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On the Chaotic Behaviour of Discontinuous Systems

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Abstract We follow a functional analytic approach to study the problem of chaotic behaviour in time-perturbed discontinuous systems whose unperturbed part has a piecewise C^1 homoclinic solution that crosses transversally the discontinuity manifold. We show that if a certain Melnikov function has a simple zero at some point, then the system has solutions that behave chaotically. Application of this result to quasi periodic systems are also given.

Keywords Bernouilli shift · Chaotic behaviour · Discontinuous systems

Mathematics Subject Classification (2000) 34C23 · 34C37 · 37G20

1 Introduction

One of fascinating phenomenon which may occur in nonlinear dynamical systems (NDSs) is the existence of chaotic orbits with the consequent sensitive dependence of orbits on initial conditions. Then of course it is rather difficult to predict asymptotic behaviour of orbits in the future for such NDSs. Such a chaotic behaviour of solutions can be explained mathematically by showing the existence of transversal homoclinic point of the time map of NDS with the

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Dedicated to Professor Russell Allan Johnson on the occasion of his 60th birthday.

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corresponding invariant Smale horseshoe set. This chaos theory is well known [12,26] for smooth NDSs. In general, however, it is not easy to show the existence of a transversal homoclinic point for a general NDSs. To this end the perturbation approach is a powerful method, which is by now known as the Melnikov method for the persistence/bifurcation of either periodics or homoclinics/heteroclinics [5,11,12,22,26]. Thus, bifurcation from homoclinic orbits in perturbed smooth differential equations is well developed for hyperbolic equilibria.

On the other hand, non-smooth/discontinuous differential equations, i.e. equations where the vector field is only piecewise smooth, occur in various situations as, for example, in mechanical systems with dry frictions or with impacts. They also appear in control theory, electronics, economics, medicine and biology (see [6,7,15–17] for more references). Recently several papers appeared to extend the theory of chaos to differential equations with piecewise smooth right-hand sides. To handle this kind of problem one has to face with the new problem that stable and unstable manifolds may only be Lipschitz in the state variable (even if they are possibly smooth with respect to parameters). So it is not clear what the notion of transverse intersection of invariant manifolds would be.

Planar discontinuous differential equations are investigated in [1,14–16] using geometric, analytic and numeric approaches. Piecewise linear three dimensional discontinuous differential equations are investigated in [4,20] where the reader can find more details. Weakly discontinuous systems are studied in [10]. In [4] bifurcations of bounded solutions from homoclinic orbits is investigated for time perturbed discontinuous differential equations in any finite dimensional space. We anticipated that under the conditions of [4] not only the existence of bounded solutions on $\mathbb{R}$, but also chaotic solutions could occur. The purpose of this paper is to justify this conjecture about the existence of chaotic solutions.

The plan of our paper is as follows. In Sect. 2, we introduce the problem together with basic assumptions and state our main result (Theorem 2.2). The preparatory Sects. 3 and 4 deal with local analysis of dynamics close to homoclinic solutions of the unperturbed discontinuous differential equations. Basically Propositions 3.6, 3.8, 4.1 state that the piecewise C^1-smooth homoclinic orbit of the unperturbed equation is approximated/shadowed by a family $\{z_m(t)\}, m \in \mathbb{Z}$, of piecewise C^1-smooth solutions of the perturbed equation with small jumps at the discontinuity surface. A bifurcation function is then derived in the Sect. 5 by solving, essentially with the help of the Lyapunov–Schmidt method, the system obtained by equating to zero all these jumps and those of two consecutive functions near the fixed point to which the homoclinic solution is asymptotic. To apply Lyapunov–Schmidt method we need to show few smoothness properties of the functions defining this system. However, since the proof of these properties is quite technical, we decided to postpone it in Appendix A.

We obtain then the result that if the perturbed system satisfies some kind of recurrence condition (cf. (2.6)), the existence of a simple zero of a certain Melnikov-like function guarantees existence of a continuous, piecewise C^1-smooth solutions of the perturbed system shadowed by the homoclinic orbit (see Theorem 2.2). Using the results of Sect. 5, chaos is derived in Sect. 6 for general perturbations with applications to almost periodic, quasiperiodic and periodic cases. Piecewise linear three dimensional discontinuous differential equations motivated by [4,20] with quasiperiodic perturbations are finally studied in Sect. 7.

2 Setting of the Problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ and $G(z)$ be a C^r-function on $\bar{\Omega}$, with $r \geq 2$. We set $\Omega_\pm = \{z \in \Omega \mid \pm G(z) > 0\}$, $\Omega_0 := \{z \in \Omega \mid G(z) = 0\}$. Let $f_\pm(z) \in C_b^r(\bar{\Omega}_\pm)$ and $g \in C_b^r(\mathbb{R} \times \bar{\Omega} \times \mathbb{R})$, i.e. $f_\pm$ and g have uniformly bounded derivatives up to the rth order

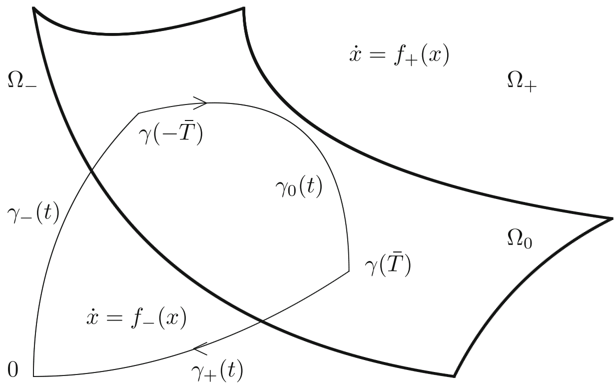


Fig. 1 The homoclinic cycle $\gamma(t)$ of $\dot{x} = f_\pm(x)$ to equilibrium $x = 0$

on $\bar{\Omega}_\pm$ and $\mathbb{R} \times \bar{\Omega} \times \mathbb{R}$, respectively. We also assume that the rth order derivatives of $f_\pm$ and g are uniformly continuous. Let $\varepsilon_0 \in (0, 1)$. Throughout this paper ε will denote a real parameter such that $|\varepsilon| \le \varepsilon_0$. In particular ε is bounded.

Remark 2.1 For technical purposes, we C_b^r-smoothly extend $f_\pm$ on $\mathbb{R}^n$, g on $\mathbb{R}^{n+2}$ and $\gamma_\pm, \gamma_0$ on $\mathbb{R}$ in such a way that

$$\sup\{|f_\pm(z)| \mid z \in \mathbb{R}^n\} \le 2\sup\{|f_\pm(z)| \mid z \in \bar{\Omega}_\pm\},$$
$$\sup\{|g(t, z, \varepsilon)| \mid (t, z, \varepsilon) \in \mathbb{R}^{n+2}\} \le 2\sup\{|g(t, z, \varepsilon)| \mid t \in \mathbb{R}, z \in \bar{\Omega}, |\varepsilon| \le \varepsilon_0\}.$$

We also assume that up to the rth order all the derivatives of the extended $f_\pm$ and g are uniformly continuous and continue to keep the same notations for extended mappings and functions.

We say that a function $z(t)$ is a solution of the equation

$$\dot{z} = f_\pm(z) + \varepsilon g(t, z, \varepsilon), \quad z \in \bar{\Omega}_\pm, \tag{2.1}$$

if it is continuous, piecewise C^1, satisfies Eq. (2.1) on $\Omega_\pm$ and, moreover, the following holds: if for some t_0 we have $z(t_0) \in \Omega_0$, then there exists $r > 0$ such that for any $t \in (t_0 - r, t_0 + r)$ with $t \neq t_0$, we have $z(t) \in \Omega_- \cup \Omega_+$. Moreover, if, for example $z(t) \in \Omega_-$ for any $t \in (t_0 - r, t_0)$, then the left derivative of $z(t)$ at $t = t_0$ satisfies: $\dot{z}(t_0^-) = f_-(z(t_0)) + \varepsilon g(t_0, z(t_0), \varepsilon)$; similarly, if $z(t) \in \Omega_-$ for any $t \in (t_0, t_0 + r)$, then $\dot{z}(t_0^+) = f_-(z(t_0)) + \varepsilon g(t_0, z(t_0), \varepsilon)$. A similar meaning is assumed when $z(t) \in \Omega_+$ for either $t \in (t_0 - r, t_0)$ or $t \in (t_0, t_0 + r)$. Note that, since $z(t) \notin \Omega_0$ for $t \in (t_0 - r, t_0 + r) \setminus \{t_0\}$ we have either $z(t) \in \Omega_-$ or $z(t) \in \Omega_+$ when $t \in (t_0 - r, t_0)$ or $t \in (t_0, t_0 + r)$.

We assume (see Fig. 1)

(H1) for $\varepsilon = 0$ Eq. (2.1) has the hyperbolic equilibrium $x = 0 \in \Omega_-$ and a continuous (not necessarily C^1) solution $\gamma(t)$ which is homoclinic to $x = 0$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_-(t) & \text{if } t \le -\bar{T} \\ \gamma_0(t) & \text{if } -\bar{T} \le t \le \bar{T} \\ \gamma_+(t) & \text{if } t \ge \bar{T} \end{cases}$$

where $\gamma_\pm(t) \in \Omega_-$ for $|t| > \bar{T}$, $\gamma_0(t) \in \Omega_+$ for $|t| < \bar{T}$ and

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}) \in \Omega_0, \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T}) \in \Omega_0;$$

(H2) it results

$$G'(\gamma(-\bar{T}))f_{\pm}(\gamma(-\bar{T})) > 0, \quad \text{and} \quad G'(\gamma(\bar{T}))f_{\pm}(\gamma(\bar{T})) < 0.$$

According to (H1) and because of roughness of exponential dichotomies (see [8,22]) the linear systems $\dot{x} = f'_{-}(\gamma_{-}(t))x$ and $\dot{x} = f'_{-}(\gamma_{+}(t))x$ have exponential dichotomies on $(-\infty, -\bar{T}]$ and $[\bar{T}, \infty)$ respectively, that is projections $P_{\pm} : \mathbb{R}^n \to \mathbb{R}^n$ and positive numbers $k \geq 1$ and $\delta > 0$ exist such that the following hold:

$$\begin{array}{ll} \|X_{-}(t)P_{-}X_{-}^{-1}(s)\| \leq k\,\mathrm{e}^{-\delta(t-s)} & \text{if } s \leq t \leq -\bar{T} \\ \|X_{-}(t)(\mathbb{I} - P_{-})X_{-}^{-1}(s)\| \leq k\,\mathrm{e}^{\delta(t-s)} & \text{if } t \leq s \leq -\bar{T} \\ \|X_{+}(t)P_{+}X_{+}^{-1}(s)\| \leq k\,\mathrm{e}^{-\delta(t-s)} & \text{if } \bar{T} \leq s \leq t \\ \|X_{+}(t)(\mathbb{I} - P_{+})X_{+}^{-1}(s)\| \leq k\,\mathrm{e}^{\delta(t-s)} & \text{if } \bar{T} \leq t \leq s \end{array} \tag{2.2}$$

where $X_{-}(t)$ and $X_{+}(t)$ are the fundamental matrices of the linear systems $\dot{x} = f'_{-}(\gamma_{-}(t))x, \dot{x} = f'_{-}(\gamma_{+}(t))x$, respectively, such that $X_{-}(-\bar{T}) = X_{+}(\bar{T}) = \mathbb{I}$. Later in this paper we will need to extend the validity of (2.2) to a larger set of values of s, t. So, let us take, for example, $u(t) = X_{+}(t)(\mathbb{I} - P_{+})X_{+}^{-1}(s)$, with $\bar{T} \leq s \leq t \leq s + 2$. Then:

$$u(t) = u(s) + \int_s^t f'_{-}(\gamma_{+}(\tau))u(\tau)\,d\tau$$

and hence (using also $|u(s)| \leq k$, see (2.2))

$$|u(t)| \leq k + K_{-}\int_s^t |u(\tau)|\,d\tau$$

where $K_{-} = \sup\{f'_{-}(\gamma_{+}(t)) \mid t \geq \bar{T}\}$. From Gronwall Lemma we obtain:

$$|X_{+}(t)(\mathbb{I} - P_{+})X_{+}^{-1}(s)| \leq k\,\mathrm{e}^{K_{-}(t-s)} \leq \hat{k}\,\mathrm{e}^{\delta(t-s)} \quad \text{if } \bar{T} \leq s \leq t \leq s + 2$$

where, for example $\hat{k} = k\max\{1, \mathrm{e}^{2(K_{-}\delta)}\}$. By similar arguments we prove that, possibly replacing k with a larger value:

$$\begin{array}{ll} \|X_{-}(t)P_{-}X_{-}^{-1}(s)\| \leq k\,\mathrm{e}^{-\delta(t-s)} & \text{if } s - 2 \leq s, t \leq -\bar{T} \\ \|X_{-}(t)(\mathbb{I} - P_{-})X_{-}^{-1}(s)\| \leq k\,\mathrm{e}^{\delta(t-s)} & \text{if } t - 2 \leq s, t \leq -\bar{T} \\ \|X_{+}(t)P_{+}X_{+}^{-1}(s)\| \leq k\,\mathrm{e}^{-\delta(t-s)} & \text{if } \bar{T} \leq s, t \leq t + 2 \\ \|X_{+}(t)(\mathbb{I} - P_{+})X_{+}^{-1}(s)\| \leq k\,\mathrm{e}^{\delta(t-s)} & \text{if } \bar{T} \leq s, t \leq s + 2. \end{array} \tag{2.3}$$

We now state our third assumption. It is a kind of nondegeneracy condition of the homoclinic orbit $\gamma(t)$ with respect to $\dot{x} = f_{\pm}(x)$, that reduces to the known notion of nondegeneracy in the smooth case [5,22]. This is discussed in more details in [4, Sect. 3].

Let $R_0 : \mathbb{R}^n \to \mathbb{R}^n$ be the projection onto $\mathcal{N}G'(\gamma(\bar{T}))$ along the direction of $\dot{\gamma}_0(\bar{T})$, i.e.

$$R_0 w = w - \frac{G'(\gamma(\bar{T}))w}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\dot{\gamma}_0(\bar{T})$$

and $X_0(t)$ be the fundamental solution of the linear system $\dot{z} = f'_{+}(\gamma_0(t))z$, $-\bar{T} \leq t \leq \bar{T}$, satisfying $X_0(-\bar{T}) = \mathbb{I}$.

Then let

$$\mathcal{S}' = \mathcal{N}P_{-} \cap \mathcal{N}G'(\gamma(-\bar{T})) \quad \text{and} \quad \mathcal{S}'' = \mathcal{R}P_{+} \cap \mathcal{N}G'(\gamma(\bar{T})).$$

Since $\dot{\gamma}_-(-\bar{T}) \notin \mathcal{N}G'(\gamma(-\bar{T}))$, $\dim \mathcal{N}G'(\gamma(-\bar{T})) = n-1$ and $\dot{\gamma}_-(-\bar{T}) \in \mathcal{N}P_-$, we have $\dim[\mathcal{N}P_- + \mathcal{N}G'(\gamma(-\bar{T}))] = n$ and hence:

$$\begin{aligned}\dim \mathcal{S}' &= \dim[\mathcal{N}P_- \cap \mathcal{N}G'(\gamma(-\bar{T}))]\\ &= \dim \mathcal{N}P_- + \dim \mathcal{N}G'(\gamma(-\bar{T})) - n = \dim \mathcal{N}P_- - 1.\end{aligned}$$

Similarly, from $\dot{\gamma}_+(\bar{T}) \notin \mathcal{N}G'(\gamma(\bar{T}))$, $\dot{\gamma}_+(\bar{T}) \in \mathcal{R}P_+$ and $\dim \mathcal{N}G'(\gamma(\bar{T})) = n-1$, we see that

$$\begin{aligned}\dim \mathcal{S}'' &= \dim[\mathcal{R}P_+ \cap \mathcal{N}G'(\gamma(\bar{T}))]\\ &= \dim \mathcal{R}P_+ + \dim \mathcal{N}G'(\gamma(\bar{T})) - n = \dim \mathcal{R}P_+ - 1.\end{aligned}$$

We assume the following condition holds:

(H3) $\mathcal{S}'' + R_0[X_0(\bar{T})\mathcal{S}']$ has codimension 1 in $\mathcal{R}R_0$.

According to [4, Lemma 2.11] the linear subspaces $\mathcal{S}''$ and $\mathcal{S}''' = R_0[X_0(\bar{T})\mathcal{S}']$ intersect transversally in $\mathcal{R}R_0$. Moreover, we have $\dim \mathcal{S}''' = \dim \mathcal{S}'$ and a unitary vector $\psi \in \mathcal{R}R_0$ exists such that

$$\mathbb{R}^n = \mathrm{span}\,\{\psi\} \oplus \mathcal{N}R_0 \oplus \mathcal{S}'' \oplus \mathcal{S}''' \tag{2.4}$$

and

$$\langle \psi, v\rangle = 0, \quad \text{for any } v \in \mathcal{S}'' \oplus \mathcal{S}'''. \tag{2.5}$$

The main result of this paper is the following:

Theorem 2.2 *Assume that $f_\pm(z)$ and $g(t,z,\varepsilon)$ are C^2-functions with bounded derivatives and that their second order derivatives are uniformly continuous. Let conditions* (H1), (H2) *and* (H3) *hold. Then there exists a C^2-function $\mathcal{M}(\alpha)$ of the real variable α such that if $\mathcal{M}(\alpha^0) = 0$ and $\mathcal{M}'(\alpha^0) \neq 0$ for some $\alpha^0 \in \mathbb{R}$, then the following holds: there exists $\widetilde{c}_1 > 0$, $\rho > 0$ and $\widetilde{\varepsilon} > 0$ such that for any $0 \neq \varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, there exists $\nu_\varepsilon \in (0, |\varepsilon|)$ (cf.* (5.41)*) such that for any increasing sequence $\mathcal{T} = \{T_m\}_{m\in\mathbb{Z}}$ that satisfies*

$$T_{m+1} - T_m > \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon| \ for \ any \ m \in \mathbb{Z}$$

along with the following recurrence condition

$$|g(t + T_{2m}, z, 0) - g(t, z, 0)| < \nu_\varepsilon \quad \text{for any } (t, z, m) \in \mathbb{R}^{n+1} \times \mathbb{Z}, \tag{2.6}$$

there exist unique sequences $\hat{\alpha} = \{\hat{\alpha}_m\}_{m\in\mathbb{Z}}$, $\hat{\beta} = \{\hat{\beta}_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ (depending on $\mathcal{T}$ and ε, i.e. $\hat{\alpha} = \hat{\alpha}_{\mathcal{T}}(\varepsilon)$, $\hat{\beta} = \hat{\beta}_{\mathcal{T}}(\varepsilon)$) such that $\sup_{m\in\mathbb{Z}} |\hat{\alpha}_m - \alpha^0| < \widetilde{c}_1|\varepsilon|$, $\sup_{m\in\mathbb{Z}} |\hat{\beta}_m - \alpha^0| < \widetilde{c}_1|\varepsilon|$ and a unique solution $z(t, \mathcal{T}, \varepsilon)$ of Eq. (2.1) *satisfying*

$$\begin{aligned}\sup_{t\in[T_{2m-1}+\hat{\beta}_{m-1}, T_{2m}-\bar{T}+hat\alpha_m]} |z(t) - \gamma_-(t - T_{2m} - \hat{\alpha}_m)| &< \rho\\ \sup_{t\in[T_{2m}-\bar{T}+\hat{\alpha}_m, T_{2m}+\bar{T}+\hat{\beta}_m]} |z(t) - \gamma_0(t - T_{2m} - \hat{\alpha}_m)| &< \rho\\ \sup_{t\in[T_{2m}+\bar{T}+\hat{\beta}_m, T_{2m+1}+\hat{\beta}_m]} |z(t) - \gamma_+(t - T_{2m} - \hat{\beta}_m)| &< \rho.\end{aligned} \tag{2.7}$$

We conclude this section with a remark on the projections of the dichotomies of the systems $\dot{x} = f'(\gamma_\pm(t))x$ on $[\bar{T}, \infty)$ and $(-\infty, -\bar{T}]$. Let us set

$$P_\pm(t) = X_\pm(\pm t)P_\pm X_\pm^{-1}(\pm t) \tag{2.8}$$

and let P_0 be the projection of the dichotomy of the linear system $\dot{x} = f'(0)x$ on $\mathbb{R}$. We have (see [22])

$$\lim_{t\to\infty} \|P_\pm(t) - P_0\| = 0.$$

Thus $T > \bar{T}$ exists such that:

$$\mathcal{N}P_+(t') \oplus \mathcal{R}P_-(t'') = \mathbb{R}^n \quad \text{for any } t', t'' \geq T. \tag{2.9}$$

We prove that a positive constant $\widetilde{c}$ exists such that

$$\max\{|x_+|, |x_-|\} \leq \widetilde{c}|x_+ + x_-| \quad \forall (x_+, x_-) \in \mathcal{N}P_+(t') \times \mathcal{R}P_-(t''). \tag{2.10}$$

Since it is clear that $|x_+ + x_-| \leq 2\max\{|x_+|, |x_-|\}$ we get, then, that the two norms $|x_+ + x_-|$ and $\max\{|x_+|, |x_-|\}$ are equivalent.

To prove the statement (2.10) take $0 < \upsilon < 1/2$ and fix $T > \bar{T}$ such that for any $t', t'' \geq T > \bar{T}$ we have

$$\|P_0 - P_+(t')\| \leq \upsilon, \quad \|P_0 - P_-(t'')\| \leq \upsilon.$$

Next consider a linear mapping $A_\upsilon : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$A_\upsilon z := (\mathbb{I} - P_+(t'))z + P_-(t'')z.$$

Note

$$A_\upsilon z := z - \left[(P_+(t') - P_0) + (P_0 - P_-(t''))\right] z.$$

Since $\|(P_+(t') - P_0) + (P_0 - P_-(t''))\| \leq 2\upsilon < 1$, A_υ is invertible and

$$\|A_\upsilon\| \leq 1 + 2\upsilon, \quad \|A_\upsilon^{-1}\| \leq 1/(1 - 2\upsilon).$$

So for any $x \in \mathbb{R}^n$ there is a unique $z \in \mathbb{R}^n$ such that

$$x = A_\upsilon z = x_+ + x_-$$

where

$$x_+ = (\mathbb{I} - P_+(t'))z \in \mathcal{N}P_+(t') \quad \text{and} \quad x_- = P_-(t'')z \in \mathcal{R}P_-(t'').$$

Then

$$|x_+| \leq \|\mathbb{I} - P_+(t')\||z| \leq \|\mathbb{I} - P_+(t')\|\|A_\upsilon^{-1}\||x| \leq \frac{\|\mathbb{I} - P_0\| + \upsilon}{1 - 2\upsilon}|x|,$$
$$|x_-| \leq \|P_-(t'')\||z| \leq \|P_-(t'')\|\|A_\upsilon^{-1}\||x| \leq \frac{\|P_0\| + \upsilon}{1 - 2\upsilon}|x|.$$

This proves (2.10) with, for example,

$$\widetilde{c} = \frac{\max\{\|\mathbb{I} - P_0\| + \upsilon, \|P_0\| + \upsilon\|\}}{1 - 2\upsilon} \leq \frac{1 + \|P_0\| + \upsilon}{1 - 2\upsilon} \leq 2(1 + \|P_0\|)$$

for $\upsilon \leq \frac{1+\|P_0\|}{1+4(1+\|P_0\|)} < \frac{1}{2}$.

3 Orbits Close to the Branches $\gamma_\pm(t)$

Let $\rho > 0$ be a positive, sufficiently small number, $\alpha, \beta \in \mathbb{R}$ two real numbers such that $|\beta - \alpha| < \min\{1, 2\bar{T}\}$, and $\ell_T^\infty(\mathbb{R})$ be the space of doubly infinity sequences $\{T_m\}_{m\in\mathbb{Z}}$ such that $T_{m+1} - T_m \geq T + 1$ where T is chosen so that (2.9) holds. Note that $T_m - T_0 \geq mT$ if m is positive and $T_m - T_0 \leq mT$ if m is negative.

