Universality of Crystallographic Pinning

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Abstract   We study traveling waves for reaction diffusion equations on the spatially discrete domain $\mathbb{Z}^2$. The phenomenon of crystallographic pinning occurs when traveling waves become pinned in certain directions despite moving with non-zero wave speed in nearby directions. In [19] it was shown that crystallographic pinning occurs for all rational directions, so long as the nonlinearity is close to the sawtooth, which itself was considered in [6]. In this paper we show that crystallographic pinning holds in the horizontal and vertical directions for bistable nonlinearities which satisfy a specific computable generic condition. The proof is based on dynamical systems. In particular, it relies on an examination of the heteroclinic chains which occur as singular limits of wave profiles on the boundary of the pinning region.

Keywords   Lattice differential equation · Traveling wave · Pinning · Heteroclinic orbit · Bogdanov–Takens singularity

1 Introduction

The setting for this paper is traveling waves for lattice differential equations of reaction diffusion type. A lattice differential equation (LDE) is an infinite system of coupled ordinary differential equations, where each ODE represents the dynamics at a single point on a spatial lattice. A simple LDE is

Dedicated to Professor Jack K. Hale on the occasion of his 80th birthday.

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$$\dot{u}_i = d(u_{i+1} + u_{i-1} - 2u_i) - f(u_i) \tag{1.1}$$

where $i \in \mathbb{Z}$ is a spatial index, $d \in \mathbb{R}$ is the coupling constant, $f : \mathbb{R} \to \mathbb{R}$ is a given function, and each u_i is a function of a single variable t. If $d > 0$ then Eq. 1.1 may be regarded as a reaction-diffusion equation: The first term $d(u_{i+1} + u_{i-1} - 2u_i)$ is a discrete second derivative which provides the diffusion, while the second term $f(u_i)$ is the reaction term.

If we denote $d = \frac{1}{h^2}$, then Eq. 1.1 is obtained from the PDE

$$u_t = u_{xx} - f(u) \tag{1.2}$$

upon replacing the term u_{xx} by a standard central difference approximation with grid size h. The limit $h \to 0$, that is $d \to \infty$ in Eq. 1.1, corresponds at least formally to Eq. 1.2. In this study we are interested in spatially discrete systems such as (1.1) which are not necessarily close to the PDE limit in this sense, namely d need not be large, although we do assume $d > 0$. In fact, without loss we may take $d = 1$ by rescaling time and redefining f. This normalization serves to emphasize that the grid is not particularly small and is not meant to approximate a continuum. Thus we study the equation

$$\dot{u}_i = u_{i+1} + u_{i-1} - 2u_i - f(u_i) \tag{1.3}$$

and its higher-dimensional analogs in what follows.

Lattice models are widely used in applications such as solid state physics, materials science, and physiology; see [4,12,14,21,26] and the references therein.

We assume the nonlinearity f is of *bistable* type, in particular that

$$
\begin{aligned}
& f(\pm 1) = 0, \quad f(a) = 0, \quad f'(\pm 1) > 0, \quad f'(a) < 0, \\
& f(u) > 0 \text{ for } u \in (-1, a) \cup (1, \infty), \\
& f(u) < 0 \text{ for } u \in (-\infty, -1) \cup (a, 1),
\end{aligned}
\tag{1.4}
$$

for some $a \in (-1, 1)$. Generally, only the values of u for $|u| \leq 1$ will be relevant for our arguments, but in several places (in particular some arguments by contradiction) it will be convenient to assume that $f(u)$ is smoothly extended for $|u| > 1$, with the sign condition as in (1.4). We in fact take a family of bistable functions $f = f(u, a)$ parameterized by $a \in (-1, 1)$, in addition satisfying the monotonicity condition

$$\frac{\partial f}{\partial a}(u, a) > 0 \text{ for } u \in (-1, 1) \text{ and } a \in (-1, 1) \tag{1.5}$$

in a. To be precise, let us define a set $\mathcal{N}$ of functions by

$$
\begin{aligned}
\mathcal{N} = \{ f : [-1, 1] \times (-1, 1) \to \mathbb{R} \mid & f(\cdot, \cdot) \text{ is } C^2 \text{ smooth, with } f(\cdot, a) \\
& \text{satisfying (1.4) for every } (u, a) \in [-1, 1] \times (-1, 1), \text{ and (1.5) holding}\}.
\end{aligned}
$$

We say that $f : [-1, 1] \times (-1, 1) \to \mathbb{R}$ is a *normal family* if $f \in \mathcal{N}$. We do not endow the set $\mathcal{N}$ with a topology, although we shall do so with certain subsets of $\mathcal{N}$. The parameter $a \in (-1, 1)$ is known as the *detuning parameter*. The function

$$f(u, a) = (u^2 - 1)(u - a)$$

furnishes a simple example of a normal family.

The bistable reaction term f in Eq. 1.3 pushes the system toward spatial heterogeneity; it forces u_i toward $+1$ when $u_i > a$ and toward -1 when $u_i < a$. By contrast, the diffusion term $u_{i+1} + u_{i-1} - 2u_i$ promotes spatial homogeneity, forcing u_i toward the average

$\frac{1}{2}(u_{i+1} + u_{i-1})$ of its neighbors. This competition between reaction and diffusion, spatial heterogeneity and spatial homogeneity, is what gives reaction diffusion equations of bistable type the richness to support both spatially chaotic patterns [9] and traveling waves [19].

By a traveling wave solution of Eq. 1.3 we mean a solution of the form

$$u_i(t) = \phi(i - ct), \qquad i \in \mathbb{Z}, \tag{1.6}$$

for some function $\phi : \mathbb{R} \to \mathbb{R}$, the so-called *wave profile*, and some $c \in \mathbb{R}$, the *wave speed*. In this paper we consider monotone traveling waves which connect the spatially homogeneous equilibria at ± 1, that is, ϕ satisfies the boundary conditions

$$\phi(-\infty) = -1, \qquad \phi(\infty) = 1, \tag{1.7}$$

and $\phi(\xi)$ is monotone in ξ.

Substitution of (1.6) into (1.3) shows that the function ϕ must satisfy the *wave profile equation*

$$-c\phi'(\xi) = \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi) - f(\phi(\xi), a) \tag{1.8}$$

for $\xi \in \mathbb{R}$ if $c \neq 0$, or for $\xi \in \mathbb{Z}$ if $c = 0$. Conversely, any solution of (1.8) generates a traveling wave solution of (1.3). Note that Eq. 1.8 is a differential-difference equation if $c \neq 0$, and a difference equation if $c = 0$. This difference in character of the wave profile equation between $c = 0$ and $c \neq 0$ is one of the chief reasons that the LDE (1.3) exhibits behavior which is not present in the PDE (1.2).

In the PDE case (1.2) with $x \in \mathbb{R}$, traveling wave solutions have the form $u(t, x) = \phi(x - ct)$ where ϕ satisfies the ordinary differential equation

$$-c\phi' = \phi'' - f(\phi, a). \tag{1.9}$$

For both the LDE (1.3) and PDE (1.2) problems above, it is known that there is a unique wavespeed c at which a monotone traveling wave exists, that is, a monotone solution to either Eqs. 1.8 or 1.9, which satisfies the boundary conditions (1.7). Of course the wave speed $c = c(a)$ depends on the parameter $a \in (-1, 1)$ as does the wave profile $\phi(\xi) = \phi(\xi; a)$. Moreover, the wave speed is continuous and nondecreasing in a. In the PDE case $c(a)$ is strictly increasing, in fact $c'(a) > 0$ for all $a \in (-1, 1)$. The LDE case differs from the PDE case in that the wave speed may be zero for an open set of a. In particular, there are quantities

$$-1 \leq a_- \leq a_+ \leq 1$$

such that

$$c(a) = 0 \text{ for } a \in [a_-, a_+] \cap (-1, 1),$$
$$c(a) > 0 \text{ for } a \in (a_+, 1),$$
$$c(a) < 0 \text{ for } a \in (-1, a_-),$$

with $c(a)$ depending smoothly on a and $c'(a) > 0$, both whenever $c(a) \neq 0$. In case a strict inequality $a_- < a_+$ holds, we say that *pinning* or *propagation failure* occurs; the wave is pinned and cannot propagate when a is between these values. The interval (a_-, a_+) is called the *pinning interval* and its length measures the severity of the pinning. Pinning was observed by Bell [4], Bell and Cosner [5], and Keener [18] and has been studied extensively (see for example [3,7,13,22]).

To see why pinning can occur in the LDE case, observe that when $c = 0$ Eq. 1.8 reduces to a pure difference equation

$$0 = \phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi) - f(\phi(\xi), a), \qquad \xi \in \mathbb{Z},$$

which is equivalent to the discrete-time dynamical system

$$q_{n+1} = 2q_n - r_n + f(q_n, a),$$
$$r_{n+1} = q_n$$

under the transformation $r_n = \phi(n)$ and $q_n = \phi(n + 1)$. A solution of (1.8) satisfying the boundary conditions (1.7) thus corresponds to a heteroclinic connection between the equilibria $(-1, -1) \in \mathbb{R}^2$ and $(1, 1) \in \mathbb{R}^2$ in this dynamical system. As both these equilibria are saddles, such a heteroclinic connection may lie on a transverse intersection of their stable and unstable manifolds, in which case this connection will persist as a varies in some interval, with $c(a) = 0$ throughout this interval. The maximal interval on which the connection persists is thus $[a_-, a_+]$.

By contrast, pinning does not occur for the PDE (1.2) because here the traveling wave corresponds to a saddle-saddle connection in the continuous time planar dynamical system (1.9). Such heteroclinic solutions do not generally persist under perturbations, and in fact $c(a) = 0$ for a unique value of a.

Our interest is in propagation failure for LDE's as it occurs in higher-dimensional lattices such as $\mathbb{Z}^k \subseteq \mathbb{R}^k$. The phenomena here are more subtle because all of the quantities mentioned thus far—the wave speed c, the wave profile ϕ, and the pinning interval (a_-, a_+)—depend now on the direction of propagation. We take the lattice $\mathbb{Z}^2$, for which the analog of Eq. 1.3 is

$$\dot{u}_{i,j} = (\Delta u)_{i,j} - f(u_{i,j}, a), \tag{1.10}$$

with the nonlinearity f as before. Here the coordinate (i, j) indexes a spatial point in $\mathbb{Z}^2$, each $u_{i,j}$ is a function of t as before, and the discrete laplacian Δ is given by

$$(\Delta u)_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}. \tag{1.11}$$

A traveling wave solution of (1.10) is a solution of the form

$$u_{i,j}(t) = \phi(i\kappa + j\sigma - ct)$$

for some $(\kappa, \sigma) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, termed the *direction vector*, and some function $\phi : \mathbb{R} \to \mathbb{R}$. The wave profile equation satisfied by ϕ now takes the form

$$-c\phi'(\xi) = \phi(\xi + \sigma) + \phi(\xi - \sigma) + \phi(\xi + \kappa) + \phi(\xi - \kappa) - 4\phi(\xi) - f(\phi(\xi), a). \tag{1.12}$$

It is known that for each direction vector $(\kappa, \sigma) \in \mathbb{R}^2 \setminus \{0\}$ and each $a \in (-1, 1)$, there is a unique wave speed $c = c(a, (\kappa, \sigma))$ such that Eq. 1.12 admits a monotone solution satisfying the boundary conditions (1.7). Moreover, when $c \neq 0$ this solution $\phi = \phi(\xi; a, (\kappa, \sigma))$ is unique up to translation [20]. The wave speed $c(a, (\kappa, \sigma))$ depends continuously on a and on (κ, σ). For each (κ, σ) it is nondecreasing in a, and as before is smooth in a and satisfies $\frac{\partial c(a, (\kappa, \sigma))}{\partial a} > 0$ when $c(a, (\kappa, \sigma)) \neq 0$. Also as before, we have quantities $a_\pm = a_\pm(\kappa, \sigma)$ characterized by

$$[a_-(\kappa, \sigma), \ a_+(\kappa, \sigma)] \cap (-1, 1) = \{a \in (-1, 1) \mid c(a, (\kappa, \sigma)) = 0\}.$$

Writing $(\kappa, \sigma) = (r \cos \theta, r \sin \theta)$ for some $r > 0$ and $\theta \in \mathbb{R}$, one easily checks by rescaling the independent variable ξ in (1.12) by a factor r that

$$c(a, (r \cos \theta, r \sin \theta)) = rc(a, (\cos \theta, \sin \theta)), \qquad a_\pm(r \cos \theta, r \sin \theta) = a_\pm(\cos \theta, \sin \theta).$$

We will sometimes abuse notation by writing $a_\pm(\theta)$ instead of $a_\pm(r \cos \theta, r \sin \theta)$.

The functions $a_\pm(\theta)$ are the central objects of study in this paper. For definiteness we study $a_+(\theta)$. Although $c(a, (\kappa, \sigma))$ depends continuously on both a and (κ, σ), the function $a_+(\theta)$ need not depend continuously on θ. However, $a_+(\theta)$ is upper semi-continuous in θ, that is,

$$\limsup_{\theta \to \theta_0} a_+(\theta) \le a_+(\theta_0) \qquad (1.13)$$

holds for every θ_0. This is an immediate consequence of the continuity of the function $c(\cdot, \cdot)$.

Definition We say that *crystallographic pinning* occurs for the system (1.10) in the direction θ_0 in case either the inequality (1.13) is strict, or the analogous inequality for $a_-(\theta)$ is strict.

For the sawtooth function $f_0(u, a) = u - \mathrm{sgn}(u - a)$ it was shown in [6] that crystallographic pinning occurs in every direction θ_0 for which $\tan \theta_0$ is rational. It was then established in [19] that for any such direction, if f is sufficiently close to the sawtooth function then crystallographic pinning occurs in that direction. Numerical studies [16] suggest that crystallographic pinning occurs in the vertical and horizontal directions when f is the cubic nonlinearity; see also [11].

The goal of this paper is to give a specific generic condition on a general nonlinearity f under which crystallographic pinning occurs at $\theta_0 = 0$, that is, with $(\kappa, \sigma) = (1, 0)$. This is given in Theorem 1.1 below. Before stating that result, let us introduce two conditions which will be needed. These conditions do not necessarily hold for an arbitrary f, but rather will be taken as hypotheses in our main result.

Condition A There exists $p \in \ell^\infty(\mathbb{Z})$, denoted $p = \{p_n\}_{n \in \mathbb{Z}}$, satisfying

$$p_{n+1} + p_{n-1} - 2p_n = f(p_n, a_+(0)), \qquad n \in \mathbb{Z}, \qquad (1.14)$$

and which also satisfies the boundary and monotonicity conditions

$$\lim_{n \to \pm\infty} p_n = \pm 1, \qquad p_n \le p_{n+1} \text{ for } n \in \mathbb{Z}. \qquad (1.15)$$

Moreover, such p is unique up to a shift in the index n.

Condition B Condition A holds. Further, if $v \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$ satisfies

$$v_{n+1} + v_{n-1} - 2v_n = f'(p_n, a_+(0))v_n, \qquad n \in \mathbb{Z}, \qquad (1.16)$$

where p is as in Condition A and where we denote $f'(u, a) = \frac{\partial f(u,a)}{\partial u}$, then the series

$$B = \frac{1}{2} \sum_{n=-\infty}^{\infty} f''(p_n, a_+(0))v_n^3 \qquad (1.17)$$

converges and its limit satisfies satisfies $B \ne 0$.

We now state the main results of this paper.

Theorem 1.1 *Assume that Condition B holds. Then the inequality (1.13) is strict at $\theta_0 = 0$ and so crystallographic pinning occurs in the direction $\theta_0 = 0$.*

Theorem 1.2 *Condition B is generic in the following sense. Fix any $f_0 \in \mathcal{N}$ and define the set*

$$C_+^2 = \{\gamma \in C^2[-1, 1] \mid \gamma(u) > 0 \text{ for every } u \in [-1, 1]\}, \qquad (1.18)$$

noting that for every $\gamma \in C_+^2$ we have $f \in \mathcal{N}$, where $f(u, a) = \gamma(u) f_0(u, a)$. Let the set C_+^2 be endowed with the usual C^2 topology, and so making it an open subset of the Banach space $C^2[-1, 1]$. Then the set $\mathcal{G}(f_0) \subseteq C_+^2$ defined as

$$\mathcal{G}(f_0) = \{\gamma \in C_+^2 \mid \gamma f_0 \text{ satisfies Condition B}\}$$

is a residual subset of C_+^2.

The next two propositions provide a partial verification of Conditions A and B under general conditions. In particular, the existence of at least one (although not necessarily unique) $p \in \ell^\infty(\mathbb{Z})$ satisfying (1.14) and (1.15) is shown in Proposition 1.3, while in Proposition 1.4 the convergence of the series (1.17) is shown for any $v \in \ell^\infty(\mathbb{Z})$ as in the statement of Condition B.

Proposition 1.3 *Assume that $c(a, (1, 0)) = 0$, equivalently, that $a \in [a_-(0), a_+(0)]$. Then there exists $p \in \ell^\infty(\mathbb{Z})$ satisfying*

$$p_{n+1} + p_{n-1} - 2p_n = f(p_n, a), \qquad n \in \mathbb{Z}, \tag{1.19}$$

along with (1.15). Moreover, any such monotone p is strictly monotone, that is $p_n < p_{n+1}$ for all $n \in \mathbb{Z}$.

Proof The difference Eq. 1.19 is nothing more than the wave profile Eq. 1.12 with $c = 0$ and $\theta = 0$, that is, $(\kappa, \sigma) = (1, 0)$. As $c(a, (1, 0)) = 0$, this equation has a monotone heteroclinic solution joining ± 1. It only remains to show that this solution is strictly monotone.

To show strict monotonicity, suppose to the contrary that $p_{m+1} = p_m$ for some $m \in \mathbb{Z}$. Then $f(p_m, a) = p_{m-1} - p_m \leq 0$. But $f(p_m, a) = f(p_{m+1}, a) = p_{m+2} - p_{m+1} \geq 0$. Therefore $f(p_m, a) = f(p_{m+1}, a) = 0$ and $p_{m-1} = p_m = p_{m+1} = p_{m+2}$. Continuing in this fashion, we see that p_n is constant in n, which contradicts the fact that it is a heteroclinic connection between -1 and $+1$. □

Equation 1.16 can be expressed as $Lv = 0$, with the operator $L \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ given by

$$L = S + S^{-1} - 2I - f'(p, a_+(0)). \tag{1.20}$$

Here $S \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ is the shift operator defined as

$$(SX)_n = x_{n+1} \text{ for } X = \{x_n\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}),$$

and by a slight abuse of notation $f'(p, a_+(0)) \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ denotes the diagonal operator with entries $f'(p_n, a_+(0))$.

Proposition 1.4 *Assume that Condition A holds, with p as stated there, and let $L \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ be as in (1.20). Then there exists $v \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$ satisfying (1.16), that is, $Lv = 0$. The vector v is unique up to scalar multiple, and thus*

$$\ker(L) = \{av \mid a \in \mathbb{R}\}.$$

Further, v can be chosen to satisfy

$$v_n > 0, \qquad n \in \mathbb{Z},$$

and its coordinate enjoys the estimate

$$v_n \leq K\mu^{|n|}, \qquad n \in \mathbb{Z}, \tag{1.21}$$

for some $K > 0$ and $0 < \mu < 1$. Thus $v \in \ell^1(\mathbb{Z})$, and we may normalize v to satisfy $\langle v, v \rangle = 1$ where $\langle \cdot, \cdot \rangle$ denotes the duality (dot product) between $\ell^1(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$. The operator L is Fredholm with index zero, with range

$$\text{ran}(L) = \{w \in \ell^\infty(\mathbb{Z}) \mid \langle v, w \rangle = 0\},$$

and its spectrum satisfies

$$\sigma(L) \cap (0, \infty) = \emptyset.$$

Note that the exponential estimate (1.21) on v_n implies the absolute convergence of the sum (1.17) defining the quantity B in Condition B.

