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On Growth of Sobolev Norms in Linear Schrödinger Equations with Time Dependent Gevrey Potential

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Abstract We improve Delort's method to show that solutions of linear Schrödinger equations with a time dependent Gevrey potential on the torus, have at most logarithmically growing Sobolev norms. In particular, it contains the result of Wang (Commun Partial Differ Equ 33:2164–2179, 2008), which deals with analytic potentials in dimension 1.

Keywords Sobolev norms · Time dependent Schrödinger equation · Gevrey Potential

1 Introduction and Statement of the Theorem

The main goal of this paper is to obtain logarithmic growth of Sobolev norms of solutions of linear Schrödinger equations with a time dependent Gevrey potential on the torus, using the method of Delort [4]. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and let $\mathbb{T}^d$ denote the standard torus, where $d \in \mathbb{N}^*$. We consider the time dependent linear Schrödinger equations:

$$i\partial_t u - \Delta u + V(x,t)u = 0 \tag{1.1}$$

on $\mathbb{T}^d \times \mathbb{R}$. We assume that the potential V is a real smooth function on $\mathbb{T}^d \times \mathbb{R}$. Let $\mu, \lambda \in [1, +\infty)$. We further assume that V is a Gevrey-μ function in time t and Gevrey-λ in every space variable, i.e., $V(x,t)$ satisfies estimates

$$\sup_{t\in\mathbb{R}} \sup_{x\in\mathbb{T}^d} |\partial_t^k \partial_x^\alpha V(x,t)| \le C^{k+|\alpha|+1}(k!)^\mu (\alpha!)^\lambda \tag{1.2}$$

for any $k \in \mathbb{N}$, for any $\alpha \in \mathbb{N}^d$ and for some constant C independent of k and α.

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We prove the following result:

Theorem 1 *There exists* $\zeta > 0$ *independent of* μ *and* λ *such that for any* $s > 0$*, there is a constant* $C_{s,\lambda,d} > 0$ *such that*

$$\|u(t)\|_{H^s} \leq C_{s,\lambda,d}\big[\log(2+|t|)\big]^{\zeta\mu\lambda s}\|u(0)\|_{H^s}, \tag{1.3}$$

where $u(t)$ *is the solution to (1.1) with the initial condition* $u_0 \overset{def}{=} u(0) \in H^s(\mathbb{T}^d)$.

Remark 1.1 Wang [8] obtained (1.3) with the exponent '$\zeta\mu\lambda s$' replaced by 'ζs' and $C_{s,\lambda,d}$ replaced by C_s under the assumption that the dimension $d = 1$ and that the potential $V(x,t)$ is bounded and analytic in space and time on $\Omega_{\tilde{\rho}}$ ($\tilde{\rho} > 0$ is a constant) when V is identified with a periodic function on $\mathbb{R}^d \times \mathbb{R}$, where

$$\Omega_{\tilde{\rho}} = \{(x,t) \in \mathbb{C} \times \mathbb{C} \; : \; |\operatorname{Im} x| < \tilde{\rho}, \; |\operatorname{Im} t| < \tilde{\rho}\}.$$

When $d = 1$, the assumption we made here on the potential V is weaker than the assumption that V is analytic both in space and time on the strip $\Omega_{\tilde{\rho}}$, since the latter implies that V is a function of Gevrey-1 in time and Gevrey-1 in space. Moreover, our result concerns the case of any dimension $d \in \mathbb{N}^*$ instead of just $d = 1$.

Remark 1.2 One may assume that $V(x,t)$ is a Gevrey-μ function in time and Gevrey-λ_i in space variable x_i for $1 \leq i \leq d$ with $\mu, \lambda_i \in [1, +\infty)$. However, this leads to (1.2) if we take $\lambda = \max\{\lambda_i \; : \; 1 \leq i \leq d\}$, and thus we may obtain the same result.

The problem of finding optimal bounds for $\|u(t,\cdot)\|_{H^s}$ has been addressed by Nenciu [7] and Barbaroux and Joye [1], in the abstract framework of an operator P (instead of $-\Delta$) and a perturbation $V(t)$ acting on elements of a Hilbert space, when the spectrum of P is discrete and has increasing gaps. This condition is satisfied by the Laplacian on the circle. It follows from the results of [1,7], that solutions of (1.1) verify

$$\|u(t,\cdot)\|_{H^s} \leq C_\epsilon |t|^\epsilon \|u(0,\cdot)\|_{H^s} \tag{1.4}$$

when t goes to infinity, for any $\epsilon > 0$. Later, Bourgain [3] proved that a similar bound holds for solutions of (1.1) on the torus $\mathbb{T}^d$. The increasing gap condition of Nenciu [7] and Barbaroux and Joye [1] is no longer satisfied, and has to be replaced by a convenient decomposition of $\mathbb{Z}^d$ in well separated clusters. Delort [4] recently published a simpler proof of the results of Bourgain (included for other examples of compact manifolds than the torus), which is close to the one of Nenciu and Barbaroux and Joye. If one further assumes that V is analytic, and quasi-periodic in t, then it was showed by Bourgain [2] that (1.4) holds with $(1+|t|)^\epsilon$ replaced by some power of $\log t$ when $t > 2$. When the dimension $d = 1$, for any real analytic potential, whose holomorphic extension to $\Omega_{\tilde{\rho}}$ is bounded, Wang [8] showed that one may still obtain such a logarithmic bound, using the method of [3]. In this paper, we improve the method of Delort [4] to provide a new proof of the result of Wang [8] and extend it to any dimension $d \geq 1$ and to Gevrey regularity.

There are also some results about uniformly bounded Sobolev norms. Eliasson and Kuksin [5] have shown that if the potential V on $\mathbb{T}^d \times \mathbb{R}$ is analytic in space, quasi-periodic in time, and small enough, then for most values of the parameter of quasi-periodicity, the equation reduces to an autonomous one. Consequently, the Sobolev norm of the solution is uniformly bounded. A similar result for the harmonic oscillator has been obtained by Grébert and Thomann recently [6]. For Schrödinger equations on the circle with a small time periodic potential, Wang [9] showed that the solutions of the corresponding equation have bounded Sobolev norms.

Now let us give a picture of the proof of Theorem 1. For any given $N \in \mathbb{N}^*$, one first finds for every fixed time t an operator $Q^N(\cdot, t)$, which extends as a bounded linear operator from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$ such that

$$\big(I + Q^N(\cdot, t)\big)^*(i\partial_t - \Delta + V)\big(I + Q^N(\cdot, t)\big) = i\partial_t - \Delta + V'_N(\cdot, t) + R'_N(\cdot, t) \quad (1.5)$$

with self-adjoint operator V'_N exactly commuting to the modified Laplacian $\tilde{\Delta}$ (see (2.4) for its precise definition) and R'_N a remainder operator which is essentially a bounded linear map from $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. Moreover, we also require that the adjoint of Q^N in the usual L^2 paring (denoted by $Q^N(\cdot, t)^*$) extends as a bounded linear operator from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. In order to obtain the estimate for the solution u of (1.1), one needs to 'invert' the operator $I + Q^N$, that is, to find an operator P^N, which extends as a bounded linear operator not only from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$, but also from $L^2(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$, such that

$$\big(I + Q^N(\cdot, t)\big)\big(I + P^N(\cdot, t)\big) = I + R_N(\cdot, t) \quad (1.6)$$

where R_N is a remainder operator such that $[i\partial_t - \Delta + V, R_N]$ sends $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. Now by setting

$$v = (I + P^N)u, \quad (1.7)$$

we deduce from (1.5), (1.6) and (1.1)

$$(i\partial_t - \Delta + V'_N)v = (I + Q^N)^*[i\partial_t - \Delta + V, R_N]u - R'_N v. \quad (1.8)$$

Remarking that the modified Laplacian has the property that

$$C^{-N}\|(1-\Delta)^{\frac{N}{2}}u\|_{L^2} \le \|(1-\tilde{\Delta})^{\frac{N}{2}}u\|_{L^2} \le C^N\|(1-\Delta)^{\frac{N}{2}}u\|_{L^2}$$

holds for any $u \in H^N(\mathbb{T}^d)$ and for some uniform constant C, then we let the operator $(1-\tilde{\Delta})^{\frac{N}{2}}$ act on both sides of (1.8) and deduce from the energy inequality

$$\|v(t)\|_{H^N} \le C_N\|v(0)\|_{H^N} + C_N\int_0^t \|(I + Q^N)^*[i\partial_t - \Delta + V, R_N]u(t)\|_{H^N} + \|R'_N v(t)\|_{H^N}\,dt,$$

which together with (1.7), the conservation law of the L^2-norm of (1.1) and the properties of those operators we have constructed, implies

$$\|v(t)\|_{H^N} \le C_N\|v(0)\|_{H^N} + C_N|t|\|u(0)\|_{L^2}. \quad (1.9)$$

We then use (1.6), (1.7) and the properties of the operators to deduce

$$\|u(t)\|_{H^N} \le C_N\Big(\|u(0)\|_{H^N} + (2 + |t|)\|u(0)\|_{L^2}\Big). \quad (1.10)$$

Remark that the above constants C_N may be different in different lines and they depend on the norms of operators which appear in the above process. Since (1.10) holds for any $N \in \mathbb{N}^*$, if we have good estimates for C_N (we shall finally see that C_N can be controlled by C^N times a power of the factorial of N), then the theorem will follow by interpolation just as we shall do in the last section. There are two difficulties. The first one is that we have to carefully choose those operators Q^N so that the above process can go on. The second is to obtain proper estimates for C_N, which means that we have to estimate the norms of operators and remainders for every $N \in \mathbb{N}^*$ in the above process.

The paper is organized as follows. In Sect. 2, we introduce the spaces and give their properties we shall use. Then we construct the operator in these spaces to conjugate the original equation in Sect. 3. The last section is dedicated to the proof of the main theorem.

2 Definitions of Operator Spaces and Their Properties

Let us introduce some notation.

Notation 1 *We denote by* Π_n *the spectral projector on* $L^2(\mathbb{T}^d)$ *defined by*

$$\Pi_n u = \frac{e^{inx}}{(2\pi)^{d/2}}\Big\langle u, \frac{e^{inx}}{(2\pi)^{d/2}}\Big\rangle, \quad n \in \mathbb{Z}^d. \tag{2.1}$$

For $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^d$*, we set*

$$a_+ = \max\{a, 0\}, \quad \langle b\rangle = (1 + |b|^2)^{1/2}. \tag{2.2}$$

By $A \lesssim B$ *we mean that there is an absolute constant* $C > 0$ *such that* $A \le CB$*. For* $s \in \mathbb{R}$*, denote by* $H^s(\mathbb{T}^d)$ *the Sobolev space consisting of* $u \in L^2(\mathbb{T}^d)$ *with its norm*

$$\|u\|_{H^s} = \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \|\Pi_n u\|_{L^2}^2\right)^{1/2} < +\infty. \tag{2.3}$$

Using the following proposition which is just Lemma 3.2 in [4], we shall give an equivalent characterization of the Sobolev space $H^s(\mathbb{T}^d)$ when $s > 0$.

Proposition 2.1 (Bourgain) *Let* $\sigma \in (0, 1/10)$*. Then there are* $\tau_0 \in (0, \sigma)$, $\gamma > 0$ *and a partition* $(A_\alpha)_{\alpha\in\Lambda}$ *of* $\mathbb{Z}^d$ *such that*

- $\forall \alpha \in \Lambda, \forall n \in A_\alpha, \forall n' \in A_\alpha, |n - n'| + \||n|^2 - |n'|^2| < \gamma + \max(|n|, |n'|)^\sigma$*;*
- $\forall \alpha, \beta \in \Lambda, \alpha \ne \beta, \forall n \in A_\alpha, \forall n' \in A_\beta, |n - n'| + \||n|^2 - |n'|^2| > \max(|n|, |n'|)^{\tau_0}$.

Notation 2 *We denote for* $\alpha \in \Lambda$

$$\widetilde{\Pi}_\alpha = \sum_{n\in A_\alpha} \Pi_n.$$

For any $\alpha \in \Lambda$*, we choose* $n(\alpha) \in A_\alpha$ *and define*

$$\tilde{\Delta} u = -\sum_{\alpha\in\Lambda} |n(\alpha)|^2 \widetilde{\Pi}_\alpha u. \tag{2.4}$$

By definition we know that

$$[\Delta, \tilde{\Delta}] = 0, \quad [i\partial_t, \tilde{\Delta}] = 0. \tag{2.5}$$

For $s \in \mathbb{R}$*, let* $\widetilde{H}^s(\mathbb{T}^d)$ *be the space consisting of those elements* $u \in L^2(\mathbb{T}^d)$ *with its norm*

$$\|u\|_{\widetilde{H}^s} = \left(\sum_{\alpha\in\Lambda} \langle n(\alpha)\rangle^{2s} \|\widetilde{\Pi}_\alpha u\|_{L^2}^2\right)^{1/2} < +\infty. \tag{2.6}$$

By the first condition in Proposition 2.1, we deduce that there is a constant $C_0 > 0$ such that for any $s > 0$, for any $u \in \widetilde{H}^s(\mathbb{T}^d)$

$$C_0^{-s}\|u\|_{\widetilde{H}^s} \leq \|u\|_{H^s} \leq C_0^s\|u\|_{\widetilde{H}^s}. \tag{2.7}$$

We introduce some operator spaces which will be used in the next section. Let μ, λ defined in (1.2) be fixed throughout the paper. We also fix throughout the paper any $\rho \in (0, \frac{1}{3C}]$, where the constant C is the same as in (1.2).

Definition 2.1 Let $M > 0$, $\tau \in (0, 1]$, $\delta \in \{0, 1\}$ and $j \in \mathbb{N}$. We denote by $\mathcal{L}_\tau^{-j}(M, \delta)$ the space of smooth families in time of linear operators $Q(\cdot, t)$ from $C^\infty(\mathbb{T}^d)$ to $\mathscr{D}'(\mathbb{T}^d)$ such that there is a constant $B > 0$ independent of M and ρ, for which one has

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|\Pi_n \partial_t^k Q(\cdot, t)\Pi_{n'}\|_{\mathcal{L}(L^2)} &\leq BM^{k+(j+\delta-1)_+}\Big[\Big(k + (j + \delta - 1)_+\Big)!\Big]^{\max(2,\,\mu)} \\ &\times e^{-\rho\,|n-n'|^{\frac{1}{\lambda}}}\big\langle n - n'\big\rangle^{-(d+2)}\Big(1 + \max(|n|, |n'|)\Big)^{-j\,\tau}\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}}\end{aligned} \tag{2.8}$$

for any $k \in \mathbb{N}$, any $n, n' \in \mathbb{Z}^d$. The best constant B will be denoted by $\|Q\|_{j,\delta}^{(M,\tau)}$. This defines a seminorm of $\mathcal{L}_\tau^{-j}(M, \delta)$.

The notation $\|Q\|_{j,\delta}^{(M,\tau)}$ will be abbreviated to $\|Q\|_{j,\delta}$ when M, τ are fixed and there is no confusion.

Remark 2.1 In comparison with the space introduced in Delort [4], we have added a cut-off in the definition, which depends on the size of j. This ensures that the composition of two elements in the space is essentially in the same space and the seminorm can be controlled by an absolute constant times the product of those of the original two operators. This will be described precisely in Proposition 2.7 and it is important to obtain the logarithmic growth of Sobolev norms.

Remark 2.2 As we shall see in Proposition 3.2, we chose the quantity $M^{k+(j+\delta-1)_+}\big[(k + (j + \delta - 1))_+!\big]^{\max(2,\,\mu)}$ to ensure that all the operators which will be used to conjugate the equation (1.1) are in the same type of space, i.e., $\mathcal{L}_\tau^{-j}(M, \delta)$.

