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Quasistatic Evolution in Perfect Plasticity as Limit of Dynamic Processes

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Abstract We introduce a model of dynamic visco-elasto-plastic evolution in the linearly elastic regime and prove an existence and uniqueness result. Then we study the limit of (a rescaled version of) the solutions when the data vary slowly. We prove that they converge, up to a subsequence, to a quasistatic evolution in perfect plasticity.

Keywords Visco-elasto-plasticity · Perfect plasticity · Dynamic evolution · Quasistatic evolution · Discrete time approximation · Implicit euler scheme · Incremental minimum problems

1 Introduction

The quasistatic evolution of rate independent systems has been often obtained as the limit case of a viscosity driven evolution (see [7–10,14,15,17,18,20–23,26,27,30]). In this paper we present a case study on the approximation of a quasistatic evolution by dynamic evolutions, in a mechanical problem governed by partial differential equations. For a similar problem in finite dimension we refer to [1].

More precisely, we approximate the solutions of the quasistatic evolution in linearly elastic perfect plasticity (see [6,27]) by the solutions of suitable dynamic visco-elasto-plastic problems, when a parameter connected with the speed of the process tends to 0.

In the first part of the paper we consider a model of dynamic visco-elasto-plastic evolution in the linearly elastic regime. In a sense it couples dynamic visco-elasticity with Perzyna visco-plasticity. In [27] the quasistatic evolution for perfect plasticity is obtained as a vanishing viscosity limit of Perzyna visco-plasticity. This is the main reason for our choice of this particular dynamic model.

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We now describe the model in more detail. The reference configuration is a bounded open set $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. The linearized strain Eu, defined as the symmetric part of the gradient of the displacement u, is decomposed as $Eu = e + p$, where e is the elastic part and p is the plastic part. The stress $\sigma = A_0 e + A_1 \dot{e}$ is the sum of an elastic part $A_0 e$ and a viscous part $A_1 \dot{e}$, where A_0 is the elasticity tensor, A_1 is the viscosity tensor, and $\dot{e}$ is the derivative of e with respect to time. As usual we assume that A_0 is symmetric and positive definite, while we only assume that A_1 is symmetric and positive semidefinite, so that we are allowed to consider also $A_1 = 0$, which corresponds to the purely visco-plastic case.

The balance of momentum gives the equation

$$\ddot{u} - \operatorname{div}\sigma = f,$$

where f is the volume force, and we have supposed, for simplicity, that the mass density is identically equal to 1. As in Perzyna visco-plasticity, the evolution of the plastic part is governed by the flow rule

$$\dot{p} = \sigma_D - \pi_K \sigma_D,$$

where σ_D is the deviatoric part of σ and π_K is the projection onto a prescribed convex set K in the space of deviatoric symmetric matrices, which can be interpreted as the domain of visco-elasticity. Indeed, if σ_D belongs to K during the evolution, then there is no production of plastic strain, so that, if $p = 0$ at the initial time, then $p = 0$ for every time and the solution is purely visco-elastic.

The complete system of equations is then

$$Eu = e + p, \tag{1.1a}$$

$$\sigma = A_0 e + A_1 \dot{e}_{A_1}, \tag{1.1b}$$

$$\ddot{u} - \operatorname{div}\sigma = f, \tag{1.1c}$$

$$\dot{p} = \sigma_D - \pi_K \sigma_D, \tag{1.1d}$$

where e_{A_1} denotes the projection of e into the image of A_1. This system is supplemented by initial and boundary conditions. Other dynamic models of elasto-plasticity with viscosity have been considered in [2] and [3]. The main difference with respect to our model is that they couple visco-elasticity with perfect plasticity, while we couple visco-elasticity with visco-plasticity.

Under natural assumptions on A_0, A_1, f, and K we prove existence and uniqueness of a solution to (1.1) with initial and boundary conditions (Theorem 1). In analogy with the energy method for rate independent processes developed by Mielke (see [20] and the references therein), we first prove that system (1.1) has a weak formulation expressed in terms of an energy balance together with a stability condition (Theorem 2). The proof of the existence of a solution to this weak formulation is obtained by time discretization. In the discrete formulation we solve suitable incremental minimum problems and then we pass to the limit as the time step tends to 0.

In the second part of the work we analyze the behavior of the solution to system (1.1) as the data of the problem become slower and slower. After rescaling time, as described at the beginning of Sect. 6, we are led to study the solutions to the system

$$Eu^\epsilon = e^\epsilon + p^\epsilon, \tag{1.2a}$$
$$\sigma^\epsilon = A_0 e^\epsilon + \epsilon A_1 \dot{e}^\epsilon_{A_1}, \tag{1.2b}$$
$$\epsilon^2 \ddot{u}^\epsilon - \mathrm{div}\sigma^\epsilon = f, \tag{1.2c}$$
$$\epsilon \dot{p}^\epsilon = \sigma^\epsilon_D - \pi_K \sigma^\epsilon_D, \tag{1.2d}$$

as ϵ tends to 0.

Under suitable assumptions we show (Theorem 6) that these solutions converge, up to a subsequence, to a weak solution of the quasistatic evolution problem in perfect plasticity (see [27] and [6]), whose strong formulation is given by

$$Eu = e + p, \tag{1.3a}$$
$$\sigma = A_0 e, \tag{1.3b}$$
$$-\mathrm{div}\sigma = f, \tag{1.3c}$$
$$\sigma_D \in K \quad \text{and} \quad \dot{p} \in N_K \sigma_D, \tag{1.3d}$$

where $N_K\sigma_D$ denotes the normal cone to K at σ_D.

The proof of this convergence result is obtained using the weak formulation of (1.1) expressed by energy balance and stability condition (see Theorem 2). We show that we can pass to the limit obtaining the energy formulation of (1.3) developed in [6]. A remarkable difficulty in this proof is due to the fact that problems (1.1) and (1.3) are formulated in completely different function spaces (see Theorem 1 and Definition 1).

2 Visco-Elasto-Plastic Evolution

2.1 Notation

Vectors and Matrices If $a, b \in \mathbb{R}^n$, their scalar product is defined by $a \cdot b := \sum_i a_i b_i$, and $|a| := (a \cdot a)^{1/2}$ is the norm of a. If $\eta = (\eta_{ij})$ and $\xi = (\xi_{ij})$ belong to the space $\mathbb{M}^{n\times n}$ of $n \times n$ matrices with real entries, their scalar product is defined by $\eta \cdot \xi := \sum_{ij} \eta_{ij}\xi_{ij}$. Similarly $|\eta| := (\eta \cdot \eta)^{1/2}$ is the norm of η. $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ is the subspace of $\mathbb{M}^{n\times n}$ composed of symmetric matrices. Moreover $\mathbb{M}^{n\times n}_D$ denotes the subspace of symmetric matrices with null trace, i.e., $\eta \in \mathbb{M}^{n\times n}_D$ if η is symmetric and $\mathrm{tr}\eta = \sum_i \eta_{ii} = 0$. The space $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ can be split as

$$\mathbb{M}^{n\times n}_{\mathrm{sym}} = \mathbb{M}^{n\times n}_D \oplus \mathbb{R}I,$$

where I is the identity matrix, so that every $\eta \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ can be written as $\eta = \eta_D + \frac{\mathrm{tr}\eta}{n} I$, where η_D, called the deviatoric part of η, is the projection of η into $\mathbb{M}^{n\times n}_D$.

Duality and Norms If X is a Banach space and $u \in X$, we usually denote the norm of u by $\|u\|_X$. If X is $L^p(\Omega)$, $L^p(\Omega; \mathbb{R}^n)$, $L^p(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$, or $L^p(\Omega; \mathbb{M}^{n\times n}_D)$ the norm is denoted by $\|u\|_{L^p}$. In general, if X is a Banach space, X' is its dual space and $\langle u, v\rangle_X$ denotes the duality product between $u \in X'$ and $v \in X$. The subscript X is sometimes omitted, if it is clear from the context.

2.2 Kinematic Setting

The Reference Configuration The reference configuration is a bounded connected open set Ω in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. We suppose that $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \partial\Gamma$, where Γ_0, Γ_1, and $\partial\Gamma$ are pairwise disjoint, Γ_0 and Γ_1 are relatively open in $\partial\Omega$, and $\partial\Gamma$ is the relative boundary in $\partial\Omega$ both of Γ_0 and Γ_1. We assume that $\Gamma_0 \neq \emptyset$ and that $\mathcal{H}^{n-1}(\partial\Gamma) = 0$, where $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. On Γ_0 we will prescribe a Dirichlet condition on the displacement u, while on Γ_1 we will impose a Neumann condition on the stress σ.

Elastic and Plastic Strain If u is the displacement, the linearized strain Eu is its symmetrized gradient, defined as the $\mathbb{M}^{n\times n}_{\mathrm{sym}}$-valued distribution with components $E_{ij}u = \frac{1}{2}(D_i u_j + D_j u_i)$. The linearized strain is decomposed as the sum of the elastic strain e and the plastic strain p. Given $w \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple (u, e, p) is kinematically admissible for the visco-elasto-plastic problem with boundary datum w if $u \in H^1(\Omega; \mathbb{R}^n)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$, $p \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$, and

$$Eu = e + p \quad \text{on } \Omega, \tag{2.1a}$$

$$u|_{\Gamma_0} = w \quad \text{on } \Gamma_0. \tag{2.1b}$$

We denote the set of these triples by $A(w)$. It is convenient to introduce the subspace of $H^1(\Omega; \mathbb{R}^n)$ defined by

$$H^1_{\Gamma_0}(\Omega; \mathbb{R}^n) := \{u \in H^1(\Omega; \mathbb{R}^n) : u|_{\Gamma_0} = 0\}$$

and its dual space, denoted by $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$. It is clear that $(u, e, p) \in A(w)$ if and only if $u - w \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$ and $Eu = e + p$, with $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $p \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$.

Stress and External Forces In the visco-elasto-plastic model the stress σ depends linearly on the elastic part e of the strain Eu and on its time derivative $\dot{e}$. To express this dependence we introduce the elastic tensor A_0 and the visco-elastic tensor A_1, which are symmetric linear operators of $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ into itself. We assume that there exist positive constants α_0, β_0, and β_1 such that

$$|A_i\xi| \leq \beta_i|\xi|, \quad \text{for } i = 1, 2, \tag{2.2a}$$

$$A_0\xi \cdot \xi \geq \alpha_0|\xi|^2 \quad \text{and } A_1\xi \cdot \xi \geq 0, \tag{2.2b}$$

for every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$. Note that $A_1 = 0$ is allowed. Inequalities (2.2) imply

$$|A_i\xi|^2 \leq \beta_i A_i\xi \cdot \xi, \tag{2.2c}$$

for every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ and for $i = 1, 2$.

For every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ let ξ_{A_1} be the orthogonal projection of ξ onto the image of A_1. Then there exists a constant $\alpha_1 > 0$ such that

$$A_1\xi \cdot \xi \geq \alpha_1|\xi_{A_1}|^2 \tag{2.3}$$

for every $\xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$.

The stress satisfies the constitutive relation

$$\sigma = A_0 e + A_1\dot{e}. \tag{2.4}$$

The term $A_1\dot{e}$ in the equation above is the component of the stress due to internal frictions. To express the energy balance it is useful to introduce the quadratic forms

$$Q_0(\xi) = \frac{1}{2}A_0\xi \cdot \xi \quad \text{and} \quad Q_1(\xi) = A_1\xi \cdot \xi.$$

For every $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ we define

$$\mathcal{Q}_0(e) = \int_\Omega Q_0(e)dx \quad \text{and} \quad \mathcal{Q}_1(e) = \int_\Omega Q_1(e)dx.$$

These function turn out to be lower semicontinuous with respect to the weak topology of $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$. $\mathcal{Q}_0(e)$ represents the *stored elastic energy* associated to $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ while $\mathcal{Q}_1(\dot{e})$ represents the rate of visco-elastic dissipation.

We assume that the time dependent body force $f(t)$ belongs to $L^2(\Omega; \mathbb{R}^n)$ and that the time dependent surface force $g(t)$ belongs to $L^2(\Gamma_1, \mathcal{H}^{n-1}; \mathbb{R}^n)$. It is convenient to introduce the total load $\mathcal{L}(t) \in H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ of external forces acting on the body, defined by

$$\langle \mathcal{L}(t), u\rangle := \langle f(t), u\rangle_\Omega + \langle g(t), u\rangle_{\Gamma_1}, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ and $H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$, $\langle\cdot,\cdot\rangle_\Omega$ denotes the scalar product in $L^2(\Omega; \mathbb{R}^n)$, while $\langle\cdot,\cdot\rangle_{\Gamma_1}$ denotes the scalar product in $L^2(\Gamma_1, \mathcal{H}^{n-1}; \mathbb{R}^n)$.

When dealing with the visco-elasto-plastic problem, we will only suppose that the total load $\mathcal{L}(t)$ belongs to $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$, without assuming the particular form (2.5). The hypotheses on the functions $t \mapsto \mathcal{L}(t)$ and $t \mapsto w(t)$ and the regularity of $t \mapsto (u(t), e(t), p(t))$ will be made precise in the statement of Theorems 1 and 2 below.

The law which expresses the second law of dynamic is

$$\ddot{u}(t) - \mathrm{div}\sigma(t) = f(t) \quad \text{in } \Omega, \tag{2.6}$$

where we assume that the mass density of the elasto-plastic body is 1. Equation (2.6) is supplemented with the boundary conditions

$$u(t) = w(t) \quad \text{on } \Gamma_0, \tag{2.7a}$$

$$\sigma(t)\nu = g(t) \quad \text{on } \Gamma_1. \tag{2.7b}$$

To deal with (2.6) and (2.7), it is convenient to introduce the continuous linear operator $\mathrm{div}_{\Gamma_0} : L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \to H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ defined by

$$\langle \mathrm{div}_{\Gamma_0}\sigma, \varphi\rangle := -\langle \sigma, E\varphi\rangle \tag{2.8}$$

for every $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and every $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$.

If $f(t)$, $g(t)$, $\sigma(t)$, $u(t)$, Γ_0, and Γ_1 are sufficiently regular and $\mathcal{L}(t)$ is the total external load defined by (2.5), then we can prove, using integration by parts, that (2.6) and (2.7b) are equivalent to

$$\ddot{u}(t) - \mathrm{div}_{\Gamma_0}\sigma(t) = \mathcal{L}(t), \tag{2.9}$$

interpreted as equality between elements of $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$. In other words (2.9) is satisfied if and only if

$$\langle \ddot{u}(t), \varphi\rangle + \langle \sigma(t), E\varphi\rangle = \langle \mathcal{L}(t), \varphi\rangle \tag{2.10}$$

for every $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$. In the irregular case, Eq. (2.10) represents the weak formulation of problem (2.6) with boundary condition (2.7b).

Plastic Dissipation The elastic domain K is a convex and compact set in $\mathbb{M}^{n\times n}_D$. We will suppose that there exist two positive real numbers $r_1 < R_1$ such that

$$B(0, r_1) \subseteq K \subseteq B(0, R_1). \tag{2.11}$$

It is convenient to introduce the set

$$\mathcal{K}(\Omega) := \{\xi \in L^2(\Omega; \mathbb{M}^{n\times n}_D) : \xi(x) \in K \text{ for a.e. } x \in \Omega\}. \tag{2.12}$$

If π_K denotes the minimal distance projection of $\mathbb{M}^{n\times n}_D$ into K, and $\pi_{\mathcal{K}(\Omega)}$ denotes the projection of $L^2(\Omega; \mathbb{M}^{n\times n}_D)$ into $\mathcal{K}(\Omega)$, then it is easy to check that

$$(\pi_{\mathcal{K}(\Omega)}\xi)(x) = \pi_K \xi(x) \qquad \text{for a.e. } x \in \Omega, \tag{2.13}$$

for every $\xi \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$.

The evolution of the plastic strain $p(t, x)$ will be expressed by the Maximum Dissipation Principle (Hill's Principle of Maximum Work, see, e.g., [12,19,27]): if σ is the stress, then p will satisfy the following

$$\begin{aligned} &(\sigma_D(t, x) - \xi) \cdot \dot{p}(t, x) \geq 0 \quad \text{for every } \xi \in K \text{ and a.e. } x \text{ in } \Omega \\ &\sigma_D(t, x) - \dot{p}(t, x) \in K, \quad \text{for a.e. } x \text{ in } \Omega, \end{aligned}$$

where we assume for simplicity that the viscosity coefficient is 1. Thanks to the characterization of the projection onto convex sets (see, e.g., [13]), this condition is satisfied if and only if $\sigma_D(t, x) - \dot{p}(t, x)$ coincides with $\pi_K \sigma_D(t, x)$, for a.e. $x \in \Omega$. By (2.13), this can be written as

$$\dot{p}(t) = \sigma_D(t) - \pi_{\mathcal{K}(\Omega)}\sigma_D(t). \tag{2.14}$$

We define the support function $H : \mathbb{M}^{n\times n}_D \to [0, +\infty[$ of K by

$$H(\xi) = \sup_{\zeta\in K} \zeta \cdot \xi. \tag{2.15}$$

It turns out that H is convex and positively homogeneous of degree one. In particular it satisfies the triangle inequality

$$H(\xi + \zeta) \leq H(\xi) + H(\zeta)$$

and the following inequality, due to (2.11):

$$r_1|\xi| \leq H(\xi) \leq R_1|\xi|. \tag{2.16}$$

We define $\mathcal{H} : L^2(\Omega; \mathbb{M}^{n\times n}_D) \to \mathbb{R}$ by

$$\mathcal{H}(p) = \int_\Omega H(p(x))dx. \tag{2.17}$$

If $p \in H^1([0, T]; L^2(\Omega; \mathbb{M}^{n\times n}_D))$ and $\dot{p}(t)$ is its time derivative, then $\mathcal{H}(\dot{p})$ represents the rate of plastic dissipation, so that,

$$\int_0^T \mathcal{H}(\dot{p})dt \tag{2.18}$$

is the total plastic dissipation in the time interval $[0, T]$.

We notice that, by the definition of H, the subdifferential of H satisfies (see e.g. [25, Theorem 13.1])

$$\partial H(0) = K. \tag{2.19}$$

From (2.19), it easily follows

$$\partial\mathcal{H}(0) = \mathcal{K}(\Omega), \tag{2.20}$$

where $\partial\mathcal{H}(\xi)$ denotes the subdifferential of $\mathcal{H}$ at ξ.

2.3 Existence Results for Elasto-Visco-Plastic Evolutions

Given an elasto-visco-plastic body satisfying all the properties described in the previous section, we fix an external load $\mathcal{L}$ and a Dirichlet boundary datum w, and look for a solution of the dynamic Eq. (2.9) and of the flow rule (2.14), with stress σ defined by (2.4) and strain satisfying Eq. (2.1). Our existence result for an elasto-visco-plastic evolution is given by the following theorem.