The proof of Proposition 1.4 will be given in Sect. 3. We remark that even if Condition A does not hold, so p in Proposition 1.4 is not unique, we believe the statement of this result to be true, with v depending on p.

To prove crystallographic pinning as in Theorem 1.1 we must prove that the inequality (1.13) is strict. In our proof of Theorem 1.1 we shall assume that (1.13) is an equality for $\theta_0 = 0$ and that Condition A holds, and proceed to show that $B = 0$. An important step in this proof is the analysis of the second order difference equation

$$X_{m+1} + X_{m-1} - 2X_m + (S + S^{-1} - 2I)X_m - f(X_m, a_+(0)) = 0, \qquad m \in \mathbb{Z}, \quad (1.22)$$

where $X_m \in \ell^\infty(\mathbb{Z})$ for each m. We write the vector X_m in coordinate form as $X_m = \{x_{n,m}\}_{n \in \mathbb{Z}}$ where each $x_{n,m}$ is a scalar, and the operator S acts on the index n so that $(SX_m)_n = x_{n+1,m}$. In this way, Eq. 1.22 is effectively a difference equation in $x_{n,m}$ involving both indices n and m. The function $f(\cdot, a) : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ in (1.22) is again a slight abuse of notation, where we evaluate the nonlinear scalar function $f(\cdot, a) : \mathbb{R} \to \mathbb{R}$ coordinatewise, that is, $f(X_m, a)_n = f(x_{n,m}, a)$.

Equation 1.22 arises when we consider the monotone traveling wave ϕ^ε which propagates in the direction $(1, \varepsilon)$ with speed c, and we adjust the detuning parameter $a = a^\varepsilon$ so that c is appropriately small. This provides an infinite family of transition layers appearing at the integers $\xi = n$ (after an appropriate shift), and to capture these layers we let $\phi_n^\varepsilon(\zeta) = \phi^\varepsilon(\varepsilon\zeta + n)$. Under this rescaling the differential-difference Eq. 1.12 becomes the infinite system

$$-\frac{c}{\varepsilon}(\phi_n^\varepsilon)'(\zeta) = \phi_n^\varepsilon(\zeta + 1) + \phi_n^\varepsilon(\zeta - 1) + \phi_{n+1}^\varepsilon(\zeta) + \phi_{n-1}^\varepsilon(\zeta) - 4\phi_n^\varepsilon(\zeta) - f(\phi_n^\varepsilon(\zeta), a),$$

$$(1.23)$$

where each $\phi_n^\varepsilon(\zeta)$ corresponds to the layer at $\xi = n$. By choosing $a = a^\varepsilon$ so that $c = \varepsilon^2$, the system (1.23) again develops an infinite family of transition layers spaced (in the limit) a unit distance apart, that is, we have "layers within layers." (This secondary scaling in our analysis takes the form (2.11).) Let us also remark that solutions of PDE's with nested families of transition layers has been related to solutions of variational problems on tori [2,23,27]. The vector $X_m = \{\lim_{\varepsilon \to 0} \phi_n^\varepsilon(m)\}_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ captures the values of the plateaus between these secondary layers, and if it is the case that (1.13) is an equality, then the sequence $\{X_m\}_{m \in \mathbb{Z}}$ satisfies Eq. 1.22. The vectors X_m form a monotone heteroclinic orbit, having limits $X_{\pm\infty}$ which are approached from above as $m \to -\infty$ and below as $m \to \infty$, and in the event that Condition A holds, it is the case that $X_\infty = SX_{-\infty} = p$, with p as in Condition A. Furthermore, the limiting point p possesses a two-dimensional center manifold as an equilibrium of the difference Eq. 1.22. Theorem 1.1 is proved by showing that if $B \neq 0$, then the dynamics on the center manifold has a Bogdanov–Takens normal form, specifically (4.32), (4.33). This in turn is shown to preclude the possibility of monotone limits to p from both

above and below, and thus to preclude the monotone heteroclinic orbit $\{X_m\}_{m\in\mathbb{Z}}$. It follows that $B = 0$, as desired.

We remark that we restrict attention to the special angle $\theta_0 = 0$, corresponding to wave motion in the horizontal direction, as the case of other directions involves analysis which is beyond the scope of this paper. In particular, for the case of a general rational direction, namely that $\tan\theta_0$ is either rational or infinite so that we may choose $(\kappa, \sigma) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ for the direction vector, the analog of (1.22) is a difference equation of order $2\max\{|\kappa|, |\sigma|\}$. In the case of an irrational direction, namely where $\tan\theta_0$ is an irrational number, there is no finite-dimensional analog of Eq. 1.22.

This paper is organized as follows. In Sect. 2 we construct the heteroclinic solution $\{X_m\}_{m\in\mathbb{Z}}$ to Eq. 1.22, assuming that (1.13) is an equality and that Condition A holds. In Sect. 3 we prove Proposition 1.4, which develops information about the linearization of Eq. 1.22 at the equilibrium p. This proof relies on properties of resolvent positive operators. In Sect. 4 the center manifold reduction along with associated shadowing properties is given for Eq. 1.22. In Sect. 5 we prove Theorem 1.1 by showing that $B = 0$ must hold if (1.13) is an equality. Finally, in Sect. 6 we use transversality methods to prove Theorem 1.2, namely that Conditions A and B hold generically for normal families of operators.

2 Doubly Heteroclinic Orbits

Throughout this section we assume that f is a normal family, in particular satisfying (1.4) and (1.5). We study the LDE (1.10), keeping the notation as before.

Lemma 2.1 *The inequalities $-1 < a_-(0)$ and $a_+(0) < 1$ are strict. That is, there is a $\delta > 0$ such that $c(a, (1, 0)) < 0$ for $a \in (-1, -1 + \delta]$ and $c(a, (1, 0)) > 0$ for $a \in [1 - \delta, 1)$.*

Proof Without loss we show that $a_+(0) < 1$. The conclusions about c follow directly from this. We proceed by contradiction. Assume that $a_+(0) = 1$. Thus $c(a) \leq 0$ for every $a \in (-1, 1)$, where we denote $c(a) = c(a, (\kappa, \sigma))$ with the horizontal direction $(\kappa, \sigma) = (1, 0)$. Thus the corresponding wave profile $\phi(\xi) = \phi(\xi; a, (1, 0))$ satisfies

$$\phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi) - f(\phi(\xi), a) \geq 0 \tag{2.1}$$

by (1.12). Without loss we may assume that

$$\phi(\xi) \leq 0 \text{ for } \xi < 0, \qquad \phi(\xi) \geq 0 \text{ for } \xi > 0, \tag{2.2}$$

after an appropriate translation of ξ. Now take any sequence $a_n \to 1$ and let $\phi_n(\xi)$ be the corresponding wave profile, which is nondecreasing in ξ and takes values in $[-1, 1]$. Then Helly's Selection Theorem asserts that there exists a subsequence (which we still denote by ϕ_n) such that ϕ_n converges pointwise on $\mathbb{R}$. That is,

$$\lim_{n\to\infty} \phi_n(\xi) = \phi_*(\xi)$$

holds for every $\xi \in \mathbb{R}$, for some nondecreasing function $\phi_* : \mathbb{R} \to [-1, 1]$. Fixing such a subsequence, from (2.1) we see that

$$\phi_n(\xi + 1) + \phi_n(\xi - 1) - 2\phi_n(\xi) \geq \alpha_n = \min_{u\in[-1,1]} f(u, a_n),$$

for every $\xi \in \mathbb{R}$, where the above equality serves as the definition of α_n. Clearly $\alpha_n \to 0$ as $n \to \infty$, so upon taking this limit we have

$$\phi_*(\xi + 1) + \phi_*(\xi - 1) - 2\phi_*(\xi) \geq 0, \qquad \xi\phi_*(\xi) \geq 0, \tag{2.3}$$

for $\xi \in \mathbb{R}$, where (2.2) has been used for the second inequality. Moreover, the first inequality in (2.3) is strict for those ξ for which $\phi_*(\xi) \neq \pm 1$. One sees this from (2.1), since upon fixing any such ξ we have that $f(\phi_n(\xi), a_n)$ is strictly positive and bounded away from zero for large n. Now take any $\xi \in \mathbb{R}$ and denote $\delta_n = \phi_*(\xi + n) - \phi_*(\xi + n - 1)$. Then the first inequality in (2.3) implies that $\delta_{n+1} \geq \delta_n$ for every $n \in \mathbb{Z}$. Thus if $\delta_1 > 0$ we have

$$\phi_*(\xi + n) - \phi_*(\xi) = \sum_{k=1}^{n} \delta_k \geq n\delta_1 \to \infty$$

as $n \to \infty$, which contradicts the boundedness of ϕ_*. It follows that $\delta_1 = 0$, and thus $\phi_*(\xi + 1) = \phi_*(\xi)$ for every $\xi \in \mathbb{R}$. But ϕ_* is monotone and thus constant on $\mathbb{R}$, and with the second inequality in (2.3) we conclude that $\phi_*(\xi) = 0$ for every $\xi \in \mathbb{R}$. Thus the first inequality in (2.3) is an equality for every $\xi \in \mathbb{R}$, however, this contradicts our assertion above that the first inequality in (2.3) is strict whenever $\phi_*(\xi) \neq \pm 1$. This contradiction completes the proof. $\qquad\square$

Lemma 2.2 *Let $q \in \ell^\infty(\mathbb{Z})$ satisfy*

$$q_{n+1} + q_{n-1} - 2q_n = f(q_n, a), \qquad n \in \mathbb{Z}, \tag{2.4}$$

for some $a \in (-1, 1)$. Assume also that $q_n \leq q_{n+1}$ for every $n \in \mathbb{Z}$. Then either

$$\lim_{n \to \pm\infty} q_n = \pm 1, \tag{2.5}$$

or else q is a constant sequence with $q_n = q_0 \in \{-1, a, 1\}$ for every n.

Proof Denoting $q_{\pm\infty} = \lim_{n \to \pm\infty} q_n$, which are finite quantities, we have $f(q_{\pm\infty}, a) = 0$ from (2.4), and hence $q_{\pm\infty} \in \{-1, a, 1\}$. It is thus enough to prove that (a) it is impossible for both $q_{-\infty} = -1$ and $q_\infty = a$ to hold simultaneously; and (b) it is impossible for both $q_{-\infty} = a$ and $q_\infty = 1$ to hold simultaneously. Without loss we only prove (a), so assume to the contrary that $q_{-\infty} = -1$ and $q_\infty = a$. Noting that $f(q_n, a) \geq 0$ for every $n \in \mathbb{Z}$ and denoting $\delta_n = q_n - q_{n-1}$, we have from (2.4) that

$$\delta_{n+1} = \delta_n + f(q_n, a) \geq \delta_n \geq 0. \tag{2.6}$$

As $\{q_n\}_{n\in\mathbb{Z}}$ is not a constant sequence, necessarily $\delta_m > 0$ for some m. But then $\delta_{m+k} \geq \delta_m$ for every $k \geq 0$ by (2.6), hence

$$q_{m+k} - q_{m-1} = \sum_{i=0}^{k} \delta_{m+i} \geq (k+1)\delta_m \to \infty$$

as $k \to \infty$, contradicting boundedness of q_{m+k}. $\qquad\square$

Proposition 2.3 *Assume that Condition A holds, with p as stated there. Then the $\ell^\infty(\mathbb{Z})$-valued ODE*

$$-\Phi' = (S + S^{-1} - 2I)\Phi - f(\Phi, a_+(0)) \tag{2.7}$$

admits a heteroclinic solution $\Phi : \mathbb{R} \to \ell^\infty(\mathbb{Z})$ with limits

$$\lim_{\zeta \to -\infty} \Phi(\zeta) = S^{-1}p, \qquad \lim_{\zeta \to \infty} \Phi(\zeta) = p.$$

Moreover, there exists such a solution $\Phi(\zeta) = \{\phi_n(\zeta)\}_{n\in\mathbb{Z}}$ satisfying the monotonicity condition

$$\phi_n(\zeta) < \phi_n(\zeta + \delta) \text{ for } \delta > 0,$$

for every $n \in \mathbb{Z}$ and $\zeta \in \mathbb{R}$.

Proof Choose $b \in (-1, 1) \setminus \{a\}$ so that also $b \neq p_j$ for every $j \in \mathbb{Z}$, with p as in Condition A. We shall consider the monotone traveling wave $\phi^\varepsilon = \phi^\varepsilon(\xi) = \phi^\varepsilon(\xi; a^\varepsilon, (1, 0))$ traveling in the direction $(1, 0)$ with appropriately chosen a^ε and normalized by a translation so that $\phi^\varepsilon(0) = b$. For sufficiently small ε choose a^ε so that $c = c(a^\varepsilon, (1, 0)) = \varepsilon$. Such a choice is possible by the continuity of the function c and by Lemma 2.1. Note that $\lim_{\varepsilon\to 0} a^\varepsilon = a_+(0)$.

Let $\phi_n^\varepsilon(\zeta) = \phi^\varepsilon(\varepsilon\zeta + n)$, and substitute into (1.8) to obtain

$$-(\phi_n^\varepsilon)'(\zeta) = \phi_{n+1}^\varepsilon(\zeta) + \phi_{n-1}^\varepsilon(\zeta) - 2\phi_n^\varepsilon(\zeta) - f(\phi_n^\varepsilon(\zeta), a^\varepsilon).$$

For each n, as ε varies ϕ_n^ε constitute a bounded equicontinuous family and thus by Ascoli's Theorem and a diagonalization argument, we have the limit

$$\lim_{\varepsilon\to 0} \phi_n^\varepsilon(\zeta) = \phi_n(\zeta)$$

for some subsequence of ε, holding uniformly on compact ζ-intervals for every $n \in \mathbb{Z}$. Letting $\Phi(\zeta) = \{\phi_n(\zeta)\}_{n\in\mathbb{Z}}$ we obtain (2.7). The monotonicity of the original wave profile ϕ^ε together with the scaling $\zeta = \varepsilon\xi + n$ implies the monotonicity condition

$$\phi_n(\zeta) \leq \phi_n(\zeta + \delta) \leq \phi_{n+1}(\zeta), \qquad \delta > 0, \tag{2.8}$$

holds for all $\zeta \in \mathbb{R}$ and all $n \in \mathbb{Z}$. We now show that this monotonicity is strict, as in the statement of the proposition. Suppose to the contrary that $\phi_n(\zeta_0) = \phi_n(\zeta_0 + \delta)$ for some $n \in \mathbb{Z}$ and some $\delta > 0$, and some $\zeta_0 \in \mathbb{R}$. Since each ϕ_n is non-decreasing we have $\phi_n(\zeta) = \phi_n(\zeta_0)$ for every $\zeta \in [\zeta_0, \zeta_0 + \delta]$. It is then a consequence of (2.7) that $\phi_{n+1}(\zeta) + \phi_{n-1}(\zeta)$ is constant in $[\zeta_0, \zeta_0 + \delta]$, and so both $\phi_{n\pm 1}(\zeta)$ are constant in that interval as these functions are non-decreasing. By induction we see that the vector $\Phi(\zeta)$ is in fact constant in this interval, and so (2.7) at $\zeta = \zeta_0$ rewrites as $(S + S^{-1} - 2I)\Phi(\zeta_0) - f(\Phi(\zeta_0), a_+(0)) = 0$. Thus the point $\Phi(\zeta_0) \in \ell^\infty(\mathbb{Z})$ is an equilibrium of Eq. 2.7, and by uniqueness we must have $\Phi(\zeta) = \Phi(\zeta_0)$ for every $\zeta \in \mathbb{R}$. Lemma 2.2 now applies to the vector $q = \Phi(\zeta_0)$ due to the monotonicity of $\phi_n(\zeta_0)$ in n. Because $q_0 = \phi_0(0) = b \notin \{-1, a, 1\}$ it follows that (2.5) holds. Thus by the uniqueness (up to translation) of the vector p in Condition A, we have that $q = S^j p$ for some $j \in \mathbb{Z}$. But then $b = q_0 = p_j$ contradicts the choice of b. Thus the first inequality in (2.8) is strict, as desired. Note that the second inequality in (2.8) is also strict.

We now establish the boundary conditions for $\Phi(\zeta)$. It is a consequence of (2.8) that the limits $\phi_n(\pm\infty) = \lim_{\zeta\to\pm\infty} \phi_n(\zeta)$ exist. Taking these limits in (2.7) gives $(S + S^{-1} - 2I)\Phi(\pm\infty) - f(\Phi(\pm\infty), a_+(0)) = 0$. Again Lemma 2.2 applies, and this time due to the strict monotonicity we have (2.5) for both vectors $\Phi(\pm\infty)$. As $\Phi(-\infty) \leq \Phi(\infty)$ and $\Phi(-\infty) \neq \Phi(\infty)$ it follows that $\Phi(-\infty) = S^j p$ and $\Phi(\infty) = S^k p$ for some integers $j < k$, with p given by Condition A. However, we also have that $\Phi(\infty) \leq S\Phi(-\infty)$ from the second inequality in (2.8), and so $k \leq j + 1$. Thus $k = j + 1$. By reindexing n, namely by replacing $\Phi(\zeta)$ with $S^{-k}\Phi(\zeta)$, we obtain the desired boundary conditions. □

Proposition 2.4 *Assume that the inequality (1.13) is an equality at $\theta_0 = 0$. Also assume that Condition A holds, with p as stated there. Then the $\ell^\infty(\mathbb{Z})$-valued difference Eq. 1.22 admits a heteroclinic orbit $\{X_m\}_{m\in\mathbb{Z}}$ with limits*

$$\lim_{m \to \pm\infty} X_m = X_{\pm\infty}, \qquad X_{-\infty} = S^{-1}p, \qquad X_\infty = p, \tag{2.9}$$

in the space $\ell^\infty(\mathbb{Z})$. Moreover $X_m = \{x_{n,m}\}_{n \in \mathbb{Z}}$ satisfies the lexicographic monotonicity property

$$p_{n-1} = x_{n,-\infty} < x_{n,m} < x_{n,m+1} < x_{n,\infty} = p_n, \qquad n, m \in \mathbb{Z}. \tag{2.10}$$

Proof We shall consider the monotone traveling wave $\phi^\varepsilon = \phi^\varepsilon(\xi) = \phi(\xi; a^\varepsilon, (1, \varepsilon))$, which propagates in direction $(\kappa, \sigma) = (1, \varepsilon)$, taking an appropriately chosen $a = a^\varepsilon$. Note that the angle θ corresponding to this direction is $\theta = \arctan \varepsilon$, and so the upper value of the pinning range is located at $a = a_+(\theta) = a_+(\arctan \varepsilon)$.

