+j}(\hat{\sigma}_f^1) = k - j \geq 0, \tag{55}$$

for  $j = 1, \dots, k$ . Therefore,  $\hat{\sigma}_f^1$  is a Morse permutation. We are then able to apply Theorem 1 and ensure the existence of a function  $h$  realizing  $\hat{\sigma}_f^1$ .

Let  $D \subset \mathbb{R}^2$  be the large open set defined in (35). Our aim is to change  $f$  outside  $[0, \pi] \times D$  in such a way that the modified semiflow coincides with the original one on the bounded subset  $\mathcal{A}_f^c$  of  $\mathcal{A}_f$ . We let  $h$  be such that

$$h(x, u, p) = f(x, u, p), \text{ for all } (x, u, p) \in [0, \pi] \times D. \tag{56}$$

This ensures the existence of a large ball  $B \subset X^\alpha \subset C^1$  where the semiflow generated by (1) coincides with the semiflow generated by

$$\begin{cases} u_t = u_{xx} + h(x, u, u_x), & x \in [0, \pi] \\ u_x(t, 0) = u_x(t, \pi) = 0. \end{cases} \tag{57}$$

We will define  $h$  for  $(x, u, p)$  outside  $[0, \pi] \times D$  in such a way that the dynamical system induced by (57) is dissipative. Hence (57) will possess a compact global attractor  $\mathcal{A}_h$ . Nevertheless, we can decompose  $\mathcal{A}_h$  into a maximal compact invariant subset  $\mathcal{A}_h^c$  contained in  $B$  and its complement, that is,

$$\mathcal{A}_h = \mathcal{A}_h^c \cup \{\mathcal{A}_h \setminus \mathcal{A}_h^c\}, \text{ with } \mathcal{A}_h^c \subset B.$$

We notice that the set  $E_f^c$  of equilibria of (1) coincides with the subset of equilibria of Eq. (57) that are contained in  $B$ . This follows directly from the fact that  $h = f$  on  $B$ . As a consequence, the set of all equilibria of (57) is decomposed as follows

$$E_h = E_f^c \cup \{E_h \setminus E_f^c\}.$$

The subset  $\mathcal{A}_h^c \subset B$  is the set of all orbits that never leave  $B$ . Therefore,  $\mathcal{A}_h^c$  is composed of the set  $E_f^c$  of equilibria in  $B$  and their heteroclinic orbits, which coincides with the definition of  $\mathcal{A}_f^c$ . In this way, we have  $\mathcal{A}_f^c = \mathcal{A}_h^c$ .

Also let  $\hat{D} \subset \mathbb{R}^2$  be a larger set containing  $D$ . We require that

$$h(x, u, p) = cu, \text{ for all } (x, u, p) \notin [0, \pi] \times \hat{D}, \tag{58}$$

for some constant  $c < 0$ . To define the function  $h$  on the remaining portion of the domain  $[0, \pi] \times \mathbb{R}^2$  we appeal to the permutation  $\hat{\sigma}_f^1$ . We use the fact that  $\hat{\sigma}_f^1$  is realizable by a function  $h$  and then verify that, in Theorem 1, it is possible to choose  $h$  satisfying the requirements (56) and (58). In this way, we have defined a cut-off function  $h$  that coincides with  $f$  on the large ball  $B$  containing the bounded subset  $\mathcal{A}_f^c$  and such that the corresponding semiflow is dissipative.

Before imposing on  $h$  the requirements (56) and (58) we remark that, on both meanders  $\gamma = \gamma_f$  and  $\gamma_h$ , the arcs joining the intersection points with the  $u$ -axis can be isotopically

transformed into semicircles. Therefore, we are allowed to work with meanders  $\gamma$  in canonical form, i.e., essentially composed of semicircle arcs (see [13]).