Definition 2.2 Let $M > 0$, $\tau \in (0, 1]$, $\delta \in \{0, 1\}$ and $j \in \mathbb{N}$. We denote by $\widetilde{\mathcal{L}}_\tau^{-j}(M, \delta)$ the subspace of $\mathcal{L}_\tau^{-j}(M, \delta)$ consisting of those elements $Q(\cdot, t) \in \mathcal{L}_\tau^{-j}(M, \delta)$ such that (2.8) holds with the cut-off $\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}}$ replaced by $\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(2+j)}\}}$. We also denote by $\overline{\mathcal{L}}_\tau^{-j}(M, \delta)$ the set of those $Q(\cdot, t) \in \widetilde{\mathcal{L}}_\tau^{-j}(M, \delta)$ such that (2.8) holds with the cut-off $\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}}$ replaced by
$\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(2+j)},\ ||n|^2-|n'|^2|>\frac{1}{4}(|n|+|n'|)^{\tau_0}\}}$, where τ_0 is given by Proposition 2.1.

We shall also define some other convenient subspaces of $\widetilde{\mathcal{L}}_\tau^{-j}(M, \delta)$ and $\overline{\mathcal{L}}_\tau^{-j}(M, \delta)$.

Definition 2.3 Let $M > 0$, $\tau \in (0, 1]$, $\delta \in \{0, 1\}$ and $j \in \mathbb{N}$. We denote by $\widetilde{\mathcal{L}}_{\tau,D}^{-j}(M, \delta)$ (resp. $\widetilde{\mathcal{L}}_{\tau,ND}^{-j}(M, \delta)$) the subspace of $\widetilde{\mathcal{L}}_\tau^{-j}(M, \delta)$ given by those operators $Q(\cdot, t) \in \widetilde{\mathcal{L}}_\tau^{-j}(M, \delta)$ such that for any $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ (resp. any $\alpha \in \Lambda$) $\widetilde{\Pi}_\alpha Q\widetilde{\Pi}_\beta \equiv 0$ (resp. $\widetilde{\Pi}_\alpha Q\widetilde{\Pi}_\alpha \equiv 0$). We also set

$$\overline{\mathcal{L}}_{\tau,D}^{-j}(M, \delta) = \overline{\mathcal{L}}_\tau^{-j}(M, \delta) \cap \widetilde{\mathcal{L}}_{\tau,D}^{-j}(M, \delta),$$
$$\overline{\mathcal{L}}_{\tau,ND}^{-j}(M, \delta) = \overline{\mathcal{L}}_\tau^{-j}(M, \delta) \cap \widetilde{\mathcal{L}}_{\tau,ND}^{-j}(M, \delta).$$

Proposition 2.2 *It follows by definition that if* Q *is an element of* $\widetilde{\mathcal{L}}^{-j}_{\tau,D}(M,\delta)$ *or* $\overline{\mathcal{L}}^{-j}_{\tau,D}(M,\delta)$, *then we have* $[\tilde{\Delta}, Q] = 0$.

Notation 3 *Let* $M > 0$, $\tau \in (0,1]$, $\delta \in \{0,1\}$ *and* $j \in \mathbb{N}$. *If* Q *is an element of* $\mathcal{L}^{-j}_{\tau}(M,\delta)$ *(resp.* $\widetilde{\mathcal{L}}^{-j}_{\tau}(M,\delta)$, $\overline{\mathcal{L}}^{-j}_{\tau}(M,\delta)$*), we denote*

$$Q_D = \sum_{\alpha\in\Lambda} \widetilde{\Pi}_\alpha Q \widetilde{\Pi}_\alpha, \quad Q_{ND} = \sum_{\substack{\alpha,\beta\in\Lambda\\ \alpha\neq\beta}} \widetilde{\Pi}_\alpha Q \widetilde{\Pi}_\beta. \tag{2.9}$$

By definition we immediately have

$$\begin{aligned} &\|Q_D\|_{j,\delta} \le \|Q\|_{j,\delta}, \quad \|Q_{ND}\|_{j,\delta} \le \|Q\|_{j,\delta},\\ &Q_D \in \mathcal{L}^{-j}_{\tau,D}(M,\delta) \ \big(\text{resp. } \widetilde{\mathcal{L}}^{-j}_{\tau,D}(M,\delta),\ \overline{\mathcal{L}}^{-j}_{\tau,D}(M,\delta)\big),\\ &Q_{ND} \in \mathcal{L}^{-j}_{\tau,ND}(M,\delta) \ \big(\text{resp. } \widetilde{\mathcal{L}}^{-j}_{\tau,ND}(M,\delta),\ \overline{\mathcal{L}}^{-j}_{\tau,ND}(M,\delta)\big). \end{aligned} \tag{2.10}$$

Proposition 2.3 *Let* $M > 0$, $\tau \in (0,\tau_0]$, $\delta \in \{0,1\}$ *and* $j \in \mathbb{N}^*$. *Here* τ_0 *is given by Proposition 2.1. Assume* $S \in \overline{\mathcal{L}}^{-(j-1)}_{\tau,ND}(M,\delta)$. *Then the equation* $[Q,\Delta] = -S$ *defines an element* $Q \in \mathcal{L}^{-j}_{\tau}(M,0)$ *with* $\|Q\|_{j,0} \lesssim \|S\|_{j-1,\delta}$. *If* S *is self-adjoint, then* $Q^* = -Q$, *where* Q^* *denote the adjoint of* Q *(at fixed time, for the usual* L^2*-pairing).*

Proof The equation $[Q,\Delta] = -S$ may be written

$$(|n'|^2 - |n|^2)\Pi_n Q \Pi_{n'} = \Pi_n S \Pi_{n'}. \tag{2.11}$$

To define $Q \in \mathcal{L}^{-j}_{\tau}(M,0)$, we only need to estimate $\|\Pi_n \partial_t^k Q \Pi_{n'}\|_{\mathcal{L}(L^2)}$ when it is non zero. So we may assume both sides of (2.11) are non zero. Since $S \in \overline{\mathcal{L}}^{-(j-1)}_{\tau,ND}(M,\delta)$, we then have

$$|n-n'| \leqslant \frac{\max(|n|,|n'|)}{10(1+j)}, \quad \big||n|^2 - |n'|^2\big| > \frac{1}{4}(|n|+|n'|)^{\tau_0},$$

which, together with (2.11) and the fact $\tau \le \tau_0$, allows us to deduce

$$\begin{aligned}\sup_t \|\Pi_n \partial_t^k Q \Pi_{n'}\|_{\mathcal{L}(L^2)} &\lesssim \|S\|_{j-1,\delta} M^{k+(j-1)_+}\Big[\big(k+(j-1)_+\big)!\Big]^{\max(2,\mu)}\\ &\quad\times e^{-\rho\,|n-n'|^{\frac{1}{\lambda}}}\langle n-n'\rangle^{-(d+2)}\Big(1+\max(|n|,|n'|)\Big)^{-j\tau}\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}}.\end{aligned}$$

This means $Q \in \mathcal{L}^{-j}_{\tau}(M,0)$ and $\|Q\|_{j,0} \lesssim \|S\|_{j-1,\delta}$. If S is self-adjoint, then by (2.11) we see that $Q^* = -Q$. This concludes the proof. □

We shall also need the following remainder operators which raise the order of regularity as much as we want.

Definition 2.4 Let $M > 0$, $\tau \in (0,1]$ and $j \in \mathbb{N}$. We denote by $\mathcal{R}^{-\infty}_j(M,\tau)$ the space of smooth families in time of linear operators $R(\cdot,t)$ from $C^\infty(\mathbb{T}^d)$ to $\mathscr{D}'(\mathbb{T}^d)$ such that there is a constant $B > 0$ independent of M, for which one has

$$\begin{aligned}\sup_{t\in\mathbb{R}} \|\Pi_n \partial_t^k R(\cdot,t)\Pi_{n'}\|_{\mathcal{L}(L^2)} &\le B M^{N+j+k}\big((j+k)!\big)^{\max(2,\mu)} N!\\ &\quad\times \langle n-n'\rangle^{-(d+2)}\Big(1+\max(|n|,|n'|)\Big)^{-\tau N}\end{aligned} \tag{2.12}$$

for any $k, N \in \mathbb{N}$, any $n, n' \in \mathbb{Z}^d$. The best constant B will be denoted by $|R|_j^{(M,\tau)}$. This defines a seminorm of $\mathcal{R}_j^{-\infty}(M, \tau)$.

Similarly as before, the notation $|R|_j^{(M,\tau)}$ will be abbreviated to $|R|_j$ when M, τ are fixed and there is no confusion.

By definition, we immediately have the following proposition.

Proposition 2.4 *Let $M > 1$, $\tau \in (0, 1]$ and $j \in \mathbb{N}^*$. If $Q \in \mathcal{L}_\tau^{-j}(M, 0)$, then*

$$[i\partial_t, Q] = i\partial_t Q \in \mathcal{L}_\tau^{-j}(M, 1) \quad and \quad \|[i\partial_t, Q]\|_{j,1} \le \|Q\|_{j,0}. \tag{2.13}$$

The elements defined in the above definitions may be extended as bounded linear operators acting on Sobolev spaces.

Proposition 2.5 *Let $M > 0$, $\tau \in (0, 1]$, $\delta \in \{0, 1\}$ and $j \in \mathbb{N}$. Let $Q \in \mathcal{L}_\tau^{-j}(M, \delta)$. Then for any $k \in \mathbb{N}$, $\partial_t^k Q$ extends as a bounded linear operator from $H^s(\mathbb{T}^d)$ to $H^{s+j\tau}(\mathbb{T}^d)$ for any $s \in \mathbb{R}$. Moreover, its operator norm, denoted by $\|\partial_t^k Q\|_{\mathcal{L}(H^s, H^{s+j\tau})}$, satisfies*

$$\|\partial_t^k Q\|_{\mathcal{L}(H^s, H^{s+j\tau})} \lesssim C_1^{|s|} \|Q\|_{j,\delta} M^{k+(j+\delta-1)_+} \Big(\big(k + (j+\delta-1)_+\big)!\Big)^{\max(2,\mu)}, \tag{2.14}$$

where $C_1 > 1$ is an absolute constant. Recall that by $A \lesssim B$ we mean that there is a constant C independent of any other quantities such that $A \le CB$.

Proof Assume $u \in H^s(\mathbb{T}^d)$. Since $|n - n'| \leqslant \frac{\max(|n|,|n'|)}{10(1+j)}$ implies $C_1^{-1}\langle n'\rangle \le \langle n\rangle \le C_1\langle n'\rangle$ for some absolute constant C_1, we compute using (2.8)

$$\begin{aligned}
\|\partial_t^k Qu\|_{H^{s+j\tau}}^2 &= \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2(s+j\tau)} \|\Pi_n \partial_t^k Qu\|_{L^2}^2 \\
&\le \sum_{n\in\mathbb{Z}^d} \left(\sum_{n'\in\mathbb{Z}^d} \langle n\rangle^{s+j\tau} \|\Pi_n \partial_t^k Q \Pi_{n'} u\|_{L^2} \right)^2 \\
&\le \sum_{n\in\mathbb{Z}^d} \Big(\sum_{n'\in\mathbb{Z}^d} \langle n\rangle^{s+j\tau} \|Q\|_{j,\delta} M^{k+(j+\delta-1)_+} \big[\big(k+(j+\delta-1)_+\big)!\big]^{\max(2,\mu)} \\
&\quad \times \langle n-n'\rangle^{-(d+2)} \big(1+\max(|n|,|n'|)\big)^{-j\tau} \mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}} \|\Pi_{n'} u\|_{L^2} \Big)^2 \\
&\le C_1^{2|s|} \|Q\|_{j,\delta}^2 M^{2(k+(j+\delta-1)_+)} \big[\big(k+(j+\delta-1)_+\big)!\big]^{2\max(2,\mu)} \\
&\quad \times \sum_{n\in\mathbb{Z}^d} \left(\sum_{n'\in\mathbb{Z}^d} \langle n-n'\rangle^{-(d+2)} \langle n'\rangle^s \|\Pi_{n'} u\|_{L^2} \right)^2 \\
&\lesssim C_1^{2|s|} \|Q\|_{j,\delta}^2 M^{2(k+(j+\delta-1)_+)} \big[\big(k+(j+\delta-1)_+\big)!\big]^{2\max(2,\mu)} \|u\|_{H^s}^2,
\end{aligned}$$

where in the last step we used Young inequality. The conclusion follows by taking the square root of both sides. □

Proposition 2.6 *Let $M > 0$, $\tau \in (0, 1]$ and $j \in \mathbb{N}$. Let $R \in \mathcal{R}_j^{-\infty}(M, \tau)$. Then operators $R(\cdot, t)$, $\partial_t^k R$ and $[\Delta, R]$ may be extended as bounded linear operators from $H^{-s}(\mathbb{T}^d)$ to $H^{\tau m}(\mathbb{T}^d)$ for any $s \ge 0$ and any $k, m \in \mathbb{N}$. Moreover,*

$$\begin{aligned} \|\partial_t^k R\|_{\mathcal{L}(H^{-s},H^{\tau m})} &\lesssim |R|_j M^{m+\left[\frac{s+1}{\tau}\right]+j+k} \big((j+k)!\big)^{\max(2,\mu)} \left(m+\left[\frac{s+1}{\tau}\right]\right)!,\\ \|[\Delta, R]\|_{\mathcal{L}(H^{-s},H^{\tau m})} &\lesssim |R|_j M^{m+\left[\frac{s+2}{\tau}\right]+j} \big(j!\big)^{\max(2,\mu)} \left(m+\left[\frac{s+2}{\tau}\right]\right)!, \end{aligned} \tag{2.15}$$

where $[\cdot]$ *means the integer part of a real number.*

Proof Let $s \geq 0$, $m \in \mathbb{N}$, $u \in H^{-s}(\mathbb{T}^d)$. For $k \in \mathbb{N}$, we have by (2.12) with $N = m + \left[\frac{s+1}{\tau}\right]$

$$\begin{aligned} \|\partial_t^k Ru\|^2_{H^{\tau m}} &= \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2\tau m} \|\Pi_n \partial_t^k Ru\|^2_{L^2}\\ &\leq \sum_{n\in\mathbb{Z}^d} \Bigg[\sum_{n'\in\mathbb{Z}^d} \langle n\rangle^{\tau m} |R|_j M^{m+\left[\frac{s+1}{\tau}\right]+j+k} \big((j+k)!\big)^{\max(2,\mu)} \left(m+\left[\frac{s+1}{\tau}\right]\right)!\\ &\qquad \times \langle n-n'\rangle^{-(d+2)} \big(1+\max(|n|,|n'|)\big)^{-\tau\left(m+\left[\frac{s+1}{\tau}\right]\right)} \|\Pi_{n'}u\|_{L^2}\Bigg]^2\\ &\leq |R|_j^2 M^{2\left(m+\left[\frac{s+1}{\tau}\right]+j+k\right)} \big((j+k)!\big)^{2\max(2,\mu)} \left[\left(m+\left[\frac{s+1}{\tau}\right]\right)!\right]^2\\ &\quad \times \sum_{n\in\mathbb{Z}^d} \Bigg[\sum_{n'\in\mathbb{Z}^d} \langle n-n'\rangle^{-(d+2)} \langle n'\rangle^{-s} \|\Pi_{n'}u\|_{L^2}\Bigg]^2\\ &\lesssim |R|_j^2 M^{2\left(m+\left[\frac{s+1}{\tau}\right]+j+k\right)} \big((j+k)!\big)^{2\max(2,\mu)} \left[\left(m+\left[\frac{s+1}{\tau}\right]\right)!\right]^2 \|u\|^2_{H^{-s}}, \end{aligned}$$

where in the last step we used Young inequality. The first inequality of (2.15) follows by taking the square root of both sides. The second inequality follows by a similar argument and by noting that $\||n|^2 - |n'|^2| \lesssim \langle n-n'\rangle \big(1+\max(|n|,|n'|)\big)$ and taking $N = m + \left[\frac{s+2}{\tau}\right]$ in (2.12). □

When one conjugates the original equation, one needs to compute the composition of two elements in $\mathcal{L}_\tau^{-j}(M,\delta)$ and the commutator $[i\partial_t, Q]$ for $Q \in \mathcal{L}_\tau^{-j}(M,0)$. First of all let us introduce some notation before we give a precise description of that.