In this section we show how to construct solutions $z_m^-(t)$ and $z_m^+(t)$ of (2.1) in the intervals $[T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha]$ and $[T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1]$ respectively, in such a way that

$$\begin{aligned} \sup_{t\in[T_{2m-1}-1, T_{2m}-\bar{T}]} |z_m^-(t+\alpha) - \gamma_-(t - T_{2m})| < \rho \\ \sup_{t\in[T_{2m}+\bar{T}, T_{2m+1}+1]} |z_m^+(t+\beta) - \gamma_+(t - T_{2m})| < \rho. \end{aligned} \tag{3.1}$$

Note $T_{2m-1} + \alpha - 1 < T_{2m} - \bar{T} + \alpha < T_{2m} + \bar{T} + \beta < T_{2m+1} + \beta + 1$. We show how to construct $z_m^-(t)$ for $t \in [T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha]$, the construction of $z_m^+(t)$ for $t \in [T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1]$ being similar.

Let

$$\begin{aligned} I_m^- &:= [T_{2m-1} - 1, T_{2m} - \bar{T}], \quad I_m^+ := [T_{2m} + \bar{T}, T_{2m+1} + 1], \\ I_{m,\alpha}^- &:= [T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha], \\ I_{m,\beta}^+ &:= [T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1] \end{aligned} \tag{3.2}$$

and set, for $t \in I_m^-$

$$x(t) = z_m^-(t+\alpha) - \gamma_-(t - T_{2m})$$

and

$$\begin{aligned} h_m^-(t, x, \alpha, \varepsilon) = {} & f_-(x + \gamma_-(t - T_{2m})) - f_-(\gamma_-(t - T_{2m})) \\ & - f'_-(\gamma_-(t - T_{2m}))x + \varepsilon g(t + \alpha, x + \gamma_-(t - T_{2m}), \varepsilon). \end{aligned} \tag{3.3}$$

Then $z_m^-(t)$ satisfies Eq. (2.1) for $t \in I_{m,\alpha}^-$ together with (3.1) if and only if $x(t)$ is a solution, in I_m^-, of the equation

$$\dot{x} - f'_-(\gamma_-(t - T_{2m}))x = h_m^-(t, x, \alpha, \varepsilon), \tag{3.4}$$

such that $\sup_{t\in I_m^-} |x(t)| < \rho$.

Remark 3.1 According to Remark 2.1, we see that up to the rth order all derivatives of $h_m^-(t, x, \alpha, \varepsilon)$ with respect to (x, α, ε) are bounded and uniformly continuous in (x, α, ε) uniformly with respect to $t \in I_m^-$ and $m \in \mathbb{Z}$. This statement easily follows from the fact that, for $t \leq -\bar{T}$, one has $h_m^-(t+T_{2m}, x, \alpha, \varepsilon) = f_-(x+\gamma_-(t)) - f_-(\gamma_-(t)) - f'_-(\gamma_-(t))x + \varepsilon g(t + T_{2m}+\alpha, x+\gamma_-(t), \varepsilon)$ and the conclusion holds as far as $f(x)$ and $g(t+T_{2m}+\alpha, x+\gamma_-(t), \varepsilon)$ are concerned.

We will need the following Lemma (see [2,19])

Lemma 3.2 *Let the linear system $\dot{x} = A(t)x$ have an exponential dichotomy on $(-\infty, -\bar{T}]$ with projection P, and let $X(t)$ be its fundamental matrix such that $X(-\bar{T}) = \mathbb{I}$. Set $P(t) := X(t)PX^{-1}(t)$. Then for any continuous function $h(t) \in C^0([-T, -\bar{T}])$, $\xi_- \in \mathcal{N}P$ and $\varphi_- \in \mathcal{R}P(-T)$ the linear non homogeneous system*

$$\dot{x} = A(t)x + h(t)$$

has a unique solution $x(t)$ such that

$$\begin{aligned}(\mathbb{I}-P)x(-\bar{T}) &= \xi_- \\ P(-T)x(-T) &= \varphi_-\end{aligned} \tag{3.5}$$

and this solution satisfies

$$x(t) = X(t)\xi_- + X(t)PX^{-1}(-T)\varphi_- + \int_{-T}^{t} X(t)PX^{-1}(s)h(s)ds - \int_{t}^{\bar{T}} X(t)(\mathbb{I}-P)X^{-1}(s)h(s)ds. \tag{3.6}$$

Remark 3.3 From (2.2) and (3.6) we immediately obtain the following estimate for $|x(t)|$:

$$\sup_{-T\le t\le -\bar{T}} |x(t)| \le k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}\sup_{-T\le t\le -\bar{T}} |h(t)|\right]. \tag{3.7}$$

We apply Lemma 3.2 and Remark 3.3 with $A(t) = f'_-(\gamma_-(t-T_{2m}))$ in the interval I_m^- (instead of $[-T, -\bar{T}]$). Note that the fundamental matrix $X(t)$ and the projection P of the dichotomy on $(-\infty, T_{2m}-\bar{T}]$ of the linear system $\dot{x} = f'_-(\gamma_-(t-T_{2m}))x$ are $X_-(t-T_{2m})$ and P_-, respectively. Thus, in the notation of (2.8) and Lemma 3.2 we have

$$\begin{aligned}P_{-,m} &:= P(T_{2m-1}-1) = X_-(T_{2m-1}-T_{2m}-1)P_-X_-^{-1}(T_{2m-1}-T_{2m}-1) \\ &= P_-(T_{2m}-T_{2m-1}+1).\end{aligned}$$

Set:

$$\|x\|_{I_m^-} = \sup_{t\in I_m^-} |x(t)|.$$

Then a trivial application of Lemma 3.2 and (3.7) gives the following

Corollary 3.4 *Let $h(t) \in C^0(I_m^-)$, $\xi_- \in \mathcal{N}P_-$ and $\varphi_- \in \mathcal{R}P_{-,m}$. Then the linear non homogeneous system*

$$\dot{x} = f'_-(\gamma_-(t-T_{2m}))x + h(t)$$

has a unique solution $x(t) \in C^1(I_m^-)$ such that

$$\begin{aligned}(\mathbb{I}-P_-)x(T_{2m}-\bar{T}) &= \xi_- \\ P_{-,m}x(T_{2m-1}-1) &= \varphi_-.\end{aligned} \tag{3.8}$$

Moreover this solution satisfies (see (3.7))

$$\|x(t)\|_{I_m^-} \le k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}\|h(t)\|_{I_m^-}\right] \tag{3.9}$$

and

$$x(t) = X_-(t - T_{2m})\xi_- + X_-(t - T_{2m})P_-X_-^{-1}(T_{2m-1} - 1 - T_{2m})\varphi_- + \int_{T_{2m-1}-1}^{t} X_-(t - T_{2m})P_-X_-^{-1}(s - T_{2m})h(s)ds - \int_{t}^{T_{2m}-\bar{T}} X_-(t - T_{2m})(\mathbb{I} - P_-)X_-^{-1}(s - T_{2m})h(s)ds. \tag{3.10}$$

Using Corollary 3.4 we define a map from $C^0(I_m^-) \times \mathcal{N}P_- \times \mathcal{R}P_{-,m} \times \mathbb{R}^2$ into $C^0(I_m^-)$ as

$$(x(t), \xi_-, \varphi_-, \alpha, \varepsilon) \mapsto \hat{x}(t) \tag{3.11}$$

where $y(t) = \hat{x}(t)$ is the unique solution given by Corollary 3.4 of the equation

$$\dot{y}(t) - f'_-(\gamma_-(t - T_{2m}))y(t) = h_m^-(t, x(t), \alpha, \varepsilon)$$

that satisfies conditions (3.8). We observe that the map

$$(x(t), \alpha, \varepsilon) \mapsto h_m^-(t, x(t), \alpha, \varepsilon)$$

is a C^r map from $C^0(I_m^-) \times \mathbb{R}^2$ into $C^0(I_m^-)$ (see [9]) and hence, from (3.10) we see that so is the map (3.11) from $C^0(I_m^-) \times \mathcal{N}P_- \times \mathcal{R}P_{-,m} \times \mathbb{R}^2$ into $C^0(I_m^-)$.

Next, from (3.3) we obtain immediately:

$$\|h_m^-(\cdot, x, \alpha, \varepsilon)\| \le \Delta_-(|x|)|x| + N|\varepsilon| \tag{3.12}$$

where

$$\Delta_-(r) = \sup\left\{\left|f'_-(x + \gamma_-(t)) - f'_-(\gamma_-(t))\right| \mid t \le -\bar{T}, |x| \le r\right\}$$

is an increasing function such that $\Delta_-(0) = 0$ and

$$N = \sup\left\{|g(t, z, \varepsilon)| \mid (t, z, \varepsilon) \in \mathbb{R}^{n+2}\right\}$$

and hence, using (3.9) we get:

$$\|\hat{x}\|_{I_m^-} \le k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}\Delta_-(\|x\|_{I_m^-})\|x\|_{I_m^-} + 2\delta^{-1}N|\varepsilon|\right]. \tag{3.13}$$

Similarly, for fixed $(\xi_-, \varphi_-, \alpha, \varepsilon) \in \mathcal{N}P_- \times \mathcal{R}P_{-,m} \times \mathbb{R}^2$ and $x_1(t), x_2(t) \in C^0(I_m^-)$ we see that

$$\|\hat{x}_2 - \hat{x}_1\|_{I_m^-} \le 2k\delta^{-1}\left[\Delta_-(\bar{r}) + N'|\varepsilon|\right]\|x_2 - x_1\|_{I_m^-}, \tag{3.14}$$

where $\bar{r} = \max\{\|x_1\|_{I_m^-}, \|x_2\|_{I_m^-}\}$ and

$$N' = \sup\left\{\left|\frac{\partial g}{\partial x}(t, z, \varepsilon)\right| \mid (t, z, \varepsilon) \in \mathbb{R}^{n+2}\right\}.$$

Thus if $\rho > 0$, $|\xi_-|$, $|\varphi_-|$ and $|\varepsilon|$ are sufficiently small, the map (3.11) is a C^r-contraction in the ball of center $x(t) = 0$ and radius ρ in $C^0(I_m^-)$, which is uniform with respect to the other parameters $(\xi_-, \varphi_-, \alpha, \varepsilon)$ and $m \in \mathbb{Z}$.

Hence we obtain the following

Proposition 3.5 *Assume that conditions* (H1), (H2) *hold and let* $(\xi_-, \varphi_-, \alpha, \varepsilon) \in \mathcal{N}P_- \times \mathcal{R}P_{-,m} \times \mathbb{R}^2$ *and* $\rho > 0$ *be such that* $2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \le \rho$ *and* $4k\delta^{-1}\left[\Delta_-(\rho) + N'|\varepsilon|\right] < 1$. *Then, for* $t \in I_m^-$, *Eq.* (3.4) *has a unique bounded solution* $x_m^-(t) = x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *which is* C^r *in the parameters* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *and* $m \in \mathbb{Z}$, *and satisfies*

$$\|x_m^-(\cdot, \xi_-, \varphi_-, \alpha, \varepsilon)\|_{I_m^-} \le 2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \le \rho \tag{3.15}$$

together with

$$(\mathbb{I} - P_-)x_m^-(T_{2m} - \bar{T}) = \xi_-, \qquad P_{-,m}x_m^-(T_{2m-1} - 1) = \varphi_-.$$

Moreover the derivatives of $x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *with respect to* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *are also bounded in* I_m^- *uniformly with respect to* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *and* $m \in \mathbb{Z}$ *and they are uniformly continuous in* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *uniformly with respect to* m *and* $t \in I_m^-$.

Proof Only the last part of the statement needs to be proved. We know that $x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ is the unique fixed point of the map given by the right hand side of Eq. (3.10) with $h_m^-(t, x(t), \alpha, \varepsilon)$ instead of $h(t)$. Since $\xi_- \in \mathcal{N}P_-$ we have $|X_-(t - T_{2m})\xi_-| = |X_-(t - T_{2m})(\mathbb{I} - P_-)X_-(-\bar{T})\xi_-| \le k\,\mathrm{e}^{\delta(t - T_{2m} - \bar{T})}|\xi_-| \le k|\xi_-|$ for any $t \in I_m^-$. A similar argument shows that $|X_-(t - T_{2m})P_-X_-^{-1}(T_{2m-1} - 1 - T_{2m})\varphi_-| \le k|\varphi_-|$ for any $t \in I_m^-$. As a consequence the right hand side of (3.10) consists of a bounded linear map in (ξ_-, φ_-), with bound independent of $m \in \mathbb{Z}$, and the nonlinear map from $C_b^0(I_m^-) \times \mathbb{R} \times \mathbb{R}$:

$$(x(\cdot), \alpha, \varepsilon) \mapsto \int_{T_{2m-1}-1}^{t} X_-(t - T_{2m})P_-X_-^{-1}(s - T_{2m})h_m^-(s, x(s), \alpha, \varepsilon)ds$$
$$- \int_{t}^{T_{2m}-\bar{T}} X_-(t - T_{2m})(\mathbb{I} - P_-)X_-^{-1}(s - T_{2m})h_m^-(s, x(s), \alpha, \varepsilon)ds$$

whose derivatives up to the rth order are bounded and uniformly continuous in (x, α, ε) uniformly with respect to m because of the properties of $h_m^-(t, x, \alpha, \varepsilon)$ (see Remark 3.1) and (2.2). The proof is complete. □

We are now ready to prove the main result of this section:

Proposition 3.6 *Assume that conditions* (H1), (H2) *hold and let* $(\xi_-, \varphi_-, \alpha, \varepsilon) \in \mathcal{N}P_- \times \mathcal{R}P_{-,m} \times \mathbb{R}^2$ *and* $\rho > 0$ *be such that* $2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \le \rho$ *and* $4k\delta^{-1}\left[\Delta_-(\rho) + N'|\varepsilon|\right] < 1$. *Then, for* $t \in I_{m,\alpha}^-$, *Eq.* $\dot{z} = f_-(z) + \varepsilon g(t, z, \varepsilon)$ *has a unique bounded solution* $z_m^-(t) = z_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *which is* C^r *in the parameters* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *and satisfies*

$$\|z_m^-(\cdot + \alpha, \xi_-, \varphi_-, \alpha, \varepsilon) - \gamma_-(\cdot - T_{2m})\|_{I_m^-} \le 2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \le \rho \tag{3.16}$$

together with

$$(\mathbb{I} - P_-)[z_m^-(T_{2m} - \bar{T} + \alpha) - \gamma_-(-\bar{T})] = \xi_-,$$
$$P_{-,m}[z_m^-(T_{2m-1} + \alpha - 1) - \gamma_-(T_{2m-1} - T_{2m} - 1)] = \varphi_-.$$

Moreover $x_m^-(t) := z_m^-(t + \alpha, \xi_-, \varphi_-, \alpha, \varepsilon) - \gamma_-(t - T_{2m})$ *is the unique fixed point of the map* (3.10) *and* $z_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *and its derivatives with respect to* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *are also* *bounded in I_m^- uniformly with respect to $(\xi_-, \varphi_-, \alpha, \varepsilon)$ and $m \in \mathbb{Z}$, uniformly continuous in $(\xi_-, \varphi_-, \alpha, \varepsilon)$ uniformly with respect to (t, m) with $t \in I_m^-$, $m \in \mathbb{Z}$ and satisfy:*

$$\begin{aligned}
\frac{\partial z_m^-}{\partial \xi_-}(t+\alpha, 0, 0, \alpha, 0) &= X_-(t - T_{2m})(\mathbb{I} - P_-) \\
\frac{\partial z_m^-}{\partial \varphi_-}(t+\alpha, 0, 0, \alpha, 0)\varphi_- &= X_-(t - T_{2m})P_- X_-^{-1}(T_{2m-1} - T_{2m} - 1)\varphi_- \\
\frac{\partial z_m^-}{\partial \varepsilon}(t+\alpha, 0, 0, \alpha, 0) &= \int_{T_{2m-1}-1}^{t} X_-(t - T_{2m})P_- X_-^{-1}(s - T_{2m}) \\
&\quad g(s+\alpha, \gamma_-(s - T_{2m}), 0)ds \\
&\quad - \int_{t}^{T_{2m}-\bar{T}} X_-(t - T_{2m})\,(\mathbb{I} - P_-)\,X_-^{-1}(s - T_{2m}) \\
&\quad g(s+\alpha, \gamma_-(s - T_{2m}), 0)ds. \qquad (3.17)
\end{aligned}$$

Proof Setting $x(t) := z_m^-(t+\alpha) - \gamma_-(t - T_{2m})$ the existence of $z_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ satisfying (3.16) follows from Proposition 3.5. Thus we only need to prove (3.17).

From (3.13) we see that $x_m^-(t, 0, 0, \alpha, 0) = 0$ and then differentiating equation (3.10) with $x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ instead of $x(t)$ and $h_m^-(t, x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon), \alpha, \varepsilon)$ instead of $h(t)$ we see that

$$\frac{\partial z_m^-}{\partial \xi_-}(t+\alpha, 0, 0, \alpha, 0)\xi_- = \frac{\partial x_m^-}{\partial \xi_-}(t, 0, 0, \alpha, 0)\xi_- = X_-(t - T_{2m})\xi_-.$$

Similarly we obtain the rest of (3.17). □

Remark 3.7 The function $z_m^-(t) = z_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ is a bounded solution of Eq. (2.1) in the interval $I_{m,\alpha}^-$ as long as it remains in Ω_- for $t \in I_{m,\alpha}^-$, and satisfies (3.16). However in order that $z_m^-(t) \in \Omega_-$ for $t \in I_{m,\alpha}^-$ it is sufficient that $G(z_m^-(T_{2m} - \bar{T} + \alpha)) = 0$ (see [4, p. 345] for details).

Next, let

$$\begin{aligned}
\Delta_+(r) :&= \sup\left\{\left|f'_-(x + \gamma_+(t)) - f'_-(\gamma_+(t))\right| \mid \bar{T} \le t, |x| \le r\right\}, \\
P_{+,m} :&= P_+(T_{2m+1} - T_{2m} + 1) \\
&= X_+(T_{2m+1} - T_{2m} + 1)P_+ X_+(T_{2m+1} - T_{2m} + 1)^{-1}, \qquad (3.18) \\
h_m^+(t, x, \beta, \varepsilon) &= f_-(x + \gamma_+(t - T_{2m})) - f_-(\gamma_+(t - T_{2m})) \\
&\quad - f'_-(\gamma_+(t - T_{2m}))x + \varepsilon g(t+\beta, x + \gamma_+(t - T_{2m}), \varepsilon).
\end{aligned}$$

By an almost identical argument we show the following

Proposition 3.8 *Assume that conditions* (H1), (H2) *hold and let $(\xi_+, \varphi_+, \beta, \varepsilon) \in \mathcal{R}P_+ \times \mathcal{N}P_{+,m} \times \mathbb{R}^2$ and $\rho > 0$ be such that $2k\left[|\xi_+| + |\varphi_+| + 2\delta^{-1}N|\varepsilon|\right] \le \rho$ and $4k\delta^{-1}\left[\Delta_+(\rho) + N'|\varepsilon|\right] < 1$. Then, for $t \in I_{m,\beta}^+$, equation $\dot{z} = f_+(z) + \varepsilon g(t, z, \varepsilon)$ has a unique bounded solution $z_m^+(t) = z_m^+(t, \xi_+, \varphi_+, \beta, \varepsilon)$ which is C^r in the parameters $(\xi_+, \varphi_+, \beta, \varepsilon)$ and satisfies*

$$\|z_m^+(\cdot + \beta, \xi_+, \varphi_+, \beta, \varepsilon) - \gamma_+(\cdot - T_{2m})\|_{I_m^+} \le 2k\left[|\xi_+| + |\varphi_+| + 2\delta^{-1}N|\varepsilon|\right] \le \rho \quad (3.19)$$

together with

$$P_+[z_m^+(T_{2m}+\bar{T}+\beta)-\gamma_+(\bar{T})]=\xi_+,$$
$$(\mathbb{I}-P_{+,m})[z_m^+(T_{2m+1}+\beta+1)-\gamma_+(T_{2m+1}-T_{2m}+1)]=\varphi_+.$$

Moreover $x_m^+(t):=z_m^+(t+\beta,\xi_+,\varphi_+,\beta,\varepsilon)-\gamma_+(t-T_{2m})$ *is the unique fixed point of the map*

$$\begin{aligned}(x(t),\xi_+,\varphi_+,\beta,\varepsilon)\mapsto\ & X_+(t-T_{2m})\xi_+\\ &+X_+(t-T_{2m})\,(\mathbb{I}-P_+)\,X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_+\\ &+\int_{T_{2m}+\bar{T}}^{t}X_+(t-T_{2m})P_+X_+^{-1}(s-T_{2m})h_m^+(s,x(s),\beta,\varepsilon)ds\\ &-\int_{t}^{T_{2m+1}+1}X_+(t-T_{2m})\,(\mathbb{I}-P_+)\,X_+^{-1}(s-T_{2m})h_m^+(s,x(s),\beta,\varepsilon)ds,\end{aligned}\tag{3.20}$$

and $z_m^+(t,\xi_+,\varphi_+,\beta,\varepsilon)$ *and its derivatives with respect to* $(\xi_+,\varphi_+,\beta,\varepsilon)$ *are also bounded in* I_m^+ *uniformly with respect to* $(\xi_+,\varphi_+,\beta,\varepsilon)$ *and* $m\in\mathbb{Z}$, *uniformly continuous in* $(\xi_+,\varphi_+,\beta,\varepsilon)$ *uniformly with respect to* (t,m) *with* $t\in I_m^+$, $m\in\mathbb{Z}$ *and satisfy:*

$$\begin{aligned}\frac{\partial z_m^+}{\partial\xi_+}(t+\beta,0,0,\beta,0)&=X_+(t-T_{2m})P_+\\ \frac{\partial z_m^+}{\partial\varphi_+}(t+\beta,0,0,\beta,0)\varphi_+&=X_+(t-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_+\\ \frac{\partial z_m^+}{\partial\varepsilon}(t+\beta,0,0,\beta,0)&=\int_{T_{2m}+\bar{T}}^{t}X_+(t-T_{2m})P_+X_+^{-1}(s-T_{2m})\\ &\qquad g(s+\beta,\gamma_+(s-T_{2m}),0)ds\\ &\quad-\int_{t}^{T_{2m+1}+1}X_+(t-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(s-T_{2m})\\ &\qquad g(s+\beta,\gamma_+(s-T_{2m}),0)ds.\end{aligned}\tag{3.21}$$

Remark 3.9 Note that $z_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ (resp. $z_m^+(t,\xi_+,\varphi_+,\alpha,\varepsilon)$) depends on m by means of T_{2m-1} and T_{2m} (resp. T_{2m} and T_{2m+1}). Thus we may also write $x^-(t,\xi_-,\varphi_-,\alpha,\varepsilon,T_{2m},T_{2m-1})$, $x^+(t,\xi_+,\varphi_+,\alpha,\varepsilon,T_{2m},T_{2m+1})$ instead of $x_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$, $x_m^+(t,\xi_+,\varphi_+,\alpha,\varepsilon)$ and say that $x^-(t,\xi_-,\varphi_-,\alpha,\varepsilon,T_{2m},T_{2m-1})$, resp. $x^+(t,\xi_+,\varphi_+,\alpha,\varepsilon,T_{2m},T_{2m+1})$, is uniformly continuous with respect to $(\xi_-,\varphi_-,\alpha,\varepsilon)$, resp. $(\xi_+,\varphi_+,\beta,\varepsilon)$, uniformly with respect to T_{2m},T_{2m-1}, resp. T_{2m},T_{2m+1}, and $t\in I_m^-$, (resp. $t\in I_m^+$).

4 Orbits Close to the Branch $\gamma_0(t)$

As in [4, Proposition 2.9] we can prove the following.