The first requirement (56) is quite simple since we notice that  $\gamma_f$  and  $\gamma_h$  share the same permutation on the set  $E_f^c$ . The second requirement (58) follows from the realization results of canonical meanders by boundary value problems (see [11, 13]), and the condition that for  $|u|$  sufficiently large the projection  $(u, p) \mapsto (u, 0)$  is a local diffeomorphism. We also have that the canonical meander  $\gamma_h$ , outside the semicircles shared with  $\gamma_f$ , is composed of semicircles corresponding to the suspensions  $\hat{\sigma}_f^k, \hat{\sigma}_f^{k-1}, \dots, \hat{\sigma}_f^1$ . The realization of the canonical meander  $\gamma_h$  by second order boundary value problems then completes the definition of the smooth function  $h$  on the missing set  $[0, \pi] \times (\hat{D} \setminus D)$ .

Therefore, the requirements (56) and (58) on  $h$  imply that the dynamical system induced by (57) is dissipative and preserves the equilibria in  $E_f^c$  and the heteroclinic connections between them.

### 6 Connecting Orbit Structure

The aim of this section is to prove our main result, Theorem 2, presented next. This theorem provides a simple criterion for the existence of heteroclinic connections between the equilibria in  $E_f$ .

**Theorem 2** *Let  $u$  and  $v$  be equilibria in  $E_f$  satisfying  $z(u - v) = m$ , with the extended interpretation of the zero number for the equilibria at infinity given by (20) and (21). Then there exists a heteroclinic connection between  $u$  and  $v$  if, and only if, they are  $m$ -adjacent. Moreover, if the equilibria  $v$  and  $w$  are connected, then the one with higher Morse index is the source of the connection.*

Let  $E_h = \{w_1, w_2, \dots, w_N\}$  be the set of equilibria of (57). The permutation  $\sigma_h = \hat{\sigma}_f^1$  being Sturm implies that the Morse indices  $i(w_j)$  can be obtained in terms of  $\sigma_h$  as in (32). Also, the intersection numbers  $z(w_j - w_m)$  are given explicitly in terms of  $\sigma_h$ , for all  $j, m \in \{1, \dots, N\}$ , as we can see in [11, 12, 28] and as it is presented in the next Proposition. We may equivalently use the recursions formulae in [12, Proposition 2.1].

**Proposition 1** *Under the above setting and considering the Sturm permutation  $\sigma_h \in S(N)$ , for  $1 \leq m < l \leq N$ , the Morse indices are given by*

$$i(w_m) = \sum_{j=1}^{m-1} (-1)^{j+1} \text{sign}(\sigma_h^{-1}(j+1) - \sigma_h^{-1}(j))$$

and the intersection numbers by

$$z(w_l - w_m) = i(w_m) + \frac{1}{2} [(-1)^l \text{sign}(\sigma_h^{-1}(l) - \sigma_h^{-1}(m)) - 1] + \sum_{j=m+1}^{l-1} (-1)^j \text{sign}(\sigma_h^{-1}(j) - \sigma_h^{-1}(m)),$$

where empty sums denote zero.

We collect in the following lemmas the results previously obtained from  $\hat{\sigma}_f^1$  for the Morse indices  $i(w_j)$ ,  $w_j \in E_h$ . Lemmas 11 and 12 follow respectively from (55) and (54).

**Lemma 11** *The Morse indices of the exterior equilibria in  $E_h$ , i.e., the equilibria  $w_j \in E_h$  with  $j \in \{1, \dots, k\} \cup \{n + k + 1, \dots, n + 2k\}$ , satisfy:*

$$i(w_j) = j - 1, \text{ for } 1 \leq j \leq k,$$

and

$$i(w_{n+2k-j}) = j, \text{ for } 0 \leq j \leq k - 1.$$

**Lemma 12** *The Morse indices of the interior equilibria in  $E_h$ , i.e., the equilibria  $w_j \in E_h$  with  $j \in \{k + 1, \dots, k + n\}$ , satisfy:*

$$i(w_{k+l}) = i(v_l), \text{ for } 1 \leq l \leq n.$$

The previous lemma displays the relation between the Morse indices  $i(w_j)$  of  $w_j \in E_h$  and the Morse indices  $i(v_l)$  of  $v_l \in E_h^c$ , and is also a direct consequence of (56).