Notation 4 *Recall that* Q^* *denote the adjoint of* $Q \in \mathcal{L}_\tau^{-j}(M,\delta)$ ($\delta \in \{0,1\}$*, at fixed time, for the usual* L^2*-pairing). If* $Q_i \in \mathcal{L}_\tau^{-j_i}(M,\delta_i)$, $j_i \in \mathbb{N}$, $\delta_i \in \{0,1\}$, $i = 1,2$*, we then denote*

$$\begin{aligned} \mathscr{M}(Q_1,Q_2) &= \sum_{n,n'\in\mathbb{Z}^d} \Pi_n\big(Q_1\circ Q_2\big)\Pi_{n'}\mathbf{1}_{\left\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(2+j_1+j_2)}\right\}},\\ \mathscr{R}(Q_1,Q_2) &= \sum_{n,n'\in\mathbb{Z}^d} \Pi_n\big(Q_1\circ Q_2\big)\Pi_{n'}\mathbf{1}_{\left\{|n-n'|> \frac{\max(|n|,|n'|)}{10(2+j_1+j_2)}\right\}}. \end{aligned} \tag{2.16}$$

We shall also denote

$$\begin{aligned} \mathscr{M}'(Q_1,Q_2) &= \mathscr{M}(Q_1,Q_2) + \mathscr{M}(Q_1,Q_2)^*,\\ \mathscr{R}'(Q_1,Q_2) &= \mathscr{R}(Q_1,Q_2) + \mathscr{R}(Q_1,Q_2)^*. \end{aligned} \tag{2.17}$$

Note that $\mathscr{M}(Q_1,Q_2)^* = \mathscr{M}(Q_2^*,Q_1^*)$ and the operator $\mathscr{M}(Q_1,Q_2)$ is the main part of the operator obtained by composing Q_1 and Q_2. As we shall see, it essentially falls into the same operator class as the original ones. The remainder part, i.e., $\mathscr{R}(Q_1,Q_2)$ is a regularizing operator. Moreover, $\mathscr{M}'(Q_1, Q_2)$ and $\mathscr{R}'(Q_1, Q_2)$ are obviously self-adjoint. Remark that for $\rho > 0$, $a \in [1, +\infty)$, $\tau \in (0, 1]$, denoting

$$\theta_0(\rho, a, \tau) = \min\left(\frac{2}{60^{1/(2a)}}\left(\log\frac{162}{100}\right)^{\frac{1}{2}}(\rho\tau)^{\frac{1}{2}}, \frac{\rho}{\left(30\sqrt{2}\,\right)^{1/a}}\right), \tag{2.18}$$

we have that

$$\exp\left\{-\rho\left(\frac{x}{10(2+t)}\right)^{\frac{1}{a}}\right\}(1+x)^{-t\tau}\left(\frac{100}{81}\right)^{t\tau} \le \exp\left\{-\theta_0(\rho, a, \tau)(x+1)^{\frac{1}{2a}}\right\} \tag{2.19}$$

holds for any $x \ge 1$, any $t \ge 0$. Denote

$$\theta_1(\rho, \tau) = 1 + \max_{a \ge 1}\,[\theta_0(\rho, a, \tau)]^{-1}. \tag{2.20}$$

Proposition 2.7 *Let $\tau \in (0, 1]$ and $j_1, j_2 \in \mathbb{N}$. Let $M > \theta_1(\rho, \tau)$ and $j = j_1 + j_2$. Assume $Q_1 \in \mathcal{L}_\tau^{-j_1}(M, 0)$ and $Q_2 \in \mathcal{L}_\tau^{-j_2}(M, 0)$. Then one has*

$$Q_1 \circ Q_2 = \mathscr{M}(Q_1, Q_2) + \mathscr{R}(Q_1, Q_2) \tag{2.21}$$

with

$$\begin{aligned}\mathscr{M}(Q_1, Q_2) &\in \widetilde{\mathcal{L}}_\tau^{-j}(M, 0), \quad \|\mathscr{M}(Q_1, Q_2)\|_{j,0} \lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0},\\ \mathscr{R}(Q_1, Q_2) &\in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda}), \quad |\mathscr{R}(Q_1, Q_2)|_j \lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}.\end{aligned} \tag{2.22}$$

Proof We only need to check (2.22). For $k \in \mathbb{N}$, we have by (2.8)

$$\begin{aligned}&\|\Pi_n \partial_t^k \mathscr{M}(Q_1, Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)}\\ &\le \sum_{k_1+k_2=k}\binom{k}{k_1}\sum_{\ell\in\mathbb{Z}^d}\|\Pi_n\partial_t^{k_1}Q_1\Pi_\ell\|_{\mathcal{L}(L^2)}\|\Pi_\ell\partial_t^{k_2}Q_2\Pi_{n'}\|_{\mathcal{L}(L^2)}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(2+j)}\}}\\ &\le \sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{k+(j_1-1)_++(j_2-1)_+}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(2+j)}\}}\\ &\times\Big[\big(k_1+(j_1-1)_+\big)!\Big]^{\max(2,\mu)}\Big[\big(k_2+(j_2-1)_+\big)!\Big]^{\max(2,\mu)}\\ &\times e^{-\rho|n-n'|^{\frac{1}{\lambda}}}\langle n-\ell\rangle^{-(d+2)}\langle\ell-n'\rangle^{-(d+2)}\\ &\times\Big(1+\max(|n|,|\ell|)\Big)^{-j_1\tau}\Big(1+\max(|\ell|,|n'|)\Big)^{-j_2\tau}\\ &\times\mathbf{1}_{\{|n-\ell|\leqslant\frac{\max(|n|,|\ell|)}{10(1+j_1)}\}}\mathbf{1}_{\{|n'-\ell|\leqslant\frac{\max(|n'|,|\ell|)}{10(1+j_2)}\}}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(2+j)}\}},\end{aligned} \tag{2.23}$$

where we have used the following inequality:

$$|n-\ell|^{\frac{1}{\lambda}} + |\ell-n'|^{\frac{1}{\lambda}} \ge |n-n'|^{\frac{1}{\lambda}} \quad \text{when } \lambda \ge 1.$$

We need to estimate the following two terms:

$$\mathbf{I} \overset{def}{=} \sum_{k_1+k_2=k}\binom{k}{k_1}\Big[\big(k_1+(j_1-1)_+\big)!\Big]^{\max(2,\mu)}\Big[\big(k_2+(j_2-1)_+\big)!\Big]^{\max(2,\mu)},$$

$\mathbf{II} \overset{def}{=}$ the last two lines of (2.23).

To obtain an estimate for **I**, let us first estimate

$$\mathbf{I}' = \sum_{k_1+k_2=k} \binom{k}{k_1} \Big[\big(k_1 + (j_1-1)_+\big)!\Big]^2 \Big[\big(k_2 + (j_2-1)_+\big)!\Big]^2.$$

If neither of $(j_1 - 1)_+$ and $(j_2 - 1)_+$ is larger than 0, then

$$\mathbf{I}' = \sum_{k_1+k_2=k} k!k_1!k_2! \le 3(k!)^2.$$

If only one of $(j_1 - 1)_+$ and $(j_2 - 1)_+$ is larger than 0, for instance, $(j_2 - 1)_+ > 0$, then

$$\begin{aligned}
\mathbf{I}' &= \sum_{k_1+k_2=k} \binom{k}{k_1} [k_1!]^2 \big[(k_2 + j_2 - 1)!\big]^2 \\
&\le 2\big[(k + j - 1)!\big]^2 + \sum_{\substack{k_1+k_2=k \\ k_1\ge 1, k_2 \ge 1}} k!k_1!\big(k_2 + j_2 - 1 + k_1 - 1\big)\dots(2 + k_1 - 1) \\
&\qquad\qquad \times (k_2 + j_2 - 1 + k_1)\dots(k_2 + 1 + k_1) \\
&\le 3\big[(k + j - 1)!\big]^2,
\end{aligned}$$

while if both of $(j_1 - 1)_+$ and $(j_2 - 1)_+$ are larger than 0, then

$$\begin{aligned}
\mathbf{I}' &= \sum_{k_1+k_2=k} \binom{k}{k_1} \big[(k_1 + j_1 - 1)!\big]^2 \big[(k_2 + j_2 - 1)!\big]^2 \\
&= \sum_{k_1+k_2=k} k!(k_1 + j_1 - 1)!(k_2 + j_2 - 1)! \\
&\quad \times (k_1 + j_1 - 1)\dots(k_1 + 1)(k_2 + j_2 - 1)\dots(k_2 + 1) \\
&\le \sum_{k_1+k_2=k} k!(k_1 + j_1 - 1 + k_2 + j_2 - 1)\dots(1 + k_2 + j_2 - 1)(k_2 + j_2 - 1)! \\
&\quad \times (k_1 + j_1 - 1 + k_2 + j_2 - 1)\dots(k_1 + 1 + k_2 + j_2 - 1) \\
&\quad \times (k_2 + j_2 - 1 + k_1)\dots(k_2 + 1 + k_1) \\
&\le \big[(k + j - 1)!\big]^2.
\end{aligned}$$

Thus we always have

$$\mathbf{I}' \le 3\big[\big(k + (j - 1)_+\big)!\big]^2. \tag{2.24}$$

Since

$$\Big[\big(k_1 + (j_1 - 1)_+\big)!\Big]^{(\mu-2)_+} \Big[\big(k_2 + (j_2 - 1)_+\big)!\Big]^{(\mu-2)_+} \le \big[(k + (j - 1)_+)!\big]^{(\mu-2)_+},$$

we have by (2.24)

$$\mathbf{I} \le 3\big[\big(k + (j - 1)_+\big)!\big]^{\max(2,\mu)}. \tag{2.25}$$

We first assume $|n'| \ge |n|$ when estimating **II**. From the cut-offs we deduce $|n'| \le 2|n|$ so that

$$|n - n'| \le \frac{\max(|n|, |n'|)}{10(2 + j)} = \frac{|n'|}{10(2 + j)} \le \frac{|n|}{5(1 + j)}.$$

Therefore

$$\begin{aligned}\mathbf{II} &\le \big(1+|n|\big)^{-j_1\tau}\big(1+|n'|\big)^{-j_2\tau}\mathbf{1}_{\{|n-n'|\leqslant \frac{|n|}{5(1+j)}\}}\\ &\le \big(1+|n'|\big)^{-j\,\tau}\big(1+\frac{|n-n'|}{1+|n|}\big)^{j_1\tau}\mathbf{1}_{\{|n-n'|\leqslant \frac{|n|}{5(1+j)}\}}\\ &\le \big(1+|n'|\big)^{-j\,\tau}\big(1+\frac{1}{5(1+j)}\big)^{j\,\tau}\\ &\le 3\big(1+|n'|\big)^{-j\,\tau}.\end{aligned}$$

We may get an analogue when $|n|\ge|n'|$ and thus we obtain

$$\mathbf{II}\le 3\big(1+\max(|n|,|n'|)\big)^{-j\,\tau}. \tag{2.26}$$

Plugging (2.25), (2.26) into (2.23) and using the fact that

$$\sum_{\ell\in\mathbb{Z}^d}\langle n-\ell\rangle^{-(d+2)}\langle \ell-n'\rangle^{-(d+2)}\lesssim\langle n-n'\rangle^{-(d+2)}, \tag{2.27}$$

we obtain

$$\begin{aligned}\|\Pi_n\partial_t^k\mathscr{M}(Q_1,Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)} &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{k+(j-1)_+}\big[\big(k+(j-1)_+\big)!\big]^{\max(2,\mu)}\\ &\quad\times e^{-\rho|n-n'|^{\frac{1}{\lambda}}}\langle n-n'\rangle^{-(d+2)}\big(1+\max(|n|,|n'|)\big)^{-j\,\tau}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(2+j)}\}},\end{aligned}$$

which implies the claims in the first line of (2.22).

We are left with estimating the remainder operator. We have for $k\in\mathbb{N}$

$$\begin{aligned}&\|\Pi_n\partial_t^k\mathscr{R}(Q_1,Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)}\\ &\le \sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|\Pi_n\partial_t^{k_1}Q_1\Pi_\ell\|_{\mathcal{L}(L^2)}\|\Pi_\ell\partial_t^{k_2}Q_2\Pi_{n'}\|_{\mathcal{L}(L^2)}\mathbf{1}_{\{|n-n'|>\frac{\max(|n|,|n'|)}{10(2+j)}\}}\\ &\le \sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{k+(j_1-1)_++(j_2-1)_+}\mathbf{1}_{\{|n-n'|>\frac{\max(|n|,|n'|)}{10(2+j)}\}}\\ &\quad\times\Big[\big(k_1+(j_1-1)_+\big)!\Big]^{\max(2,\mu)}\Big[\big(k_2+(j_2-1)_+\big)!\Big]^{\max(2,\mu)}\\ &\quad\times e^{-\rho|n-n'|^{\frac{1}{\lambda}}}\langle n-\ell\rangle^{-(d+2)}\langle\ell-n'\rangle^{-(d+2)}\\ &\quad\times\Big(1+\max(|n|,|\ell|)\Big)^{-j_1\tau}\Big(1+\max(|\ell|,|n'|)\Big)^{-j_2\tau}\\ &\quad\times\mathbf{1}_{\{|n-\ell|\leqslant\frac{\max(|n|,|\ell|)}{10(1+j_1)}\}}\mathbf{1}_{\{|n'-\ell|\leqslant\frac{\max(|n'|,|\ell|)}{10(1+j_2)}\}}\mathbf{1}_{\{|n-n'|>\frac{\max(|n|,|n'|)}{10(2+j)}\}}.\end{aligned} \tag{2.28}$$

We only deal with the case $|n'|\ge|n|$. The other will be the same. Thus we assume

$$\max(|n|,|n'|)=|n'|. \tag{2.29}$$

We denote by **III** the last two lines of (2.28). We may assume that **III** is non-zero when estimating it. Thus by the cut-offs we deduce

$$|n|\ge\frac{9}{10}|\ell|\ge\frac{81}{100}|n'|>0$$

so that

$$\mathbf{III} \le (1+|n|)^{-j_1\tau}(1+|n'|)^{-j_2\tau}\mathbf{1}_{\{\frac{81}{100}|n'|\le|n|\}} \le (1+|n'|)^{-j\,\tau}\Big(\frac{100}{81}\Big)^{j_1\tau}.$$

Thus by the assumption (2.29), (2.19), (2.20) and the assumption on M, we have

$$\begin{aligned} e^{-\rho|n-n'|^{\frac{1}{\lambda}}}\mathbf{1}_{\{|n-n'|>\frac{\max(|n|,|n'|)}{10(2+j)}\}}\mathbf{III} &\le e^{-\rho|n-n'|^{\frac{1}{\lambda}}}(1+|n'|)^{-j\,\tau}\Big(\frac{100}{81}\Big)^{j_1\tau}\mathbf{1}_{\{|n-n'|>\frac{|n'|}{10(2+j)}\}} \\ &\le e^{-\rho(\frac{|n'|}{10(2+j)})^{\frac{1}{\lambda}}}(1+|n'|)^{-j\,\tau}\Big(\frac{100}{81}\Big)^{j\,\tau} \\ &\le e^{-\theta_0(\rho,\lambda,\tau)(|n'|+1)^{\frac{1}{2\lambda}}} \\ &\le \Big(\frac{1}{\theta_0(\rho,\lambda,\tau)}\Big)^N N!\big(1+\max(|n|,|n'|)\big)^{-\frac{N}{2\lambda}} \\ &\le M^N N!\big(1+\max(|n|,|n'|)\big)^{-\frac{N}{2\lambda}}. \end{aligned} \tag{2.30}$$

Plugging (2.30), (2.27), (2.25) into (2.28), we obtain for $k\in\mathbb{N}$ and for any $N\in\mathbb{N}$

$$\begin{aligned} \sup_t \|\Pi_n\partial_t^k\mathscr{R}(Q_1,Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)} &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{N+k+j} \\ &\quad\times\big((j+k)!\big)^{\max(2,\mu)}N!\langle n-n'\rangle^{-(d+2)}\big(1+\max(|n|,|n'|)\big)^{-\frac{N}{2\lambda}}, \end{aligned}$$

which gives the claims in the second line of (2.22) and concludes the proof. □

We also have the following proposition.