Proposition 4.1 *Assume that conditions* (H1), (H2) *hold. Then there exist positive constants* c, ε_0 *and* $\widetilde{\rho}_0$ *such that for any* $\alpha, \beta, \varepsilon \in \mathbb{R}$ *and* $\bar{\xi} \in \mathbb{R}^n$ *such that* $|\beta-\alpha| < \min\{1, 2\bar{T}\}$, $|\varepsilon| \le \varepsilon_0$ *and* $|\bar{\xi} - \gamma_0(-\bar{T})| < \widetilde{\rho}_0$, *there exists a unique solution* $z_m^0(t) = z_m^0(t, \bar{\xi}, \alpha, \beta, \varepsilon)$ *of equation* $\dot{z} = f_+(z) + \varepsilon g(t, z, \varepsilon)$, *for* $t \in [T_{2m} - \bar{T} + \alpha, T_{2m} + \bar{T} + \beta]$ *such that*

$$z_m^0(T_{2m} - \bar{T} + \alpha) = \bar{\xi}$$

and

$$\|z_m^0(t) - \gamma_0(t - T_{2m} - \alpha)\|_{[T_{2m}-\bar{T}+\alpha, T_{2m}+\bar{T}+\beta]} \le c[|\bar{\xi} - \gamma_0(-\bar{T})| + 2N\delta^{-1}|\varepsilon|]. \quad (4.1)$$

Moreover $z_m^0(t, \bar{\xi}, \alpha, \beta, \varepsilon)$ *and its derivatives with respect to* $(\bar{\xi}, \alpha, \beta, \varepsilon)$ *are bounded in* $[T_{2m} - \bar{T} + \alpha, T_{2m} + \bar{T} + \beta]$ *uniformly with respect to* $m \in \mathbb{Z}$, *uniformly continuous in* $(\bar{\xi}, \alpha, \beta, \varepsilon)$, *uniformly with respect to* $t \in [T_{2m} - \bar{T} + \alpha, T_{2m} + \bar{T} + \beta], m \in \mathbb{Z}$, *and have the following properties:*

(i) $x_m^0(t) = z_m^0(t + \alpha, \bar{\xi}, \alpha, \beta, \varepsilon) - \gamma_0(t - T_{2m})$ *is a fixed point of the map*

$$\begin{aligned} x(t) \mapsto\ & X_0(t - T_{2m})\left[\bar{\xi} - \gamma_0(-\bar{T})\right] \\ & + \int_{T_{2m}-\bar{T}}^{t} X_0(t - T_{2m}) X_0^{-1}(s - T_{2m}) h_m^0(s, x(s), \alpha, \varepsilon) ds \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} h_m^0(t, x, \alpha, \varepsilon) = \ & f_+(x + \gamma_0(t - T_{2m})) - f_+(\gamma_0(t - T_{2m})) - f'_+(\gamma_0(t - T_{2m}))x \\ & +\varepsilon g(t + \alpha, x + \gamma_0(t - T_{2m}), \varepsilon); \end{aligned}$$

(ii) *the following equalities hold:*

$$\begin{aligned} \frac{\partial z_m^0}{\partial \alpha}(t, \gamma_0(\bar{T}), \alpha, \beta, 0) &= -\dot{\gamma}_0(t - T_{2m} - \alpha), \\ \frac{\partial z_m^0}{\partial \beta}(t, \gamma_0(-\bar{T}), \alpha, \beta, 0) &= 0 \\ \frac{\partial z_m^0}{\partial \bar{\xi}}(t, \gamma_0(-\bar{T}), \alpha, \beta, 0) &= X_0(t - T_{2m} - \alpha) \\ \frac{\partial z_m^0}{\partial \varepsilon}(t + \alpha, \gamma_0(-\bar{T}), \alpha, \beta, 0) &= \int_{T_{2m}-\bar{T}}^{t} X_0(t - T_{2m}) X_0^{-1}(s - T_{2m}) g(s + \alpha, \\ & \qquad \gamma_0(s - T_{2m}), 0)\, ds. \end{aligned} \quad (4.3)$$

Sketch of the proof The statement concerning the existence of the solution $z_m^0(t) = z_m^0(t, \xi, \alpha, \beta, \varepsilon)$ such that (4.1) holds, follows from the continuous dependence on the data. Moreover the fact that $x_m^0(t)$ is a fixed point of the map (4.2) follows from the variation of constants formula. The boundedness and continuity properties of $z_m^0(t, \xi, \alpha, \beta, \varepsilon)$ follow from the similar properties of $h_m^0(t, x, \alpha, \varepsilon)$ as in Propositions 3.6, 3.8. Then, because of uniqueness of fixed points we also get:

$$z_m^0(t, \gamma_0(-\bar{T}), \alpha, \beta, 0) = \gamma_0(t - T_{2m} - \alpha)$$

from which the first two equalities of point (ii) easily follow. Differentiating (4.2) with respect to ξ, ε respectively and using the fact that $h_m^0(t, x, \alpha, 0)$ is of the second order in x, we derive the other two equalities in (ii). □

Note that if

$$c[\widetilde{\rho}_0 + 2N\delta^{-1}\varepsilon_0] < \rho$$

from (4.1) we obtain:

$$\sup\{|z_m^0(t+\alpha) - \gamma_0(t - T_{2m})| \mid t \in [T_{2m} - \bar{T}, T_{2m} + \bar{T} + \beta - \alpha]\} < \rho. \tag{4.4}$$

Remark 4.2 Note that $z_m^0(t, \bar{\xi}, \alpha, \beta, \varepsilon))$ depends on m by means of T_{2m}. Thus we may also write $z^0(t, \bar{\xi}, \alpha, \beta, \varepsilon, T_{2m})$ instead of $z_m^0(t, \bar{\xi}, \alpha, \beta, \varepsilon)$ and say that $z^0(t, \bar{\xi}, \alpha, \beta, \varepsilon, T_{2m})$ is uniformly continuous in $(\bar{\xi}, \alpha, \beta, \varepsilon)$ uniformly with respect to T_{2m} and $t \in [T_{2m} - \bar{T} + \alpha, T_{2m} + \bar{T} + \beta]$.

5 The Bifurcation Equation

Let $\varepsilon_0 > 0$, $\widetilde{\rho}_0 > 0$ and $c > 0$ be constants as in Proposition 4.1, $C := \max\{c, 2k\}$, $\chi < 1$ a positive constant that will be specified and fixed later and $\rho_0 \leq c\widetilde{\rho}_0$ be the largest positive number satisfying

$$4k\delta^{-1}\left[\Delta_{\pm}(\rho_0) + \frac{N'\delta}{2NC}\rho_0\right] \leq 1.$$

Next, let $0 < \rho < \rho_0$ and $\varepsilon_\rho := \min\left\{\frac{\rho\delta}{2CN}, \varepsilon_0\right\}$. For any $\alpha = \{\alpha_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ and $\varepsilon \in (-\varepsilon_\rho, \varepsilon_\rho)$ we set

$$\begin{aligned}\ell^\infty_{\rho,\alpha,\varepsilon} := \Big\{\theta := &\left\{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\right\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}^{5n+1}) :\\ &(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m) \in \mathcal{R}P_{-,m} \times \mathcal{N}P_{+,m} \times \mathcal{N}P_- \times \mathcal{R}P_+ \times \mathbb{R}^{n+1},\\ &2k\left[|\xi_m^\pm| + |\varphi_m^\pm| + 2\delta^{-1}N|\varepsilon|\right] < \rho, \quad c[|\bar{\xi}_m - \gamma_0(-\bar{T})| + 2N\delta^{-1}|\varepsilon|] < \rho,\\ &\sup_{m\in\mathbb{Z}} |\alpha_{m+1} - \beta_m| < \chi\Big\}\end{aligned}$$

and

$$\ell^\infty_\rho = \left\{(\theta, \alpha, \varepsilon) \in \ell^\infty_{\rho,\alpha,\varepsilon} \times \ell^\infty(\mathbb{R}) \times (-\varepsilon_\rho, \varepsilon_\rho) : \alpha \in \ell^\infty_\chi\right\}$$

where

$$\ell^\infty_\chi = \left\{\alpha \in \ell^\infty(\mathbb{R}) : \sup_{m\in\mathbb{Z}} |\alpha_m - \alpha_{m-1}| < \chi\right\}.$$

Note that, because of the choice of ρ and ε_ρ, $\ell^\infty_{\rho,\alpha,\varepsilon}$, ℓ^∞_ρ and ℓ^∞_χ are open nonempty subsets of $\ell^\infty(\mathcal{R}P_{-,m} \times \mathcal{N}P_{+,m} \times \mathcal{N}P_- \times \mathcal{R}P_+ \times \mathbb{R}^n \times \mathbb{R})$, $\ell^\infty(\mathcal{R}P_{-,m} \times \mathcal{N}P_{+,m} \times \mathcal{N}P_- \times \mathcal{R}P_+ \times \mathbb{R}^n \times \mathbb{R}) \times \ell^\infty(\mathbb{R}) \times (-\varepsilon_\rho, \varepsilon_\rho)$ and $\ell^\infty(\mathbb{R})$, respectively. In $\ell^\infty_{\rho,\alpha,\varepsilon}$ we take the norm

$$\begin{aligned}\|\theta\| &= \left\|\left\{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\right\}_{m\in\mathbb{Z}}\right\|\\ &= \sup_{m\in\mathbb{Z}} \max\left\{|\varphi_m^- + \varphi_m^+|, |\xi_m^-|, |\xi_m^+|, |\bar{\xi}_m|, |\beta_m|\right\}.\end{aligned}$$

Let $\mathcal{T} = \{T_m\}_{m\in\mathbb{Z}}$ be given as in Sect. 3 and take $(\theta, \alpha, \varepsilon) \in \ell^\infty_\rho$. In this section we want to find conditions so that system (2.1) has a solution $z(t)$ defined on $\mathbb{R}$ such that for any $m \in \mathbb{Z}$ satisfies:

$$\|z(t) - \gamma_-(t - T_{2m} - \alpha_m)\|_{\widetilde{I}_m^-} < \rho$$
$$\|z(t) - \gamma_0(t - T_{2m} - \alpha_m)\|_{\widetilde{I}_m^0} < \rho$$
$$\|z(t) - \gamma_+(t - T_{2m} - \beta_m)\|_{\widetilde{I}_m^+} < \rho$$

where $\widetilde{I}_m^- = [T_{2m-1} + \alpha_m - 1, T_{2m} - \bar{T} + \alpha_m]$, $I_m^0 = [T_{2m} - \bar{T} + \alpha_m, T_{2m} + \bar{T} + \beta_m]$ and $\widetilde{I}_m^+ = [T_{2m} + \bar{T} + \beta_m, T_{2m+1} + \beta_m]$.

We note that for any $(\theta, \alpha, \varepsilon) \in \ell_\rho^\infty$ assumptions of Propositions 3.6, 3.8 and 4.1 are satisfied. Indeed we have

$$4k\delta^{-1}\left[\Delta_\pm(\rho) + N'|\varepsilon|\right] < 4k\delta^{-1}\left[\Delta_\pm(\rho) + N'\varepsilon_\rho\right] < 4k\delta^{-1}\left[\Delta_\pm(\rho_0) + \frac{N'\delta}{2NC}\rho_0\right] \le 1$$

along with $|\varepsilon| < \varepsilon_0$ and

$$|\bar{\xi} - \gamma_0(-\bar{T})| < \frac{\rho}{c} < \frac{\rho_0}{c} \le \widetilde{\rho}_0.$$

So according to the previous sections and because of uniqueness of the solutions $z_m^+(t, \xi_m^+, \varphi_m^+, \beta_m, \varepsilon)$, $z_m^-(t, \xi_m^-, \varphi_m^-, \alpha_m, \varepsilon)$ and $z_m^0(t, \bar{\xi}_m, \alpha_m, \beta_m, \varepsilon)$ we see that such a solution can be found if and only if we are able to solve the infinite set of equations ($m \in \mathbb{Z}$):

$$\begin{cases} z_m^+(T_{2m+1} + \beta_m, \xi_m^+, \varphi_m^+, \beta_m, \varepsilon) - z_{m+1}^-(T_{2m+1} + \beta_m, \xi_{m+1}^-, \varphi_{m+1}^-, \alpha_{m+1}, \varepsilon) = 0 \\ z_m^0(T_{2m} - \bar{T} + \alpha_m, \bar{\xi}_m, \alpha_m, \beta_m, \varepsilon) - z_m^-(T_{2m} - \bar{T} + \alpha_m, \xi_m^-, \varphi_m^-, \alpha_m, \varepsilon) = 0 \\ z_m^0(T_{2m} + \bar{T} + \beta_m, \bar{\xi}_m, \alpha_m, \beta_m, \varepsilon) - z_m^+(T_{2m} + \bar{T} + \beta_m, \xi_m^+, \varphi_m^+, \beta_m, \varepsilon) = 0 \\ G(z_m^-(T_{2m} - \bar{T} + \alpha_m, \xi_m^-, \varphi_m^-, \alpha_m, \varepsilon)) = 0 \\ G(z_m^0(T_{2m} + \bar{T} + \beta_m, \bar{\xi}_m, \alpha_m, \beta_m, \varepsilon)) = 0 \\ G(z_m^+(T_{2m} + \bar{T} + \beta_m, \xi_m^+, \varphi_m^+, \beta_m, \varepsilon)) = 0. \end{cases} \tag{5.1}$$

Since $T_{2m+1} + \alpha_{m+1} - 1 < T_{2m+1} + \beta_m$, system (5.1) is well-posed. Note that, from Proposition 4.1, the second of the above equations reads:

$$\bar{\xi}_m = z_m^-(T_{2m} - \bar{T} + \alpha_m, \xi_m^-, \varphi_m^-, \alpha_m, \varepsilon)$$

and gives the sequence $\{\bar{\xi}_m\}_{m\in\mathbb{Z}}$ in terms of the sequences $\{\xi_m^-\}_{m\in\mathbb{Z}}$, $\{\varphi_m^-\}_{m\in\mathbb{Z}}$, $\{\alpha_m\}_{m\in\mathbb{Z}}$, and ε. Moreover, if ρ is sufficiently small, $z_m^0(T_{2m} + \bar{T} + \beta_m, \bar{\xi}_m, \alpha_m, \beta_m, \varepsilon)$ is close to $\gamma_0(\bar{T} + \beta_m - \alpha_m)$, while $z_m^+(T_{2m} + \bar{T} + \beta_m, \xi_m^+, \varphi_m^+, \beta_m, \varepsilon)$ is close to $\gamma_+(\bar{T}) = \gamma_0(\bar{T})$. So there is a positive constant $\chi < \min\left\{1, 2\bar{T}\right\}$ such that the 5th and the 6th equation in (5.1) imply that the 3rd equation is equivalent to (see [4, Lemma 2.10])

$$R_0[z_m^0(T_{2m} + \bar{T} + \beta_m, \bar{\xi}_m, \alpha_m, \beta_m, \varepsilon) - z_m^+(T_{2m} + \bar{T} + \beta_m, \xi_m^+, \varphi_m^+, \beta_m, \varepsilon)] = 0,$$

where $R_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the projection defined in Sect. 2. From now on, we fix such a χ. Here we use the fact $|\beta_m - \alpha_m| < 2\chi$ for any $m \in \mathbb{Z}$, so $\gamma_0(\bar{T} + \beta_m - \alpha_m)$ and $\gamma_0(\bar{T})$ are sufficiently close for χ small enough uniformly for any $m \in \mathbb{Z}$.

Let

$$\ell_1^\infty = \ell^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{R}R_0 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$$

with the norm

$$\sup_{m\in\mathbb{Z}} \max\{|a_m|, |b_m|, |c_m|, |d_m|, |e_m|, |f_m|\}$$

for $\{(a_m, b_m, c_m, d_m, e_m, f_m)\}_{m\in\mathbb{Z}} \in \ell_1^\infty$.

We define a map

$$\mathcal{G}_\mathcal{T} \in C^r\left(\ell_\rho^\infty, \ell_1^\infty\right)$$

as

$$\mathcal{G}_{\mathcal{T}}(\theta,\alpha,\varepsilon)=\mathcal{G}_{\mathcal{T}}(\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar{\xi}_m,\beta_m)\}_{m\in\mathbb{Z}},\{\alpha_m\}_{m\in\mathbb{Z}},\varepsilon):=$$
$$\left\{\begin{array}{c} z_m^+(T_{2m+1}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)-z_{m+1}^-(T_{2m+1}+\beta_m,\xi_{m+1}^-,\varphi_{m+1}^-,\alpha_{m+1},\varepsilon)\\ \bar{\xi}_m-z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)\\ R_0[z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)-z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)]\\ G(z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon))\\ G(z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon))\\ G(z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)) \end{array}\right\}_{m\in\mathbb{Z}}$$

so that Eq. (5.1) reads

$$\mathcal{G}_{\mathcal{T}}(\theta,\alpha,\varepsilon)=0. \tag{5.2}$$

From [9] it follows that $\mathcal{G}_{\mathcal{T}}$ is C^r and has bounded derivatives. More precisely, from the continuity properties of the solutions $z_m^+(t,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$, $z_m^-(t,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$, and $z_m^0(t,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)$ we see that $\mathcal{G}_{\mathcal{T}}(\theta,\alpha,\varepsilon)$ and its derivatives are bounded and uniformly continuous in $(\theta,\alpha,\varepsilon)$ uniformly with respect to $\mathcal{T}\in\ell_T^\infty(\mathbb{R})$.

We also need to introduce further maps. For $\alpha\in\ell_\chi^\infty$ and $m\in\mathbb{Z}$ we define $\mathcal{L}_{\alpha,m}$: $\mathcal{R}P_{-,m+1}\oplus\mathcal{N}P_{+,m}\to\mathcal{N}P_+(T_{2m+1}-T_{2m})\oplus\mathcal{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1})$ as

$$\begin{aligned}\mathcal{L}_{\alpha,m}:(\varphi_{m+1}^-,\varphi_m^+)\mapsto\ & X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\\ &-X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi_{m+1}^-\end{aligned}$$

and $\mathcal{H}_\alpha:\ell_{\rho,\alpha,\varepsilon}^\infty\to\ell_1^\infty$ as

$$\mathcal{H}_\alpha\theta=\left\{\begin{array}{c}\mathcal{L}_{\alpha,m}(\varphi_{m+1}^-,\varphi_m^+)\\ \bar{\xi}_m-\xi_m^-\\ R_0[X_0(\bar{T})\bar{\xi}_m-\xi_m^+]\\ G'(\gamma_0(-\bar{T}))\xi_m^-\\ G'(\gamma_0(\bar{T}))\cdot[X_0(\bar{T})\bar{\xi}_m+\dot{\gamma}_0(\bar{T})\beta_m]\\ G'(\gamma_+(\bar{T}))\cdot\xi_m^+\end{array}\right\}_{m\in\mathbb{Z}}.$$

Note that since $T_{m+1}-T_m\geq T+1$ and $|\alpha_{m+1}-\alpha_m|<\chi<1$, we have (see (2.9))

$$\mathcal{R}P_{-,m+1}\oplus\mathcal{N}P_{+,m}=\mathcal{N}P_+(T_{2m+1}-T_{2m})\oplus\mathcal{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1})=\mathbb{R}^n.$$

Before giving our main result we state few properties of the maps here introduced. For any $\alpha\in\ell_\chi^\infty$, we set:

$$\theta_\alpha=\left\{(0,0,0,0,\gamma_0(-\bar{T}),\alpha_m)\right\}_{m\in\mathbb{Z}}.$$

Then

$G_1)$ $\|\mathcal{G}_{\mathcal{T}}(\theta_\alpha,\alpha,0)\|\leq 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\max\{|\dot{\gamma}_-(-\bar{T})|,|\dot{\gamma}_+(\bar{T})|\}$;

$G_2)$ $\left\|\frac{d}{d\alpha}[\mathcal{G}_{\mathcal{T}}(\theta_\alpha,\alpha,0)]\right\|\leq 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}|\dot{\gamma}_-(-\bar{T})|$;

$G_3)$ $\mathcal{L}_{\alpha,m}:\mathbb{R}^n\to\mathbb{R}^n$ is a linear isomorphism satisfying:

$$\|\mathcal{L}_{\alpha,m}\|\leq k\widetilde{c},\qquad \|\mathcal{L}_{\alpha,m}^{-1}\|\leq k\widetilde{c}\,\mathrm{e}^{2\delta}$$

and

$$\left\|\frac{\partial}{\partial\alpha}\mathcal{L}_{\alpha,m}\right\|\leq 2N_-k\quad\text{and}\quad\left\|\frac{\partial}{\partial\alpha}\mathcal{L}_{\alpha,m}^{-1}\right\|\leq 2N_-k^3\widetilde{c}^2\,\mathrm{e}^{4\delta}$$

for a suitable constant $\widetilde{c}$;

G_4) $\mathcal{H}_\alpha : \ell^\infty_{\rho,\alpha,\varepsilon} \to \ell^\infty_1$ is a bounded linear map and $\left\| \frac{\partial}{\partial\alpha}\mathcal{H}_\alpha \right\| \le 2N_- k$;
G_5) $\|D_1\mathcal{G}_T(\theta_\alpha, \alpha, 0) - \mathcal{H}_\alpha\| \le \widetilde{c}_3 k\, \mathrm{e}^{-\delta(T-\bar{T})}$ for a suitable constant $\widetilde{c}_3$.

Properties (G_1)–(G_5) will be proved in the Appendix A.