Regarding the information on the zero numbers contained on the permutation  $\sigma_h$  we have two results. In Lemma 13 we present the zero numbers of the differences between the  $k$  first/last equilibria and the remaining  $n$  middle equilibria in  $E_h$ , i.e., the number of intersections between the exterior and the interior equilibria. In Lemma 14 we obtain the zero number results for the  $n$  middle equilibria  $w_j \in E_h$ , i.e., the interior equilibria.

**Lemma 13** *For any  $1 \leq l \leq n$ , the following holds*

$$z(w_{k+l} - w_j) = j - 1, \text{ for } 1 \leq j \leq k, \tag{59}$$

$$z(w_{n+2k-j} - w_{k+l}) = j, \text{ for } 0 \leq j \leq k - 1. \tag{60}$$

*Proof* By [12, Proposition 2.1] we are also provided with the following expressions

$$z(w_{j+1} - w_j) = \min\{i(w_{j+1}), i(w_j)\}, \quad 1 \leq j \leq n + 2k, \tag{61}$$

$$\begin{aligned} z(w_{l+1} - w_j) &= z(w_l - w_j) + \frac{1}{2}[(-1)^{l+1} \text{sign}(\sigma_h^{-1}(l + 1) - \sigma_h^{-1}(j)) \\ &\quad + (-1)^l \text{sign}(\sigma_h^{-1}(l) - \sigma_h^{-1}(j))], \quad 1 \leq j < l \leq n + 2k - 1, \end{aligned} \tag{62}$$

$$z(w_j - w_1) = z(w_{n+2k} - w_j) = 0, \quad 2 \leq j \leq n + 2k - 1. \tag{63}$$

In order to prove the first statement of the lemma, we claim that for any  $1 \leq j \leq k$  we have

$$z(w_{k+l} - w_j) = z(w_{k+1} - w_j), \text{ for } 2 \leq l \leq n. \tag{64}$$

Indeed, from (62) we obtain

$$\begin{aligned} z(w_{k+l} - w_j) &= z(w_{k+l-1} - w_j) + \frac{1}{2}[(-1)^{k+l} \text{sign}(\sigma_h^{-1}(k + l) - \sigma_h^{-1}(j)) \\ &\quad + (-1)^{k+l-1} \text{sign}(\sigma_h^{-1}(k + l - 1) - \sigma_h^{-1}(j))], \end{aligned} \tag{65}$$

for all  $1 \leq j \leq k$  and  $2 \leq l \leq n$ . But, since

$$\sigma_h^{-1}(k + l) \in \{k + 1, \dots, k + n\} \text{ for all } 1 \leq l \leq n, \tag{66}$$

we have

$$\text{sign}(\sigma_h^{-1}(k + l) - \sigma_h^{-1}(j)) = \text{sign}(\sigma_h^{-1}(k + l - 1) - \sigma_h^{-1}(j)) = (-1)^{j+1},$$

and conclude from (65) that

$$z(w_{k+l} - w_j) = z(w_{k+l-1} - w_j)$$

for all  $1 \leq j \leq k$  and  $2 \leq l \leq n$ . In particular, this shows (64).

It is then sufficient to compute  $z(w_{k+1} - w_j)$ . For that, using (62) we write

$$z(w_{j+m} - w_j) = z(w_{j+m-1} - w_j) + \frac{1}{2}[(-1)^{j+m} \text{sign}(\sigma_h^{-1}(j+m) - \sigma_h^{-1}(j)) + (-1)^{j+m-1} \text{sign}(\sigma_h^{-1}(j+m-1) - \sigma_h^{-1}(j))],$$

for  $2 \leq m \leq k - j$  and  $1 \leq j \leq k$ . We further notice that by (51) we have

$$\text{sign}(\sigma_h^{-1}(j+m) - \sigma_h^{-1}(j)) = \text{sign}(\sigma_h^{-1}(j+m-1) - \sigma_h^{-1}(j)) = (-1)^{j+1}.$$