Proposition 2.8 *Let* $\tau\in(0,1]$ *and* $j_1, j_2\in\mathbb{N}^*$. *Let* $M>\theta_1(\rho,\tau)$ *and* $j=j_1+j_2$. *Assume* $Q_1\in\mathcal{L}_\tau^{-j_1}(M,0)$ *and* $Q_2\in\mathcal{L}_\tau^{-j_2}(M,1)$. *Then one has*

$$\begin{aligned} Q_1\circ Q_2 &= \mathscr{M}(Q_1,Q_2)+\mathscr{R}(Q_1,Q_2), \\ Q_2\circ Q_1 &= \mathscr{M}(Q_2,Q_1)+\mathscr{R}(Q_2,Q_1), \end{aligned} \tag{2.31}$$

with

$$\begin{aligned} \mathscr{M}(Q_1,Q_2),\mathscr{M}(Q_2,Q_1) &\in \widetilde{\mathcal{L}}_\tau^{-j}(M,0), \\ \mathscr{R}(Q_1,Q_2),\mathscr{R}(Q_2,Q_1) &\in \mathcal{R}_j^{-\infty}\Big(M,\frac{1}{2\lambda}\Big), \\ \|\mathscr{M}(Q_1,Q_2)\|_{j,0}+\|\mathscr{M}(Q_2,Q_1)\|_{j,0} &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,1}, \\ |\mathscr{R}(Q_1,Q_2)|_j+|\mathscr{R}(Q_2,Q_1)|_j &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,1}. \end{aligned} \tag{2.32}$$

Proof The proof is the same as that of Proposition 2.7 except that instead of estimating **I**, we have to estimate

$$\mathbf{I}'' \stackrel{def}{=} \sum_{k_1+k_2=k}\binom{k}{k_1}\Big((k_1+j_1-1)!\Big)^{\max(2,\mu)}\Big(\big(k_2+j_2\big)!\Big)^{\max(2,\mu)},$$

which is less or equals $3\big[\big(k+(j-1)_+\big)!\big]^{\max(2,\mu)}$ if $j_1, j_2\in\mathbb{N}^*$. Note that $M^{k+(j_1-1)_++j_2}\le M^{k+(j-1)_+}$ fails when $j_1=0$ and $j_2\in\mathbb{N}^*$. However, we shall only need to use the result for $j_1, j_2\in\mathbb{N}^*$. □

The following two corollaries are immediate consequences of Proposition 2.7 and 2.8.

Corollary 2.1 *Under the hypotheses of Proposition 2.7, one has*

$$Q_1 \circ Q_2 + (Q_1 \circ Q_2)^* = \mathscr{M}'(Q_1, Q_2) + \mathscr{R}'(Q_1, Q_2). \tag{2.33}$$

Moreover, $\mathscr{M}'(Q_1, Q_2), \mathscr{R}'(Q_1, Q_2)$ *are self-adjoint and we have*

$$\begin{aligned} &\mathscr{M}'(Q_1, Q_2) \in \widetilde{\mathcal{L}}_\tau^{-j}(M, 0), \quad \mathscr{R}'(Q_1, Q_2) \in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda}), \\ &\|\mathscr{M}'(Q_1, Q_2)\|_{j,0} \lesssim \Big(\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0} + \|Q_1^*\|_{j_1,0}\|Q_2^*\|_{j_2,0}\Big), \\ &|\mathscr{R}'(Q_1, Q_2)|_j \lesssim \Big(\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0} + \|Q_1^*\|_{j_1,0}\|Q_2^*\|_{j_2,0}\Big). \end{aligned} \tag{2.34}$$

Corollary 2.2 *Under the hypotheses of Proposition 2.8, one has*

$$\begin{aligned} Q_1 \circ Q_2 + (Q_1 \circ Q_2)^* &= \mathscr{M}'(Q_1, Q_2) + \mathscr{R}'(Q_1, Q_2), \\ Q_2 \circ Q_1 + (Q_2 \circ Q_1)^* &= \mathscr{M}'(Q_2, Q_1) + \mathscr{R}'(Q_2, Q_1). \end{aligned} \tag{2.35}$$

Moreover, $\mathscr{M}'(Q_1, Q_2), \mathscr{R}'(Q_1, Q_2)$ *are self-adjoint and (2.34) holds.* $\mathscr{M}'(Q_2, Q_1)$, $\mathscr{R}'(Q_2, Q_1)$ *respectively have the same properties as that of* $\mathscr{M}'(Q_1, Q_2), \mathscr{R}'(Q_1, Q_2)$.

Proposition 2.9 *Let* $\tau \in \big(0, 1\big]$, $M > \theta_1(\rho, \tau)$ *and* $j \in \mathbb{N}^*$. *Let* $Q \in \mathcal{L}_\tau^{-j}(M, 1)$. *Then one may decompose*

$$Q = \widetilde{Q} + \widetilde{R} \tag{2.36}$$

with

$$\begin{aligned} &\widetilde{Q} \in \widetilde{\mathcal{L}}_\tau^{-j}(M, 1), \quad \|\widetilde{Q}\|_{j,1} \le \|Q\|_{j,1}, \\ &\widetilde{R} \in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda}), \quad |\widetilde{R}|_j \le \|Q\|_{j,1}. \end{aligned} \tag{2.37}$$

Moreover, if we further assume that Q is a self-adjoint operator (for fixed t, Q extends as a bounded linear operator on $L^2(\mathbb{T}^d)$ by Proposition 2.5), so are $\widetilde{Q}$ and $\widetilde{R}$.

Proof Defining

$$\begin{aligned} \widetilde{Q} &= \sum_n \sum_{n'} \Pi_n Q \Pi_{n'} \mathbf{1}_{\{|n-n'| \leqslant \frac{\max(|n|,|n'|)}{10(2+j)}\}}, \\ \widetilde{R} &= \sum_n \sum_{n'} \Pi_n Q \Pi_{n'} \mathbf{1}_{\{|n-n'| > \frac{\max(|n|,|n'|)}{10(2+j)}\}}, \end{aligned}$$

we see that (2.36) holds and that the claims in the first line of (2.37) hold true. For $k, N \in \mathbb{N}$, we have by (2.19) and (2.20)

$$\begin{aligned} &\|\Pi_n \partial_t^k \widetilde{R} \Pi_{n'}\|_{\mathcal{L}(L^2)} \\ &\le \|Q\|_{j,1} M^{k+j} \big((k+j)!\big)^{\max(2,\mu)} \langle n - n'\rangle^{-(d+2)} \\ &\qquad \times e^{-\rho\, |n-n'|^{\frac{1}{\lambda}}} \big(1 + \max(|n|, |n'|)\big)^{-j\tau} \mathbf{1}_{\{\frac{\max(|n|,|n'|)}{10(2+j)} < |n-n'| \leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}} \\ &\le \|Q\|_{j,1} M^{k+j} \big((k+j)!\big)^{\max(2,\mu)} \langle n - n'\rangle^{-(d+2)} e^{-\rho\, (\frac{\max(|n|,|n'|)}{10(2+j)})^{\frac{1}{\lambda}}} \big(1 + \max(|n|, |n'|)\big)^{-j\tau} \\ &\le \|Q\|_{j,1} M^{k+j} \big((k+j)!\big)^{\max(2,\mu)} \langle n - n'\rangle^{-(d+2)} e^{-\theta_0(\rho,\lambda,\tau)(1+\max(|n|,|n'|))^{\frac{1}{2\lambda}}} \\ &\le \|Q\|_{j,1} M^{N+k+j} \big((k+j)!\big)^{\max(2,\mu)} N!\, \langle n - n'\rangle^{-(d+2)} \big(1 + \max(|n|, |n'|)\big)^{-\frac{N}{2\lambda}}. \end{aligned}$$

This gives the claims in the second line of (2.37). The last claim in the proposition follows by the construction of $\widetilde{Q}$ and $\widetilde{R}$. This concludes the proof. □

We shall also need to compute the composition of three elements in $\mathcal{L}_\tau^{-j}(M,0)$. To do that, we first have to compute the composition of one element in $\mathcal{L}_\tau^{-j}(M,0)$ and one in $\mathcal{R}_j^{-\infty}(M,\tau)$.

Proposition 2.10 *Let $\tau,\tau'\in(0,1]$ and $j_1,j_2\in\mathbb{N}$. Let $M>1$ and $j=j_1+j_2$. Assume $Q\in\mathcal{L}_{\tau'}^{-j_1}(M,0)$ and $R\in\mathcal{R}_{j_2}^{-\infty}(M,\tau)$. Then*

$$\begin{aligned} &Q\circ R\in\mathcal{R}_j^{-\infty}(2M,\tau),\quad R\circ Q\in\mathcal{R}_j^{-\infty}(2M,\tau),\\ &|Q\circ R|_j^{(2M,\tau)}+|R\circ Q|_j^{(2M,\tau)}\lesssim\|Q\|_{j_1,0}|R|_{j_2}. \end{aligned}\tag{2.38}$$

Recall the notation $|R|_j^{(M,\tau)}$ in Definition 2.4.

Proof We need to estimate $\|\Pi_n\partial_t^k(Q\circ R)\Pi_{n'}\|_{\mathcal{L}(L^2)}$ and $\|\Pi_n\partial_t^k(R\circ Q)\Pi_{n'}\|_{\mathcal{L}(L^2)}$ for $k\in\mathbb{N}$ and for any $n,n'\in\mathbb{Z}^d$. By definition, the estimate for $\mathbf{I}''$, (2.27)

$$\begin{aligned} &\sup_t\|\Pi_n\partial_t^k(Q\circ R)\Pi_{n'}\|_{\mathcal{L}(L^2)}\\ &\quad\le\sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|\Pi_n\partial_t^{k_1}Q\Pi_\ell\|_{\mathcal{L}(L^2)}\|\Pi_\ell\partial_t^{k_2}R\Pi_{n'}\|_{\mathcal{L}(L^2)}\\ &\quad\le\sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|Q\|_{j_1,0}|R|_{j_2}M^{N+j+k}\big[\big(k_1+(j_1-1)_+\big)!\big]^{\max(2,\mu)}\\ &\quad\times\big[(k_2+j_2)!\big]^{\max(2,\mu)}N!\,e^{-\rho|\ell-n|^{\frac{1}{\lambda}}}\langle n-\ell\rangle^{-(d+2)}\langle\ell-n'\rangle^{-(d+2)}\\ &\quad\times\big(1+\max(|n|,|\ell|)\big)^{-j_1\tau'}\big(1+\max(|\ell|,|n'|)\big)^{-\tau N}\mathbf{1}_{\{|\ell-n|\leqslant\frac{\max(|n|,|\ell|)}{10(1+j_1)}\}}\\ &\quad\lesssim\|Q\|_{j_1,0}|R|_{j_2}(2M)^{N+j+k}\big((k+j)!\big)^{\max(2,\mu)}N!\\ &\quad\times\langle n-n'\rangle^{-(d+2)}\big(1+\max(|n|,|n'|)\big)^{-\tau N} \end{aligned}\tag{2.39}$$

holds for any $N\in\mathbb{N}$, any $n,n'\in\mathbb{Z}^d$, where in the last step we have used

$$\big(1+\max(|\ell|,|n'|)\big)^{-\tau N}\mathbf{1}_{\{|\ell-n|\leqslant\frac{\max(|n|,|\ell|)}{10(1+j_1)}\}}\le 2^N\big(1+\max(|n|,|n'|)\big)^{-\tau N}.$$

With the same reasoning, we see that the quantity after the last sign of inequality in (2.39) is also an upper bound of $\|\Pi_n\partial_t^k(R\circ Q)\Pi_{n'}\|_{\mathcal{L}(L^2)}$. Thus (2.38) holds and this concludes the proof. □

Combining Propositions 2.7 and 2.10 and remarking that $\mathcal{R}_j^{-\infty}(M,\tau)\subset\mathcal{R}_j^{-\infty}(2M,\tau)$, we obtain:

Proposition 2.11 *Let $\tau\in(0,1]$ and $M>\theta_1(\rho,\tau)$ with θ_1 defined by (2.20). Let $j_1,j_2,j_3\in\mathbb{N}$ and $j=j_1+j_2+j_3$. Assume $Q_i\in\mathcal{L}_\tau^{-j_i}(M,0)$, $i=1,2,3$. Then one may decompose*

$$Q_1\circ Q_2\circ Q_3=Q+R\tag{2.40}$$

with

$$Q \in \widetilde{\mathcal{L}}_\tau^{-j}(M,0), \quad \|Q\|_{j,0} \lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}\|Q_3\|_{j_3,0},$$
$$R \in \mathcal{R}_j^{-\infty}(2M, \frac{1}{2\lambda}), \quad |R|_j^{(2M,\frac{1}{2\lambda})} \lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}\|Q_3\|_{j_3,0}, \tag{2.41}$$

where the notation $|R|_j^{(2M,\frac{1}{2\lambda})}$ *is indicated in Definition 2.4.*

By (2.40), its adjoint equation and (2.41) we have the following corollary which is an analogue of Corollary 2.1.

Corollary 2.3 *Under the hypotheses of Proposition 2.11, one may find self-adjoint operators* $Q \in \widetilde{\mathcal{L}}_\tau^{-j}(M,0)$, $R \in \mathcal{R}_j^{-\infty}(2M, \frac{1}{2\lambda})$ *such that*

$$Q_1 \circ Q_2 \circ Q_3 + (Q_1 \circ Q_2 \circ Q_3)^* = Q + R \tag{2.42}$$

with

$$\|Q\|_{j,0} \lesssim \Big(\prod_{i=1}^{3}\|Q_i\|_{j_i,0} + \prod_{i=1}^{3}\|Q_i^*\|_{j_i,0}\Big),$$
$$|R|_j^{(2M,\frac{1}{2\lambda})} \lesssim \Big(\prod_{i=1}^{3}\|Q_i\|_{j_i,0} + \prod_{i=1}^{3}\|Q_i^*\|_{j_i,0}\Big). \tag{2.43}$$

3 Conjugating the Equation

The goal of this section is to obtain the following: Roughly speaking, for any given $N \in \mathbb{N}^*$, we want to conjugate the operator $i\partial_t - \Delta + V$ into $i\partial_t - \Delta + V_N' + R_N'$ with V_N' exactly commuting with the modified Laplacian $\tilde{\Delta}$ and R_N' essentially being a bounded linear operator from $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. The process is essentially an induction. Before giving the precise description of the statement, we first present the following proposition.