Theorem 1 *Let $T > 0$, let $\mathcal{L} \in AC([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, and let w be a function such that*

$$w \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.21a}$$

$$\dot w \in C^0([0,T]; L^2(\Omega;\mathbb{R}^n)) \cap L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.21b}$$

$$\ddot w \in L^2([0,T]; L^2(\Omega;\mathbb{R}^n)). \tag{2.21c}$$

Then for every $(u_0, e_0, p_0) \in A(w(0))$ and $v_0 \in L^2(\Omega;\mathbb{R}^n)$ there exists a unique quadruple (u, e, p, σ) of functions, with

$$u \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.22a}$$

$$\dot u \in L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{2.22b}$$

$$\ddot u \in L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{2.22c}$$

$$e \in L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{2.22d}$$

$$p \in L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{2.22e}$$

$$\dot e_{A_1} \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{2.22f}$$

$$\dot p \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{2.22g}$$

$$\sigma \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{2.22h}$$

such that for a.e. $t \in [0,T]$ we have

$$Eu(t) = e(t) + p(t), \tag{2.23a}$$

$$\sigma(t) = A_0 e(t) + A_1 \dot e_{A_1}(t), \tag{2.23b}$$

$$\ddot u(t) - div_{\Gamma_0}\sigma(t) = \mathcal{L}(t), \tag{2.23c}$$

$$\dot p(t) = \sigma_D(t) - \pi_{\mathcal{K}(\Omega)}\sigma_D(t), \tag{2.23d}$$

and

$$u(t) = w(t) \quad on \quad \Gamma_0, \tag{2.24}$$

$$u(0) = u_0, \quad p(0) = p_0, \tag{2.25a}$$

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^h \|e(t) - e_0\|^2_{L^2} dt = 0, \quad \lim_{h\to 0^+} \frac{1}{h}\int_0^h \|\dot u(t) - v_0\|^2_{L^2} dt = 0. \tag{2.25b}$$

In (2.22f) and in the rest of the paper the symbol $\dot e_{A_1}$ denotes the time derivative (in the sense of distributions) of the function e_{A_1} defined before (2.3).

Moreover (u, e, p, σ) *satisfies the equilibrium condition*

$$-\mathcal{H}(q) \leq \langle \sigma(t), \eta \rangle + \langle \dot{p}(t), q \rangle + \langle \ddot{u}(t), \varphi \rangle - \langle \mathcal{L}(t), \varphi \rangle \leq \mathcal{H}(-q), \tag{2.26}$$

for a.e. $t \in [0, T]$ *and for every* $(\varphi, \eta, q) \in A(0)$, *where* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ *and* $H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$ *in the terms containing* $\ddot{u}$ *and* $\mathcal{L}$, *while it denotes the scalar product in* L^2 *in all other terms.*

Remark 1 In view of (2.21) and (2.22) we see that u, w, $\dot{u}$, $\dot{w}$, e_{A_1}, and p are absolutely continuous in time, more precisely,

$$w \in AC([0, T]; H^1(\Omega; \mathbb{R}^n)), \tag{2.27a}$$

$$u, \dot{w} \in AC([0, T]; L^2(\Omega; \mathbb{R}^n)), \tag{2.27b}$$

$$\dot{u} \in AC([0, T]; H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)), \tag{2.27c}$$

$$e_{A_1} \in AC([0, T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{2.27d}$$

$$p \in AC([0, T]; L^2(\Omega; \mathbb{M}^{n\times n}_D)) \tag{2.27e}$$

(see, e.g., [5], Proposition A.3 and following Corollary). Properties (2.27b) and (2.27e) give a precise meaning to the initial conditions (2.25a).

Moreover since u is bounded in $H^1(\Omega; \mathbb{R}^n)$ by (2.22a), we deduce from (2.27b) that $t \mapsto u(t)$ is weakly continuous into $H^1(\Omega; \mathbb{R}^n)$. Similarly, thanks to (2.27c) and since $\dot{u} \in L^\infty([0, T]; L^2(\Omega; \mathbb{R}^n))$ by (2.22b), it follows that $t \mapsto \dot{u}(t)$ is weakly continuous into $L^2(\Omega; \mathbb{R}^n)$. Moreover, $e = Eu - p \in H^1([0, T]; H^{-1}(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}))$ by (2.22a), (2.22b), (2.22e), and (2.22g), thus $e \in AC([0, T]; H^{-1}(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}))$. Since we have also $e \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}))$ by (2.22d), we conclude that $t \mapsto e(t)$ is weakly continuous into $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$. In particular for every $t \in [0, T]$ the functions $u(t)$, $e(t)$, $p(t)$, $\dot{u}(t)$ are univocally defined as elements of $H^1(\Omega; \mathbb{R}^n)$, $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$, $L^2(\Omega; \mathbb{M}^{n\times n}_D)$, and $L^2(\Omega; \mathbb{R}^n)$, respectively.

Remark 2 From (2.22), (2.23a), and (2.25) it follows that

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h \|u(t) - u_0\|^2_{H^1} dt = 0. \tag{2.28}$$

Indeed, by (2.27b) we have $\frac{1}{h}\int_0^h \|u(t) - u_0\|^2_{L^2} dt \to 0$, and (2.22g), (2.23a), while (2.25b) give $\frac{1}{h}\int_0^h \|Eu(t) - Eu_0\|^2_{L^2} dt \to 0$.

Before proving Theorem 1 we will first state the following result, which characterizes the solutions of Eqs. (2.23c) and (2.23d).

Theorem 2 *Under the hypotheses of Theorem* 1, *we assume that* (u, e, p, σ) *satisfies* (2.22), (2.23a), (2.23b), (2.24), *and* (2.25). *Then* (u, e, p, σ) *satisfies* (2.23c) *and* (2.23d) *for a.e.* $t \in [0, T]$ *if and only if both the following conditions hold:*

(a) *Energy balance: for a.e $t \in [0,T]$ we have*

$$\begin{aligned}&\mathcal{Q}_0(e(t)) + \frac{1}{2}\|\dot u(t) - \dot w(t)\|^2_{L^2} + \int_0^t \mathcal{Q}_1(\dot e_{A_1})ds + \int_0^t \|\dot p\|^2_{L^2} ds + \int_0^t \mathcal{H}(\dot p)ds \\ &= \mathcal{Q}_0(e_0) + \frac{1}{2}\|v_0 - \dot w(0)\|^2_{L^2} + \int_0^t \langle \sigma, E\dot w\rangle ds - \int_0^t \langle \ddot w, \dot u - \dot w\rangle ds \\ &\quad + \langle \mathcal{L}(t), u(t) - w(t)\rangle - \langle \mathcal{L}(0), u_0 - w(0)\rangle - \int_0^t \langle \dot{\mathcal{L}}, u - w\rangle ds, \end{aligned} \tag{2.29}$$

(b) *For a.e. $t \in [0,T]$ the equilibrium condition* (2.26) *holds for every $(\varphi, \eta, q) \in A(0)$.*

Moreover, if the two previous conditions are satisfied, then

$$\langle \sigma_D(t) - \dot p(t), \dot p(t)\rangle = \mathcal{H}(\dot p(t)) \quad \textit{for a.e. } t \in [0,T]. \tag{2.30}$$

Remark 3 If A_1 is positive definite, then (2.27d), (2.27e), and the Korn inequality, imply that $u \in AC([0,T]; H^1(\Omega;\mathbb{R}^n))$. If moreover the data w and $\mathcal{L}$ are sufficiently regular, $\mathcal{L}$ has the form (2.5), then we can integrate by parts the terms $\int_0^t \langle \ddot w, \dot u\rangle ds$ and $\int_0^t \langle \ddot w, \dot w\rangle ds$ obtaining that we can rewrite the energy balance as follows:

$$\begin{aligned}&\mathcal{Q}_0(e(t)) + \frac{1}{2}\|\dot u(t)\|^2_{L^2} + \int_0^t \mathcal{Q}_1(\dot e)ds + \int_0^t \|\dot p\|^2_{L^2} ds + \int_0^t \mathcal{H}(\dot p)ds \\ &= \int_0^t \langle \sigma, E\dot w\rangle ds + \int_0^t \langle f, \dot u - \dot w\rangle ds + \int_0^t \langle g, \dot u - \dot w\rangle_{\Gamma_1} ds \\ &\quad + \int_0^t \langle \ddot u, \dot w\rangle ds + \mathcal{Q}_0(e_0) + \frac{1}{2}\|v_0\|^2_{L^2}, \end{aligned}$$

which becomes, using $\ddot u = \mathrm{div}_{\Gamma_0}\sigma + \mathcal{L}$:

$$\begin{aligned}&\mathcal{Q}_0(e(t)) + \frac{1}{2}\|\dot u(t)\|^2_{L^2} + \int_0^t \mathcal{Q}_1(\dot e)ds + \int_0^t \|\dot p\|^2_{L^2} ds + \int_0^t \mathcal{H}(\dot p)ds \\ &= \int_0^t \langle \sigma\nu, \dot u\rangle_{\Gamma_0} ds + \int_0^t \langle f, \dot u\rangle ds + \int_0^t \langle g, \dot u\rangle_{\Gamma_1} ds + \mathcal{Q}_0(e_0) + \frac{1}{2}\|v_0\|^2_{L^2}, \end{aligned}$$

where we have used $\dot u = \dot w$ on Γ_0. This is the usual formulation of the energy balance. Indeed $\mathcal{Q}_0(e(t))$ is the stored elastic energy, $\frac{1}{2}\|\dot u(t)\|^2_{L^2}$ is the kinetic energy, $\int_0^t \mathcal{Q}_1(\dot e(t))ds$ is the visco-elastic dissipation, $\int_0^t \|\dot p\|^2_{L^2} ds$ is the visco-plastic dissipation, and $\int_0^t \mathcal{H}(\dot p)ds$ is the plastic dissipation. On the right-hand side the terms $\int_0^t \langle \sigma\nu, \dot u\rangle_{\Gamma_0} ds$, $\int_0^t \langle g, \dot u\rangle_{\Gamma_1} ds$, and $\int_0^t \langle f, \dot u\rangle ds$ represent the work done by the external forces on the Dirichlet boundary, on the Neumann boundary, and on the body itself, while the two terms $\mathcal{Q}_0(e_0)$ and $\frac{1}{2}\|v_0\|^2_{L^2}$ are the stored elastic energy and the kinetic energy at the initial time.

Lemma 1 *Let $T > 0$, let $\mathcal{L} \in AC([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, let w satisfy* (2.21), *and let (u, e, p, σ) be a quadruple satisfying* (2.22), (2.23a), (2.23b), (2.23c), (2.24), *and* (2.25). *Then*

$$\begin{aligned}&\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \int_0^t \mathcal{Q}_1(\dot e_{A_1})ds - \int_0^t \langle \sigma, E\dot w\rangle ds + \int_0^t \langle \sigma_D, \dot p\rangle ds \\ &+ \frac{1}{2}\|\dot u(t) - \dot w(t)\|^2_{L^2} - \frac{1}{2}\|v_0 - \dot w(0)\|^2_{L^2} = -\int_0^t \langle \ddot w, \dot u - \dot w\rangle ds \\ &+ \langle \mathcal{L}(t), u(t) - w(t)\rangle - \langle \mathcal{L}(0), u_0 - w(0)\rangle - \int_0^t \langle \dot{\mathcal{L}}, u - w\rangle ds, \end{aligned} \tag{2.31}$$

for a.e. $t \in [0, T]$.

Proof Given a function ϑ from $[0, T]$ into a Banach space X, for all $h > 0$ we define the difference quotient $s^h\vartheta : [0, T-h] \to X$ as $s^h\vartheta(t) := \frac{1}{h}(\vartheta(t+h)-\vartheta(t))$. By (2.21), (2.22), and (2.24) for a.e. $t \in [0, T]$ the function $\varphi := s^h u(t) - s^h w(t)$ belongs to $H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$. We use this function in (2.10) first at time t and then at time $t+h$. Summing the two expressions we get

$$\begin{aligned}
&\langle \ddot{u}(t+h)-\ddot{w}(t+h)+\ddot{u}(t)-\ddot{w}(t), s^hu(t)-s^hw(t)\rangle + \langle hs^h\sigma(t), s^hp(t)-s^hEw(t)\rangle \\
&\quad + \langle A_0e(t+h) + A_1\dot{e}_{A_1}(t+h) + A_0e(t) + A_1\dot{e}_{A_1}(t), s^he(t)\rangle \\
&= -\langle \ddot{w}(t+h) + \ddot{w}(t), s^hu(t) - s^hw(t)\rangle + \langle \mathcal{L}(t+h) + \mathcal{L}(t), s^hu(t) - s^hw(t)\rangle. \qquad (2.32)
\end{aligned}$$

We now integrate in time on the interval $[0, t]$. An integration by parts in time gives that the first term is equal to

$$\begin{aligned}
&\langle \dot{u}(t+h) - \dot{w}(t+h), s^hu(t) - s^hw(t)\rangle + \langle \dot{u}(t) - \dot{w}(t), s^hu(t) - s^hw(t)\rangle \\
&- \langle \dot{u}(h) - \dot{w}(h), s^hu(0) - s^hw(0)\rangle - \langle \dot{u}(0) - \dot{w}(0), s^hu(0) - s^hw(0)\rangle \\
&- \frac{1}{h}\int_t^{t+h} \|\dot{u}(r) - \dot{w}(r)\|^2_{L^2}dr + \frac{1}{h}\int_0^h \|\dot{u}(r) - \dot{w}(r)\|^2_{L^2}dr. \qquad (2.33)
\end{aligned}$$

As for the third term we find that it is equal to

$$\frac{2}{h}\int_t^{t+h} \mathcal{Q}_0(e(r))dr - \frac{2}{h}\int_0^h \mathcal{Q}_0(e(r))dr + \int_0^t \langle hA_1s^h\dot{e}_{A_1}(r), s^he_{A_1}(r)\rangle dr, \qquad (2.34)$$

while the last one is equal to

$$\begin{aligned}
&\frac{2}{h}\int_t^{t+h} \langle \mathcal{L}(r), u(r) - w(r)\rangle dr - \frac{2}{h}\int_0^h \langle \mathcal{L}(r), u(r) - w(r)\rangle dr \\
&- \int_0^t \langle s^h\mathcal{L}(r), u(r+h) - w(r+h) + u(r) - w(r)\rangle dr. \qquad (2.35)
\end{aligned}$$

Now (2.21), (2.25b), (2.27b), the weak continuity of $\dot{u}$ on $[0, T]$ into $L^2(\Omega; \mathbb{R}^n)$ (see Remark 1), and the Lebesgue mean value Theorem, allow us to pass to the limit as $h \to 0$ in (2.33) for a.e. $t \in [0, T]$. By similar arguments, using (2.21), (2.22), (2.25b), (2.28), and the weak continuity of u on $[0, T]$ into $H^1(\Omega; \mathbb{R}^n)$ (see Remark 1), we pass to the limit in (2.34), (2.35), and in the other terms of (2.32), so that we obtain (2.31) for a.e. $t \in [0, T]$.

Proof (Theorem 2) Let us suppose that the quadruple (u, e, p, σ) satisfies (2.26) and (2.29); let us prove (2.23c). Let $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$; since $(\varphi, E\varphi, 0) \in A(0)$, we can choose $\eta = E\varphi$ and $q = 0$ in (2.26) and for a.e. $t \in [0, T]$ we get

$$\langle A_0e(t) + A_1\dot{e}_{A_1}(t), E\varphi\rangle + \langle \ddot{u}(t), \varphi\rangle - \langle \mathcal{L}(t), \varphi\rangle = 0, \qquad (2.36)$$

which is equivalent to (2.23c), thanks to (2.10) and (2.23b).

It remains to prove (2.23d). Choosing $(0, q, -q) \in A(0)$ in (2.26) for some $q \in L^2(\Omega, \mathbb{M}^{n\times n}_D)$, for a.e. $t \in [0, T]$ we get

$$-\mathcal{H}(-q) \le \langle A_0e(t) + A_1\dot{e}_{A_1}(t), q\rangle - \langle \dot{p}(t), q\rangle \le \mathcal{H}(q), \qquad (2.37)$$

which, by (2.23b), says that

$$\sigma_D(t) - \dot{p}(t) \in \partial\mathcal{H}(0) = \mathcal{K}(\Omega) \qquad (2.38)$$

thanks to the arbitrariness of q.

Now we observe that (u, e, p, σ) satisfies the hypotheses of Lemma 1, so (2.31) holds for a.e. $t \in [0, T]$. This, together with the energy balance (2.29), implies that (2.30) holds for a.e. $t \in [0, T]$. As a consequence, by the definition of $\mathcal{H}$, we deduce that for a.e. $t \in [0, T]$ and for every $\xi \in \mathcal{K}(\Omega)$ we have

$$\langle \sigma_D(t) - \dot{p}(t), \dot{p}(t) \rangle \geq \langle \xi, \dot{p}(t) \rangle,$$

which is equivalent to

$$\langle \sigma_D(t) - (\sigma_D(t) - \dot{p}(t)), \xi - (\sigma_D(t) - \dot{p}(t)) \rangle \leq 0.$$

Thanks to (2.38), $\sigma_D(t) - \dot{p}(t)$ belongs to $\mathcal{K}(\Omega)$; therefore the arbitrariness of ξ and the well-known characterization of the projection onto convex sets (see, e.g., [13], Chap. 1.2) give that $\sigma_D(t) - \dot{p}(t) = \pi_{\mathcal{K}(\Omega)} \sigma_D(t)$ for a.e. $t \in [0, T]$.

Conversely, suppose (u, e, p, σ) to be a solution of the system of Eq. (2.23). Then (2.23d) implies (2.38), which in turn gives (2.37). On the other hand (2.23b) and (2.23c) give (2.36). Subtracting (2.37) from (2.36) term by term and taking into account (2.1a), we get (2.26).

In order to obtain the energy balance we first prove that, if a function (u, e, p, σ) satisfies (2.23), then (2.30) holds. Indeed, if $\xi \in \mathcal{K}(\Omega)$, then from the properties of convex sets it follows that for a.e. $t \in [0, T]$

$$\begin{aligned}
&(\sigma_D - \dot{p}) \cdot \dot{p} = \pi_K \sigma_D \cdot (\sigma_D - \pi_K \sigma_D) \\
&\geq \pi_K \sigma_D \cdot (\sigma_D - \pi_K \sigma_D) + (\xi - \pi_K \sigma_D) \cdot (\sigma_D - \pi_K \sigma_D) = \xi \cdot (\sigma_D - \pi_K \sigma_D)
\end{aligned}$$

almost everywhere in Ω, that is $(\sigma_D - \dot{p}) \cdot \dot{p} \geq H(\sigma_D - \pi_K \sigma_D) = H(\dot{p})$ thanks to the definition of H. Since $\sigma_D - \dot{p} \in K$ a.e. in Ω and for a.e. $t \in [0, T]$ by (2.23d), the definition of H gives also the opposite inequality. So integrating on Ω we get (2.30).

Now since (u, e, p, σ) satisfies the hypotheses of Lemma 1, we obtain (2.31), which together with (2.30) gives the energy balance (2.29) for a.e. $t \in [0, T]$.