Then, we obtain

$$z(w_{j+m} - w_j) = z(w_{j+m-1} - w_j), \tag{67}$$

for  $2 \leq m \leq k - j$ . Also, for  $m = k + 1 - j$  we have from (62)

$$z(w_{k+1} - w_j) = z(w_k - w_j) + \frac{1}{2}[(-1)^{k+1} \text{sign}(\sigma_h^{-1}(k+1) - \sigma_h^{-1}(j)) + (-1)^k \text{sign}(\sigma_h^{-1}(k) - \sigma_h^{-1}(j))]$$

which, by (66) implies that  $z(w_{k+1} - w_j) = z(w_k - w_j)$ . Therefore we conclude that (67) holds for  $2 \leq m \leq k + 1 - j$ .

Hence, using (67) iteratively from  $m = k + 1 - j$  to  $m = 2$  we obtain

$$z(w_{k+1} - w_j) = z(w_{j+1} - w_j).$$

Thus, by (61) and Lemma 11 we conclude that  $z(w_{k+1} - w_j) = j - 1$ . Therefore, (64) implies (59) for  $1 \leq l \leq n$ .

To ensure (60), we first prove that

$$i(w_{k+l}) = k - \sum_{m=l+1}^n (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) \tag{68}$$

for all  $1 \leq l \leq n$ . We remark that for  $l = n$  the above sum is empty, which means that  $i(w_{k+n}) = k$  as expected. In order to prove (68), we first notice that  $z(w_{n+2k} - w_{k+l}) = 0$  by (63), and

$$z(w_{n+2k} - w_{k+l}) = i(w_{k+l}) + \frac{1}{2}[(-1)^{n+2k} \text{sign}(\sigma_h^{-1}(n+2k) - \sigma_h^{-1}(k+l)) - 1] + \sum_{m=l+1}^{n+k-1} (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l))$$

by Proposition 1. These equalities imply

$$i(w_{k+l}) = 1 - \sum_{m=l+1}^{n+k-1} (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)), \tag{69}$$

since  $(-1)^{n+2k} \text{sign}(\sigma_h^{-1}(n+2k) - \sigma_h^{-1}(k+l)) = -1$  by (51). Noticing that, again by (51), the last  $(k-1)$  terms of the sum in (69) add-up to

$$\sum_{m=n+1}^{n+k-1} (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) = -(k-1),$$

we conclude that

$$i(w_{k+l}) = k - \sum_{m=l+1}^n (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)),$$

which shows (68).

We are then able to obtain (60) using (68). From Proposition 1 we have

$$\begin{aligned} z(w_{n+2k-j} - w_{k+l}) &= i(w_{k+l}) \\ &+ \frac{1}{2} [(-1)^{n+2k-j} \text{sign}(\sigma_h^{-1}(n+2k-j) - \sigma_h^{-1}(k+l)) - 1] \\ &+ \sum_{m=l+1}^{n+k-j-1} (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) \end{aligned} \tag{70}$$

for  $1 \leq l \leq n$  and  $0 \leq j \leq k-1$ . Again by (51) we obtain

$$(-1)^{n+2k-j} \text{sign}(\sigma_h^{-1}(n+2k-j) - \sigma_h^{-1}(k+l)) = -1,$$

and the last  $(k-j-1)$  terms of the sum in (70) add-up to

$$\sum_{m=n+1}^{n+k-j-1} (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) = -(k-j-1).$$

By (68) this yields

$$\begin{aligned} z(w_{n+2k-j} - w_{k+l}) &= i(w_{k+l}) - 1 \\ &+ \sum_{m=l+1}^n (-1)^{k+m} \text{sign}(\sigma_h^{-1}(k+m) - \sigma_h^{-1}(k+l)) \\ &- (k-j-1) = k-1 - (k-j-1) = j, \end{aligned}$$

Then (60) holds for  $1 \leq l \leq n$ . □

The next result on the zero numbers of the differences between interior equilibria in  $E_h$ , follows immediately from (56). It provides the obvious relation between the zero numbers of the differences between equilibria  $w_j \in E_h$  with  $k+1 \leq j \leq k+n$  and the zero numbers of the differences between equilibria  $v_l \in E_f^c$ .

