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Criteria for the Existence and Lower Bounds of Principal Eigenvalues of Time Periodic Nonlocal Dispersal Operators and Applications

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Abstract The current paper is concerned with the spectral theory, in particular, the principal eigenvalue theory, of nonlocal dispersal operators with time periodic dependence, and its applications. Nonlocal and random dispersal operators are widely used to model diffusion systems in applied sciences and share many properties. There are also some essential differences between nonlocal and random dispersal operators, for example, a smooth random dispersal operator always has a principal eigenvalue, but a smooth nonlocal dispersal operator may not have a principal eigenvalue. In this paper, we first establish criteria for the existence of principal eigenvalues of time periodic nonlocal dispersal operators with Dirichlet type, Neumann type, or periodic type boundary conditions. It is shown that a time periodic nonlocal dispersal operator possesses a principal eigenvalue provided that the nonlocal dispersal distance is sufficiently small, or the time average of the underlying media satisfies some vanishing condition with respect to the space variable at a maximum point or is nearly globally homogeneous with respect to the space variable. Next we obtain lower bounds of the principal spectrum points of time periodic nonlocal dispersal operators in terms of the corresponding time averaged problems. Finally we discuss the applications of the established principal eigenvalue theory to time periodic Fisher or KPP type equations with nonlocal dispersal and prove that such equations are of monostable feature, that is, if the trivial solution is linearly unstable, then there is a unique time periodic positive solution which is globally asymptotically stable.

Keywords Nonlocal dispersal · Random dispersal · Principal eigenvalue · Principal spectrum point · Vanishing condition · Lower bound · Monostable equation

Mathematics Subject Classifications 35K55 · 35K57 · 45C05 · 45M15 · 45M20 · 47G10 · 92D25

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1 Introduction

Both random dispersal evolution equations and nonlocal dispersal evolution equations are widely used to model diffusive systems in applied sciences. Classically, one assumes that the internal interaction of organisms in a diffusive system is infinitesimal or the internal dispersal is random, which leads to a diffusion operator, e.g., Δu as dispersal operator. Many diffusive systems in real world exhibit long range internal interaction or dispersal, which can be modeled by nonlocal dispersal operators such as $\int_{\mathbb{R}^N} \kappa(y-x)\big(u(t,y)-u(t,x)\big)dy$, here $\kappa(\cdot)$ is a convolution kernel supported on the ball centered at the origin with radius r, the interaction range. As a basic technical tool for the study of nonlinear evolution equations with random and nonlocal dispersals, it is of great importance to investigate aspects of spectral theory for random and nonlocal dispersal operators.

The current paper is devoted to the study of principal eigenvalues of the following three eigenvalue problems associated to nonlocal dispersal operators with time periodic dependence,

$$\begin{cases} -\partial_t u + \nu_1[\int_D \kappa(y-x)u(t,y)dy - u(t,x)] + a_1(t,x)u = \lambda u, & x \in \bar{D} \\ u(t+T,x) = u(t,x) \end{cases} \tag{1.1}$$

where $D \subset \mathbb{R}^N$ is a smooth bounded domain and $a_1(t,x)$ is a continuous function with $a_1(t+T,x) = a_1(t,x)$,

$$\begin{cases} -\partial_t u + \nu_2[\int_D \kappa(y-x)(u(t,y) - u(t,x))dy] + a_2(t,x)u = \lambda u, & x \in \bar{D} \\ u(t+T,x) = u(t,x) \end{cases} \tag{1.2}$$

where $D \subset \mathbb{R}^N$ is as in (1.1) and $a_2(t,x)$ is a continuous function with $a_2(t+T,x) = a_2(t,x)$, and

$$\begin{cases} -\partial_t u + \nu_3[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + a_3(t,x)u = \lambda u, & x \in \mathbb{R}^N \\ u(t+T,x) = u(t,x+p_j\mathbf{e_j}) = u(t,x), & x \in \mathbb{R}^N \end{cases} \tag{1.3}$$

where $p_j > 0$, $\mathbf{e_j} = (\delta_{j1}, \delta_{j2}, \cdots, \delta_{jN})$ with $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$, and $a_3(t,x)$ is a continuous function with $a_3(t+T,x) = a_3(t,x+p_j\mathbf{e_j}) = a_3(t,x)$, $j = 1, 2, \cdots, N$. $\kappa(\cdot)$ in (1.1)–(1.3) is a nonnegative C^1 function with compact support, $\kappa(0) > 0$, and $\int_{\mathbb{R}^N} \kappa(z)dz = 1$.

The eigenvalue problems (1.1), (1.2), and (1.3) can be viewed as the nonlocal dispersal counterparts of the following eigenvalue problems associated to random dispersal operators,

$$\begin{cases} -\partial_t u + \nu_1 \Delta u + a_1(t,x)u = \lambda u, & x \in D \\ u(t+T,x) = u(t,x), & x \in D \\ u = 0, & x \in \partial D, \end{cases} \tag{1.4}$$

$$\begin{cases} -\partial_t u + \nu_2 \Delta u + a_2(t,x)u = \lambda u, & x \in D \\ u(t+T,x) = u(t,x), & x \in D \\ \frac{\partial u}{\partial n} = 0, & x \in \partial D, \end{cases} \tag{1.5}$$

and

$$\begin{cases} -\partial_t u + \nu_3 \Delta u + a_3(t,x)u = \lambda u, & x \in \mathbb{R}^N \\ u(t+T,x) = u(t,x+p_j\mathbf{e_j}) = u(t,x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

respectively. It is in fact proved in [29] that the principal eigenvalues of (1.4), (1.5), and (1.6) can be approximated by the principal spectrum points of (1.1), (1.2), and (1.3) with properly rescaled kernels, respectively (see Definition 2.1 for the definition of principal spectrum points of (1.1), (1.2), and (1.3)). The reader is referred to [6,7], and [29] about the approximations of the initial value problems of the random dispersal operators associated to (1.4), (1.5), and (1.6) by the initial value problems of the nonlocal dispersal operators with properly rescaled kernels associated to (1.1), (1.2), and (1.3), respectively. We may hence say that (1.1), (1.2), and (1.3) are of the Dirichlet type boundary condition, Neumann type bounday condition, and periodic boundary condition, respectively.

The eigenvalue problems (1.4), (1.5), and (1.6), in particular, their associated principal eigenvalue problems, are well understood. For example, it is known that there is $\lambda_{R,1} \in \mathbb{R}$ such that $\lambda_{R,1}$ is an isolated algebraically simple eigenvalue of (1.4) with a positive eigenfunction, and for any other eigenvalues λ of (1.4), $\mathrm{Re}\lambda \le \lambda_{R,1}$ ($\lambda_{R,1}$ is called the *principal eigenvalue* of (1.4)) (see [17]).

The principal eigenvalue problem for time independent nonlocal dispersal operators with Dirichlet type, or Neumann type, or periodic boundary condition has been recently studied by many people (see [9,15,18,22,31,30], and references therein) and is quite well understood now. For example, among others, the following criteria for the existence of principal eigenvalues for nonlocal dispersal operators are established in [30] and [31] (see Definition 2.1 for the definition of principal eigenvalues of nonlocal dispersal operators),

(i) *If* $a_1(t,x) \equiv a_1(x)$ (*resp.* $a_2(t,x) \equiv a_2(x)$, $a_3(t,x) \equiv a_3(x)$) *is* C^N *and there is some* $x_0 \in \mathrm{Int}(D)$ (*resp.* $x_0 \in \mathrm{Int}(D)$, $x_0 \in \mathbb{R}^N$) *satisfying that* $a_1(x_0) = \max_{x\in\bar{D}} a_1(x)$ (*resp.* $-\nu_2 \int_D \kappa(y-x_0)dy + a_2(x_0) = \max_{x\in\bar{D}}(-\nu_2 \int_D \kappa(y-x)dy + a_2(x))$, $a_3(x_0) = \max_{x\in\mathbb{R}^N} a_3(x)$) *and the partial derivatives of* $a_1(x)$ (*resp.* $-\nu_2 \int_D \kappa(y-x)dy + a_2(x)$, $a_3(x)$) *up to order* $N-1$ *at* x_0 *are zero, then* (1.1) (*resp.* (1.2), (1.3)) *admits a principal eigenvalue*.

(ii) *If* $a_1(t,x) \equiv a_1(x)$ (*resp.* $a_2(t,x) \equiv a_2(x)$, $a_3(t,x) \equiv a_3(x)$) *and* $\max_{x\in\bar{D}} a_1(x) - \min_{x\in\bar{D}} a_1(x) < \nu_1 \inf_{x\in\bar{D}} \int_D \kappa(y-x)dy$ (*resp.* $\max_{x\in\bar{D}} a_2(x) - \min_{x\in\bar{D}} a_2(x) < \nu_2 \inf_{x\in\bar{D}} \int_D \kappa(y-x)dy$, $\max_{x\in\mathbb{R}^N} a_3(x) - \min_{x\in\mathbb{R}^N} a_3(x) < \nu_3$), *then* (1.1) (*resp.* (1.2), (1.3)) *admits a principal eigenvalue*.

(iii) *If* $a_1(t,x) \equiv a_1(x)$ (*resp.* $a_2(t,x) \equiv a_2(x)$, $a_3(t,x) \equiv a_3(x)$) *and* $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$ *for some* $\delta > 0$ *and* $\tilde{\kappa}(\cdot)$ *with* $\tilde{\kappa}(z) \ge 0$, $\mathrm{supp}(\tilde{k}) = B(0,1) := \{z \in \mathbb{R}^N \mid \|z\| < 1\}$, $\int_{\mathbb{R}^N} \tilde{\kappa}(z)dz = 1$, *and* $\tilde{\kappa}(\cdot)$ *being symmetric with respect to* 0 (*i.e.* $\tilde{k}(-z) = \tilde{k}(z)$), *then* (1.1) (*resp.* (1.2), (1.3)) *admits a principal eigenvalue provided that* $0 < \delta \ll 1$.

It should be pointed out that [9] contains some similar result to (i) and [22] contains some similar result to (iii) in the Dirichlet type boundary condition case. The work [31] includes (i)–(iii) in the periodic boundary condition case and is also concerned with the spatial spreading dynamics of nonlocal monostable equations in spatially periodic habitats. The work [30] includes (i)–(iii) in the Dirichlet type and Nuemann type boundary condition cases and also deals with the effects of spatial variations, dispersal rates, and dispersal distance on the principal eigenvalues. The conditions in (i), (ii), and (iii) can be viewed as the spatial inhomogeneity satisfying the vanishing condition (i.e. the partial derivatives up to order $N-1$ are zero) at some maximum point, the spatial inhomogeneity being nearly globally homogeneous, and the nonlocal dispersal distance being sufficiently small, respectively. It should also be pointed out that a nonlocal dispersal operator may not have a principal eigenvalue (see [31] for an example), which reveals some essential difference between nonlocal and random dispersal operators. Methologically, due to the lack of regularity and compactness of the solutions of nonlocal evolution equations, some difficulties, which do not arise in the study of spectral theory of random dispersal operators, arise in the study of spectral theory of nonlocal dispersal operators.

Regarding nonlocal dispersal operators with time periodic dependence, in [21], the authors studied the existence of principal eigenvalue of (1.1) in the case that $N = 1$. In [21] and [28], the influence of temporal variation on the principal eigenvalue of (1.1) (if exists) is investigated. In general, the understanding to the principal eigenvalue problems associated to (1.1), (1.2), and (1.3) is very little.