Proof (Theorem 1)

The proof is reminiscent of that of [3, Theorem 3.1], with some important differences. In [3, Theorem 3.1] only Dirichlet conditions are considered and the data of the problem are more regular than ours: the external load f belongs to $AC([0, T]; L^2(\Omega; \mathbb{R}^n))$ and the boundary datum w belongs to $H^2([0, T]; H^1(\Omega; \mathbb{R}^n)) \cap H^3([0, T]; L^2(\Omega; \mathbb{R}^n))$. Moreover, the model discussed in [3] is slightly different from ours: in [3] the plastic component of the strain plays a role in the viscous part of the stress, while we assume that the component $\dot{p}$ of the strain rate does not affect the viscous stress, which only depends on $\dot{e}$. This leads to a different flow rule, whose strong form cannot be proved directly from the approximate flow rules as in [3]; for this reason we prefer to prove first the energy balance and then to derive the flow rule from it.

As in [3] we will obtain the solution by time discretization, considering the limit of approximate solutions constructed by solving incremental minimum problems. Given an integer $N > 0$ we define $\tau = T/N$ and subdivide the interval $[0, T)$ into N subintervals $[t_i, t_{t+1})$, $i = 0, \ldots, N-1$ of length τ, with $t_i = i\tau$. Let us set

$$\begin{aligned}
&u_{-1} = u_0 - \tau v_0, \quad w_{-1} = w_0 - \tau \dot{w}(0), \\
&w_i = w(t_i), \qquad \mathcal{L}_i = \frac{1}{\tau} \int_{t_i}^{t_{i+1}} \mathcal{L}(s) ds.
\end{aligned}$$

We construct a sequence (u_i, e_i, p_i) with $i = 0, 1, \dots, N$ by induction. First (u_0, e_0, p_0) coincides with the initial data in (2.25). Let us fix i and let us suppose $(u_j, e_j, p_j) \in A(w_j)$ to have been defined for $j = 0, \dots, i$. Then $(u_{i+1}, e_{i+1}, p_{i+1})$ is defined as the unique minimizer on $A(w_{i+1})$ of the functional

$$\begin{aligned} V_i(u, e, p) =& \frac{1}{2}\langle A_0 e, e\rangle + \frac{1}{2\tau}\langle A_1(e - e_i), e - e_i\rangle + \frac{1}{2\tau}\|p - p_i\|^2_{L^2} \\ &+ \mathcal{H}(p - p_i) + \frac{1}{2}\|\frac{u - u_i}{\tau} - \frac{u_i - u_{i-1}}{\tau}\|^2_{L^2} - \langle \mathcal{L}_i, u\rangle, \end{aligned} \tag{2.39}$$

which turns out to be coercive and strictly convex on $A(w_{i+1})$.

To obtain the Euler conditions we observe that $(u_{i+1}, e_{i+1}, p_{i+1}) + \lambda(\varphi, \eta, q)$ belongs to $A(w_{i+1})$ for every $(\varphi, \eta, q) \in A(0)$, and for every $\lambda \in \mathbb{R}$. Evaluating V_i in this point and differentiating with respect to λ at $0^\pm$ we get

$$\begin{aligned} -\mathcal{H}(q) \leq& \langle A_0 e_{i+1}, \eta\rangle + \frac{1}{\tau}\langle A_1(e_{i+1} - e_i), \eta\rangle + \frac{1}{\tau}\langle p_{i+1} - p_i, q\rangle \\ &+ \frac{1}{\tau}\langle v_{i+1} - v_i, \varphi\rangle - \langle \mathcal{L}_i, \varphi\rangle \leq \mathcal{H}(-q), \end{aligned} \tag{2.40}$$

where we have set

$$v_j = \frac{1}{\tau}(u_j - u_{j-1}). \tag{2.41}$$

We now define the piecewise affine interpolation $u_\tau, e_\tau, p_\tau, w_\tau$ on $[0, T]$ by

$$u_\tau(t) = u_i + \frac{u_{i+1} - u_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{2.42a}$$

$$e_\tau(t) = e_i + \frac{e_{i+1} - e_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{2.42b}$$

$$p_\tau(t) = p_i + \frac{p_{i+1} - p_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{2.42c}$$

$$w_\tau(t) = w_i + \frac{w_{i+1} - w_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{2.42d}$$

To simplify the notation we also set $\omega_i = \frac{1}{\tau}(w_i - w_{i-1}) = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} \dot{w}(s)ds$ and define, for $t \in [0, T]$,

$$\omega_\tau(t) = \omega_i + (\omega_{i+1} - \omega_i)\frac{t - t_i}{\tau} \qquad \text{if } t \in [t_i, t_{i+1}), \tag{2.43a}$$

$$v_\tau(t) = v_i + (v_{i+1} - v_i)\frac{t - t_i}{\tau} \qquad \text{if } t \in [t_i, t_{i+1}). \tag{2.43b}$$

The proof now is divided into five steps: in the first one we prove that a subsequence of (u_τ, e_τ, p_τ) has a limit (u, e, p) as $\tau \to 0$, and we show that this limit satisfies the regularity conditions (2.22). In the second step we pass to the limit in (2.40), obtaining the equilibrium condition (2.26). In the third step we obtain the energy balance (2.29) for (u, e, p). In the fourth step we prove that (u, e, p) satisfies the initial conditions (2.25). From this and Theorem 2 it will follow that (u, e, p) satisfies the required Eq. (2.23). In the last step we prove the uniqueness.

Step 1 Since $\ddot{w} \in L^2([0,T]; L^2(\Omega;\mathbb{R}^n))$ and $\dot{w} \in L^2([0,T]; H^1(\Omega;\mathbb{R}^n))$, we see that

$$w_\tau \to w \text{ strongly in } L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.44a}$$
$$\dot{w}_\tau \to \dot{w} \text{ strongly in } L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.44b}$$
$$\omega_\tau \to \dot{w} \text{ strongly in } L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.44c}$$
$$\dot{\omega}_\tau \to \ddot{w} \text{ strongly in } L^2([0,T]; L^2(\Omega;\mathbb{R}^n)). \tag{2.44d}$$

The proof of the first three properties is straightforward. To prove (2.44d) we first put $\tilde{w}_\tau(t) := \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \ddot{w}(s)ds \in L^2(\Omega;\mathbb{R}^n)$ for $t \in [t_i, t_{i+1})$. Since $\tilde{w}_\tau$ tends to $\ddot{w}$, it suffices to show that $\tilde{w}_\tau - \dot{\omega}_\tau$ tends to 0 strongly in $L^2([0,T]; L^2(\Omega;\mathbb{R}^n))$. So we write

$$\begin{aligned}
\|\dot{\omega}_\tau - \tilde{w}_\tau\|^2_{L^2(L^2)} &= \sum_{i=0}^{N-1} \tau \left\| \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \Big(\frac{1}{\tau}\int_{s-\tau}^{s} \ddot{w}(r)dr - \ddot{w}(s)\Big)ds \right\|^2_{L^2} \\
&\le \frac{1}{\tau}\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}}\int_{s-\tau}^{s} \|\ddot{w}(r)-\ddot{w}(s)\|^2_{L^2} dr ds \\
&\le \frac{1}{\tau}\sum_{i=0}^{N-1} \int_{t_{i-1}}^{t_{i+1}}\int_{t_{i-1}}^{t_{i+1}} \|\ddot{w}(r)-\ddot{w}(s)\|^2_{L^2} dr ds,
\end{aligned}$$

where we set $\ddot{w}(s) = 0$ for $s < 0$. Defining $W(r,s) = \|\ddot{w}(r)-\ddot{w}(s)\|^2_{L^2}$, we see that the integral in the last line is bounded by

$$\frac{2}{\tau}\int_{-2\tau}^{2\tau} dh \int_0^T W(r, r+h)dr,$$

that turns out to go to 0 as $\tau \to 0$, because $h \mapsto \int_0^T W(r,r+h)dr$ is continuous and vanishes at $h = 0$.

Therefore we can argue as in [3, Proposition 3.4], using (2.44) for the boundary conditions w, and the duality between $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ for the load $\mathcal{L}$. We obtain that

$$u_\tau \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.45a}$$
$$\dot{u}_\tau \in L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{2.45b}$$
$$e_\tau \in L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{2.45c}$$
$$(\dot{e}_\tau)_{A_1} \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{2.45d}$$
$$p_\tau \in L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{2.45e}$$
$$\dot{p}_\tau \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{2.45f}$$

and these functions are bounded in these spaces uniformly with respect to τ. Moreover from the same estimate we find that

$$\tau^{\frac{1}{2}}\dot{e}_\tau \in L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{2.46}$$

uniformly with respect to τ. With the same arguments of [3, Proposition 3.4] we pass to the limit as τ tends to 0 in a subsequence, not relabeled, and prove that

$$u_\tau \rightharpoonup u \quad \text{weakly* in} \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{2.47a}$$

$$\dot u_\tau \rightharpoonup \dot u \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{R}^n)), \tag{2.47b}$$

$$e_\tau \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{2.47c}$$

$$(\dot e_\tau)_{A_1} \rightharpoonup \dot e_{A_1} \quad \text{weakly in } L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{2.47d}$$

$$p_\tau \rightharpoonup p \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{2.47e}$$

$$\dot p_\tau \rightharpoonup \dot p \quad \text{weakly in } L^2([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_D)). \tag{2.47f}$$

Moreover we can prove that

$$Eu(t) = e(t) + p(t) \tag{2.48}$$

for a.e. $t \in [0,T]$.

Let $\varphi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Putting $\eta = E\varphi$ and $q = 0$ in (2.40) we get

$$-\text{div}_{\Gamma_0}(A_0 e_{i+1}) - \text{div}_{\Gamma_0}\left(A_1 \frac{e_{i+1}-e_i}{\tau}\right) + \frac{v_{i+1}-v_i}{\tau} = \mathcal{L}_i,$$

which allows us to deduce from (2.45c) and (2.45d) that $\dot v_\tau = \frac{v_{i+1}-v_i}{\tau}$ is bounded in $L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$ uniformly with respect to τ, thanks to the continuity of the operator div_{Γ_0}.

So, using the Hölder inequality, we estimate

$$\|v_\tau(t) - v_\tau(t_{i+1})\|_{H^{-1}_{\Gamma_0}} \le \tau^{1/2} M \quad \text{for } t \in [t_i, t_{i+1}),$$

for some positive constant M independent of τ, t, and i. Since $\dot u_\tau(t) = v_\tau(t_{i+1})$ for $t \in [t_i, t_{i+1})$ we have

$$\|v_\tau(t) - \dot u_\tau(t)\|_{H^{-1}_{\Gamma_0}} \le \tau^{1/2} M,$$

so that $v_\tau - \dot u_\tau$ tends to 0 strongly in $L^\infty([0,T], H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$. From this it easily follows that the two sequences v_τ and $\dot u_\tau$ must have the same weak* limit in $L^\infty([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, so

$$v_\tau \rightharpoonup \dot u \quad \text{weakly* in } L^\infty([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{2.49}$$

The boundness condition proved above implies that $\dot v_\tau$ tends, up to a subsequence, to a function ζ weakly in $L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, and it easily follows that $\zeta = \ddot u$. Therefore

$$\dot v_\tau \rightharpoonup \ddot u \quad \text{weakly in } L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{2.50}$$

We now define $\sigma(t) := A_0 e(t) + A_1 \dot e_{A_1}(t)$. The results proved so far imply that (u, e, p, σ) satisfies (2.22).

Step 2 In order to show that the functions above satisfy (2.23) we need to pass to the limit in (2.40). We consider the piecewise constant interpolation $\tilde e_\tau$ defined by

$$\tilde e_\tau(t) = e_{i+1} \quad \text{if} \quad t \in [t_i, t_{i+1}).$$

We want to prove that

$$\tilde e_\tau \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})). \tag{2.51}$$

Since $\tilde{e}_\tau$ is bounded in $L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ it is not restrictive to assume that $\tilde{e}_\tau \rightharpoonup \tilde{e}$ weakly* in $L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$. Since $e_\tau = Eu_\tau - p_\tau$, by (2.45) we have that

$$(e_\tau)_{\tau>0} \text{ is bounded } H^1([0,T];H^{-1}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})). \tag{2.52}$$

Therefore, using the Hölder inequality, we obtain

$$\|e_\tau(t) - e_\tau(t_{i+1})\|_{H^{-1}} \le \tau^{1/2} M \quad \text{for } t \in [t_i, t_{i+1}),$$

for some constant $M > 0$ independent of τ, t, and i. Since $\tilde{e}_\tau(t) = e_\tau(t_{i+1})$ for $t \in [t_i, t_{i+1})$, we have

$$\|e_\tau(t) - \tilde{e}_\tau(t)\|_{H^{-1}} \le \tau^{1/2} M \quad \text{for all } t \in [0,T].$$

This implies $e = \tilde{e}$ and concludes the proof of (2.51).

We also define the piecewise affine interpolation $\mathcal{L}_\tau$ by

$$\mathcal{L}_\tau(t) = \mathcal{L}_i + (\mathcal{L}_{i+1} - \mathcal{L}_i)\frac{t - t_i}{\tau} \quad \text{if} \quad t \in [t_i, t_{i+1}),$$

where $\mathcal{L}_i := \mathcal{L}(t_i)$. By standard properties of L^2 functions and of their approximation by averaging on subintervals, we have that

$$\mathcal{L}_\tau \to \mathcal{L} \qquad \text{strongly in } L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{2.53a}$$

$$\dot{\mathcal{L}}_\tau \to \dot{\mathcal{L}} \qquad \text{strongly in } L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{2.53b}$$

For fixed τ (2.40) says that for a.e. $t \in [0,T]$ we have

$$-\mathcal{H}(q) \le \langle A_0 \tilde{e}_\tau, \eta\rangle + \langle A_1 (\dot{e}_\tau)_{A_1}, \eta\rangle + \langle \dot{p}_\tau, q\rangle + \langle \dot{v}_\tau, \varphi\rangle - \langle \mathcal{L}_\tau, \varphi\rangle \le \mathcal{H}(-q)$$

for every $(\varphi, \eta, q) \in A(0)$. All terms in the formula above converge weakly in $L^1([0,T])$ as $\tau \to 0$, thanks to (2.47d), (2.47e), (2.50), (2.51), and (2.53). So for every $(\varphi, \eta, q) \in A(0)$ we can pass to the limit obtaining

$$-\mathcal{H}(q) \le \langle A_0 e, \eta\rangle + \langle A_1 \dot{e}_{A_1}, \eta\rangle + \langle \dot{p}, q\rangle + \langle \ddot{u}, \varphi\rangle - \langle \mathcal{L}, \varphi\rangle \le \mathcal{H}(-q) \tag{2.54}$$

for a.e. $t \in [0,T]$. Since the space $A(0)$ is separable, we can construct a set of full measure in $[0,T]$ such that (2.54) holds in this set for every $(\varphi, \eta, q) \in A(0)$, which gives (2.26).

Step 3 We will now prove the energy balance (2.29). We shall use the three following identities:

$$\langle A_0 e_{i+1}, e_{i+1} - e_i\rangle = \int_{t_i}^{t_{i+1}} \langle A_0 e_\tau, \dot{e}_\tau\rangle ds + \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \langle A_0 \dot{e}_\tau, \dot{e}_\tau\rangle ds, \tag{2.55}$$

$$\begin{aligned}&\langle A_0 e_{i+1}, Ew_{i+1} - Ew_i\rangle \\ &\quad = \int_{t_i}^{t_{i+1}} \langle A_0 e_\tau, E\dot{w}_\tau\rangle ds + \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \langle A_0 \dot{e}_\tau, E\dot{w}_\tau\rangle ds,\end{aligned} \tag{2.56}$$

$$\begin{aligned}&\langle (v_{i+1} - v_i) - (\omega_{i+1} - \omega_i), v_{i+1} - \omega_{i+1}\rangle \\ &\quad = \frac{1}{2}\|v_{i+1} - \omega_{i+1}\|^2_{L^2} - \frac{1}{2}\|v_i - \omega_i\|^2_{L^2} + \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \|\dot{v}_\tau - \dot{\omega}_\tau\|^2_{L^2} ds.\end{aligned} \tag{2.57}$$

Let $\lambda \in (0,1)$ and put $\varphi = u_{i+1} - \lambda(u_{i+1} - u_i) + \lambda(w_{i+1} - w_i)$, $\eta = e_{i+1} - \lambda(e_{i+1} - e_i) + \lambda(Ew_{i+1} - Ew_i)$, and $q = p_{i+1} - \lambda(p_{i+1} - p_i)$, so by the minimality of $(u_{i+1}, e_{i+1}, p_{i+1})$

for the functional V_i defined by (2.39) we have $V_i(u_{i+1}, e_{i+1}, p_{i+1}) \leq V_i(\varphi, \eta, q)$. This implies

$$\begin{aligned}
&\frac{1}{2}\langle A_0 e_{i+1}, e_{i+1}\rangle + \frac{1}{2\tau}\langle A_1(e_{i+1}-e_i), e_{i+1}-e_i\rangle + \frac{1}{2\tau}\|p_{i+1}-p_i\|^2_{L^2} \\
&\quad + \mathcal{H}(p_{i+1}-p_i) + \frac{1}{2}\|v_{i+1}-v_i\|^2_{L^2} - \langle \mathcal{L}_i, u_{i+1}\rangle \\
&\leq \frac{(1-\lambda)^2}{2}\langle A_0 e_{i+1}, e_{i+1}\rangle + \lambda(1-\lambda)\langle A_0 e_{i+1}, e_i\rangle + \frac{\lambda^2}{2}\langle A_0 e_i, e_i\rangle \\
&\quad + \frac{\lambda^2}{2}\langle A_0(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle + \lambda\langle A_0 e_{i+1}, Ew_{i+1}-Ew_i\rangle \\
&\quad - \lambda^2\langle A_0(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle + \frac{(1-\lambda)^2}{2\tau}\langle A_1(e_{i+1}-e_i), e_{i+1}-e_i\rangle \\
&\quad + \frac{\lambda^2}{2\tau}\langle A_1(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle \\
&\quad + \frac{\lambda(1-\lambda)}{\tau}\langle A_1(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle \\
&\quad + \frac{(1-\lambda)^2}{2\tau}\|p_{i+1}-p_i\|^2_{L^2} + (1-\lambda)\mathcal{H}(p_{i+1}-p_i) + \frac{1}{2}\|v_{i+1}-v_i\|^2_{L^2} \\
&\quad + \frac{\lambda^2}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2} - \lambda\langle v_{i+1}-v_i-(\omega_{i+1}-\omega_i), v_{i+1}-\omega_{i+1}\rangle \\
&\quad - \langle \mathcal{L}_i, u_{i+1}\rangle + \lambda\tau\langle \mathcal{L}_i - \frac{\omega_{i+1}-\omega_i}{\tau}, v_{i+1}-\omega_{i+1}\rangle.
\end{aligned}$$