**Lemma 14** *For all  $1 \leq m < l \leq n$ , the following holds:*

$$z(w_{k+l} - w_{k+m}) = z(v_l - v_m). \tag{71}$$

The following lemma establishes expressions for these zero numbers in terms of the permutation  $\sigma_f$ .

**Lemma 15** For any  $1 \leq m < l \leq n$ , the zero number  $z(v_l - v_m)$  is given by

$$z(v_l - v_m) = i(v_m) + \frac{1}{2}[(-1)^{l+k} \text{sign}(\sigma_f^{-1}(l) - \sigma_f^{-1}(m)) - 1] + \sum_{j=m+1}^{l-1} (-1)^{j+k} \text{sign}(\sigma_f^{-1}(j) - \sigma_f^{-1}(m)). \tag{72}$$

*Proof* From Proposition 2, using Lemmas 14 and 12, we have

$$z(v_l - v_m) = i(v_m) + \frac{1}{2}[(-1)^{k+l} \text{sign}(\sigma_h^{-1}(k+l) - \sigma_h^{-1}(k+m)) - 1] + \sum_{j=m+1}^{l-1} (-1)^{k+j} \text{sign}(\sigma_h^{-1}(k+j) - \sigma_h^{-1}(k+m)). \tag{73}$$

Then, recalling that  $\sigma_h = \hat{\sigma}_f^1$  we obtain from (52)

$$\sigma_h(k+r) = k + \sigma_f(r), \text{ for } r = 1, \dots, n. \tag{74}$$

If we let  $j = \sigma_f(r)$  for  $r \in \{1, \dots, n\}$  we have that  $j \in \{1, \dots, n\}$ , and taking inverse permutations we rewrite (74) as

$$\sigma_h^{-1}(k+j) = k + \sigma_f^{-1}(j), \text{ for } j = 1, \dots, n.$$

This implies the following equalities

$$\sigma_h^{-1}(k+j) - \sigma_h^{-1}(k+m) = \sigma_f^{-1}(j) - \sigma_f^{-1}(m)$$

for all  $m$  and  $j$  in  $\{1, \dots, n\}$ . By substitution into (73) this yields (72) as wanted. □

Remarkably, Lemmas 11 and 13 lead to the following convenient relations:

$$i(w_{j+1}) = i(\Phi_j^{\infty,-})$$

$$i(w_{n+2k-j}) = i(\Phi_j^{\infty,+})$$

for  $0 \leq j \leq k-1$ , and

$$z(w_{k+l} - w_{j+1}) = z(v_l - \Phi_j^{\infty,-})$$

$$z(w_{n+2k-j} - w_{k+l}) = z(\Phi_j^{\infty,+} - v_l)$$

for  $0 \leq j \leq k-1$  and  $1 \leq l \leq n$ , that is, for all bounded equilibria  $v_l \in E_f^c$  and all equilibria at infinity  $\Phi_j^{\infty,\pm}$ . Moreover, as seen in Lemmas 12 and 14, the Morse indices and zero numbers for the differences between interior equilibria  $w_j \in E_h$ , with  $j \in \{k+1, \dots, k+n\}$ , coincide with the corresponding Morse indices and zero numbers for the equilibria  $v_l \in E_f^c$ .

Under the dissipative setting of Eq. (28), we recall from [29] the adjacency notion and the associated heteroclinic connection result.