The first objective of the current paper is to develop criteria for the existence of principal eigenvalues of (1.1), (1.2), and (1.3) and to explore fundamental properties of principal eigenvalues of (1.1), (1.2), and (1.3). Many existing results on principal eigenvalues of time independent and some special time periodic nonlocal dispersal operators are extended to general time periodic nonlocal dispersal operators. To be a little more specific, let $\hat{a}_i(x)$ be the time average of $a_i(t,x)$ $(i = 1, 2, 3)$, that is,

$$\hat{a}_i(x) = \frac{1}{T}\int_0^T a_i(t,x)dt. \tag{1.7}$$

Let $s_1(a_1)$ (resp. $s_2(a_2),\ s_3(a_3)$) be the *principal spectrum point* (i.e. the largest real part of the spectrum) of the spectral problem (1.1) (resp. (1.2), (1.3)) (see Definition 2.1 for detail). $s_1(a_1)$ (resp $s_2(a_2),\ s_3(a_3)$) is called the *principal eigenvalue* of (1.1) (resp. (1.2), (1.3)) if it is an isolated eigenvalue with a positive eigenfunction (see Definition 2.1 again for detail). Note that $s_i(a_i)$ $(i = 1, 2, 3)$ may not be an eigenvalue of its corresponding eigenvalue problem. Among others, the following criterion is established in this paper, which extends (i) in the above for time independent nonlocal dispersal operators to time periodic ones,

- *If* $\hat{a}_1(x)$ *(resp.* $-\nu_2\int_D \kappa(y-x)dy + \hat{a}_2(x),\ \hat{a}_3(x)$*) is in* C^N *in* x *and there is some* $x_0 \in \mathrm{Int}(D)$ *(resp.* $x_0 \in \mathrm{Int}(D)$, $x_0 \in \mathbb{R}^N$*) such that* $\hat{a}_1(x_0) = \max_{x\in\bar{D}} \hat{a}_1(x)$ *(resp.* $-\nu_2\int_D \kappa(y-x_0)dy + \hat{a}_2(x_0) = \max_{x\in\bar{D}}\big(-\int_D \kappa(y-x)dy + \hat{a}_2(x)\big)$, $\hat{a}_3(x_0) = \max_{x\in\mathbb{R}^N}\hat{a}_3(x)$*) and the partial derivatives of* $\hat{a}_1(x)$ *(resp.* $-\nu_2\int_D \kappa(y-x)dy + \hat{a}_2(x)$, $\hat{a}_3(x)$*) up to order* $N-1$ *at* x_0 *are zero, then* (1.1) *(resp.* (1.2), (1.3)*) admits a principal eigenvalue, i.e.* $s_1(a_1)$ *(resp.* $s_2(a_2),\ s_3(a_3)$*) is the principal eigenvalue of* (1.1) *(resp.* (1.2), (1.3)*)* (see Theorem B(2) in Sect. 2).

We obtain the following result for the lower bound of $s_i(a_i)$, which extends [21, Theorem 4.1] for the lower bound of $s_1(a_1)$ in the case that $s_1(a_1)$ is the principal eigenvalue of (1.1).

- *For given* $1 \le i \le 3$, $s_i(a_i) \ge s_i(\hat{a}_i)$*. Moreover, if* $s_i(a_i)$ *is the principal eigenvalue of (1.i), then* $s_i(a_i) = s_i(\hat{a}_i)$ *iff* $a_i(t,x) - \hat{a}_i(x)$ *is independent of* x*, that is,* $a_i(t,x) = \hat{a}_i(x) + \tilde{a}_i(t)$ *for some time periodic function* $\tilde{a}_i(t)$ *with* $\int_0^T \tilde{a}_i(t)dt = 0$ *(see Theorem C in Sect. 2).*

The reader is referred to Theorems A–C in Sect. 2 for the principal eigenvalue theories established in this paper for general time periodic nonlocal dispersal operators.

The second objective of the current paper is to consider applications of the established principal eigenvalue theories to the following time periodic KPP type or Fisher type equations with nonlocal dispersal,

$$\partial_t u = \nu_1\left[\int_D \kappa(y-x)u(t,y)dy - u(t,x)\right] + uf_1(t,x,u), \quad x \in \bar{D}, \tag{1.8}$$

$$\partial_t u = \nu_2\left[\int_D \kappa(y-x)(u(t,y) - u(t,x))dy\right] + uf_2(t,x,u), \quad x \in \bar{D}, \tag{1.9}$$

and

$$\begin{cases} \partial_t u = \nu_3 \left[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)\right] + uf_3(t,x,u), & x \in \mathbb{R}^N \\ u(t, x + p_j \mathbf{e_j}) = u(t,x), & x \in \mathbb{R}^N, \end{cases} \tag{1.10}$$

where $f_i(t,x)$ $(i = 1,2,3)$ are C^1 functions, $f_i(t+T,x,u) = f_i(t,x,u)$ $(i = 1,2,3)$, $f_3(t, x + p_j\mathbf{e_j}, u) = f_3(t,x,u)$ $(j = 1,2,\cdots,N)$, and $f_i(t,x,u) < 0$ for $u \gg 1$ and $\partial_u f_i(t,x,u) < 0$ for $u \ge 0$ $(i = 1,2,3)$.

Equations (1.8), (1.9), and (1.10) are the nonlocal counterparts of the following reaction diffusion equations,

$$\begin{cases} \partial_t u = \nu_1 \Delta u + uf_1(t,x,u), & x \in D \\ u(t,x) = 0, & x \in \partial D, \end{cases} \tag{1.11}$$

$$\begin{cases} \partial_t u = \nu_2 \Delta u + uf_2(t,x,u), & x \in D \\ \frac{\partial u}{\partial n} = 0, & x \in \partial D, \end{cases} \tag{1.12}$$

and

$$\begin{cases} \partial_t u = \nu_3 \Delta u + uf_3(t,x,u), & x \in \mathbb{R}^N \\ u(t, x + p_j\mathbf{e_j}) = u(t,x), & x \in \mathbb{R}^N, \end{cases} \tag{1.13}$$

respectively (see [29] for the approximations of the solutions of (1.11), (1.12), and (1.13) by the solutions of (1.8), (1.9), and (1.10) with properly rescaled kernels, respectively).

Equations (1.8)–(1.10) and (1.11)–(1.13) are widely used to model population dynamics of species exhibiting nonlocal internal interactions and random internal interactions, respectively. Thanks to the pioneering works of Fisher [14] and Kolmogorov et al. [23] on the following special case of (1.13),

$$\partial_t u = u_{xx} + u(1-u), \quad x \in \mathbb{R},$$

(1.8)–(1.10) and (1.11)–(1.13) are referred to as Fisher type or KPP type equations.

One of the central problems for (1.8)–(1.10) and (1.11)–(1.13) is about the existence, uniqueness, and stability of positive time periodic solutions. This problem has been extensively studied and is well understood for (1.11)–(1.13). For example, it is known that (1.11) exhibits the following *monostable feature*: if the trivial solution $u \equiv 0$ is a linearly unstable solution of (1.11), then (1.11) has a unique stable time periodic positive solution. Again, some difficulties, which do not arise in the study of (1.11)–(1.13), aries in the study of (1.8)–(1.10) due to the lack of compactness and regularities of the solutions of nonlocal dispersal evolution equations. In [33], the authors proved that time independent KPP equations with nonlocal dispersal also exhibit monostable feature (see also [2,9] for the study of positive stationary solutions of time independent KPP equations with nonlocal dispersal). But it is hardly studied whether a general time periodic KPP equation with nonlocal dispersal is of the monostable feature. In this paper, by applying the established principal eigenvalue theories for time periodic nonlocal dispersal operators, we prove

- *A time periodic KPP equations with nonlocal dispersal is of the monostable feature, that is, if $u \equiv 0$ is a linearly unstable solution of a time periodic KPP equation with nonlocal dispersal, then the equation has a unique stable time periodic positive solution* (see Theorem E in Sect. 2).

Nonlocal evolution equations have been attracting more and more attention due to the presence of nonlocal interaction in many diffusive systems in applied sciences. The reader is referred to [5,8,10,11,13,15,16,19,20,22,24–26,28,32], etc., for the study of various aspects of evolution equations with nonlocal dispersal. The reader is also referred to [1,3,34], etc. for the study of evolution equations with nonlocal reaction.

The rest of the paper is organized as follows. In Sect. 2, we introduce standing notations and definitions and state the main results of the paper. We present in Sect. 3 some preliminary materials to be used in the proofs of the main results. The main results are proved in Sects. 4 and 5.

2 Notations, Definitions, and Main Results

In this section, we first introduce the standing notations to be used throughout the paper and the definitions of principal spectrum points and principal eigenvalues of (1.1), (1.2), and (1.3). We then state the main results of the paper.

Let

$$\mathcal{X}_1 = \mathcal{X}_2 = \big\{u \in C(\mathbb{R} \times \bar{D}, \mathbb{R}) \,|\, u(t+T, x) = u(t, x)\big\}$$

with norm $\|u\|_{\mathcal{X}_i} = \sup_{t\in\mathbb{R}, x\in\bar{D}} |u(t, x)|$ $(i = 1, 2)$,

$$\mathcal{X}_3 = \Big\{u \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) \,|\, u(t+T, x) = u(t, x + p_i\mathbf{e_i}) = u(t, x)\Big\}$$

with norm $\|u\|_{\mathcal{X}_3} = \sup_{t\in\mathbb{R}, x\in\mathbb{R}^N} |u(t, x)|$, and

$$\mathcal{X}_i^+ = \{u \in \mathcal{X}_i \,|\, u \geq 0\}$$

$(i = 1, 2, 3)$. For given $a_i \in \mathcal{X}_i$, let $L_i(a_i) : \mathcal{D}(L_i(ai)) \subset \mathcal{X}_i \to \mathcal{X}_i$ be defined as follows,

$$(L_1(a_1)u)(t, x) = -\partial_t u(t, x) + \nu_1 \left[\int_D \kappa(y - x)u(t, y)dy - u(t, x)\right] + a_1(t, x)u(t, x),$$

$$(L_2(a_2)u)(t, x) = -\partial_t u(t, x) + \nu_2 \left[\int_D \kappa(y - x)(u(t, y) - u(t, x))dy\right] + a_2(t, x)u(t, x),$$

and

$$(L_3(a_3)u)(t, x) = -\partial_t u(t, x) + \nu_3 \left[\int_{\mathbb{R}^N} \kappa(y - x)u(t, y)dy - u(t, x)\right] + a_3(t, x)u(t, x).$$

Definition 2.1 Let

$$s_i(a_i) = \sup\{\text{Re}\lambda \,|\, \lambda \in \sigma(L_i(a_i))\}$$

for $i = 1, 2, 3$. $s_i(a_i)$ is called the principal spectrum point of $L_i(a_i)$ $(i = 1, 2, 3)$. If $s_i(a_i)$ is an isolated eigenvalue of $L_i(a_i)$ with a positive eigenfunction ϕ (i.e. $\phi \in \mathcal{X}_i^+$), then $s_i(a_i)$ is called the principal eigenvalue of $L_i(a_i)$ or it is said that $L_i(a_i)$ has a principal eigenvalue $(i = 1, 2, 3)$.

Remark 2.1 If $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$, then it is geometrically simple (see Proposition 3.9).

For given $1 \le i \le 3$ and $a_i \in \mathcal{X}_i$, let $\hat{a}_i$ be as in (1.7). Let

$$b_i(x) = \begin{cases} -\nu_i & \text{for } i = 1, 3 \\ -\nu_2 \int_D \kappa(y-x)dy & \text{for } i = 2. \end{cases} \tag{2.1}$$

Let

$$D_i = \begin{cases} \bar{D} & \text{for } i = 1, 2 \\ [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N] & \text{for } i = 3. \end{cases} \tag{2.2}$$

Our main results on the principal spectrum points and principal eigenvalues of nonlocal dispersal operators can then be stated as follows.

Theorem A (Necessary and sufficient condition)
Let $1 \le i \le 3$ be given. If $\lambda \in \mathbb{R}$ is an eigenvalue of $L_i(a_i)$ with a positive eigenfunction $\phi(t, x)$, then $\lambda = s_i(a_i) > \max_{x \in D_i}(b_i(x) + \hat{a}_i(x))$ and λ is the principal eigenvalue of $L_i(a_i)$. Conversely, if $s_i(a_i) > \max_{x \in D_i}(b_i(x)+\hat{a}_i(x))$, then $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$ (hence $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$ iff $s_i(a_i) > \max_{x \in D_i}(b_i(x) + \hat{a}_i(x))$).