Dividing by λ we get

$$\begin{aligned}
&\frac{2-\lambda}{2}(A_0 e_{i+1}, e_{i+1}) - (1-\lambda)\langle A_0 e_{i+1}, e_i\rangle \\
&- \langle A_0 e_{i+1}, Ew_{i+1}-Ew_i\rangle + \lambda\langle A_0(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle \\
&- \frac{\lambda}{2}\langle A_0(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle + \frac{2-\lambda}{2\tau}\langle A_1(e_{i+1}-e_i), e_{i+1}-e_i\rangle \\
&+ \frac{2-\lambda}{2\tau}\|p_{i+1}-p_i\|^2_{L^2} + \mathcal{H}(p_{i+1}-p_i) - \frac{\lambda}{2\tau}\langle A_1(Ew_{i+1}-Ew_i), Ew_{i+1}-Ew_i\rangle \\
&- \frac{1-\lambda}{\tau}\langle A_1(e_{i+1}-e_i), Ew_{i+1}-Ew_i\rangle + \langle v_{i+1}-v_i-(\omega_{i+1}-\omega_i), v_{i+1}-\omega_{i+1}\rangle \\
&- \tau\langle \mathcal{L}_i - \frac{\omega_{i+1}-\omega_i}{\tau}, v_{i+1}-\omega_{i+1}\rangle \leq \frac{\lambda}{2}\langle A_0 e_i, e_i\rangle + \frac{\lambda}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2}.
\end{aligned}$$

Since $\langle A_0 e_{i+1}, e_{i+1}\rangle \geq 0$ and $\lambda \in (0, 1)$ it follows that

$$\begin{aligned}
&(1-\lambda)\langle A_0 e_{i+1}, e_{i+1}-e_i\rangle + \frac{2-\lambda}{2}\tau\langle A_1\frac{e_{i+1}-e_i}{\tau}, \frac{e_{i+1}-e_i}{\tau}\rangle \\
&\quad - \langle A_0 e_{i+1}, Ew_{i+1}-Ew_i\rangle + \lambda\tau^2\langle A_0\frac{e_{i+1}-e_i}{\tau}, \frac{Ew_{i+1}-Ew_i}{\tau}\rangle \\
&\quad - \tau^2\frac{\lambda}{2}\langle A_0\frac{Ew_{i+1}-Ew_i}{\tau}, \frac{Ew_{i+1}-Ew_i}{\tau}\rangle \\
&\quad - (1-\lambda)\tau\langle A_1\frac{e_{i+1}-e_i}{\tau}, \frac{Ew_{i+1}-Ew_i}{\tau}\rangle + \frac{2-\lambda}{2}\tau\|\frac{p_{i+1}-p_i}{\tau}\|^2_{L^2} \\
&\quad + \tau\mathcal{H}(\frac{p_{i+1}-p_i}{\tau}) + \langle (v_{i+1}-v_i)-(\omega_{i+1}-\omega_i), v_{i+1}-\omega_{i+1}\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \tau \langle \mathcal{L}_i - \frac{\omega_{i+1}-\omega_i}{\tau}, v_{i+1}-\omega_{i+1}\rangle \\
&\qquad + \frac{\lambda}{2}\langle A_0 e_i, e_i\rangle + \frac{\lambda}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2} + \frac{\lambda\tau}{2}\langle A_1 \frac{Ew_{i+1}-Ew_i}{\tau}, \frac{Ew_{i+1}-Ew_i}{\tau}\rangle.
\end{aligned}$$

Now, thanks to (2.55)–(2.57), from the last inequality we get

$$\begin{aligned}
&(1-\lambda)\int_{t_i}^{t_{i+1}} \langle A_0 e_\tau, \dot{e}_\tau\rangle ds + \frac{2-\lambda}{2}\int_{t_i}^{t_{i+1}} \langle A_1 \dot{e}_\tau, \dot{e}_\tau\rangle ds \\
&\qquad + \frac{2-\lambda}{2}\int_{t_i}^{t_{i+1}} \|\dot{p}_\tau\|^2_{L^2} ds + \int_{t_i}^{t_{i+1}} \mathcal{H}(\dot{p}_\tau) ds \\
&\qquad + \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \|\dot{v}_\tau - \dot{\omega}_\tau\|^2_{L^2} ds + \frac{1}{2}\|v_{i+1}-\omega_{i+1}\|^2_{L^2} - \frac{1}{2}\|v_i - \omega_i\|^2_{L^2} \\
&\le -\int_{t_i}^{t_{i+1}} \langle \dot{\omega}_\tau, \dot{u}_\tau - \dot{w}_\tau\rangle ds - \int_{t_i}^{t_{i+1}} \langle \dot{\mathcal{L}}, u_\tau - w_\tau\rangle ds \\
&\qquad + \langle \mathcal{L}(t_{i+1}), u_{i+1}-w_{i+1}\rangle - \langle \mathcal{L}(t_i), u_i - w_i\rangle - \frac{6-7\lambda}{12}\tau\int_{t_i}^{t_{i+1}} \langle A_0 \dot{e}_\tau, \dot{e}_\tau\rangle ds \\
&\qquad + \frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}} \|\dot{u}_\tau - \dot{w}_\tau\|^2_{L^2} ds + \frac{\lambda}{2}\int_{t_i}^{t_{i+1}} \langle \tau A_0 E\dot{w}_\tau + A_1 E\dot{w}_\tau, E\dot{w}_\tau\rangle ds \\
&\qquad + \int_{t_i}^{t_{i+1}} \langle A_0 e_\tau + A_1 \dot{e}_\tau, E\dot{w}_\tau\rangle ds + \int_{t_i}^{t_{i+1}} \langle (\frac{\tau}{2} - \lambda\tau) A_0 \dot{e}_\tau - \lambda A_1 \dot{e}_\tau, E\dot{w}_\tau\rangle ds \\
&\qquad + \frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}} \langle A_0 e_\tau, e_\tau\rangle ds - \frac{\lambda}{2}\int_{t_i}^{t_{i+1}} \langle A_0 e_\tau, \dot{e}_\tau\rangle ds,
\end{aligned}$$

where we have used that

$$\begin{aligned}
\frac{\lambda}{2}\langle A_0 e_i, e_i\rangle &= \frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}} \langle A_0 e_\tau, e_\tau\rangle ds \\
&\quad - \frac{\lambda}{2}\int_{t_i}^{t_{i+1}} \langle A_0 e_\tau, \dot{e}_\tau\rangle ds + \frac{\lambda\tau}{12}\int_{t_i}^{t_{i+1}} \langle A_0 \dot{e}_\tau, \dot{e}_\tau\rangle ds.
\end{aligned}$$

We now sum over $i = 0, \ldots, j$ and we obtain

$$\begin{aligned}
&\frac{1-\lambda}{2}\langle A_0 e_\tau(t_{j+1}), e_\tau(t_{j+1})\rangle - \frac{1-\lambda}{2}\langle A_0 e_0, e_0\rangle \\
&\qquad + \frac{2-\lambda}{2}\int_0^{t_{j+1}} \langle A_1 (\dot{e}_\tau)_{A_1}, (\dot{e}_\tau)_{A_1}\rangle ds + \frac{2-\lambda}{2}\int_0^{t_{j+1}} \|\dot{p}_\tau\|^2_{L^2} ds + \int_0^{t_{j+1}} \mathcal{H}(\dot{p}_\tau) ds \\
&\qquad + \frac{\tau}{2}\int_0^{t_{j+1}} \|\dot{v}_\tau - \dot{\omega}_\tau\|^2_{L^2} ds + \frac{1}{2}\|v_{j+1}-\omega_{j+1}\|^2_{L^2} - \frac{1}{2}\|v_0 - \omega_0\|^2_{L^2} \\
&\le \int_0^{t_{j+1}} \langle \dot{\omega}_\tau, \dot{u}_\tau - \dot{w}_\tau\rangle ds - \int_0^{t_{j+1}} \langle \dot{\mathcal{L}}_\tau, u_\tau - w_\tau\rangle ds + \langle \mathcal{L}_\tau(t_{j+1}), u_{j+1} - w_{j+1}\rangle \\
&\qquad - \langle \mathcal{L}(0), u_0 - w(0)\rangle + \int_0^{t_{j+1}} \langle A_0 e_\tau + A_1 (\dot{e}_\tau)_{A_1}, E\dot{w}_\tau\rangle ds \\
&\qquad + \frac{\lambda}{2\tau}\int_0^{t_{j+1}} \langle A_0 e_\tau, e_\tau\rangle ds + \frac{\lambda}{2\tau}\int_0^{t_{j+1}} \|\dot{u}_\tau - \dot{w}_\tau\|^2_{L^2} ds \\
&\qquad - \frac{6-7\lambda}{12}\tau\int_0^{t_{j+1}} \langle A_0 \dot{e}_\tau, \dot{e}_\tau\rangle ds \qquad + \frac{\lambda}{2}\int_0^{t_{j+1}} \langle \tau A_0 E\dot{w}_\tau + A_1 E\dot{w}_\tau, E\dot{w}_\tau\rangle ds
\end{aligned}$$

$$+\int_0^{t_{j+1}}\langle(\frac{\tau}{2}-\lambda\tau)A_0\dot e_\tau-\lambda A_1(\dot e_\tau)_{A_1},E\dot w_\tau\rangle ds-\frac{\lambda}{2}\int_0^{t_{j+1}}\langle A_0e_\tau,\dot e_\tau\rangle ds.$$

We now take $\lambda = o(\tau)$ and then pass to the limit as $\tau \to 0$. To this aim we fix $t \in [0, T]$ and, for every $\tau > 0$, we define $\hat t_\tau = t_{j+1}$, where j is the unique index such that $t_j \le t < t_{j+1}$. For the third, fourth, and fifth term in the left-hand side of the previous inequality we just use the lower semicontinuity with respect to the convergences in (2.47); the sixth term is nonnegative; to deal with the first and the seventh term we apply Lemma 2 below taking into account (2.44c), (2.44d), (2.47c), (2.49), (2.50), and (2.52), obtaining

$$e_\tau(t_{j+1})=e_\tau(\hat t_\tau)\rightharpoonup e(t)\ \text{weakly in}\ H^{-1}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}),$$
$$v_{j+1}-\omega_{j+1}=v_\tau(\hat t_\tau)-\omega_\tau(\hat t_\tau)\rightharpoonup \dot u(t)-\dot w(t)\ \text{weakly in}\ H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n).$$

Since the L^2 norm is lower semicontinuous with respect to weak convergence in H^{-1} and $H^{-1}_{\Gamma_0}$ (this can be proved by a duality argument), we obtain a lower semicontinuity inequality also for these terms.

As for the right-hand side of the previous inequality, we can pass to the limit in the first two terms thanks to (2.44), (2.47a), (2.47b), and (2.53), which implies also that $u_\tau \rightharpoonup u$ weakly in $H^1([0,T]; L^2(\Omega;\mathbb{R}^n))$. This implies by Lemma 2 that $u_{j+1} = u_\tau(\hat t_j) \rightharpoonup u(t)$ weakly in $L^2(\Omega;\mathbb{R}^n)$. Since u_{j+1} is bounded in $H^1(\Omega;\mathbb{R}^n)$ by (2.47a) we deduce that $u_{j+1} \rightharpoonup u(t)$ weakly in $H^1(\Omega;\mathbb{R}^n)$. We can now pass to the limit in the third term of the right-hand side thanks to (2.44) and (2.52), and in the fifth term thanks to (2.44b), (2.47c), and (2.47d). The eighth has a negative coefficient, while all other terms tend to 0 by (2.44), (2.45), and (2.46). Thus we obtain

$$\begin{aligned}&\mathcal{Q}_0(e(t))-\mathcal{Q}_0(e(0))+\frac12\|\dot u(t)-\dot w(t)\|^2_{L^2}-\frac12\|v_0-\dot w(0)\|^2_{L^2}+\int_0^t\mathcal{Q}_1(\dot e_{A_1})ds\\&+\int_0^t\|\dot p\|^2_{L^2}ds+\int_0^t\mathcal{H}(\dot p)ds-\int_0^t\langle A_0e+A_1\dot e_{A_1},E\dot w\rangle ds+\int_0^t\langle\ddot w,\dot u-\dot w\rangle ds\\&+\int_0^t\langle\dot{\mathcal{L}},u-w\rangle ds-\langle\mathcal{L}(t),u(t)-w(t)\rangle+\langle\mathcal{L}(0),u_0-w(0)\rangle\le 0.\end{aligned}\tag{2.58}$$

To prove the energy balance (2.29) we need to show that also the opposite inequality holds. We use the notation of the proof of Lemma 1. For a.e. $t \in [0, T]$, we consider the first inequality of (2.26) with $\varphi = s^h u(t) - s^h w(t)$, $\eta = s^h e(t) - s^h Ew(t)$, $q = s^h p(t)$, and we sum this expression to the one obtained from (2.26) at time $t + h$ with the same test functions. Then, using an argument similar to the one employed in (2.31), we get the opposite inequality in (2.58) for a.e. $t \in [0, T]$.

Step 4 Equalities (2.25a) follow easily from (2.47) and from the initial conditions satisfied by the approximate solutions (u_τ, e_τ, p_τ). Moreover by (2.52) the functions e_τ converge to e weakly in $H^1([0,T]; H^{-1}(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ as $\tau \to 0$. Since $e_\tau(0) = e_0$ for all τ, we conclude that $e(0) = e_0$. Since $t \to e(t)$ is weakly continuous into $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ by Remark 1, we deduce that

$$e(t)\rightharpoonup e_0\quad\text{weakly in}\ L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\quad\text{as}\ t\to 0.\tag{2.59}$$

Similarly, using (2.49) and (2.50), we obtain that $v_\tau \to \dot u$ weakly in $H^1([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$. Since $v_\tau(0) = v_0$ for every τ, we conclude that $\dot u(0) = v_0$. Since $t \to \dot u(t)$ is weakly continuous into $L^2(\Omega; \mathbb{R}^n)$ by Remark 1, we deduce that

$$\dot{u}(t) \rightharpoonup v_0 \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n) \quad \text{as} \quad t \to 0. \tag{2.60}$$

In order to deduce from (2.59) and (2.60) the stronger conditions (2.25b) we use the energy equality (2.29). Let t_k be a sequence in $[0, T]$ converging to 0 such that (2.29) holds for $t = t_k$. Then

$$\frac{1}{2}\|\dot{u}(t_k) - \dot{w}(t_k)\|^2_{L^2} + \mathcal{Q}_0(e(t_k)) \to \frac{1}{2}\|v_0 - \dot{w}(0)\|^2_{L^2} + \mathcal{Q}_0(e_0). \tag{2.61}$$

Since $\dot{w} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$, the weak convergence (2.59) and (2.60) together with (2.61) imply that $e(t_k) \to e_0$ strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ and $\dot{u}(t_k) \to v_0$ strongly in $L^2(\Omega; \mathbb{R}^n)$. Equalities (2.25b) follow now from the arbitrariness of the sequence t_k.

We are now in a position to apply Theorem 2: since the quadruple (u, e, p, σ) satisfies (2.26) and (2.29), it satisfies also Eqs. (2.23c) and (2.23d) .

Step 5 It only remains to prove that the solution is unique. Let us suppose that $(u_1, e_1, p_1, \sigma_1)$ and $(u_2, e_2, p_2, \sigma_2)$ are solutions. We set $u := u_2 - u_1$, $e := e_2 - e_1$, $p := p_2 - p_1$, $\sigma := \sigma_2 - \sigma_1$, and observe that the quadruple (u, e, p, σ) satisfies the hypotheses of Lemma 1, implying that (2.31) holds for a.e. $t \in [0, T]$. Since the map $\xi \to \xi - \pi_K\xi$ is a monotone operator from $\mathbb{M}^{n\times n}_D$ into itself (see, e.g., [5, Chap. 2]), it follows from (2.23d) that

$$\langle \sigma_D(t), \dot{p}(t)\rangle ds \geq 0$$

for a.e. $t \in [0, T]$. Using this inequality in (2.31) we obtain that

$$\mathcal{Q}_0(e(t)) + \int_0^t \mathcal{Q}_1(\dot{e}_{A_1})ds + \frac{1}{2}\|\dot{u}(t)\|^2_{L^2} = 0$$

for a.e. $t \in [0, T]$, taking into account the initial and boundary conditions satisfied by u. This implies by standard arguments that $u(t) = 0$ for all $t \in [0, T]$, concluding the proof.

Here we prove the lemma we have used in the previous proof.

Lemma 2 *Let X be a Banach space. Assume that q_τ tends to q_0 weakly in $H^1([0, T]; X)$ as τ tends to zero. Then*

$$q_\tau(t_\tau) \rightharpoonup q_0(t_0) \quad \text{weakly in } X \tag{2.62}$$

for every $t_\tau, t_0 \in [0, T]$ with $t_\tau \to t_0$ as $\tau \to 0$.

Proof Since $H^1([0, T]; X)$ is continuously embedded in $C^{0,1/2}([0, T]; X)$, we have $q_\tau \rightharpoonup q_0$ weakly in $C^{0,1/2}([0, T]; X)$. This implies in particular that

$$q_\tau(t) \rightharpoonup q_0(t) \text{ weakly in } X \tag{2.63}$$

for all $t \in [0, T]$. If $t_\tau \to t_0$ we have

$$\|q_\tau(t_\tau) - q_\tau(t_0)\| \leq \int_{t_0}^{t_\tau} \|\dot{q}_\tau\| dt \leq M(t_\tau - t_0)^{1/2},$$

where $\|\cdot\|$ is the norm in X and M is an upper bound for the norm of q_τ in $H^1([0, T]; X)$. Now (2.62) follows from the previous inequality and (2.63).

Theorem 3 *Let (u, e, p, σ) be the solution of the problem considered in Theorem 1. Then $u \in C^0([0, T]; H^1(\Omega; \mathbb{R}^n))$, $e \in C^0([0, T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}))$, $\dot{u} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$, and the energy balance (2.29) holds for all $t \in [0, T]$.*

Proof We may assume that w and $\mathcal{L}$ are defined on $[0, T+1]$ and satisfy the hypotheses of Theorem 1 with T replaced by $T+1$. As for w, it is enough to set $w(t) := w(T)+(t-T)\dot{w}(T)$ for $t \in (T, T+1]$, noticing that $\dot{w}(T)$ can be univocally defined as an element of $H^1(\Omega;\mathbb{R}^n)$ arguing as in Remark 1. By Theorem 1 the solution on $[0, T]$ can be extended to a solution on $[0, T+1]$ still denoted by (u, e, p, σ).