Next, given $\{(a_m, b_m, c_m, d_m, e_m, f_m)\}_{m\in\mathbb{Z}} \in \ell^\infty_1$ we want to solve the linear equation

$$\mathcal{H}_\alpha\theta = \left\{ \begin{matrix} a_m \\ b_m \\ c_m \\ d_m \\ e_m \\ f_m \end{matrix} \right\}_{m\in\mathbb{Z}} \tag{5.3}$$

that is the set of equations:

$$\begin{cases} \mathcal{L}_{\alpha,m}(\varphi^-_{m+1}, \varphi^+_m) = a_m \\ \bar{\xi}_m - \xi^-_m = b_m \\ R_0[X_0(\bar{T})\bar{\xi}_m - \xi^+_m] = c_m \\ G'(\gamma_0(-\bar{T}))\xi^-_m = d_m \\ G'(\gamma_0(\bar{T}))\cdot[X_0(\bar{T})\bar{\xi}_m + \dot{\gamma}_0(\bar{T})\beta_m] = e_m \\ G'(\gamma_+(\bar{T}))\cdot\xi^+_m = f_m. \end{cases} \tag{5.4}$$

To solve (5.4) we write:

$$\begin{aligned} \xi^-_m &= \eta^\perp_m + \mu^-_m\dot{\gamma}_-(-\bar{T}), \quad m \in \mathbb{Z}, \\ \xi^+_m &= \zeta^\perp_m + \mu^+_m\dot{\gamma}_+(\bar{T}), \quad m \in \mathbb{Z}, \end{aligned} \tag{5.5}$$

where

$$\{\eta^\perp_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathcal{S}'), \quad \{\zeta^\perp_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathcal{S}''), \quad \{\mu^\pm_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}),$$

and plug (5.5) into (5.4). We derive

$$\begin{aligned} (\varphi^-_{m+1}, \varphi^+_m) &= \mathcal{L}^{-1}_{\alpha,m} a_m \\ \mu^-_m &= \frac{d_m}{G'(\gamma_-(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \\ \mu^+_m &= \frac{f_m}{G'(\gamma_+(\bar{T}))\dot{\gamma}_+(\bar{T})} \\ \bar{\xi}_m &= \eta^\perp_m + \mu^-_m\dot{\gamma}_-(-\bar{T}) + b_m \\ \beta_m &= \frac{e_m - G'(\gamma_0(\bar{T}))X_0(\bar{T})\bar{\xi}_m}{G'(\gamma_0(\bar{T}))\dot{\gamma}_0(\bar{T})} \\ R_0X_0(\bar{T})\eta^\perp_m - \zeta^\perp_m &= c_m - \mu^-_m R_0X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) \\ &\quad - R_0X_0(\bar{T})b_m + \mu^+_m R_0\dot{\gamma}_+(\bar{T}). \end{aligned} \tag{5.6}$$

Now we denote with $\Pi : \mathcal{R}R_0 \to \mathcal{S}'' \oplus \mathcal{S}''' \subset \mathcal{R}R_0$ the orthogonal projection onto $\mathcal{S}'' \oplus \mathcal{S}'''$ along span$\{\psi\}$ (recall that $\psi \in \mathcal{R}R_0 = \mathcal{N}G'(\gamma(T))$ is a unitary vector such that (2.4) and (2.5) hold). In other words:

$$(\mathbb{I} - \Pi)w = \langle \psi, w \rangle \psi \tag{5.7}$$

for any $w \in \mathcal{R}R_0$. Assumption (H3) implies that the linear mapping $\mathcal{S}'' \oplus \mathcal{S}' \mapsto \mathcal{S}'' \oplus \mathcal{S}''' = \mathcal{R}\Pi$ defined as $(\zeta^\perp, \eta^\perp) \to -\zeta^\perp + R_0 X_0(\bar{T})\eta^\perp$ is invertible. So in order to solve (5.6), we need to suppose

$$\{(a_m, b_m, c_m, d_m, e_m, f_m)\}_{m \in \mathbb{Z}} \in \ell^\infty(\mathcal{S}^{iv})$$

where

$$\mathcal{S}^{iv} = \left\{(a, b, c, d, e, f) \in \mathbb{R}^{2n} \times \mathcal{R}R_0 \times \mathbb{R}^3 : (\mathbb{I} - \Pi)L(a, b, c, d, e, f) = 0\right\}$$

and $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{R}R_0 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathcal{R}R_0$ is the linear map given by:

$$\begin{aligned} L(a, b, c, d, e, f) = c - \frac{d}{G'(\gamma_-(-\bar{T}))\dot{\gamma}_-(-\bar{T})} R_0 X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) \\ - R_0 X_0(\bar{T})b + \frac{f}{G'(\gamma_+(\bar{T}))\dot{\gamma}_+(\bar{T})} R_0 \dot{\gamma}_+(\bar{T}). \end{aligned} \tag{5.8}$$

Note $\mathcal{S}^{iv}$ is a codimension 1 linear subspace of $\mathbb{R}^{2n} \times \mathcal{R}R_0 \times \mathbb{R}^3$. Hence $\widetilde{\psi} \in \mathbb{R}^{2n} \times \mathcal{R}R_0 \times \mathbb{R}^3$ exists such that

$$\text{span}\,\{\widetilde{\psi}\} \oplus \mathcal{S}^{iv} = \mathbb{R}^{2n} \times \mathcal{R}R_0 \times \mathbb{R}^3.$$

Of course we can be more precise and take $\widetilde{\psi}$ so that $\langle \widetilde{\psi}, v \rangle = 0$ for any $v \in \mathcal{S}^{iv}$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^{3n+3}$. To construct such a $\widetilde{\psi}$ we note that, from (5.7), it follows that $(\mathbb{I} - \Pi)Lv = \langle \psi, Lv \rangle \psi = \langle L^*\psi, v \rangle \psi$, where we take the natural restriction of $\langle \cdot, \cdot \rangle$ onto $\mathcal{R}R_0 \subset \mathbb{R}^n$. Thus $v = (a, b, c, d, e, f) \in \mathcal{S}^{iv}$ if and only if $\langle L^*\psi, v \rangle = 0$ or $v \in \{L^*\psi\}^\perp$ and we can take

$$\widetilde{\psi} = L^*\psi / |L^*\psi|.$$

Let $\widetilde{\Pi} : \mathbb{R}^{2n} \times \mathcal{R}R_0 \times \mathbb{R}^3 \to \mathcal{S}^{iv}$ be the orthogonal projection onto $\mathcal{S}^{iv}$ along span $\{\widetilde{\psi}\}$. Then

$$(\mathbb{I} - \widetilde{\Pi})v = \langle \widetilde{\psi}, v \rangle \widetilde{\psi} = \frac{\langle L^*\psi, v \rangle}{|L^*\psi|} \widetilde{\psi} = \frac{\langle \psi, Lv \rangle}{|L^*\psi|} \widetilde{\psi}.$$

We set

$$\ell_\psi^\infty = \ell^\infty(\text{span}\{\widetilde{\psi}\}) \subset \ell_1^\infty.$$

Let $\Pi_\psi : \ell_1^\infty \to \ell^\infty(\mathcal{S}^{iv})$ be the projection onto $\ell^\infty(\mathcal{S}^{iv})$ along ℓ_ψ^∞ given by

$$\Pi_\psi \left(\{(a_m, b_m, c_m, d_m, e_m, f_m)\}_{m \in \mathbb{Z}}\right) = \left\{\widetilde{\Pi}(a_m, b_m, c_m, d_m, e_m, f_m)\right\}_{m \in \mathbb{Z}}.$$

Summarizing, we see from (5.6) that there is a continuous inverse $\mathcal{H}_\alpha^{-1} : \ell^\infty(\mathcal{S}^{iv}) \mapsto \ell_2^\infty$, where

$$\begin{aligned} \ell_2^\infty = \Big\{ &\left\{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\right\}_{m \in \mathbb{Z}} \in \ell^\infty\left(\mathbb{R}^{5n+1}\right) : \\ &(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m) \in \mathcal{R}P_{-,m} \times \mathcal{N}P_{+,m} \times \mathcal{N}P_- \times \mathcal{R}P_+ \times \mathbb{R}^{n+1},\ \forall m \in \mathbb{Z}\Big\}. \end{aligned}$$

Note that from (5.6) and it easily follows that $\left\|\mathcal{H}_\alpha^{-1}\right\|$ and $\left\|\frac{\partial}{\partial\alpha}\mathcal{H}_\alpha^{-1}\right\| \le \left\|\frac{\partial}{\partial\alpha}\mathcal{H}_\alpha\right\| \, \|\mathcal{H}_\alpha^{-1}\|^2$ are uniformly bounded with respect to α.

Finally, we define projections onto $\mathcal{R}G'(\gamma(\bar{T}))$ and $\mathcal{R}G'(\gamma(-\bar{T}))$ respectively, as

$$(\mathbb{I}-R_+)w = \frac{G'(\gamma(\bar{T}))w}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T})$$
$$(\mathbb{I}-R_-)w = \frac{G'(\gamma(-\bar{T}))w}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\dot{\gamma}_-(-\bar{T}). \tag{5.9}$$

Note that R_+ is the projection onto $\mathcal{N}G'(\gamma(\bar{T}))$ along $\dot{\gamma}_+(\bar{T})$ whereas R_- is the projection onto $\mathcal{N}G'(\gamma(-\bar{T}))$ along $\dot{\gamma}_-(-\bar{T})$. Moreover, from [4, p. 358] we know that

$$P_+^* R_+^* \psi = 0, \quad \left(\mathbb{I}-P_-^*\right) R_-^* X_0(\bar{T})^* R_0^* \psi = 0. \tag{5.10}$$

Next we set:

$$\psi(t) = \begin{cases} X_-^{-1*}(t) R_-^* X_0(\bar{T})^* R_0^* \psi & \text{if } t \le -\bar{T} \\ X_0^{-1*}(t) X_0(\bar{T})^* R_0^* \psi & \text{if } -\bar{T} < t \le \bar{T} \\ X_+^{-1*}(t) R_+^* \psi & \text{if } t > \bar{T} \end{cases} \tag{5.11}$$

and

$$\mathcal{M}(\alpha) = \int_{-\infty}^{\infty} \psi^*(t) g(t+\alpha, \gamma(t), 0) dt. \tag{5.12}$$

Using (5.10), we easily obtain:

$$\begin{aligned} &|\psi(t)| \le \|X_+^{-1*}(t)(\mathbb{I}-P_+^*)X_+^*(\bar{T})\||R_+^*\psi| \le k\|R_+\| \mathrm{e}^{-\delta(t-\bar{T})} && \text{if } t \ge \bar{T} \\ &|\psi(t)| \le k\|R_0 X_0(\bar{T}) R_-\| \mathrm{e}^{\delta(t+\bar{T})} && \text{if } t \le -\bar{T}. \end{aligned} \tag{5.13}$$

Thus $\mathcal{M}(\alpha)$ is a well defined C^2 function because of Lebesgue theorem.

We are now ready to state the following result.

Theorem 5.1 *Assume $f_\pm(z)$ and $g(t,z,\varepsilon)$ are C^r-functions with bounded derivatives and that their r-order derivatives are uniformly continuous. Assume, moreover, that conditions* (H1), (H2) *and* (H3) *hold.*

Then given $c_0 > 0$ there exist constants $\rho_0 > 0$, $\chi > 0$ and $c_1 > 0$ such that for any $0 < \rho < \rho_0$, there is $\bar{\varepsilon}_\rho > 0$ such that for any ε, $0 < |\varepsilon| < \bar{\varepsilon}_\rho$, for any increasing sequence $\mathcal{T} = \{T_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ with $T_m - T_{m-1} > \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon|$ and such that

$$\mathcal{M}\left(T_{2m}+\alpha_m^0\right) = 0 \forall m \in \mathbb{Z} \text{ and } \inf_{m\in\mathbb{Z}} |\mathcal{M}'\left(T_{2m}+\alpha_m^0\right)| > c_0 \tag{5.14}$$

for some $\alpha_0 = \{\alpha_m^0\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$, there exist unique sequences $\{\hat{\alpha}_m\}_{m\in\mathbb{Z}} = \{\hat{\alpha}_m(\mathcal{T},\varepsilon)\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty(\mathbb{R})$ and $\{\hat{\beta}_m\}_{m\in\mathbb{Z}} = \{\hat{\beta}_m(\mathcal{T},\varepsilon)\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ with $|\hat{\alpha}_m(\mathcal{T},\varepsilon) - \alpha_m^0| < c_1|\varepsilon|$ and $|\hat{\beta}_m(\mathcal{T},\varepsilon) - \alpha_m^0| < c_1|\varepsilon| \forall m \in \mathbb{Z}$, and a unique bounded solution $z(t) = z(\mathcal{T},\varepsilon)(t)$ of system (2.1) such that

$$\sup_{t\in[T_{2m-1}+\hat{\beta}_{m-1}, T_{2m}-\bar{T}+\hat{\alpha}_m]} |z(t) - \gamma_-(t - T_{2m} - \hat{\alpha}_m)| < \rho$$
$$\sup_{t\in[T_{2m}-\bar{T}+\hat{\alpha}_m, T_{2m}+\bar{T}+\hat{\beta}_m]} |z(t) - \gamma_0(t - T_{2m} - \hat{\alpha}_m)| < \rho$$
$$\sup_{t\in[T_{2m}+\bar{T}+\hat{\beta}_m, T_{2m+1}+\hat{\beta}_m]} |z(t) - \gamma_+(t - T_{2m} - \hat{\beta}_m)| < \rho$$

for any $m \in \mathbb{Z}$ (cf. (2.7)). Hence $z(t)$ is orbitally close to $\gamma(t)$ in the sense that

$$\operatorname{dist}(z(t), \Gamma) < \rho,$$

where $\Gamma = \{\gamma(t) \mid t \in \mathbb{R}\}$ is the orbit of $\gamma(t)$ and $\operatorname{dist}(z, \Gamma) = \inf_{t\in\mathbb{R}} |z - \gamma(t)|$ is the distance of z from Γ.

Proof If ρ and $\bar{\varepsilon}_\rho < \varepsilon_\rho$ are sufficiently small then, for $t \in I^-_{m,\alpha}$, the solution $z(t)$ we look for must satisfy $z(t) = z^-_m(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ for some value of the parameters $(\xi_-, \varphi_-, \alpha, \varepsilon)$ and similarly in the others intervals $[T_{2m} - \bar{T} + \alpha, T_{2m} + \bar{T} + \beta]$ and $I^+_{m,\beta}$.

So, we solve Eq. (5.2) for $(\theta, \alpha) \in \ell^\infty_{\rho,\alpha,\varepsilon} \times \ell^\infty_\chi$ in terms of $\mathcal{T}$ and $\varepsilon \in (-\bar{\varepsilon}_\rho, \bar{\varepsilon}_\rho)$. Set

$$\begin{aligned}\mathcal{F}_\mathcal{T}(\theta, \alpha, \varepsilon) = \mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon) - \mathcal{H}_\alpha(\theta - \theta_\alpha) &= \mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0)\\ &+ [\mathcal{G}_\mathcal{T}(\theta, \alpha, 0) - \mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0) - D_1\mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0)(\theta - \theta_\alpha)]\\ &+ (D_1\mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0) - \mathcal{H}_\alpha)(\theta - \theta_\alpha) + \varepsilon \int_0^1 D_3\mathcal{G}_\mathcal{T}(\theta, \alpha, \tau\varepsilon)\, d\tau\end{aligned}$$

where $D_3\mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon)$ denotes the derivative of $\mathcal{G}_\mathcal{T}$ with respect to ε. It is easy to see that

$$\begin{aligned}\mathcal{F}_\mathcal{T}(\theta_\alpha, \alpha, \varepsilon) &= \mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, \varepsilon)\\ D_1\mathcal{F}_\mathcal{T}(\theta, \alpha, \varepsilon) &= D_1\mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon) - \mathcal{H}_\alpha\\ D_1\mathcal{F}_\mathcal{T}(\theta_1, \alpha, \varepsilon) - D_1\mathcal{F}_\mathcal{T}(\theta_2, \alpha, \varepsilon) &= D_1\mathcal{G}_\mathcal{T}(\theta_1, \alpha, \varepsilon) - D_1\mathcal{G}_\mathcal{T}(\theta_2, \alpha, \varepsilon)\\ D_2\mathcal{F}_\mathcal{T}(\theta, \alpha, \varepsilon) &= D_2\mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon) - \frac{\partial\mathcal{H}_\alpha}{\partial\alpha}(\theta - \theta_\alpha) - \mathcal{H}_\alpha\frac{\partial\theta_\alpha}{\partial\alpha}.\end{aligned} \tag{5.15}$$

For simplicity we also set:

$$\mu = \mathrm{e}^{-\delta(T-\bar{T})}.$$

From the definition of $\mathcal{F}_\mathcal{T}(\theta, \alpha, \varepsilon)$ we see that Eq. (5.2) has the form

$$\theta - \theta_\alpha + \mathcal{H}^{-1}_\alpha \Pi_\psi \mathcal{F}_\mathcal{T}(\theta, \alpha, \varepsilon) = 0 \tag{5.16}$$

and

$$\left(\mathbb{I} - \Pi_\psi\right)\mathcal{F}_\mathcal{T}(\theta, \alpha, \varepsilon) = 0. \tag{5.17}$$

We denote with $c^{(1)}_\mathcal{G}$, resp. $c^{(2)}_\mathcal{G}$, upper bounds for the norms of the first order, resp. second order, derivatives of $\mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon)$, in ℓ^∞_ρ. Thus for example

$$c^{(1)}_\mathcal{G} = \sup_{(\theta,\alpha,\varepsilon)\in\ell^\infty_\rho} \{\|D_1\mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon)\|, \|D_2\mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon)\|, \|D_3\mathcal{G}_\mathcal{T}(\theta, \alpha, \varepsilon)\|\}$$

and similarly for $c^{(2)}_\mathcal{G}$. Then

$$\begin{aligned}&\mathcal{G}_\mathcal{T}(\theta, \alpha, 0) - \mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0) - D_1\mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0)(\theta - \theta_\alpha)\\ &= \int_0^1 [D_1\mathcal{G}_\mathcal{T}(\tau\theta + (1-\tau)\theta_\alpha, \alpha, 0) - D_1\mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0)]\, d\tau(\theta - \theta_\alpha)\\ &\eta(\theta, \theta_\alpha, \alpha)(\theta - \theta_\alpha),\end{aligned}$$

where

$$\|\eta(\theta, \theta_\alpha, \alpha)\| \le c^{(2)}_\mathcal{G}\|\theta - \theta_\alpha\|.$$

Hence, since

$$\begin{aligned}\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)-\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)&=\int_0^1[D_1\mathcal{F}_{\mathcal{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon)]\,d\tau(\theta-\theta_\alpha)\\&=\int_0^1[D_1\mathcal{F}_{\mathcal{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon)-D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)]d\tau(\theta-\theta_\alpha)\\&\quad+D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)(\theta-\theta_\alpha)\\&=\int_0^1[D_1\mathcal{G}_{\mathcal{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon)-D_1\mathcal{G}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)]d\tau(\theta-\theta_\alpha)\\&\quad+[D_1\mathcal{G}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathcal{H}_\alpha](\theta-\theta_\alpha)\end{aligned}\tag{5.18}$$

(see also (5.15)) we derive, using also (G_5) and recalling $\mu=\mathrm{e}^{-\delta(T-\bar{T})}$:

$$\|\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)-\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)\|\le\frac{1}{2}c_{\mathcal{G}}^{(2)}\|\theta-\theta_\alpha\|^2+(k\widetilde{c}_3\mu+c_{\mathcal{G}}^{(2)}|\varepsilon|)\|\theta-\theta_\alpha\|\tag{5.19}$$

and (see also (G_1), (5.15))

$$\|\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)\|\le\frac{c_{\mathcal{G}}^{(2)}}{2}\|\theta-\theta_\alpha\|^2+(k\widetilde{c}_3\mu+c_{\mathcal{G}}^{(2)}|\varepsilon|)\|\theta-\theta_\alpha\|+c_{\mathcal{G}}^{(1)}|\varepsilon|+c_\gamma\mu\tag{5.20}$$

where $c_\gamma=2k\delta^{-1}\max\{|\dot{\gamma}_-(-\bar{T})|,|\dot{\gamma}_+(\bar{T})|\}$. Note that $c_\gamma, c_{\mathcal{G}}^{(1)}, c_{\mathcal{G}}^{(2)}$ and $\widetilde{c}_3$ do not depend on $(\alpha,\mathcal{T},\varepsilon)\in\ell^\infty_\chi\times\ell^\infty_{\mathcal{T}}(\mathbb{R})\times\mathbb{R}$.

Next, from (G_4), (G_5) and (5.15) we get

$$\begin{aligned}\|D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,0)\|&\le k\widetilde{c}_3\mu\\\|D_1\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)-D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)\|&\le c_{\mathcal{G}}^{(2)}\|\theta-\theta_\alpha\|\\\|D_2\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)-D_2\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)\|&\le\left(c_{\mathcal{G}}^{(2)}+2kN_-\right)\|\theta-\theta_\alpha\|.\end{aligned}\tag{5.21}$$

From (5.20) and (5.21) we conclude that

$$\lim_{(\theta,\varepsilon,\mu)\to(\theta_\alpha,0,0)}\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)=0$$
$$\lim_{(\theta,\varepsilon,\mu)\to(\theta_\alpha,0,0)}D_1\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)=0$$

uniformly with respect to α.

Thus, if $\bar{\rho}_0>0$, $\mu_0>0$ and $0<\bar{\varepsilon}_0\le\varepsilon_\rho$ are sufficiently small and $0<\mu<\mu_0$, $|\varepsilon|<\bar{\varepsilon}_0$, from the Implicit Function Theorem the existence follows of a unique solution $\theta=\theta_{\mathcal{T}}(\alpha,\varepsilon)$ of (5.16) which is defined for any $\alpha\in\ell^\infty_\chi$, $|\varepsilon|<\bar{\varepsilon}_0$, $0<\mu\le\mu_0$ and $\mathcal{T}=\{T_m\}_{m\in\mathbb{Z}}$ such that $T_{m+1}-T_m>T+1$ where $T-\bar{T}=-\delta^{-1}\ln\mu$. Moreover $\theta_{\mathcal{T}}(\alpha,\varepsilon)$ satisfies

$$\sup_{\alpha,\mathcal{T},\varepsilon}\|\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha\|<\bar{\rho}_0\tag{5.22}$$

the sup being taken over all α, $\mathcal{T}$ and ε satisfying the above conditions. Next, using (5.16) with $\theta_{\mathcal{T}}(\alpha,\varepsilon)$ instead of θ and (5.20), we see that:

$$\begin{aligned}&\|\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha\|\le\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|\|\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\|\\&\le\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|\left(\frac{c_{\mathcal{G}}^{(2)}}{2}\|\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha\|^2+(k\widetilde{c}_3\mu+c_{\mathcal{G}}^{(2)}|\varepsilon|)\|\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha\|+c_{\mathcal{G}}^{(1)}|\varepsilon|+c_\gamma\mu\right).\end{aligned}$$

Hence if $\bar{\rho}_0$, μ_0 and ε_0 are so small that

$$\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|[c_{\mathcal{G}}^{(2)}(\bar{\rho}_0+2\varepsilon_0)+2k\widetilde{c}_3\mu_0]<1 \tag{5.23}$$

we obtain:

$$\|\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha\|\le 2\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma\mu+c_{\mathcal{G}}^{(1)}|\varepsilon|). \tag{5.24}$$

Note that, since $\widetilde{\Pi}$ is an orthogonal projection, it is enough that μ_0, ε_0 and $\bar{\rho}_0$ are chosen in such a way that $c_{\mathcal{G}}^{(2)}(\bar{\rho}_0+2\varepsilon_0)+2k\widetilde{c}_3\mu_0<\|\mathcal{H}_\alpha^{-1}\|^{-1}$.