Theorem B (Sufficient conditions)
Let $1 \le i \le 3$ be given.

(1) *The principal eigenvalue of $L_i(a_i)$ exists if $b_i(x) + \hat{a}_i(\cdot)$ is C^N, there is some $x_0 \in \text{Int}(D_i)$ in the case $i = 1, 2$ and $x_0 \in D_i$ in the case $i = 3$ satisfying that $b_i(x_0) + \hat{a}_i(x_0) = \max_{x \in D_i}(b_i(x) + \hat{a}_i(x))$, and the partial derivatives of $b_i(x) + \hat{a}_i(x)$ up to order $N-1$ at x_0 are zero.*
(2) *The principal eigenvalue of $L_i(a_i)$ exists if $\max_{x \in D_i} \hat{a}_i(x) - \min_{x \in D_i} \hat{a}_i(x) < \nu_i \inf_{x \in D_i} \int_{D_i} \kappa(y-x)dy$ in the case $i = 1, 2$ and $\max_{x \in D_i} \hat{a}_i(x) - \min_{x \in D_i} \hat{a}_i(x) < \nu_i$ in the case $i = 3$.*
(3) *Suppose that $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$ for some $\delta > 0$ and $\tilde{\kappa}(\cdot)$ with $\tilde{\kappa}(z) \ge 0$, $\text{supp}(\tilde{\kappa}) = B(0, 1) := \{z \in \mathbb{R}^N \mid \|z\| < 1\}$, $\int_{\mathbb{R}^N} \tilde{\kappa}(z)dz = 1$, and $\tilde{\kappa}(\cdot)$ being symmetric with respect to 0. Then the principal eigenvalue of $L_i(a_i)$ exists for $0 < \delta \ll 1$.*

Theorem C (Influence of temporal variation)
For given $1 \le i \le 3$, $s_i(a_i) \ge s_i(\hat{a}_i) \ge \max_{x \in D_i}(b_i(x) + \hat{a}_i(x))$. Moreover, if $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$, then $s_i(a_i) = s_i(\hat{a}_i)$ if and only if $a_i(t, x) - \hat{a}_i(x)$ is independent of x.

Remark 2.2 If $a_i(t, x) - \hat{a}_i(x)$ is independent of x, then $s_i(a_i) = s_i(\hat{a}_i)$ no matter $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$ or not, which follows from the proof of Theorem C in Sect. 4. Conversely, if $s_i(a_i) = s_i(\hat{a}_i)$ and $s_i(a_i)$ is not the principal eigenvalue of $L_i(a_i)$, then it may not be true that $a_i(t, x) - \hat{a}_i(x)$ is independent of x (see Example 4.1 in Sect. 4).

Corollary D *If $s_i(\hat{a}_i)$ is the principal eigenvalue of $L_i(\hat{a}_i)$, then $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$.*

Proof Assume that $s_i(\hat{a}_i)$ is the principal eigenvalue of $L_i(\hat{a}_i)$. Then by Theorem A,

$$s_i(\hat{a}_i) > \max_{x \in D_i} \left(b_i(x) + \hat{a}_i(x)\right).$$

This together with Theorem C implies that

$$s_i(a_i) > \max_{x \in D_i} \left(b_i(x) + \hat{a}_i(x)\right).$$

Then by Theorem A again, $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$. □

Observe that when $a_i(t,x) \equiv a_i(x)$ $(i = 1, 2, 3)$, Theorems A and B recover the existing results for time independent nonlocal dispersal operators (see [30,31], and references therein). The conditions in Theorem B (1)–(3) can be viewed as the time average of the underlying time periodic medium satisfying the vanishing condition with respect to the space variable (i.e. the partial derivatives of $b_i(x) + \hat{a}_i(x)$ up to order $N-1$ are zero) at some maximum point of $b_i(x) + \hat{a}_i(x)$, the time average of the underlying time periodic medium is nearly globally spatially homogeneous, and the nonlocal dispersal distance being sufficiently small, respectively. Theorem B (1) extends a result in [21] for the case $i = 1$ and $N = 1$ to time periodic nonlocal dispersal operators in higher space dimension domains. In the case $i = 1$ and both $s_i(a_i)$ and $s_i(\hat{a}_i)$ are eigenvalues of $L_i(a_i)$ and $L_i(\hat{a}_i)$, it is shown in [21] that $s_i(a_i) \ge s_i(\hat{a}_i)$. Theorem C extends this result to general time periodic nonlocal dispersal operators and shows that temporal variation does not reduce the principal spectrum point of a general time periodic nonlocal dispersal operator.

Theorems A–C and Corollary D establish some fundamental principal eigenvalue theory for general time periodic nonlocal dispersal operators and provide a basic tool for the study of nonlinear evolution equations with nonlocal dispersal. In the following, we consider their applications to the study of the asymptotic dynamics of (1.8)–(1.10).

Let

$$X_1 = X_2 = \{u \in C(\bar{D}, \mathbb{R})\}$$

with norm $\|u\|_{X_i} = \sup_{x\in\bar{D}} |u(x)|$ $(i = 1, 2)$,

$$X_3 = \left\{u \in C\left(\mathbb{R}^N, \mathbb{R}\right) \mid u(x + p_j \mathbf{e_j}) = u(x)\right\}$$

with norm $\|u\|_{X_3} = \sup_{x\in\mathbb{R}^N} |u(x)|$, and

$$X_i^+ = \{u \in X_i \mid u \ge 0\}, \quad i = 1, 2, 3,$$

$$X_i^{++} = \begin{cases} \{u \in X_i^+ \mid u(x) > 0 \quad \forall\, x \in \bar{D}\}, & i = 1, 2 \\ \{u \in X_i^+ \mid u(x) > 0 \quad \forall x \in \mathbb{R}^N\}, & i = 3. \end{cases}$$

By general semigroup theory (see [27]), for any $s \in \mathbb{R}$ and $u_0 \in X_1$ (resp. $u_0 \in X_2$, $u_0 \in X_3$), (1.8) (resp. (1.9), (1.10)) has a unique (local) solution $u_1(t, x; s, u_0)$ (resp. $u_2(t, x; s, u_0)$,
$u_3(t, x; s, u_0)$) with $u_1(s, x; s, u_0) = u_0(x)$ (reps. $u_2(s, x; s, u_0) = u_0(x)$, $u_3(s, x; s, u_0) = u_0(x)$). Moreover, if $u_0 \in X_i^+$, then $u_i(t, x; s, u_0)$ exists and $u_i(t, \cdot; s, u_0) \in X_i^+$ for all $t \ge s$ $(i = 1, 2, 3)$ (see Proposition 3.1).

Theorem E (Existence, uniqueness, and stability of time periodic positive solutions)
Let $a_i(t,x) = f_i(t,x,0)$ $(i = 1, 2, 3)$. If $s_1(a_1) > 0$ (resp. $s_2(a_2) > 0$, $s_3(a_3) > 0$), then (1.8) (resp. (1.9), (1.10)) has a unique time periodic solution $u_1^*(t,\cdot) \in X_1^{++}$ (resp. $u_2^*(t,\cdot) \in X_2^{++}$, $u_3^*(t,\cdot) \in X_3^{++}$). Moreover, $u_i^*(\cdot,\cdot)$ is locally stable and is also globally asymptotically stable in the sense that for any $u_0 \in X_i^+ \setminus \{0\}$,

$$\|u_i(t, \cdot; 0, u_0) - u_i^*(t, \cdot)\|_{X_i} \to 0$$

as $t \to \infty$ $(i = 1, 2, 3)$.

Corollary F *Let $a_i(t,x) = f_i(t,x,0)$ $(i = 1, 2, 3)$. If $s_1(\hat{a}_1) > 0$ (resp. $s_2(\hat{a}_2) > 0$, $s_3(\hat{a}_3) > 0$), then (1.8) (resp. (1.9), (1.10)) has a unique time periodic solution $u_1^*(t,\cdot) \in X_1^{++}$ (resp. $u_2^*(t,\cdot) \in X_2^{++}$, $u_3^*(t,\cdot) \in X_3^{++}$). Moreover, $u_i^*(\cdot,\cdot)$ is locally stable and is also globally asymptotically stable in the sense that for any $u_0 \in X_i^+ \setminus \{0\}$,*

$$\|u_i(t,\cdot;0,u_0)-u_i^*(t,\cdot)\|_{X_i}\to 0$$

as $t\to\infty$ $(i=1,2,3)$.

Proof Assume $s_1(\hat{a}_1)>0$ (resp. $s_2(\hat{a}_2)>0,\ s_3(\hat{a}_3)>0$). By Theorem C, $s_1(a_1)>0$ (resp. $s_2(a_2)>0,\ s_3(a_3)>0$). The corollary then follows from Theorem E. □

3 Preliminary

In this section, we present some basic properties for solutions of nonlocal evolution equations and some basic properties of principal spectrum points of nonlocal dispersal operators.

Throughout this section, i denotes any integer with $1\le i\le 3$, unless specified otherwise and $\mathcal{X}_i,\ \mathcal{X}_i^+$, and $X_i,\ X_i^+,\ X_i^{++}$ are as in Sect. 2. D_i is as in (2.2). For $u_1,u_2\in\mathcal{X}_i$, we define

$$u_1\le u_2\ (u_1\ge u_2)\quad \text{if } u_2-u_1\in\mathcal{X}_i^+(u_1-u_2\in\mathcal{X}_i^+).$$

For $u_1,u_2\in X_i$, we define

$$u_1\le u_2\ (u_1\ge u_2)\quad \text{if } u_2-u_1\in X_i^+(u_1-u_2\in X_i^+),$$

and

$$u_1\ll u_2\ (u_1\gg u_2)\quad \text{if } u_2-u_1\in X_i^{++}(u_1-u_2\in X_i^{++}).$$

3.1 Basic Properties for Solutions of Nonlocal Evolution Equations

In this subsection, we present some basic properties for solutions of (1.8)–(1.10) and linear nonlocal evolution equations,

$$\partial_t u=\nu_1\left[\int_D \kappa(y-x)u(t,y)dy-u(t,x)\right]+a_1(t,x)u,\quad x\in\bar{D},\tag{3.1}$$

$$\partial_t u=\nu_2\left[\int_D \kappa(y-x)(u(t,y)-u(t,x))dy\right]+a_2(t,x)u,\quad x\in\bar{D},\tag{3.2}$$

and

$$\partial_t u=\nu_3\left[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy-u(t,x)\right]+a_3(t,x)u,\quad x\in\mathbb{R}^N,\tag{3.3}$$

where $a_i\in\mathcal{X}_i$ $(i=1,2,3)$.

As in Sect. 2, $u_1(t,x;s,u_0)$ (resp. $u_2(t,x;s,u_0),\ u_3(t,x;s,u_0)$) denotes the solution of (1.8) (resp. (1.9), (1.10)) with $u_1(s,\cdot;s,u_0)=u_0(\cdot)\in X_1$ (resp. $u_2(s,\cdot;s,u_0)=u_0(\cdot)\in X_2,\ u_3(s,\cdot;s,u_0)=u_0(\cdot)\in X_3$). By general semigroup theory, (3.1) (resp. (3.2), (3.3)) generates evolution families $\{\Phi_1(t,s;a_1)\}$ (resp. $\{\Phi_2(t,s;a_2)\},\ \{\Phi_3(t,s;a_3)\}$) on X_1 (resp. $X_2,\ X_3$), that is, for any $u_0\in X_1$ (resp. $u_0\in X_2,\ u_0\in X_3$), $u(t,x;s,u_0):=(\Phi_1(t,s;a_1)u_0)(x)$ (resp. $u(t,x;s,u_0):=(\Phi_2(t,s;a_2)u_0)(x),\ u(t,x;s,u_0):=(\Phi_3(t,s;a_3)u_0)(x)$) is the solution of (3.1) (resp. (3.2), (3.3)) with $u(s,x;s,u_0)=u_0(x)$.