Proposition 3.1 *Let* $V(x,t)$ *be the potential in the equation (1.1) so that it satisfies (1.2). Let* $\tau \in (0,1]$. *Then one may find* $\overline{M} > 0$ *such that for any* $M > \overline{M}$, *the multiplication operator generated by* $V(x,t)$ *may be written as* $Q_V + R_V$ *with self-adjoint operators* $Q_V \in \widetilde{\mathcal{L}}_\tau^0(M,0)$, $R_V \in \mathcal{R}_0^{-\infty}(M, \frac{1}{\lambda})$. *Moreover,*

$$\|Q_V\|_{0,0} \le h(\lambda,\, d), \quad |R_V|_0 \le h(\lambda,\, d), \tag{3.1}$$

where $h(\lambda, d)$ *is a constant depending only on* λ, d.

Proof By (1.2), we know that

$$\left|\int_{\mathbb{T}^d} n^\alpha \partial_t^k V(x,t)e^{-inx}dx\right| \le (2\pi)^d C^{k+|\alpha|+1}(k!)^\mu(\alpha!)^\lambda$$

holds for any $n = (n_1,\dots,n_d) \in \mathbb{Z}^d$, any $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$, any $k \in \mathbb{N}$, any $t \in \mathbb{R}$. From this inequality we deduce

$$\frac{1}{\alpha_1!}\cdots\frac{1}{\alpha_d!}\Big(\frac{|n_1|}{C}\Big)^{\frac{\alpha_1}{\lambda}}\cdots\Big(\frac{|n_d|}{C}\Big)^{\frac{\alpha_d}{\lambda}}\|\Pi_n\partial_t^k V(x,t)\|_{L^\infty}^{\frac{1}{\lambda}} \le C^{\frac{k+1}{\lambda}}\big(k!\big)^{\frac{\mu}{\lambda}}. \tag{3.2}$$

Multiplying $2^{-(\alpha_1+\cdots+\alpha_d)}$ in both sides and then taking a sum over $\alpha_1, \ldots, \alpha_d \in \mathbb{N}$, using the fact $|n_1|^{\frac{1}{\lambda}} + \cdots + |n_d|^{\frac{1}{\lambda}} \geq (|n_1| + \cdots + |n_d|)^{\frac{1}{\lambda}} \geq |n|^{\frac{1}{\lambda}}$ for $\lambda \geq 1$, we obtain after some simple calculation

$$\|\Pi_n \partial_t^k V(x,t)\|_{L^\infty} \leq 2^{\lambda d} C^{k+1} (k!)^\mu e^{-\rho_0(\lambda)|n|^{\frac{1}{\lambda}}}, \tag{3.3}$$

where

$$\rho_0(\lambda) = \lambda\,(2C^{\frac{1}{\lambda}})^{-1}.$$

Since $\rho_0(\lambda) - \frac{1}{3C} \geq \frac{1}{6C}$ if $\lambda \geq 1$ and

$$\sup_{r\geq 1} \langle r\rangle^{d+2} \exp\{-\frac{1}{6C} r^{\frac{1}{\lambda}}\} \leq 2^{d+2}\Big(\frac{6C\lambda(d+2)}{e}\Big)^{\lambda(d+2)},$$

we have

$$C2^{\lambda d} \exp\{-\rho_0(\lambda)|n|^{\frac{1}{\lambda}}\} \leq h(\lambda,d) \exp\{-\frac{1}{3C}|n|^{\frac{1}{\lambda}}\}\langle n\rangle^{-(d+2)},$$

where $h(\lambda, d)$ is given by (3.4). Thus by (3.3),

$$\|\Pi_n \partial_t^k V(x,t)\|_{L^\infty} \leq h(\lambda,d) C^k (k!)^\mu e^{-\frac{1}{3C}|n|^{\frac{1}{\lambda}}} \langle n\rangle^{-(d+2)}.$$

Therefore if $\rho \in (0, \frac{1}{3C}]$, by the fact

$$\begin{aligned}\|\Pi_n \partial_t^k V(x,t)\Pi_{n'} u\|_{L^2} &= \Big|\Big\langle \frac{\partial_t^k V e^{in'x}}{(2\pi)^{d/2}}, \frac{e^{inx}}{(2\pi)^{d/2}}\Big\rangle\Big| \Big|\Big\langle u, \frac{e^{in'x}}{(2\pi)^{d/2}}\Big\rangle\Big| \\ &= \Big|\Big\langle \frac{\partial_t^k V}{(2\pi)^{d/2}}, \frac{e^{i(n-n')x}}{(2\pi)^{d/2}}\Big\rangle\Big| \Big|\Big\langle u, \frac{e^{in'x}}{(2\pi)^{d/2}}\Big\rangle\Big| \\ &\leq \|\Pi_{n-n'} \partial_t^k V(x,t)\|_{L^\infty} \|u\|_{L^2}, \quad \forall u \in L^2,\end{aligned}$$

we then have

$$\|\Pi_n \partial_t^k V \Pi_{n'}\|_{\mathcal{L}(L^2)} \leq \|\Pi_{n-n'} \partial_t^k V(x,t)\|_{L^\infty} \leq h(\lambda,d) C^k (k!)^\mu\, e^{-\rho|n-n'|^{\frac{1}{\lambda}}} \langle n-n'\rangle^{-(d+2)}.$$

We define

$$Q_V = \sum_{n\in\mathbb{Z}^d} \sum_{n'\in\mathbb{Z}^d} \Pi_n V \Pi_{n'} \mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{20}\}},$$

$$R_V = \sum_{n\in\mathbb{Z}^d} \sum_{n'\in\mathbb{Z}^d} \Pi_n V \Pi_{n'} \mathbf{1}_{\{|n-n'|> \frac{\max(|n|,|n'|)}{20}\}}.$$

By the above formulas, for any $M \geq C$, we have $Q_V \in \widetilde{\mathcal{L}}_\tau^0(M,0)$ with $\|Q_V\|_{0,0} \leq h(\lambda,\ d)$. For $k \in \{0, 1\}$, we know that

$$\begin{aligned}\|\Pi_n \partial_t^k R_V \Pi_{n'}\|_{\mathcal{L}(L^2)} &\leq h(\lambda,\ d) C^k e^{-\frac{1}{3C}|n-n'|^{\frac{1}{\lambda}}} \langle n-n'\rangle^{-(d+2)} \mathbf{1}_{\{|n-n'|> \frac{\max(|n|,|n'|)}{20}\}} \\ &\leq h(\lambda,\ d) C^k \big(120C\big)^N N! \langle n-n'\rangle^{-(d+2)} \Big(1+\max(|n|,|n'|)\Big)^{-\frac{N}{\lambda}}\end{aligned}$$

holds for any $N \in \mathbb{N}$, where we have used

$$\max(|n|,|n'|)\mathbf{1}_{\{|n-n'|> \frac{\max(|n|,|n'|)}{20}\}} \geq \frac{1}{2}\big(1+\max(|n|,|n'|)\big).$$

If $M > \overline{M} \overset{def}{=} 120C$, then $R_V \in \mathcal{R}_0^{-\infty}(M, \frac{1}{\lambda})$ and $|R_V|_0 \leq h(\lambda, d)$. This concludes the proof. □

Remark 3.1 As we have already seen in the proof,

$$h(\lambda, d) = C2^{\lambda d + d + 2}\Big(\frac{6C\lambda(d+2)}{e}\Big)^{\lambda(d+2)}. \tag{3.4}$$

But this explicit expression is not important in obtaining logarithmic growth of Sobolev norms.

Remark 3.2 Let $\sigma \in (0, \frac{1}{10})$ and $\tau_0 \in (0, \sigma)$ be given by Proposition 2.1. From now on, we fix $\tau = \min(\frac{\tau_0}{\lambda}, \frac{1}{2\lambda}) = \frac{\tau_0}{\lambda}$ and fix $\rho \in (0, \frac{1}{3C}]$. We also fix $M > \max(\overline{M}, 2\theta_1(\rho, \tau)) \geq \frac{2}{\rho}$ so that all the conclusions in Sect. 2 and Proposition 3.1 hold, where $\theta_1(\rho, \tau)$ is given by (2.20). We choose those M, τ because they will be used in the argument of the following proposition. Note that M depends on λ, but this dependence does not matter in the sequel.

The main result of this section is the following:

Proposition 3.2 *Let $m \in \mathbb{N}^*$ and denote $P_0 = i\partial_t - \Delta$. Let K be a large constant. There are sequences $(Q'_j)_{1\leq j\leq m}, (Q''_j)_{1\leq j\leq m}$ satisfying*

$$Q'_j \in \mathcal{L}_\tau^{-j}(M, 0), \quad Q'^*_j = -Q'_j, \quad \|Q'_j\|_{j,0} \leq \frac{K^{j-\frac{1}{2}}}{j^2} h(\lambda, d)^j; \tag{3.5}$$

$$[Q'_j, \Delta] \in \mathcal{L}_\tau^{-(j-1)}(M, 0), \qquad \|[Q'_j, \Delta]\|_{j-1,0} \leq \frac{K^{j-1}}{j^2} h(\lambda, d)^j; \tag{3.6}$$

$$Q''_j \in \mathcal{L}_\tau^{-(j+1)}(M, 0), \quad Q''^*_j = Q''_j, \quad \|Q''_j\|_{j+1,0} \leq \frac{K^{j+\frac{1}{2}}}{(j+1)^2} h(\lambda, d)^{j+1}; \tag{3.7}$$

$$[Q''_j, \Delta] \in \mathcal{L}_\tau^{-j}(M, 0), \qquad \|[Q''_j, \Delta]\|_{j,0} \leq \frac{K^j}{(j+1)^2} h(\lambda, d)^{j+1} \tag{3.8}$$

such that if we set $Q_j = Q'_j + Q''_j$, $Q^m = \sum_{j=1}^m Q_j$

$$\begin{aligned} &\big(I+Q^m\big)^*(P_0+V)\big(I+Q^m\big) \\ &= i\partial_t - \Delta + V^m + \frac{1}{2}\sum_{j=m+1}^{2m+1}\big(S_j P_0 + P_0 S_j\big) + \frac{1}{2}\sum_{j=1}^{2m+1}\big(R_j P_0 + P_0 R_j\big) \\ &\quad + \widetilde{S}_{m+1} + \sum_{j=m+1}^{2m+3}\overline{S}_j + \sum_{j=2}^{2m+3}\overline{R}_j + \sum_{j=1}^{m}\widehat{R}_j \end{aligned} \tag{3.9}$$

where the terms in the right hand side of (3.9) satisfy the following conditions:

- $V^m, S_j, R_j, \widetilde{S}_j, \overline{S}_j, \overline{R}_j, \widehat{R}_j$ *are self-adjoint;*
- $[V^m, \tilde{\Delta}] = 0$;
- $S_j \in \mathcal{L}_\tau^{-(j+1)}(M, 0), \quad \|S_j\|_{j+1,0} \lesssim \frac{K^j}{(j+1)^2} h(\lambda, d)^{j+1}, \quad m+1 \leq j \leq 2m+1;$
- $[\Delta, S_j] \in \widetilde{\mathcal{L}}_\tau^{-j}(M, 0), \quad \|[\Delta, S_j]\|_{j,0} \lesssim \frac{K^{j-\frac{1}{2}}}{(j+1)^2} h(\lambda, d)^{j+1}, \quad m+1 \leq j \leq 2m+1$

- $R_j \in \mathcal{R}^{-\infty}_{j+1}(M,\tau)$, $\quad |R_j|_{j+1} \lesssim \frac{K^j}{(j+1)^2} h(\lambda,d)^{j+1}$, $\quad 1 \le j \le 2m+1$;
- $\widetilde{S}_{m+1} \in \widetilde{\mathcal{L}}^{-m}_\tau(M,1)$, $\quad \|\widetilde{S}_{m+1}\|_{m,1} \lesssim \frac{K^{m-\frac{1}{2}}}{(m+1)^2} h(\lambda,d)^m$;
- $\overline{S}_j \in \widetilde{\mathcal{L}}^{-(j-1)}_\tau(M,0)$, $\quad \|\overline{S}_j\|_{j-1,0} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^j$, $\quad m+1 \le j \le 2m+3$;
- $\overline{R}_j \in \mathcal{R}^{-\infty}_{j-1}(4M,\tau)$, $\quad |\overline{R}|^{(4M,\tau)}_{j-1} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^j$, $\quad 2 \le j \le 2m+3$;
- $\widehat{R}_j \in \mathcal{R}^{-\infty}_{j-1}(M,\tau)$, $\quad |\widehat{R}_j|_{j-1} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^j$, $\quad 1 \le j \le m$.

The notation $|\overline{R}|^{(4M,\tau)}_{j-1}$ is explained in Definition 2.4 and by $A \lesssim B$ we mean that there is an absolute constant C such that $A \le CB$.

Let us first compute the left hand side of (3.9).

Lemma 3.1 *Let Q'_j, Q''_j be given operators satisfying (3.5)–(3.8) for $1 \le j \le m$. Denote $Q'^m = \sum_{j=1}^m Q'_j$, $Q''^m = \sum_{j=1}^m Q''_j$. Then one may find*

- *Elements $(S_j)_{1\le j\le 2m+1}$, $(R_j)_{1\le j\le 2m+1}$ satisfying*
  (1) $S_j \in \widetilde{\mathcal{L}}^{-(j+1)}_\tau(M,0)$, $\quad \|S_j\|_{j+1,0} \lesssim \frac{K^j}{(j+1)^2} h(\lambda,d)^{j+1}$, $1 \le j \le 2m+1$;
  (2) $[\Delta, S_j] \in \widetilde{\mathcal{L}}^{-j}_\tau(M,0)$, $\|[\Delta,S_j]\|_{j,0} \lesssim \frac{K^{j-\frac{1}{2}}}{(j+1)^2} h(\lambda,d)^{j+1}$, $1 \le j \le 2m+1$;
  (3) $R_j \in \mathcal{R}^{-\infty}_{j+1}(M,\tau)$, $\quad |R_j|_{j+1} \lesssim \frac{K^j}{(j+1)^2} h(\lambda,d)^{j+1}$, $\quad 1 \le j \le 2m+1$;
  (4) S_j , R_j *are self-adjoint and depend only on* Q'_ℓ, $1 \le \ell \le \min(j,m)$, Q''_ℓ, $1 \le \ell < \min(j,m+1)$;
- *Elements $(\widetilde{S}_j)_{2\le j\le m+1}$, $(\overline{S}_j)_{2\le j\le 2m+3}$, $(\overline{R}_j)_{2\le j\le 2m+3}$ satisfying*
  (5) $\widetilde{S}_j \in \widetilde{\mathcal{L}}^{-(j-1)}_\tau(M,1)$, $\quad \|\widetilde{S}_j\|_{j-1,1} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^{j-1}$, $\quad 2 \le j \le m+1$;
  (6) $\overline{S}_j \in \widetilde{\mathcal{L}}^{-(j-1)}_\tau(M,0)$, $\quad \|\overline{S}_j\|_{j-1,0} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^j$, $\quad 2 \le j \le 2m+3$;
  (7) $\overline{R}_j \in \mathcal{R}^{-\infty}_{j-1}(4M,\tau)$, $\quad |\overline{R}_j|^{(4M,\tau)}_{j-1} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^j$, $\quad 2 \le j \le 2m+3$;
  (8) $\widetilde{S}_j$, $\overline{S}_j$, $\overline{R}_j$ *are self-adjoint and depend only on* Q'_ℓ, Q''_ℓ, $1 \le \ell < \min(j,m+1)$,

such that

$$
\begin{aligned}
&(I+Q^m)^*(P_0+V)(I+Q^m) \\
&\quad = i\partial_t - \Delta + V + [Q'^m,\Delta] + Q''^m P_0 + P_0 Q''^m \\
&\qquad + \frac{1}{2}\sum_{j=1}^{2m+1}\left(S_j P_0 + P_0 S_j\right) + \frac{1}{2}\sum_{j=1}^{2m+1}\left(R_j P_0 + P_0 R_j\right) \\
&\qquad + \sum_{j=2}^{m+1}\widetilde{S}_j + \sum_{j=2}^{2m+3}\overline{S}_j + \sum_{j=2}^{2m+3}\overline{R}_j .
\end{aligned}
\tag{3.10}
$$

Proof of Lemma 3.1 Using that $(Q'^m)^* = -Q'^m$, $(Q''^m)^* = Q''^m$, we write

$$\begin{aligned}(I+Q^m)^*(P_0+V)(I+Q^m) = i\partial_t - \Delta + V \\ + [Q'^m, \Delta] - [Q'^m, i\partial_t] + Q''^m P_0 + P_0 Q''^m \quad (3.11)\\ + \frac{1}{2}\Big((Q^m)^* Q^m P_0 + P_0 (Q^m)^* Q^m\Big) \quad (3.12)\\ + \frac{1}{2}\Big((Q^m)^*[i\partial_t, Q^m] + [(Q^m)^*, i\partial_t]Q^m\Big) \quad (3.13)\\ + \frac{1}{2}\Big((Q^m)^*[-\Delta, Q^m] + [(Q^m)^*, -\Delta]Q^m\Big) \quad (3.14)\\ + (Q^m)^* V + V Q^m + (Q^m)^* V Q^m. \quad (3.15)\end{aligned}$$

Let us show how the right hand side contributes to that of (3.10). We deal with it term by term.