Moreover, plugging (5.24) into (5.19) we obtain

$$\begin{aligned}\|\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)\| &\le 2c_{\mathcal{G}}^{(2)}\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|^2(c_\gamma\mu+c_{\mathcal{G}}^{(1)}|\varepsilon|)^2\\ +2(k\widetilde{c}_3\mu+c_{\mathcal{G}}^{(2)}|\varepsilon|)\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma\mu+c_{\mathcal{G}}^{(1)}|\varepsilon|) &\le \Lambda_1(\mu+|\varepsilon|)^2\end{aligned} \tag{5.25}$$

where $\Lambda_1>0$ is independent of $(\mathcal{T},\alpha,\mu,\varepsilon)$. For example:

$$\Lambda_1=2\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|\max\{c_\gamma,c_{\mathcal{G}}^{(1)},c_{\mathcal{G}}^{(2)},k\widetilde{c}_3\}^2\left[\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|c_{\mathcal{G}}^{(2)}+1\right].$$

Next, differentiating the equality

$$\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha+\mathcal{H}_\alpha^{-1}\Pi_\psi\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=0$$

with respect to α we obtain:

$$\begin{aligned}\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha] &= -\mathcal{H}_\alpha^{-1}\Pi_\psi\frac{\partial}{\partial\alpha}\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\\ &\quad-\left[\frac{\partial}{\partial\alpha}\mathcal{H}_\alpha^{-1}\Pi_\psi\right]\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\\ &= -\mathcal{H}_\alpha^{-1}\Pi_\psi\left\{\frac{\partial}{\partial\alpha}[\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)]\right.\\ &\quad\left.+\frac{\partial}{\partial\alpha}[\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,0)]+\frac{\partial}{\partial\alpha}\mathcal{G}_{\mathcal{T}}(\theta_\alpha,\alpha,0)\right\}\\ &\quad-\left[\frac{\partial}{\partial\alpha}\mathcal{H}_\alpha^{-1}\Pi_\psi\right]\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon).\end{aligned} \tag{5.26}$$

Then note that

$$\begin{aligned}&\frac{\partial}{\partial\alpha}[\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)]\\ &=\frac{\partial}{\partial\alpha}\int_0^1 D_1\mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau(\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha)\\ &=\left\{\int_0^1 D_1^2\mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\tau\,d\tau\right.\\ &\quad+\int_0^1 D_1^2\mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{d}{d\alpha}\theta_\alpha\,d\tau\end{aligned}$$

$$+\int_0^1 D_1 D_2 \mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\Bigg\}(\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha)$$
$$+\int_0^1 D_1 \mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]. \tag{5.27}$$

First we derive

$$\left\|\int_0^1 D_1^2 \mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\tau\,d\tau\right\|$$
$$\le\int_0^1 c_{\mathcal{G}}^{(2)}\tau d\tau\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|=\frac{1}{2}c_{\mathcal{G}}^{(2)}\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|.$$

Next, from (5.21) we obtain

$$\left\|\int_0^1 D_1 \mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$
$$\le\Bigg(\int_0^1\|D_1\mathcal{F}_{\mathcal{T}}(\tau\theta_{\mathcal{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)-D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)]\|\,d\tau$$
$$+\|D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)-D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,0)\|+\|D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,0)\|\Bigg)\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$
$$\le\left(\int_0^1 c_{\mathcal{G}}^{(2)}\|\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha\|\,\tau\,d\tau+c_{\mathcal{G}}^{(2)}|\varepsilon|+k\widetilde{c}_3\mu\right)\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$
$$\le\left(c_{\mathcal{G}}^{(2)}\left(\frac{1}{2}\|\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha\|+|\varepsilon|\right)+k\widetilde{c}_3\mu\right)\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|.$$

Finally, using (G_4), (5.22) and (5.24), the identity

$$\frac{d\theta_\alpha}{d\alpha}=(0,0,0,0,0,\mathbb{I}) \tag{5.28}$$

and $D_1D_2\mathcal{F}_{\mathcal{T}}(\theta,\alpha,\varepsilon)=D_1D_2\mathcal{G}_{\mathcal{T}}(\theta,\alpha,\varepsilon)-\frac{\partial\mathcal{H}_\alpha}{\partial\alpha}$, we conclude

$$\left\|\frac{\partial}{\partial\alpha}[\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathcal{F}_{\mathcal{T}}(\theta_\alpha,\alpha,\varepsilon)]\right\|$$
$$\le[c_{\mathcal{G}}^{(2)}(\bar{\rho}_0+\varepsilon_0)+k\widetilde{c}_3\mu_0]\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathcal{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$
$$+4\left(c_{\mathcal{G}}^{(2)}+kN_-\right)\|\mathcal{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma\mu+c_{\mathcal{G}}^{(1)}|\varepsilon|). \tag{5.29}$$

Similarly, we obtain

$$\left\| \frac{\partial}{\partial \alpha} [\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon) - \mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)] \right\| = |\varepsilon| \left\| \frac{\partial}{\partial \alpha} \int_0^1 D_3 \mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \tau\varepsilon)\, d\tau \right\| \le 2c_{\mathcal{G}}^{(2)} |\varepsilon|. \tag{5.30}$$

Now, since

$$\left\| \frac{\partial}{\partial \alpha} \mathcal{H}_\alpha^{-1} \Pi_\psi \right\| \le \|\mathcal{H}_\alpha^{-1} \Pi_\psi\|^2 \left\| \frac{\partial}{\partial \alpha} \mathcal{H}_\alpha \right\| \le 2kN_- \|\mathcal{H}_\alpha^{-1} \Pi_\psi\|^2,$$

we derive, using also (G_1), (5.25):

$$\begin{aligned} \left\| \left[\tfrac{\partial}{\partial \alpha} \mathcal{H}_\alpha^{-1} \Pi_\psi \right] \mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) \right\| &\le 2kN_- \|\mathcal{H}_\alpha^{-1} \Pi_\psi\|^2 \cdot \\ &\cdot \{ \|\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) - \mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon)\| + \|\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon)\| \} \\ &\le 2kN_- \|\mathcal{H}_\alpha^{-1} \Pi_\psi\|^2 \left[\Lambda_1 (\mu + |\varepsilon|)^2 + c_\gamma \mu + c_{\mathcal{G}}^{(1)} |\varepsilon| \right]. \end{aligned} \tag{5.31}$$

Plugging (5.29), (5.30), (5.31) into (5.26) and assuming, instead of (5.23), that

$$2\|\mathcal{H}_\alpha^{-1} \Pi_\psi\| [c_{\mathcal{G}}^{(2)} (\bar{\rho}_0 + \bar{\varepsilon}_0) + k\widetilde{c}_3 \mu_0] \le 1$$

we obtain

$$\begin{aligned} \left\| \tfrac{\partial}{\partial \alpha} [\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha] \right\| &\le 2\|\mathcal{H}_\alpha^{-1} \Pi_\psi\| \left\{ 4 \left(c_{\mathcal{G}}^{(2)} + kN_- \right) \|\mathcal{H}_\alpha^{-1} \Pi_\psi\| (c_\gamma \mu + c_{\mathcal{G}}^{(1)} |\varepsilon|) \right. \\ &\left. + 2c_{\mathcal{G}}^{(2)} |\varepsilon| + c_\gamma \mu + 2kN_- \|\mathcal{H}_\alpha^{-1} \Pi_\psi\| \left[\Lambda_1 (\mu + |\varepsilon|)^2 + c_\gamma \mu + c_{\mathcal{G}}^{(1)} |\varepsilon| \right] \right\} \le \Lambda_2 (\mu + |\varepsilon|) \end{aligned} \tag{5.32}$$

where Λ_2 is a positive constant that does not depend on $(\mathcal{T}, \alpha, \mu, \varepsilon)$. We now take

$$\mu = \varepsilon^2$$

that is $T = \bar{T} - 2\delta^{-1} \ln |\varepsilon|$. Note that, from (5.24), we get:

$$\|\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha\| \le 2\|\mathcal{H}_\alpha^{-1} \Pi_\psi\| (c_\gamma |\varepsilon| + c_{\mathcal{G}}^{(1)}) |\varepsilon|. \tag{5.33}$$

Then, if we can solve the equation $\left(\mathbb{I} - \Pi_\psi \right) \mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) = 0$ for $\alpha = \alpha_{\mathcal{T}}(\varepsilon) = \{\alpha_{m,\mathcal{T}}(\varepsilon)\}_{m \in \mathbb{Z}}$ and define $z_{m,\mathcal{T}}^{\pm}(t, \varepsilon), z_{m,\mathcal{T}}^{0}(t, \varepsilon)$ as $z_m^+(t, \xi_m^+, \varphi_m^+, \beta_m, \varepsilon), z_m^-(t, \xi_m^-, \varphi_m^-, \alpha_m, \varepsilon)$ and $z_m^0(t, \bar{\xi}_m, \alpha_m, \beta_m, \varepsilon)$, with

$$\theta_{\mathcal{T}}(\varepsilon) = \theta_{\mathcal{T}}(\alpha_{\mathcal{T}}(\varepsilon), \varepsilon)$$

instead of $\theta = \left\{ (\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m) \right\}_{m\mathbb{Z}}$ and with $\mu = \varepsilon^2$, we see that condition (2.7) follows from (3.16), (3.19) and (4.4) provided $|\varepsilon| < \varepsilon_\rho$, taking ε_ρ smaller if necessary.

Thus to complete the proof of Theorem 5.1 we only need to show that the equation

$$(\mathbb{I} - \Pi_\psi) \mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) = 0$$

can be solved for α in terms of $\varepsilon \in (-\varepsilon_\rho, \varepsilon_\rho)$ and $\mathcal{T}$ satisfying the conditions of Theorem 5.1. Now, from (5.33) we see that

$$\lim_{\varepsilon \to 0} (\mathbb{I} - \Pi_\psi) \mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) = \lim_{\varepsilon \to 0} (\mathbb{I} - \Pi_\psi) \mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) = 0$$

uniformly with respect to $(\alpha, \mathcal{T})$ (since (G_1) gives $\|\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\| \le c_\gamma \mu = c_\gamma \varepsilon^2$). Hence we are led to prove that the bifurcation function

$$\frac{1}{\varepsilon}(\mathbb{I} - \Pi_\psi)\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) = 0 \tag{5.34}$$

can be solved for α in terms of $\varepsilon \in (-\varepsilon_\rho, \varepsilon_\rho)$, $\varepsilon \neq 0$, and $\mathcal{T}$ satisfying the conditions of Theorem 5.1.

We observe that, with $\mu = \varepsilon^2$, (5.25) reads:

$$\|\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) - \mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon)\| \le \Lambda_1(1 + |\varepsilon|)^2\varepsilon^2.$$

Hence, using also (5.15) and (G_1) with $\mu = \mathrm{e}^{-\delta(T-\bar{T})} = \varepsilon^2$:

$$\begin{aligned} B_{\mathcal{T}}(\alpha, \varepsilon) &= \frac{1}{\varepsilon}\left(\mathbb{I} - \Pi_\psi\right)\left\{\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon) + O(\varepsilon^2)\right\} \\ &= \frac{1}{\varepsilon}\left(\mathbb{I} - \Pi_\psi\right)[\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon) - \mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)] + O(\varepsilon) \\ &= \left(\mathbb{I} - \Pi_\psi\right) D_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) + O(\varepsilon) \end{aligned}$$

where $O(\varepsilon)$ is uniform with respect to $(\mathcal{T}, \alpha)$.

Now we look at:

$$D_1 B_{\mathcal{T}}(\alpha, \varepsilon) = \frac{1}{\varepsilon}(\mathbb{I} - \Pi_\psi)\frac{\partial}{\partial\alpha}\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon). \tag{5.35}$$

Subtracting

$$\begin{aligned} &\left(D_1^2\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\frac{d\theta_\alpha}{d\alpha} + D_1 D_2\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\right)(\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha) \\ &\quad = \frac{d}{d\alpha}[D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)](\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha) \end{aligned}$$

from both sides of (5.27) and using the uniform continuity of $D_1^2\mathcal{F}_{\mathcal{T}}(\theta, \alpha, \varepsilon)$, $D_1 D_2 \mathcal{F}_{\mathcal{T}}(\theta, \alpha, \varepsilon)$ in $(\theta, \alpha, \varepsilon)$, uniformly with respect to $\mathcal{T}$ we see that:

$$\begin{aligned} &\Bigg\| \frac{\partial}{\partial\alpha}\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) - \frac{\partial}{\partial\alpha}\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon) \\ &\quad - \left(D_1^2\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\frac{d\theta_\alpha}{d\alpha} + D_1 D_2\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\right)(\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha)\Bigg\| \\ &\le \left(\left(c_{\mathcal{G}}^{(2)}(\|\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha\| + |\varepsilon|\right) + k\widetilde{c}_3\varepsilon^2\right)\left\|\frac{\partial}{\partial\alpha}(\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha)\right\| \\ &\quad + \eta(\|\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha\| + |\varepsilon|)\|\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha\| \end{aligned}$$

where $\eta(r) \to 0$ as $r \to 0$, uniformly with respect to $(\mathcal{T}, \alpha, \varepsilon)$, So, using (5.33) and (5.32) with $\mu = \varepsilon^2$ we obtain:

$$\frac{\partial}{\partial\alpha}\mathcal{F}_{\mathcal{T}}(\theta_{\mathcal{T}}(\alpha, \varepsilon), \alpha, \varepsilon) - \frac{\partial}{\partial\alpha}\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon) - \frac{d}{d\alpha}[D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)](\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha) = o(\varepsilon) \tag{5.36}$$

uniformly with respect to $(\alpha, \mathcal{T})$. So, plugging (5.36) into (5.35), using (5.15) and (G_2) with $\mu = \mathrm{e}^{-\delta(T-\bar{T})} = \varepsilon^2$, we obtain:

$$
\begin{aligned}
D_1 B_{\mathcal{T}}(\alpha, \varepsilon) &= (\mathbb{I} - \Pi_\psi)\frac{\partial}{\partial\alpha}\frac{\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, \varepsilon) - \mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)}{\varepsilon} \\
&\quad + (\mathbb{I} - \Pi_\psi)\left\{\varepsilon^{-1}\frac{d}{d\alpha}[D_1\mathcal{F}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)][\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha]\right\} + o(1) \\
&= \frac{d}{d\alpha}(\mathbb{I} - \Pi_\psi)D_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) \\
&\quad + (\mathbb{I} - \Pi_\psi)\left\{\varepsilon^{-1}\frac{d}{d\alpha}[D_1\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) - \mathcal{H}_\alpha][\theta_{\mathcal{T}}(\alpha, \varepsilon) - \theta_\alpha]\right\} + o(1)
\end{aligned}
$$

$o(1)$ being uniform with respect to α. But, differentiating (A.6) (cf. Appendix A) we see that

$$
\frac{d}{d\alpha}\left(D_1\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) - \mathcal{H}_\alpha\right) = \{(\mathcal{L}_m^\alpha, 0, 0, 0, 0, 0)\}_{m\in\mathbb{Z}}
$$

where

$$
\begin{aligned}
\mathcal{L}_m^\alpha(\widetilde{\alpha})(\theta) &= \mathcal{L}_m^\alpha(\widetilde{\alpha})(\varphi_{m+1}^-, \varphi_m^+, \xi_{m+1}^-, \xi_m^+, \bar{\xi}_m, \beta_m) \\
&= \left[\dot{X}_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})(\widetilde{\alpha}_{m+1} - \widetilde{\alpha}_m)\right]\xi_{m+1}^- \\
&\quad + [\ddot{\gamma}_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})(\widetilde{\alpha}_{m+1} - \widetilde{\alpha}_m)]\beta_m \\
&\le 2N_-k\delta^{-1}(\delta + |\dot{\gamma}_-(-\bar{T})|)\mu\|\theta\|\,\|\widetilde{\alpha}\| = O(\varepsilon^2)\|\theta\|\|\widetilde{\alpha}\|
\end{aligned}
$$

and hence

$$
\left\|\frac{d}{d\alpha}[D_1\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) - \mathcal{H}_\alpha]\right\| = O(\varepsilon^2).
$$

Summarizing, we obtain:

$$
D_1 B_{\mathcal{T}}(\alpha, \varepsilon) = \frac{d}{d\alpha}[(\mathbb{I} - \Pi_\psi)D_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)] + o(1) \tag{5.37}
$$

uniformly with respect to α and $\mathcal{T}$. We have then

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} B_{\mathcal{T}}(\alpha, \varepsilon) = (\mathbb{I} - \Pi_\psi)D_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) = \frac{1}{|L^*\psi|}\langle\psi, LD_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\rangle\widetilde{\psi} \\
&\lim_{\varepsilon\to 0} D_1 B_{\mathcal{T}}(\alpha, \varepsilon) = \frac{d}{d\alpha}\frac{1}{|L^*\psi|}\langle\psi, LD_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\rangle\widetilde{\psi}
\end{aligned}
$$

uniformly with respect to α and $\mathcal{T}$ (recall that L has been defined in (5.8)). To conclude the proof of Theorem 5.1 we evaluate $\langle\psi, LD_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\rangle$. We have:

$$
D_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) = \left\{\begin{array}{c}
\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m+1} + \alpha_m, 0, 0, \alpha_m, 0) - \frac{\partial z_{m+1}^-}{\partial\varepsilon}(T_{2m+1} + \alpha_m, 0, 0, \alpha_{m+1}, 0) \\
-\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m} - \bar{T} + \alpha_m, 0, 0, \alpha_m, 0) \\
R_0[\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m} + \bar{T} + \alpha_m, \gamma_0(-\bar{T}), \alpha_m, \alpha_m, 0) - \frac{\partial z_m^+}{\partial\varepsilon}(T_{2m} + \bar{T} + \alpha_m, 0, 0, \alpha_m, 0)] \\
G'(\gamma(-\bar{T}))\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m} - \bar{T} + \alpha_m, 0, 0, \alpha_m, 0)) \\
G'(\gamma(\bar{T}))\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m} + \bar{T} + \alpha_m, \gamma_0(-\bar{T}), \alpha_m, \alpha_m, 0)) \\
G'(\gamma(\bar{T}))\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m} + \bar{T} + \alpha_m, 0, 0, \alpha_m, 0))
\end{array}\right\}_{m\in\mathbb{Z}}
$$

Thus:

$$
\begin{aligned}
LD_\varepsilon \mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) &= R_0 \left\{ \frac{\partial z_m^0}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, \gamma_0(-\bar{T}), \alpha_m, \alpha_m, 0) \right. \\
&\quad - \frac{\partial z_m^+}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, 0, 0, \alpha_m, 0) \\
&\quad - \frac{G'(\gamma(-\bar{T}))\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m} - \bar{T} + \alpha_m, 0, 0, \alpha_m, 0))}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) \\
&\quad + X_0(\bar{T})\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m} - \bar{T} + \alpha_m, 0, 0, \alpha_m, 0) \\
&\quad \left. + \frac{G'(\gamma(\bar{T}))\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, 0, 0, \alpha_m, 0))}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \dot{\gamma}_+(\bar{T}) \right\} \\
&= R_0 \left\{ \frac{\partial z_m^0}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, \gamma_0(-\bar{T}), \alpha_m, \alpha_m, 0) \right. \\
&\quad + X_0(\bar{T}) R_- \frac{\partial z_m^-}{\partial \varepsilon}(T_{2m} - \bar{T} + \alpha_m, 0, 0, \alpha_m, 0)) \\
&\quad \left. - R_+ \frac{\partial z_m^+}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, 0, 0, \alpha_m, 0))\dot{\gamma}_+(\bar{T}) \right\} \\
&= R_0 \left\{ \frac{\partial z_m^0}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, \gamma_0(-\bar{T}), \alpha_m, \alpha_m, 0) \right. \\
&\quad \left. + X_0(\bar{T}) R_- \frac{\partial z_m^-}{\partial \varepsilon}(T_{2m} - \bar{T} + \alpha_m, 0, 0, \alpha_m, 0)) \right\} \\
&\quad - R_+ \frac{\partial z_m^+}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, 0, 0, \alpha_m, 0))\dot{\gamma}_+(\bar{T})
\end{aligned}
$$

since $\mathcal{R}R_+ \subset \mathcal{R}R_0$. Next from Eqs. (3.17), (3.21), (4.3) we get:

$$
\begin{aligned}
&\frac{\partial z_m^0}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, \gamma_0(-\bar{T}), \alpha_m, \alpha_m, 0) \\
&= \int_{-\bar{T}}^{\bar{T}} X_0(\bar{T}) X_0^{-1}(t) g(t + T_{2m} + \alpha_m, \gamma_0(t), 0) dt, \\
&\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m} - \bar{T} + \alpha_m, 0, 0, \alpha_m, 0)) \\
&= \int_{T_{2m-1} - T_{2m} - 1}^{-\bar{T}} P_- X_-^{-1}(t) g(t + T_{2m} + \alpha_m, \gamma_-(t), 0) dt, \\
&\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m} + \bar{T} + \alpha_m, 0, 0, \alpha_m, 0)) \\
&= - \int_{\bar{T}}^{T_{2m+1} - T_{2m} + 1} (\mathbb{I} - P_+) X_+^{-1}(t) g(t + T_{2m} + \alpha_m, \gamma_+(t), 0) dt.
\end{aligned}
\tag{5.38}
$$

As a consequence, using also (5.10), we get:

$$
\begin{aligned}
&\langle \psi, LD_3\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0)\rangle \\
&= \psi^* \Bigg[\int_{T_{2m-1}-T_{2m}-1}^{-\bar{T}} R_0 X_0(\bar{T}) R_- P_- X_-^{-1}(t) g(t + T_{2m} + \alpha_m, \gamma_-(t), 0) dt \\
&\quad + \int_{-\bar{T}}^{\bar{T}} R_0 X_0(\bar{T}) X_0^{-1}(t) g(t + T_{2m} + \alpha_m, \gamma_0(t), 0) dt \\
&\quad + \int_{\bar{T}}^{T_{2m+1}-T_{2m}+1} R_+(\mathbb{I} - P_+) X_+^{-1}(t) g(t + T_{2m} + \alpha_m, \gamma_+(t), 0) dt \Bigg] \qquad (5.39)\\
&= \int_{T_{2m-1}-T_{2m}-1}^{T_{2m+1}-T_{2m}+1} \psi^*(t) g(t + T_{2m} + \alpha_m, \gamma(t), 0) dt \\
&= \int_{-\infty}^{\infty} \psi^*(t) g(t + T_{2m} + \alpha_m, \gamma(t), 0) dt + O(\mathrm{e}^{-\delta(T+1)}) \\
&= \int_{-\infty}^{\infty} \psi^*(t) g(t + T_{2m} + \alpha_m, \gamma(t), 0) dt + O(\varepsilon^2)
\end{aligned}
$$

where $\psi(t)$ has been defined in (5.11). Thus we proved that

$$
\begin{aligned}
B_{\mathcal{T}}(\alpha, \varepsilon) &= \frac{1}{|L^*\psi|}\{\mathcal{M}(\alpha_m + T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}} + O(\varepsilon) \\
D_1 B_{\mathcal{T}}(\alpha, \varepsilon) &= \frac{1}{|L^*\psi|}\{\mathcal{M}'(\alpha_m + T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}} + o(1)
\end{aligned}
$$

where $O(\varepsilon)$ and $o(1)$ are uniform with respect to α and $\mathcal{T}$. Now assume that $\mathcal{T} = \{T_m\}_{m\in\mathbb{Z}}$ and $\alpha_0 = \{\alpha_m^0\}_{m\in\mathbb{Z}}$ satisfy the assumptions of Theorem 5.1. We have:

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} B_{\mathcal{T}}(\alpha_0, \varepsilon) = 0 \\
&\lim_{\varepsilon\to 0} D_1 B_{\mathcal{T}}(\alpha_0, \varepsilon) = \frac{1}{|L^*\psi|}\{\mathcal{M}'(\alpha_m^0 + T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}}
\end{aligned}
$$

uniformly with respect to $\mathcal{T}$. That is $\|D_1 B_{\mathcal{T}}(\alpha_0, \varepsilon)\| > \frac{c_0}{2|L^*\psi|}$ provided $|\varepsilon|$ is sufficiently small. From the Implicit Function Theorem we deduce the existence of $0 < \bar{\varepsilon}_\rho < \varepsilon_0$ such that for any $0 \neq \varepsilon \in (-\bar{\varepsilon}_\rho, \bar{\varepsilon}_\rho)$ and any sequence $\mathcal{T} = \{T_m\}_{m\in\mathbb{Z}}$ that satisfy the assumption of Theorem 5.1 there exists a unique sequence $\alpha(\mathcal{T}, \varepsilon) = \{\alpha_m(\mathcal{T}, \varepsilon)\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$ such that $\alpha(\mathcal{T}, 0) = \alpha_0$ and

$$
B_{\mathcal{T}}(\alpha(\mathcal{T}, \varepsilon), \varepsilon) = 0.
$$

Taking $\theta_{\mathcal{T}}(\varepsilon) = \theta_{\mathcal{T}}(\alpha(\mathcal{T}, \varepsilon), \varepsilon)$ and

$$
z(t) = \begin{cases}
z_{m,\mathcal{T}}^-(t, \varepsilon) & \text{if } t \in [T_{2m-1} + \beta_{m-1,\mathcal{T}}(\varepsilon), T_{2m} - \bar{T} + \alpha_{m,\mathcal{T}}(\varepsilon)] \\
z_{m,\mathcal{T}}^0(t, \varepsilon) & \text{if } t \in [T_{2m} - \bar{T} + \alpha_{m,\mathcal{T}}(\varepsilon), T_{2m} + \bar{T} + \beta_{m,\mathcal{T}}(\varepsilon)] \\
z_{m,\mathcal{T}}^+(t, \varepsilon) & \text{if } t \in [T_{2m} + \bar{T} + \beta_{m,\mathcal{T}}(\varepsilon), T_{2m+1} + \beta_{m,\mathcal{T}}(\varepsilon)]
\end{cases}
$$

we see that $z(t)$ satisfies the conclusion of Theorem 5.1 with $\hat{\alpha}_m = \alpha_m(\mathcal{T}, \varepsilon)$ and $\hat{\beta}_m = \beta_m(\alpha(\mathcal{T}, \varepsilon), \varepsilon)$. The proof is complete. □

Remark 5.2 Functions $\mathcal{M}, \mathcal{M}' : \mathbb{R} \to \mathbb{R}$ are bounded.

Remark 5.3 Following the above arguments, we can consider also cases when $\bar{m} \in \mathbb{Z}$ exists such that either $T_j = -\infty \forall j \le 2\bar{m} - 1$ or $T_j = \infty \forall j \ge 2\bar{m} + 1$. Then Theorem 5.1 is obviously modified (see (6.6), (6.7) and (6.8) below).

Remark 5.4 Here we emphasize that, during the proof of Theorem 5.1, we only used the fact that f and g are C^2 with bounded and uniformly continuous derivatives. We should need higher derivatives if α_0 is a degenerate root of $\mathcal{M}_{\mathcal{T}}(\alpha) = \{\mathcal{M}(T_{2m} + \alpha_m)\}_{m \in \mathbb{Z}}$, i.e. when condition (5.14) fails.

We are now able to give the proof of Theorem 2.2. First we show the following preparatory results.

Lemma 5.5 *For any $\varepsilon \neq 0$ there exists $|\varepsilon| > \nu_\varepsilon > 0$ such that if a sequence $\mathcal{T} = \{T_m\}_{m \in \mathbb{Z}}$ satisfies (2.6) then also it holds*

$$|D_1 g(t + T_{2m}, z, 0) - D_1 g(t, z, 0)| < |\varepsilon| \tag{5.40}$$

for any $(t, z, m) \in \mathbb{R}^{n+1} \times \mathbb{Z}$.