Definition 3.1 A continuous function $u(t,x)$ on $[0,\tau)\times\bar{D}$ is called a super-solution (or sub-solution) of (1.8) if for any $x\in\bar{D}$, u(t,x) is differentiable on $[0,\tau)$ and satisfies that for each $x\in\bar{D}$,

$$\frac{\partial u}{\partial t}\geq (or\leq)\nu_1\left[\int_D \kappa(y-x)u(t,y)dy-u(t,x)\right]+u(t,x)f_1(t,x,u)$$

for $t\in[0,\tau)$.

Super-solutions and sub-solutions of (1.9), (1.10), and (3.1)–(3.3) are defined in an analogous way.

Proposition 3.1 (Comparison principle)

(1) *If* $u^1(t,x)$ *and* $u^2(t,x)$ *are bounded sub- and super-solution of* (3.1) *(resp.* (3.2)*,* (3.3)*) on* $[0,\tau)$*, respectively, and* $u^1(0,\cdot)\leq u^2(0,\cdot)$*, then* $u^1(t,\cdot)\leq u^2(t,\cdot)$ for $t\in[0,\tau)$.
(2) *If* $u^1(t,x)$ *and* $u^2(t,x)$ *are bounded sub- and super-solutions of* (1.8) *(resp.* (1.9)*,* (1.10)*) on* $[0,\tau)$*, respectively, and* $u^1(0,\cdot)\leq u^2(0,\cdot)$*, then* $u^1(t,\cdot)\leq u^2(t,\cdot)$ *for* $t\in[0,\tau)$.
(3) *Given* $1\leq i\leq 3$*, for every* $u_0\in X_i^+$*,* $u_i(t,x;s,u_0)$ *exists for all* $t\geq s$.

Proof It follows from the arguments in [31, Proposition 2.1]. □

Proposition 3.2 (Strong monotonicity) *Let* $1\leq i\leq 3$ *be given.*

(1) *If* $u^1,u^2\in X_i$*,* $u^1\leq u^2$ *and* $u^1\neq u^2$*, then* $\Phi_i(t,s;a_i)u^1\ll\Phi_i(t,s;a_i)u^2$ *for all* $t>s$.
(2) *If* $u^1,u^2\in X_i$*,* $u^1\leq u^2$ *and* $u^1\neq u^2$*, then* $u_i(t,\cdot;s,u^1)\ll u_i(t,\cdot;s,u^2)$ *for every* $t>s$ *at which both* $u_i(t,\cdot;s,u^1)$ *and* $u_i(t,\cdot;s,u^2)$ *exist.*

Proof It follows from the arguments in [31, Proposition 2.2]. □

For simplicity in notation, put

$$\Phi_i(T;a_i)=\Phi_i(T,0;a_i),\quad i=1,2,3.$$

Let $r(\Phi_i(T;a_i))$ be the spectral radius of $\Phi_i(T;a_i)$ $(i=1,2,3)$.

Proposition 3.3 *For given* $1\leq i\leq 3$,

$$\frac{\ln r(\Phi_i(T;a_i))}{T}=\limsup_{t-s\to\infty}\frac{\ln\|\Phi_i(t,s;a_i)\|}{t-s}.$$

Proof First, by $(\Phi_i(T;a_i))^n=\Phi_i(nT,0;a_i)$, it is clear that

$$\frac{\ln r(\Phi_i(T;a_i))}{T}=\frac{\ln\left\{\lim_{n\to\infty}\left(\|(\Phi_i(T;a_i))^n\|\right)^{1/n}\right\}}{T}\leq\limsup_{t-s\to\infty}\frac{\ln\|\Phi_i(t,s;a_i)\|}{t-s}.$$

Next, for any $\epsilon>0$, there is $K\geq 1$ such that

$$\left\|(\Phi_i(T;a_i))^n\right\|=\|\Phi_i(nT,0;a_i)\|\leq(r(\Phi_i(T;a_i))+\epsilon)^n\quad\forall\, n\geq K.$$

Note that there is $M>0$ such that

$$\|\Phi_i(t,s;a_i)\|\leq M\quad\forall t>s,\ t-s<T.$$

For any $s < t$ with $t - s \geq (K+2)T$, let $n_1, n_2 \in \mathbb{Z}$ be such that $0 \leq n_1 T - s < T$ and $0 \leq t - n_2 T < T$. Then

$$n_2 - n_1 \geq K$$

and

$$\begin{aligned}\|\Phi_i(t,s;a_i)\| &= \|\Phi_i(t,n_2T;a_i) \circ \Phi_i(n_2T,n_1T;a_i) \circ \Phi_i(n_1T,s;a_i)\| \\ &\leq \|\Phi_i(t,n_2T;a_i)\| \cdot \|\Phi_i((n_2-n_1)T,0;a_i)\| \cdot \|\Phi_i(n_1T,s;a_i)\| \\ &\leq M^2(r(\Phi_i(T;a_i))+\epsilon)^{n_2-n_1}.\end{aligned}$$

This implies that

$$\frac{\ln\|\Phi_i(t,s;a_i)\|}{t-s} \leq \frac{\ln M^2 + (n_2-n_1)\ln(r(\Phi_i(T;a_i))+\epsilon)}{(n_2-n_1)T}$$

and hence

$$\limsup_{t-s\to\infty} \frac{\ln\|\Phi_i(t,s;a_i)\|}{t-s} \leq \frac{\ln(r(\Phi_i(T;a_i))+\epsilon)}{T}.$$

Let $\epsilon \to 0$, we have

$$\limsup_{t-s\to\infty} \frac{\ln\|\Phi_i(t,s;a_i)\|}{t-s} \leq \frac{\ln r(\Phi_i(T;a_i))}{T}.$$

□

3.2 Basic Properties of Principal Spectrum Points

In this subsection, we present some basic properties of principal spectrum points of nonlocal dispersal operators.

First of all, let $K_i : \mathcal{X}_i \to \mathcal{X}_i$ and $H_i(a_i) : \mathcal{D}(H_i(a_i)) \subset \mathcal{X}_i \to \mathcal{X}_i$ be as follows,

$$\begin{aligned}(K_1u)(t,x) &= (K_2u)(t,x) = \int_D \kappa(y-x)u(t,y)dy, \\ (K_3u)(t,x) &= \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy, \\ (H_1(a_1)u)(t,x) &= -\partial_t u(t,x) - \nu_1 u(t,x) + a_1(t,x)u(t,x), \\ (H_2(a_2)u)(t,x) &= -\partial_t u(t,x) - \nu_2 \int_D \kappa(y-x)dy u(t,x) + a_2(t,x)u(t,x),\end{aligned}$$

and

$$(H_3(a_3)u)(t,x) = -\partial_t u(t,x) - \nu_3 u(t,x) + a_3(t,x)u(t,x).$$

Then

$$L_i(a_i)u = (\nu_i K_i + H_i(a_i))u, \quad i = 1,2,3.$$

We denote I as an identity map from $\mathcal{X}_i$ to $\mathcal{X}_i$ and may write $\alpha I u$ as αu and $\alpha I - H_i(a_i)$ as $\alpha - H_i(a_i)$, etc.. If no confusion occurs, we may write $L_i(a_i)$ and $H_i(a_i)$ as L_i and H_i, respectively.

Observe that if $\alpha \in \mathbb{C}$ is such that $(\alpha - H_i)^{-1}$ exists, then

$$(\nu_i K_i + H_i)u = \alpha u$$

has nontrivial solutions in $\mathcal{X}_i \oplus i\mathcal{X}_i$ is equivalent to

$$\nu_i K_i(\alpha - H_i)^{-1} v = v$$

has nontrivial solutions in $\mathcal{X}_i \oplus i\mathcal{X}_i$, where

$$\mathcal{X}_i \oplus i\mathcal{X}_i = \{u + iv \mid u, v \in \mathcal{X}_i\}.$$

Let

$$\lambda_i(x) = b_i(x) + \hat{a}_i(x), \tag{3.4}$$

where $\hat{a}_i(\cdot)$ and $b_i(\cdot)$ are as in (1.7) and (2.1), respectively, and

$$\lambda_{i,\max} = \max_{x\in D_i} \lambda_i(x), \quad \lambda_{i,\min} = \min_{x\in D_i} \lambda_i(x) \tag{3.5}$$

for $i = 1, 2, 3$.

Proposition 3.4 *Let* $1 \le i \le 3$ *be given.* $[\lambda_{i,\min}, \lambda_{i,\max}] \subset \sigma(H_i)$.

Proof It follows from the arguments in [21, Lemma 3.7]. For the reader's convenience, we provide a proof in the following.

Fix any $x_0 \in D_i$. By Floquet theory for time periodic ordinary differential equations, the equation

$$\dot{\phi} = b_i(x_0)\phi + a_i(t, x_0)\phi - \lambda_i(x_0)\phi \tag{3.6}$$

has a nontivial solution $\phi^*(t)$ with $\phi^*(t + T) = \phi^*(t)$. Similarly, the equation

$$\dot{\psi} = -b_i(x_0)\psi - a_i(t, x_0)\psi + \lambda_i(x_0)\psi \tag{3.7}$$

has a nontivial solution $\psi^*(t)$ with $\psi^*(t + T) = \psi^*(t)$.

Assume that $\lambda_i(x_0) \in \rho(H_i)$. Then for any $v \in \mathcal{X}_i$ with $v(t, x) \equiv v(t)$, there is a unique $u(\cdot, \cdot; v) \in \mathcal{X}_i$ such that

$$\partial_t u(t, x; v) = b_i(x)u(t, x; v) + a_i(t, x)u(t, x; v) - \lambda_i(x_0)u(t, x; v) + v(t) \tag{3.8}$$

This implies that

$$\partial_t u(t, x_0; \psi^*) = b_i(x_0)u(t, x_0; \psi^*) + a_i(t, x_0)u(t, x_0; \psi^*) - \lambda_i(x_0)u(t, x_0; \psi^*) + \psi^*(t). \tag{3.9}$$

Put

$$\tilde{\phi}^*(t) = u(t, x_0; \psi^*).$$

By (3.7) and (3.9),

$$\begin{aligned}\int_0^T \psi^*(t)\psi^*(t)dt &= \int_0^T \left[\frac{d\tilde{\phi}^*(t)}{dt} - b_i(x_0)\tilde{\phi}^*(t) - a_i(t, x_0)\tilde{\phi}^*(t) + \lambda_i(x_0)\tilde{\phi}^*(t)\right]\psi^*(t)dt \\ &= \int_0^T \left[-\frac{d\psi^*(t)}{dt} - b_i(x_0)\psi^*(t) - a_i(t, x_0)\psi^*(t) + \lambda_i(x_0)\psi^*(t)\right]\tilde{\phi}^*(t)dt \\ &= 0,\end{aligned}$$

which is a contradiction. Therefore $\lambda_i(x_0) \in \sigma(H_i)$ and the proposition follows. □

Proposition 3.5 *Let* $1 \le i \le 3$ *be given. For any* $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}\,\alpha > \lambda_{i,\max}$, $(\alpha - H_i)^{-1}$ *exists. Moreover,*

$$\left((\alpha - H_i)^{-1}v\right)(t,x) \ge \frac{M}{\alpha - \lambda_i(x)} v(x)$$

for any $\lambda_{i,\max} < \alpha \le \lambda_{i,\max} + 1$ *and any* $v \in \mathcal{X}_i^+$ *with* $v(t,x) \equiv v(x)$*, where*

$$M = \inf_{s \le t \le s+T, s,t \in \mathbb{R}} \exp\left(\int_s^t \left(\min_{x \in D_i}(b_i(x) + a_i(\tau,x)) - \lambda_{i,\max} - 1\right) d\tau\right).$$

Proof First of all, by Floquet theory for periodic ordinary differential equations, for any $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > \lambda_{i,\max}$, $(\alpha - H_i)^{-1}$ exists. Moreover, for any $v \in \mathcal{X}_i \oplus i\mathcal{X}_i$, we have

$$\left((\alpha - H_i)^{-1}v\right)(t,x) = \int_{-\infty}^t \exp\left(\int_s^t (b_i(x) + a_i(\tau,x) - \alpha)v(\tau,x)d\tau\right) ds.$$