We write by Corollary 2.1 and Notation 4

$$\begin{aligned}(Q^m)^* Q^m =& \frac{1}{2}\sum_{j=1}^{2m-1} \sum_{\substack{j_1+j_2=j+1\\ 1\le j_1, j_2\le m}} \mathscr{M}'(Q'^*_{j_1}, Q'_{j_2}) + \mathscr{R}'(Q'^*_{j_1}, Q'_{j_2})\\ &+ \sum_{j=2}^{2m} \sum_{\substack{j_1+j_2=j\\ 1\le j_1, j_2\le m}} \mathscr{M}'(Q''^*_{j_1}, Q'_{j_2}) + \mathscr{R}'(Q''^*_{j_1}, Q'_{j_2}) \qquad (3.16)\\ &+ \frac{1}{2}\sum_{j=3}^{2m+1} \sum_{\substack{j_1+j_2=j-1\\ 1\le j_1, j_2\le m}} \mathscr{M}'(Q''^*_{j_1}, Q''_{j_2}) + \mathscr{R}'(Q''^*_{j_1}, Q''_{j_2})\\ =& \sum_{j=1}^{2m-1}(S_j^{(1)} + R_j^{(1)}) + \sum_{j=2}^{2m}(S_j^{(2)} + R_j^{(2)}) + \sum_{j=3}^{2m+1}(S_j^{(3)} + R_j^{(3)})\end{aligned}$$

for self-adjoint operators $S_j^{(i)} \in \widetilde{\mathcal{L}}_\tau^{-(j+1)}(M,0)$, $R_j^{(i)} \in \mathcal{R}_{j+1}^{-\infty}(M, \frac{1}{2\lambda}) \subset \mathcal{R}_{j+1}^{-\infty}(M,\tau)$, $i = 1, 2, 3$, $j = 1, \dots, 2m+1$. We make the following convention: *we set the terms that do not appear to be zero*. For instance, here we set

$$\begin{aligned} S_j^{(1)} &= R_j^{(1)} = 0, \quad j = 2m, 2m+1,\\ S_j^{(2)} &= R_j^{(2)} = 0, \quad j = 1, 2m+1\\ S_j^{(3)} &= R_j^{(3)} = 0, \quad j = 1, 2.\end{aligned}$$

We shall use such a convention throughout the proof of Lemma 3.1. Using (2.34), (3.5), (3.7) and the fact that

$$\sum_{\substack{j_1+j_2=j+1\\ 1\le j_1, j_2\le m}} \frac{1}{j_1^2}\cdot\frac{1}{j_2^2} + 2\sum_{\substack{j_1+j_2=j\\ 1\le j_1, j_2\le m}} \frac{1}{j_1^2}\cdot\frac{1}{j_2^2} + \sum_{\substack{j_1+j_2=j-1\\ 1\le j_1, j_2\le m}} \frac{1}{j_1^2}\cdot\frac{1}{j_2^2} \lesssim \frac{1}{(j+1)^2}, \quad (3.17)$$

we obtain

$$\sum_{i=1}^{3}\Big(\|S_j^{(i)}\|_{j+1,0} + |R_j^{(i)}|_{j+1}\Big) \lesssim \frac{K^j}{(j+1)^2} h(\lambda, d)^{j+1}, \quad 1 \le j \le 2m+1.$$

Defining

$$S_j=\sum_{i=1}^{3}S_j^{(i)},\quad 1\le j\le 2m+1;\quad R_j=\sum_{i=1}^{3}R_j^{(i)},\quad 1\le j\le 2m+1,$$

we know by the construction that S_j, R_j satisfy (1), (3) and (4). Moreover, by expressions (2.16) and (2.17), we get

$$\begin{aligned}[\Delta,S_j]&=\sum_{i=1}^{3}[\Delta,S_j^{(i)}]\\&=\sum_{\substack{j_1+j_2=j+1\\1\le j_1,j_2\le m}}\left[\mathscr{M}([\Delta,Q'^*_{j_1}],Q'_{j_2})+\mathscr{M}(Q'^*_{j_1},[\Delta,Q'_{j_2}])\right]\\&\quad+\sum_{\substack{j_1+j_2=j\\1\le j_1,j_2\le m}}\Big[\mathscr{M}\big([\Delta,Q'^*_{j_1}],Q''_{j_2}\big)+\mathscr{M}\big(Q'^*_{j_1},[\Delta,Q''_{j_2}]\big)\\&\qquad\qquad+\mathscr{M}\big([\Delta,Q''^*_{j_1}],Q'_{j_2}\big)+\mathscr{M}\big(Q''^*_{j_1},[\Delta,Q'_{j_2}]\big)\Big]\\&\quad+\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}}\left[\mathscr{M}([\Delta,Q''^*_{j_1}],Q''_{j_2})+\mathscr{M}(Q''^*_{j_1},[\Delta,Q''_{j_2}])\right],\end{aligned}$$

so we know from (3.5) to (3.8), Proposition 2.7 and (3.17) that $[\Delta,S_j]\in\widetilde{\mathcal{L}}_\tau^{-j}(M,0)$ and $\|[\Delta,S_j]\|_{j,0}\lesssim\frac{K^{j-\frac12}}{(j+1)^2}h(\lambda,d)^{j+1}$. Therefore, (3.12) contributes to the third line of (3.10).

By Propositions 2.4, 2.9, one may decompose

$$-[Q'_{j-1},i\partial_t]=\widetilde{S}_j+\widetilde{R}_j,\quad 2\le j\le m+1 \tag{3.18}$$

with

$$\begin{aligned}&\widetilde{S}_j\in\widetilde{\mathcal{L}}_\tau^{-(j-1)}(M,1),\quad \|\widetilde{S}_j\|_{j-1,1}\le\frac{K^{j-\frac32}}{(j-1)^2}h(\lambda,d)^{j-1},\ 2\le j\le m+1;\\&\widetilde{R}_j\in\mathcal{R}_{j-1}^{-\infty}(M,\frac{1}{2\lambda})\subset\mathcal{R}_{j-1}^{-\infty}(M,\tau),\quad |\widetilde{R}_j|_{j-1}\le\frac{K^{j-\frac32}}{(j-1)^2}h(\lambda,d)^{j-1},\quad 2\le j\le m+1.\end{aligned}\tag{3.19}$$

Since $-[Q'_{j-1},i\partial_t]$ is self-adjoint, so are $\widetilde{S}_j$ and $\widetilde{R}_j$. Thus this determines the first term in the fourth line of (3.10) and $\widetilde{R}_j$ contributes to $\overline{R}_j$.

According to Proposition 2.4, Corollary 2.2 and Notation 4, we may write

$$\begin{aligned}(3.13)&=\frac12\sum_{j=3}^{2m+1}\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q'^*_{j_1},[i\partial_t,Q'_{j_2}]\big)+\mathscr{R}'\big(Q'^*_{j_1},[i\partial_t,Q'_{j_2}]\big),\\&\quad+\frac12\sum_{j=4}^{2m+2}\sum_{\substack{j_1+j_2=j-2\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q''^*_{j_1},[i\partial_t,Q'_{j_2}]\big)+\mathscr{R}'\big(Q''^*_{j_1},[i\partial_t,Q'_{j_2}]\big)\\&\quad+\frac12\sum_{j=4}^{2m+2}\sum_{\substack{j_1+j_2=j-2\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q'^*_{j_1},[i\partial_t,Q''_{j_2}]\big)+\mathscr{R}'\big(Q'^*_{j_1},[i\partial_t,Q''_{j_2}]\big)\end{aligned}$$

$$+\frac{1}{2}\sum_{j=5}^{2m+3}\sum_{\substack{j_1+j_2=j-3\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q''^{*}_{j_1},[i\partial_t,Q''_{j_2}]\big)+\mathscr{R}'\big(Q''^{*}_{j_1},[i\partial_t,Q''_{j_2}]\big)$$

$$=\sum_{j=3}^{2m+1}\big(\overline{S}^{(1)}_j+\overline{R}^{(1)}_j\big)+\sum_{j=4}^{2m+2}\big(\overline{S}^{(2)}_j+\overline{R}^{(2)}_j\big)+\sum_{j=5}^{2m+3}\big(\overline{S}^{(3)}_j+\overline{R}^{(3)}_j\big) \tag{3.20}$$

for self-adjoint operators $\overline{S}^{(i)}_j\in\widetilde{\mathcal{L}}^{-(j-1)}_{\tau}(M,0)$, $\overline{R}^{(i)}_j\in\mathcal{R}^{-\infty}_{j-1}(M,\frac{1}{2\lambda})\subset\mathcal{R}^{-\infty}_{j-1}(M,\tau)$, $1\le i\le 3$, $3\le j\le 2m+3$. Here we have used the convention made on Sect. 3. By the inequalities which are contained in the statement of Corollary 2.2, (2.13), (3.5), (3.7) and (3.17), we obtain

$$\sum_{i=1}^{3}\Big(\|\overline{S}^{(i)}_j\|_{j-1,0}+|\overline{R}^{(i)}_j|_{j-1}\Big)\lesssim\frac{K^{j-2}}{j^2}h(\lambda,d)^{j-1},\quad 3\le j\le 2m+3. \tag{3.21}$$

Now we turn to the term (3.14). Using Notation 4, Corollary 2.1, (3.5)–(3.8), we write

$$\begin{aligned}(3.14)=&\frac{1}{2}\sum_{j=2}^{2m}\sum_{\substack{j_1+j_2=j\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q'^{*}_{j_1},[-\Delta,Q'_{j_2}]\big)+\mathscr{R}'\big(Q'^{*}_{j_1},[-\Delta,Q'_{j_2}]\big)\\
&+\frac{1}{2}\sum_{j=3}^{2m+1}\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q''^{*}_{j_1},[-\Delta,Q'_{j_2}]\big)+\mathscr{R}'\big(Q''^{*}_{j_1},[-\Delta,Q'_{j_2}]\big)\\
&+\frac{1}{2}\sum_{j=3}^{2m+1}\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q'^{*}_{j_1},[-\Delta,Q''_{j_2}]\big)+\mathscr{R}'\big(Q'^{*}_{j_1},[-\Delta,Q''_{j_2}]\big)\\
&+\frac{1}{2}\sum_{j=4}^{2m+2}\sum_{\substack{j_1+j_2=j-2\\1\le j_1,j_2\le m}}\mathscr{M}'\big(Q''^{*}_{j_1},[-\Delta,Q''_{j_2}]\big)+\mathscr{R}'\big(Q''^{*}_{j_1},[-\Delta,Q''_{j_2}]\big)\\
=&\sum_{j=2}^{2m}\Big(\overline{S}^{(4)}_j+\overline{R}^{(4)}_j\Big)+\sum_{j=3}^{2m+1}\Big(\overline{S}^{(5)}_j+\overline{R}^{(5)}_j\Big)+\sum_{j=4}^{2m+2}\Big(\overline{S}^{(6)}_j+\overline{R}^{(6)}_j\Big)\end{aligned}$$

for self-adjoint operators $\overline{S}^{(i)}_j\in\widetilde{\mathcal{L}}^{-(j-1)}_{\tau}(M,0)$, $\overline{R}^{(i)}_j\in\mathcal{R}^{-\infty}_{j-1}(M,\frac{1}{2\lambda})\subset\mathcal{R}^{-\infty}_{j-1}(M,\tau)$, $4\le i\le 6$, $2\le j\le 2m+2$, using the convention made on Sect. 3. By (2.34), (3.5)– (3.8) and (3.17) we have

$$\sum_{i=4}^{6}\Big(\|\overline{S}^{(i)}_j\|_{j-1,0}+|\overline{R}^{(i)}_j|_{j-1}\Big)\lesssim\frac{K^{j-\frac{3}{2}}}{j^2}h(\lambda,d)^{j},\quad 2\le j\le 2m+2.$$

Let us now analyze (3.15). By Proposition 3.1, Corollary 2.1 and Proposition 2.10, we write

$$\begin{aligned}(Q^m)^*V + VQ^m &= \sum_{j=2}^{m+1} \left[Q'^*_{j-1}(Q_V + R_V) + (Q_V + R_V)Q'_{j-1} \right] \\ &\quad + \sum_{j=3}^{m+2} \left[\left(Q''_{j-2}\right)^*(Q_V + R_V) + (Q_V + R_V)Q''_{j-2} \right] \\ &= \sum_{j=2}^{m+1} \left(\overline{S}_j^{(7)} + \overline{R}_j^{(7)} \right) + \sum_{j=3}^{m+2} \left(\overline{S}_j^{(8)} + \overline{R}_j^{(8)} \right)\end{aligned}$$

for self-adjoint operators $\overline{S}_j^{(i)} \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M, 0)$, $\overline{R}_j^{(i)} \in \mathcal{R}_{j-1}^{-\infty}(2M, \frac{1}{2\lambda}) \subset \mathcal{R}_{j-1}^{-\infty}(2M, \tau)$, $7 \le i \le 8$, $2 \le j \le m+2$. In this case the convention reads that

$$\overline{S}_{m+2}^{(7)} = \overline{R}_{m+2}^{(7)} = \overline{S}_2^{(8)} = \overline{R}_2^{(8)} = 0.$$

Moreover, by (2.34), (3.1), (3.5), (3.7) and (3.17)

$$\sum_{i=7}^{8} \left(\|\overline{S}_j^{(i)}\|_{j-1,0} + |\overline{R}_j^{(i)}|_{j-1}^{(2M,\tau)} \right) \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda, d)^j, \quad 2 \le j \le m+2. \tag{3.22}$$