Proof of Lemma 5.5 Let $\varepsilon \neq 0$. Take $n_\varepsilon \in \mathbb{N}$ and $\nu_\varepsilon > 0$ as

$$n_\varepsilon = 2\left[\frac{\|D_{11} g\|}{|\varepsilon|}\right] + 1, \quad \nu_\varepsilon := \frac{|\varepsilon|}{4 n_\varepsilon} \tag{5.41}$$

and let $\mathcal{T} = \{T_m\}_{m \in \mathbb{Z}}$ be a sequence satisfying (2.6). Then we derive (see [18]):

$$\begin{aligned}
&|D_1 g(t + T_{2m}, z, 0) - D_1 g(t, z, 0)| \\
&\le \left| D_1 g(t + T_{2m}, z, 0) - n_\varepsilon \left[g\left(t + T_{2m} + \frac{1}{n_\varepsilon}, z, 0\right) - g(t + T_{2m}, z, 0) \right] \right| \\
&\quad + \left| D_1 g(t, z, 0) - n_\varepsilon \left[g\left(t + \frac{1}{n_\varepsilon}, z, 0\right) - g(t, z, 0) \right] \right| \\
&\quad + n_\varepsilon \left| g\left(t + T_{2m} + \frac{1}{n_\varepsilon}, z, 0\right) - g\left(t + \frac{1}{n_\varepsilon}, z, 0\right) \right| \\
&\quad + n_\varepsilon \left| g(t + T_{2m}, z, 0) - g(t, z, 0) \right| \\
&\le n_\varepsilon \int_0^{1/n_\varepsilon} |D_1 g(t + T_{2m} + \eta, z, 0) - D_1 g(t + T_{2m}, z, 0)| \, d\eta \\
&\quad + n_\varepsilon \int_0^{1/n_\varepsilon} |D_1 g(t + \eta, z, 0) - D_1 g(t, z, 0)| \, d\eta + 2 n_\varepsilon \nu_\varepsilon \\
&\le \frac{\|D_{11} g\|}{n_\varepsilon} + 2 n_\varepsilon \nu_\varepsilon < |\varepsilon|.
\end{aligned}$$

The proof of Lemma 5.5 is complete. □

Lemma 5.6 *If $\varepsilon \neq 0$ is sufficiently small then for any given sequence $\{T_m\}_{m \in \mathbb{Z}}$ with the properties of Lemma 5.5, a sequence $\{\alpha_m^0\}_{m \in \mathbb{Z}} \in \ell_\chi^\infty$ exists satisfying (5.14) for some $c_0 > 0$.*

Proof of Lemma 5.6 Let $|\mathcal{M}'(\alpha^0)| = 4c_0$. We have:

$$\mathcal{M}(T_{2m}+\alpha) = \mathcal{M}(\alpha) + \int_{-\infty}^{\infty} \psi^*(t)[g(t+T_{2m}+\alpha,\gamma(t),0) - g(t+\alpha,\gamma(t),0)]dt$$

and hence:

$$|\mathcal{M}(T_{2m}+\alpha) - \mathcal{M}(\alpha)| \le |\varepsilon| \int_{-\infty}^{\infty} |\psi^*(t)|dt \le 2K\delta^{-1}|\varepsilon|$$

since $|\psi^*(t)| \le K\,\mathrm{e}^{-\delta|t|}$ for some $K \ge 1$ (see (5.13)). Similarly, from (5.40) we get

$$|\mathcal{M}'(T_{2m}+\alpha) - \mathcal{M}'(\alpha)| \le 2K\delta^{-1}|\varepsilon|.$$

Let $\chi/2 > \delta_1 > 0$ be so small that $\mathcal{M}(\alpha^0-\delta_1)\mathcal{M}(\alpha^0+\delta_1) < 0$ and $|\mathcal{M}'(\alpha)| \ge 2c_0$ for $\alpha \in [\alpha^0-\delta_1, \alpha^0+\delta_1]$. Then, there is an $\widetilde{\varepsilon}_0 > 0$ such that for $0 < |\varepsilon| < \widetilde{\varepsilon}_0$ and for any $m \in \mathbb{Z}$ the equation $\mathcal{M}(T_{2m}+\alpha) = 0$ has a unique solution $\alpha_m^0 = \alpha(T_{2m}) \in (\alpha^0-\delta_1, \alpha^0+\delta_1)$ along with $|\mathcal{M}'(T_{2m}+\alpha)| \ge c_0$ for $\alpha \in [\alpha^0-\delta_1, \alpha^0+\delta_1]$. The proof of Lemma 5.6 is complete. □

Now we proceed with the proof of Theorem 2.2. Using Lemma 5.6, assumptions of Theorem 5.1 are verified and consequently, we obtain sequences $\{\hat{\alpha}_{m,\mathcal{T}}(\varepsilon)\}$, $\{\hat{\beta}_{m,\mathcal{T}}(\varepsilon)\}$, and a unique solution $z(t)$ of Eq. (2.1) that satisfies (2.7). To prove that $\sup_{m\in\mathbb{Z}} |\hat{\alpha}_{m,\mathcal{T}}(\varepsilon) - \alpha^0| < c_1|\varepsilon|$ and $\sup_{m\in\mathbb{Z}} |\hat{\beta}_{m,\mathcal{T}}(\varepsilon) - \alpha^0| < c_1|\varepsilon|$ assume for simplicity that $\mathcal{M}'(\alpha^0) = 4c_0$ (a similar argument applies when $\mathcal{M}'(\alpha^0) = -4c_0$). Then we have, since $\mathcal{M}'(T_{2m}+\alpha) > c_0$ for any $\alpha \in [\alpha_0-\delta_1, \alpha_0+\delta_1]$:

$$2K\delta^{-1}|\varepsilon| \ge \left| \int_{\alpha^0}^{\alpha_m^0} \mathcal{M}'(T_{2m}+\tau)d\tau \right| \ge c_0|\alpha_m^0 - \alpha^0|$$

hence

$$|\hat{\alpha}_{m,\mathcal{T}}(\varepsilon) - \alpha^0| \le |\hat{\alpha}_m(\mathcal{T},\varepsilon) - \alpha_m^0| + |\alpha_m^0 - \alpha^0| \le c_1|\varepsilon| + \frac{2K|\varepsilon|}{\delta c_0} = \widetilde{c}_1|\varepsilon|.$$

Similarly we get (possibly changing $\widetilde{c}_1$):

$$|\hat{\beta}_{m,\mathcal{T}}(\varepsilon) - \alpha^0| \le \widetilde{c}_1|\varepsilon|.$$

The proof of Theorem 2.2 is complete.

Remark 5.7 By (5.41), we get $\nu_\varepsilon \sim \varepsilon^2$ in Theorem 2.2.

6 Chaotic Behaviour

Let $\mathcal{E} := \{e : \mathbb{Z} \to \{0,1\}\}$ be the set of doubly infinite sequences of 0 and 1. We write an element $e \in \mathcal{E}$ as $e = \{e_m\}_{m\in\mathbb{Z}}$. It is well known that $\mathcal{E}$ becomes a totally disconnected compact metric space with the distance

$$d(e',e'') = \sum_{m\in\mathbb{Z}} \frac{|e'_m - e''_m|}{2^{|m|+1}}.$$

Let $\sigma : \mathcal{E} \to \mathcal{E}$ be the Bernoulli shift that is $\sigma(e) := \{e_{m+1}\}_{m\in\mathbb{Z}}$. Set

$$\begin{aligned}\hat{\mathcal{E}} &: = \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} = -\infty,\ \sup\{m \in \mathbb{Z} \mid e_m = 1\} = \infty\}\\ \mathcal{E}_+ &: = \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty,\ \sup\{m \in \mathbb{Z} \mid e_m = 1\} = \infty\}\\ \mathcal{E}_- &: = \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} = -\infty,\ \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty\}\\ \mathcal{E}_0 &: = \{e \in \mathcal{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty,\ \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty\}.\end{aligned}$$

Note that $\hat{\mathcal{E}}, \mathcal{E}_-, \mathcal{E}_+, \mathcal{E}_0$ are invariant under the Bernoulli shift.

In this section we suppose for simplicity that assumptions of Theorem 5.1 are satisfied with a technical condition $\|\alpha_0\| < \chi/2$, i.e the following holds:

C) For any $\varepsilon \neq 0$ sufficiently small there is a sequence $\mathcal{T} = \{T_m\}_{m\in\mathbb{Z}}$ such that $T_{m+1} - T_m > \bar{T} + 1 - 2\delta^{-1} \ln|\varepsilon|$ along with the existence of an $\alpha_0 = \{\alpha_m^0\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$ with $\|\alpha_0\| < \chi/2$ and satisfying (5.14).

Let $\mathcal{T} = \{T_m\}_{m\in\mathbb{Z}}$ be as in assumption (C). Assume, first, that $e \in \hat{\mathcal{E}}$. Let $\{n_m^e\}_{m\in\mathbb{Z}}$ be a fixed increasing doubly-infinite sequence of integers such that $e_k = 1$ if and only if $k = n_m^e$. We define sequences $\mathcal{T}^e = \{T_m^e\}_{m\in\mathbb{Z}}$ and $\alpha_0^e = \{\alpha_m^{0e}\}_{m\in\mathbb{Z}}$ as

$$T_m^e := \begin{cases} T_{2n_k^e} & \text{if } m = 2k \\ T_{2n_k^e - 1} & \text{if } m = 2k - 1 \end{cases} \tag{6.1}$$

and similarly

$$\alpha_m^{0e} := \alpha_{n_m^e}^0. \tag{6.2}$$

Note $T_{m+1}^e - T_m^e > \bar{T} + 1 - 2\delta^{-1} \ln|\varepsilon|$ for any $m \in \mathbb{Z}$ and $\mathcal{M}_{\mathcal{T}^e}(\alpha)$ has a simple zero α_0^e, i.e. (5.14) holds with exchanges $\mathcal{T}^e \leftrightarrow \mathcal{T}$ and $\alpha_0^e \leftrightarrow \alpha_0$. Since $|\alpha_{m+1}^{0e} - \alpha_m^{0e}| < \chi$ for any $m \in \mathbb{Z}$, assumptions of Theorem 5.1 are satisfied by $\mathcal{M}_{\mathcal{T}^e}(\alpha)$, $\mathcal{T}^e$ and α_0^e. Let $z(t) = z(t, \mathcal{T}^e)$ be the corresponding solution of Eq. (2.1) whose existence is stated in Theorem 5.1. Then $z(t)$ satisfies

$$\begin{aligned} \sup_{t\in[T_{2m-1}^e + \beta_{m-1}^e,\, T_{2m}^e - \bar{T} + \alpha_m^e]} &|z(t) - \gamma_-(t - T_{2m}^e - \alpha_m^e)| < \rho \\ \sup_{t\in[T_{2m}^e - \bar{T} + \alpha_m^e,\, T_{2m}^e + \bar{T} + \beta_m^e]} &|z(t) - \gamma_0(t - T_{2m}^e - \alpha_m^e)| < \rho \\ \sup_{t\in[T_{2m}^e + \bar{T} + \beta_m^e,\, T_{2m+1}^e + \beta_m^e]} &|z(t) - \gamma_+(t - T_{2m}^e - \beta_m^e)| < \rho, \end{aligned} \tag{6.3}$$

where the sequences $\alpha^e = \{\alpha_m^e\}_{m\in\mathbb{Z}}$ and $\beta^e = \{\beta_m^e\}_{m\in\mathbb{Z}}$ are determined as in Theorem 5.1 (note here we remove hats for notational simplicity).

Now, consider the sequence $\widetilde{n}_m^e := n_{m+1}^e$ instead of n_m^e and denote with $\widetilde{\mathcal{T}}^e, \widetilde{\alpha}^e, \widetilde{\beta}^e$ and $\widetilde{\alpha}_0^e$ the corresponding sequences:

$$\widetilde{T}_m^e = T_{m+2}^e, \quad \widetilde{\alpha}_m^e = \alpha_{m+1}^e, \quad \widetilde{\beta}_m^e = \beta_{m+1}^e, \quad \widetilde{\alpha}_m^{0e} = \alpha_{m+1}^{0e}. \tag{6.4}$$

Then $\mathcal{M}_{\widetilde{\mathcal{T}}^e}(\alpha)$ has a simple zero $\widetilde{\alpha}_0^e$ and Theorem 5.1 is applicable. But clearly $\widetilde{z}(t) := z(t, \widetilde{\mathcal{T}}^e)$ satisfies the same set of estimates (6.3) and hence, because of uniqueness, $z(t, \widetilde{\mathcal{T}}^e) = z(t, \mathcal{T}^e)$ depends only on e and $\mathcal{T}$ (and not on the choice of n_m^e). So we will write $z(t, \mathcal{T}, e)$ instead of $z(t, \mathcal{T}^e)$.

Now, assume that $e_j = 1$. Then $j = n_m^e$ for some $m \in \mathbb{Z}$ and (6.3) gives, provided $|\varepsilon|$ is sufficiently small:

$$|z(T_{2j}) - \gamma_0(-\alpha_j^0)| \le |z(T_{2j}) - \gamma_0(-\alpha_m^e)| + |\gamma_0(-\alpha_m^e) - \gamma_0(-\alpha_j^0)|$$
$$< \rho + \sup_{t\in\mathbb{R}} |\dot\gamma_0(t)|\,|\alpha_m^e - \alpha_j^0| < \rho + c_1|\varepsilon| \sup_{t\in\mathbb{R}} |\dot\gamma_0(t)| < \frac{3}{2}\rho$$

since $T_{2m}^e = T_{2j}$. On the other hand, if $e_j = 0$, let $m \in \mathbb{Z}$ be such that $n_m^e < j < n_{m+1}^e$. Then $n_{m+1}^e - 1 \ge j \ge n_m^e + 1$ and so

$$T_{2j} - T_{2n_m^e} - \bar T - \beta_m^e \ge T_{2n_m^e+2} - T_{2n_m^e} - \bar T - \|\alpha_0\| - c_1|\varepsilon|$$
$$\ge \bar T + 2 - 4\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| > 0$$

and

$$T_{2m+1}^e + \beta_m^e - T_{2j} \ge T_{2n_{m+1}^e-1} - T_{2n_{m+1}^e-2} - \|\alpha_0\| - c_1|\varepsilon|$$
$$\ge \bar T + 1 - 2\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| > 0$$

for $0 < |\varepsilon| \ll 1$. Consequently, we have $T_{2j} \in \left[T_{2m}^e + \bar T + \beta_m^e, T_{2m+1}^e + \beta_m^e\right]$, and using (6.3), we get

$$|z(T_{2j})| \le |z(T_{2j}) - \gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)| + |\gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)| < \frac{3}{2}\rho$$

since $T_{2j} - T_{2n_m^e} - \beta_m^e \ge T_{2n_m^e+2} - T_{2n_m^e} - \|\alpha_0\| - c_1|\varepsilon| > 2\bar T + 2 - 4\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| \gg 1$ for $0 < |\varepsilon| \ll 1$, and thus $\left|\gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)\right| < \rho/2$. So $z(t, \mathcal{T}, e)$ has the following property

$$\begin{array}{ll} |z(T_{2j}) - \gamma_0(-\alpha_j^0)| < \frac{3}{2}\rho & \text{if } e_j = 1 \\ |z(T_{2j})| < \frac{3}{2}\rho & \text{if } e_j = 0 \end{array} \tag{6.5}$$

Next, assume $e \in \mathcal{E}_+$ and let again $\{n_m^e\}_{m\in\mathbb{Z}}$ be a fixed increasing sequence of integers such that $e_k = 1$ if and only if $k = n_m^e$ and $\lim_{m\to\infty} n_m^e = \infty$. Corresponding to this sequence, we define $\mathcal{T}^e$ as in (6.1) and then we obtain α^e and β^e as in (6.3) with the difference that $T_m^e = -\infty$ and $\alpha_m^e = \beta_m^e = 0$ for any $m < 2\bar m$ where $\bar m$ is such that $e_{n_{\bar m}^e} = 1$ and $e_j = 0$ for any $j < n_{\bar m}^e$. According to this choice, by Remark 5.3, we obtain a solution $z(t) = z(t, \mathcal{T}^e)$ of Eq. (2.1) that satisfies (6.3) when $m > \bar m$ whereas for $m = \bar m$ it satisfies:

$$\begin{aligned} &\sup_{t\in(-\infty, T_{2\bar m}^e - \bar T + \alpha_{\bar m}^e]} |z(t) - \gamma_-(t - T_{2\bar m}^e - \alpha_{\bar m}^e)| < \rho \\ &\sup_{t\in[T_{2\bar m}^e - \bar T + \alpha_{\bar m}^e, T_{2\bar m}^e + \bar T + \beta_{\bar m}^e]} |z(t) - \gamma_0(t - T_{2\bar m}^e - \alpha_{\bar m}^e)| < \rho \\ &\sup_{t\in[T_{2\bar m}^e + \bar T + \beta_{\bar m}^e, T_{2\bar m+1}^e + \beta_{\bar m}^e]} |z(t) - \gamma_+(t - T_{2\bar m}^e - \beta_{\bar m}^e)| < \rho. \end{aligned} \tag{6.6}$$

Note, then, that if we take, as in the previous case, $\widetilde n_m^e = n_{m+1}^e$ and $\widetilde{\mathcal{T}}^e, \widetilde\alpha^e, \widetilde\beta^e$ as in (6.4), then (6.3) holds with $\widetilde{\mathcal{T}}^e$ instead $\mathcal{T}^e$, provided $m > \bar m - 1$ whereas (6.6) holds with $\widetilde T_{2(\bar m-1)}^e$ and $\widetilde T_{2\bar m-1}^e$ instead of $T_{2\bar m}^e$ and $T_{2\bar m+1}^e$ respectively. So in this case too we see that $z(t, \mathcal{T}^e) = z(t, \mathcal{T}, e)$ depends only on $(\mathcal{T}, e)$ and not on the choice of the sequence n_m^e. Moreover, (6.5) holds also in this case. In fact if either $e_j = 1$ or $e_j = 0$ and there exists $m \in \mathbb{Z}$ such that $n_m^e < j < n_{m+1}^e$ the same proof as before goes through. If, instead, $e_j = 0$ and $j < n_{\bar m}^e$, then the estimate $|z(T_{2j})| < \frac{3}{2}\rho$ follows from the first estimate in (6.6) since $2j \le 2n_{\bar m}^e - 2$ and then

$T^e_{2j} - T^e_{2\bar m} - \alpha^e_{\bar m} \le T^e_{2n^e_{\bar m}-2} - T_{2n^e_{\bar m}} + \|\alpha_0\| + c_1|\varepsilon| \le -2\bar T - 2 - 4\delta^{-1}\ln|\varepsilon| + \|\alpha_0\| + c_1|\varepsilon| \ll 0$ for $0 < |\varepsilon| \ll 1$.

Similarly, if $e \in \mathcal{E}_-$ then by Remark 5.3, we obtain a solution $z(t) = z(t, \mathcal{T}^e)$ of Eq. (2.1) that satisfies (6.3) when $m < \bar m$ whereas for $m = \bar m$ it satisfies

$$\begin{aligned} &\sup_{t\in(T^e_{2\bar m-1}, T^e_{2\bar m}-\bar T+\alpha^e_{\bar m}]} |z(t) - \gamma_-(t - T^e_{2\bar m} - \alpha^e_{\bar m})| < \rho \\ &\sup_{t\in[T^e_{2\bar m}-\bar T+\alpha^e_{\bar m}, T^e_{2\bar m}+\bar T+\beta^e_{\bar m}]} |z(t) - \gamma_0(t - T^e_{2\bar m} - \alpha^e_{\bar m})| < \rho \\ &\sup_{t\in[T^e_{2\bar m}+\bar T+\beta^e_{\bar m}, \infty)} |z(t) - \gamma_+(t - T^e_{2\bar m} - \beta^e_{\bar m})| < \rho. \end{aligned} \tag{6.7}$$

An argument similar to the previous one (in this case, we can take for example $\widetilde{n}^e_m = n^e_{m-1}$) we see that $z(t, \mathcal{T}^e) = z(t, \mathcal{T}, e)$ depends only on $(\mathcal{T}, e)$ and not on the choice of the sequence n^e_m and (6.5) holds.

Next, assume that $e \in \mathcal{E}_0$ with $e \ne 0$. Then there are $\bar m_- < \bar m_+$ such that $e_k = 0$ if either $k < n^e_{m_-}$ or $k > n^e_{m_+}$ and Eq. (2.1) has a unique solution $z(t, \mathcal{T}^e)$ such that (6.3) holds when $\bar m_- < m < \bar m_+$ whereas when either $m = m_-$ or $m = m_+$ it satisfies

$$\begin{aligned} &\sup_{t\in(-\infty, T^e_{2\bar m_-}-\bar T+\alpha^e_{\bar m_-}]} |z(t) - \gamma_-(t - T^e_{2\bar m_-} - \alpha^e_{\bar m_-})| < \rho \\ &\sup_{t\in[T^e_{2\bar m_-}-\bar T+\alpha^e_{\bar m_-}, T^e_{2\bar m_-}+\bar T+\beta^e_{\bar m_-}]} |z(t) - \gamma_0(t - T^e_{2\bar m_-} - \alpha^e_{\bar m})| < \rho \\ &\sup_{t\in[T^e_{2\bar m_-}+\bar T+\beta^e_{\bar m_-}, T^e_{2\bar m_-+1}+\beta^e_{\bar m_-}]} |z(t) - \gamma_+(t - T^e_{2\bar m_-} - \beta^e_{\bar m_-})| < \rho \\ &\sup_{t\in(T^e_{2\bar m_+-1}, T^e_{2\bar m_+}-\bar T+\alpha^e_{\bar m_+}]} |z(t) - \gamma_-(t - T^e_{2\bar m_+} - \alpha^e_{\bar m_+})| < \rho \\ &\sup_{t\in[T^e_{2\bar m_+}-\bar T+\alpha^e_{\bar m_+}, T^e_{2\bar m_+}+\bar T+\beta^e_{\bar m_+}]} |z(t) - \gamma_0(t - T^e_{2\bar m_+} - \alpha^e_{\bar m_+})| < \rho \\ &\sup_{t\in[T^e_{2\bar m_+}+\bar T+\beta^e_{\bar m_+}, \infty)} |z(t) - \gamma_+(t - T^e_{2\bar m_+} - \beta^e_{\bar m_+})| < \rho. \end{aligned} \tag{6.8}$$

Moreover $z(t, \mathcal{T}^e) = z(t, \mathcal{T}, e)$ depends only on $(\mathcal{T}, e)$ and not on the choice of n^e_m and (6.5) holds.

Finally, if $e = 0$, that is $e_k = 0$ for any $k \in \mathbb{Z}$, by [4, Remark 2.14] we define $z(t, \mathcal{T}, 0) = u(t)$ as the unique bounded solution of (2.1) such that

$$\sup_{t\in\mathbb{R}} |u(t)| < \rho. \tag{6.9}$$

Now we are able to prove the following theorem:

Theorem 6.1 *Let assumptions* (H1), (H2), (H3) *and* (C) *be satisfied. Let* $\rho > 0$ *be small. Then for any* $\varepsilon \ne 0$ *sufficiently small and for any* $e \in \mathcal{E}$, *Eq.* (2.1) *has a unique solution* $z(t, \mathcal{T}, e, \varepsilon)$ *that satisfies one among* (6.3), *(6.6),* (6.7) *or* (6.8) *and hence* (6.5). *Moreover, setting* $\mathcal{T}^{(k)} := \{T_{n+2k}\}_{n\in\mathbb{Z}}$, *we have*

$$z(t, \mathcal{T}^{(k+1)}, \sigma(e), \varepsilon) = z(t, \mathcal{T}^{(k)}, e, \varepsilon) \tag{6.10}$$

for any $t \in \mathbb{R}$ *and* $e \in \mathcal{E}$.