Hence for any $v \in \mathcal{X}_i$ with $v(t,x) \equiv v(x)$, we have

$$\left((\alpha - H_i)^{-1}v\right)(t,x) = \left\{\int_{-\infty}^t \exp\left(\int_s^t (b_i(x) + a_i(\tau,x) - \alpha)d\tau\right) ds\right\} v(x).$$

If $\lambda_{i,\max} < \alpha \le \lambda_{i,\max} + 1$, then

$$\int_{-\infty}^t \exp\left(\int_s^t (b_i(x) + a_i(\tau,x) - \alpha)d\tau\right) ds \ge \frac{M}{\alpha - \lambda_i(x)},$$

where

$$M = \inf_{s \le t \le s+T, s,t \in \mathbb{R}} \exp\left(\int_s^t \left(\min_{x \in D_i}(b_i(x) + a_i(\tau,x)) - \lambda_{i,\max} - 1\right) d\tau\right)$$

(see the arguments of [21, Lemma 3.6]). It then follows that for any $\lambda_{i,\max} < \alpha \le \lambda_{i,\max} + 1$ and $v \in \mathcal{X}_i^+$ with $v(t,x) \equiv v(x)$,

$$\left((\alpha - H_i)^{-1}v\right)(t,x) \ge \frac{M}{\alpha - \lambda_i(x)} v(x).$$

The proposition is thus proved. □

Proposition 3.6 *Let* $1 \le i \le 3$ *be given.* $H_i - \max_{x \in D_i, t \in \mathbb{R}}\left(b_i(x) + a_i(t,x)\right)$ *generates a positive semigroup of contractions on* $\mathcal{X}_i$ *and for any* $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}\,\alpha > \lambda_{i,\max}$, $\nu_i K_i(\alpha - H_i)^{-1}$ *is a compact operator on* $\mathcal{X}_i \oplus i\mathcal{X}_i$.

Proof First, by the arguments in [21, Lemma 3.4], $H_i - \max_{x \in D_i, t \in \mathbb{R}}\left(b_i(x) + a_i(t,x)\right)$ generates a positive semigroup of contractions on $\mathcal{X}_i$. By Proposition 3.5, for any α with

$\operatorname{Re}\alpha > \lambda_{i,\max}$, $(\alpha - H_i)^{-1}$ exists. Moreover, by the arguments of Proposition 3.5, for any α with $\operatorname{Re}\alpha > \lambda_{i,\max}$ and any $v \in \mathcal{X}_i \oplus i\mathcal{X}_i$,

$$\begin{aligned}&\left(\nu_i K_i(\alpha - H_i)^{-1} v\right)(t,x)\\&= \nu_i \int_{\tilde{D}} \left\{ k(y-x) \int_{-\infty}^{t} \exp\left(\int_s^t (b_i(y) + a_i(\tau, y) - \alpha) v(\tau, y) d\tau \right) ds \right\} dy,\end{aligned}$$

where $\tilde{D} = D$ in the case $i = 1, 2$ and $\tilde{D} = \mathbb{R}^N$ in the case $i = 3$. It then follows that for any bounded subset $E \subset \mathcal{X}_i \oplus i\mathcal{X}_i$, $\nu_i K_i(\alpha - H_i)^{-1} E$ is a relatively compact subset of $\mathcal{X}_i \oplus i\mathcal{X}_i$ and hence $\nu_i K_i(\alpha - H)^{-1}$ is a compact operator on $\mathcal{X}_i \oplus i\mathcal{X}_i$. □

Proposition 3.7 *For given* $1 \le i \le 3$, $s_i(a_i) > \lambda_{i,\max}$ *iff there is* $\alpha > \lambda_{i,\max}$ *such that* $r(\nu_i K_i(\alpha - H_i)^{-1}) > 1$, *where* $r(\nu_i K_i(\alpha - H_i)^{-1})$ *is the spectral radius of* $\nu_i K_i(\alpha - H_i)^{-1}$.

Proof By Propositions 3.4 and 3.5,

$$\lambda_{i,\max} = \sup \sigma(H_i).$$

By Proposition 3.6, $\nu_i K_i(\alpha - H_i)^{-1}$ is a compact operator for any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \lambda_{i,\max}$. It then follows from [4, Theorem 2.2] that $s_i(a_i) > \lambda_{i,\max}$ iff there is $\alpha > \lambda_{i,\max}$ such that $r(\nu_i K_i(\alpha - H_i)^{-1}) > 1$. □

Proposition 3.8 *For given* $1 \le i \le 3$, *if there is* $\alpha_0 > \lambda_{i,\max}$ *such that* $r(\nu_i K_i(\alpha_0 - H)^{-1}) > 1$, *then* $s_i(a_i) > \lambda_{i,\max}$, $r(\nu_i K_i(s_i(a_i) - H)^{-1}) = 1$, *and* $s_i(a_i)$ *is an isolated eigenvalue of* $\nu_i K_i + H_i$ *of finite multiplicity with a positive eigenfunction.*

Proof Suppose that there is $\alpha_0 > \lambda_{i,\max}$ such that $r(\nu_i K_i(\alpha_0 - H)^{-1}) > 1$. Then by Proposition 3.7, $s_i(a_i) > \lambda_{i,\max}$. Moreover, by [4, Theorem 2.2], $r(\nu_i K_i(s_i(a_i) - H)^{-1}) = 1$, and $s_i(a_i)$ is an isolated eigenvalue of $\nu_i K_i + H_i$ of finite multiplicity with a positive eigenfunction. □

Proposition 3.9 *For given* $1 \le i \le 3$, *if* $\lambda \in \mathbb{R}$ *is an eigenvalue of* $L_i(a_i)$ *with a positive eigenfunction, then it is geometrically simple.*

Proof Suppose that $\phi(t, x)$ is a positive eigenfunction of L_i associated with λ. By Proposition 3.2, $\phi(t, x) > 0$ for $t \in \mathbb{R}$ and $x \in \bar{D}$. Assume that $\psi(t, x)$ is also an eigenfunction of L_i associated with λ. Then there is $a \in \mathbb{R}$ such that $w(t, x) = \phi(t, x) - a\psi(t, x)$ satisfies

$$w(t, x) \ge 0 \quad \forall t \in \mathbb{R},\ x \in \bar{D} \quad \text{and } w(t_0, x_0) = 0$$

for some $t_0 \in \mathbb{R}$ and $x_0 \in \bar{D}$. By Proposition 3.2 again, $w(t, x) \equiv 0$ and then $\phi(t, x) = a\psi(t, x)$. This implies that λ is geometrically simple. □

Proposition 3.10 *For* $1 \le i \le 3$, $s_i(a_i) = \frac{\ln r(\Phi_i(T; a_i))}{T}$.

Proof By the arguments in [21, Proposition 2.5 and Theorem 3.2],

$$s_i(a_i) = \limsup_{t-s\to\infty} \frac{\ln \|\Phi_i(t, s; a_i)\|}{t-s}.$$

By Proposition 3.3,

$$\limsup_{t-s\to\infty} \frac{\ln \|\Phi_i(t, s; a_i)\|}{t-s} = \frac{\ln r(\Phi_i(T; a_i))}{T}.$$

The proposition thus follows. □

Proposition 3.11 *For $1 \le i \le 3$, if $a_i^n \in \mathcal{X}_i$ and $a_i^n \to a_i$ in $\mathcal{X}_i$ as $n \to \infty$, then*

$$s_i\left(a_i^n\right) \to s_i(a_i) \quad \text{as} \quad n \to \infty.$$

Proof It follows from the arguments in [21, Proposition 2.6]. □

4 Principal Eigenvalues of Nonlocal Dispersal Operators

In this section, we investigate the existence and lower bounds of principal eigenvalues of nonlocal dispersal operators with time periodic dependence and prove Theorems A–C.

First of all, we prove an important technical lemma, which will also be used in next section.

Lemma 4.1 *For any $a_i \in \mathcal{X}_i$ and any $\epsilon > 0$, there is $a_{i,\epsilon} \in \mathcal{X}_i$ satisfying that*

$$\|a_i - a_{i,\epsilon}\|_{\mathcal{X}_i} < \epsilon,$$

$b_i + \hat{a}_{i,\epsilon}$ is C^N, $b_i + \hat{a}_{i,\epsilon}$ attains its maximum at some point $x_0 \in \text{Int}(D_i)$, and the partial derivatives of $b_i + \hat{a}_{i,\epsilon}$ up to order $N-1$ at x_0 are zero, where $\hat{a}_{i,\epsilon}(x) = \frac{1}{T}\int_0^T a_{i,\epsilon}(t,x)dt$.

Proof We prove the case $i = 1$ or 2. The case $i = 3$ can be proved similarly.

First, let $\tilde{x}_0 \in D_i$ be such that

$$\lambda_i(\tilde{x}_0) = \max_{x \in D_i} \lambda_i(x).$$

For any $\epsilon > 0$, there is $\tilde{x}_\epsilon \in \text{Int}(D_i)$ such that

$$\lambda_i(\tilde{x}_0) - \lambda(\tilde{x}_\epsilon) < \epsilon. \tag{4.1}$$

Let $\tilde{\sigma} > 0$ be such that

$$B(\tilde{x}_\epsilon, \tilde{\sigma}) \subset\subset D_i,$$

where $B(\tilde{x}_\epsilon, \tilde{\sigma})$ denotes the open ball with center $\tilde{x}_\epsilon$ and radius $\tilde{\sigma}$.

Note that there is $\tilde{h}_i \in C(D_i)$ such that $0 \le \tilde{h}_i(x) \le 1$, $\tilde{h}_i(\tilde{x}_\epsilon) = 1$, and $\text{supp}(\tilde{h}_i) \subset B(\tilde{x}_\epsilon, \tilde{\sigma})$. Let

$$\tilde{a}_{i,\epsilon}(t,x) = a_i(t,x) + \epsilon\tilde{h}_i(x)$$

and

$$\tilde{\lambda}_{i,\epsilon}(x) = b_i(x) + \hat{a}_i(x) + \epsilon\tilde{h}_i(x).$$

Then $\tilde{a}_{i,\epsilon}$ and $\tilde{\lambda}_{i,\epsilon}$ are continuous on D_i,

$$\|\tilde{a}_{i,\epsilon} - a_i\| \le \epsilon \tag{4.2}$$

and $\tilde{\lambda}_{i,\epsilon}$ attains its maximum in $\text{Int}(D_i)$.

Let $\tilde{D}_i \subset \mathbb{R}^N$ be such that $D_i \subset\subset \tilde{D}_i$. Note that $\tilde{\lambda}_{i,\epsilon}$ can be continuously extended to $\tilde{D}_i$. Without loss of generality, we may then assume that $\tilde{\lambda}_{i,\epsilon}$ is a continuous function on $\tilde{D}_i$ and assume that $x_0 \in \text{Int}(D_i)$ is such that $\tilde{\lambda}_{i,\epsilon}(x_0) = \sup_{x \in \tilde{D}_i} \tilde{\lambda}_{i,\epsilon}(x)$ (since $\tilde{\lambda}_{i,\epsilon}$ attains its maximum in $\text{Int}(D_i)$).