To be precise, fix a sequence $\varepsilon_k \to 0$ with $\varepsilon_k \neq 0$ such that

$$\lim_{k \to \infty} a_+(\theta_k) = a_+(0)$$

for $\theta_k = \arctan \varepsilon_k$, by the equality assumption in (1.13). For the remainder of this proof we shall select ε only from the sequence ε_k, suppressing the index k for notational simplicity. For sufficiently small ε choose $a = a^\varepsilon$ so that the wave speed is given by $c(a^\varepsilon, (1, \varepsilon)) = \varepsilon^2$. Such a choice is possible by the continuity of the function $c = c(a, (\kappa, \sigma))$ and by Lemma 2.1. Indeed, one has $c(a_+(\arctan \varepsilon), (1, \varepsilon)) = 0$ while $c(1 - \delta, (1, \varepsilon)) > 0$ is bounded away from zero as $\varepsilon \to 0$, by Lemma 2.1. Let $\bar{a} = \lim_{\varepsilon \to 0} a^\varepsilon$, where the limit is taken along a subsequence if necessary. Certainly $a^\varepsilon > a_+(\arctan \varepsilon)$ for all ε and therefore $\bar{a} \geq \lim_{\theta \to 0} a_+(\arctan \varepsilon) = a_+(0)$. Similarly, continuity of the function c implies that $c(\bar{a}, (1, 0)) = 0$, so $\bar{a} \leq a_+(0)$. Thus $\bar{a} = a_+(0)$.

Now fix $b \in (p_{-1}, p_0)$ so that $b \neq a_+(0)$. Of course $b \neq p_n$ for every n. By translating the argument ξ of ϕ^ε, we may assume without loss that $\phi^\varepsilon(0) = b$. Let

$$\phi^\varepsilon_{n,m}(\zeta) = \phi^\varepsilon(\varepsilon^2 \zeta + \varepsilon m + n). \tag{2.11}$$

As $\phi^\varepsilon(\xi)$ satisfies Eq. 1.12 with $c = \varepsilon^2$, with $a = a^\varepsilon$, and with $(\kappa, \sigma) = (1, \varepsilon)$, we see that the functions $\phi^\varepsilon_{n,m}(\zeta)$ satisfy the LDE

$$-(\phi^\varepsilon_{n,m})' = \phi^\varepsilon_{n,m+1} + \phi^\varepsilon_{n,m-1} + \phi^\varepsilon_{n+1,m} + \phi^\varepsilon_{n-1,m} - 4\phi^\varepsilon_{n,m} - f(\phi^\varepsilon_{n,m}, a^\varepsilon). \tag{2.12}$$

Note in particular that all arguments of the functions $\phi^\varepsilon_{i,j}$ in (2.12) are evaluated at the same point ζ, and so (2.12) is an infinite-dimensional ODE. By Ascoli's Theorem and a diagonalization argument, we have the limit

$$\lim_{\varepsilon \to 0} \phi^\varepsilon_{n,m}(\zeta) = \phi_{n,m}(\zeta)$$

for some subsequence of ε, holding uniformly on compact ζ-intervals for every n and m. The limiting function $\phi_{n,m}$ satisfies the same LDE (2.12), but with $a_+(0)$ in the argument of f. Of course, $\phi_{0,0}(0) = b$. The monotonicity of the original wave profile $\phi^\varepsilon(\xi)$, and the particular scaling $\xi = \varepsilon^2 \zeta + \varepsilon m + n$ of the argument, imply that our limiting functions enjoy the lexicographic montonicity condition

$$\phi_{n,m}(\zeta) \leq \phi_{n,m}(\zeta + \delta) \leq \phi_{n,m+k}(\zeta) \leq \phi_{n+1,m}(\zeta - \delta) \text{ for } k \geq 1 \text{ and } \delta > 0, \tag{2.13}$$

and every $n, m \in \mathbb{Z}$ and $\zeta \in \mathbb{R}$. Indeed, one sees that for any given such k, δ, n, m, and ζ, the inequalities (2.13) hold for $\phi^\varepsilon_{n,m}$ for small ε. Define now

$$x_{n,m} = \lim_{\zeta \to \infty} \phi_{n,m}(\zeta), \qquad X_m = \{x_{n,m}\}_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}).$$

Certainly these limits exist, and of course $|x_{n,m}| \leq 1$. Additionally $\lim_{\zeta \to \infty} \phi'_{n,m}(\zeta) = 0$ from the differential Eq. 2.12. From this one easily sees that taking the limit $\zeta \to \infty$ in (2.12) yields the difference Eq. 1.22, which may also be written as

$$(\Delta x)_{n,m} = f(x_{n,m}, a_+(0)), \qquad (n,m) \in \mathbb{Z}^2, \tag{2.14}$$

where Δ is the discrete laplacian (1.11) on $\mathbb{Z}^2$. The ordering $x_{n,m} \leq x_{n,m+k} \leq x_{n+1,m}$ for $k \geq 1$, which follows from (2.13), implies that $x_{n,\pm\infty} = \lim_{m \to \pm\infty} x_{n,m}$ exist and that

$$x_{n,-\infty} \leq x_{n,m} \leq x_{n,m+1} \leq x_{n,\infty}, \qquad x_{n,\infty} \leq x_{n+1,-\infty}, \tag{2.15}$$

for every n and m. One sees that the vectors $X_{-\infty} = \{x_{n,-\infty}\}_{n \in \mathbb{Z}}$ and $X_\infty = \{x_{n,\infty}\}_{n \in \mathbb{Z}}$ both satisfy the conditions of Lemma 2.2 with $a = a_+(0)$.

Also observe that

$$x_{0,-1} \leq b \leq x_{0,0}. \tag{2.16}$$

The second inequality in (2.16) holds because $\phi_{0,0}(\zeta) \geq \phi_{0,0}(0) = b$ for $\zeta \geq 0$. The first inequality in (2.16) follows by taking $(n,m) = (0,-1)$ with $k = 1$, and $\zeta = 0$, in the second inequality of (2.13), to give $\phi_{0,-1}(\delta) \leq \phi_{0,0}(0) = b$. One then lets $\delta \to \infty$.

We next show that the first three inequalities in (2.15) are strict. Suppose that $x_{n,m} = x_{n,m+1}$ for some n and m. Then by Eq. 2.14

$$\begin{aligned}
0 &= (\Delta x)_{n,m+1} - (\Delta x)_{n,m} \\
&= (x_{n+1,m+1} - x_{n+1,m}) + (x_{n-1,m+1} - x_{n-1,m}) + (x_{n,m+2} - x_{n,m-1}).
\end{aligned}$$

Each of the three differences in the final line of the above equation are nonnegative by (2.15), and so we have that

$$x_{n+1,m} = x_{n+1,m+1}, \qquad x_{n-1,m} = x_{n-1,m+1},$$
$$x_{n,m-1} = x_{n,m} = x_{n,m+1} = x_{n,m+2}.$$

It is clear by repeating the procedure that we may conclude that

$$x_{n,m} = x_{n,0} \tag{2.17}$$

for every n and m, namely, that $X_m = X_0$ is constant in m. Thus $X_{\pm\infty} = X_0$, and as noted this vector satisfies the conditions of Lemma 2.2 with $a = a_+(0)$. We also have, from (2.16) and (2.17), that $x_{0,0} = b \in (-1,1) \setminus \{a_+(0)\}$, and so it follows from Lemma 2.2 that $\lim_{n \to \pm\infty} x_{n,0} = \pm 1$. By the uniqueness assumption on p in the statement of the proposition, we have that $X_0 = S^k p$ for some $k \in \mathbb{Z}$. But then $b = x_{0,0} = p_k$, which contradicts the choice of b. We thus conclude that $x_{n,m} < x_{n,m+1}$ for every n and m, and hence the first three inequalities in (2.15) are strict.

One thus has $x_{n,-\infty} < x_{n,\infty} \leq x_{n+1,-\infty} < x_{n+1,\infty}$ for every n, and this implies that $x_{n,-\infty} < x_{n+1,-\infty}$ and $x_{n,\infty} < x_{n+1,\infty}$. Applying Lemma 2.2 again, to both $X_{-\infty}$ and X_∞, we conclude that each of these vectors are shifts of p, say

$$X_{-\infty} = S^j p, \qquad X_\infty = S^k p,$$

for some j and k. As $x_{n,-\infty} < x_{n,\infty}$ we have that $S^j p \leq S^k p$ with $S^j p \neq S^k p$, and so $j < k$. As $x_{n,\infty} \leq x_{n+1,-\infty}$ we have that $S^k p \leq S^{j+1} p$, hence $k \leq j + 1$. Thus $j = k - 1$. Further,

$$p_{k-1} = (S^{k-1} p)_0 = (S^j p)_0 = x_{0,-\infty} < x_{0,0} = b < x_{0,\infty} = (S^k p)_0 = p_k,$$

and so $k = 0$ by the choice of b. This implies that $x_{n,-\infty} = p_{n-1}$ and $x_{n,\infty} = p_n$, to give the result.   □

3 The Proof of Proposition 1.4

For each pair $(q, a) \in \ell^\infty(\mathbb{Z}) \times (-1, 1)$ we may linearize Eq. 2.4 about q to obtain an associated operator $L_q(a) \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ given by

$$L_q(a) = S + S^{-1} - 2I - f'(q, a), \tag{3.1}$$

where $f'(q, a)$ denotes the diagonal operator with entries $f'(q_n, a)$. Proposition 1.4 concerns the operator $L_q(a)$ with $a = a_+(0)$ and with $q = p$ as in (1.14), (1.15). We first examine the spatial dynamics associated with the difference equation $L_q(a)v = 0$.

Generally, we shall denote $L_q = L_q(a)$ when the value of a is clear.

Lemma 3.1 *Let $D \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ be the diagonal operator $D = \mathrm{diag}\{d_n\}_{n \in \mathbb{Z}}$, where we assume the existence of the limits and the sign conditions*

$$\lim_{n \to \pm\infty} d_n = d_{\pm\infty}, \quad d_{\pm\infty} > 0.$$

Then the operator $T = S + S^{-1} - 2I - D$ is Fredholm of index zero. Moreover, the dimension of the kernel of T is either zero or one. Furthermore, elements $v \in \ker(T)$ of the kernel enjoy the estimate

$$|v_n| \le K\mu^{|n|}, \quad n \in \mathbb{Z}, \tag{3.2}$$

for some $K > 0$ and $0 < \mu < 1$, and in addition,

$$v_n \ne 0 \text{ for all large } |n| \tag{3.3}$$

for nontrivial elements of the kernel. If v is a nontrivial element of the kernel then

$$\mathrm{ran}(T) = \{w \in \ell^\infty(\mathbb{Z}) \mid \langle v, w \rangle = 0\}, \tag{3.4}$$

with $\mathrm{ran}(T) = \ell^\infty(\mathbb{Z})$ if $\ker(T) = \{0\}$.

If $q \in \ell^\infty(\mathbb{Z})$ satisfies $\lim_{n \to \pm\infty} q_n = \pm 1$ and $a \in (-1, 1)$, then all the above claims hold for the operator $T = \lambda I - L_q$, where $L_q = L_q(a)$, provided that $\lambda > \max\{-f'(1, a), -f'(-1, a)\}$.

Finally, the same conclusions hold if we consider the operator T, or $\lambda I - L_q$, lying in the space $\mathcal{L}(Q_0)$ instead of $\mathcal{L}(\ell^\infty(\mathbb{Z}))$, where

$$Q_0 = \{x \in \ell^\infty(\mathbb{Z}) \mid \lim_{n \to \pm\infty} x_n = 0\}.$$

Proof The arguments below apply equally to operators in the space $\mathcal{L}(\ell^\infty(\mathbb{Z}))$, and in the space $\mathcal{L}(Q_0)$.

Consider the second order difference equation

$$x_{n+1} + x_{n-1} - (2 + d_n)x_n = 0 \tag{3.5}$$

associated to the operator T. This difference equation is asymptotically hyperbolic, that is, all roots μ of the limiting characteristic equations

$$\mu + \mu^{-1} - (2 + d_{\pm\infty}) = 0 \tag{3.6}$$

satisfy $|\mu| \neq 1$. Indeed, one easily checks this by noting the left-hand side of (3.6) is negative whenever $\mu = e^{i\theta}$. In fact the two roots of this equation have the form $\mu = \mu_\pm$ and $\mu = \mu_\pm^{-1}$, satisfying $0 < \mu_\pm < 1 < \mu_\pm^{-1}$, for each choice of sign $+$ or $-$, and so this equation is of saddle type.

Asymptotic hyperbolicity of Eq. 3.5 now implies that this equation admits dichotomies whose stable and unstable subspaces are each one-dimensional, both for $n \to \infty$ and for $n \to -\infty$. This and the above remarks about the roots of (3.6) in turn imply that the kernel of T is at most one-dimensional, that any kernel element necessarily decays at the rate (3.2) for some $K > 0$ and $0 < \mu < 1$, and that T is Fredholm.

In particular, the proof that T is Fredholm follows by adapting the techniques of Palmer [25] for ODE's to the setting of difference equations. Palmer proved that a differential operator of the form $Lx(t) = \dot{x}(t) - A(t)x(t)$, where $A(t)$ is a bounded continuous coefficient matrix, is Fredholm if and only if it admits exponential dichotomies on both half lines. (Here the operator L acts from the space of bounded C^1 functions on the line with bounded first derivative, into the space of bounded continuous functions.) The techniques of Palmer's proof carry over to difference operators of the form (3.5), with appropriate modifications. We omit the details.

To prove that the Fredholm index of T is zero, consider the operator $T - \lambda I$. For every $\lambda \geq 0$ this operator satisfies the same conditions in the statement of the lemma as T does, and so it too is Fredholm, necessarily with the same index as T. But $T - \lambda I$ is invertible for $\lambda > \|T\|$, and so has index zero. Thus the index of T also is zero. In particular, if $\ker(T) = \{0\}$ then $\mathrm{ran}(T) = \ell^\infty(\mathbb{Z})$.

To prove (3.4) in the case that $\ker(T)$ contains a nontrivial element v, note that if $w \in \mathrm{ran}(T)$, say $w = Ty$, then $\langle v, w \rangle = \langle v, Ty \rangle = \langle T^* v, y \rangle = 0$, where $T^* \in \mathcal{L}(\ell^1(\mathbb{Z}))$ has the same formula as does T, but considered in the space $\ell^1(\mathbb{Z})$. This now proves an inclusion for the formula (3.4). But both spaces in this formula have the same codimension, in particular $\mathrm{codim}\,\mathrm{ran}(T) = \dim \ker(T) = 1$, as the Fredholm index of T is zero. We conclude that (3.4) is valid, as claimed.

To prove that (3.3) holds for any nontrivial element of $\ker(T)$, consider such an element v. Suppose for some $n \geq n_0$ that $v_{n+1} > v_n \geq 0$, where n_0 is large enough that $d_m \geq 0$ for every $m > n_0$. Then from the difference Eq. 3.5 at $n + 1$ we have that $v_{n+2} - v_{n+1} \geq v_{n+1} - v_n$, and so $v_{n+2} > v_{n+1} \geq 0$. Upon repeating this procedure, we conclude that $v_{n+k} - v_{n+k-1}$ is a nondecreasing and strictly positive sequence for $k \geq 1$. But this forces $v_m \to \infty$ as $m \to \infty$, which is false. Thus the assumption that $v_{n+1} > v_n \geq 0$ for some $n \geq n_0$ is impossible. Similarly, $v_{n+1} < v_n \leq 0$ cannot occur for any $n \geq n_0$. Thus if $v_n = 0$ for some $n \geq n_0$, necessarily $v_{n+1} = 0$. However, in light of the difference Eq. 3.5, this forces $v_m = 0$ for every $m \in \mathbb{Z}$, which contradicts the assumption that v is nontrivial. We thus conclude that $v_n \neq 0$ for all $n \geq n_0$. Similarly, $v_n \neq 0$ for all sufficiently large negative n, as claimed.

To establish the penultimate claim of the lemma, we note that the operator $L_q - \lambda I$ has the same form as T in the statement of the lemma, with

$$d_{\pm\infty} = f'(q_{\pm\infty}, a) + \lambda = f'(\pm 1, a) + \lambda.$$

Thus $L_q - \lambda I$ is Fredholm with index zero so long as $\lambda > \max\{-f'(1, a), -f'(-1, a)\}$, as desired. $\qquad\square$

The proof of Proposition 1.4 relies on the theory of resolvent positive operators, which we now briefly outline.

Recall that a closed convex subset $\mathcal{K}$ of a Banach Space $\mathcal{X}$ is called a *cone* if it is closed under positive linear combinations and if also $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. A cone $\mathcal{K}$ is called a *total cone* if in addition, its linear span $\{x_1 - x_2 \mid x_1, x_2 \in \mathcal{K}\}$ is dense in $\mathcal{X}$.

Let $\mathcal{K} \subseteq \mathcal{X}$ be a cone. An element $x \in \mathcal{X}$ is called *positive* with respect to $\mathcal{K}$, denoted $x \geq 0$, if $x \in \mathcal{K}$, and we write $x_1 \geq x_2$ to mean $x_1 - x_2 \geq 0$, for $x_1, x_2 \in \mathcal{X}$. The relation $\geq$ thus defines a partial order on $\mathcal{X}$. A linear operator $T \in \mathcal{L}(\mathcal{X})$ is called *positive* if $T(\mathcal{K}) \subseteq \mathcal{K}$, that is, $Tx \geq 0$ whenever $x \geq 0$. We similarly write $T \geq 0$ to denote that T is a positive operator, and $T_1 \geq T_2$ when $T_1 - T_2 \geq 0$. An operator T is called *resolvent positive* if there exists $\Lambda \in \mathbb{R}$ such that $\lambda \in \rho(T)$ and $(\lambda I - T)^{-1} \geq 0$ whenever $\lambda \geq \Lambda$.

Resolvent positive operators satisfy the following nice property [19,24,29]. A proof of the following result can be found, for example, in Proposition 4.7 of [19].

Proposition 3.2 *Let $\mathcal{K}$ be a total cone in a Banach Space $\mathcal{X}$ and let $T \in \mathcal{L}(\mathcal{X})$ be resolvent positive with respect to $\mathcal{K}$. Then*

$$\sigma(T) \cap \mathbb{R} \neq \emptyset.$$

Further, let

$$\lambda_0 = \sup(\sigma(T) \cap \mathbb{R}).$$

Then either λ_0 is an eigenvalue of T and there exists an associated positive eigenvector v, that is

$$Tv = \lambda_0 v, \qquad v \in \mathcal{K} \setminus \{0\},$$

or else the operator $\lambda_0 I - T$ is not Fredholm.

In what follows, whenever we take $\mathcal{X} = \ell^\infty(\mathbb{Z})$, then we let $\mathcal{K}$ be the cone of non-negative sequences, namely those $x = \{x_n\}_{n \in \mathbb{Z}}$ with $x_n \geq 0$ for every n.

Lemma 3.3 *If $T \in \mathcal{L}(\mathcal{X})$ satisfies $T \geq -\sigma I$ for some $\sigma \in \mathbb{R}$, then T is resolvent positive. In particular, the operator $L_q(a)$ in Eq. 3.1 is of this form for any $q \in \ell^\infty(\mathbb{Z})$ and $a \in (-1, 1)$, and hence is resolvent positive.*

Proof For a general operator T as in the statement of the lemma, write $T = B - \sigma I$ where $B \geq 0$. Then

$$(\lambda I - T)^{-1} = ((\lambda + \sigma)I - B)^{-1} = \frac{1}{\lambda + \sigma} \sum_{k=0}^{\infty} \frac{B^k}{(\lambda + \sigma)^k}$$

converges for $\lambda + \sigma > \|B\|$, and is a sum of positive operators hence is positive.