Let us fix $t^* \in [0, T]$. Thanks to Remark 1, the functions $u(t^*)$, $e(t^*)$, $p(t^*)$, $\dot{u}(t^*)$ are univocally defined as elements of $H^1(\Omega;\mathbb{R}^n)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, $L^2(\Omega;\mathbb{M}^{n\times n}_{D})$, and $L^2(\Omega;\mathbb{R}^n)$, respectively. Therefore we can consider the solution $(u^*, e^*, p^*, \sigma^*)$ of the problem of Theorem 1, with $[0, T]$ replaced by $[t^*, T+1]$ and initial data $u(t^*)$, $e(t^*)$, $p(t^*)$, and $\dot{u}(t^*)$ in the sense of (2.25), with 0 replaced by t^*. It is easy to see that the function defined by (u, e, p, σ) on $[0, t^*)$ and by $(u^*, e^*, p^*, \sigma^*)$ on $[t^*, T+1]$ is a solution of the problem considered in Theorem 1 on $[0, T+1]$, with initial data u_0, e_0, p_0, and v_0. By uniqueness $(u^*, e^*, p^*, \sigma^*) = (u, e, p, \sigma)$ on $[t^*, T+1]$.

In view of Theorem 2, we can fix $\hat{t} \in (t^*, T+1]$ such that the energy balance (2.29) between 0 and $\hat{t}$ holds for (u, e, p, σ) and the energy balance between t^* and $\hat{t}$ holds for $(u^*, e^*, p^*, \sigma^*)$. Since $(u^*, e^*, p^*, \sigma^*) = (u, e, p, \sigma)$ on $[t^*, \hat{t}]$, by difference we obtain the energy balance for (u, e, p, σ) between $[0, t^*]$. Since t^* is arbitrary, this implies that the energy balance holds for all $t \in [0, T]$.

Now the energy balance, together with the continuity of $\mathcal{L}$ and the weak continuity of $u - w$, implies that the term $\mathcal{Q}_0(e) + \|\dot{u} - \dot{w}\|^2_{L^2}$ is a continuous function on $[0, T]$. Then for all $t \in [0, T]$ and any sequence $t_k \to t \in [0, T]$ we have

$$\mathcal{Q}_0(e(t)) + \|\dot{u}(t) - \dot{w}(t)\|^2_{L^2} = \lim_{k\to\infty} \mathcal{Q}_0(e(t_k)) + \|\dot{u}(t_k) - \dot{w}(t_k)\|^2_{L^2}.$$

This and the weak continuity of e and $\dot{u} - \dot{w}$, thanks to the fact that $\mathcal{Q}_0$ is equivalent to the norm on $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, imply that $e(t_k) \to e(t)$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, and $\dot{u}(t_k) - \dot{w}(t_k) \to \dot{u}(t) - \dot{w}(t)$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Thanks to (2.21b), this implies that $e \in C^0([0, T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ and $\dot{u} \in C^0([0, T]; L^2(\Omega;\mathbb{R}^n))$. We conclude that $u \in C^0([0, T]; H^1(\Omega;\mathbb{R}^n))$ by (2.27e).

3 Perfect Plasticity

In this and in the next sections we study the behavior of the solutions of (2.23) when the data of the problem, i.e., the external load and the boundary conditions, vary very slowly. We are going to prove that the inertial and viscosity terms become negligible in the limit, and that the solutions of the dynamic problems actually approach the quasistatic evolution for perfect plasticity. To this aim we provide in this section the mathematical setting and tools to formulate and solve the perfect plasticity problem.

3.1 Preliminary Tools

Space BD In perfect plasticity the displacement u belongs to the space of functions with bounded deformation on Ω, defined as

$$BD(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^n) : Eu \in \mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})\}.$$

Here and henceforth, if V is a finite dimensional vector space and A is a locally compact subset of $\mathbb{R}^n$, the symbol $\mathcal{M}_b(A; V)$ denotes the space of V-valued bounded Radon measures on A, endowed with the norm $\|\lambda\|_{\mathcal{M}_b} := |\lambda|(A)$, where $|\lambda|$ is the variation of λ.

The space $BD(\Omega)$ is endowed with the norm

$$\|u\|_{BD} = \|u\|_{L^1} + \|Eu\|_{\mathcal{M}_b}.$$

Besides the strong convergence, we shall also consider a notion of weak* convergence in $BD(\Omega)$. We say that a sequence u_k converges to u weakly* in $BD(\Omega)$ if and only if u_k converges to u weakly in $L^1(\Omega;\mathbb{R}^n)$ and Eu_k converges to Eu weakly* in $\mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. Every function u in $BD(\Omega)$ has a trace in $L^1(\partial\Omega;\mathbb{R}^n)$, that we will still denote by u, or sometimes by $u|_{\partial\Omega}$. By [28, Proposition 2.4 and Remark 2.5] there exists a constant C depending only on Ω such that

$$\|u\|_{L^1(\Omega)} \le C(\|u\|_{L^1(\Gamma_0)} + \|Eu\|_{\mathcal{M}_b(\Omega)}). \tag{3.1}$$

For technical reasons related to the stress-strain duality, in addition to the assumption already introduced in Sect. 2.1, we now suppose that

$$\partial\Omega \text{ and } \partial\Gamma \text{ are of class } C^2. \tag{3.2}$$

Elastic and Plastic Strain In perfect plasticity the plastic strain p belongs to $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$. The singular part of this measure describes plastic slips. Given $w\in H^1(\Omega;\mathbb{R}^n)$, we say that a triple (u,e,p) is kinematically admissible for the perfectly plastic problem with boundary datum w if $u\in BD(\Omega;\mathbb{R}^n)$, $e\in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, $p\in\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, and

$$Eu = e + p \quad \text{on } \Omega, \tag{3.3a}$$

$$p = (w-u)\odot \nu\mathcal{H}^{n-1} \quad \text{on } \Gamma_0, \tag{3.3b}$$

where ν denotes the outer unit normal to $\partial\Omega$ and $\odot$ denotes the symmetrized tensor product.

The set of these triples will be denoted by $A_{BD}(w)$. Note that in this definition of kinematic admissibility, the Dirichlet boundary condition (2.1b) is replaced by the relaxed condition (3.3b), which represents a plastic slip occurring at Γ_0. It is also easily seen that the inclusion $A(w)\subset A_{BD}(w)$ holds, so that every admissible triple for the visco-elasto-plastic problem is also admissible for the perfectly plastic problem.

The following closure property is proved in [6, Lemma 2.1].

Lemma 3 *Let w_k be a sequence in $H^1(\Omega;\mathbb{R}^n)$ and $(u_k,e_k,p_k)\in A_{BD}(w_k)$. Let us suppose that $w_k \rightharpoonup w_\infty$ weakly in $H^1(\Omega;\mathbb{R}^n)$, $u_k \rightharpoonup u_\infty$ weakly* in $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, and $p_k \rightharpoonup p_\infty$ weakly* in $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$. Then $(u_\infty,e_\infty,p_\infty)\in A_{BD}(w_\infty)$.*

Stress In addition to the assumptions of Sect. 2.1, we now suppose that the elastic tensor A_0 maps the orthogonal spaces $\mathbb{M}^{n\times n}_D$ and $\mathbb{R}I$ into themselves. This is equivalent to require that there exist a positive definite symmetric operator $A_{0D}:\mathbb{M}^{n\times n}_D\to\mathbb{M}^{n\times n}_D$ and a positive constant κ^0 such that

$$A_0\xi = A_{0D}\xi_D + \kappa^0(\mathrm{tr}\xi)I. \tag{3.4}$$

In the perfectly plastic model the stress σ is related to the strain by the equation

$$\sigma = A_0 e \tag{3.5}$$

where e is the elastic component of the strain Eu. Therefore if (u,e,p) is kinematically admissible, then σ belongs to $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$.

In perfect plasticity the stress satisfies the constraint

$$\sigma_D \in \mathcal{K}(\Omega), \tag{3.6}$$

where $\mathcal{K}(\Omega)$ is defined in (2.12). In particular

$$\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D). \tag{3.7}$$

Convex Functions of Measures In perfect plasticity we need to define the functional (2.17) for $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. This is done by using the theory of convex functions of measures (see [11,24,30]): for every $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ we consider the nonnegative Radon measure $H(p)$ on $\Omega \cup \Gamma_0$ defined by

$$H(p)(B) := \int_B H(p/|p|)d|p| \tag{3.8}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$, where $p/|p|$ is the Radon-Nikodym derivative of p with respect to its variation $|p|$. We also define

$$\mathcal{H}(p) := H(p)(\Omega \cup \Gamma_0) = \int_{\Omega\cup\Gamma_0} H(p/|p|)d|p|.$$

The function $p \mapsto \mathcal{H}(p)$ turns out to be lower semicontinuous with respect to the weak* topology of $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and satisfies the triangle inequality. Moreover if $p_k \rightharpoonup p$ weakly* and $|p_k|(\Omega \cup \Gamma_0) \to |p|(\Omega \cup \Gamma_0)$, then $\mathcal{H}(p_k) \to \mathcal{H}(p)$.

Stress-Strain Duality If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$, with $\text{div}\sigma \in L^2(\Omega; \mathbb{R}^n)$, we define the distribution $[\sigma\nu]$ on $\partial\Omega$ by setting

$$\langle[\sigma\nu], \varphi\rangle_{\partial\Omega} := \langle\text{div}\sigma, \varphi\rangle + \langle\sigma, E\varphi\rangle, \tag{3.9}$$

for each $\varphi \in H^1(\Omega; \mathbb{R}^n)$. It turns out that $[\sigma\nu] \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ (see e.g. [28, Theorem 1.2, Chap. I]). We define the normal and tangential part of $[\sigma\nu]$ by

$$[\sigma\nu]_\nu := ([\sigma\nu]\cdot\nu)\nu, \qquad [\sigma\nu]^\perp_\nu := [\sigma\nu] - [\sigma\nu]_\nu, \tag{3.10}$$

and we have that $[\sigma\nu]_\nu$ and $[\sigma\nu]^\perp_\nu$ belong to $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ thanks to the regularity assumption (3.2) on $\partial\Omega$. If $\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$, by [16, Lemma 2.4] we also have that $[\sigma\nu]^\perp_\nu \in L^\infty(\partial\Omega; \mathbb{R}^n)$ and

$$\|[\sigma\nu]^\perp_\nu\|_{\infty,\partial\Omega} \le \frac{1}{\sqrt{2}}\|\sigma_D\|_{L^\infty}. \tag{3.11}$$

The set of admissible stresses for the perfectly plastic problem is defined by

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}) : \text{div}\sigma \in L^n(\Omega; \mathbb{R}^n) \text{ and } \sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)\}.$$

The set of admissible plastic strains $\Pi_{\Gamma_0}(\Omega)$ is the set of all $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ such that there exist $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfying $(u, e, p) \in A_{BD}(w)$.

If $\sigma \in \Sigma(\Omega)$ it turns out that $\sigma \in L^r(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ for all $r < +\infty$ (see [29, Proposition 2.5]). For every $u \in BD(\Omega)$ with $\text{div}u \in L^2(\Omega)$ we define the distribution $[\sigma_D \cdot E_D u]$ by

$$\langle[\sigma_D \cdot E_D u], \varphi\rangle = -\langle\text{div}\sigma, \varphi u\rangle - \frac{1}{n}\langle\text{tr}\sigma, \varphi\text{div}u\rangle - \langle\sigma, u \odot \nabla\varphi\rangle \tag{3.12}$$

for every $\varphi \in C_c^\infty(\Omega)$. As proved in [29, Theorem 3.2] the distribution $[\sigma_D \cdot E_D u]$ is a bounded Radon measure in Ω.

As in [6], if $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we define the bounded Radon measure $[\sigma_D \cdot p]$ on $\Omega \cup \Gamma_0$ by setting

$$\begin{aligned} [\sigma_D \cdot p] &:= [\sigma_D \cdot E_D u] - \sigma_D \cdot e_D \qquad \text{on } \Omega, \\ [\sigma_D \cdot p] &:= [\sigma \nu]_\nu^\perp \cdot (w - u)\mathcal{H}^{n-1} \qquad \text{on } \Gamma_0, \end{aligned}$$

where $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfy $(u, e, p) \in A_{BD}(w)$, and we notice that this definition does not depend on the particular choice of u, e, w (see [6, page 250]). We also define the duality pairing between $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$ by

$$\langle \sigma_D, p \rangle := [\sigma_D \cdot p](\Omega \cup \Gamma_0). \tag{3.13}$$

The following inequalities between measures hold (see [6, (2.33) and Proposition 2.4]):

$$|[\sigma_D \cdot p]| \le \|\sigma_D\|_{L^\infty} |p| \quad \text{on } \Omega \cup \Gamma_0, \tag{3.14}$$

$$[\sigma_D \cdot p] \le H(p) \quad \text{on } \Omega \cup \Gamma_0, \tag{3.15}$$

where $H(p)$ is the measure introduced in (3.8). The following integration by parts formula is proved in [6, Proposition 2.2] when $\varphi \in C^1(\bar{\Omega})$. The extension to Lipschitz functions is straightforward.

Proposition 1 *Let $\sigma \in \Sigma(\Omega)$, $f \in L^n(\Omega; \mathbb{R}^n)$, $g \in L^\infty(\Gamma_1; \mathbb{R}^n)$ and suppose $(u, e, p) \in A_{BD}(w)$ with $w \in H^1(\Omega; \mathbb{R}^n)$. If $-\text{div}\sigma = f$ on Ω and $[\sigma \nu] = g$ on Γ_1, then it holds*

$$\langle \sigma_D, p \rangle + \langle \sigma, e - Ew \rangle = \langle f, u - w \rangle + \langle g, u - w \rangle_{\Gamma_1}. \tag{3.16}$$

Moreover

$$\begin{aligned} &\langle [\sigma_D \cdot p], \varphi \rangle + \langle \sigma \cdot (e - Ew), \varphi \rangle + \langle \sigma, \nabla\varphi \odot (u - w) \rangle \\ &= \langle f, \varphi(u - w) \rangle + \langle g, \varphi(u - w) \rangle_{\Gamma_1}, \end{aligned} \tag{3.17}$$

for every $\varphi \in C^{0,1}(\bar{\Omega})$.

As a consequence of the formula above we obtain the following lemma.

Lemma 4 *Let $\sigma_k, \sigma \in \Sigma(\Omega)$, $w_k, w \in H^1(\Omega; \mathbb{R}^n)$, $(u_k, e_k, p_k) \in A_{BD}(w_k)$, and $(u, e, p) \in A_{BD}(w)$ be such that*

$$\begin{aligned} &\sigma_k \to \sigma \text{ strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}), \\ &\text{div}\sigma_k \to \text{div}\sigma \text{ strongly in } L^n(\Omega; \mathbb{R}^n), \\ &(\sigma_k)_D \text{ are uniformly bounded in } L^\infty(\Omega; \mathbb{M}^{n\times n}_D), \\ &u_k \rightharpoonup u \text{ weakly in } L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n), \\ &w_k \rightharpoonup w \text{ weakly in } H^1(\Omega; \mathbb{R}^n), \\ &e_k \rightharpoonup e \text{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}), \end{aligned}$$

then $\langle [(\sigma_k)_D \cdot p_k], \varphi \rangle \to \langle [\sigma \cdot p], \varphi \rangle$ for every $\varphi \in C_c^{0,1}(\Omega \cup \Gamma_0)$.

Proof Our hypotheses imply that $\sigma_k \to \sigma$ strongly in $L^n(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$ by [29, Proposition 2.5]. The conclusion follows now from (3.17).

3.2 Hypotheses on the Data

We discuss here the hypotheses on the data for the quasistatic evolution problem in perfect plasticity.

External Load In contrast to the dynamic case, in perfect plasticity it is not enough to assume that the total load $\mathcal{L}(t)$ belongs to $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Instead, we assume that $\mathcal{L}(t)$ takes the form (2.5), with $f(t)\in L^n(\Omega;\mathbb{R}^n)$ and $g(t)\in L^\infty(\Gamma_1;\mathbb{R}^n)$, so that now the duality $\langle\mathcal{L}(t),u\rangle$ is well defined by (2.5) for every $u\in BD(\Omega)$.

The balance equations for the forces are

$$-\operatorname{div}\sigma(t)=f(t)\quad\text{in }\Omega,\tag{3.18}$$

$$[\sigma(t)\nu]=g(t)\quad\text{on }\Gamma_1,\tag{3.19}$$

where $[\sigma(t)\nu]$ denotes the normal component of $\sigma(t)$, which can be defined as a distribution according to (3.9), since $\operatorname{div}\sigma(t)\in L^2(\Omega;\mathbb{R}^n)$ by (3.18). As for the time dependence, we assume that

$$f\in AC([0,T];L^n(\Omega;\mathbb{R}^n)),\tag{3.20a}$$

$$g\in AC([0,T];L^\infty(\Gamma_1;\mathbb{R}^n)).\tag{3.20b}$$

This implies that for a.e. $t\in[0,T]$ there exists an element of the dual of $BD(\Omega)$, denoted by $\dot{\mathcal{L}}(t)$, such that

$$\langle\dot{\mathcal{L}}(t),u\rangle=\lim_{s\to t}\langle\frac{\mathcal{L}(s)-\mathcal{L}(t)}{s-t},u\rangle\tag{3.21}$$

for every $u\in BD(\Omega)$ (see [6, Remark 4.1]).

As usual in perfect plasticity problems, we assume a uniform safe-load condition: there exist a function $\varrho:[0,T]\to L^2(\Omega,\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and a positive constant δ such that for every $t\in[0,T]$ we have

$$-\operatorname{div}\varrho(t)=f(t)\quad\text{on }\Omega,\tag{3.22a}$$

$$[\varrho(t)\nu]=g(t)\quad\text{on }\Gamma_1,\tag{3.22b}$$

and

$$\varrho_D(t)+\xi\in\mathcal{K}(\Omega)\quad\text{for every }\xi\in\mathbb{M}^{n\times n}_D\text{ with }|\xi|\le\delta.\tag{3.23}$$

Moreover we require that

$$t\mapsto\varrho(t)\quad\text{and}\quad t\mapsto\varrho_D(t)\quad\text{are absolutely continuous}\tag{3.24}$$

from $[0,T]$ to $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $L^\infty(\Omega;\mathbb{M}^{n\times n}_D)$ respectively, so that the function $t\mapsto\dot\varrho(t)$ belongs to $L^1([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ and

$$\frac{\varrho_D(t)-\varrho_D(s)}{t-s}\to\dot\varrho_D(s)\text{ weakly* in }L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\quad\text{as }t\to s,\tag{3.25}$$

for a.e. $s\in[0,T]$, and

$$t\mapsto\|\dot\varrho(t)\|_{L^\infty}\text{ belongs to }L^1([0,T])\tag{3.26}$$

(see [6, Theorem 7.1]).

Using (3.14) and (3.24) we see that for every $p\in\Pi_{\Gamma_0}(\Omega)$ the function

$$t\mapsto\langle\varrho_D(t),p\rangle\quad\text{belongs to }AC([0,T]).\tag{3.27}$$

Moreover, by (3.20a), (3.22a), (3.23), and (3.24), we obtain

$$\tfrac{d}{dt}\langle \varrho_D(t), p\rangle = \langle \dot{\varrho}_D(t), p\rangle \quad \text{for a.e. } t \in [0, T], \tag{3.28}$$

thanks to [6, formula (2.38)].