Similarly, by Proposition 3.1, Proposition 2.10, Corollary 2.3, we also have

$$\begin{aligned}(Q^m)^*VQ^m &= \frac{1}{2} \sum_{j=3}^{2m+1} \sum_{\substack{j_1+j_2=j-1 \\ 1 \le j_1, j_2 \le m}} Q'^*_{j_1}(Q_V + R_V)Q'_{j_2} + Q'^*_{j_2}(Q_V + R_V)Q'_{j_1} \\ &\quad + \sum_{j=4}^{2m+2} \sum_{\substack{j_1+j_2=j-2 \\ 1 \le j_1, j_2 \le m}} Q''^*_{j_1}(Q_V + R_V)Q'_{j_2} + Q'^*_{j_2}(Q_V + R_V)Q''_{j_1} \\ &\quad + \frac{1}{2} \sum_{j=5}^{2m+3} \sum_{\substack{j_1+j_2=j-3 \\ 1 \le j_1, j_2 \le m}} Q''^*_{j_1}(Q_V + R_V)Q''_{j_2} + Q''^*_{j_2}(Q_V + R_V)Q''_{j_1} \\ &= \sum_{j=3}^{2m+1} \left(\overline{S}_j^{(9)} + \overline{R}_j^{(9)} \right) + \sum_{j=4}^{2m+2} \left(\overline{S}_j^{(10)} + \overline{R}_j^{(10)} \right) + \sum_{j=5}^{2m+3} \left(\overline{S}_j^{(11)} + \overline{R}_j^{(11)} \right)\end{aligned}$$

for self-adjoint operators $\overline{S}_j^{(i)} \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M, 0)$, $\overline{R}_j^{(i)} \in \mathcal{R}_{j-1}^{-\infty}(4M, \frac{1}{2\lambda}) \subset \mathcal{R}_{j-1}^{-\infty}(4M, \tau)$, $9 \le i \le 11$, $3 \le j \le 2m+3$ and by (2.43), (2.38), (3.5), (3.7) and (3.17)

$$\sum_{i=9}^{11} \left(\|\overline{S}_j^{(i)}\|_{j-1,0} + |\overline{R}_j^{(i)}|_{j-1}^{(4M,\tau)} \right) \lesssim \frac{K^{j-2}}{j^2} h(\lambda, d)^j, \quad 3 \le j \le 2m+3. \tag{3.23}$$

Using the convention made on Sect. 3, we set

$$\overline{S}_j = \sum_{i=1}^{11} \overline{S}_j^{(i)}, \quad \overline{R}_j = \widetilde{R}_j + \sum_{i=1}^{11} \overline{R}_j^{(i)}, \quad 2 \le j \le 2m+3.$$

Since $\mathcal{R}^{-\infty}_{j-1}(M,\tau) \subset \mathcal{R}^{-\infty}_{j-1}(4M,\tau)$, we see from (3.19) to (3.23) that $(\overline{S}_j)_{2\le j\le 2m+3}$, $(\overline{R}_j)_{2\le j\le 2m+3}$ satisfy the conditions listed in the lemma and contribute respectively to the second and last terms in the last line of (3.10). This concludes the proof. □

Proof of Proposition 3.2 We shall recursively construct $Q'_1, Q''_1, \dots, Q'_m, Q''_m$ with the required estimates so that the left hand side of (3.9) may be written for $r = 1, \dots, m+1$

$$\begin{aligned} i\partial_t - \Delta + V^{r-1} &+ \sum_{j=r}^{m}[Q'_j, \Delta] + \sum_{j=r}^{m}\left(Q''_j P_0 + P_0 Q''_j\right) \\ &+ \frac{1}{2}\sum_{j=r}^{2m+1}\left(S_j P_0 + P_0 S_j\right) + \frac{1}{2}\sum_{j=1}^{2m+1}\left(R_j P_0 + P_0 R_j\right) \\ &+ \sum_{j=r}^{m+1}\widetilde{S}_j + \sum_{j=r}^{2m+3}\overline{S}_j + \sum_{j=1}^{2m+3}\overline{R}_j + \sum_{j=1}^{r-1}\widehat{R}_j, \end{aligned} \tag{3.24}$$

where $V^0 = 0$, $\left(V^j\right)^* = V^j$ and $[\tilde{\Delta}, V^j] = 0$ for $j \ge 1$, $\widetilde{S}_1 = 0$, $\overline{S}_1 = Q_V$, $\overline{R}_1 = R_V$. Here Q_V, R_V are defined in Proposition 3.1. Remark that without regard to all the estimates, (3.24) with $r = 1$ is the conclusion of Lemma 3.1 and (3.24) with $r = m+1$ is the conclusion we want to reach. Assume that (3.24) has been obtained at rank r and we have already had the estimates (3.5)–(3.8) for $Q'_1, \dots, Q'_{r-1}$, $Q''_1, \dots, Q''_{r-1}$. By Lemma 3.1, we have determined S_ℓ, R_ℓ, $1 \le \ell \le r-1$, $\widetilde{S}_\ell$, $\overline{S}_\ell$, $\overline{R}_\ell$, $1 \le \ell \le r$ and they also satisfy the estimates listed in Lemma 3.1. Using Notation 3, we set $V^r = V^{r-1} + (\widetilde{S}_r)_D + (\overline{S}_r)_D$ and denote

$$\begin{aligned} (\widetilde{S}_r)^M_{ND} &= \sum_{n,n'\in\mathbb{Z}^d} \Pi_n(\widetilde{S}_r)_{ND}\Pi_{n'}\mathbf{1}_{\{\||n|^2-|n'|^2|>\frac{1}{4}(|n|+|n'|)^{\tau_0}\}}, \\ (\overline{S}_r)^M_{ND} &= \sum_{n,n'\in\mathbb{Z}^d} \Pi_n(\overline{S}_r)_{ND}\Pi_{n'}\mathbf{1}_{\{\||n|^2-|n'|^2|>\frac{1}{4}(|n|+|n'|)^{\tau_0}\}}, \end{aligned} \tag{3.25}$$

with τ_0 given by Proposition 2.1. We now deduce from (2.10) and Proposition 2.2 that $[\tilde{\Delta}, V^r] = 0$, $(\widetilde{S}_r)^M_{ND} \in \overline{\mathcal{L}}^{-(r-1)}_{\tau,ND}(M,1)$ and $(\overline{S}_r)^M_{ND} \in \overline{\mathcal{L}}^{-(r-1)}_{\tau,ND}(M,0)$. We let Q'_r satisfy

$$[Q'_r, \Delta] = -(\widetilde{S}_r)^M_{ND} - (\overline{S}_r)^M_{ND}. \tag{3.26}$$

Since $\tau_0 \ge \tau$ by Remark 3.2, according to Proposition 2.3 this equation defines an element $Q'_r \in \mathcal{L}^{-r}_\tau(M,0)$ with

$$\|Q'_r\|_{r,0} \lesssim \|(\widetilde{S}_r)^M_{ND}\|_{r-1,1} + \|(\overline{S}_r)^M_{ND}\|_{r-1,0} \lesssim \frac{K^{r-\frac{3}{2}}}{r^2}h(\lambda,d)^r \le \frac{K^{r-\frac{1}{2}}}{r^2}h(\lambda,d)^r, \tag{3.27}$$

if K is larger than the implicit constant and since $(\widetilde{S}_r)^M_{ND}$, $(\overline{S}_r)^M_{ND}$ are self-adjoint, $Q'^*_r = -Q'_r$. (3.6) with $j = r$ follows from (3.26), (5) and (6) with $j = r$ if K is larger than the square of the implicit constant. Thus Q'_r satisfies (3.5) and (3.6). We then claim that $(\widetilde{S}_r)_{ND} - (\widetilde{S}_r)^M_{ND}$ and $(\overline{S}_r)_{ND} - (\overline{S}_r)^M_{ND}$ contribute to $\widehat{R}_r$. But

$$\Pi_n\left((\widetilde{S}_r)_{ND} - (\widetilde{S}_r)^M_{ND}\right)\Pi_{n'} = \Pi_n(\widetilde{S}_r)_{ND}\Pi_{n'}\mathbf{1}_{\{\||n|^2-|n'|^2|\le\frac{1}{4}(|n|+|n'|)^{\tau_0}\}} \tag{3.28}$$

and since $(\widetilde{S}_r)_{ND} \in \widetilde{\mathcal{L}}^{-(r-1)}_{\tau,ND}(M,1)$, this expression is non zero only when n and n' belong to A_α and A_β with $\alpha \ne \beta$, where A_α and A_β are defined in Proposition 2.1. So the second condition in Proposition 2.1, together with the cut-off in (3.28), implies that $|n - n'| \ge$

$\frac{1}{2}(1+\max(|n|,|n'|))^{\tau_0}$. Then it follows by (2.8) and the assumption $M > \frac{2}{\rho}$ stated in Remark 3.2 that

$$\begin{aligned}&\|\Pi_n\partial_t^k\big((\widetilde{S}_r)_{ND}-(\widetilde{S}_r)_{ND}^M\big)\Pi_{n'}\|_{\mathcal{L}(L^2)}\\&\quad\lesssim\|(\widetilde{S}_r)_{ND}\|_{r-1,1}M^{N+k+r-1}[(k+r-1)!]^{\max(2,\mu)}N!\\&\qquad\times\langle n-n'\rangle^{-(d+2)}\big(1+\max(|n|,|n'|)\big)^{-\frac{\tau_0N}{\lambda}}\end{aligned}\tag{3.29}$$

for any $k, N\in\mathbb{N}$, any $n, n'\in\mathbb{Z}^d$. With the same reasoning we can get a similar estimate for $\|\Pi_n\big((\overline{S}_r)_{ND}-(\overline{S}_r)_{ND}^M\big)\Pi_{n'}\|_{\mathcal{L}(L^2)}$. We then set

$$\widehat{R}_r=(\widetilde{S}_r)_{ND}-(\widetilde{S}_r)_{ND}^M+(\overline{S}_r)_{ND}-(\overline{S}_r)_{ND}^M$$

and deduce from (3.29), a similar estimate to (3.29) for $\|\Pi_n\big((\overline{S}_r)_{ND}-(\overline{S}_r)_{ND}^M\big)\Pi_{n'}\|_{\mathcal{L}(L^2)}$, (2.10), (5) and (6) with $j=r$ and the fact $\tau=\frac{\tau_0}{\lambda}$ that $\widehat{R}_r$ satisfies the required properties in Proposition 3.2.

We also have to find Q''_r satisfying (3.7) and (3.8) such that

$$Q''_rP_0+P_0Q''_r=-\frac{1}{2}[S_rP_0+P_0S_r].$$

Since by Lemma 3.1, S_r depends only on $Q'_1,\ldots,Q'_r,Q''_1,\ldots,Q''_{r-1}$ which have been already determined, we may define $Q''_r=-\frac{1}{2}S_r$. We see by Lemma 3.1 that Q''_r obeys (3.7) and (3.8) if K is chosen to be much larger than the square of the implicit constant. Therefore we obtain (3.24) at rank $r+1$ with terms satisfying the corresponding estimates. This concludes the proof. □

4 Proof of the Main Theorem

For any given $N\in\mathbb{N}^*$, once one has conjugated the operator $i\partial_t-\Delta+V$ into $i\partial_t-\Delta+V'_N+R'_N$ with V'_N exactly commuting with the modified Laplacian $\tilde{\Delta}$ and R'_N essentially being a bounded linear operator from $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$, which has already been done in the previous section when m is taken to be so large that $m\tau\gg N$, we need to invert the transformation in order to get an estimate for the solution of the original Cauchy problem. Moreover, we have to compute the norms of the operators in order to obtain logarithmic growth of Sobolev norms from the energy inequality. To realize this, we begin with the following lemma.

Lemma 4.1 *Let $m\in\mathbb{N}^*$ and assume $\overline{Q}_j\in\mathcal{L}_\tau^{-j}(M,0)$, $j=1,2,\ldots,m$. Then there are sequences $P_j\in\mathcal{L}_\tau^{-j}(M,0)$, $1\le j\le m$, $T_j\in\mathcal{L}_\tau^{-j}(M,0)$, $m+1\le j\le 2m$, $R'_j\in\mathcal{R}_j^{-\infty}(M,\frac{1}{2\lambda})$, $2\le j\le 2m$ such that*

$$(I+\overline{Q}_1+\cdots+\overline{Q}_m)(I+P_1+\cdots+P_m)=I+\sum_{j=m+1}^{2m}T_j+\sum_{j=2}^{2m}R'_j\tag{4.1}$$

with

$$\|P_j\|_{j,0}\le\sum_{\ell=1}^{j}\sum_{\substack{j_1+\cdots+j_\ell=j\\1\le j_1,\ldots,j_\ell\le m}}C_2^{\ell-1}\|\overline{Q}_{j_1}\|_{j_1,0}\ldots\|\overline{Q}_{j_\ell}\|_{j_\ell,0},\quad 1\le j\le m,$$

$$\|T_j\|_{j,0} \le \sum_{\ell=2}^{j} \sum_{\substack{j_1+\cdots+j_\ell=j\\ 1\le j_1,\ldots,j_\ell\le m}} C_2^{\ell-1}\|\overline{Q}_{j_1}\|_{j_1,0}\ldots\|\overline{Q}_{j_\ell}\|_{j_\ell,0}, \quad m+1\le j\le 2m,$$

$$|R'_j|_j \le \sum_{\ell=2}^{j} \sum_{\substack{j_1+\cdots+j_\ell=j\\ 1\le j_1,\ldots,j_\ell\le m}} C_2^{\ell-1}\|\overline{Q}_{j_1}\|_{j_1,0}\ldots\|\overline{Q}_{j_\ell}\|_{j_\ell,0}, \quad 2\le j\le 2m, \tag{4.2}$$

where C_2 is an absolute constant.

Proof Let $\overline{Q}_1,\ldots,\overline{Q}_m$ be given. We set $P_1 = -\overline{Q}_1$ and by Proposition 2.7 we may recursively determine $P_j \in \mathcal{L}_\tau^{-j}(M,0)$ and $R'_j \in \mathcal{R}_j^{-\infty}(M,\frac{1}{2\lambda})$ for $j = 2,\ldots,m$ such that

$$-\overline{Q}_j - \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \overline{Q}_i P_k = P_j + R'_j \tag{4.3}$$

with

$$\begin{aligned} \|P_j\|_{j,0} &\lesssim \|\overline{Q}_j\|_{j,0} + \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad 2\le j\le m,\\ |R'_j|_j &\lesssim \|\overline{Q}_j\|_{j,0} + \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad 2\le j\le m. \end{aligned} \tag{4.4}$$

Consequently, we have

$$\begin{aligned} &(I+\overline{Q}_1+\cdots+\overline{Q}_m)(I+P_1+\cdots+P_m)\\ &= I + P_1 + \overline{Q}_1 + \sum_{j=2}^{m}\Big(P_j + \overline{Q}_j + \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \overline{Q}_i P_k\Big) + \sum_{j=m+1}^{2m} \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \overline{Q}_i P_k\\ &= I + \sum_{j=m+1}^{2m} \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \overline{Q}_i P_k + \sum_{j=2}^{m} R'_j\,. \end{aligned} \tag{4.5}$$

Moreover by induction we obtain from (4.4) the required inequalities for P_j, $1\le j\le m$ and the third inequality in (4.2) holds when $2\le j\le m$, if C_2 is chosen to be larger than the implicit constant. Since $P_1,\ldots,P_m$ have already been determined, by Proposition 2.7, we may also find $T_j \in \mathcal{L}_\tau^{-j}(M,0)$, $R'_j \in \mathcal{R}_j^{-\infty}(M,\frac{1}{2\lambda})$, $m+1\le j\le 2m$, such that

$$\sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \overline{Q}_i P_k = T_j + R'_j, \quad m+1\le j\le 2m, \tag{4.6}$$

with

$$\begin{aligned} \|T_j\|_{j,0} &\lesssim \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad m+1\le j\le 2m,\\ |R'_j|_j &\lesssim \sum_{\substack{i+k=j\\ 1\le i,\, k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad m+1\le j\le 2m. \end{aligned} \tag{4.7}$$

Thus (4.1) follows by (4.5) and (4.6). The required estimates for T_j, R'_j, $m+1 \le j \le 2m$, follow by (4.7) and the estimates of P_j, $1 \le j \le m$, which we have already obtained. This concludes the proof. □