For the specific operator L_q we see that taking $\sigma \geq 2 + \max\{f'(u, a) \mid -1 \leq u \leq 1\}$ gives $L_q \geq -\sigma I$, as desired.   □

Proposition 3.4 *Suppose that $q \in \ell^\infty(\mathbb{Z})$ and $a \in (-1, 1)$ are as in the statement of Lemma 3.1. Then $\sigma(L_q) \cap \mathbb{R} \neq \emptyset$. Assume additionally that the quantity*

$$\lambda_0 = \sup(\sigma(L_q) \cap \mathbb{R}) \tag{3.7}$$

satisfies

$$\lambda_0 > \max\{-f'(1, a), -f'(-1, a)\}. \tag{3.8}$$

Then λ_0 is an eigenvalue of L_q and $\lambda_0 I - L_q$ has a one-dimensional kernel, and so

$$\ker(\lambda_0 I - L_q) = \{av \mid a \in \mathbb{R}\}$$

for some $v \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$. Further, the eigenvector v can be chosen to satisfy $v_n > 0$ for all $n \in \mathbb{Z}$, and there are constants $K > 0$ and $0 < \mu < 1$ such that $v_n \le K\mu^{|n|}$ for all $n \in \mathbb{Z}$, and so also $v \in \ell^1(\mathbb{Z})$. Finally, the operator $\lambda_0 I - L_q$ is Fredholm with index zero, with range

$$\operatorname{ran}(\lambda_0 I - L_q) = \{w \in \ell^\infty(\mathbb{Z}) \mid \langle v, w \rangle = 0\}. \tag{3.9}$$

Proof The operator L_q is resolvent positive by Lemma 3.3 and so Proposition 3.2 implies that $\sigma(L_q) \cap \mathbb{R} \ne \emptyset$. Now assume that the inequality (3.8) holds. Then $\lambda_0 I - L_q$ is Fredholm with index zero, by Lemma 3.1. Thus Proposition 3.2 implies that λ_0 is an eigenvalue of L_q with an associated eigenvector $v \in \mathcal{K} \setminus \{0\}$, that is, with $v \ne 0$, and $v_n \ge 0$ for every $n \in \mathbb{Z}$. It follows from Lemma 3.1 that λ_0 is simple and that v enjoys exponential decay in n, and that moreover the formula (3.9) for the range holds.

It remains to prove that the strict inequality $v_n > 0$ holds for every n. If this is false, then there exists some $k \in \mathbb{Z}$ for which $v_k = 0$, and with either $v_{k+1} > 0$ or $v_{k-1} > 0$. In any case, both $v_{k\pm 1} \ge 0$. But then

$$0 = ((\lambda_0 I - L_q)v)_k = \lambda_0 v_k - v_{k+1} - v_{k-1} + (2 + f'(q_k, a))v_k < 0,$$

a contradiction.   □

Proof of Proposition 1.4 It follows from Proposition 3.4 that it suffices to show that $\lambda_0 = 0$ for the quantity in (3.7), taking $q = p$ and $a = a_+(0)$, and thus with the operator $L = L_p(a_+(0))$ as in Eq. 1.20.

Let us first show that $\lambda_0 \le 0$. Assuming to the contrary that $\lambda_0 > 0$, let $v \in \ell^\infty(\mathbb{Z})$ be the associated positive eigenvector as in Proposition 3.4. Also let $\Psi(\zeta) = S\Phi(\zeta) - p$, with $\Phi(\zeta)$ as in Proposition 2.3 , so that $\Psi(\zeta) \ge 0$ with $\Psi(\zeta) \ne 0$. Then from Eq. 2.7 we have

$$-\Psi' = (S + S^{-1} - 2I)\Psi - f(\Psi + p, a_+(0)) + f(p, a_+(0))$$

which can be rewritten as

$$-\Psi' = L\Psi - E(\zeta),$$
$$E(\zeta) = f(\Psi(\zeta) + p, a_+(0)) - f(p, a_+(0)) - f'(p, a_+(0))\Psi(\zeta).$$

Now $v \in \ell^1(\mathbb{Z})$ by the exponential decay of v_n, and so we may take the inner (duality) product $\langle \cdot, \cdot \rangle$ of the above equation with v. Denoting this by $w(\zeta) = \langle v, \Psi(\zeta) \rangle$, we have that

$$\begin{aligned}
-w'(\zeta) &= \langle v, L\Psi(\zeta) \rangle - \langle v, E(\zeta) \rangle \\
&= \langle L^*v, s\Psi(\zeta) \rangle - \langle v, E(\zeta) \rangle \\
&= \lambda_0 w(\zeta) - e(\zeta),
\end{aligned} \tag{3.10}$$

where $e(\zeta) = \langle v, E(\zeta) \rangle$. Also note that $w(\zeta) > 0$ for every ζ. Now

$$E_n(\zeta) = f(\psi_n(\zeta) + p_n, a_+(0)) - f(p_n, a_+(0)) - f'(p_n, a_+(0))\psi_n(\zeta)$$

and so there exists $K > 0$ such that

$$|E_n(\zeta)| \le K\psi_n(\zeta)^2 \le K\|\Psi(\zeta)\|\psi_n(\zeta)$$

with $\| \cdot \|$ denoting the norm in $\ell^\infty(\mathbb{Z})$. Thus

$$|e(\zeta)| \leq \sum_{n=-\infty}^{\infty} v_n |E_n(\zeta)| \leq K\|\Psi(\zeta)\| \sum_{n=-\infty}^{\infty} v_n \psi_n(\zeta) = K\|\Psi(\zeta)\| w(\zeta),$$

where the positivity of v and of $\Psi(\zeta)$ is crucial in obtaining this inequality. From this and from (3.10) we have that

$$-w'(\zeta) \geq (\lambda_0 - K\|\Psi(\zeta)\|) w(\zeta),$$

and in particular, $-w'(\zeta) \geq \frac{\lambda_0}{2} w(\zeta)$ for all sufficiently negative ζ, since $\Psi(\zeta) \to 0$ as $\zeta \to -\infty$. However, this is impossible as $w(\zeta) \to 0$ as $\zeta \to -\infty$ with $w(\zeta) > 0$ for every ζ. This contradiction proves that $\lambda_0 \leq 0$.

To complete the proof that $\lambda_0 = 0$, we prove that $\lambda_0 \geq 0$. Assume to the contrary that $\lambda_0 < 0$. Then $0 \notin \sigma(L)$ from the definition of λ_0 and so L is invertible. This fact together with the Implicit Function Theorem allows us to solve Eq. 2.4 for $q = q(a) \in \ell^\infty(\mathbb{Z})$ depending smoothly on a, for a near $a_+(0)$, and with $q(a) = p$ at $a = a_+(0)$. To be specific, assume that such $q(a)$ is defined for $a \in U$, where $U \subseteq (-1, 1)$ is an open interval containing the point $a_+(0)$. (We reserve the right below to decrease U by taking a smaller neighborhood of $a_+(0)$.) Let us recall here the strict inequalities $-1 < a_+(0) < 1$ from Lemma 2.1, which will also be used below.

It is enough to show that $q_n(a)$ depends monotonically on n, that is, $q_n(a) \leq q_{n+1}(a)$ for every $n \in \mathbb{Z}$ and $a \in U$. Indeed, if this holds then for every $a \in U$ we have a monotone solution of Eq. 1.12 at $(\kappa, \sigma) = (1, 0)$ with $c = 0$, and this implies that U is a subset of the pinning region $(a_-(0), a_+(0))$. However, this contradicts the fact that U is a neighborhood of $a_+(0)$.

Let us prove strict monotonicity in n for $n \geq 0$, the proof of monotonicity for $n \leq 0$ being similar. Fix a sufficiently large integer N and sufficiently small $\varepsilon > 0$, such that

$$p_N > \frac{1 + a_+(0)}{2} > a + \varepsilon, \qquad \|q(a) - p\| < \varepsilon, \qquad p_{n+1} - p_n > 2\varepsilon, \qquad (3.11)$$

holds for every $a \in U$ and for $0 \leq n < N$. One easily does this by first choosing N to satisfy the first inequality in (3.11), then choosing ε, and then possibly decreasing the neighborhood U. We claim that $q_{n+1}(a) > q_n(a)$ for every $n \geq 0$ and $a \in U$. For such a, first observe that

$$q_{n+1}(a) - q_n(a) > p_{n+1} - p_n - 2\varepsilon > 0, \qquad 0 \leq n < N,$$

by (3.11). Also observe that

$$q_n(a) > p_n - \varepsilon \geq p_N - \varepsilon > \frac{1 + a_+(0)}{2} - \varepsilon > a, \qquad n \geq N, \qquad (3.12)$$

again by (3.11). Now assume for some $a \in U$ that there exists $m \geq N$ such that $q_{m+1}(a) - q_m(a) \leq 0$. Take m to be the least such integer, and so $q_m(a) - q_{m-1}(a) > 0$. Then from Eq. 2.4 at $n = m$ we have that $f(q_m(a), a) < 0$, and so from (3.12) we have that $q_m(a) \in (a, 1)$. Two cases now arise, and we shall rule out each of them in turn. For the first case, there exists $j > m$ such that $q_{j+1}(a) - q_j(a) > 0$. Take j to be the least such integer, and so $q_j(a) - q_{j-1}(a) \leq 0$, which with Eq. 2.4 implies that $f(q_j(a), a) > 0$. However, from the choice of j and from (3.12) it follows that $a < q_j(a) \leq q_m(a) < 1$, which implies that $f(q_j(a), a) < 0$, a contradiction. We now consider the second case, in which $q_{j+1}(a) - q_j(a) \leq 0$ for every $j > m$. Then the limit $q_j(a) \to q_\infty(a)$ exists as a nonincreasing sequence for $j \geq m$, and upon taking the limit $n \to \infty$ in Eq. 2.4 we have

that $f(q_\infty(a), a) = 0$, hence $q_\infty(a) \in \{-1, a, 1\}$. But (3.12) implies that $q_\infty(a) > a$, and so $q_\infty(a) = 1$. On the other hand, $q_\infty(a) \le q_m(a) < 1$, and this is a contradiction. With this, the proof of the proposition is complete.   □

The fact that $\lambda_0 = 0$ will allow us to make a center manifold reduction. The positivity of the kernel element v will allow us to make use of the center manifold reduction to induce a sign condition on f. The next section is devoted to constructing the center manifold.

4 A Center Manifold Reduction

So long as Condition A holds and (1.13) is an equality at $\theta_0 = 0$, then Proposition 2.4 guarantees that the difference Eq. 1.22 admits a monotone increasing solution X_m which converges to p as $m \to \infty$. Similarly, the solution of (1.22) given by SX_m is also monotone increasing in m and converges (downward) to p as $m \to -\infty$. Our aim is to close the argument by showing that such dynamical behavior can occur only when $B = 0$ for the quantity in Condition B, and thus can only happen when Condition B fails. In other words, when Condition B holds and so $B \ne 0$, then the inequality (1.13) is strict at $\theta_0 = 0$ and so crystallographic pinning occurs in the direction $\theta_0 = 0$.

Our approach is to consider the difference equation

$$Y_{m+1} = 2Y_m - Y_{m-1} - (S + S^{-1} - 2I)Y_m$$
$$+ f(p + Y_m, a_+(0)) - f(p, a_+(0)), \tag{4.1}$$

which is satisfied by both $Y_m = X_m - p$ and also by $Y_m = SX_{-m} - p$ whenever X_m satisfies (1.22). We shall show that when Condition A holds, then the system (4.1) possesses a two-dimensional center manifold at the origin. Moreover, this center manifold contains two monotone orbits which approach zero from below and above, and toward which the orbits $X_m - p$ and $SX_{-m} - p$, respectively, obtained in Proposition 2.4 converge exponentially fast as $m \to \infty$. A convexity argument then shows that the existence of one of these orbits forces $B \le 0$ and the other forces $B \ge 0$, thus $B = 0$ must hold.

The first step in this process is the construction of a smooth center manifold. The following theorem is a discrete-time version of Theorem 4.1 in [31]. For details see [17] in the case where $\mathcal{Z}$ is a Hilbert space.

Theorem 4.1 *[Center Manifold Theorem] Let $\mathcal{Z}$ be a Banach space and let $A \in \mathcal{L}(\mathcal{Z})$ satisfy the following spectral gap condition: There exists $\alpha > 0$ such that for each $\lambda \in \sigma(A)$ either $|\lambda| = 1$ or $|\log|\lambda|| > \alpha$. Denote by $\mathcal{Z}^c$ and $\mathcal{Z}^h$ the associated center and hyperbolic subspaces of $\mathcal{Z}$, corresponding to spectra with $|\lambda| = 1$ and $|\lambda| \ne 1$, respectively. Assume that $\mathcal{Z}^c$ is finite dimensional. Let $g : \mathcal{Z} \to \mathcal{Z}$ be a C^k-smooth function for some $1 \le k < \infty$ and satisfy $g(0) = 0$ and $Dg(0) = 0$. Then there exist neighborhoods $\Omega^c \subseteq \mathcal{Z}^c$ and $\Omega^h \subseteq \mathcal{Z}^h$ of the origin in these subspaces, and a C^k mapping $\psi : \Omega^c \to \Omega^h$ with $\psi(0) = 0$ and $D\psi(0) = 0$, such that the following properties hold:*

- *If $z_n^c \in \Omega^c$ for $0 \le n \le N + 1$ satisfies the reduced system*

$$z_{n+1}^c = A^c z_n^c + \pi^c g(z_n^c + \psi(z_n^c)) \tag{4.2}$$

for $0 \le n \le N$, and if we let $z_n = z_n^c + \psi(z_n^c)$, then z_n satisfies the full system

$$z_{n+1} = Az_n + g(z_n) \tag{4.3}$$

for $0 \le n \le N$.

- *If z_n satisfies the full system (4.3) and also $z_n \in \Omega$ for all $n \in \mathbb{Z}$, then $\pi^h z_n = \psi(\pi^c z_n)$ and $z_n^c = \pi^c z_n$ satisfies the reduced system (4.2) for every $n \in \mathbb{Z}$.*

In the above statements, we let π^c and π^h denote the canonical projections with respect to the decomposition $\mathcal{Z} = \mathcal{Z}^c \oplus \mathcal{Z}^h$, and $A^c = \pi^c A$ and $A^h = \pi^h A$ are the corresponding operators on these subspaces. We also denote $\Omega = \{z^c + z^h \mid z^c \in \Omega^c \text{ and } z^h \in \Omega^h\}$, which is a neighborhood of the origin in $\mathcal{Z}$.

When Theorem 4.1 holds, we call the set

$$W^c = \{z^c + \psi(z^c) \mid z^c \in \Omega^c\} \subseteq \Omega$$

a *local center manifold* for the full system (4.3), and the system (4.2) restricted to this set is called the *reduced system*. We only outline the proof of this result. For more details, see [17,30,31].

Sketch of Proof The full evolution Eq. 4.3 may be written in terms of its center, stable, and unstable parts, where we decompose $\mathcal{Z} = \mathcal{Z}^c \oplus \mathcal{Z}^h = \mathcal{Z}^c \oplus (\mathcal{Z}^s \oplus \mathcal{Z}^u)$ in this fashion. We generally denote $z^\circ = \pi^\circ z$, with $A^\circ = \pi^\circ A$ and $g^\circ(z) = \pi^\circ g(z)$, where π° denotes the spectral projection onto $\mathcal{Z}^\circ$, with $\circ$ representing $c, h, s,$ or u. We thus have

$$z_{n+1}^c = A^c z_n^c + g^c(z_n),$$
$$z_{n+1}^s = A^s z_n^s + g^s(z_n),$$
$$z_{n+1}^u = A^u z_n^u + g^u(z_n),$$

for the full Eq. 4.3, and we have that $z_n = z_n^c + z_n^s + z_n^u$. The corresponding variation of constants formulae are

$$z_n^c = (A^c)^n z_0^c + \sum_{j=0}^{n-1} (A^c)^{n-1-j} g^c(z_j),$$

$$z_n^s = (A^s)^n z_0^s + \sum_{j=0}^{n-1} (A^s)^{n-1-j} g^s(z_j),$$

$$z_n^u = (A^u)^n z_0^u + \sum_{j=0}^{n-1} (A^u)^{n-1-j} g^u(z_j), \tag{4.4}$$

where if $n \leq 0$ we interpret $\sum_{j=0}^{n-1} = -\sum_{j=n}^{-1}$, with $\sum_{j=0}^{-1}$ the empty sum. We begin the proof by restricting attention to the case where the orbit $\{z_n\}_{n \in \mathbb{Z}}$ is bounded for all $n \in \mathbb{Z}$. In this case, upon multiplying the second (stable) equation in (4.4) by $(A^s)^{-n}$ and letting $n \to -\infty$, and also multiplying the third (unstable) equation in (4.4) by $(A^u)^{-n}$ and letting $n \to \infty$, we obtain

$$z_0^s = \sum_{j=-\infty}^{-1} (A^s)^{-1-j} g^s(z_j),$$

$$z_0^u = -\sum_{j=0}^{\infty} (A^u)^{-1-j} g^u(z_j), \tag{4.5}$$

and then substituting (4.5) into the second and third equations of (4.4) gives

$$z_n^s = \sum_{j=-\infty}^{n-1} (A^s)^{n-1-j} g^s(z_j),$$

$$z_n^u = -\sum_{j=n}^{\infty} (A^u)^{n-1-j} g^u(z_j). \tag{4.6}$$

Combining (4.6) with the first equation in (4.4) thus gives

$$z_n = (A^c)^n z_0^c + \sum_{j=0}^{n-1} (A^c)^{n-1-j} g^c(z_j)$$

$$+ \sum_{j=-\infty}^{n-1} (A^s)^{n-1-j} g^s(z_j) - \sum_{j=n}^{\infty} (A^u)^{n-1-j} g^u(z_j). \tag{4.7}$$

We regard (4.7) as a fixed point equation for the trajectory $\{z_n\}_{n\in\mathbb{Z}}$ with z_0^c as a parameter. For each $\zeta > 0$ define the Banach space $\mathcal{Y}_\zeta$ of sequences $z = \{z_n\}_{n\in\mathbb{Z}}$ in $\mathcal{Z}$ by

$$\mathcal{Y}_\zeta = \{z : \mathbb{Z} \to \mathcal{Z} \mid \|z\|_\zeta < \infty\}, \qquad \|z\|_\zeta = \sup_{n\in\mathbb{Z}} e^{-\zeta|n|} \|z_n\|_\mathcal{Z},$$

with $\|\cdot\|_\zeta$ being the norm in $\mathcal{Y}_\zeta$. Also define $G : \mathcal{Z}^c \times \mathcal{Y}_\alpha \to \mathcal{Y}_\alpha$ to be the right hand side of Eq. 4.7, with arguments $z_0^c \in \mathcal{Z}^c$ and $z = \{z_n\}_{n\in\mathbb{Z}} \in \mathcal{Y}_\alpha$, with α as in the statement of the theorem. Thus the fixed points of $G(z_0^c, \cdot)$ are solutions of (4.7). It is not hard to show that $G(z_0^c, \cdot)$ is a contraction mapping, so long as the Lipschitz constant of g is sufficiently small. Assuming this, denote the unique fixed point of $G(z_0^c, \cdot)$ by $\Psi(z_0^c) \in \mathcal{Y}_\alpha$, and let $\psi : \mathcal{Z}^c \to \mathcal{Z}^h$ be given by $\psi(z_0^c) = \pi^h[\Psi(z_0^c)_0]$. Observe that $z_n^c = \pi^c[\Psi(z_0^c)_n]$ solves the reduced system (4.2) and that $z_n = \pi^c[\Psi(z_0^c)_n] + \psi(\pi^c[\Psi(z_0^c)_n])$ solves the full system (4.3).