Proof We only need to prove that (6.10) holds. Since $z(t, \mathcal{T}, e, \varepsilon)$ does not depend on the choice of $\{n^e_m\}_{m\in\mathbb{Z}}$ we see that we can take $n^{\sigma(e)}_m = n^e_m - 1$ and then, setting $\mathcal{T}' = \{T_{m+2}\}_{m\in\mathbb{Z}}$, we have, if $m = 2k$:

$$T'^{\sigma(e)}_{2k} = T_{2n^{\sigma(e)}_k+2} = T_{2n^e_k} = T^e_{2k}$$

and, if $m = 2k-1$:

$$T'^{\sigma(e)}_{2k-1} = T_{2n^{\sigma(e)}_k+1} = T_{2n^e_k-1} = T^e_{2k-1}$$

that is

$$\mathcal{T}'^{\sigma(e)} = \mathcal{T}^e. \tag{6.11}$$

Hence we see that, for any $t \in \mathbb{R}$ and any $e \in \mathcal{E}$, the following holds

$$z(t, \mathcal{T}', \sigma(e), \varepsilon) = z(t, \mathcal{T}, e, \varepsilon). \tag{6.12}$$

Now, from the definition of $\mathcal{T}^{(k)}$ we see that $\mathcal{T}^{(k+1)} = \mathcal{T}^{(k)\prime}$ thus (6.10) follows from (6.12). The proof is complete. □

Now we define $F_k : \mathbb{R}^n \to \mathbb{R}^n$ so that $F_k(\xi)$ is the value at time $T_{2(k+1)}$ of the solution $z(t)$ of Eq. (2.1) such that $z(T_{2k}) = \xi$:

$$\dot{z} = f_\pm(z) + \varepsilon g(t, z, \varepsilon), \quad z(T_{2k}) = \xi \tag{6.13}$$

and let $\Phi_k(e) := z(T_{2k}, \mathcal{T}^{(k)}, e, \varepsilon)$. Note that, according to assumption (H2), F_k is certainly defined in neighborhood of the compact set $\{\gamma(t) \mid t \in \mathbb{R}\} \cup \{0\}$. Then we have:

$$\begin{aligned}\Phi_{k+1} \circ \sigma(e) &= z(T_{2(k+1)}, \mathcal{T}^{(k+1)}, \sigma(e), \varepsilon) = z(T_{2(k+1)}, \mathcal{T}^{(k)}, e, \varepsilon)\\ &= F_k(z(T_{2k}, \mathcal{T}^{(k)}, e, \varepsilon)) = F_k \circ \Phi_k(e).\end{aligned} \tag{6.14}$$

Note that (6.14) can be stated in the following way. Let

$$\mathcal{S}_k = \left\{(z(T_{2k}, \mathcal{T}^{(k)}, e, \varepsilon) \mid e \in \mathcal{E}\right\}, \quad k \in \mathbb{Z}.$$

It is standard to prove (see [22]) that $\mathcal{S}_k$ are compact in $\mathbb{R}^n$ and $\Phi_k : \mathcal{E} \mapsto \mathcal{S}_k$ are continuous and clearly onto. Moreover, by (6.14), all $F_k : \mathcal{S}_k \to \mathcal{S}_{k+1}$ are homeomorphisms.

Next, let $e, e' \in \mathcal{E}$ be two different sequences in $\mathcal{E}$. Then there exists $j \in \mathbb{Z}$ such that, for example, $e'_j = 0$ and $e_j = 1$. From $[-\chi/2, \chi/2] \subset [-\bar{T}, \bar{T}]$ and (6.5) we see that

$$\begin{aligned}\left|z(T_{2j}, \mathcal{T}, e, \varepsilon) - z(T_{2j}, \mathcal{T}, e', \varepsilon)\right| &\ge \left|\gamma_0(-\alpha^0_j)\right|\\ &\quad - \left|z(T_{2j}, \mathcal{T}, e, \varepsilon) - \gamma_0(-\alpha^0_j)\right| - \left|z(T_{2j}, \mathcal{T}, e', \varepsilon)\right|\\ &\ge \left|\gamma_0(-\alpha^0_m)\right| - 3\rho \ge \min_{t\in[-\bar{T},\bar{T}]} |\gamma_0(t)| - 3\rho > 0\end{aligned}$$

provided ρ is sufficiently small. As a consequence $z(T_{2j}, \mathcal{T}, e, \varepsilon) \ne z(T_{2j}, \mathcal{T}, e', \varepsilon)$ and, since both are solutions of the same Eq. (2.1):

$$z(t, \mathcal{T}, e, \varepsilon) \ne z(t, \mathcal{T}, e', \varepsilon) \tag{6.15}$$

for any $t \in \mathbb{R}$. Thus we have proved that the map $e \mapsto z(t, \mathcal{T}, e, \varepsilon)$ is one-to-one. Hence if $\Phi_k(e) = \Phi_k(e')$ then $e = e'$ since otherwise:

$$\Phi_k(e) = z(T_{2k}, \mathcal{T}^{(k)}, e, \varepsilon) \ne z(T_{2k}, \mathcal{T}^{(k)}, e', \varepsilon) = \Phi_k(e').$$

So $\Phi_k : \mathcal{E} \to \mathcal{S}_k$ is one-to-one and a homeomorphism for any $k \in \mathbb{Z}$.

Summarizing, we get the next result.

Theorem 6.2 *Assume* (H1), (H2), (H3) *and* (C) *hold. Then for any* $\varepsilon \neq 0$ *sufficiently small, the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\sigma\quad} & \mathcal{E} \\ {\scriptstyle \Phi_k}\downarrow & & \downarrow{\scriptstyle \Phi_{k+1}} \\ \mathcal{S}_k & \xrightarrow[\quad F_k\quad]{} & \mathcal{S}_{k+1} \end{array}$$

for all $k \in \mathbb{Z}$. *Moreover, all* Φ_k *are homeomorphisms.*

Sequences of 2-dimensional maps are also studied in [26].

Remark 6.3 We improve (6.3) as follows. First, assume that $e_j = 1$, and $e_{j+1} = 0$. Then, if $j = n_k^e$, we have $n_{k+1}^e > n_k^e + 1$ and then if

$$t \in [T_{2n_k^e+1} + \beta_k^e, T_{2n_{k+1}^e-1} + \beta_k^e] = \cup_{j=2n_k^e+1}^{2(n_{k+1}^e-1)} [T_j + \beta_k^e, T_{j+1} + \beta_k^e],$$

we have $t \in [T_{2k}^e + \bar{T} + \beta_k^e, T_{2k+1}^e + \beta_k^e]$ and

$$t - T_{2k}^e - \beta_k^e \in [T_{2n_k^e+1} - T_{2n_k^e}, T_{2n_{k+1}^e-1} - T_{2n_k^e}] \subset (\bar{T} + 1 - 2\delta^{-1} \ln |\varepsilon|, \infty)$$

and hence if ε is small enough that $|\gamma_-(t)| < \rho$ for any $t \geq \bar{T} + 1 - 2\delta^{-1} \ln |\varepsilon|$, by (6.3) we get:

$$\sup_{t \in [T_j + \beta_k^e, T_{j+1} + \beta_k^e]} |z(t) - u(t)| < 3\rho$$

for any $j \in \{2n_k^e + 1, \ldots, 2(n_{k+1}^e - 1)\}$. On the other hand

$$\sup_{t \in [T_{2n_k^e-1} + \beta_{k-1}^e, T_{2n_k^e} + \bar{T} + \beta_k^e]} |z(t) - \gamma(t - T_{2n_k^e} - \alpha_k^e)| < \rho$$
$$\sup_{t \in [T_{2n_k^e} + \bar{T} + \beta_k^e, T_{2n_k^e+1} + \beta_k^e]} |z(t) - \gamma(t - T_{2n_k^e} - \beta_k^e)| < \rho$$

that is for $t \in [T_j + \beta_k^e, T_{j+1} + \beta_k^e]$ the solution $z(t)$ is close either to the homoclinic orbit $\gamma(t)$ or to the bounded solution according to $e_j = 1$ or $e_j = 0$. The cases $e_j = 0$, $e_{j+1} = 1$ and $e_j = e_{j+1} = 1$ can be similarly handled.

6.1 The Almost and Quasi Periodic Cases

In this section we assume that $g(t, x, \varepsilon)$ is almost periodic in t uniformly in (x, ε) that is the following holds:

(H4) For any $\nu > 0$ there exists $L_\nu > 0$ such that in any interval of a length greater than L_ν there exists T_ν which is an *almost period* for ν that is satisfying:

$$|g(t + T_\nu, x, \varepsilon) - g(t, x, \varepsilon)| < \nu$$

for any $(t, x, \varepsilon) \in \mathbb{R}^{n+2}$.

In this section we suppose the existence of a simple zero α^0 of $\mathcal{M}(\alpha)$. Then following the arguments of the proof of Theorem 2.2 we see that for any $\varepsilon \neq 0$ sufficiently small there is a sequence $\mathcal{T}^\varepsilon = \{T_m^\varepsilon\}_{m\in\mathbb{Z}}$ such that $T_{m+1}^\varepsilon - T_m^\varepsilon > \bar{T} + 1 + 4|\alpha^0| - 2\delta^{-1}\ln|\varepsilon|$ along with the existence of $\alpha^\varepsilon = \{\alpha_m^\varepsilon\}_{m\in\mathbb{Z}} \in \ell^\infty$ with $\|\alpha^\varepsilon\| \le 2|\alpha^0|$ and satisfying $\mathcal{M}(T_{2m}^\varepsilon + \alpha_m^\varepsilon) = 0$ for any $m \in \mathbb{Z}$ and $\inf_{m\in\mathbb{Z}} |\mathcal{M}'(T_{2m}^\varepsilon + \alpha_m^\varepsilon)| > c_0$ for some $c_0 > 0$. Then taking $T_{2m} = T_{2m}^\varepsilon + \alpha_m^\varepsilon$, $T_{2m-1} = T_{2m-1}^\varepsilon$, and $\alpha_0 = 0$, assumption (C) is satisfied. So applying Theorem 6.2, system (2.1) is chaotic for any $\varepsilon \neq 0$ small. Summarizing we obtain the following theorem.

Theorem 6.4 *Assume that* (H1)–(H4) *hold and that the almost periodic Melnikov function* $\mathcal{M}(\alpha)$ *has a simple zero. Then system (2.1) is chaotic for any* $\varepsilon \neq 0$ *sufficiently small.*

Next, it is well known (see [13,21,23,25]) that near the hyperbolic equilibrium $x = 0$ of the equation $\dot{x} = f_-(x)$ there exists a unique almost periodic solution of $\dot{x} = f_-(x)+\varepsilon g(t, x, \varepsilon)$. More precisely, given $\rho > 0$ there exists $\bar{\varepsilon} > 0$ such that for any $|\varepsilon| < \bar{\varepsilon}$ equation $\dot{x} = f_-(x) + \varepsilon g(t, x, \varepsilon)$ has a solution $u(t) = u(t, \varepsilon)$ such that $|u(t)| < \rho$ for any $t \in \mathbb{R}$ and it is almost periodic with common almost periods as $g(t, x, \varepsilon)$, i.e. assumption (H4) holds in addition with

$$|u(t + T_\nu) - u(t)| < \hat{c}_0\nu \quad \forall m \in \mathbb{Z}$$

for a positive constant $\hat{c}_0$. Note that $u(t)$ is the bounded solution of $\dot{x} = f_-(x) + \varepsilon g(t, x, \varepsilon)$ mentioned in (6.9). Thus the conclusion of Remark 6.3 holds with the further property that $u(t)$ is almost periodic.

Results of this section generalize the deterministic chaos of [21,23,25,26] to the discontinuous almost periodic system (2.1).

Finally, if $g(t, x, \varepsilon)$ is quasi periodic in t that is the following holds:

H5) $g(t, x, \varepsilon) = q(\omega_1 t, \dots, \omega_m t, x, \varepsilon)$ for $\omega_1, \dots, \omega_m \in \mathbb{R}$ with $q \in C_b^r(\mathbb{R}^{m+n+1}, \mathbb{R}^n)$ and $q(\theta_1, \dots, \theta_m, x, \varepsilon)$ is 1-periodic in each θ_i, $i = 1, 2, \dots, m$. Moreover, ω_i, $i = 1, 2, \dots, m$ are linearly independent over $\mathbb{Z}$, i.e. if $\sum_{i=1}^m l_i\omega_i = 0$, $l_i \in \mathbb{Z}$, $i = 1, 2, \dots, m$, then $l_i = 0$, $i = 1, 2, \dots, m$.

Then $g(t, x, \varepsilon)$ satisfies assumption (H4) [18,21] and hence the conclusion of Theorem 6.4 holds.

6.2 The Periodic Case

Here we assume that $g(t + p, z, \varepsilon) = g(t, z, \varepsilon)$ that is $g(t, z, \varepsilon)$ is p-periodic. Then $\mathcal{M}(\alpha)$ is also p-periodic. We suppose the existence of a simple zero α^0 of $\mathcal{M}(\alpha)$. Then Theorem 2.2 is applicable with $T_m = mT$ and $2T = rp$ for $r \gg 1$, $r \in \mathbb{N}$. So:

$$T_m^e = \begin{cases} 2n_k^e T & \text{if } m = 2k \\ (2n_k^e - 1)T & \text{if } m = 2k - 1. \end{cases}$$

Since we can take $n_m^{\sigma(e)} = n_m^e - 1$ we see that

$$T_m^{\sigma(e)} = \begin{cases} 2n_k^e T - 2T & \text{if } m = 2k \\ (2n_k^e - 1)T - 2T & \text{if } m = 2k - 1 \end{cases} = T_m^e - 2T$$

for any $m \in \mathbb{Z}$. Again we denote with $z(t) = z(t, \mathcal{T}, e)$ the solution of Eq. (2.1) corresponding to the sequence $\mathcal{T}^e$. Then $Z(t) := z(t + 2T)$ satisfies the equation

$$\dot{z} = f_\pm(z) + \varepsilon g(t, z, \varepsilon)$$

together with the estimates:

$$\begin{aligned}
&\sup_{t\in[T_{2m-1}^{\sigma(e)}+\beta_{m-1}^e,\,T_{2m}^{\sigma(e)}-\bar{T}+\alpha_m^e]} |Z(t)-\gamma_-(t-T_{2m}^{\sigma(e)}-\alpha_m^e)|<\rho\\
&\sup_{t\in[T_{2m}^{\sigma(e)}-\bar{T}+\alpha_m^e,\,T_{2m}^{\sigma(e)}+\bar{T}+\beta_m^e]} |Z(t)-\gamma_0(t-T_{2m}^{\sigma(e)}-\alpha_m^e)|<\rho\\
&\sup_{t\in[T_{2m}^{\sigma(e)}+\bar{T}+\beta_m^e,\,T_{2m-1}^{\sigma(e)}+\beta_m^e]} |Z(t)-\gamma_+(t-T_{2m}^{\sigma(e)}-\beta_m^e)|<\rho.
\end{aligned} \tag{6.16}$$

Thus, because of uniqueness:

$$\alpha(\mathcal{T}^e,\varepsilon)=\alpha(\mathcal{T}^{\sigma(e)},\varepsilon)\in\ell^\infty(\mathbb{R}),\quad \beta(\mathcal{T}^e,\varepsilon)=\beta(\mathcal{T}^{\sigma(e)},\varepsilon)\in\ell^\infty(\mathbb{R})$$

and $z(t+2T,\mathcal{T},e,\varepsilon)=z(t,\mathcal{T},\sigma(e),\varepsilon)$. Thus, using (6.10) and recalling that $T_k=kT$:

$$z(T_{2(k+1)},\mathcal{T}^{(k+1)},e,\varepsilon)=z(T_{2k},\mathcal{T}^{(k+1)},\sigma(e),\varepsilon)=z(T_{2k},\mathcal{T}^{(k)},e,\varepsilon)$$

that is we see that

$$\Phi_k(e)=\Phi(e),\quad \mathcal{S}_k=\mathcal{S}$$

are independent of k. Similarly, because of uniqueness and periodicity, the value at the time $T_{2(k+1)}=2(k+1)T$ of the solution of (6.13) is the same as the value at time $2T$ of the solution of

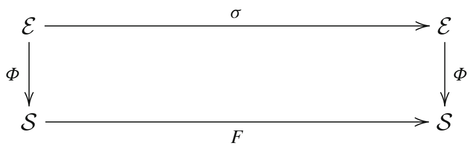
$$\dot{z}=f_\pm(z)+\varepsilon g(t,z,\varepsilon),\quad z(0)=\xi$$

that is also $F_k(\xi)=F(\xi)$ are independent of k and we have:

$$\Phi\circ\sigma=F\circ\Phi.$$

Summarizing we arrive at the following result.

Theorem 6.5 *Assume $g(t+p,z,\varepsilon)=g(t,z,\varepsilon)$ that is $g(t,z,\varepsilon)$ is p-periodic. If $\varepsilon\neq 0$ is sufficiently small and there is a simple zero α^0 of $\mathcal{M}(\alpha)$ then the following diagram commutes:*



Here $F=\varphi_\varepsilon^r=\varphi_\varepsilon\circ\cdots\circ\varphi_\varepsilon$ (r times) is the rth iterate of the p-period map φ_ε of (2.1), *$r\in\mathbb{N}$, $r\gg 1$.*

Theorem 6.5 generalizes the deterministic chaos of [12,22] to the discontinuous periodic system (2.1).

7 Quasi Periodic Piecewise Linear Systems

In in section, we consider the example

$$\dot{x}=\begin{cases} Ax+\varepsilon\,(g_1\sin\omega_1 t+g_2\sin\omega_2 t) & \text{for } \widetilde{a}\cdot x<d,\\ Ax+b+\varepsilon\,(g_1\sin\omega_1 t+g_2\sin\omega_2 t) & \text{for } \widetilde{a}\cdot x>d \end{cases} \tag{7.1}$$

of a quasi periodically perturbed piecewise linear 3-dimensional differential equation. Here $d > 0, \omega_{1,2} > 0, \widetilde{a}, x, g_{1,2} \in \mathbb{R}^3, \widetilde{a} \cdot x$ is the scalar product in $\mathbb{R}^3$.

Moreover, we consider the system (7.1) under the following assumptions

(i) A is an 3×3-matrix with semisimple eigenvalues $\lambda_1, \lambda_2 > 0 > \lambda_3$ and with the corresponding eigenvectors e_1, e_2, e_3.
(ii) Let $b = \sum_{i=1}^{3} b_i e_i$ and $a_i := \widetilde{a} \cdot e_i, i = 1, 2, 3$. Then $a_1, b_3 \geq 0, a_2, a_3 > 0$ and $b_1, b_2 < 0$.

Remark 7.1 Certainly we can study more general systems

$$\dot{x} = \begin{cases} Ax + \varepsilon \sum_{k=1}^{m} g_k \sin \omega_k t & \text{for } \widetilde{a} \cdot x < d, \\ Ax + b + \varepsilon \sum_{k=1}^{m} g_k \sin \omega_k t & \text{for } \widetilde{a} \cdot x > d \end{cases}$$

but for simplicity we concentrate on (7.1) in this paper.

If either $g_1 = 0$ or $g_2 = 0$ or the ratio $\frac{\omega_1}{\omega_2}$ is rational, then we get the periodic case studied in [4]. Theorem 6.5 of this paper, however, improve the result in [4] in the sense that here we obtain chaotic behaviour of the solutions. Thus, we focus here on the case

(iii) $g_1 \neq 0, g_2 \neq 0$ and ω_1/ω_2 is irrational.

Given the vectors in $\mathbb{R}^3$: $x = \sum_{i=1}^{3} x_i e_i$ and $y = \sum_{i=1}^{3} y_i e_i$ we define

$$\langle x, y \rangle = \sum_{i=1}^{3} x_i y_i.$$

Then $\langle x, y \rangle$ is a scalar product in $\mathbb{R}^3$ that makes $\{e_1, e_2, e_3\}$ an orthonormal basis of $\mathbb{R}^3$. From now on we will write also (x_1, x_2, x_3) for the vector $x = \sum_{i=1}^{3} x_i e_i$ and hence we identify e_1, e_2, e_3 with $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ respectively.

Writing $x = \sum_{i=1}^{3} x_i e_i$ and $g_j = \sum_{i=1}^{3} g_{ji} e_i, j = 1, 2,$ (7.1) has the form

$$\dot{x}_i = \begin{cases} \lambda_i x_i + \varepsilon (g_{1i} \sin \omega_1 t + g_{2i} \sin \omega_2 t) & \text{for } \langle a, x \rangle < d, \\ \lambda_i x_i + b_i + \varepsilon (g_{1i} \sin \omega_1 t + g_{2i} \sin \omega_2 t) & \text{for } \langle a, x \rangle > d, \\ i = 1, 2, 3 \end{cases} \tag{7.2}$$

where $a = \sum_{i=1}^{3} a_i e_i$ and $G(x) = \langle a, x \rangle - d := \sum_{j=1}^{3} a_j x_j - d$. Hence

$$\Omega_- = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} a_i x_i < d \right\},$$
$$\Omega_+ = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} a_i x_i > d \right\}.$$

The following result is proved in [4].

Proposition 7.2 *If conditions (i)–(ii) and the next ones*

$$\begin{aligned} & a_3 b_3 (\mathrm{e}^{2\lambda_3 \bar{T}} - 1) = d\lambda_3 \\ & \sum_{j=1}^{2} \frac{a_j b_j}{\lambda_j} (\mathrm{e}^{-2\lambda_j \bar{T}} - 1) = d \end{aligned} \tag{7.3}$$

hold, then system

$$\dot{x}_i = \begin{cases} \lambda_i x_i & \text{for} \quad \langle a, x\rangle < d, \\ \lambda_i x_i + b_i & \text{for} \quad \langle a, x\rangle > d, \\ i = 1, 2, 3 \end{cases} \tag{7.4}$$

has a homoclinic orbit to $x = 0$*:*

$$\gamma(t) = \begin{cases} \gamma_-(t) & \text{if } t \leq -\bar{T} \\ \gamma_0(t) & \text{if } -T \leq t \leq \bar{T} \\ \gamma_+(t) & \text{if } t \geq \bar{T} \end{cases}$$

where

$$\begin{aligned} \gamma_-(t) &= \left(e^{\lambda_1(t+\bar{T})} \left(e^{-2\lambda_1 \bar{T}} - 1 \right) \frac{b_1}{\lambda_1},\ e^{\lambda_2(t+\bar{T})} \left(e^{-2\lambda_2 \bar{T}} - 1 \right) \frac{b_2}{\lambda_2}, 0 \right), \\ \gamma_0(t) &= \left(\left(e^{\lambda_1(t-\bar{T})} - 1 \right) \frac{b_1}{\lambda_1}, \left(e^{\lambda_2(t-\bar{T})} - 1 \right) \frac{b_2}{\lambda_2}, \left(e^{\lambda_3(t+\bar{T})} - 1 \right) \frac{b_3}{\lambda_3} \right), \\ \gamma_+(t) &= \left(0, 0, \frac{d}{a_3} e^{\lambda_3(t-\bar{T})} \right), \end{aligned}$$

and conditions (H1)*,* (H2) *and* (H3) *are satisfied.*

Moreover, we have [4]

$$\mathcal{S}' = \text{span}\{(a_2, -a_1, 0)\},$$

$$X_0(t) = X_-(t) = \begin{pmatrix} e^{\lambda_1(t+\bar{T})} & 0 & 0 \\ 0 & e^{\lambda_2(t+\bar{T})} & 0 \\ 0 & 0 & e^{\lambda_3(t+\bar{T})} \end{pmatrix}, \tag{7.5}$$

$$X_+(t) = \begin{pmatrix} e^{\lambda_1(t-\bar{T})} & 0 & 0 \\ 0 & e^{\lambda_2(t-\bar{T})} & 0 \\ 0 & 0 & e^{\lambda_3(t-\bar{T})} \end{pmatrix}, \tag{7.6}$$

$$X_0(T)\mathcal{S}' = \text{span}\{w_0\}, \quad \text{with} \quad w_0 := \begin{pmatrix} a_2 e^{2\lambda_1 \bar{T}} \\ -a_1 e^{2\lambda_2 \bar{T}} \\ 0 \end{pmatrix}, \tag{7.7}$$

$$\begin{aligned} R_0 w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_0(\bar{T})\rangle} \dot{\gamma}_0(\bar{T}), \\ R_+ w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_+(\bar{T})\rangle} \dot{\gamma}_+(\bar{T}), \\ R_- w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_-(-\bar{T})\rangle} \dot{\gamma}_-(-\bar{T}). \end{aligned} \tag{7.8}$$

Using the above formulas we obtain (see [4])

Proposition 7.3 *Let assumptions (i)–(ii) hold and suppose* (7.3) *is satisfied. Then the function* $\psi(t)$ *of* (5.11) *for the system* (7.4) *reads*

$$\psi(t) = \begin{cases} e^{-\lambda_3(t+\bar{T})} \langle A_3, a \wedge R_0 w_0\rangle e_3 & \text{if } t \leq -\bar{T} \\ \frac{|a|^2}{\langle a, \dot{\gamma}_0(\bar{T})\rangle} X_0(-t)[\dot{\gamma}_0(\bar{T}) \wedge R_0 w_0] & \text{if } -\bar{T} < t \leq \bar{T} \\ \frac{|a|^2}{a_3} X_+^{-1}(t)[e_3 \wedge R_0 w_0] & \text{if } t > \bar{T} \end{cases} \tag{7.9}$$

where $X_0(t)$, $X_+(t)$, w_0, R_0 *are given by* (7.5), (7.6), (7.7), (7.8), *respectively, and*

$$A_3 = R_0 X_0(\bar{T}) R_- e_3. \tag{7.10}$$

The computation of the vector A_3 is really messy even in an example as simple as this, so we don't proceed further with its computation now, but will do it later when we fix some particular values of the parameters.