Proof of the main theorem Recall that $\tau = \frac{\tau_0}{\lambda} < \frac{1}{2\lambda}$, where τ_0 is given by Proposition 2.1 and λ given by (1.2). For any $N \in \mathbb{N}^*$, let m be an integer such that

$$N + 3 \le (m+2)\tau < N + 4, \tag{4.8}$$

which implies

$$m > \frac{3}{\tau}, \quad m\tau > N. \tag{4.9}$$

Let the operators Q'_j, Q''_j, $1 \le j \le m$ be given by Proposition 3.2. Applying Lemma 4.1 to $\overline{Q}_1 = Q'_1$, $\overline{Q}_j = Q'_j + Q''_{j-1}$, $2 \le j \le m$, $\overline{Q}_{m+1} = Q''_m$, we may find

$$\begin{aligned} &P_j \in \mathcal{L}_\tau^{-j}(M, 0), \quad 1 \le j \le m+1, \\ &T_j \in \mathcal{L}_\tau^{-j}(M, 0), \quad m+2 \le j \le 2m+2, \\ &R'_j \in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda}), \quad 2 \le j \le 2m+2 \end{aligned}$$

such that if we set $P^{m+1} = \sum_{j=1}^{m+1} P_j$, $Q^m = \sum_{j=1}^{m+1} \overline{Q}_j$

$$(I + Q^m)(I + P^{m+1}) = I + \sum_{j=m+2}^{2m+2} T_j + \sum_{j=2}^{2m+2} R'_j. \tag{4.10}$$

Moreover, (4.2) with m replaced by $m+1$ are satisfied by those operators. Since by (3.5), (3.7)

$$\|\overline{Q}_j\|_{j,0} \le \frac{2K^{j-\frac{1}{2}}}{j^2} h(\lambda, d)^j, \quad 1 \le j \le m+1, \tag{4.11}$$

we get by (4.2)

$$\begin{aligned} \|P_j\|_{j,0} &\le \sum_{\ell=1}^{j} \sum_{j_1+\cdots+j_\ell=j} C_2^{\ell-1} \frac{2K^{j_1-\frac{1}{2}}}{j_1^{\,2}} \cdots \frac{2K^{j_\ell-\frac{1}{2}}}{j_\ell^{\,2}} h(\lambda, d)^j \\ &\le C_{\lambda,d}^j, \quad 1 \le j \le m+1, \\ \|T_j\|_{j,0} &\le C_{\lambda,d}^j, \quad m+2 \le j \le 2m+2, \\ |R'_j|_j &\le C_{\lambda,d}^j, \quad 2 \le j \le 2m+2 \end{aligned} \tag{4.12}$$

if $K > (2C_2)^2$ and it is large enough so that Proposition 3.2 holds, where $C_{\lambda,d}$ is some constant depending only on λ, d. Keep in mind that from now on the meaning of constant $C_{\lambda,d}$ depending only on λ and d may change from line to line. For the solution u of (1.1), we set

$$v = (I + P^{m+1})u. \tag{4.13}$$

Then by Proposition 2.5, (4.12), for any $\sigma \in \mathbb{R}$,

$$\begin{aligned} \|v(t)\|_{H^\sigma} &\lesssim \left(1 + C_1^{|\sigma|} \sum_{j=1}^{m+1} \|P_j\|_{j,0} M^{j-1} \big[(j-1)!\big]^{\max(2,\mu)}\right) \|u(t)\|_{H^\sigma} \\ &\lesssim C_1^{|\sigma|} C_{\lambda,d}^{m+2} (m!)^{\max(2,\mu)} \|u(t)\|_{H^\sigma}. \end{aligned} \tag{4.14}$$

Similarly, by Proposition 2.5, (4.12), for any $\sigma \in \mathbb{R}$,

$$\begin{aligned} &\|\partial_t v(t)\|_{H^\sigma} \\ &\quad \le \|\partial_t u(t)\|_{H^\sigma} + \sum_{j=1}^{m+1} \|[\partial_t, P_j] u(t)\|_{H^\sigma} + \sum_{j=1}^{m+1} \|P_j \partial_t u(t)\|_{H^\sigma} \\ &\quad \lesssim C_1^{|\sigma|} C_{\lambda,d}^{m+2} \big((m+1)!\big)^{\max(2,\mu)} \|u(t)\|_{H^\sigma} + C_1^{|\sigma|} C_{\lambda,d}^{m+2} (m!)^{\max(2,\mu)} \|\partial_t u(t)\|_{H^\sigma}, \end{aligned} \tag{4.15}$$

and by (4.10), (4.13), (4.9), Proposition 2.5, Proposition 2.6, (4.11), (4.12)

$$\begin{aligned} &\|u(t)\|_{H^N} \\ &\quad \le \|(I + Q^m) v(t)\|_{H^N} + \sum_{j=m+2}^{2m+2} \|T_j u(t)\|_{H^{(m+2)\tau}} + \sum_{j=2}^{2m+2} \|R'_j u(t)\|_{H^{m\tau}} \\ &\quad \lesssim C_{\lambda,d}^{m+1} (m!)^{\max(2,\mu)} \|v(t)\|_{H^N} + C_{\lambda,d}^{4m+3} \big[(2m+2)!\big]^{\max(2,\mu)+1} \|u(t)\|_{L^2}. \end{aligned} \tag{4.16}$$

By (3.9), (4.10) and (1.1)

$$(i\partial_t - \Delta + V^m) v = f + g, \tag{4.17}$$

where

$$\begin{aligned} f = &-\Bigg[\frac{1}{2} \sum_{j=m+1}^{2m+1} \big(S_j P_0 + P_0 S_j\big) v + \frac{1}{2} \sum_{j=1}^{2m+1} \big(R_j P_0 + P_0 R_j\big) v \\ &+ \left(\widetilde{S}_{m+1} + \sum_{j=m+1}^{2m+3} \overline{S}_j + \sum_{j=2}^{2m+3} \overline{R}_j + \sum_{j=1}^{m} \widehat{R}_j\right) v\Bigg], \end{aligned} \tag{4.18}$$

$$g = (I + Q^m)^* \left[i\partial_t - \Delta + V, \sum_{j=m+2}^{2m+2} T_j + \sum_{j=2}^{2m+2} R'_j\right] u. \tag{4.19}$$

Therefore by (2.5) and the property of V^m, we have

$$\big(i\partial_t - \Delta + V^m\big)\big(1 - \tilde{\Delta}\big)^{\frac{N}{2}} v = w,$$

where by Lemma 4.2 below

$$\|w\|_{L^2} \le C_{\lambda,d}^{5m+6} \big[(2m+3)!\big]^{2\max(2,\mu)} \|u_0\|_{L^2},$$

if $C_{\lambda,d}$ is in addition larger than the implicit constants of (4.24) and (4.25). Since V^m is self-adjoint, this implies the energy inequality

$$\begin{aligned} \|v(t)\|_{\widetilde{H}^N} &\leq \|v(0)\|_{\widetilde{H}^N} + \int_0^t C_{\lambda,d}^{5m+6}\big[(2m+3)!\big]^{2\max(2,\mu)}\|u_0\|_{L^2}dt \\ &\leq \|v(0)\|_{\widetilde{H}^N} + |t|C_{\lambda,d}^{5m+6}\big[(2m+3)!\big]^{2\max(2,\mu)}\|u_0\|_{L^2}. \end{aligned} \tag{4.20}$$

Now using (4.16), (2.7), (4.20), (4.14), the conservation law of the L^2-norm of (1.1) and (4.9), we deduce for some constant $C_{\lambda,d}$ independent of m and N

$$\|u(t)\|_{H^N} \leq C_{\lambda,d}^N\big[(2m+3)!\big]^{\frac{5}{2}\max(2,\mu)}(2+|t|)\|u_0\|_{H^N}, \tag{4.21}$$

if we use

$$(m!)^{\max(2,\mu)} \leq [(2m)!]^{\frac{1}{2}\max(2,\mu)}.$$

Since by (4.8), $2m+3 \leq ([\frac{10}{\tau}]+1)N$, we deduce from (4.21)

$$\|u(t)\|_{H^N} \leq C_{\lambda,d}^N\Big[\big(([\frac{10}{\tau}]+1)N\big)!\Big]^{\frac{5}{2}\max(2,\mu)}(2+|t|)\|u_0\|_{H^N}. \tag{4.22}$$

By Stirling's approximation $N! \sim \sqrt{2\pi N}(\frac{N}{e})^N$ for any $N \in \mathbb{N}$, there is a constant p_λ which depends on τ and thus on λ such that $\big(([\frac{10}{\tau}]+1)N\big)! \leq p_\lambda^N(N!)^{[\frac{10}{\tau}]+1}$, which, together with the fact that $\tau = \frac{\tau_0}{\lambda}$, allows us to rewrite (4.22) for some constant $C_{\lambda,d}$ independent of m, N, μ and for some constant ζ independent of m, N, μ and λ as

$$\|u(t)\|_{H^N} \leq C_{\lambda,d}^N(N!)^{\zeta\mu\lambda}(2+|t|)\|u_0\|_{H^N}. \tag{4.23}$$

Since (4.23) holds for any $N \in \mathbb{N}^*$, we deduce, for any $s > 0$, from the conservation law of the L^2-norm and interpolation

$$\|u(t)\|_{H^s} \leq C_{\lambda,d}^{\theta N}(N!)^{\zeta\mu\lambda\theta}(2+|t|)^\theta\|u_0\|_{H^s}$$

where θ satisfies $s = \theta N, \theta \in [0,1]$. Assuming $\|u_0\|_{H^s} \neq 0$, we obtain for any $N \in \mathbb{N}$ and for some other constant $C_{s,\lambda,d}$ independent of N

$$\Big(\frac{1}{C_{s,\lambda,d}}\Big(\frac{\|u(t)\|_{H^s}}{\|u_0\|_{H^s}}\Big)^{\frac{1}{\zeta\mu\lambda s}}\Big)^N \leq N!\,(2+|t|)^{\frac{1}{\zeta\mu\lambda}}.$$

This gives immediately for some other constant $C_{s,\lambda,d}$

$$\|u(t)\|_{H^s} \leq C_{s,\lambda,d}\big[\log(2+|t|)\big]^{\zeta\mu\lambda s}\|u_0\|_{H^s},$$

thus concludes the proof of the main theorem. □

Lemma 4.2 *Let f, g be the quantities defined respectively by* (4.18) *and* (4.19). *Then*

$$\|f\|_{\widetilde{H}^N} \lesssim C_{\lambda,d}^{5m+5}\big[(2m+3)!\big]^{2\max(2,\mu)}\|u_0\|_{L^2}, \tag{4.24}$$

$$\|g\|_{\widetilde{H}^N} \lesssim C_{\lambda,d}^{5m+5}\big[(2m+3)!\big]^{2\max(2,\mu)}\|u_0\|_{L^2}. \tag{4.25}$$

Proof We have by (2.7), (4.8), the properties of S_j listed in Proposition 3.2, Proposition 2.4, Proposition 2.5, (4.15), (4.14), (1.1) and the conservation law of the L^2-norm of (1.1)

$$\begin{aligned}
&\| \sum_{j=m+1}^{2m+1} \big(S_j P_0 + P_0 S_j\big) v(t)\|_{\widetilde{H}^N} \\
&\lesssim C_0^N \sum_{j=m+1}^{2m+1} \Big(\| S_j \partial_t v(t)\|_{H^{-2+(m+2)\tau}} + \| S_j \Delta v(t)\|_{H^{-2+(m+2)\tau}} \\
&+ \|[i\partial_t, S_j] v(t)\|_{H^{(m+2)\tau}} + \|[\Delta, S_j] v(t)\|_{H^{(m+1)\tau}} \Big) \\
&\lesssim C_0^N \sum_{j=m+1}^{2m+1} \frac{K^j}{(j+1)^2} h(\lambda, d)^{j+1} M^j \big(j!\big)^{\max(2,\mu)} \Big(\|\partial_t v(t)\|_{H^{-2}} + \|\Delta v(t)\|_{H^{-2}} \Big) \\
&+ C_0^N \sum_{j=m+1}^{2m+1} \frac{K^j}{(j+1)^2} h(\lambda, d)^{j+1} M^{j+1} \big[(j+1)!\big]^{\max(2,\mu)} \|v(t)\|_{L^2} \\
&\lesssim C_0^N C_{\lambda,d}^{2m+2} [(2m+1)!]^{\max(2,\mu)} \\
&\times \Big(C_{\lambda,d}^{m+2} \big((m+1)!\big)^{\max(2,\mu)} \|u(t)\|_{L^2} + C_{\lambda,d}^{m+2} (m!)^{\max(2,\mu)} \|\partial_t u(t)\|_{H^{-2}} \Big) \\
&+ C_0^N C_{\lambda,d}^{2m+2} \big[(2m+2)!\big]^{\max(2,\mu)} C_{\lambda,d}^{m+2} (m!)^{\max(2,\mu)} \|u(t)\|_{L^2} \\
&\lesssim C_{\lambda,d}^{3m+4} \big[(2m+2)!\big]^{2\max(2,\mu)} \|u_0\|_{L^2}.
\end{aligned}$$

Using, in addition, (4.9) and Proposition 2.6, we similarly have

$$\| \sum_{j=1}^{2m+1} \big(R_j P_0 + P_0 R_j\big) v(t)\|_{\widetilde{H}^N} \lesssim C_{\lambda,d}^{5m+5} \big[(2m+3)!\big]^{2\max(2,\mu)} \|u_0\|_{L^2}.$$

By Proposition 2.5, Proposition 2.6 and Proposition 3.2, we easily deduce that the other terms in the expression of f can be controlled by the right hand side of (4.24). What is important here is that $C_{\lambda,d}$ does not depend on m, N.

Next we want to show (4.25). First notice that by Proposition 2.5, (4.11), (4.9)

$$\begin{aligned}
\|(I + Q^m)^*\|_{\mathcal{L}(H^N, H^N)} &\lesssim C_1^N \sum_{j=1}^{m+1} \frac{2K^{j-\frac{1}{2}}}{j^2} h(\lambda, d)^j M^{j-1} \big[(j-1)!\big]^{\max(2,\mu)} \\
&\lesssim C_{\lambda,d}^{m+1} (m!)^{\max(2,\mu)}.
\end{aligned} \tag{4.26}$$

On the other hand, by (2.7), (4.9), Proposition 2.5, Proposition 2.6, (4.12), the conservation law of the L^2-norm of (1.1),

$$\begin{aligned}
&\|[i\partial_t, \sum_{j=m+2}^{2m+2} T_j + \sum_{j=2}^{2m+2} R'_j] u(t)\|_{\widetilde{H}^N} \\
&\le C_0^N \sum_{j=m+2}^{2m+2} \|[i\partial_t, T_j] u(t)\|_{H^{m\tau}} + C_0^N \sum_{j=2}^{2m+2} \|[i\partial_t, R'_j] u(t)\|_{H^{m\tau}} \\
&\lesssim C_0^N \sum_{j=m+2}^{2m+2} C_{\lambda,d}^j (j!)^{\max(2,\mu)} \|u(t)\|_{L^2}
\end{aligned}$$

$$+C_0^N \sum_{j=2}^{2m+2} C_{\lambda,d}^{2m+j+1}\big((j+1)!\big)^{\max(2,\mu)}(2m)!\|u(t)\|_{L^2}$$
$$\lesssim C_{\lambda,d}^{4m+4}\big[(2m+3)!\big]^{\max(2,\mu)+1}\|u_0\|_{L^2}, \tag{4.27}$$

and

$$\|[-\Delta, \sum_{j=m+2}^{2m+2} T_j + \sum_{j=2}^{2m+2} R'_j]u(t)\|_{\widetilde{H}^N} \lesssim C_{\lambda,d}^{4m+3}\big[(2m+2)!\big]^{\max(2,\mu)+1}\|u_0\|_{L^2}. \tag{4.28}$$

Since the quantity $\|[V, \sum_{j=m+2}^{2m+2} T_j + \sum_{j=2}^{2m+2} R'_j]u(t)\|_{\widetilde{H}^N}$ is also less than a constant times the last line of (4.27), by (4.26)-(4.28) we see that (4.25) holds true. □

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