The difficult part is proving that ψ is smooth. Since we are using the uniform contraction mapping principle, we obtain that the fixed point Ψ is only as smooth in z_0^c as the uniform contraction mapping G is. The Nemytskii operator associated to g is in general not smooth as a mapping from $\mathcal{Y}_\alpha$ to itself. However, it is C^k-smooth as a mapping from $\mathcal{Y}_\zeta$ to $\mathcal{Y}_\alpha$ whenever $k\zeta < \alpha$. Using this fact, one may show with a clever application of the Fiber Contraction Theorem to the map G that $\Psi : \mathcal{Z}^c \to \mathcal{Y}_\alpha$ is C^k-smooth, and thus $\psi : \mathcal{Z}^c \to \mathcal{Z}^h$ is C^k-smooth.

To complete the theorem, it remains only to remove the assumption that the Lipschitz constant of g is small. This is done by considering the modified system

$$z_{n+1} = Az_n + \tilde{g}(z_n) \tag{4.8}$$

where $\tilde{g}(z) = g(z)\chi(z)$. Here $\chi(z) = \chi_0(\frac{\|z^c\|}{\varepsilon})\chi_0(\frac{\|z^h\|}{\varepsilon})$ where $\chi_0 : [0,\infty) \to \mathbb{R}$ is a smooth cutoff function which is identically $+1$ on $[0, 1]$ and vanishes identically on $[2, \infty)$. By choosing ε small enough, we can make the Lipschitz constant of g as small as we wish on the region on which $\chi(z)$ is nonzero, and it follows from what we have sketched above that the system (4.8) has a global center manifold. As for smoothness of this manifold, recall that in general Banach spaces need not have smooth norms. However, as we have assumed that $\mathcal{Z}^c$ is finite dimensional, we may assume that its norm is smooth. Thus $\chi(z)$ depends smoothly on z^c, but in general is only Lipschitz in z^h. It follows that χ and thus $G(z_0^c, \cdot)$ is smooth in the region $\|z^h\| < \varepsilon$ where the cutoff function in the hyperbolic direction is

constant. One now uses exponential estimates on the map G to show that $\|\Psi(z_0^c)_n\| < \varepsilon$ for all coordinates of the fixed point $\Psi(z_0^c)$ of $G(z_0^c, \cdot)$, whenever $\|z_0^c\| < \varepsilon$. With this and the above remarks about smoothness, it follows that $\psi(z_0^c)$ depends smoothly on z_0^c for $\|z_0^c\| < \varepsilon$. Taking Ω^c and Ω^h to be the balls of radius ε in $\mathcal{Z}^c$ and $\mathcal{Z}^h$ centered at the origin, the result now follows.   □

In addition to the existence of a smooth local center manifold, our proof of Theorem 1.1 requires the following shadowing lemma which guarantees that each orbit which stays close to the center manifold approaches an orbit on the center manifold exponentially fast. This is classical, and generally a consequence of the existence of a center-stable foliation. We do not require the full apparatus here so we provide an independent proof for the specific estimate that we need. The proof is similar to the construction of a stable manifold.

Lemma 4.2 *[Shadowing Lemma] Consider the setting of Theorem 4.1, with $\alpha > 0$ as in the statement of that result. Then there exists a neighborhood $\Omega \subseteq \mathcal{Z}$ of the origin such that any forward solution $z = \{z_n\}_{n \geq 0} \subseteq \Omega$ to the full system (4.3) which lies in Ω possesses an exponentially close shadow on the center manifold. More precisely, there exists a positive constant K such that for any such solution to Eq. 4.3, there exists a sequence $y^c = \{y_n^c\}_{n \geq 0} \subseteq \mathcal{Z}^c$ which satisfies the reduced system (4.2) and such that*

$$\|z_n - y_n^c - \psi(y_n^c)\| \leq K e^{-\alpha n} \|z_0 - z_0^c - \psi(z_0^c)\|, \qquad n \geq 0.$$

Proof The proof breaks up naturally into two parts. First we show that z_n approaches the center manifold exponentially fast. Then we show that z_n differs from a particular orbit on the center manifold by an exponentially decreasing amount. Throughout, we work in a sufficiently small neighborhood Ω of the origin in $\mathcal{Z}$. In what follows we shall let $\tilde{\Omega}$ denote the neighborhood which was denoted by Ω in the statement of Theorem 4.1, and which contains the center manifold. We shall let Ω denote a (possibly) smaller neighborhood $\Omega \subseteq \tilde{\Omega}$ which will be constructed below and for which the statement of the present lemma is valid. We shall otherwise keep the same notation as in the proof of Theorem 4.1.

We first introduce new coordinates (z^c, w) replacing (z^c, z^h) in a neighborhood of the origin, where $w \in \mathcal{Z}^h$ replaces z^h and is defined by $w = z^h - \psi(z^c)$, and with the coordinate z^c unchanged. Thus $w = 0$ if and only if z lies on the center manifold, in this neighborhood, and we have $z = z^c + \psi(z^c) + w$. The evolution equation for the full system (4.3) written in the new coordinates takes the form

$$\begin{aligned}
z_{n+1}^c &= A^c z_n^c + g^c(z_n^c + \psi(z_n^c) + w_n), \\
w_{n+1} &= A^h w_n + A^h \psi(z_n^c) + g^h(z_n^c + \psi(z_n^c) + w_n) \\
&\quad - \psi(A^c z_n^c + g^c(z_n^c + \psi(z_n^c) + w_n)).
\end{aligned} \qquad (4.9)$$

It is a consequence of the invariance of the center manifold that ψ satisfies the functional relation

$$\psi(A^c u + g^c(u + \psi(u))) = A^h \psi(u) + g^h(u + \psi(u)),$$

and it follows from this that we may rewrite the system (4.9) as

$$\begin{aligned}
z_{n+1}^c &= A^c z_n^c + \tilde{g}^c(z_n^c, w_n), \\
w_{n+1} &= A^h w_n + \tilde{g}^h(z_n^c, w_n),
\end{aligned}$$

where $\tilde{g}^c$ and $\tilde{g}^h$ are given by

$$
\begin{aligned}
\tilde{g}^c(u, w) &= g^c(u + \psi(u) + w), \\
\tilde{g}^h(u, w) &= g^h(u + \psi(u) + w) - g^h(u + \psi(u)) \\
&\quad + \psi(A^c u + g^c(u + \psi(u))) - \psi(A^c u + g^c(u + \psi(u) + w)).
\end{aligned}
$$

Observe that $\tilde{g}^h(u, 0) = 0$ identically, and also that $D\tilde{g}^h(0, 0) = 0$ since $Dg(0) = 0$ and $D\psi(0) = 0$.

Now let $w^s = \pi^s w$ and $w^u = \pi^u w$ denote the projections of w onto the stable and unstable subspaces, and so $w = w^s + w^u$. Let $\tilde{g}^s(u, w) = \pi^s \tilde{g}^h(u, w)$ and $\tilde{g}^u(u, w) = \pi^u \tilde{g}^h(u, w)$, and note that $\tilde{g}^s(u, 0) = 0$ and $\tilde{g}^u(u, 0) = 0$ identically, and that $D\tilde{g}^s(0, 0) = 0$ and $D\tilde{g}^u(0, 0) = 0$. Much as in the proof of Theorem 4.1, we may write the variation of constants formulae

$$
w_n^s = (A^s)^n w_0^s + \sum_{j=0}^{n-1} (A^s)^{n-1-j} \tilde{g}^s(z_j^c, w_j),
$$

$$
w_n^u = -\sum_{j=n}^{\infty} (A^u)^{n-1-j} \tilde{g}^u(z_j^c, w_j), \tag{4.10}
$$

although we take a finite sum here for the stable part in contrast to the proof of Theorem 4.1. (We have also omitted the formula for z_n^c as it will not be needed here.) Adding the two formulae in (4.10) gives

$$
w_n = (A^s)^n w_0^s + \sum_{j=0}^{n-1} (A^s)^{n-1-j} \tilde{g}^s(z_j^c, w_j) - \sum_{j=n}^{\infty} (A^u)^{n-1-j} \tilde{g}^u(z_j^c, w_j), \tag{4.11}
$$

for every $n \geq 0$, and which is valid for any forward orbit $z = \{z_n\}_{n \geq 0}$ lying in $\tilde{\Omega}$.

We wish to show that the coordinates w_n decay exponentially to zero provided the sequence z_n lies in a sufficiently small neighborhood Ω of the origin. Our approach is to regard the coordinates z_n^c and also w_0^s as known, and to consider Eq. 4.11 as a fixed point problem for the bounded sequence $w = \{w_n\}_{n \geq 0}$. More precisely, we shall show this sequence is in fact a fixed point both in a space of bounded sequences, and also in a space of exponentially decaying sequences.

To this end, for any $\zeta \geq 0$ define the Banach space $\mathcal{R}_\zeta^h$ of one-sided sequences in $\mathcal{Z}^h$

$$
\mathcal{R}_\zeta^h = \{r : \mathbb{N}_0 \to \mathcal{Z}^h \mid \|r\|_\zeta < \infty\}, \qquad \|r\|_\zeta = \sup_{n \geq 0} e^{\zeta n} \|r_n\|_{\mathcal{Z}}, \tag{4.12}
$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. (Contrast the definition of this space of one-sided decaying sequences, with that of $\mathcal{Y}_\zeta$ which has two-sided growing sequences. We trust that the same notation $\|\cdot\|_\zeta$ for the two norms will not lead to confusion.) Next, with α as in the statement of the theorem, there exist $\gamma > \alpha$ and $C > 0$ such that

$$
\|(A^s)^n\|, \|(A^u)^{-n}\| \leq Ce^{-\gamma n}, \qquad n \geq 0. \tag{4.13}
$$

Now fix a neighborhood $\Omega \subseteq \tilde{\Omega}$ of the origin and take small enough quantities ε and δ such that the following all hold. First,

$$
\|w^s\| \leq \frac{\varepsilon}{2C} \quad \text{whenever } z \in \Omega, \tag{4.14}
$$

where as usual $w^s = \pi^s w$ with $w = z^h - \psi(z^c)$. Next,

$$\|D_r \tilde{g}^s(z^c, r)\|, \|D_r \tilde{g}^u(z^c, r)\| \leq \delta$$
$$\text{whenever } z \in \Omega \text{ and } r \in \mathcal{Z}^h \text{ with } \|r\| \leq \varepsilon, \qquad (4.15)$$

with D_r denoting the derivative with respect to the second argument. Here r need not be the w-coordinate arising from z as above, but rather is an arbitrary point in the closed ε-ball about the origin in $\mathcal{Z}^h$. Finally, we require that the inequality (4.18) below with (4.16) should hold both for $\zeta = \alpha$ and for $\zeta = 0$. It is easily seen that all this can be accomplished by first fixing δ, and then ε, and then taking Ω sufficiently small.

Now let $\{z_n\}_{n\geq 0} \subseteq \Omega$ be as in the statement of the lemma, with $w_n = z_n^h - \psi(z_n^c)$ as usual. Denote the closed ε-ball about the origin in $\mathcal{R}_\zeta^h$ by

$$B_\zeta(\varepsilon) = \{r = \{r_n\}_{n\geq 0} \in \mathcal{R}_\zeta^h \mid \|r\|_\zeta \leq \varepsilon\},$$

and define a map $G : B_\zeta(\varepsilon) \to \mathcal{R}_\zeta^h$ by setting

$$G(r)_n = (A^s)^n w_0^s + \sum_{j=0}^{n-1} (A^s)^{n-1-j} \tilde{g}^s(z_j^c, r_j) - \sum_{j=n}^{\infty} (A^u)^{n-1-j} \tilde{g}^u(z_j^c, r_j)$$

for $n \geq 0$. Of course it must be shown that $G(r)$ actually lies in $\mathcal{R}_\zeta^h$. We show this, and will also bound the Lipschitz constant of the map G, which will give conditions under which G is a contraction mapping.

Fix ζ satisfying $0 \leq \zeta < \gamma$ and take any $r, \bar{r} \in B_\zeta(\varepsilon)$. Then $\|r_n - \bar{r}_n\| \leq e^{-\zeta n}\|r - \bar{r}\|_\zeta$, and using (4.13) and (4.15) we have that

$$\|G(r)_n - G(\bar{r})_n\|$$
$$\leq C\delta \sum_{j=0}^{n-1} e^{-\gamma(n-1-j)} e^{-\zeta j}\|r - \bar{r}\|_\zeta + C\delta \sum_{j=n}^{\infty} e^{-\gamma(j+1-n)} e^{-\zeta j}\|r - \bar{r}\|_\zeta$$

$$< \tilde{C}(\zeta) C\delta e^{-\zeta n}\|r - \bar{r}\|_\zeta$$

after a short calculation, where

$$\tilde{C}(\zeta) = \frac{e^\zeta}{1 - e^{-(\gamma-\zeta)}} + \frac{e^{-\gamma}}{1 - e^{-(\gamma+\zeta)}}. \qquad (4.16)$$

Thus we have that

$$\|G(r) - G(\bar{r})\|_\zeta \leq \tilde{C}(\zeta) C\delta\|r - \bar{r}\|_\zeta.$$

Next note that the sequence $(A^s)^n w_0^s$, which is $G(0)$, lies in the space $\mathcal{R}_\zeta^h$ with the bound

$$\|G(0)\|_\zeta \leq C\|w_0^s\| \leq \frac{\varepsilon}{2} \qquad (4.17)$$

on its norm, from (4.13) and (4.14). It follows immediately that if

$$\tilde{C}(\zeta) C\delta \leq \frac{1}{2} \qquad (4.18)$$

holds, then G is a contraction mapping of $B_\zeta(\varepsilon)$ into itself, and thus has a fixed point. In fact, we have assumed this to be the case for the choices $\zeta = 0$ and $\zeta = \alpha$. Moreover, we have that $B_\alpha(\varepsilon) \subseteq B_0(\varepsilon)$ and so the fixed points in these two balls are identical, namely, the bounded

sequence $\{w_n\}_{n\geq 0}$ obtained from the original orbit $\{z_n\}_{n\geq 0}$ and which satisfies (4.11). Thus $\{w_n\}_{n\geq 0}$ in fact lies in $B_\alpha(\varepsilon)$. Additionally, as the contraction constant of G in this ball is bounded by $\frac{1}{2}$, it follows that $\|w\|_\alpha \leq 2\|G(0)\|_\alpha$, which with (4.17) yields the bound

$$\|w_n\| \leq 2Ce^{-\alpha n}\|w_0^s\|.$$

We now show the existence of a shadow orbit $\{y_n^c\}_{n\geq 0}$, as in the statement of the lemma. As this part of the proof is very similar to the part above, and in particular involves a contraction mapping in a similar space $\mathcal{R}_\zeta^c$, we only sketch the argument. Here $\mathcal{R}_\zeta^c$ and its norm $\|\cdot\|_\zeta$ are defined as before (4.12), except that the elements of $\mathcal{R}_\zeta^c$ are sequences $r : \mathbb{N}_0 \to \mathcal{Z}^c$ in the space $\mathcal{Z}^c$. We write $u_n = y_n^c - z_n^c$. As z_n^c satisfies the first equation in (4.9) and we wish y_n^c to satisfy the reduced Eq. 4.2, then u_n must satisfy

$$u_{n+1} = A^c u_n + \hat{g}(z_n^c, u_n) + q_n \tag{4.19}$$

where

$$\hat{g}(z^c, u) = \tilde{g}^c(z^c + u, 0) - \tilde{g}^c(z^c, 0), \qquad q_n = \tilde{g}^c(z^c, 0) - \tilde{g}^c(z^c, w_n).$$

A sufficient condition for (4.19) to hold for $n \geq 0$, with u_n decaying exponentially, is that

$$u_n = Q_n - \sum_{j=n}^{\infty} (A^c)^{n-1-j} \hat{g}(z_j^c, u_j) \tag{4.20}$$

where

$$Q_n = -\sum_{j=n}^{\infty} (A^c)^{n-1-j} q_j.$$

In particular, one has the (growing) estimate

$$\|(A^c)^{-n}\| \leq Ce^{\kappa n}, \qquad n \geq 0,$$

where $\kappa > 0$ can be taken arbitrarily small, and where $C = C(\kappa)$. One checks that $\{q_n\}_{n\geq 0}$ and thus $\{Q_n\}_{n\geq 0}$ belong to the space $\mathcal{R}_\alpha^c$, and also that Eq. 4.20 possesses a fixed point near the origin in this space. This concludes the sketch of the proof of the existence of a shadow orbit and hence the proof of the lemma.   □

The existence of a center manifold for our particular system (1.22) follows from the Center Manifold Theorem upon checking that we have a spectral gap at the unit circle. The following lemma establishes the existence of a spectral gap and in addition characterizes the center subspace for a class of difference equations which includes (1.22). Here and below, we let $\mathrm{col}(\cdot, \cdot)$ denote a column vector in the usual fashion.

Lemma 4.3 *Let $L \in \mathcal{L}(\mathcal{X})$ be an operator for which $0 \in \sigma(L)$ is an isolated point of the spectrum, and let $\pi^0 \in \mathcal{L}(\mathcal{X})$ denote the spectral projection onto the spectral subspace of $\mathcal{X}$ corresponding to this point. Let $\mathcal{Z} = \mathcal{X} \times \mathcal{X}$ and define $A \in \mathcal{L}(\mathcal{Z})$ by*

$$A = \begin{pmatrix} 2I - L & -I \\ I & 0 \end{pmatrix}.$$

Then $1 \in \sigma(A)$ is an isolated point in the spectrum of A, and the spectral projection $\Pi \in \mathcal{L}(\mathcal{Z})$ corresponding to this point is given by

$$\Pi = \begin{pmatrix} \pi^0 & 0 \\ 0 & \pi^0 \end{pmatrix}. \tag{4.21}$$

Also, if there exists $\beta_1 > 0$ such that

$$\sigma(L) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda < -\beta_1\} \cup \{0\} \tag{4.22}$$

then there exists $\beta_2 > 0$ such that

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid ||\lambda| - 1| > \beta_2\} \cup \{1\}. \tag{4.23}$$

Finally, if the operator L is Fredholm of index zero and has a one-dimensional generalized kernel spanned by $v \in \mathcal{X} \setminus \{0\}$, that is $\ker(L) = \{av \mid a \in \mathbb{R}\}$ with $v \notin \operatorname{ran}(L)$, then

$$\operatorname{ran}(\pi^0) = \{av \mid a \in \mathbb{R}\}, \qquad \operatorname{ran}(\Pi) = \{\operatorname{col}(av, bv) \mid (a, b) \in \mathbb{R}^2\},$$

are the above-mentioned spectral subspaces of L and A, and they have dimension one and two, respectively.

Proof We first show that

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} \setminus \{0\} \mid 2 - \lambda - \lambda^{-1} \in \sigma(L)\}. \tag{4.24}$$

Suppose that $\lambda \neq 0$ is such that $2 - \lambda - \lambda^{-1} \notin \sigma(L)$, and denote $\Gamma(\lambda) = ((2 - \lambda - \lambda^{-1})I - L)^{-1}$. We claim that

$$(\lambda I - A)^{-1} = \begin{pmatrix} -\lambda I & I \\ -I & (2 - \lambda)I - L \end{pmatrix} \begin{pmatrix} \lambda^{-1}\Gamma(\lambda) & 0 \\ 0 & \lambda^{-1}\Gamma(\lambda) \end{pmatrix}, \tag{4.25}$$

and thus $\lambda \notin \sigma(A)$. Indeed, this is straightforwardly proved by multiplying the above matrix product by $\lambda I - A$, to obtain the identity matrix after a brief calculation. We omit the details. Thus (4.24) follows from this, and from the fact that $0 \notin \sigma(A)$, which holds because

$$A^{-1} = \begin{pmatrix} 0 & I \\ -I & 2I - L \end{pmatrix}.$$

It now follows from (4.24), and from the easily-checked fact that $2 - \lambda - \lambda^{-1} \neq 0$ whenever $\lambda \neq 1$, that if $\lambda = 1$ belongs to the spectrum of A then it is an isolated point of the spectrum.