So we are in position to apply Theorem 5.1. Writing g_j for the vector $(g_{j1}\ g_{j2}\ g_{j3})$, we get the Melnikov function (5.12)

$$\begin{aligned}\mathcal{M}(\alpha) &= \int_{-\infty}^{\infty} \left[\sin\omega_1(t+\alpha)\psi^*(t)g_1 + \sin\omega_2(t+\alpha)\psi^*(t)g_2\right] dt \\ &= \sin(\alpha\omega_1)\int_{-\infty}^{\infty} \cos(\omega_1 t)\psi^*(t)g_1\,dt + \cos(\alpha\omega_1)\int_{-\infty}^{\infty} \sin(\omega_1 t)\psi^*(t)g_1\,dt \\ &\quad + \sin(\alpha\omega_2)\int_{-\infty}^{\infty} \cos(\omega_2 t)\psi^*(t)g_2\,dt + \cos(\alpha\omega_2)\int_{-\infty}^{\infty} \sin(\omega_2 t)\psi^*(t)g_2\,dt \\ &= A_1(\omega_1)\sin(\omega_1\alpha + \varpi_1(\omega_1)) + A_2(\omega_2)\sin(\omega_2\alpha + \varpi_2(\omega_2))\end{aligned} \tag{7.11}$$

where

$$A_i(\omega_i) := \sqrt{\left(\int_{-\infty}^{\infty} \cos\omega_i t\psi^*(t)g_i\,dt\right)^2 + \left(\int_{-\infty}^{\infty} \sin\omega_i t\psi^*(t)g_i\,dt\right)^2}$$

for $i = 1, 2$. Now we consider the following two possibilities:

1. Either $A_1(\omega_1) \neq 0$, $A_2(\omega_2) = 0$ or $A_1(\omega_1) = 0$, $A_2(\omega_2) \neq 0$. Then $\mathcal{M}(\alpha)$ has the simple zero $\alpha_0 = -\varpi_i(\omega_i)/\omega_i$, $i = 1, 2$, respectively.
2. $A_1(\omega_1) \neq 0$ and $A_2(\omega_2) \neq 0$. Let $s_i := \operatorname{sgn} A_i(\omega_i) \in \{-1, 1\}, i = 1, 2$. Then $s_1\omega_1 A_1(\omega_1) + s_2\omega_2 A_2(\omega_2) = \omega_1|A_1(\omega_1)| + \omega_2|A_2(\omega_2)| > 0$. Since $\cos\frac{1-s_i}{2}\pi = s_i$ and $\sin\frac{1-s_i}{2}\pi = 0$ for $i = 1, 2$ and ω_1/ω_2 is irrational, from [18] the existence follows of α_0 (as a matter of fact infinitely many α_0 exists) such that $\omega_i\alpha_0 + \varpi_i(\omega_i)$ are near to $\frac{1-s_i}{2}\pi$ modulo 2π, $i = 1, 2$, and $\mathcal{M}(\alpha_0) = 0$ while $\mathcal{M}'(\alpha_0) \geq \frac{s_1\omega_1 A_1(\omega_1) + s_2\omega_2 A_2(\omega_2)}{2} > 0$. Hence also in this case we have a simple zero of $\mathcal{M}(\alpha)$.

Consequently if $A_1(\omega_1)$ and $A_1(\omega_1)$ do not vanish simultaneously, Theorem 6.4 applies and we conclude that (7.1) behaves chaotically for any $\varepsilon \neq 0$ sufficiently small. Next, we note that $A_i(\omega_i) \neq 0$ if and only if

$$\Phi_i(\omega_i) := \int_{-\infty}^{\infty} \mathrm{e}^{-\omega_i t \imath}\psi^*(t)g_i\,dt \neq 0. \tag{7.12}$$

We know that $\Phi_i(\omega)$ are analytic. Consequently, when functions $\Phi_i(\omega)$ are not identically zero, they have at most countable many positive zeroes with possible accumulations at $+\infty$ [24]. Summarizing, we get the following result.

Theorem 7.4 *Let assumptions* (i)–(iii) *hold and suppose* (7.3) *holds. When* $\Phi_1(\omega)$ *and* $\Phi_2(\omega)$ *are not both identically zero, then there is an at most countable set* $\{\widetilde{\omega}_j\} \subset (0, \infty)$ *with possible accumulating point at* $+\infty$ *such that if* $\omega_1, \omega_2 \in (0, \infty) \setminus \{\widetilde{\omega}_j\}$ *then system* (7.1) *is chaotic for any* $\varepsilon \neq 0$ *sufficiently small.*

Since in general, the above formulas are rather difficult, now we consider the following concrete example still following [4].

Example 7.5 We take

$$\begin{aligned} a_1 = 0, \quad a_2 = a_3 = 1, \quad \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1 \\ b_1 = b_2 = -1, \quad b_3 = 1, \quad d = 3/4. \end{aligned} \tag{7.13}$$

Then we have (see [4]) $\bar{T} = \ln 2$ and :

$$\Phi_i(\omega) = -\frac{256 \sin(\omega \ln 2)}{3(\omega^2+1)} \left(\omega(g_{i2}+g_{i3}) + \imath(g_{i2}-g_{i3})\right). \tag{7.14}$$

So for the parameters (7.13), $\Phi_i(\omega)$ is identically zero if and only if $g_{i2} = g_{i3} = 0$. Otherwise, it has only the simple positive zeroes $\widetilde{\omega}_j = \pi j/\ln 2$, $j \in \mathbb{N}$. As a consequence of Theorem 7.4 we get the following

Corollary 7.6 *Consider* (7.1) *with parameters* (7.13) *and iii) holds. If either* $g_{i2} \neq 0$ *or* $g_{i3} \neq 0$ *for some* $i \in \{1, 2\}$ *and* $\omega_1, \omega_2 \neq \pi j/\ln 2, \forall j \in \mathbb{N}$ *then system* (7.1) *is chaotic for any* $\varepsilon \neq 0$ *small.*

Example 7.7 On the other hands, for the following set of parameters

$$\begin{aligned} a_1 = a_2 = a_3 = 1, \quad b_1 = b_2 = -1, \quad b_3 = 13/8, \\ \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1, \quad d = 39/32, \end{aligned} \tag{7.15}$$

we get (see [4]) $T = \ln 2$ and

$$\Phi_i(\omega) = \frac{2^{6-\imath\omega}(13 \cdot 4^{\imath\omega} - 10)(g_{i1} + 2g_{i2} + g_{i1}\omega^2 + g_{i2}\omega^2 - 2g_{i3} + \omega^2 g_{i3} - \imath(g_{i2} + 3g_{i3})\omega)}{17(\omega - \imath)(\omega - 2\imath)(1 - \imath\omega)}$$

for $i = 1, 2$. Clearly, for the parameters (7.15), $\Phi_i(\omega)$ is not identically zero. If $g_{i2} \neq -3g_{i3}$ then $\Phi_i(\omega)$ has no positive roots. If $g_{i2} = -3g_{i3}$ then $\Phi_i(\omega)$ has the only positive root $\omega_{i1} = \sqrt{\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}}}$ provided $\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}} > 0$. As a consequence of Theorem 7.4 we obtain the following

Corollary 7.8 *Consider* (7.1) *with parameters* (7.15) *and iii) holds. If one of the following conditions is satisfied*

- $g_{i2} \neq -3g_{i3}$,
- $g_{i2} = -3g_{i3}$, $g_{i1} = 2g_{i3} \neq 0$,
- $g_{i2} = -3g_{i3}$, $g_{i1} \neq 2g_{i3}$, $\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}} < 0$,
- $g_{i2} = -3g_{i3}$, $g_{i1} \neq 2g_{i3}$, $\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}} > 0$ *and* $\omega_i \neq \sqrt{\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}}}$,

for some $i \in \{1, 2\}$ *then system* (7.1) *is chaotic for any* $\varepsilon \neq 0$ *small.*

Remark 7.9 Parameters (7.13) and (7.15) give Examples 7.5 and 7.7 for which $\Phi_i(\omega)$ is either identically zero, or it has infinitely many positive roots, or it has no positive roots, or it has finitely many positive roots.

Remark 7.10 If $\Phi_1(\omega_1) = 0$ and $\Phi_2(\omega_2) = 0$ then $\mathcal{M}(\alpha)$ is identically zero then a second-order Melnikov function must be derived as in [3]. But those computations should be very awkward for (7.1), so we omit them.

Finally when $g_1 \neq 0$, $g_2 \neq 0$ and ω_1/ω_2 is rational, then we get different situation. For instance, consider Example 7.5 with $\omega_1 = 1, \omega_2 = 3$ and $g_{i2} = g_{i3}, i = 1, 2$. Thus (7.1) is 2π-periodic and

$$\mathcal{M}(\alpha) = \Phi_1(1)\sin\alpha + \Phi_2(3)\sin 3\alpha = \sin\alpha - \frac{1}{3}\sin 3\alpha = \frac{4}{3}\sin^3\alpha$$

provided $\Phi_1(1) = 1$ and $\Phi_2(3) = -\frac{1}{3}$. From (7.14) we derive

$$\begin{aligned} g_{12} + g_{13} &= -\frac{3}{128\sin(\ln 2)}, \\ g_{22} + g_{23} &= \frac{5}{384\sin(3\ln 2)}. \end{aligned}$$

Then the Melnikov function is $\mathcal{M}(\alpha) = \frac{4}{3}\sin^3\alpha$ and it has only the zero $\alpha_0 = 0$ in $[-\pi, \pi]$ which is nonsimple but it has Brouwer index 1. So Theorem 6.4 is not applicable, but we still could get a chaos for (7.1) with $\varepsilon \neq 0$ small as in [2].

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Appendix A

Proof of the Properties (G_1)–(G_5)

Here we prove the properties (G_1)–(G_5) of the maps $\mathcal{G}_{\mathcal{T}}(\theta, \alpha, \varepsilon)$, $\mathcal{L}_{\alpha,m}$ and $\mathcal{H}_\alpha$. First we prove G_1). From (3.16), (3.19), (4.1), and $G(\gamma_\pm(\pm\bar{T})) = 0$, $\gamma_\pm(\pm\bar{T}) = \gamma_0(\pm\bar{T})$, we get

$$\mathcal{G}_{\mathcal{T}}(\theta_\alpha, \alpha, 0) = \left\{\begin{matrix} \gamma_+(T_{2m+1} - T_{2m}) - \gamma_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\}_{m\in\mathbb{Z}}.$$

Now, for $t \geq T$ we have

$$|\gamma_+(t)| \leq \int_t^\infty |\dot{\gamma}_+(s)|ds \leq \int_t^\infty k\,\mathrm{e}^{-\delta(s-\bar{T})}|\dot{\gamma}_+(\bar{T})|ds = k\delta^{-1}\,\mathrm{e}^{-\delta(t-\bar{T})}|\dot{\gamma}_+(\bar{T})|$$

and similarly

$$|\gamma_-(t)| \leq k\delta^{-1}\,\mathrm{e}^{\delta(t+\bar{T})}|\dot{\gamma}_-(-\bar{T})|$$

for any $t \le -\bar{T}$. Thus

$$\begin{aligned}&|\gamma_+(T_{2m+1}-T_{2m})-\gamma_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})| \le \\ &k\delta^{-1}\,\mathrm{e}^{-\delta(T_{2m+1}-T_{2m}-\bar{T})}|\dot{\gamma}_+(\bar{T})|+k\delta^{-1}\,\mathrm{e}^{\delta(T_{2m+1}-T_{2m+2}+\bar{T}+1)}|\dot{\gamma}_-(-\bar{T})| \le \\ &2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\max\{|\dot{\gamma}_-(-\bar{T})|,|\dot{\gamma}_+(\bar{T})|\}\end{aligned}$$

from which (G_1) easily follows. Similarly we get:

$$\frac{d}{d\alpha}[\mathcal{G}_{\mathcal{T}}(\theta_\alpha,\alpha,0)]\widetilde{\alpha} = \left\{\begin{matrix}\dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})(\widetilde{\alpha}_{m+1}-\widetilde{\alpha}_m)\\0\\0\\0\\0\\0\end{matrix}\right\}_{m\in\mathbb{Z}}$$

that proves (G_2). Next, from Propositions 3.6, 3.8, 4.1, the equality $R_0\dot{\gamma}_0(\bar{T})=0$ and the identities

$$\begin{aligned}P_-X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^- &= X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-,\\ (\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+ &= X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\end{aligned} \tag{A.1}$$

(that follow from $\varphi_m^-\in\mathcal{R}P_{-,m}$, $\varphi_m^+\in\mathcal{N}P_{+,m}$), we see that the derivative $D_1\mathcal{G}_{\mathcal{T}}$ of $\mathcal{G}_{\mathcal{T}}$ with respect to $\theta\in\ell^\infty_{\rho,\alpha,\varepsilon}$ at the point $(\theta_\alpha,\alpha,0)$ is given by

$$D_1\mathcal{G}_{\mathcal{T}}(\theta_\alpha,\alpha,0)\theta = \left\{\begin{matrix}\mathcal{L}_\alpha(\varphi_{m+1}^-,\varphi_m^+,\xi_{m+1}^-,\xi_m^+,\bar{\xi}_m,\beta_m)\\ \bar{\xi}_m-\xi_m^--X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-\\ R_0[X_0(\bar{T})\bar{\xi}_m-\xi_m^+-X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+]\\ G'(\gamma_0(-\bar{T}))[\xi_m^-+X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-]\\ G'(\gamma_0(\bar{T}))\cdot[X_0(\bar{T})\bar{\xi}_m+\dot{\gamma}_0(\bar{T})\beta_m]\\ G'(\gamma_+(\bar{T}))\cdot[\xi_m^++X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+]\end{matrix}\right\}_{m\in\mathbb{Z}}$$

where $\theta=\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar{\xi}_m,\beta_m)\}_{m\in\mathbb{Z}}$ and

$$\begin{aligned}&\mathcal{L}_\alpha(\varphi_{m+1}^-,\varphi_m^+,\xi_{m+1}^-,\xi_m^+,\bar{\xi}_m,\beta_m)=X_+(T_{2m+1}-T_{2m})\xi_m^+\\ &\quad -X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\xi_{m+1}^- -\dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\beta_m\\ &\quad +X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\\ &\quad -X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi_{m+1}^-.\end{aligned}$$

Then, using again (A.1) we obtain:

$$\begin{aligned}\left|X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\right| &\le k\,\mathrm{e}^{-\delta(T_{2m+1}-T_{2m}-\bar{T}+1)}|\varphi_m^+| \le k\,\mathrm{e}^{-\delta(T-\bar{T}+2)}|\varphi_m^+|\\ \left|X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-\right| &\le k\,\mathrm{e}^{-\delta(T_{2m}-T_{2m-1}+1-\bar{T})}|\varphi_m^-| \le k\,\mathrm{e}^{-\delta(T-\bar{T}+2)}|\varphi_m^-|.\end{aligned} \tag{A.2}$$

Moreover,

$$|X_+(T_{2m+1}-T_{2m})\xi_m^+| = |X_+(T_{2m+1}-T_{2m})P_+X_+^{-1}(\bar{T})\xi_m^+| \le k\,\mathrm{e}^{-\delta(T-\bar{T}+1)}|\xi_m^+| \tag{A.3}$$

and, since $|\alpha_m - \alpha_{m+1}| < 1$ implies that $T_{2m+2} - T_{2m+1} - \alpha_m + \alpha_{m+1} \geq T > \bar{T}$:

$$\begin{aligned}&|X_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\xi^-_{m+1}|\\ &\quad = |X_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})(\mathbb{I} - P_-)X_-^{-1}(-\bar{T})\xi^-_{m+1}|\\ &\quad \leq k\,\mathrm{e}^{-\delta(T-\bar{T})}|\xi^-_{m+1}|,\\ &\quad |\dot{\gamma}_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})| \leq k\,\mathrm{e}^{-\delta(T-\bar{T})}|\dot{\gamma}_-(-\bar{T})|\end{aligned} \tag{A.4}$$

for any $m \in \mathbb{Z}$. Next,

$$\begin{aligned}&X_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})P_-X_-^{-1}(T_{2m+1} - T_{2m+2} - 1)\varphi^-_{m+1}\\ &\quad \in \mathcal{R}P_-(T_{2m+2} - T_{2m+1} - \alpha_m + \alpha_{m+1}),\\ &\quad X_+(T_{2m+1} - T_{2m})(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi^+_m\\ &\quad \in \mathcal{N}P_+(T_{2m+1} - T_{2m}),\end{aligned}$$

and (see (2.9))

$$\mathcal{N}P_+(T_{2m+1} - T_{2m}) \oplus \mathcal{R}P_-(T_{2m+2} - T_{2m+1} - \alpha_m + \alpha_{m+1}) = \mathbb{R}^n.$$

Hence the linear map

$$\begin{aligned}\mathcal{L}_{\alpha,m} : (\varphi^-_{m+1}, \varphi^+_m) \mapsto\ & X_+(T_{2m+1} - T_{2m})(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi^+_m\\ & -X_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})P_-X_-^{-1}(T_{2m+1} - T_{2m+2} - 1)\varphi^-_{m+1}\end{aligned}$$

is a linear isomorphism from $\mathcal{R}P_{-,m+1} \oplus \mathcal{N}P_{+,m} = \mathbb{R}^n$ into $\mathcal{N}P_+(T_{2m+1} - T_{2m}) \oplus \mathcal{R}P_-(T_{2m+2} - T_{2m+1} - \alpha_m + \alpha_{m+1}) = \mathbb{R}^n$ whose inverse is given by:

$$\begin{aligned}\mathcal{L}^{-1}_{\alpha,m} : (\tilde{\varphi}^-_{m+1}, \tilde{\varphi}^+_m) \mapsto\ & X_+(T_{2m+1} - T_{2m} + 1)(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m})\tilde{\varphi}^+_m\\ & -X_-(T_{2m+1} - T_{2m+2} - 1)P_-X_-^{-1}(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\tilde{\varphi}^-_{m+1}.\end{aligned}$$

Then we note that (see (2.3)):

$$\begin{aligned}&|X_-(T_{2m+1} - T_{2m+2} - 1)P_-X_-^{-1}(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\tilde{\varphi}^-_{m+1}|\\ &\quad \leq k\,\mathrm{e}^{\delta(1+\alpha_m-\alpha_{m+1})}|\tilde{\varphi}^-_{m+1}| \leq k\,\mathrm{e}^{\delta(1+\chi)}|\tilde{\varphi}^-_{m+1}|;\\ &|X_+(T_{2m+1} - T_{2m} + 1)(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m})\tilde{\varphi}^-_{m+1}| \leq k\,\mathrm{e}^{\delta}|\tilde{\varphi}^-_{m+1}|\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\alpha}\mathcal{L}_{\alpha,m}(\varphi^-_{m+1}, \varphi^+_m)\alpha =\ & -f'_-(\gamma_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1}))\cdot\\ & X_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})P_-X_-^{-1}(T_{2m+1} - T_{2m+2} - 1)\varphi^-_{m+1}(\alpha_m - \alpha_{m+1}).\end{aligned}$$

Thus we obtain (see also (2.10)):

$$\begin{aligned}&|\mathcal{L}_{\alpha,m}(\varphi^-_{m+1}, \varphi^+_m)| \leq k\,\mathrm{e}^{-\delta}|\varphi^+_m| + k\,\mathrm{e}^{-\delta(1-\chi)}|\varphi^-_{m+1}| \leq k\widetilde{c}|\varphi^+_m + \varphi^-_{m+1}|\\ &|\mathcal{L}^{-1}_{\alpha,m}(\widetilde{\varphi}^-_{m+1}, \varphi^+_m)| \leq k\,\mathrm{e}^{\delta}|\varphi^+_m| + k\,\mathrm{e}^{\delta(1+\chi)}|\widetilde{\varphi}^-_{m+1}| \leq k\widetilde{c}\,\mathrm{e}^{2\delta}|\varphi^+_m + \varphi^-_{m+1}|\\ &|\frac{\partial}{\partial\alpha}\mathcal{L}_{\alpha,m}(\varphi^-_{m+1}, \varphi^+_m)| \leq 2N_-k|\varphi^-_{m+1}|\end{aligned}$$

for $N_- := \sup_{x\in\mathbb{R}^n}|f_-(x)|$. So, using also $\frac{\partial}{\partial\alpha}\mathcal{L}^{-1}_{\alpha,m} = \mathcal{L}^{-1}_{\alpha,m} \circ \frac{\partial}{\partial\alpha}\mathcal{L}_{\alpha,m} \circ \mathcal{L}^{-1}_{\alpha,m}$ we see that (G_3) holds. Next, using (A.3), (A.4):

$$\begin{aligned}&\left|\mathcal{L}_\alpha(\varphi^-_{m+1}, \varphi^+_m, \xi^-_{m+1}, \xi^+_m, \bar{\xi}_m, \beta_m) - \mathcal{L}_{\alpha,m}(\varphi^-_{m+1}, \varphi^+_m)\right|\\ &\quad \leq k\,\mathrm{e}^{-\delta(T-\bar{T})}(2 + |\dot{\gamma}_-(-\bar{T})|)\|\theta\|\end{aligned} \tag{A.5}$$

(recall $\theta = \{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\}_{m\in\mathbb{Z}}$). Furthermore it is easily seen that

$$\frac{\partial}{\partial\alpha}\mathcal{H}_\alpha\theta = \begin{Bmatrix} \frac{\partial}{\partial\alpha}\mathcal{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{m\in\mathbb{Z}}$$

and so (G_4) readily follows from (G_3). Finally, (G_5) follows from

$$\begin{aligned} &[D_1\mathcal{G}_\mathcal{T}(\theta_\alpha, \alpha, 0) - \mathcal{H}_\alpha]\theta \\ &= \begin{Bmatrix} \mathcal{L}_\alpha(\varphi_{m+1}^-, \varphi_m^+, \xi_{m+1}^-, \xi_m^+, \bar{\xi}_m, \beta_m) - \mathcal{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+) \\ -X_-^{-1}(T_{2m-1} - T_{2m} - 1)\varphi_m^- \\ -R_0X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi_m^+ \\ G'(\gamma_0(-\bar{T}))X_-^{-1}(T_{2m-1} - T_{2m} - 1)\varphi_m^- \\ 0 \\ G'(\gamma_+(\bar{T}))X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi_m^+ \end{Bmatrix}_{m\in\mathbb{Z}}, \end{aligned} \tag{A.6}$$

(A.2) and (A.5) with

$$\widetilde{c}_3 := \max\left\{2 + |\dot{\gamma}_-(-\bar{T})|, \|R_0\|\,\mathrm{e}^{-2\delta}, |G'(\gamma_+(\bar{T}))|\,\mathrm{e}^{-2\delta}, |G'(\gamma_0(-\bar{T}))|\,\mathrm{e}^{-2\delta}\right\}.$$

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