Observe that there is $\sigma > 0$ and $\bar{\lambda}_{i,\epsilon} \in C(\tilde{D}_i)$ such that $B(x_0, \sigma) \subset\subset D_i$,

$$\begin{aligned} &0 \le \bar{\lambda}_{i,\epsilon}(x) - \tilde{\lambda}_{i,\epsilon}(x) \le \epsilon \quad \forall\ x \in \tilde{D}_i, \\ &\bar{\lambda}_{i,\epsilon}(x) = \tilde{\lambda}_{i,\epsilon}(x_0) \quad \forall\ x \in B(x_0, \sigma), \end{aligned} \tag{4.3}$$

and

$$\bar{\lambda}_{i,\epsilon}(x) \le \tilde{\lambda}_{i,\epsilon}(x_0) \quad \forall x \in \tilde{D}_i.$$

Let

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{\|x\|^2-1}\right) & \text{if} \quad \|x\| < 1 \\ 0 \quad \text{if} \quad \|x\| \ge 1, \end{cases}$$

where $C > 0$ is such that $\int_{\mathbb{R}^N} \eta(x)dx = 1$. For given $\delta > 0$, set

$$\eta_\delta(x) = \frac{1}{\delta^N}\eta\left(\frac{x}{\delta}\right).$$

Let

$$\lambda_{i,\epsilon,\delta}(x) = \int\limits_{\tilde{D}_i} \eta_\delta(y-x)\bar{\lambda}_{i,\epsilon}(y)dy.$$

By [12, Theorem 6, Appendix C], $\lambda_{i,\epsilon,\delta}$ is in $C^\infty(\tilde{D}_i)$ and when $0 < \delta \ll 1$,

$$\left|\lambda_{i,\epsilon,\delta}(x) - \bar{\lambda}_{i,\epsilon}(x)\right| < \epsilon \quad \forall\, x \in D_i.$$

It is not difficulty to see that for $0 < \delta \ll 1$,

$$\lambda_{i,\epsilon,\delta}(x) = \bar{\lambda}_{i,\epsilon}(x_0) \quad \forall x \in B(x_0, \sigma/2)$$

and

$$\lambda_{i,\epsilon,\delta}(x) \le \bar{\lambda}_{i,\epsilon}(x_0) \quad \forall x \in \tilde{D}_i.$$

Fix $0 < \delta \ll 1$. Let

$$\lambda_{i,\epsilon}(x) = \lambda_{i,\epsilon,\delta}(x).$$

Then $\lambda_{i,\epsilon}$ attains its maximum at some $x_0 \in \text{Int}(D_i)$, and the partial derivatives of $\lambda_{i,\epsilon}$ up to order $N-1$ at x_0 are zero. Let

$$a_{i,\epsilon} = \tilde{a}_{i,\epsilon} + \lambda_{i,\epsilon} - \tilde{\lambda}_{i,\epsilon}.$$

Then $a_{i,\epsilon} \in \mathcal{X}_i$,

$$\|a_i - a_{i,\epsilon}\| \le \|a_i - \tilde{a}_{i,\epsilon}\| + \|\lambda_{i,\epsilon} - \bar{\lambda}_{i,\epsilon}\| + \|\bar{\lambda}_{i,\epsilon} - \tilde{\lambda}_{i,\epsilon}\| < 3\epsilon$$

and

$$b_i(x) + \hat{a}_{i,\epsilon}(x) = \lambda_{i,\epsilon}(x).$$

Therefore, $b_i + \hat{a}_{i,\epsilon}$ is C^N, attains its maximum at some point $x_0 \in \text{Int}(D)$, and the partial derivatives of $b_i + \hat{a}_{i,\epsilon}$ up to order $N-1$ at x_0 are zero. The lemma is thus proved. □

Proof of Theorem A First of all, assume that $\lambda \in \mathbb{R}$ is an eigenvalue of $L_i(a_i)$ with a positive eigenfunction $\phi(t,x)$. We first prove that $s_i(a_i) = \lambda$. By direct computation, we have

$$\left(\Phi_i(t, 0; a_i)\phi(0, \cdot)\right)(t, x) = e^{\lambda t}\phi(t, x).$$

By Proposition 3.2, we have

$$\phi(t, x) > 0 \quad \forall\, t \in \mathbb{R}, \;\; x \in D_i.$$

Then for any $u_0 \in X_i^+$,

$$u_0(x) \leq M_0\phi(0, x) \quad \forall\, x \in D_i$$

where $M_0 = \frac{\|u_0\|}{\min_{x\in D_i} \phi(0,x)}$. It then follows that

$$\Phi_i(t, 0; a_i)u_0 \leq M_0\Phi_i(t, 0; \phi(0, \cdot)) = M_0 e^{\lambda t}\phi(t, \cdot) \quad \forall\, t > 0.$$

This together with Proposition 3.10 implies that

$$s_i(a_i) = \lambda.$$

We now prove that $s_i(a_i) > \max_{x\in D_i}(b_i(x)+\hat{a}_i(x))$ and $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$ for the case $i = 1$. Other cases can be proved similarly. Observe that

$$-\frac{\phi_t(t, x)}{\phi(t, x)} + \frac{\nu_1 \int_D \kappa(y - x)\phi(t, y)dy}{\phi(t, x)} - \nu_1 + a_1(t, x) = s_1(a_1) \quad \forall x \in \bar{D},\ t \in \mathbb{R}.$$

This implies that

$$s_1(a_1) = -\nu_1 + \hat{a}_1(x) + \frac{\nu_1}{T}\int_0^T \frac{\int_D \kappa(y - x)\phi(t, y)dy}{\phi(t, x)}dt \quad \forall x \in \bar{D}$$

and hence

$$s_1(a_1) > -\nu_1 + \max_{x\in\bar{D}} \hat{a}_1(x) \left(= \max_{x\in D_1}(b_1(x) + \hat{a}_1(x)) = \lambda_{1,\max}\right).$$

By Propositions 3.7 and 3.8, $s_1(a_1)$ is the principal eigenvalue of $L_1(a_1)$.

Conversely, assume that $s_i(a_i) > \max_{x\in D_i}(b_i(x)+\hat{a}_i(x))(= \lambda_{i,\max})$. By Propositions 3.7 and 3.8, $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$. □

Next, we prove Theorem B(1).

Proof of Theorem B (1) We prove the case that $i = 2$. The other cases can be proved similarly. By Proposition 3.5, there is $M > 0$ such that for any $\alpha > \lambda_{2,\max}$ with $\alpha < \lambda_{2,\max} + 1$,

$$\left((\alpha - H_2)^{-1}v\right)(t, x) \geq \frac{M}{\alpha - \lambda_2(x)}v(x)$$

where $v(t, x) \equiv c(x) \succeq 0$. This implies that

$$\left(\nu_2 K_2(\alpha - H_2)^{-1}v\right)(t, x) \geq \int_D \frac{\nu_2 M\kappa(y - x)}{\alpha - \lambda_2(y)}v(y)dy.$$

By the arguments in [31, Theorem B (2)], for $0 < \alpha - \lambda_{2,\max} \ll 1$, there is $v(x) \geq 0$ such that

$$\nu_2 K_2(\alpha - H_2)^{-1}v > v.$$

Hence there is $\epsilon > 0$ such that

$$\nu_2 K_2(\alpha - H_2)^{-1}v \geq (1 + \epsilon)v$$

and then

$$\left(\nu_2 K_2(\alpha - H_2)^{-1}\right)^n v \geq (1 + \epsilon)^n v \quad \forall\, n \geq 1.$$

This implies that $r(\nu_2 K_2(\alpha - H_2)^{-1}) > 1$. By Proposition 3.8, $s_2(a_2)$ is the principal eigenvalue of $L_2(a_2)$. □

Before proving Theorem B (2) and (3), we first prove Theorem C.

Proof of Theorem C We prove the case $i = 2$. Other cases can be proved similarly.

First, we prove

$$s_2(a_2) \geq s_2(\hat{a}_2) \geq \lambda_{2,\max}. \tag{4.4}$$

In the case that both $L_2(a_2)$ and $L_2(\hat{a}_2)$ have principal eigenvalues, the first inequality in (4.4) follows from the arguments in [21, Theorem 4.1]. Regarding the second inequality, by Proposition 3.4, $\lambda_{2,\max} \in \sigma_{\text{ess}}(H_2(\hat{a}_2))$ ($\sigma_{\text{ess}}(\cdot)$ denotes the essential spectrum of an operator). Note that K_2 is a compact operator on X_2. Hence $\lambda_{2,\max} \in \sigma_{\text{ess}}(\nu_2 K_2 + H_2(\hat{a}_2))$. This implies that $s_2(\hat{a}_1) \geq \lambda_{2,\max}$. Hence (4.4) holds.

In general, $s_2(a_2)$ (resp. $s_2(\hat{a}_2)$) may not be the principal eigenvalue of $L_2(a_2)$ (resp. $L_2(\hat{a}_2)$). By Lemma 4.1 and Theorem B (1), for any $\epsilon > 0$, there is $a_{2,\epsilon} \in \mathcal{X}_2$ such that

$$\|a_{2,\epsilon} - a_2\|_{\mathcal{X}_2} < \epsilon,$$

and $s_2(a_{2,\epsilon})$ and $s_2(\hat{a}_{3,\epsilon})$ are principal eigenvalues of $L_2(a_{2,\epsilon})$ and $L_2(\hat{a}_{2,\epsilon})$, respectively. By the arguments in [21, Theorem 4.1] again,

$$s_2(a_{2,\epsilon}) \geq s_2(\hat{a}_{3,\epsilon}). \tag{4.5}$$

By Proposition 3.1, for any $v \in X_2^+$,

$$\Phi_2(nT; a_{2,\epsilon} - \epsilon)v \leq \Phi_2(nT; a_2)v \leq \Phi_2(nT; a_{2,\epsilon} + \epsilon)v.$$

Note that

$$\Phi_2(nT; a_{2,\epsilon} \pm \epsilon) = e^{\pm \epsilon nT} \Phi_2(nT; a_{2,\epsilon}).$$

Then by Proposition 3.3,

$$r(\Phi_2(T; a_{2,\epsilon})) \cdot e^{-\epsilon T} \leq r(\Phi_2(T; a_2)) \leq r(\Phi_2(T; a_{2,\epsilon})) \cdot e^{\epsilon T}.$$

By Proposition 3.10,

$$s_2(a_{2,\epsilon}) - \epsilon \leq s_2(a_2) \leq s_2(a_{2,\epsilon}) + \epsilon.$$

Similarly, we have

$$s_2(\hat{a}_{2,\epsilon}) - \epsilon \leq s_2(\hat{a}_2) \leq s_2(\hat{a}_{3,\epsilon}) + \epsilon.$$

It then follows that

$$s_2(a_2) \geq s_2(a_{2,\epsilon}) - \epsilon, \quad s_2(\hat{a}_2) \leq s_2(\hat{a}_{3,\epsilon}) + \epsilon.$$

This together with (4.5) implies that

$$s_2(a_2) \geq s_2(\hat{a}_2) - 2\epsilon$$

for any $\epsilon > 0$ and hence $s_2(a_2) \geq s_2(\hat{a}_2)$, that is, the first inequality in (4.4) holds. The second inequality in (4.4) holds by the same reason as before.

Next, we prove that if $s_2(a_2)$ is the principal eigenvalue of $L_2(a_2)$, then $s_2(a_2) = s_2(\hat{a}_2)$ iff $a_2(t, x) - \hat{a}_2(x)$ is independent of x.

First of all, assume that $a_2(t,x)-\hat{a}_2(x)$ is independent of x and let $\tilde{a}_2(t)=a_2(t,x)-\hat{a}_2(x)$. Then

$$\Phi_2(t,0;a_2)=e^{\int_0^t \tilde{a}_2(s)ds}\Phi_2(t,0;\hat{a}_2).$$

Since $\int_0^T \tilde{a}_2(s)ds=0$, we have

$$\Phi_2(T;a_2)=\Phi_2(T;\hat{a}_2).$$

By Proposition 3.10,

$$s_2(a_2)=s_2(\hat{a}_2).$$

Conversely, assume that $s_2(a_2)$ is the principal eigenvalue of $L_2(a_2)$ and $s_2(a_2)=s_2(\hat{a}_2)$. By Theorem A, $s_2(\hat{a}_2)$ is also the principal eigenvalue of $L_2(\hat{a}_2)$. By the arguments similar to those in [21, Theorem 4.1], we can prove that $a_2(t,x)-\hat{a}_2(x)$ is independent of x. For the completeness, we provide a proof in the following.