**Definition 4** Consider any two different equilibria  $w_r$  and  $w_s$  in  $E_h$  with  $z(w_r - w_s) = m$  and  $w_s(0) < w_r(0)$ . We say that  $w_r$  and  $w_s$  are  $m$ -adjacent if there does not exist any other equilibrium  $w \in E_h$  satisfying

$$z(w_r - w) = z(w - w_s) = m$$

and

$$w_s(0) < w(0) < w_r(0).$$



**Theorem 3** [29] *Let  $w_r$  and  $w_s$  be any two different equilibria in  $E_h$  with  $z(w_r - w_s) = m$ . Then there exists a heteroclinic connection between  $w_r$  and  $w_s$  if, and only if, they are  $m$ -adjacent.*

We know that the bounded subset  $\mathcal{A}_f^c$  of  $\mathcal{A}_f$  coincides with the subset  $\mathcal{A}_h^c \subset \mathcal{A}_h$ , where  $\mathcal{A}_h^c \subset B$  and  $\mathcal{A}_h = \mathcal{A}_h^c \cup \{\mathcal{A}_h \setminus \mathcal{A}_h^c\}$ . In particular, the equilibrium  $w_{k+l} \in E_h$  coincides with the equilibrium  $v_l \in E_f$ , for any  $l \in \{1, \dots, n\}$ . Moreover, the heteroclinic connections in  $\mathcal{A}_h^c$  between the equilibria  $w_{k+l}$  are identical to the heteroclinic connections in  $\mathcal{A}_f^c$  between the equilibria  $v_l$ . All of these facts follow from (56). We thus conclude that the connections on  $\mathcal{A}_f^c$  are given as in Theorem 3, i.e., there is a heteroclinic connection between  $v_l$  and  $v_m$  if, and only if,  $v_l$  and  $v_m$  are adjacent. Then  $\sigma_h$  determines the connecting orbit structure on  $\mathcal{A}_f^c$ .

*Proof of Theorem 2* After the previous observation we are only concerned with the connections involving the equilibria at infinity. The case of intra-infinite heteroclinic connections (27) is simple to verify from the adjacency rules, since there are neither infinite nor exterior blockings. Then suppose that an interior equilibrium  $w_{k+l} = v_l$ , for some  $l \in \{1, \dots, n\}$ , has a heteroclinic connection to the exterior equilibrium  $w_j$ , for some  $j \in \{1, \dots, k\}$ . We thus know, from Lemma 13, that

$$z(w_{k+l} - w_j) = j - 1.$$

Since  $w_{k+l}$  connects to  $w_j$ , we conclude from Theorem 3, that there does not exist any equilibrium  $w \in E_h$  satisfying

$$z(w_{k+l} - w) = z(w - w_j) = j - 1 \tag{75}$$

and

$$w_j(0) < w(0) < w_{k+l}(0). \tag{76}$$

We want to see that the equilibrium  $v_l = w_{k+l}$  connects to the equilibrium in  $E_f$  corresponding to  $w_j$ , that is, the equilibrium at infinity  $\Phi_{j-1}^{\infty,-}$ . Assume by contradiction that this is not the case. Then, it follows from Lemma 8 that there exists an equilibrium  $v \in E_f^c$  satisfying

$$z(v_l - v) = z(v - \Phi_{j-1}^{\infty,-}) \tag{77}$$

and

$$\Phi_{j-1}^{\infty,-}(0) < v(0) < v_l(0). \tag{78}$$

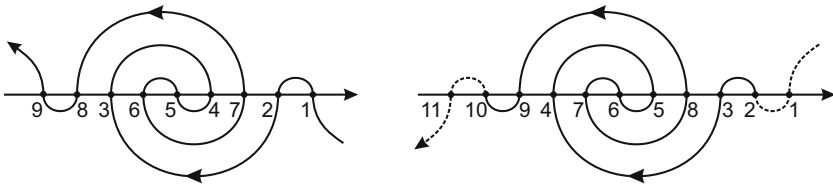
Which leads to a contradiction because  $v = w_{k+m}$  for some  $m \in \{1, \dots, n\}$  and, then,  $v$  could not satisfy (77) and (78) since there does not exist  $w \in E_h$  satisfying (75) and (76). We thus conclude that the heteroclinic connections to infinity in  $\mathcal{A}_f$  are also given as in Theorem 3, that is to say that the permutation  $\sigma_h$  determines the heteroclinic connections to infinity in  $\mathcal{A}_f$ .