It is also easy to establish (4.23) assuming that (4.22) holds. Indeed, with (4.22) holding assume that $\lambda \in \sigma(A) \setminus \{1\}$. Then $2 - \lambda - \lambda^{-1} \in \sigma(L) \setminus \{0\}$ and so $2 - \operatorname{Re}(\lambda + \lambda^{-1}) < -\beta_1$. Writing $\lambda = |\lambda|e^{i\theta}$, we see that

$$|\lambda| + |\lambda|^{-1} \geq (|\lambda| + |\lambda|^{-1}) \cos\theta = \operatorname{Re}(\lambda + \lambda^{-1}) > 2 + \beta_1.$$

Denoting the two positive roots of $r + r^{-1} = 2 + \beta_1$ by $r_\pm$ with $r_- < 1 < r_+$, we have that either $|\lambda| < r_-$ or $|\lambda| > r_+$, and thus (4.23) holds with $\beta_2 = \min\{1 - r_-, r_+ - 1\}$.

We next calculate the spectral projection Π for A corresponding to the point $\lambda = 1$. (At this point we do not yet know that $1 \in \sigma(A)$, however, this fact will follow when we show that $\Pi \neq 0$.) For sufficiently small r we have that

$$\Pi = \frac{1}{2\pi i} \int_{|\lambda - 1| = r} (\lambda I - A)^{-1} \, d\lambda.$$

We calculate the above integral for each of the four block entries of the matrix (4.25), but for simplicity we shall only provide the details for one of the entries, as the approach for the others is similar. We take the lower right-hand entry

$$\Pi_{2,2} = \frac{1}{2\pi i} \int_{|\lambda-1|=r} ((2-\lambda)I - L)\lambda^{-1}\Gamma(\lambda)\,d\lambda$$

$$= \frac{1}{2\pi i} \int_{|\nu|=r} (L + (\nu-1)I)(\nu^2 I + (\nu+1)L)^{-1}\,d\nu, \tag{4.26}$$

with $\nu = \lambda - 1$, as one sees after a short calculation using the above formula for $\Gamma(\lambda)$. We apply the projections π^0 and π^1, where we denote $\pi^1 = I - \pi^0$, to the second integral in (4.26) to get

$$\pi^j \Pi_{2,2} = \frac{1}{2\pi i} \int_{|\nu|=r} (L^j + (\nu-1)I)(\nu^2 I + (\nu+1)L^j)^{-1}\,d\nu \tag{4.27}$$

for $j = 1, 2$, where $L^j = \pi^j L$ is regarded as an operator on the spectral subspace $\pi^j \mathcal{X} \subseteq \mathcal{X}$, and we separately calculate these contour integrals. Now $0 \notin \sigma(L^1)$, and thus the integrand $(L^1 + (\nu-1)I)(\nu^2 I + (\nu+1)L^1)^{-1}$ in (4.27) is holomorphic in ν in a neighborhood of $\nu = 0$. Thus the integral vanishes by Cauchy's Theorem, and so $\pi^1 \Pi_{2,2} = 0$. On the other hand, for the case $j = 0$ we have $\sigma(L^0) = \{0\}$, and so the integrand is holomorphic for all complex ν except $\nu = 0$. In this case we may increase the radius r of the contour arbitrarily, again by Cauchy's Theorem. Upon doing so, and then scaling ν, we write the integral in the equivalent form

$$\pi^0 \Pi_{2,2} = \frac{1}{2\pi i} \int_{|\nu|=1} \nu^{-1}((r\nu)^{-1}L^0 + (1-(r\nu)^{-1})I)(I + ((r\nu)^{-1} + (r\nu)^{-2})L^0)^{-1}\,d\nu.$$

In the limit $r \to \infty$ we obtain

$$\pi^0 \Pi_{2,2} = \frac{1}{2\pi i} \int_{|\nu|=1} \nu^{-1}I\,d\nu = I,$$

which is the identity operator on the space $\pi^0 \mathcal{X}$. It thus follows that $\Pi_{2,2} = \pi^0$. The proofs for the remaining three cases are similar, in which one shows that $\Pi_{1,1} = \pi^0$ and $\Pi_{1,2} = \Pi_{2,1} = 0$. This establishes (4.21).

The final statement of the lemma follows immediately from the fact that with L having the Fredholm properties described, we have $\mathcal{X} = \ker(L) \oplus \mathrm{ran}(L)$, and so the range of the spectral projection π^0 is the one-dimensional subspace $\ker(L) \subseteq \mathcal{X}$ spanned by v. $\qquad\square$

In the following proposition we construct a center manifold for the difference Eq. 4.1 and compute the reduced equations on it. To begin, let us rewrite (4.1) as a system in the product space $\mathcal{Z} = \mathcal{X} \times \mathcal{X}$ where $\mathcal{X} = \ell^\infty(\mathbb{Z})$, namely

$$Y_{m+1} = (2I - L)Y_m - W_m + g(Y_m),$$
$$W_{m+1} = Y_m, \tag{4.28}$$

where $L = L_p(a_+(0))$ is the operator in (1.20) and where $g : \mathcal{X} \to \mathcal{X}$ is given by

$$g(Y) = f(p + Y, a_+(0)) - f(p, a_+(0)) - f'(p, a_+(0))Y. \tag{4.29}$$

Thus we have the system $Z_{m+1} = AZ_m + G(Z_m)$ with $Z \in \mathcal{Z}$, and $A \in \mathcal{L}(\mathcal{Z})$ and $G : \mathcal{Z} \to \mathcal{Z}$ given by

$$Z = \begin{pmatrix} Y \\ W \end{pmatrix}, \qquad A = \begin{pmatrix} 2I - L & -I \\ I & 0 \end{pmatrix}, \qquad G(Z) = \begin{pmatrix} g(Y) \\ 0 \end{pmatrix}.$$

The following center manifold reduction holds for this system.

Proposition 4.4 *Let Condition A hold and take v as in Proposition 1.4. Then the system (4.28), (4.29) in the space $\mathcal{Z}$ satisfies the conditions of Theorem 4.1(the Center Manifold Theorem) with the associated center and hyperbolic subspaces $\mathcal{Z}^c$ and $\mathcal{Z}^h$ of $\mathcal{Z}$ given by*

$$
\begin{aligned}
\mathcal{Z}^c &= \mathcal{X}^c \times \mathcal{X}^c, \qquad \mathcal{X}^c = \{av \mid a \in \mathbb{R}\}, \\
\mathcal{Z}^h &= \mathcal{X}^h \times \mathcal{X}^h, \qquad \mathcal{X}^h = \{x \in \mathcal{X} \mid \langle v, x \rangle = 0\},
\end{aligned}
\tag{4.30}
$$

and with a center manifold $W^c \subseteq \mathcal{Z}$ given by

$$
W^c = \{\mathrm{col}(\eta v + \psi_1(\eta, \omega), \omega v + \psi_2(\eta, \omega)) \mid (\eta, \omega) \in \Omega^c \subseteq \mathbb{R}^2\}
\tag{4.31}
$$

for smooth functions $\psi_i : \Omega^c \to \mathcal{X}^h$ satisfying $\psi_i(0) = 0$ and $D\psi_i(0) = 0$, where $\Omega^c \subseteq \mathbb{R}^2$ is a neighborhood of the origin. The reduced system on W^c, in the coordinates $(\eta, \omega) \in \Omega^c$, has the form

$$
\begin{aligned}
\eta_{m+1} &= 2\eta_m - \omega_m + \gamma(\eta_m, \omega_m), \\
\omega_{m+1} &= \eta_m,
\end{aligned}
\tag{4.32}
$$

where $\gamma : \Omega^c \to \mathbb{R}$ satisfies

$$
\gamma(\eta, \omega) = \langle v, g(\eta v + \psi_1(\eta, \omega)) \rangle = B\eta^2 + o(|\eta|^2 + |\omega|^2),
\tag{4.33}
$$

where $B \in \mathbb{R}$ is the quantity (1.17) in Condition B.

Proof Proposition 1.4 implies that the operator $L = L_p(a_+(0))$ satisfies all the conditions of Lemma 4.3, including (4.22), along with the Fredholm condition and statements about its kernel and range, and with the spectral projection $\pi^0 \in \mathcal{L}(\mathcal{X})$ given by $\pi^0 x = \langle v, x \rangle v$. It thus follows by Lemma 4.3 and because $G(0) = 0$ and $DG(0) = 0$, that the conditions of Theorem 4.1 (the Center Manifold Theorem) hold for the system (4.28) with $\mathcal{Z}^c$ and $\mathcal{Z}^h$ as in (4.30). Thus there exists a center manifold (4.31) with functions ψ_i as stated.

All that remains is to verify the form of the reduced system. We take

$$
Z_m = \begin{pmatrix} Y_m \\ W_m \end{pmatrix} = \begin{pmatrix} \eta_m v + \psi_1(\eta_m, \omega_m) \\ \omega_m v + \psi_2(\eta_m, \omega_m) \end{pmatrix}
$$

in W^c, and similarly with Z_{m+1}, in the system (4.28). We then apply the projection Π given by (4.21), that is, we take the inner product of each equation with v. In doing so we note that $\langle v, \psi_i(\eta, \omega) \rangle = 0$ identically since the range of ψ_i lies in $\mathcal{X}^h$, and also that $\langle v, x \rangle = 0$ for every $x \in \mathrm{ran}(L)$, by Proposition 1.4. This directly gives (4.32) with the first equality in (4.33). The second equality in (4.33) follows from the fact that $g(0) = 0$ and $Dg(0) = 0$, and also $\psi_1(0, 0) = 0$ and $D\psi_1(0, 0) = 0$, which implies that

$$
\langle v, g(\eta v + \psi_1(\eta, \omega)) \rangle = \frac{1}{2} \langle v, D^2 g(0)(v, v) \rangle \eta^2 + o(|\eta|^2 + |\omega|^2),
$$

where $D^2 g(0)(\cdot, \cdot)$ denotes the usual bilinear form of the second derivative. Continuing, we have

$$
\langle v, D^2 g(0)(v, v) \rangle = \langle v, f''(p, a_+(0))(v, v) \rangle = \sum_{n=-\infty}^{\infty} f''(p_n, a_+(0)) v_n^3 = 2B,
$$

to give the result as claimed. This completes the proof. $\qquad\square$

5 The Proof of Theorem 1.1

Theorem 1.1 now follows from a convexity argument applied to the reduced equations. The only remaining preparatory results guarantee that the shadow orbits corresponding to $X_m - p$ and $SX_{-m} - p$ are not identically zero.

The following lemma, which is related to an exercise in Coddington and Levinson's classic text (see Exercise 28 on p. 103 of [10]), will allow us to prove that X_m decays only polynomially fast, hence its shadow orbit on the center manifold whose existence is guaranteed by Lemma 4.2 is not the zero orbit.

Lemma 5.1 *Let d_n for $n \geq 1$ be a sequence of real numbers which satisfies*

$$\sum_{n=1}^{\infty} n^3 |d_n| < \infty, \tag{5.1}$$

and consider the difference equation

$$x_{n+1} = (2 + d_n)x_n - x_{n-1}. \tag{5.2}$$

Then given any initial condition $(x_0, x_1) \in \mathbb{R}^2$, there exist quantities $(P, Q) \in \mathbb{R}^2$ such that

$$x_n = Pn + Q + o(1) \tag{5.3}$$

as $n \to \infty$. Conversely, given any $(P, Q) \in \mathbb{R}^2$, there exists a unique initial condition $(x_0, x_1) \in \mathbb{R}^2$ such that (5.3) holds. In particular, if $\lim_{n \to \infty} x_n = 0$ then $x_n = 0$ for every n.

Proof Let $\{x_n\}_{n \geq 0}$ satisfy (5.2) and denote $u_n = \mathrm{col}(x_{n+1}, x_n)$ for $n \geq 0$. Let U_n be the transition matrix

$$U_n = U + d_{n+1}\Gamma, \qquad U = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and so $u_{n+1} = U_n u_n$. Note here that

$$U^n = \begin{pmatrix} n+1 & -n \\ n & -n+1 \end{pmatrix} \tag{5.4}$$

for every integer n, as is easily proved by induction. Now let $v_n = U^{-n}u_n$ and observe that v_n evolves according to the equation

$$v_{n+1} = (I + R_n)v_n, \qquad R_n = d_{n+1}U^{-(n+1)}\Gamma U^n = d_{n+1}\begin{pmatrix} -n(n+1) & n^2 \\ -(n+1)^2 & n(n+1) \end{pmatrix},$$

by a simple calculation. Next, if $n \geq m \geq 0$ let

$$T_{n,m} = (I + R_{n-1})(I + R_{n-2}) \cdots (I + R_m) \tag{5.5}$$

with $T_{n,n} = I$, and so $v_n = T_{n,m}v_m$. Observe that

$$\|T_{n,m}\| \leq \prod_{j=m}^{n-1}(1 + \|R_j\|) \leq \exp\left(\sum_{j=m}^{n-1} \|R_j\|\right),$$

$$\|T_{n,m} - I\| \leq \left(\prod_{j=m}^{n-1}(1 + \|R_j\|)\right) - 1 \leq \exp\left(\sum_{j=m}^{n-1} \|R_j\|\right) - 1. \tag{5.6}$$

The first inequality in the second line of (5.6) may require a brief explanation. This inequality is obtained by first expanding the matrix product in (5.5), then subtracting the term I, thereby obtaining a polynomial in the matrices R_j. Next one takes the norm of this polynomial, and passes the norm across all the terms, obtaining the same polynomial but now in the scalar quantities $\|R_j\|$. The polynomial so obtained is the second term in the second line of (5.6), as desired.

We have for some $C_1 > 0$ that $\|R_n\| \leq C_1(1 + n^2)|d_{n+1}|$ for every $n \geq 0$, and so from (5.1) and (5.6) there exists $C_2 > 0$ such that $\|T_{n,m}\| \leq C_2$ for every $n \geq m \geq 0$. Further,

$$\|T_{n,0} - T_{m,0}\| = \|(T_{n,m} - I)T_{m,0}\| \leq C_2 \left(\exp\left(\sum_{j=m}^{n-1} \|R_j\| \right) - 1 \right)$$

$$\leq C_2 \left(\exp\left(\sum_{j=m}^{\infty} C_1(1 + j^2)|d_{j+1}| \right) - 1 \right) = r_m$$

with the above equation serving as the definition of the quantity r_m. As $\lim_{m\to\infty} r_m = 0$, it follows that $T_{n,0}$ is a Cauchy sequence of matrices, so the limit $T_{\infty,0} = \lim_{n\to\infty} T_{n,0}$ exists. The vectors v_n as well possess a limit

$$v_\infty = \lim_{n\to\infty} v_n = T_{\infty,0}v_0 = T_{\infty,0}u_0.$$

Let us also note the estimate

$$m r_m \leq m C_3 \sum_{j=m}^{\infty} j^2|d_{j+1}| \leq C_3 \sum_{j=m}^{\infty} j^3|d_j|$$

for some $C_3 > 0$, which follows from the Mean Value Theorem, and which, with (5.1), implies that $\lim_{m\to\infty} m r_m = 0$. We thus have that

$$u_n = U^n v_n = U^n v_\infty + U^n(v_n - v_\infty),$$

and as $\|U^n\| \leq C_4(n + 1)$ for some $C_4 > 0$, by (5.4), we have that

$$\|U^n(v_n - v_\infty)\| \leq C_4(n + 1)\|T_{n,0} - T_{\infty,0}\|\|u_0\|$$
$$\leq C_4(n + 1)r_n\|u_0\| \to 0, \quad \text{as } n \to \infty.$$

Thus $u_n = U^n v_\infty + o(1)$, and denoting $v_\infty = \text{col}(A, B)$ it follows directly from (5.4) that the second coordinate of u_n, which is x_n, has the form

$$x_n = An - B(n - 1) + o(1).$$

This proves (5.3) with $P = A - B$ and $Q = B$.

The converse is proved more or less by following the above steps in reverse. Namely, given $(P, Q) \in \mathbb{R}^2$, then let $v_\infty = \text{col}(A, B)$ where $A = P + Q$ and $B = Q$, and let $u_0 = T_{\infty,0}^{-1}v_\infty$. We note that the matrix $T_{\infty,0}$ is invertible as $\det(I + R_j) = 1$ for every j, and thus $\det T_{n,m} = 1$ for every $n \geq m \geq 0$, and thus $\det T_{\infty,0} = 1$. The required initial condition is thus $u_0 = \text{col}(x_1, x_0)$, and we see that it is unique.

The final sentence in the statement of the lemma follows in particular from the uniqueness of (x_0, x_1) for a given (P, Q). □

Proposition 5.2 *Let x_n for $n \geq 0$ be a sequence of numbers satisfying*

$$x_{n+1} - 2x_n + x_{n-1} = Mx_n^2 + \rho(x_{n-1}, x_n, x_{n+1}), \quad n \geq 1, \tag{5.7}$$

and

$$x_n \neq 0 \text{ for infinitely many } n, \quad \lim_{n \to \infty} x_n = 0, \tag{5.8}$$

where $M \neq 0$ and where the function $\rho : \mathbb{R}^3 \to \mathbb{R}$ is C^2 and satisfies

$$\rho(\beta_-, \beta_0, \beta_+) = o(|\beta_-|^2 + |\beta_0|^2 + |\beta_+|^2) \tag{5.9}$$

at the origin. Then

$$3Mx_n > Mx_{n-1} > Mx_n > 0 \text{ for all large } n.$$

Let us remark that we expect in fact $\lim_{n \to \infty} \frac{x_n}{x_{n-1}} = 1$ should hold in the above proposition, although do not need this fact.

Proof Without loss, we shall assume that $M > 0$, as the case $M < 0$ follows by considering the sequence $-x_n$ in place of x_n.

We first show that

$$(x_{n-1}, x_n, x_{n+1}) \neq (0, 0, 0) \text{ for all large } n. \tag{5.10}$$

If (5.10) is false, then there are infinitely many n for which $x_{n-1} = x_n = x_{n+1} = 0$ but $x_{n+2} \neq 0$, in light of the first statement in (5.8). For such n, Eq. 5.7 with $n + 1$ replacing n takes the form $x_{n+2} = \rho(0, 0, x_{n+2})$, and so

$$\frac{\rho(0, 0, x_{n+2})}{x_{n+2}} = 1. \tag{5.11}$$

However, Eq. 5.11 cannot hold for infinitely many n in light of (5.9) and the second statement in (5.8). With this contradiction (5.10) is proved.