Let $\phi(t,x)$ and $\psi(x)$ be the positive principal eigenfunctions of $L_2(a_2)$ and $L_2(\hat{a}_2)$ with $\sup_{t\in\mathbb{R},x\in\bar{D}}\phi(t,x)=1$ and $\sup_{x\in\bar{D}}\psi(x)=1$, respectively. Then

$$\begin{aligned} s_2(a_2)=&-\frac{\phi_t(t,x)}{\phi(t,x)}+\nu_2\frac{\int_D\kappa(y-x)\phi(t,y)dy}{\phi(t,x)}\\ &-\nu_2\int_D\kappa(y-x)dy+a_2(t,x)\quad\forall\, t\in\mathbb{R},\ x\in\bar{D}\end{aligned}\tag{4.6}$$

and

$$s_2(\hat{a}_2)=\nu_2\frac{\int_D\kappa(y-x)\psi(y)dy}{\psi(x)}-\nu_2\int_D\kappa(y-x)dy+\hat{a}_2(x)\quad\forall\, x\in\bar{D}.\tag{4.7}$$

By (4.6),

$$\begin{aligned} s_2(a_2)=&\frac{\nu_2}{T}\int_D\kappa(y-x)\int_0^T\frac{\phi(t,y)}{\phi(t,x)}dtdy\\ &-\nu_2\int_D\kappa(y-x)dy+\hat{a}_2(x)\quad\forall\, x\in\bar{D}.\end{aligned}\tag{4.8}$$

Let

$$w(t,x)=\frac{\phi(t,x)}{\psi(x)}.$$

By the assumption that $s_2(a_2)=s_2(\hat{a}_2)$ and (4.7), (4.8), we have

$$\int_D\kappa(y-x)\frac{\psi(y)}{\psi(x)}\left[1-\frac{1}{T}\int_0^T\frac{w(t,y)}{w(t,x)}dt\right]dy=0,\quad\forall\, x\in\bar{D}.\tag{4.9}$$

By Jensen inequality,

$$\begin{aligned}\frac{1}{T}\int_0^T\frac{w(t,y)}{w(t,x)}dt&\ge\exp\left\{\frac{1}{T}\int_0^T\ln\frac{w(t,y)}{w(t,x)}dt\right\}\\ &=\frac{\exp\left\{\int_0^T\ln w(t,y)dt/T\right\}}{\exp\left\{\int_0^T\ln w(t,x)dt/T\right\}}\quad\forall\, x,y\in\bar{D}.\end{aligned}\tag{4.10}$$

and the equality in (4.10) holds for some $x_0,y_0\in\bar{D}$ iff $\frac{w(t,y_0)}{w(t,x_0)}$ is independent of t.

Let $x^* \in \bar{D}$ be such that

$$\int_0^T \ln w(t, x^*)dt = \inf_{x\in\bar{D}} \int_0^T \ln w(t, x)dt.$$

By (4.10),

$$\frac{1}{T}\int_0^T \frac{w(t, y)}{w(t, x^*)}dt \geq \frac{\exp\left\{\int_0^T \ln w(t, y)dt/T\right\}}{\exp\left\{\int_0^T \ln w(t, x^*)dt/T\right\}} \geq 1 \quad \forall\, y \in \bar{D}. \tag{4.11}$$

This together with (4.9) and $\kappa(0) > 0$ (note that $\kappa(\cdot) \geq 0$) implies that there is $\epsilon_0 > 0$ (independent of x^*) such that

$$\frac{1}{T}\int_0^T \frac{w(t, y)}{w(t, x^*)}dt = \frac{\exp\left\{\int_0^T \ln w(t, y)dt/T\right\}}{\exp\left\{\int_0^T \ln w(t, x^*)dt/T\right\}} = 1 \quad \forall\, y \in \bar{D},\ \|y - x^*\| \leq \epsilon_0. \tag{4.12}$$

This together with (4.10) implies that $\frac{w(t,y)}{w(t,x^*)}$ is independent of t for any $y \in \bar{D}$ with $\|y - x^*\| \leq \epsilon_0$.

Take any $y^* \in \bar{D}$ with $\|y^* - x^*\| < \epsilon_0$. By (4.12),

$$\int_0^T \ln w(t, y^*)dt = \inf_{x\in\bar{D}} \int_0^T \ln w(t, x)dt.$$

Repeating the above arguments, we have

$$\frac{1}{T}\int_0^T \frac{w(t, y)}{w(t, y^*)}dt = \frac{\exp\left\{\int_0^T \ln w(t, y)dt/T\right\}}{\exp\left\{\int_0^T \ln w(t, y^*)dt/T\right\}} = 1 \quad \forall\, y \in \bar{D},\ \|y - y^*\| \leq \epsilon_0 \tag{4.13}$$

and $\frac{w(t,y)}{w(t,y^*)}$ is independent of t for any $y \in \bar{D}$ with $\|y - y^*\| \leq \epsilon_0$. Hence $\frac{w(t,y)}{w(t,x^*)}$ is independent of t for any $y \in \bar{D}$ with $\|y - x^*\| < 2\epsilon_0$.

Continuing the above process, we have that $\frac{w(t,x)}{w(t,x^*)}$ is independent of t for any $x \in \bar{D}$. Let

$$p(x) = \frac{w(t, x)}{w(t, x^*)}$$

and

$$q(t) = w(t, x^*).$$

We then have

$$w(t, x) = p(x)q(t).$$

It then follows that

$$\phi(t, x) = p(x)\psi(x)q(t).$$

This together with (4.6) implies that there are $a_{2,1}(x)$ and $a_{2,2}(t)$ such that

$$a_2(t, x) = a_{2,1}(x) + a_{2,2}(t)$$

and then $a_2(t, x) - \hat{a}_2(x)$ is independent of x. □

We now prove Theorem B (2) and (3).

Theorem B (2) and (3) (2) We first claim that $s_i(\hat{a}_i)$ is the principal eigenvalue of $L_i(\hat{a}_i)$. In fact, this follows from [30, Theorem 2.1] in the case that $i = 1, 2$ and follows from [31, Theorem B(1)] in the case $i = 3$. By Theorem A,

$$s_i(\hat{a}_i) > \max_{x \in D_i} \lambda_i(x).$$

Then by Theorem C,

$$s_i(a_i) > \max_{x \in D_i} \lambda_i(x).$$

By Theorem A again, $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$.

(3) First by Theorem [30, Theorem 2.3], $s_i(\hat{a}_i)$ ($i = 1$ or 2) is the principal eigenvalue of $L_i(\hat{a}_i)$ for $0 < \delta \ll 1$ and by [31, Theorem A (1)], $s_3(\hat{a}_3)$ is the principal eigenvalue of $L_3(\hat{a}_3)$ for $0 < \delta \ll 1$. By Theorems A and C,

$$s_i(a_i) \geq s_i(\hat{a}_i) > \max_{x \in D_i} \lambda_i(x)$$

for $1 \leq i \leq 3$ and $0 < \delta \ll 1$. It then follows from Theorem A that $s_i(a_i)$ is the principal eigenvalue of $L_i(a_i)$ for $1 \leq i \leq 3$ and $0 < \delta \ll 1$. □

We end up this section with an example which shows that if $s_i(a_i)$ is not the principal eigenvalue of $L_i(a_i)$, then $s_i(a_i) = s_i(\hat{a}_i)$ may not imply that $a_i(t, x) - \hat{a}_i(x)$ is independent of x.

Example 4.1 Let $i = 1$, $D = B(0, 1)$, $N = 3$, and $\nu_1 = 1$. Let $0 < \sigma < \frac{1}{2}$ and $q(x)$ be a smooth function given by

$$q(x) = \begin{cases} e^{\frac{\|x\|^2}{\|x\|^2 - \sigma^2}} & \text{for} \quad \|x\| < \sigma \\ 0 & \text{for} \quad \sigma \leq \|x\| \leq 1. \end{cases}$$

Let $M > 1$ be a constant to be determined later and

$$a_1(t, x) = Mq(x) + (\cos t)q(x).$$

Then $a_1(t, x)$ is periodic in t with period 2π and

$$\hat{a}_1(x) = Mq(x), \quad \max_{x \in \bar{D}} \hat{a}_1(x) = M.$$

By Proposition 3.5, for any $\alpha > -1 + M$, $(\alpha - H_1(a_1))^{-1}$ exists. Moreover, by the arguments in Proposition 3.5, for $v(t, x) \equiv 1$,

$$\left((\alpha - H_1(a_1))^{-1} v\right)(t, x) = \int_{-\infty}^{t} \exp\left(\int_{s}^{t} (-1 + a_1(\tau, x) - \alpha) d\tau\right) ds$$

$$= \int_{-\infty}^{t} \exp\left((-1 + Mq(x) - \alpha)(t - s)\right) \cdot \exp\left((\sin t - \sin s)q(x)\right) ds$$

$$\leq \frac{e^2}{1 - Mq(x) + \alpha}.$$

This implies that

$$\left(K_1(\alpha - H_1(a_1))^{-1}v\right)(t,x) = \int_D \kappa(y-x)\left\{\int_{-\infty}^{t} \exp\left(\int_s^t (-1 + a_1(\tau,x) - \alpha)d\tau\right) ds\right\} dy$$
$$\le e^2 \int_D \frac{\kappa(y-x)}{1 - Mq(y) + \alpha} dy$$

Let $\alpha = -1 + M + \epsilon$. Then

$$\left(K_1(-1+M+\epsilon - H_1(a_1))^{-1}v\right)(t,x) \le e^2 \int_D \frac{\kappa(y-x)}{M(1-q(y)) + \epsilon} dy$$
$$\le \frac{e^2}{M+\epsilon} + e^2 \int_{\|y\|\le\sigma} \frac{\kappa(y-x)}{M(1 - e^{\frac{\|y\|^2}{\|y\|^2-\sigma^2}}) + \epsilon} dy$$
$$\le \frac{e^2}{M+\epsilon} + e^2 \int_{\|y\|\le\sigma} \frac{\kappa(y-x)}{M(1 - e^{-\frac{\|y\|^2}{\sigma^2}}) + \epsilon} dy$$

This implies that there is $\tilde{M} > 0$ (independent of M and ϵ) such that

$$\left(K_1(-1+M+\epsilon - H_1(a_1))^{-1}v\right)(t,x) \le \frac{e^2}{M+\epsilon} + \frac{\tilde{M}}{M}.$$

It then follows that there is $0 < \tilde{r} < 1$ such that for any $M \gg 1$ and $0 < \epsilon < 1$,

$$\|K_1(-1+M+\epsilon - H_1(a_1))^{-1}\| \le \tilde{r}.$$

By Proposition 3.7, $s_1(a_1) \le \lambda_{1,\max}$.

By Theorem C,

$$s_1(a_1) \ge s_1(\hat{a}_1) \ge \lambda_{1,\max}.$$

Hence

$$s_1(a_1) = s_1(\hat{a}_1) = \lambda_{1,\max}.$$

But

$$a_1(t,x) - \hat{a}_1(x) = (\cos t)q(x)$$

and hence $a_1(t,x) - \hat{a}_1(x)$ depends on x.

5 Time Periodic Positive Solutions of Nonlocal KPP Equations

In this section, we consider applications of the principal eigenvalue theory established in the previous section to time periodic KPP equations with nonlocal dispersal.

For given $u_1, u_2 \in X_1^{++} (= X_2^{++})$ or $u_1, u_2 \in X_3^{++}$, we define

$$\rho(u_1,u_2) = \inf\left\{\ln\alpha \mid \frac{1}{\alpha}u_1(\cdot) \le u_2(\cdot) \le \alpha u_1(\cdot),\ \alpha \ge 1\right\}. \tag{5.1}$$

Observe that for $u_1, u_2 \in X_1^{++}(= X_2^{++})$, there is $\alpha \geq 1$ such that

$$\rho(u_1, u_2) = \ln \alpha.$$

For simplicity in notation, we put

$$u_i(t, x; u_0) = u_i(t, x; 0, u_0).$$

Proposition 5.1 *For any $u_0, v_0 \in X_1^{++}(= X_2^{++})$ or $u_0, v_0 \in X_3^{++}$, $u_0 \neq v_0$, $\rho(u_i(t, \cdot; u_0), u_i(t, \cdot; v_0))$ strictly decreases as t increases.*

Proof We prove the case $i = 1$. The cases $i = 2$ and $i = 3$ can be proved similarly.