Boundary Conditions The boundary condition on Γ_0 is given in the relaxed form considered in (3.3b) with a time dependent function $t \to w(t)$. We assume that

$$w \in AC([0, T]; H^1(\Omega; \mathbb{R}^n)). \tag{3.29}$$

Plastic Dissipation In the energy formulation for the quasistatic evolution problem for perfect plasticity, it is not convenient to use formulas like (2.18), because they require the existence of the time derivative of $p(t)$. Instead, for an arbitrary function $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ we define the plastic dissipation in $[a, b] \subset [0, T]$ as

$$\mathcal{D}_H(a, b; p) := \sup \sum_{i=0}^{N-1} \mathcal{H}(p(t_{i+1}) - p(t_i)), \tag{3.30}$$

where the supremum is taken over all the possible choices of the integer $N > 0$ and of the real numbers $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b$. One can prove (see [6, Chap. 7]) that, if $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ is absolutely continuous, then

$$\mathcal{D}_H(a, b; p) = \int_a^b \mathcal{H}(\dot{p}(t))dt, \tag{3.31}$$

where $\dot{p}$ is the derivative of p defined by

$$\dot{p}(t) := w^*\text{-}\lim_{s\to t} \frac{p(s) - p(t)}{s - t}. \tag{3.32}$$

As a consequence of the safe-load condition (3.23) we can easily prove that for every $t \in [0, T]$

$$\mathcal{H}(q) - \langle \varrho(t), q\rangle \geq \gamma \|q\|_{\mathcal{M}_b}, \tag{3.33}$$

for every $q \in L^1(\Omega, \mathbb{M}^{n\times n}_D)$, where the positive constant γ is independent of q and t (see [6, Lemma 3.2]). Moreover we have that

$$H(q) - \varrho(t) \cdot q \geq 0 \text{ a.e. in } \Omega, \tag{3.34}$$

for every $q \in L^1(\Omega, \mathbb{M}^{n\times n}_D)$.

4 Quasistatic Evolution in Perfect Plasticity

We recall here the energy formulation of a perfectly plastic quasistatic evolution.

Definition 1 Suppose that f, g, $\mathcal{L}$, ϱ, and w satisfy (2.5), (3.21), (3.23), (3.23), (3.24), and (3.29). Let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})$, and $p_0 \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$. A *quasistatic evolution in perfect plasticity* with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0 is a function (u, e, p, σ) from $[0, T]$ into $BD(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{M}^{n\times n}_{\text{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_0, \mathbb{M}^{n\times n}_D) \times L^2(\Omega, \mathbb{M}^{n\times n}_{\text{sym}})$, with

$$u(0) = u_0, \quad e(0) = e_0, \quad p(0) = p_0, \tag{4.1}$$

$$\sigma(t) = A_0 e(t) \quad \text{for every } t \in [0, T], \tag{4.2}$$

such that $t \mapsto p(t)$ has bounded variation and the following two conditions are satisfied for every $t \in [0, T]$:

(a) $(u(t), e(t), p(t)) \in A_{BD}(w(t))$ and

$$\mathcal{Q}_0(e(t)) - \langle \mathcal{L}(t), u(t) \rangle \leq \mathcal{Q}_0(\eta) - \langle \mathcal{L}(t), \varphi \rangle + \mathcal{H}(q - p(t)) \tag{4.3}$$

for every $(\varphi, \eta, q) \in A_{BD}(w(t))$;

(b) $\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \mathcal{D}_H(p; 0, t) = \int_0^t \langle \sigma, E\dot{w} \rangle ds - \int_0^t \langle \mathcal{L}, \dot{w} \rangle ds$

$$+ \langle \mathcal{L}(t), u(t) \rangle - \langle \mathcal{L}(0), u_0 \rangle - \int_0^t \langle \dot{\mathcal{L}}, u \rangle ds, \tag{4.4}$$

where $\mathcal{D}_H(p; 0, t)$ is defined by (3.30).

The integrals in the right-hand side of (4.4) are well defined thanks to [6, Theorem 3.8 and Remark 4.3].

If $(u_0, e_0, p_0) \in A_{BD}(w(0))$ satisfies the following stability condition

$$\mathcal{Q}_0(e_0) - \langle \mathcal{L}(0), u_0 \rangle \leq \mathcal{Q}_0(\eta) - \langle \mathcal{L}(0), \varphi \rangle + \mathcal{H}(q - p_0) \tag{4.5}$$

for every $(\varphi, \eta, q) \in A_{BD}(w(0))$, then there exists a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0 (see [6, Theorem 4.5]). Moreover the function $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous from $[0, T]$ into $BD(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \times \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ ([6, Theorem 5.1]).

In our analysis of the behavior of the solution $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ of (1.2) as $\epsilon \to 0$ we find that $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ converges to a function (u, e, p, σ) which satisfies conditions (4.3) and (4.4) only for a.e. $t \in [0, T]$. The following theorem shows that this is enough to guarantee that (u, e, p, σ) is a quasistatic evolution, according to Definition 1.

Theorem 5 *Let u_0, e_0, p_0, f, g, $\mathcal{L}$, w, and ϱ be as in Definition 1. Let S be a subset of $[0, T]$ of full $\mathcal{L}^1$ measure containing 0 and let $(u, e, \sigma) : S \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ be a bounded and measurable function satisfying (4.1) and (4.2) for all $t \in S$. Suppose that $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation and that conditions (a) and (b) of Definition 1 are satisfied for every $t \in S$. Then there exists an absolutely continuous function $(u, e, \sigma) : [0, T] \to BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}) \times L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ which extends (u, e, σ). Moreover p is absolutely continuous and (u, e, p, σ) is a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0.*

Remark 4 Let $t \in S$, $(u(t), e(t), p(t)) \in A_{BD}(w(t))$ and $\sigma(t) := A_0 e(t)$. As shown in [6, Theorem 3.6] the following conditions are equivalent:

(a) Inequality (4.3) is satisfied for every $(\varphi, \eta, q) \in A_{BD}(w(t))$;
(b) $-\mathcal{H}(q) \leq \langle A_0 e(t), \eta \rangle - \langle \mathcal{L}(t), v \rangle \leq \mathcal{H}(-q)$ for every $(v, \eta, q) \in A_{BD}(0)$;
(c) $\sigma(t) \in \Sigma(\Omega)$, $\sigma_D(t) \in \mathcal{K}(\Omega)$, $-\mathrm{div}\sigma(t) = f(t)$ in Ω, and $[\sigma(t)\nu] = g(t)$ on Γ_1.

The following lemma gives an elementary but useful tool for the proof of Theorem 5.

Lemma 5 *Let $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ be a function with bounded variation and let $\psi(t) := \mathcal{D}_H(p; 0, t)$ for $t \in [0, T]$. Assume that there exists a set $S \subseteq [0, T]$ of full $\mathcal{L}^1$ measure such that $p|_S$ and $\psi|_S$ are absolutely continuous on S. Then p is absolutely continuous on $[0, T]$.*

Proof The absolute continuity on S implies that

$$\lim_{\substack{s\to t^-\\ s\in S}} \psi(s) = \lim_{\substack{s\to t^+\\ s\in S}} \psi(s)$$

for every $t \in [0, T]$. Since ψ is non-decreasing, we deduce that the common value of the limit coincides with $\psi(t)$. This shows that ψ is continuous on $[0, T]$. Since

$$\|p(t_1) - p(t_2)\|_{\mathcal{M}_b} \le \mathcal{D}_H(p; t_1, t_2) = \psi(t_2) - \psi(t_1)$$

for every $0 \le t_1 \le t_2 \le T$, we conclude that also p is continuous on $[0, T]$. Moreover the fact that the restriction of p to S is absolutely continuous implies that it is absolutely continuous on $[0, T]$ as well.

Proof (Proof of Theorem 5) We first prove that the functions e, p and u are absolutely continuous on S. We argue as in the proof of [6, Theorem 5.2] using only times t_1, t_2 and s in the set S, and we obtain that for any $t_1, t_2 \in S$ with $t_1 < t_2$ we have that

$$\|e(t_2) - e(t_1))\|^2_{L^2} \le \int_{t_1}^{t_2} \|e(s) - e(t_1)\|_{L^2}\phi(s)ds + \left(\int_{t_1}^{t_2} \phi(s)ds\right)^2,$$

where ϕ is a suitable nonnegative integrable function. As a consequence of [6, Lemma 5.3] we get that $\|e(t_2) - e(t_1))\|_{L^2} \le \frac{3}{2}\int_{t_1}^{t_2} \phi(s)ds$ so that $t \mapsto e(t)$ is absolutely continuous from S into $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$. Continuing as in the proof of [6, Theorem 5.2] we obtain also that p and u are absolutely continuous on S. From Eq. (4.4) it follows that $t \mapsto \mathcal{D}_H(p; 0, t)$ is absolutely continuous on S, so that, applying Lemma 5, we get that p is absolutely continuous on $[0, T]$. Now (u, e) admits an absolutely continuous extension to $[0, T]$ that we still denote by (u, e). By continuity this extension satisfies (4.3) and (4.4) for every $t \in [0, T]$. This completes the proof.

Remark 5 Under the hypotheses of Definition 1, for every $t \in [0, T]$ condition (b) of Definition 1 is equivalent to the following condition:

(b') The function $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ has bounded variation and

$$\begin{aligned} &\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p; 0, t) - \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho_D(t), p(t)\rangle \\ &= \mathcal{Q}_0(e_0) - \langle \varrho(0), e(0) - Ew(0)\rangle - \langle \varrho_D(0), p(0)\rangle + \int_0^t \langle \sigma, E\dot{w}\rangle ds \\ &- \int_0^t \langle \dot{\varrho}, e - Ew\rangle ds - \int_0^t \langle \dot{\varrho}_D, p\rangle ds. \end{aligned} \tag{4.6}$$

This is proved in [6, Theorem 4.4] using the integration by parts formula (3.16). Note that the duality product $\langle \dot{\varrho}_D(t), p(t)\rangle$ is well defined for a.e. $t \in [0, T]$ by (3.20a), (3.22a), (3.24), and (3.25).

5 Limit of Dynamic Solutions

Here we formulate in a precise way the asymptotic analysis of the dynamic problem as the data become slower and slower. This will be done by a suitable change of variables. We start from an external load $\mathcal{L}(t)$, a boundary datum $w(t)$ defined on the interval $[0, T]$, and initial conditions u_0, e_0, p_0, and v_0. We then consider the rescaled problem with external load $\mathcal{L}_\epsilon(t) = \mathcal{L}(\epsilon t)$, boundary condition $w_\epsilon(t) = w(\epsilon t)$ on the interval $[0, T/\epsilon]$, and initial conditions $u_\epsilon(0) = u_0$, $e_\epsilon(0) = e_0$, $p_\epsilon(0) = p_0$, and $\dot{u}_\epsilon(0) = \epsilon v_0$. The dynamic solutions of the corresponding systems (2.23) are denoted by $(u_\epsilon(t), e_\epsilon(t), p_\epsilon(t), \sigma_\epsilon(t))$.

To study the limit behavior of $(u_\epsilon(t), e_\epsilon(t), p_\epsilon(t), \sigma_\epsilon(t))$ on the whole interval $[0, T/\epsilon]$ it is convenient to consider the rescaled functions

$$(u^\epsilon(t), e^\epsilon(t), p^\epsilon(t), \sigma^\epsilon(t)) := (u_\epsilon(t/\epsilon), e_\epsilon(t/\epsilon), p_\epsilon(t/\epsilon), \sigma_\epsilon(t/\epsilon)),$$

defined on $[0, T]$, and to study their limit as $\epsilon \downarrow 0$. A straightforward change of variables shows that $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ will satisfy the following system of equations on $[0, T]$

$$Eu^\epsilon = e^\epsilon + p^\epsilon, \tag{5.1a}$$

$$\sigma^\epsilon = A_0 e^\epsilon + \epsilon A_1 \dot{e}^\epsilon_{A_1}, \tag{5.1b}$$

$$\epsilon^2 \ddot{u}^\epsilon - \mathrm{div}_{\Gamma_0}(\sigma^\epsilon) = \mathcal{L}, \tag{5.1c}$$

$$\epsilon \dot{p}^\epsilon = \sigma^\epsilon - \pi_K \sigma^\epsilon, \tag{5.1d}$$

with boundary and initial conditions

$$u^\epsilon(t) = w(t) \quad \text{on} \quad \Gamma_0 \quad \text{for every } t \in [0, T], \tag{5.2}$$

$$u^\epsilon(0) = u_0, \quad e^\epsilon(0) = e_0, \quad p^\epsilon(0) = p_0, \quad \dot{u}^\epsilon(0) = v_0. \tag{5.3}$$

We shall prove (Theorem 6) that, under suitable assumptions, the solutions $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ of (5.1) tend to a solution of the quasistatic evolution problem in perfect plasticity, according to Definition 1.

Hypotheses on the Data The regularity assumptions on the data considered in the dynamical problem are not sufficient to study the limit of the solutions of (5.1). Therefore we introduce a new set of hypotheses, which includes also the case of data depending on ϵ and converging in a suitable way as ϵ tends to 0.

Let $M > 0$ be a constant. For $\epsilon \in (0, 1)$ we consider the following assumptions.

(i) Hypotheses on w^ϵ and w:

$$w^\epsilon \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n)), \tag{5.4a}$$

$$\dot{w}^\epsilon \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n)) \cap L^2([0, T]; H^1(\Omega; \mathbb{R}^n)), \tag{5.4b}$$

$$\ddot{w}^\epsilon \in L^2([0, T]; L^2(\Omega; \mathbb{R}^n)), \tag{5.4c}$$

$$w \in AC([0, T]; H^1(\Omega; \mathbb{R}^n)), \tag{5.4d}$$

$$w^\epsilon \to w \text{ strongly in } W^{1,1}([0, T]; H^1(\Omega; \mathbb{R}^n)), \tag{5.4e}$$

$$\epsilon \|\dot{w}^\epsilon(0)\|_{L^2} \to 0, \tag{5.4f}$$

$$\epsilon \|\dot{w}^\epsilon(t)\|_{L^2} \leq M \text{ for all } t \in [0, T], \tag{5.4g}$$

$$\epsilon \int_0^T \|\dot{w}^\epsilon\|^2_{H^1} dt \to 0, \tag{5.4h}$$

$$\epsilon^2 \int_0^T \|\ddot{w}^\epsilon\|^2_{L^2} dt \to 0. \tag{5.4i}$$

(ii) Hypotheses on f^ϵ, g^ϵ, f, and g: we assume that there exist ϱ^ϵ and ϱ satisfying (3.23) and (3.23) with f^ϵ, g^ϵ and f, g respectively, and with δ independent of ϵ. We also suppose that

$$f^\epsilon \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{5.5a}$$

$$g^\epsilon \in AC([0,T]; H^{-\frac{1}{2}}(\Gamma_1;\mathbb{R}^n)), \tag{5.5b}$$

$$\varrho^\epsilon \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{5.5c}$$

$$f \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{5.5d}$$

$$g \in AC([0,T]; L^\infty(\Gamma_1;\mathbb{R}^n)), \tag{5.5e}$$

$$\varrho \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{5.5f}$$

$$\varrho_D \in AC([0,T]; L^\infty(\Omega;\mathbb{M}^{n\times n}_D)), \tag{5.5g}$$

$$f^\epsilon \to f \text{ strongly in } W^{1,1}([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{5.5h}$$

$$\varrho^\epsilon \to \varrho \text{ strongly in } W^{1,1}([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})). \tag{5.5i}$$

The functionals $\mathcal{L}^\epsilon(t)$ and $\mathcal{L}(t)$ are defined by (2.5) with $f^\epsilon(t)$, $g^\epsilon(t)$ and $f(t)$, $g(t)$ respectively.

(iii) Hypotheses on the initial data $(u_0^\epsilon, e_0^\epsilon, p_0^\epsilon)$, (u_0, e_0, p_0), and v_0^ϵ.

$$(u_0^\epsilon, e_0^\epsilon, p_0^\epsilon) \in A(w^\epsilon(0)), \tag{5.6a}$$

$$(u_0, e_0, p_0) \in A_{BD}(w(0)), \tag{5.6b}$$

$$(u_0, e_0, p_0) \text{ satisfies the stability condition (45)}, \tag{5.6c}$$

$$u_0^\epsilon \to u_0 \text{ strongly in } L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n), \tag{5.6d}$$

$$e_0^\epsilon \to e_0 \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}), \tag{5.6e}$$

$$p_0^\epsilon \rightharpoonup p_0 \text{ weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D), \tag{5.6f}$$

$$v_0^\epsilon \in L^2(\Omega;\mathbb{R}^n) \text{ and } \epsilon\|v_0^\epsilon\|_{L^2} \to 0. \tag{5.6g}$$

Remark 6 If we assume that

$$\varrho_D^\epsilon \in AC([0,T]; L^\infty(\Omega;\mathbb{M}^{n\times n}_D)), \tag{5.7a}$$

$$\int_0^T \|\dot\varrho_D^\epsilon - \dot\varrho_D\|_{L^\infty} dt \to 0, \tag{5.7b}$$

then we can replace (5.5c), (5.5f), and (5.5i) by the weaker conditions

$$\varrho_\epsilon,\ \varrho \in AC([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{5.7c}$$

$$\varrho^\epsilon \to \varrho \text{ strongly in } W^{1,1}([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})). \tag{5.7d}$$

Indeed using [29, Proposition 2.5] (see also [28, Chap. 2, Proposition 7.1]) from (3.26), (5.5h), and (5.7) we deduce that $\varrho^\epsilon, \varrho \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$ and that (5.5i) holds.

We now state the main result.

Theorem 6 *Assume hypotheses (i)–(iii) above. Let* $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ *be the solution of (5.1), with* $\mathcal{L}$ *replaced by* $\mathcal{L}^\epsilon$*, satisfying the boundary condition* w^ϵ *on* Γ_0 *for every* $t\in[0,T]$*, and the initial data*

$$u^\epsilon(0) = u_0^\epsilon,\ e^\epsilon(0) = e_0^\epsilon,\ p^\epsilon(0) = p_0^\epsilon\quad \dot u^\epsilon(0) = v_0^\epsilon.$$

Then there exist a quasistatic evolution in perfect plasticity (u, e, p, σ)*, with initial conditions* (u_0, e_0, p_0) *and boundary condition* w *on* Γ_0*, and a subsequence of* $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$*, not relabeled, such that*

$$u^\epsilon(t) \rightharpoonup u(t) \quad \textit{weakly* in } BD(\Omega), \tag{5.8}$$

$$e^\epsilon(t) \to e(t) \quad \textit{strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}), \tag{5.9}$$

for a.e. $t \in [0, T]$, *and*

$$p^\epsilon(t) \rightharpoonup p(t) \quad \textit{weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D), \tag{5.10}$$

for all $t \in [0, T]$. *Moreover there exists* $M > 0$ *such that*

$$\|u^\epsilon(t)\|_{L^1} + \|e^\epsilon(t)\|_{L^2} + \|p^\epsilon(t)\|_{\mathcal{M}_b} \le M \tag{5.11}$$

for every $\epsilon \in (0, 1)$ *and every* $t \in [0, T]$.