We next show that for all sufficiently large n, the two inequalities

$$x_{n+1} \leq x_n, \quad x_{n-1} \leq x_n, \tag{5.12}$$

cannot simultaneously hold. Define a function $H_1 : \mathbb{R}^3 \to \mathbb{R}$ by

$$H_1(\beta_+, \beta_0, \beta_-) = \beta_+^2 + \beta_-^2 + M\beta_0^2 + \rho(\beta_0 - \beta_-^2, \beta_0, \beta_0 - \beta_+^2),$$

and note that H_1 has a strict local minimum $H_1(0, 0, 0) = 0$ at the origin, by (5.9). Now if n is such that both inequalities in (5.12) hold, let $\beta_\pm = (x_n - x_{n\pm1})^{1/2}$ and observe that Eq. 5.7 becomes $H_1(\beta_+, x_n, \beta_-) = 0$ for this n. If n is large enough this forces $\beta_\pm = x_n = 0$ due to the strict local minimum of H_1 and the limit in (5.8), and thus $x_{n-1} = x_n = x_{n+1} = 0$. But this cannot happen for infinitely many n by (5.10). Thus the inequalities (5.12) cannot simultaneously hold for arbitrarily large n.

It follows from the failure of (5.12) for all large n that the sequence x_n is eventually monotone. In fact, either

$$x_n > x_{n+1} > 0 \text{ for all large } n, \tag{5.13}$$

or else

$$x_n < x_{n+1} < 0 \text{ for all large } n. \tag{5.14}$$

We wish to prove (5.13), so let us assume to the contrary that (5.14) holds. We claim that

$$x_{n+1} - 2x_n + x_{n-1} < 0 \text{ for infinitely many } n. \tag{5.15}$$

If (5.15) is false, then we have that $x_{n+1} - x_n \geq x_n - x_{n-1} > 0$ for all large n. Thus there exists $\delta > 0$ such that $x_{n+1} - x_n \geq \delta$ for all large n. However, this forces $x_n \to \infty$ as $n \to \infty$, which is false. This establishes (5.15).

Still assuming (5.14), for every n for which the inequality in (5.15) holds let $\beta_\pm$ and β_0 be such that

$$\beta_+ = -x_{n+1}, \qquad \beta_0 = x_{n+1} - x_n, \qquad \beta_-^2 = -x_{n+1} + 2x_n - x_{n-1}.$$

We have that $\beta_+ > 0$ and $\beta_0 > 0$, and from Eq. 5.7 that

$$\begin{aligned}
0 &= \beta_-^2 + M(\beta_0 + \beta_+)^2 + \rho(-\beta_+ - 2\beta_0 - \beta_-^2, -\beta_+ - \beta_0, -\beta_+) \\
&> \beta_-^2 + M\beta_0^2 + M\beta_+^2 + \rho(-\beta_+ - 2\beta_0 - \beta_-^2, -\beta_+ - \beta_0, -\beta_+) \\
&= H_2(\beta_+, \beta_0, \beta_-), \tag{5.16}
\end{aligned}$$

with the above formula serving as the definition of the function $H_2 : \mathbb{R}^3 \to \mathbb{R}$. The function H_2 has a strict local minimum $H_2(0, 0, 0) = 0$ at the origin by (5.9), and with (5.16) this forces $\beta_\pm = \beta_0 = 0$ if n is sufficiently large. But this contradicts $\beta_+ > 0$ as noted above, and so (5.13) is established, as desired.

To complete the proof we must show that $3x_n \geq x_{n-1}$ for all large n. Upon dividing Eq. 5.7 by x_{n-1}, we have that

$$-\frac{2x_n}{x_{n-1}} + 1 < \frac{x_{n+1} - 2x_n}{x_{n-1}} + 1 = \frac{Mx_n^2 + \rho(x_{n-1}, x_n, x_{n+1})}{x_{n-1}} \to 0$$

as $n \to \infty$, in light of the ordering $x_{n-1} > x_n > x_{n+1} > 0$ established above. Thus $1 - \frac{2x_n}{x_{n-1}} < \frac{1}{3}$ for all large n, to give the result. $\qquad \square$

It is a consequence of Lemma 5.1 that the orbits X_m guaranteed by Proposition 2.4 decay to p as $m \to \infty$, or to $S^{-1}p$ as $m \to -\infty$, at a subexponential rate, as the following result shows.

Proposition 5.3 *Assume that the inequality (1.13) is an equality at $\theta_0 = 0$. Also assume that Condition A holds, with p as stated there, and let X_m be the solution to (1.22) guaranteed by Proposition 2.4. Let $Y_m = X_m - p$. Then*

$$\sum_{m=0}^{\infty} m^3 \|Y_m\| = \infty. \tag{5.17}$$

The same conclusion holds if instead we let $Y_m = SX_{-m} - p$.

Proof For definiteness we take $Y_m = X_m - p$. Also, we shall denote the coordinates of $Y_m \in \ell^\infty(\mathbb{Z})$ by $Y_m = \{y_{n,m}\}_{n \in \mathbb{Z}}$. Let $v \in \ell^\infty(\mathbb{Z}) \setminus \{0\}$ be as in Proposition 1.4, so of course $v \in \ell^1(\mathbb{Z})$, and let $y_m = \langle v, Y_m \rangle$. From the strict positivity $v_n > 0$ of Proposition 1.4 and the strict ordering (2.10) and the limits (2.9) in Proposition 2.4, it follows that

$$y_m < 0, \qquad \lim_{m \to \infty} y_m = 0. \tag{5.18}$$

Also, Y_m satisfies (4.1), so it follows using (1.16) that y_m satisfies

$$y_{m+1} = (2 + d_m)y_m - y_{m-1}$$

where

$$d_m = \frac{\langle v, f(p + Y_m, a_+(0)) - f(p, a_+(0)) - f'(p, a_+(0))Y_m \rangle}{y_m}.$$

Here we have used the fact that

$$\langle v, (S + S^{-1} - 2I)Y_m \rangle = \langle (S + S^{-1} - 2I)v, Y_m \rangle$$
$$= \langle f'(p, a_+(0))v, Y_m \rangle = \langle v, f'(p, a_+(0))Y_m \rangle.$$

We note that the conclusion of Lemma 5.1 fails for the sequence y_m, in light of (5.18), and so necessarily

$$\sum_{m=0}^{\infty} m^3 |d_m| = \infty. \tag{5.19}$$

By the Mean Value Theorem there are quantities $\varepsilon_{m,n} \in [y_{m,n}, 0]$ such that

$$f(p_n + y_{n,m}, a_+(0)) - f(p_n, a_+(0)) - f'(p_n, a_+(0))y_{n,m}$$
$$= f''(p_n + \varepsilon_{n,m}, a_+(0))y_{n,m}^2,$$

and we note that $p_n + \varepsilon_{n,m} \in [p_n + y_{n,m}, p_n] = [x_{n,m}, p_n] \subseteq [-1, 1]$. Thus

$$|d_m| = \frac{1}{|y_m|} \left| \sum_{n=-\infty}^{\infty} v_n f''(p_n + \varepsilon_{n,m}, a_+(0))y_{n,m}^2 \right|$$

$$\leq \frac{K\|Y_m\|}{|y_m|} \sum_{n=-\infty}^{\infty} v_n |y_{n,m}| = K\|Y_m\|, \tag{5.20}$$

where $K > 0$ is an upper bound for $|f''(u, a_+(0))|$ in the interval $[-1, 1]$. Note in particular, in the final equality in (5.20), that we have used the fact that $v_n > 0$ and $y_{n,m} < 0$. With this, the desired conclusion (5.17) follows from (5.19) and (5.20). $\qquad\square$

With these rates of convergence established, we may now compare the orbit Y_m to its shadow on the center manifold.

Proof of Theorem 1.1 Suppose that Condition A holds but that the inequality (1.13) is in fact an equality, and so crystallographic pinning does not occur in the direction $\theta_0 = 0$. We will show that condition B fails, namely, that $B = 0$ for the quantity B in formula (1.17). Therefore assume to the contrary that $B \neq 0$.

With the above assumptions, let X_m denote the solution of (1.22) guaranteed by Proposition 2.4. We first let $Y_m = X_m - p$, and so Y_m satisfies (4.1), equivalently, $Z_m = \mathrm{col}(Y_m, Y_{m-1})$ satisfies the system (4.28), (4.29). Note also that Y_m approaches zero monotonically from below as $m \to \infty$. It is a consequence of Lemma 4.2 (the Shadowing Lemma) and Proposition 4.4 that there exists a sequence $\{\eta_m\}_{m \geq m_0}$ for some sufficiently large m_0, such that

$$\eta_{m+1} - 2\eta_m + \eta_{m-1} = \gamma(\eta_m, \eta_{m-1}) \tag{5.21}$$

for $m \geq m_0 + 1$, and such that

$$\|V_m\| \leq Ke^{-\alpha m}, \qquad V_m = Y_m - \eta_m v - \psi_1(\eta_m, \eta_{m-1}), \tag{5.22}$$

for some positive constants K and α, where the above equality serves as the definition of $V_m \in \ell^{\infty}(\mathbb{Z})$. Thus the sequence $\eta_m v + \psi_1(\eta_m, \eta_{m-1})$ for large m is the shadow orbit on the

center manifold to the given orbit Y_m, with the function $\psi_1(\eta, \omega)$ as in Proposition 4.4, and where Eq. 5.21 is simply (4.32) rewritten.

Now Proposition 5.3 implies that we have the divergent sum (5.17). In light of the estimate (5.22), the analogous sum for V_m converges, so it follows that

$$\delta_m = \frac{\|V_m\|}{\|Y_m\|} \to 0 \text{ for some subsequence } m = m_j \to \infty,$$

where the above equality serves to define δ_m. Note further that there exist arbitrarily large m such that $\eta_m \neq 0$. Thus Proposition 5.2 applies to the sequence η_m, using the form (4.33) of the function $\gamma(\eta, \omega)$ and the fact that $B \neq 0$ is assumed, and we thereby conclude that

$$3B\eta_m > B\eta_{m-1} > B\eta_m > 0 \text{ for all large } m. \tag{5.23}$$

Dividing the equation in (5.22) by η_m and using the inequalities immediately above gives

$$\lim_{m \to \infty} \frac{Y_m - V_m}{\eta_m} = v, \tag{5.24}$$

with the term arising from $\psi_1(\eta_m, \eta_{m-1})$ disappearing in the limit. Further, for terms with $m = m_j$ in the subsequence we have

$$\|V_{m_j}\| = \delta_{m_j} \|Y_{m_j}\| \leq \delta_{m_j} \|Y_{m_j} - V_{m_j}\| + \delta_{m_j} \|V_{m_j}\|$$

and hence

$$\|V_{m_j}\| \leq \frac{\delta_{m_j}}{1 - \delta_{m_j}} \|Y_{m_j} - V_{m_j}\| \tag{5.25}$$

and so from (5.24) and (5.25) it follows that

$$\lim_{j \to \infty} \frac{V_{m_j}}{\eta_{m_j}} = 0, \qquad \lim_{j \to \infty} \frac{Y_{m_j}}{\eta_{m_j}} = v. \tag{5.26}$$

Now let $y_m = \langle v, Y_m \rangle$. Then taking the inner product with v in the second limit in (5.26) gives $\lim_{j \to \infty} \eta_{m_j}^{-1} y_{m_j} = 1$, hence from (5.23) that

$$By_{m_j} > 0 \text{ for all large } j. \tag{5.27}$$

Noting that $X_m \leq p$ with $X_m \neq p$, hence $Y_m \leq 0$ with $Y_m \neq 0$, for every m, and because $v_n > 0$ for every n, we have that $y_m < 0$. Thus with (5.27) we conclude that $B < 0$.

We now repeat the above argument but instead taking $Y_m = SX_{-m} - p$. The only difference occurs at the end, when we note that $Y_m \geq 0$ with $Y_m \neq 0$, and so $y_m > 0$ for every m, which leads to the conclusion that $B > 0$. This is a contradiction, and with this the theorem is proved. $\square$

6 Genericity: The Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. Recall that we denote the set of normal families of bistable nonlinearities by $\mathcal{N}$. Given $f_0 \in \mathcal{N}$, we consider $f \in \mathcal{N}$ of the form $f(u, a) = \gamma(u) f_0(u, a)$ where $\gamma \in C_+^2$, and we wish to show that Condition B holds for such f, for a residual set of γ in C_+^2. Below, Propositions 6.3 and 6.4 will establish that the set $\mathcal{A}$ of γ for which Condition A holds is a residual set. Then Propositions 6.5 and 6.6

will establish that Condition B holds on a residual set of $\mathcal{A}$, thereby completing the proof of Theorem 1.2.

The basic tool used here is the Abraham Transversality Theorem (see Theorem 19.1 in [1], Theorem 10.3 in Chapter 2 of [8], and Theorem 2.7 in Chapter 3 of [15]), an extension of which is stated as Theorem 6.1 below. If $Y \subseteq X$ is a closed subspace of a Banach space X, we say Y is complemented in X if there exists a closed subspace $Z \subseteq X$ such that $X = Y \oplus Z$. Certainly any subpace of either finite dimension or finite codimension is complemented, but subspaces which are not complemented do exist. If $F : U \to W$ is a smooth map between two Banach manifolds U and W, and $M \subseteq W$ is a smooth submanifold of W (with these manifolds possibly infinite dimensional), then we say that F is transverse to M on a set $K \subseteq U$ if whenever $F(x) \in M$ for some $x \in K$, then $\mathrm{ran}(DF(x)) + T_{F(x)}M = T_{F(x)}W$ (this sum of subspaces need not be a direct sum), and the space $\Sigma_x = \{\overline{x} \in T_x U \mid DF(x)\overline{x} \in T_{F(x)}M\}$ is complemented in $T_x U$. (For purposes of this definition smooth means C^1.) If F is transverse to M as above, then for some neighborhood $O \subseteq U$ with $K \subseteq O$, the set $N = F^{-1}(M) \cap O$ is a submanifold of U, with $T_x N = \Sigma_x$ for every $x \in N$.

We also recall the Smale Density Theorem (see Theorem 1.3 of [28]), which states that if $F : U \to W$ is a C^r (for some $r \geq 1$) map between Banach manifolds U and W, with U and W Lindelöf, and if further $DF(x)$ is a Fredholm operator of index j for every $x \in U$, then the set of regular values of F is a residual subset of W provided that $r > j$.

Theorem 6.1 *[Abraham Transversality Theorem] Let $F : U \times V \to W$ be a C^r map, where U, V, and W are C^r Banach manifolds. Assume also that U and V are Lindelöf spaces (for example, affine subspaces of or open subsets of a separable Banach space). Suppose that $M \subseteq W$ is a C^r submanifold of W and that F is transverse to M on $U \times V$. Further suppose that for each $(x, \lambda) \in U \times V$ for which $F(x, \lambda) \in M$, the map*

$$\pi D_1 F(x, \lambda) \in \mathcal{L}(T_x U, \ T_{F(x,\lambda)}W / T_{F(x,\lambda)}M)$$

is Fredholm of index j, and that $r > \max\{j, 0\}$, where

$$\pi \in \mathcal{L}(T_{F(x,\lambda)}W, \ T_{F(x,\lambda)}W / T_{F(x,\lambda)}M)$$

is the canonical projection onto the quotient space, and where $D_1 F(x, \lambda)$ denotes the derivative with respect to the first argument x. Let $F_\lambda : U \to W$ denote the map $F_\lambda(x) = F(x, \lambda)$ for each $\lambda \in V$. Then

$$\{\lambda \in V \mid F_\lambda \text{ is transverse to } M \text{ on } U\}$$

is a residual subset of V.

Sketch of Proof The set $\Gamma = F^{-1}(M)$ is a submanifold of $U \times V$ with $T_{(x,\lambda)}\Gamma = \{(\overline{x}, \overline{\lambda}) \in T_x U \times T_\lambda V \mid \pi DF(x, \lambda)(\overline{x}, \overline{\lambda}) = 0\}$, and certainly Γ is Lindelöf. Consider the map $\Pi : \Gamma \to V$ given by $\Pi(x, \lambda) = \lambda$. Then one shows that $\lambda \in V$ is a regular value of Π if and only if F_λ is transverse to M on U. Further, the derivative $D\Pi(x, \lambda) \in \mathcal{L}(T_{(x,\lambda)}\Gamma, T_\lambda V)$ is Fredholm with index j. By the Smale Density Theorem, the set of regular values λ of Π is a residual subset of V. □

In what follows we shall let

$$Q_0 = \{q \in \ell^\infty(\mathbb{Z}) \mid \lim_{n \to \pm\infty} q_n = 0\},$$

$$Q_1 = \{q \in \ell^\infty(\mathbb{Z}) \mid \lim_{n \to \pm\infty} q_n = \pm 1\},$$

$$Q_2 = \{(p, q) \in Q_1 \times Q_1 \mid p \neq S^k q \text{ for every } k \in \mathbb{Z}\}.$$

Note that Q_1 is a Banach manifold which is Lindelöf, and that $T_q Q_1 = Q_0$. The following result implies that in a trivial fashion, Q_2 is also a Banach manifold which is Lindelöf, and that $T_{(p,q)} Q_2 = Q_0 \times Q_0$.

Lemma 6.2 *The set Q_2 is an open subset of $Q_1 \times Q_1$.*

Proof Take a sequence of points $(p_j, q_j) \in (Q_1 \times Q_1) \setminus Q_2$ in the complement of Q_2, where $p_j = \{p_{j,n}\}_{n \in \mathbb{Z}}$ and $q_j = \{q_{j,n}\}_{n \in \mathbb{Z}}$. Assume these converge $(p_j, q_j) \to (p_0, q_0)$ to some $(p_0, q_0) \in Q_1 \times Q_1$. We must show that $(p_0, q_0) \notin Q_2$.

We have that $p_j = S^{k_j} q_j$ for some sequence of integers k_j, from the definition of Q_2. By passing to a subsequence, we may assume that either $k_j = k \in \mathbb{Z}$ is independent of j, or else $k_j \to \infty$ as $j \to \infty$, or else $k_j \to -\infty$ as $j \to \infty$.

If $k_j = k$, then clearly $p_0 = S^k q_0$, and so $(p_0, q_0) \notin Q_2$, as desired. Thus suppose that $k_j \to \infty$ as $j \to \infty$. Then for any fixed n we have

$$|p_{j,n} - q_{0,n+k_j}| \leq |p_{j,n} - q_{j,n+k_j}| + |q_{j,n+k_j} - q_{0,n+k_j}|$$
$$\leq \|p_j - S^{k_j} q_j\| + \|S^{k_j}(q_j - q_0)\| \to 0$$

as $j \to \infty$. But also $p_{j,n} - q_{0,n+k_j} \to p_{0,n} - 1$ as $j \to \infty$ since $q_0 \in Q_1$, and so $p_{0,n} = 1$. However, this contradicts the fact that $p_0 \in Q_1$.

We omit the case in which $k_j \to -\infty$, as it is similar to the previous case. With this, the proof is complete. ☐

In what follows we shall let D_a denote the derivative with respect to the parameter a, with prime $'$ denoting the derivative with respect to u as usual, for the two arguments (u, a) of f.

Proposition 6.3 *Fix any $f \in \mathcal{N}$, and define $G_f : Q_2 \times (-1, 1) \to Q_1 \times Q_1$ by*

$$G_f(p, q, a) = \begin{pmatrix} (S + S^{-1} - 2I)p - f(p, a) \\ (S + S^{-1} - 2I)q - f(q, a) \end{pmatrix}.$$

Suppose that G_f is transverse to $\{(0, 0)\}$ on $Q_2 \times (-1, 1)$. Then Condition A holds.