Moreover, heteroclinic orbits run in the direction of decreasing Morse indices, (see [12]). These observations conclude the proof of the theorem.  $\square$

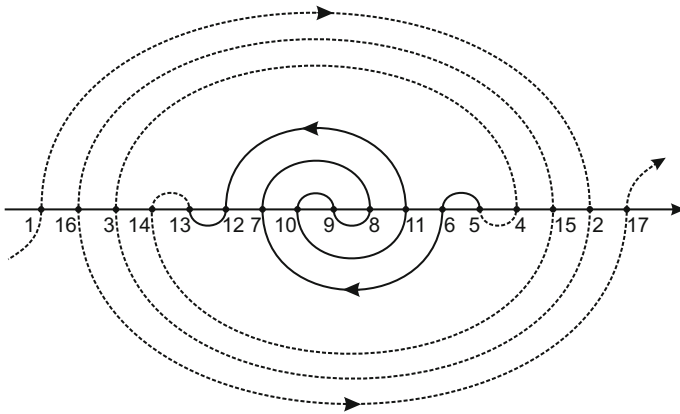
To illustrate the permutation suspension procedure we consider the following non-dissipative nonlinearity  $f = f(u)$ ,

$$f(u) = 10u + 16 \sin u.$$

In this case we have  $k = 1 + [\sqrt{10}] = 4$ . The meander  $\gamma_f$  was numerically obtained in [7]. The set  $E_f$  has  $n = 9$  equilibria, and the canonical form of the meander  $\gamma_f$  is depicted in Fig. 1 below.



**Fig. 1** Canonical representation of meander curves: (Left) the permutation  $\sigma_f = \{9, 8, 3, 6, 5, 4, 7, 2, 1\}$ ; (Right) its suspension  $\hat{\sigma}_f^4 = \{11, 10, 9, 4, 7, 6, 5, 8, 3, 2, 1\}$



**Fig. 2** Canonical representation of the meander curve of the 4th suspension  $\hat{\sigma}_f^1 = \{1, 16, 3, 14, 13, 12, 7, 10, 9, 8, 11, 6, 5, 4, 15, 2, 17\}$

In this case we have

$$\sigma_f = \{9, 8, 3, 6, 5, 4, 7, 2, 1\} ,$$

and the Morse indices computed from (38) are given by

$$(i_j(\sigma_f))_{1 \leq j \leq 9} = (4, 3, 4, 5, 6, 5, 4, 3, 4).$$

Then, the suspension process described in Definition 3 applied to  $\sigma_f$  yields

$$\hat{\sigma}_f^4 = \{11, 10, 9, 4, 7, 6, 5, 8, 3, 2, 1\}.$$

The successive application of this suspension procedure leads to

$$\hat{\sigma}_f^3 = \{1, 12, 11, 10, 5, 8, 7, 6, 9, 4, 3, 2, 13\} ,$$

$$\hat{\sigma}_f^2 = \{15, 2, 13, 12, 11, 6, 9, 8, 7, 10, 5, 4, 3, 14, 1\} ,$$

and finally,

$$\hat{\sigma}_f^1 = \{1, 16, 3, 14, 13, 12, 7, 10, 9, 8, 11, 6, 5, 4, 15, 2, 17\}.$$

This corresponds to the permutation  $\sigma_h = \hat{\sigma}_f^1$  and the canonical form of the meander  $\gamma_h$  is depicted in Fig. 2.

The total number of equilibria in  $E_h$  is  $N = 17$ , which correspond to 9 interior and 8 exterior equilibria, and the corresponding Morse indices are given by

$$(i_j(\sigma_h))_{1 \leq j \leq 17} = (0, 1, 2, 3, 4, 3, 4, 5, 6, 5, 4, 3, 4, 3, 2, 1, 0).$$

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