Proof From Theorem 2 we get the energy balance formula

$$\begin{aligned}
&\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon_{A_1})ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds \\
&= \int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds + \langle f^\epsilon(t), u^\epsilon(t) - w^\epsilon(t)\rangle - \langle f^\epsilon(0), u^\epsilon(0) - w^\epsilon(0)\rangle \\
&\quad - \int_0^t \langle \dot f^\epsilon, u^\epsilon - w^\epsilon\rangle ds + \langle g^\epsilon(t), u^\epsilon(t) - w^\epsilon(t)\rangle_{\Gamma_1} - \langle g^\epsilon(0), u^\epsilon(0) - w^\epsilon(0)\rangle_{\Gamma_1} \\
&\quad - \int_0^t \langle \dot g^\epsilon, u^\epsilon - w^\epsilon\rangle_{\Gamma_1} ds - \epsilon^2\int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds + \mathcal{Q}_0(e^\epsilon_0) + \frac{\epsilon^2}{2}\|v^\epsilon_0 - \dot w^\epsilon(0)\|^2_{L^2},
\end{aligned} \tag{5.12}$$

where $\sigma^\epsilon = A_0 e^\epsilon + \epsilon A_1 \dot e^\epsilon_{A_1}$. Using the safe-load condition (3.23) and (3.23) and integrating by parts in space, we get

$$\begin{aligned}
&\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon_{A_1})ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds \\
&= \int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds + \langle \varrho^\epsilon(t), Eu^\epsilon(t) - Ew^\epsilon(t)\rangle - \langle \varrho^\epsilon(0), Eu^\epsilon(0) - Ew^\epsilon(0)\rangle \\
&\quad - \int_0^t \langle \dot\varrho^\epsilon, Eu^\epsilon - Ew^\epsilon\rangle ds - \epsilon^2\int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds + \mathcal{Q}_0(e^\epsilon_0) + \frac{\epsilon^2}{2}\|v^\epsilon_0 - \dot w^\epsilon(0)\|^2_{L^2}.
\end{aligned} \tag{5.13}$$

By (2.2), (5.4e), (5.4g), (5.4i), (5.5i), (5.6e), and (5.6g), using the Cauchy inequality, we get a positive constant D_0 such that

$$\begin{aligned}
&\frac{\alpha_0}{2}\|e^\epsilon(t)\|^2_{L^2} + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon_{A_1})ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds \\
&\le \beta_0\int_0^t \|e^\epsilon\|_{L^2}\|E\dot w^\epsilon\|_{L^2}ds + \epsilon\int_0^t \|A_1\dot e^\epsilon_{A_1}\|_{L^2}\|E\dot w^\epsilon\|_{L^2}ds + \langle \varrho^\epsilon(t), e^\epsilon(t)\rangle \\
&\quad - \langle \varrho^\epsilon(0), e^\epsilon(0)\rangle - \int_0^t \langle \dot\varrho^\epsilon, e^\epsilon\rangle ds + \int_0^t \langle \varrho^\epsilon_D, p^\epsilon\rangle ds + \frac{\epsilon^2}{2}\int_0^t \|\dot u^\epsilon - \dot w^\epsilon\|^2_{L^2}ds + D_0,
\end{aligned} \tag{5.14}$$

for every $\epsilon \in (0, 1)$, where we have integrated by parts in time the term $\int_0^t \langle \dot\varrho^\epsilon, p^\epsilon\rangle$. Using again the Cauchy inequality and the inequality $\|e^\epsilon\|_{L^2} \le 1 + \|e^\epsilon\|^2_{L^2}$, we obtain that for every

$\lambda > 0$ the right-hand side of (5.14) can be estimated from above by

$$\beta_0\int_0^t \|e^\epsilon\|_{L^2}^2\|E\dot{w}^\epsilon\|_{L^2}ds + \epsilon\lambda\int_0^t \|A_1\dot{e}^\epsilon_{A_1}\|_{L^2}^2 ds + \lambda\|e^\epsilon(t)\|_{L^2}^2 + \int_0^t \|\dot{\varrho}^\epsilon\|_{L^2}\|e^\epsilon\|_{L^2}^2 ds$$
$$+\int_0^t \langle \varrho^\epsilon_D, \dot{p}^\epsilon\rangle ds + \frac{\epsilon^2}{2}\int_0^t \|\dot{u}^\epsilon - \dot{w}^\epsilon\|_{L^2}^2 ds + D_\lambda, \tag{5.15}$$

for a suitable constant D_λ independent of ϵ that can be obtained using (5.4e), (5.4h), (5.5i), and (5.6e). Recalling that $\|A_1\dot{e}^\epsilon_{A_1}\|_{L^2}^2 \le \beta_1\mathcal{Q}_1(\dot{e}^\epsilon_{A_1})$ by (2.2c), and taking $\lambda = \min\{\frac{\alpha_0}{4}, \frac{1}{2\beta_1}\}$, from (3.33), (5.14), and (5.15) we get

$$\frac{\alpha_0}{4}\|e^\epsilon(t)\|_{L^2}^2 + \frac{\epsilon^2}{2}\|\dot{u}^\epsilon(t) - \dot{w}^\epsilon(t)\|_{L^2}^2 + \frac{\epsilon}{2}\int_0^t \mathcal{Q}_1(\dot{e}^\epsilon_{A_1})ds + \epsilon\int_0^t \|\dot{p}^\epsilon\|_{L^2}^2 dt$$
$$+\gamma\int_0^t \|\dot{p}^\epsilon\|_{L^1}ds \le \int_0^t \psi^\epsilon\|e^\epsilon\|_{L^2}^2 ds + \frac{\epsilon^2}{2}\int_0^t \|\dot{u}^\epsilon - \dot{w}^\epsilon\|_{L^2}^2 ds + D_\lambda, \tag{5.16}$$

where $\psi^\epsilon = \beta_0\|E\dot{w}^\epsilon\|_{L^2} + \|\dot{\varrho}^\epsilon\|_{L^2}$. Since ψ^ϵ is bounded in $L^1([0,T])$ by (5.4e) and (5.5i), using the Gronwall Lemma we obtain that $\|e^\epsilon(t)\|_{L^2}$ and $\frac{\epsilon^2}{2}\|\dot{u}^\epsilon(t) - \dot{w}^\epsilon(t)\|_{L^2}^2$ are bounded by some constant independent of t and ϵ. Together with (5.4g) and (5.16), this gives

$$\|e^\epsilon(t)\|_{L^2} \le M \text{ for all } t \in [0,T], \tag{5.17a}$$
$$\epsilon\|\dot{u}^\epsilon(t)\|_{L^2} \le M \text{ for all } t \in [0,T], \tag{5.17b}$$
$$\epsilon\int_0^t \mathcal{Q}_1(\dot{e}^\epsilon_{A_1})ds \le M, \tag{5.17c}$$
$$\epsilon\int_0^T \|\dot{p}^\epsilon\|_{L^2}^2 ds \le M, \tag{5.17d}$$
$$\int_0^T \|\dot{p}^\epsilon\|_{L^1} ds \le M, \tag{5.17e}$$

for all $\epsilon \in (0,1)$ and some constant $M > 0$ independent of t and ϵ.

Since $L^1(\Omega; \mathbb{M}^{n\times n}_D)$ is naturally embedded into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, the functions p^ϵ are actually continuous from $[0,T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, and inequality (5.17e) says that the total variation of p^ϵ is bounded uniformly with respect to ϵ. Taking into account (5.6f), we can employ a generalization of Helly Theorem (see [6, Lemma 7.2] and [4, Theorem 3.5, Chap. 1]), which implies that there exist a subsequence, still denoted by p^ϵ, and a function $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, with bounded variation, such that, as $\epsilon \to 0$,

$$p^\epsilon(t) \rightharpoonup p(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \quad \text{for every } t \in [0,T]. \tag{5.18}$$

It then follows that $p(t)$ is bounded in $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ uniformly with respect to t.

From (5.17a) we also get, possibly passing to another subsequence, that there exists $e \in L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}))$ such that

$$e^\epsilon \rightharpoonup e \text{ weakly* in } L^\infty([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}})), \tag{5.19}$$

as $\epsilon \to 0$.

Writing $E(u^\epsilon - w^\epsilon) = e^\epsilon + p^\epsilon - Ew^\epsilon$, by (5.4e), (5.6f), (5.17a), and (5.17e), we see that $E(u^\epsilon - w^\epsilon)$ is bounded in $L^\infty([0,T]; L^1(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}))$ uniformly with respect to ϵ, so that, thanks to (3.1), $u^\epsilon - w^\epsilon$ is bounded in $L^\infty([0,T]; BD(\Omega, \mathbb{R}^n))$ uniformly with respect to ϵ. Then, as a consequence of the embedding $BD(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, there exists $u \in L^\infty([0,T];L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n))$ such that

$$u^\epsilon \rightharpoonup u \text{ weakly* in } L^\infty([0,T];L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)), \tag{5.20}$$

again for a suitable subsequence, as $\epsilon \to 0$. Using the equality $Eu^\epsilon = e^\epsilon + p^\epsilon$, from (5.18) and (5.19) we obtain that $u \in L^\infty([0,T];BD(\Omega))$ and $Eu = e + p$.

By (2.26) we see that the function $(u^\epsilon, e^\epsilon, p^\epsilon)$ satisfies the equilibrium condition

$$\begin{aligned} -\mathcal{H}(q) \le \langle A_0 e^\epsilon(t), \eta\rangle + \langle \epsilon A_1 \dot{e}^\epsilon_{A_1}(t), \eta\rangle + \langle \epsilon \dot{p}^\epsilon(t), q\rangle \\ + \langle \epsilon^2 \ddot{u}^\epsilon(t), \varphi\rangle - \langle f^\epsilon(t), \varphi\rangle - \langle g^\epsilon(t), \varphi\rangle_{\Gamma_1} \le \mathcal{H}(-q), \end{aligned} \tag{5.21}$$

for every $(\varphi, \eta, q) \in A(0)$ and a.e. $t \in [0,T]$.

Let us fix a smooth and nonnegative real function ψ on $[0,T]$. Multiplying the previous formula by ψ and integrating on $[0,T]$ we get

$$\begin{aligned} &-\int_0^T \mathcal{H}(q)\psi(s)ds \le \int_0^T \langle A_0 e^\epsilon(s), \eta\rangle\psi(s)ds + \int_0^T \langle \epsilon A_1 \dot{e}^\epsilon_{A_1}(s), \eta\rangle\psi(s)ds \\ &+ \int_0^T \langle \epsilon \dot{p}^\epsilon(s), q\rangle\psi(s)ds + \int_0^T \langle \epsilon^2 \ddot{u}^\epsilon(s), \varphi\rangle\psi(s)ds - \int_0^T \langle f^\epsilon(s), \varphi\rangle\psi(s)ds \\ &- \int_0^T \langle g^\epsilon(s), \varphi\rangle_{\Gamma_1}\psi(s)ds \le \int_0^T \mathcal{H}(-q)\psi(s)ds, \end{aligned} \tag{5.22}$$

for every $(\varphi, \eta, q) \in A(0)$. It is easily seen that, if ψ has compact support, thanks to (5.17b) the term

$$\int_0^T \langle \epsilon^2 \ddot{u}^\epsilon(s), \varphi\rangle\psi(s)ds = -\epsilon^2 \int_0^T \langle \dot{u}^\epsilon(s), \varphi\rangle\dot{\psi}(s)ds$$

vanishes as $\epsilon \to 0$, and the same is true for the term

$$\int_0^T \langle \epsilon \dot{p}^\epsilon(s), q\rangle\psi(s)ds$$

thanks to (5.17d).

By (2.2c) we have

$$\epsilon^2 \int_0^T \|A_1 \dot{e}^\epsilon_{A_1}\|^2_{L^2} ds \le \epsilon^2 \beta_1 \int_0^T \mathcal{Q}_1(\dot{e}^\epsilon_{A_1})ds.$$

By (5.17c) this shows that

$$\epsilon A_1 \dot{e}^\epsilon_{A_1} \to 0 \text{ strongly in } L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})), \tag{5.23}$$

as $\epsilon \to 0$. This implies that the term

$$\int_0^T \langle \epsilon A_1 \dot{e}^\epsilon_{A_1}(s), \eta\rangle\psi(s)ds$$

vanishes as $\epsilon \to 0$.

Since $(\varphi, \eta, q) \in A(0)$, by (3.23) we can write

$$\int_0^T (\langle f^\epsilon(s), \varphi\rangle + \langle g^\epsilon(s), \varphi\rangle_{\Gamma_1})\psi(s)ds = \int_0^T \langle \varrho^\epsilon(s), \eta + q\rangle\psi(s)ds,$$

and, thanks to (5.5i), we obtain that the last expression tends to

$$\int_0^T \langle \varrho(s), \eta + q\rangle \psi(s)ds = \int_0^T (\langle f(s), \varphi\rangle + \langle g(s), \varphi\rangle_{\Gamma_1})\psi(s)ds.$$

So from (5.19) and (5.22) we get

$$-\int_0^T \mathcal{H}(q)\psi(s)ds \le \int_0^T \langle A_0 e(s), \eta\rangle \psi(s)ds - \int_0^T \langle f(s), \varphi\rangle \psi(s)ds$$
$$-\int_0^T \langle g(s), \varphi\rangle_{\Gamma_1} \psi(s)ds \le \int_0^T \mathcal{H}(-q)\psi(s)ds,$$

and thanks to the arbitrariness of ψ we conclude that

$$-\mathcal{H}(q) \le \langle A_0 e(t), \eta\rangle - \langle f(t), \varphi\rangle - \langle g(t), \varphi\rangle_{\Gamma_1} \le \mathcal{H}(-q), \tag{5.24}$$

for a fixed $(\varphi, \eta, q) \in A(0)$ and for a.e. $t \in [0, T]$. The fact that $A(0)$ is separable allows us to prove that for a.e. $t \in [0, T]$ inequalities (5.24) hold for every $(\varphi, \eta, q) \in A(0)$.

Let us define $\sigma(t) := A_0 e(t)$. For each $q \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$, since $(0, q, -q) \in A(0)$, we see that

$$-\mathcal{H}(-q) \le \langle \sigma(t), q\rangle \le \mathcal{H}(q), \tag{5.25}$$

which says that $\sigma_D(t) \in \partial\mathcal{H}(0) = \mathcal{K}(\Omega)$ (see (2.20)). Moreover, since for each $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$ we have $(\varphi, E\varphi, 0) \in A(0)$, from (5.24) we obtain

$$\langle \sigma(t), E\varphi\rangle - \langle f(t), \varphi\rangle = \langle g(t), \varphi\rangle_{\Gamma_1} \quad \text{for all } \varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n). \tag{5.26}$$

From this we get $\operatorname{div}\sigma(t) = f(t)$ a.e. in Ω, and $[\sigma(t)\nu] = g(t)$ on Γ_1. Therefore, $(u(t), e(t), p(t))$ satisfies condition (c) of Remark 4. This implies that for a.e. $t \in [0, T]$, $(u(t), e(t), p(t))$ satisfies the minimality condition (4.3) for all $(\varphi, \eta, q) \in A_{BD}(w(t))$. We now set $S := \{0\} \cup \{t \in (0, T] : (4.3) \text{ is satisfied}\}$ and we define $u(0) := u_0$ and $e(0) := e_0$. Since $p(0) = p_0$ by (5.6f) and (5.18), we deduce from (5.6c) that condition (4.3) is also satisfied for $t = 0$.

Since $t \mapsto p(t)$ has bounded variation from $[0, T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, it is globally bounded and there exists a countable set $N \subset [0, T]$ such that for every $t \in [0, T] \setminus N$

$$p(s) \to p(t) \quad \text{strongly in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \quad \text{as } s \to t. \tag{5.27a}$$

By the minimality property of $(u(s), e(s), p(s))$ for $s \in S$ we can apply [6, Theorem 3.8] and for every $t \in S \setminus N$ we obtain

$$e(s) \to e(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\text{sym}}) \quad \text{as } s \to t, \tag{5.27b}$$
$$u(s) \to u(t) \quad \text{strongly in } BD(\Omega) \quad \text{as } s \to t. \tag{5.27c}$$

By the continuity of the embedding $BD(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$ we also get

$$u(s) \to u(t) \quad \text{strongly in } L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n) \quad \text{as } s \to t. \tag{5.27d}$$

In order to prove the energy balance (4.4) we fix $t \in S \setminus (N \cup \{0\})$. For every k let $0 = t^k_0 < t^k_1 < \dots < t^k_k = t$ be elements of $(S \setminus N) \cup \{0\}$ such that $\max_i (t^k_i - t^k_{i-1}) \to 0$ as $k \to \infty$. Then, since $(u(t^k_i) - (w(t^k_i) - w(t^k_{i-1})), e(t^k_i) - (Ew(t^k_i) - Ew(t^k_{i-1})), p(t^k_i)) \in$

$A_{BD}(w(t^k_{i-1}))$ by (4.3), we have

$$\begin{aligned}&\mathcal{Q}_0(e(t^k_{i-1})) - \langle f(t^k_{i-1}), u(t^k_{i-1})\rangle - \langle g(t^k_{i-1}), u(t^k_{i-1})\rangle_{\Gamma_1} \le \mathcal{Q}_0(e(t^k_i))\\ &- \langle A_0 e(t^k_i), Ew(t^k_i) - Ew(t^k_{i-1})\rangle + \mathcal{Q}_0(Ew(t^k_i)) - Ew(t^k_{i-1}))\\ &- \langle f(t^k_{i-1}), u(t^k_i) - (w(t^k_i) - w(t^k_{i-1}))\rangle\\ &- \langle g(t^k_{i-1}), u(t^k_i) - (w(t^k_i) - w(t^k_{i-1}))\rangle_{\Gamma_1} + \mathcal{H}(p(t^k_i) - p(t^k_{i-1})).\end{aligned}$$

Employing the integration by parts formula (3.16) and then summing up over $i = 1, \dots, k$, we obtain

$$\begin{aligned}&\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \sum_{i=1}^k \mathcal{H}(p(t^k_i) - p(t^k_{i-1})) + \sum_{i=1}^k \mathcal{Q}_0(Ew(t^k_i) - Ew(t^k_{i-1}))\\ &\ge \sum_{i=1}^k \langle A_0 e(t^k_i), Ew(t^k_i) - Ew(t^k_{i-1})\rangle + \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho(0), e(0) - Ew(0)\rangle\\ &\quad + \langle \varrho_D(t), p(t)\rangle - \langle \varrho_D(0), p(0)\rangle - \sum_{i=1}^k \langle \varrho(t^k_i) - \varrho(t^k_{i-1}), e(t^k_i)\rangle\\ &\quad + \sum_{i=1}^k \langle \varrho(t^k_i) - \varrho(t^k_{i-1}), Ew(t^k_i)\rangle - \sum_{i=1}^k \langle \varrho_D(t^k_i) - \varrho_D(t^k_{i-1}), p(t^k_i)\rangle. \end{aligned} \tag{5.28}$$

By (3.27), (3.28), (5.4d), (5.5f), (5.5g), and (5.27) we can apply Lemmas 6 and 7, with S replaced by $S \setminus (N \cup \{0\})$, and we obtain that the four Riemann sums in the right-hand side of (5.28) converge to

$$\int_0^t \langle \sigma, Ew\rangle ds, \quad \int_0^t \langle \dot{\varrho}, e\rangle ds, \quad \int_0^t \langle \dot{\varrho}, Ew\rangle ds, \quad \int_0^t \langle \dot{\varrho}_D, p\rangle ds.$$

Moreover we see that $\sum_{i=1}^k \mathcal{Q}_0(Ew(t^k_i) - Ew(t^k_{i-1}))$ tends to 0 as $k \to \infty$, thanks to the absolute continuity of $t \mapsto Ew(t)$. Therefore, passing to the limit in (5.28) we obtain

$$\begin{aligned}&\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p; 0, t) - \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho_D(t), p(t)\rangle\\ &\ge \mathcal{Q}_0(e_0) - \langle \varrho(0), e(0) - Ew(0)\rangle - \langle \varrho_D(0), p(0)\rangle + \int_0^t \langle \sigma, E\dot{w}\rangle ds\\ &\quad - \int_0^t \langle \dot{\varrho}, e - Ew\rangle ds - \int_0^t \langle \dot{\varrho}_D, p\rangle ds,\end{aligned} \tag{5.29}$$

for a.e. $t \in [0, T]$, where $\sigma = A_0 e$.