Proof We prove the contrapositive. Suppose that Condition A fails. Then there exists $(p, q) \in Q_2$ such that $G_f(p, q, a_+(0)) = (0, 0)$. It suffices to prove that G_f is not transverse to $\{(0, 0)\}$ at the point $(p, q, a_+(0))$, namely, that the derivative $DG_f(p, q, a_+(0))$ is not surjective. We have that

$$DG_f(p, q, a_+(0))(\overline{p}, \overline{q}, \overline{a}) = \begin{pmatrix} L_p \overline{p} - D_a f(p, a_+(0))\overline{a} \\ L_q \overline{q} - D_a f(q, a_+(0))\overline{a} \end{pmatrix} \tag{6.1}$$

where the operators $L_p = L_p(a_+(0))$ and $L_q = L_q(a_+(0))$ are as in (3.1), although here considered as an element of $\mathcal{L}(Q_0)$ rather than $\mathcal{L}(\ell^\infty(\mathbb{Z}))$. By Lemma 3.1 both L_p and L_q are Fredholm with index zero and kernels of dimension either zero or one. It follows from an argument using the Implicit Function Theorem, similar to that used in the proof of Proposition 1.4, that the dimension of the kernel of each of these operators is in fact one, and thus

$$\operatorname{codim} \operatorname{ran}(L_p) = \operatorname{codim} \operatorname{ran}(L_q) = 1$$

for the codimensions of the ranges in the space Q_0. From this and the formula (6.1), it follows immediately that the range of $DG_f(p, q, a_+(0))$ has codimension either one or two. Thus $DG_f(p, q, a_+(0))$ is not surjective, as desired. ☐

Recall now the set C_+^2 given by (1.18), which is endowed with the usual C^2 topology.

Proposition 6.4 *Fix any $f_0 \in \mathcal{N}$, and define $\tilde{G}_{f_0} : Q_2 \times (-1, 1) \times C_+^2 \rightarrow Q_1 \times Q_1$ by*

$$\tilde{G}_{f_0}(p, q, a, \gamma) = \begin{pmatrix} (S + S^{-1} - 2I)p - \gamma(p)f_0(p, a) \\ (S + S^{-1} - 2I)q - \gamma(q)f_0(q, a) \end{pmatrix}.$$

Then $\tilde{G}_{f_0}$ is transverse to $\{(0, 0)\}$ on $Q_2 \times (-1, 1) \times C_+^2$.

Proof Let us denote the dependence of the operator $L_q = L_q(a)$ in (3.1) on the nonlinearity f by

$$L_{f,q} = L_{f,q}(a) = S + S^{-1} - 2I - f'(q, a), \tag{6.2}$$

for $f \in \mathcal{N}$. Now take any $(p, q, a, \gamma) \in Q_2 \times (-1, 1) \times C_+^2$ for which $\tilde{G}_{f_0}(p, q, a, \gamma) = 0$. Denoting $f(u, a) = \gamma(u)f_0(u, a)$, we have that

$$D\tilde{G}_{f_0}(p, q, a, \gamma)(\overline{p}, \overline{q}, \overline{a}, \overline{\gamma}) = \begin{pmatrix} L_{f,p}\overline{p} - D_a f(p, a)\overline{a} - \overline{\gamma}(p)f_0(p, a) \\ L_{f,q}\overline{q} - D_a f(q, a)\overline{a} - \overline{\gamma}(q)f_0(q, a) \end{pmatrix}. \tag{6.3}$$

By Lemma 3.1 both $L_{f,p}$ and $L_{f,q}$ are Fredholm with index zero and kernels of dimension either zero or one, considered here as elements of $\mathcal{L}(Q_0)$. We must prove that the operator $D\tilde{G}_{f_0}(p, q, a, \gamma)$ is surjective. Denote by v_p any nonzero element of the kernel $\ker(L_{f,p})$, if such exists, and similarly with v_q as a nontrivial element of $\ker(L_{f,q})$.

Three cases now arise. In the first case we suppose that both $L_{f,p}$ and $L_{f,q}$ are isomorphisms, and so surjectivity is immediate.

In the second case we assume exactly one of these operators is an isomorphism, say $L_{f,q}$ for definiteness, and so the operator $L_{f,p}$ has a nontrivial kernel element v_p. Moreover, the range of $L_{f,p}$ is characterized as

$$\text{ran}(L_{f,p}) = \{w \in Q_0 \mid \langle v_p, w \rangle = 0\}$$

by Lemma 3.1. If it is not the case that $D\tilde{G}_{f_0}(p, q, a, \gamma)$ is surjective then the first line in (6.3) is annihilated by v_p, for every choice of $(\overline{p}, \overline{a}, \overline{\gamma})$. In particular, taking $\overline{p} = 0$ and $\overline{a} = 0$, and $\overline{\gamma}$ to be a function for which

$$\overline{\gamma}(p_n) = \begin{cases} 0, & n \neq k, \\ 1, & n = k, \end{cases} \tag{6.4}$$

for some integer k, we have that

$$0 = \langle v_p, \overline{\gamma}(p)f_0(p, a) \rangle = v_{p,k} f_0(p_k, a). \tag{6.5}$$

But $f_0(p_k, a) \neq 0$ for all large k, and so (6.5) implies that $v_{p,k} = 0$ for all large k. This contradicts Lemma 3.1, which states that $v_{p,k} \neq 0$ for all large k.

It remains only to consider the case where neither $L_{f,p}$ nor $L_{f,q}$ are isomorphisms, and so the vectors v_p and v_q, respectively, annihilate their ranges of these operators. Then if the derivative $D\tilde{G}_{f_0}(p, q, a, \gamma)$ is not surjective, its range is annihilated by some nontrivial combination of v_p and v_q. To be precise, there exist constants τ_p and τ_q, not both zero, such that

$$0 = \tau_p \langle v_p, \ L_{f,p}\overline{p} - D_a f(p, a)\overline{a} - \overline{\gamma}(p)f_0(p, a) \rangle$$
$$+ \tau_q \langle v_q, \ L_{f,q}\overline{q} - D_a f(q, a)\overline{a} - \overline{\gamma}(q)f_0(q, a) \rangle$$

for every $(\overline{p}, \overline{q}, \overline{a}, \overline{\gamma})$. Taking $\overline{p} = \overline{q} = 0$ and $\overline{a} = 0$ gives

$$0 = \tau_p \sum_{n=-\infty}^{\infty} v_{p,n} \overline{\gamma}(p_n) f_0(p_n, a) + \tau_q \sum_{n=-\infty}^{\infty} v_{q,n} \overline{\gamma}(q_n) f_0(q_n, a)$$

for every $\overline{\gamma}$. If one of the coefficients τ_p or τ_q is zero, say $\tau_q = 0$, then choosing a function $\overline{\gamma}$ as in (6.4) and arguing as before yields a contradiction. Thus assume that both $\tau_p \neq 0$ and $\tau_q \neq 0$. If there exist arbitrarily large k for which the point p_k is distinct from all the points q_j, that is, $p_k \neq q_j$ for every $j \in \mathbb{Z}$, then we may choose $\overline{\gamma}$ satisfying (6.4) and additionally satisfying $\overline{\gamma}(q_j) = 0$ for every $j \in \mathbb{Z}$. This now gives Eq. 6.5, and so $v_{p,k} = 0$ for such k. However, this contradicts Lemma 3.1.

Thus we have that for every sufficiently large integer k, there exists an integer j, such that $p_k = q_j$. Necessarily $j = j_k$ is uniquely determined from k, and $j_k \to \infty$ as $k \to \infty$. Reversing the roles of p and q imply that also, for every sufficiently large integer j, there exists an integer $k = k_j$ such that $q_j = p_{k_j}$. It follows immediately that there exists some integer m such that $p_k = q_{k+m}$ for all sufficiently large k. As both p and q satisfy the difference Eq. 1.19, we conclude that $p_k = q_{k+m}$ for every $k \in \mathbb{Z}$, that is, $p = S^m q$. But this contradicts the fact that $(p, q) \in Q_2$, and completes the proof. □

In what follows, let us denote

$$\text{Fred}_{1,1} = \{\Omega \in \mathcal{L}(Q_0) \mid \Omega \text{ is a Fredholm operator}$$
$$\text{of index zero, with } \dim \ker(\Omega) = 1\}.$$

It is known that $\text{Fred}_{1,1}$ is a C^∞ submanifold of $\mathcal{L}(Q_0)$ of codimension one. Moreover, if $\Omega \in \text{Fred}_{1,1}$ then the tangent space of $\text{Fred}_{1,1}$ at this point is given by

$$T_\Omega \text{Fred}_{1,1} = \{\overline{\Omega} \mid \overline{\Omega}v \in \text{ran}(\Omega) \text{ whenever } v \in \ker(\Omega)\}.$$

Proposition 6.5 *Fix any $f \in \mathcal{N}$, and suppose that Condition A holds for f. Define F_f : $Q_1 \times (-1, 1) \to Q_1 \times \mathcal{L}(Q_0)$ by*

$$F_f(p, a) = \begin{pmatrix} (S + S^{-1} - 2I)p - f(p, a) \\ S + S^{-1} - 2I - f'(p, a) \end{pmatrix}.$$

Also define the manifold

$$M = \{0\} \times \text{Fred}_{1,1} \subseteq Q_1 \times \mathcal{L}(Q_0).$$

Then f satisfies Condition B if and only if F_f is transverse to M at the point $(p, a_+(0))$, where p is as in the statement of Condition A.

Proof In what follows p is as in Condition A, with $L_p = L_p(a_+(0))$, so $L_p \in \text{Fred}_{1,1}$. Also, let $v \in Q_0$ be the kernel element as in the statement of Proposition 1.4. Then one sees immediately that

$$T_{L_p}\text{Fred}_{1,1} = \{\overline{\Omega} \in \mathcal{L}(Q_0) \mid \langle v, \overline{\Omega}v \rangle = 0\}, \qquad L_p = S + S^{-1} - 2I - f'(p, a_+(0)).$$

We now compute

$$DF_f(p, a_+(0))(\overline{p}, \overline{a}) = \begin{pmatrix} L_p\overline{p} - D_a f(p, a_+(0))\overline{a} \\ -f''(p, a_+(0))\overline{p} - D_a f'(p, a_+(0))\overline{a} \end{pmatrix}.$$

Thus F_f is transverse to M at $(p, a_+(0))$ if and only if, for every choice of $w \in Q_0$ and $\Omega \in \mathcal{L}(Q_0)$, there exists $(\overline{p}, \overline{a}) \in Q_0 \times \mathbb{R}$, and $\overline{\Omega} \in T_{L_p}\mathrm{Fred}_{1,1}$, such that

$$
\begin{aligned}
w &= L_p \overline{p} - D_a f(p, a_+(0))\overline{a}, \\
\Omega &= -f''(p, a_+(0))\overline{p} - D_a f'(p, a_+(0))\overline{a} + \overline{\Omega}.
\end{aligned}
\tag{6.6}
$$

Note that the existence of $\overline{\Omega}$ satisfying the second line in (6.6) is equivalent to the equation

$$
\langle v, \Omega v \rangle + \langle v, f''(p, a_+(0))(\overline{p}, v) \rangle + \langle v, D_a f'(p, a_+(0))(\overline{a}, v) \rangle = 0.
\tag{6.7}
$$

Further note that we have

$$
\langle v, f''(p, a_+(0))(v, v) \rangle = 2B,
$$

where B is the quantity (1.17) in Condition B.

To prove the lemma, suppose first that Condition B holds, and let w and Ω be given as above. Then the first equation in (6.6) can be solved for some $\overline{p}$ and $\overline{a}$ if and only if $\langle v, w \rangle = -\langle v, D_a f(p, a_+(0)) \rangle \overline{a}$, equivalently,

$$
\overline{a} = -\frac{\langle v, w \rangle}{\langle v, D_a f(p, a_+(0)) \rangle}.
\tag{6.8}
$$

Here we have used the fact that f is a normal family, in particular that $D_a f(u, a) > 0$ in $(-1, 1) \times (-1, 1)$, and also Proposition 1.4, in particular that $v_n > 0$, to conclude that the denominator is nonzero. With this choice of $\overline{a}$ there exists $\overline{p}$ satisfying the first equation in (6.6), and moreover, $\overline{p}$ is uniquely determined up to an additive multiple of v. Thus fixing a particular choice $\tilde{p}$ of $\overline{p}$, we see that the general form of $\overline{p}$ is $\overline{p} = \tilde{p} + \lambda v$ for arbitrary $\lambda \in \mathbb{R}$. Thus Eq. 6.7 takes the form

$$
\langle v, \Omega v \rangle + \langle v, f''(p, a_+(0))(\tilde{p}, v) \rangle + 2\lambda B + \langle v, D_a f'(p, a_+(0))(\overline{a}, v) \rangle = 0
\tag{6.9}
$$

with the above choices of $\overline{a}$ and $\overline{p}$. As $B \neq 0$ is assumed, there is a unique choice of λ solving this equation, as desired.

Now assume that Condition B does not hold, and so $B = 0$. Taking $w = 0$ above forces $\overline{a} = 0$, by (6.8), and so necessarily $\overline{p} = \lambda v$ to satisfy the first equation in (6.6). Equation 6.9 takes the form $\langle v, \Omega v \rangle = 0$. However, this equation is not satisfied for every Ω, which implies the transversality condition fails, as claimed.   □

Proposition 6.6 *Fix any $f_0 \in \mathcal{N}$, and define $\tilde{F}_{f_0} : Q_1 \times (-1, 1) \times C_+^2 \to Q_1 \times \mathcal{L}(Q_0)$ by*

$$
\tilde{F}_{f_0}(p, a, \gamma) = \begin{pmatrix} (S + S^{-1} - 2I)p - \gamma(p)f_0(p, a) \\ S + S^{-1} - 2I - (\gamma(p)f_0(p, a))' \end{pmatrix}.
$$

Then $\tilde{F}_{f_0}$ is transverse to M on $Q_1 \times (-1, 1) \times C_+^2$, where M is as in the statement of Proposition 6.5.

Proof As before, let $L_{f,q} = L_{f,q}(a)$ denote the operator (6.2). Then, denoting $f(u, a) = \gamma(u)f_0(u, a)$ in what follows, we compute the derivative

$$
D\tilde{F}_{f_0}(p, a, \gamma)(\overline{p}, \overline{a}, \overline{\gamma}) = \begin{pmatrix} L_{f,p}\overline{p} - \gamma(p)D_a f_0(p, a)\overline{a} - \overline{\gamma}(p)f_0(p, a) \\ -f''(p, a)\overline{p} - D_a f'(p, a)\overline{a} - (\overline{\gamma}(p)f_0(p, a))' \end{pmatrix}.
$$

Thus $\tilde{F}_{f_0}$ is transverse to M at a point (p, a, γ) for which $\tilde{F}_{f_0}(p, a, \gamma) \in M$ if and only if, for every choice of $w \in Q_0$ and $\Omega \in \mathcal{L}(Q_0)$, there exist $(\overline{p}, \overline{a}, \overline{\gamma}) \in Q_0 \times \mathbb{R} \times C^2[-1, 1]$, such that

$$
\begin{aligned}
w &= L_{f,p}\overline{p} - \gamma(p)D_a f_0(p, a)\overline{a} - \overline{\gamma}(p) f_0(p, a), \\
&\langle v, \Omega v \rangle + \langle v, f''(p, a)(\overline{p}, v) \rangle + \langle v, D_a f'(p, a)(\overline{a}, v) \rangle \\
&\quad + \langle v, (\overline{\gamma}(p) f_0(p, a))'v \rangle = 0,
\end{aligned}
\tag{6.10}
$$

much as in (6.6) and (6.7). Here v is a nontrivial element of the kernel of $L_{f,p}$, and is uniquely determined up to scalar multiple (which we fix) as $L_{f,p} = S + S^{-1} - 2I - f'(p, a) \in \text{Fred}_{1,1}$. We note that there is no assurance that $v_n > 0$ for the coordinates.

Given any such w and Ω, first choose $\overline{a} = 0$. Next, fix k large enough that both $v_k \neq 0$ (which can be done by Lemma 3.1) and $f(p_k, a) \neq 0$, and choose $\overline{\gamma} \in C^2[-1, 1]$ to satisfy

$$
\overline{\gamma}(p_n) = \begin{cases} 0, & n \neq k, \\ \gamma^0, & n = k, \end{cases} \qquad \overline{\gamma}'(p_n) = \begin{cases} 0, & n \neq k, \\ \gamma^1, & n = k, \end{cases}
$$

with the quantities γ^0 and γ^1 to be chosen shortly. There exists $\overline{p}$ satisfying the first equation in (6.10) if and only if $\langle v, w \rangle = -\langle v, \overline{\gamma}(p) f_0(p, a) \rangle$, or equivalently,

$$
\gamma^0 = -\frac{\langle v, w \rangle}{v_k f_0(p_k, a)}.
$$

Taking this value for γ^0, we now fix the choice of $\overline{p}$ as well. In particular, we note that although we have not yet chosen γ^1, the choice of this quantity will not affect $\overline{p}$ as the first equation in (6.10) does not involve $\overline{\gamma}'(p)$.

Finally, we make the unique choice of γ^1 so that the second equation in (6.10) holds, and one sees that the required value is

$$
\gamma^1 = -\frac{\langle v, \Omega v \rangle + \langle v, f''(p, a)(\overline{p}, v) \rangle + v_k^2 \gamma^0 f_0'(p_k, a)}{v_k^2 f_0(p_k, a)}.
$$

With this, the proof is complete. □

Proof of Theorem 1.2 With $f_0 \in \mathcal{N}$, we apply the Abraham Transversality Theorem to the maps $\tilde{G}_{f_0}$ and $\tilde{F}_{f_0}$ in Propositions 6.4 and 6.6, respectively. What must be checked is that the appropriate operators are Fredholm with the appropriate index. For the map $\tilde{G}_{f_0}$ we consider the derivative $D_{1,2,3}\tilde{G}_{f_0}(p, q, a, \gamma)$ taken with respect to the first three arguments p, q, and a, but not γ (as γ plays the role of λ in the above statement of the Transversality Theorem). As noted in the proof of Proposition 6.4, the operators $L_{f,p}$ and $L_{f,q}$ are Fredholm with index zero, and from this it follows easily, using the formula (6.3), that $D_{1,2,3}\tilde{G}_{f_0}(p, q, a, \gamma)$ is Fredholm of index one. Thus the Transversality Theorem applies as the map $\tilde{G}_{f_0}$ is C^2.

For the map $\tilde{F}_{f_0}$ we must follow the derivative $D_{1,2}\tilde{F}_{f_0}(p, a, \gamma)$ taken with respect to p and a, with the projection which annihilates the tangent space $T_{L_p}\text{Fred}_{1,1}$. This gives the operator

$$
\pi D_{1,2}\tilde{F}_{f_0}(p, a, \gamma)(\overline{p}, \overline{a}) = \begin{pmatrix} L_{f,p}\overline{p} - \gamma(p)D_a f_0(p, a)\overline{a} \\ -\langle v, f''(p, a)(\overline{p}, v) + D_a f'(p, a)(\overline{a}, v) \rangle \end{pmatrix}, \tag{6.11}
$$

where v is the kernel element of $L_{f,p}$. Again $L_{f,p}$ has index zero, and as the second coordinate in the range of (6.11) is scalar, it follows that the operator (6.11) has Fredholm index

zero. Again the Transversality Theorem applies as the map F_{f_0} is C^1. It follows directly that for a residual set of $\gamma \in C_+^2$ the maps $G_{\gamma f_0}$ and $F_{\gamma f_0}$ are transverse to $\{(0, 0)\}$ and to M, respectively, on their domains, and thus that Condition B holds for γf_0. With this Theorem 1.2 is proved.   □

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