For given $u_0, v_0 \in X_1^{++}$, there is $\alpha \geq 1$ such that

$$\frac{1}{\alpha} v_0 \leq u_0 \leq \alpha v_0$$

and

$$\rho(u_0, v_0) = \ln \alpha.$$

We first claim that $\rho(u_1(t, \cdot; u_0), u_1(t, \cdot; v_0))$ is non-increasing as t increases for $t > 0$ or equivalently for $0 < t \leq T$. In fact, by Proposition 3.1, for any $t > 0$, we have

$$u_1(t, \cdot; u_0) \leq u_1(t, \cdot; \alpha v_0). \tag{5.2}$$

Similarly, for any $t > 0$,

$$u_1(t, \cdot; \frac{1}{\alpha} v_0) \leq u_1(t, \cdot; u_0). \tag{5.3}$$

Assume $u_0 \not\equiv v_0$. Then $\alpha > 1$. Let $w(t, x) = \alpha u_1(t, x; v_0)$. Then $w(0, x) = \alpha v_0(x)$ and

$$\begin{aligned}
\partial_t w &= \int_D \kappa(y - x) w(t, y) dy - w(t, x) + w(t, x) f_1(t, x, u_1(t, x; v_0)) \\
&= \int_D \kappa(y - x) w(t, y) dy - w(t, x) + w f_1(t, x, w(t, x)) \\
&\quad + w[f_1(t, x, u_1(t, x; v_0)) - f_1(t, x, w(t, x))] \\
&\geq \int_D \kappa(y - x) w(t, y) dy - w(t, x) + w f_1(t, x, w(t, x)) + \delta_0
\end{aligned} \tag{5.4}$$

for some δ_0 and $0 \leq t \leq T$. By Proposition 3.1,

$$\alpha u_1(t, \cdot; v_0) \geq u_1(t, \cdot; \alpha v_0)$$

for $0 < t \leq T$. This together with (5.2) implies that

$$u_1(t, \cdot; u_0) \leq \alpha u_1(t, \cdot; v_0)$$

for $0 < t \leq T$. Similarly, we have

$$u_1(t, \cdot; u_0) \geq \frac{1}{\alpha} u_1(t, \cdot; v_0).$$

It then follows that

$$\rho(u_1(t, \cdot; u_0), u_1(t, \cdot; v_0)) \leq \rho(u_0, v_0)$$

for $0 < t \le T$, which implies that $\rho(u_1(t,\cdot;u_0), u_1(t,\cdot;v_0))$ is non-increasing as t increases for $0 < t \le T$.

Next, we prove that $\rho(u_1(t,\cdot;u_0), u_1(t,\cdot;v_0))$ is strictly decreasing as t increases for $t > 0$ or equivalently for $0 < t \ll 1$. By (5.4),

$$\partial_t w(0,x) \ge \partial_t u_1(0,x;\alpha v_0) + \delta_0.$$

Hence

$$\partial_t w(t,x) \ge \partial_t u_1(t,x;\alpha v_0) + \frac{\delta_0}{2}$$

and then

$$w(t,x) = \alpha u_1(t,x;v_0) \ge u_1(t,x;\alpha v_0) + \frac{\delta_0}{2}t$$

for $0 < t \ll 1$. This implies that for given $0 < t \ll 1$, there is $\tilde{\alpha}(t) < \alpha$ such that

$$\tilde{\alpha}(t)u_1(t,x;v_0) \ge u_1(t,x;\alpha v_0) \ge u_1(t,x;u_0).$$

Similarly, we can prove that for given $0 < t \ll 1$, there is $\bar{\alpha}(t) < \alpha$ such that

$$\frac{1}{\bar{\alpha}(t)} u_1(t,x;v_0) \le u_1(t,x;u_0).$$

Therefore,

$$\rho(u_1(t,\cdot;u_0), v_1(t,\cdot;v_0)) \le \ln\left(\max\{\tilde{\alpha}(t), \bar{\alpha}(t)\}\right) < \rho(u_0,v_0)$$

for $0 < t \ll 1$. This implies that $\rho(u_1(t,\cdot;u_0), u_1(t,\cdot;v_0))$ is strictly decreasing as t increases. □

Proof of Theorem E We prove the theorem in the case $i = 1$. Other cases can be proved similarly.

First of all, for given $M \gg 1$, $u(t,x) \equiv M$ is a supersolution of (1.8). This implies that $u(nT,x;M)$ decreases as t increases. Let

$$u^+(x) = \lim_{n\to\infty} u(nT,x;M) \quad \text{for} \quad x \in \bar{D}. \tag{5.5}$$

Next, by Lemma 4.1, there are $a_1^n \in \mathcal{X}_1$ such that $s_1(a_1^n)$ is the principal eigenvalue of $L_1(a_1^n)$,

$$a_1^n(t,x) < f_1(t,x,0)$$

and

$$a_1^n(t,x) \to f_1(t,x,0) \quad \text{as} \quad n \to \infty$$

uniformly in $t \in \mathbb{R}$ and $x \in \bar{D}$. By Proposition 3.11,

$$\lim_{n\to\infty} s_1\left(a_1^n\right) \to s_1(f_1(\cdot,\cdot,0))$$

as $n \to \infty$ and then

$$s_1\left(a_1^n\right) > 0 \quad \forall n \gg 1. \tag{5.6}$$

Let ϕ_1^n be the positive principal eigenfunction of $L_1(a_1^n)$ with $\|\phi_1^n\|_{\mathcal{X}_i} = 1$. Then for any $b > 0$, $u(t, x) = b\phi_1^n(t, x)$ is a solution of

$$\partial_t u = \nu_1 \left[\int_D k(y-x)u(t,y)dy - u(t,x) \right] + a_1^n(t,x)u - s_1\left(a_1^n\right)u.$$

Observe that

$$\begin{aligned}\partial_t u &= \nu_1 \left[\int_D k(y-x)u(t,y)dy - u(t,x) \right] + a_1^n(t,x)u - s_1\left(a_1^n\right)u \\ &\le \nu_1 \left[\int_D k(y-x)u(t,y)dy - u(t,x) \right] + f_1(t,x,0)u - s_1(a_1^n)u \\ &= \nu_1 \left[\int_D k(y-x)u(t,y)dy - u(t,x) \right] + uf_1(t,x,u) + [f_1(t,x,0) - f_1(t,x,u)]u \\ &\quad - s_1(a_1^n)u. \end{aligned} \tag{5.7}$$

Fix $n \gg 1$. By (5.6),

$$\left[f_1(t,x,0) - f_1\left(t,x,b\phi_1^n(t,x)\right)\right] - s_1(a_1^n) < 0 \quad \forall 0 < b \ll 1.$$

This together with (5.7) implies that $u(t, x) = b\phi_1^n(t, x)$ is a subsolution of (1.8) for $0 < b \ll 1$.

For fixed $n \gg 1$, fix $0 < b \ll 1$ such that $u(t, x) = b\phi_1^n(t, x)$ is a subsolution of (1.8). Then $u(kT, x; b\phi_i^n)$ increases as k increases. Let

$$u^-(x) = \lim_{k\to\infty} u\left(kT, x; b\phi_1^n\right) \quad \text{for} \quad x \in \bar{D}. \tag{5.8}$$

For fixed $n \gg 1$ and $0 < b \ll 1$, choose $M \gg 1$ such that

$$b\phi_1^n < M.$$

Then

$$u^-(x) \le u^+(x) \quad \forall\, x \in \bar{D}.$$

We claim that

$$u^- \equiv u^+.$$

In fact, by Proposition 5.1, $\rho\left(u_1(t, \cdot; M), u_1\left(t, \cdot; b\phi_1^n\right)\right)$ strictly decreases as t increases. Let

$$\begin{aligned}\rho^k &= \rho\left(u_1(kT, \cdot; M), u_1\left(kT, \cdot; b\phi_1^n\right)\right) \\ \rho^* &= \lim_{k\to\infty} \rho^k.\end{aligned}$$

Observe that $u^+ \equiv u^-$ iff $\rho^* = 0$. Assume that $\rho^* > 0$. Let $\alpha^* = e^{\rho^*}$. Then for any $0 < \epsilon < \alpha^*$,

$$\frac{1}{\alpha^* + \epsilon} u_1\left(kT, \cdot; b\phi_1^n\right) \le u_1(kT, \cdot; M) \le (\alpha^* + \epsilon)u_1\left(kT, \cdot; b\phi_1^n\right) \quad \forall k \gg 1.$$

Note that

$$\inf_{t\in\mathbb{R},x\in\bar{D}} \phi_1^n(t,x) > 0.$$

By the arguments in Proposition 5.1, there is $\delta_0 > 0$ such that

$$\rho^{k+1} \le \rho^k - \delta_0 \quad \forall k \gg 1.$$

This implies that

$$\rho^* \le \rho^* - \delta_0.$$

This is a contradiction. Therefore, $u^+ = u^-$.

Note that u^+ is upper semi-continuous and u^- is lower semi-continuous. Hence $u^* := u^+$ is continuous and $u^* := u^+ \in X^+ \setminus \{0\}$. By Dini's Theorem, $\lim_{k\to\infty} u(kT,\cdot;b\phi_1^n) = \lim_{k\to\infty} u(kT,\cdot;M) = u^*$ uniformly in $x \in \bar{D}$. This implies that

$$u(T,x;u^*) = \lim_{k\to\infty} u(T,x;u(kT,\cdot;M)) = \lim_{k\to\infty} u((k+1)T,x;M) = u^*(x).$$

Hence $u(t,x;u^*)$ is periodic in t. This proves the existence of time periodic positive solutions.

Now suppose that $u^1(t,x)$ and $u^2(t,x)$ are two time periodic positive solutions. Since $\rho(u^1(t,\cdot),u^2(t,\cdot))$ is strictly decreasing if $u^1 \ne u^2$, we must have $u^1 \equiv u^2$. This proves the uniqueness of time periodic positive solutions.

Finally, we show the stability of $u^*(t,x) := u(t,x;u^*)$. Observe that

$$u_t^*(t,x) = \nu_1 \left[\int_D \kappa(y-x)u^*(t,y)dy - u^*(t,x)\right] + u^*(t,x)f_1(t,x,u^*(t,x)), \quad x \in \bar{D}. \tag{5.9}$$

By Theorem A,

$$s_1(f_1(\cdot,\cdot,u^*(\cdot,\cdot))) = 0. \tag{5.10}$$

Consider the linearization of (1.8) at $u^*(t,x)$,

$$v_t(t,x) = \nu_1 \left[\int_D \kappa(y-x)v(t,y)dy - v(t,x)\right] + a_1^*(t,x)v(t,x), \quad x \in \bar{D},$$

where

$$a_1^*(t,x) = f_1(t,x,u^*(t,x)) + u^*(t,x)\partial_u f_1(t,x,u^*(t,x)).$$

By the assumption that $\partial_u f_1(t,x,u) < 0$ for $u \ge 0$,

$$a_1^*(t,x) < f_1(t,x,u^*(t,x)) \quad \forall t \in \mathbb{R},\ x \in \bar{D}.$$

This together with (5.10) implies that

$$s_1\left(a_1^*\right) < 0.$$

Then by Proposition 3.10,

$$r\left(\Phi_1(T;a_1^*)\right) < 1.$$

Therefore, $u^*(t, x)$ is locally stable. Now for any $u_0 \in X^+ \setminus \{0\}$, $u(t, \cdot; u_0) \in \mathrm{Int}(X^+)$ for $t > 0$. Fix $n \gg 1$. Then

$$b\phi_1^n \leq u(T, \cdot; u_0) \leq M$$

for $0 < b \ll 1$ and $M \gg 1$. By the above arguments,

$$\lim_{t\to\infty} \big(u(t, x; u_0) - u(t, x; u^*)\big) = 0$$

uniformly in $x \in \bar{D}$. Therefore, the unique time periodic positive solution is globally asymptotically stable. □

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