We want to show that actually equality holds. In order to prove the opposite inequality we consider Eq. (5.13).

Thanks to the semicontinuity of $\mathcal{Q}_0(\cdot)$, by (5.19) we have

$$\int_a^b \mathcal{Q}_0(e(t))dt \le \liminf_{\epsilon \to 0} \int_a^b \mathcal{Q}_0(e^\epsilon(t))dt \tag{5.30}$$

for all $0 < a < b < T$. We claim that

$$\int_a^b \Big(\mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot{\varrho}_D, p\rangle ds\Big) dt \\ \leq \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \mathcal{H}(\dot{p}^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon\rangle ds\Big) dt, \tag{5.31}$$

for all $0 < a < b < T$. This, together with (5.30), implies

$$\begin{aligned} &\int_a^b \Big(\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot{\varrho}_D, p\rangle ds\Big) dt \\ &\leq \liminf_{\epsilon\to 0} \int_a^b \Big(\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot{u}^\epsilon(t) - \dot{w}^\epsilon(t)\|_{L^2}^2 + \epsilon \int_0^t \mathcal{Q}_1(\dot{e}^\epsilon_{A_1}) ds \\ &\quad + \epsilon \int_0^t \|\dot{p}^\epsilon\|_{L^2}^2 ds + \int_0^t \mathcal{H}(\dot{p}^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon\rangle ds\Big) dt \\ &= \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \langle \sigma^\epsilon, E\dot{w}^\epsilon\rangle ds + \langle \varrho^\epsilon(t), e^\epsilon(t) - Ew^\epsilon(t)\rangle \\ &\quad - \langle \varrho^\epsilon(0), e^\epsilon(0) - Ew^\epsilon(0)\rangle - \int_0^t \langle \dot{\varrho}^\epsilon, e^\epsilon - Ew^\epsilon\rangle ds \\ &\quad - \epsilon^2 \int_0^t \langle \ddot{w}^\epsilon, \dot{u}^\epsilon - \dot{w}^\epsilon\rangle ds + \mathcal{Q}_0(e_0^\epsilon) + \frac{\epsilon^2}{2}\|v_0^\epsilon - \dot{w}^\epsilon(0)\|_{L^2}^2\Big) dt =: L, \end{aligned} \tag{5.32}$$

where the first equality follows from (5.13) after an integration by parts in time.

Using (5.4f), (5.4g), (5.4i), (5.6g), and (5.17b) it is easily seen that

$$\epsilon^2 \int_a^b \Big(\int_0^t \langle \ddot{w}^\epsilon, \dot{u}^\epsilon - \dot{w}^\epsilon\rangle ds\Big) dt \to 0, \tag{5.33a}$$

$$\epsilon^2 \|v_0^\epsilon - \dot{w}^\epsilon(0)\|_{L^2}^2 \to 0, \tag{5.33b}$$

while

$$\int_a^b \Big(\int_0^t \langle \sigma^\epsilon, E\dot{w}^\epsilon\rangle ds\Big) dt \to \int_a^b \Big(\int_0^t \langle \sigma, E\dot{w}\rangle ds\Big) dt, \tag{5.33c}$$

$$\mathcal{Q}_0(e_0^\epsilon) \to \mathcal{Q}_0(e_0), \tag{5.33d}$$

$$\int_a^b \langle \varrho^\epsilon(t), e^\epsilon(t) - Ew^\epsilon(t)\rangle dt \to \int_a^b \langle \varrho(t), e(t) - Ew(t)\rangle dt, \tag{5.33e}$$

$$\langle \varrho^\epsilon(0), e^\epsilon(0) - Ew^\epsilon(0)\rangle \to \langle \varrho(0), e(0) - Ew(0)\rangle, \tag{5.33f}$$

$$\int_a^b \Big(\int_0^t \langle \dot{\varrho}^\epsilon, e^\epsilon - Ew^\epsilon\rangle ds\Big) dt \to \int_a^b \Big(\int_0^t \langle \dot{\varrho}, e - Ew\rangle ds\Big) dt, \tag{5.33g}$$

thanks to (5.4e), (5.4h), (5.5i), (5.6e), (5.19), and (5.23). This implies that

$$\begin{aligned} &\int_a^b \Big(\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot{\varrho}_D, p\rangle ds\Big) dt \\ &\leq L = \int_a^b \Big(\int_0^t \langle \sigma, E\dot{w}\rangle ds + \mathcal{Q}_0(e_0) + \langle \varrho(t), e(t) - Ew(t)\rangle \\ &\qquad - \langle \varrho(0), e(0) - Ew(0)\rangle - \int_0^t \langle \dot{\varrho}, e - Ew\rangle ds\Big) dt. \end{aligned} \tag{5.34}$$

From the arbitrariness of a and b and from (5.29) for a.e. $t \in [0, T]$ we obtain (4.6), which is equivalent to (4.4).

It remains to prove claim (5.31). This will be done by adapting the proof of [6, Theorem 4.5]. Let $\varphi : [0, +\infty) \to \mathbb{R}$ be a nonnegative C^∞ function such that $\phi(s) = 0$ for $s \leq 1$ and $\phi(s) = 1$ for $s \geq 2$. For $\delta > 0$ we define $\psi_\delta(x) := \phi(\frac{1}{\delta}\mathrm{dist}(x, \Gamma_1))$ for $x \in \bar{\Omega}$.

Since H is positively 1-homogeneous and satisfies (3.34) we have that

$$\int_0^t \mathcal{H}(\psi_\delta \dot{p}^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon \psi_\delta \rangle ds \leq \int_0^t \mathcal{H}(\dot{p}^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon \rangle ds. \tag{5.35}$$

Integrating by parts with respect to time and using then (3.17), this is equivalent to

$$\begin{aligned}&\int_0^t \mathcal{H}(\psi_\delta \dot{p}^\epsilon)ds - \int_0^t \langle \dot{\varrho}^\epsilon, (e^\epsilon - Ew^\epsilon)\psi_\delta \rangle ds + \int_0^t \langle \dot{f}^\epsilon, \psi_\delta(u^\epsilon - w^\epsilon)\rangle ds \\ &\quad - \int_0^t \langle \dot{\varrho}^\epsilon, (u^\epsilon - w^\epsilon) \odot \nabla\psi_\delta \rangle ds - \langle [\varrho_D^\epsilon(t) \cdot p^\epsilon(t)], \psi_\delta \rangle + \langle [\varrho_D^\epsilon(0) \cdot p^\epsilon(0)], \psi_\delta \rangle \\ &\leq \int_0^t \mathcal{H}(\dot{p}^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon \rangle ds. \end{aligned} \tag{5.36}$$

The lower semicontinuity of the variation, together with (3.31) and (5.18), implies

$$\mathcal{D}_H(\psi_\delta p; 0, t) \leq \liminf_{\epsilon \to 0} \int_0^t \mathcal{H}(\psi_\delta \dot{p}^\epsilon(s))ds. \tag{5.37}$$

By (3.23), (5.4e), (5.5h), (5.5i),(5.6d), and (5.6e), using Lemma 4 we obtain

$$\langle [\varrho_D^\epsilon(0) \cdot p^\epsilon(0)], \psi_\delta \rangle \to \langle [\varrho_D(0) \cdot p(0)], \psi_\delta \rangle. \tag{5.38}$$

For what concerns the term $\langle [\varrho_D^\epsilon(t) \cdot p^\epsilon(t)], \psi_\delta \rangle$, we fix $0 \leq a < b \leq T$ and integrate on $[a, b]$ with respect to time. Using (3.17) we write

$$\begin{aligned}\int_a^b \langle [\varrho_D^\epsilon \cdot p^\epsilon], \psi_\delta \rangle ds &= -\int_a^b \langle \varrho^\epsilon \cdot (e^\epsilon - Ew^\epsilon), \psi_\delta \rangle ds \\ &+ \int_a^b \langle f^\epsilon, \psi_\delta(u^\epsilon - w^\epsilon)\rangle ds - \int_a^b \langle \varrho^\epsilon, (u^\epsilon - w^\epsilon) \odot \nabla\psi_\delta \rangle ds,\end{aligned}$$

where we have used the fact that ψ_δ is zero in a neighborhood of Γ_1. The last three terms pass to the limit thanks to (5.4e), (5.5h), (5.5i), (5.19), and (5.20). Therefore, using again (3.17) we obtain

$$\int_a^b \langle [\varrho_D^\epsilon \cdot p^\epsilon], \psi_\delta \rangle ds \to \int_a^b \langle [\varrho_D \cdot p], \psi_\delta \rangle ds. \tag{5.39}$$

We now integrate in (5.36) with respect to time. By (5.4e), (5.5h), (5.5i), (5.19), (5.20), and (5.37)-(5.39) we get

$$\begin{aligned}&\int_a^b \Big(\mathcal{D}_H(\psi_\delta p; 0, t) - \int_0^t \langle \dot{\varrho} \cdot (e - Ew), \psi_\delta \rangle ds + \int_0^t \langle \dot{f}, \psi_\delta(u - w)\rangle ds \\ &\quad - \int_0^t \langle \dot{\varrho}, (u - w) \odot \nabla\psi_\delta \rangle ds - \langle [\varrho_D(t) \cdot p(t)], \psi_\delta \rangle + \langle [\varrho_D(0) \cdot p(0)], \psi_\delta \rangle\Big)dt \\ &\leq \liminf_{\epsilon \to 0} \int_a^b \Big(\int_0^t \mathcal{H}(\dot{p}^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon \rangle ds\Big)dt.\end{aligned} \tag{5.40}$$

Using (3.17) we get

$$\int_a^b \Big(\mathcal{D}_H(\psi_\delta p; 0, t) - \langle [\varrho_D(t) \cdot p(t)], \psi_\delta \rangle + \langle [\varrho_D(0) \cdot p(0)], \psi_\delta \rangle + \int_0^t \langle [\dot{\varrho}_D \cdot p], \psi_\delta \rangle ds \Big) dt \\ \leq \liminf_{\epsilon \to 0} \int_a^b \Big(\int_0^t \mathcal{H}(\dot{p}^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot{p}^\epsilon \rangle ds \Big) dt.$$

Letting $\delta \to 0$ and using the semicontinuity of $\mathcal{D}_H$ we then obtain (5.31). This concludes the proof of (4.4) for a.e. $t \in [0, T]$.

Since (4.3) and (4.4) are satisfied for a.e. $t \in [0, T]$, and in particular for $t = 0$, we can apply Theorem 5. We obtain that $p : [0, T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n \times n})$ is absolutely continuous and we can redefine $u(t)$ and $e(t)$ on a set of times with measure zero so that $u : [0, T] \to BD(\Omega)$ and $e : [0, T] \to L^2(\Omega, \mathbb{M}_{\mathrm{sym}}^{n \times n})$ are absolutely continuous and the function (u, e, p, σ), with $\sigma(t) = A_0 e(t)$, is a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0.

From (5.34) and from the energy balance (4.4) it follows that the inequality in (5.32) is actually an equality and that the liminf is a limit. So, since

$$\int_a^b \Big(\frac{\epsilon^2}{2} \| \dot{u}^\epsilon(t) - \dot{w}^\epsilon(t) \|_{L^2}^2 + \epsilon \int_0^t \mathcal{Q}_1(\dot{e}_{A_1}^\epsilon) ds + \epsilon \int_0^t \| \dot{p}^\epsilon \|_{L^2}^2 ds \Big) dt \geq 0,$$

it follows that equality holds also in (5.30) and (5.31), and that the liminf is a limit also in these formulas. In particular

$$\int_0^T \mathcal{Q}_0(e^\epsilon(t)) dt \to \int_0^T \mathcal{Q}_0(e(t)) dt, \tag{5.41}$$

Since $e^\epsilon \rightharpoonup e$ weakly by (5.19), from (5.41) it follows that

$$e^\epsilon \to e \quad \text{strongly in } L^2([0, T]; L^2(\Omega; \mathbb{M}_{\mathrm{sym}}^{n \times n})), \tag{5.42}$$

which gives (5.9) for a suitable subsequence. From this and (5.18) we conclude that

$$E u^\epsilon(t) \rightharpoonup E u(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}_{\mathrm{sym}}^{n \times n}), \tag{5.43}$$

for a.e. $t \in [0, T]$.

Let us fix t for which (5.9) and (5.43) hold. Since $u^\epsilon(t) \in A(w^\epsilon(t))$, it follows from (3.1) that $u^\epsilon(t)$ is bounded in $BD(\Omega)$ uniformly with respect to ϵ. Up to a subsequence we may assume that $u^\epsilon(t)$ converges weakly* in $BD(\Omega)$ to a function v. By Lemma 3 it follows that $(v, e(t), p(t)) \in A_{BD}(w(t))$. Since we have also $(u(t), e(t), p(t)) \in A_{BD}(w(t))$, we deduce that $Ev = Eu(t)$ in Ω and $(w(t) - v) \odot \nu = (w(t) - u(t)) \odot \nu$ $\mathcal{H}^{n-1}$-almost everywhere on Γ_0. This implies that $v = u(t)$ $\mathcal{H}^{n-1}$ almost everywhere on Γ_0, and applying inequality (3.1) to $v - u(t)$ we obtain that $v = u(t)$ almost everywhere in Ω. This concludes the proof of (5.8).

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6 Appendix

This section contains the proof of two technical results concerning the convergence of suitable Riemann sums for functions with values in Banach spaces.

Lemma 6 *Let X be a Banach space, let $\phi \in W^{1,1}([0,T];X)$, let $S \subset (0,T]$ be a set of full measure containing T and let $\psi : S \to X'$ be a bounded weakly* continuous function. For every $k > 0$ let $\{t_i^k\}_{0\le i\le k}$ be a subset of $S \cup \{0\}$ such that $0 = t_0^k < t_1^k < \cdots < t_k^k = T$ and $\max_{i=1}^k |t_i^k - t_{i-1}^k| \to 0$ as $k \to +\infty$. Then*

$$\lim_{k\to\infty} \sum_{i=1}^{k} \langle \psi(t_i^k), \phi(t_i^k) - \phi(t_{i-1}^k)\rangle = \int_0^T \langle \psi(t), \dot\phi(t)\rangle dt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between X' and X.

Proof Let $\psi_k : [0,T] \to X'$ be the piecewise constant function defined by $\psi_k(t) = \psi(t_i^k)$ for $t_{i-1}^k < t \le t_i^k$. Then

$$\sum_{i=1}^{k} \langle \psi(t_i^k), \phi(t_i^k) - \phi(t_{i-1}^k)\rangle = \int_0^T \langle \psi_k(t), \dot\phi(t)\rangle dt.$$

Since $\psi_k(t) \rightharpoonup \psi(t)$ weakly* for every $t \in S$ we have $\langle \psi_k(t), \dot\phi(t)\rangle \to \langle \psi(t), \dot\phi(t)\rangle$ for a.e. $t \in [0,T]$. The conclusion follows from the Dominated Convergence Theorem.

The next lemma extends the previous result to the case of the duality product introduced in (3.13).

Lemma 7 *Let ϱ be the function introduced in the safe-load condition* (3.23)–(3.24) *and let $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n})$ be a bounded function. Assume that there exists a set $S \subset (0,T]$ of full measure containing T such that for every $t \in S$ the function p is continuous at t with respect to the strong topology of $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}_D^{n\times n})$ and $p(t) \in \Pi_{\Gamma_0}(\Omega)$. For every $k > 0$ let $\{t_i^k\}_{0\le i\le k}$ be a subset of $S \cup \{0\}$ such that $0 = t_0^k < t_1^k < \cdots < t_k^k = T$ and $\max_{i=1}^k |t_i^k - t_{i-1}^k| \to 0$ as $k \to +\infty$. Then*

$$\lim_{k\to\infty} \sum_{i=1}^{k} \langle \varrho_D(t_i^k) - \varrho_D(t_{i-1}^k), p(t_i^k)\rangle = \int_0^T \langle \dot\varrho_D(t), p(t)\rangle dt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product introduced in (3.13).

Proof Let $p_k : [0,T] \to \Pi_{\Gamma_0}(\Omega)$ be the piecewise constant function defined by $p_k(t) = p(t_i^k)$ for $t_{i-1}^k < t \le t_i^k$. Using (3.27) and (3.28) we obtain that

$$\sum_{i=1}^{k} \langle \varrho_D(t_i^k) - \varrho_D(t_{i-1}^k), p(t_i^k)\rangle = \int_0^T \langle \dot\varrho_D(t), p_k(t)\rangle dt =$$
$$= \int_0^T \langle \dot\varrho_D(t), p_k(t) - p(t)\rangle dt + \int_0^T \langle \dot\varrho_D(t), p(t)\rangle dt. \tag{6.1}$$

By (3.14) we have

$$\int_0^T |\langle \dot\varrho_D(t), p_k(t) - p(t)\rangle| dt \le \int_0^T \|\dot\varrho_D(t)\|_{L^\infty} \|p_k(t) - p(t)\|_{\mathcal{M}_b} dt$$

Since $\|p_k(t) - p(t)\|_{\mathcal{M}_b} \to 0$ for a.e. $t \in S$ by our continuity assumption and $t \mapsto \|\dot{\varrho}(t)\|_{L^\infty}$ belongs to $L^1([0, T])$ (see [6, Theorem 7.1]), we obtain

$$\lim_{k\to\infty} \int_0^T |\langle \dot{\varrho}_D(t), p_k(t) - p(t)\rangle| dt = 0 \tag{6.2}$$

by the Dominated Convergence Theorem. The conclusion follows from (6.1) and (6.2).

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