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Subharmonic Solutions with Prescribed Minimal Period of a Class of Nonautonomous Hamiltonian Systems

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Dedicated to Professor Zhifen Zhang on the occasion of her 80th birthday

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Abstract In this paper, by using a decomposition technique to estimate the energy (the value of the functional associated with the problem) of a solution in terms of the minimal period of the solution, we give new sufficient conditions for the existence of subharmonic solutions with prescribed minimal period of Hamiltonian systems. Our results improve some known results in the literature.

Keywords Critical point · Periodic solutions · Minimal period · Hamiltonian systems

Mathematics Subject Classifications (2000) 35J20 · 35J25 · 35J60 · 35Q55 · 47J30 · 49B60 · 58E05

1 Introduction

In this paper we study the existence of periodic solutions with prescribed minimal period for the following nonautonomous Hamiltonian systems

$$\ddot{x} + F'_x(t, x) = 0, \tag{1.1}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$, $F \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$, $F'_x(t, x) = (\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n})^T \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ and there exists a constant $T > 0$, such that for each $x \in \mathbf{R}^n$, the function $F(t, x)$ satisfies $F(t + T, x) = F(t, x)$. We will obtain some new sufficient conditions for the existence of periodic solutions of (1.1) with minimal period pT for any integer $p > 1$.

The first result in this area should go back to [1] in which the existence of a sequence of subharmonics with arbitrarily large minimal period was obtained by using perturbation type techniques, under suitable assumptions on F near the origin. A different proof of the result in [1] was given in [21]. Later, subsequently good results were obtained in [4,5,22] by

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a global approach using calculus of variations. Under rather general assumptions on F, they proved that for any integer $p > 1$, if F has a subquadratic or superquadratic growth at the origin and at infinity then one may conclude that the minimal period of periodic solutions with $x(0) = x(pT)$ of (1.1) tends towards infinity as $p \to \infty$. This kind of approach has been extensively extended in various directions, see for example [2,3,6–15,18,20,23,25–27,29–31]. The existence of periodic solutions of (1.1) was studied by many authors, see for example [16,17,19,24,28].

By the use of Morse-Conley theory, Hamiltonian systems with periodic nonlinearity were studied in [6,7]. They proved the existence of subharmonic solutions with minimal period pT for sufficiently large prime number p under some assumptions on the nondegeneracy of the solutions. In [11,13], the pendulum type equation was considered:

$$\ddot{x} + g(x) = f(t) \tag{1.2}$$

where g is a periodic function in x and f is a T-periodic function in t with mean value zero. It was proved in [13] that if (**a**) the periodic solutions of (1.2) are isolated and (**b**) every periodic solution of (1.2) having Morse index equal to zero is nondegenerate, then there exists a constant $P \geq 2$ such that, for every prime integer $p \geq P$, there is a periodic solution of (1.2) with minimal period pT. This is a good improvement of the results due to Conley and Zehnder in [7] where nondegenerate condition was assumed for all T-periodic solutions together with their iterates. But, we find that it is not convenient for us to check conditions (**a**) and (**b**). In [11], the minimal periodic problem of (1.2) was also studied by simply making some careful estimates on the critical levels of the functional associated to the problem. The following classical pendulum equation is a special form of (1.2)

$$\ddot{x} + A \sin x = f(t), \tag{1.3}$$

where $A = g/l$ is a constant with g being the gravity constant and l being the length of the pendulum, and f is a T-periodic function which is regarded as an external force.

In [29], by using the estimate of the energy (the value of the functional associated with the problem) of a solution in terms of the minimal period of the solution, the authors obtained explicit sufficient conditions for the existence of subharmonic solutions of (1.3) with minimal period pT for all $p > 1$ provided that A and the L^2-norm of f satisfy certain quantitative conditions, which are very easy to check. This approach, initially used in [9], has been successfully applied to the minimal period problem of Hamiltonian systems [9,15,20,23,26,30].

In this paper we will obtain, by using a more effective decomposition technique to estimate the energy of the periodic solutions in terms of its minimal period, some new sufficient conditions for the existence of periodic solutions with minimal period pT. When our results are applied to (1.3), the results in [29] are improved.

2 Main Results

Let $\|\cdot\|$ and $|\cdot|$ denote the norms in the spaces $L^2([0, pT], \mathbf{R}^n)$ and $\mathbf{R}^n$ respectively.

We make the following assumptions:

($\mathbf{F}_0$) $F(t,x) \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ is T-periodic in t, and for any $x \in C([0, pT], \mathbf{R}^n)$,

$$F(-t, -x) = F(t, x).$$

($\mathbf{F}_1$) There exist constants $A > 0$, $\beta > 0$ such that

$$\max \left\{ 0, \frac{1}{2} A|x|^2 - \frac{\beta}{2}|x|^4 \right\} \leq F(t,x) - F'_x(t,0)x \leq \frac{A}{2}|x|^2.$$

($\mathbf{F}_2$) There exists some constant $h \in [0, \frac{1}{2}(\frac{\omega}{p})^2)$ such that

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^2} = h, \text{ uniformly in } t.$$

($\mathbf{F}_3$) If $x = x(t)$ is a periodic function with minimal period qT, q rational, and $F'_x(t, x(t)$ is a periodic function with minimal period qT, then q is necessarily an integer.

We are now able to formulate our main result.

Theorem 2.1 *Let F satisfy ($\mathbf{F}_0$)-($\mathbf{F}_3$). Suppose that, for an integer $p > 1$,*

$$\frac{\omega^2}{A} < p^2 \le \frac{\omega^2 s_p^2}{A} \tag{2.1}$$

and

$$\|F'_x(t,0)\|^2_{L^2(0,T)} \le \frac{T(\omega^2 - A)(A - (\frac{\omega}{p})^2)^2}{6\beta}. \tag{2.2}$$

Then (1.1) has at least one periodic solution with minimal period pT.

Corollary 2.1 *Let F satisfy ($\mathbf{F}_0$)-($\mathbf{F}_3$). Suppose that*

$$0 < A < \omega^2 \tag{2.3}$$

and

$$\|F'_x(t,0)\|^2_{L^2(0,T)} \le \frac{T(\omega^2 - A)A^2}{6\beta}. \tag{2.4}$$

Then there exists a $P > 0$ such that, for any prime integer $p > P$, (1.1) has at least one periodic solution with minimal period pT.

To prove Theorem 2.1, we consider functional

$$J(x) = \frac{1}{2}\int_0^{pT} |\dot{x}(t)|^2 dt - \int_0^{pT} F(t, x(t))dt, \tag{2.5}$$

in the space

$$X = \{x \in H^1([0, pT], \mathbf{R}^n) | x(0) = x(pT)\}.$$

Without loss of generality, we may assume $F(t, 0) \equiv 0$. It can be easily found that $J \in C^1(X, \mathbf{R})$ and that the critical points of J correspond to periodic solutions of (1.1) with period pT, but not necessarily with minimal period pT. By ($\mathbf{F}_0$) we may consider the restriction of J on a subspace X^* of X:

$$X^* = \{x \in X | x \text{ is odd in } t\}.$$

Let $\{e_1, e_2, \ldots, e_n\}$ denote the canonical orthogonal basis in $\mathbf{R}^n$. By Fourier expansion, for any $x \in X$, we have

$$x(t) = \sum_{j=1}^{n}\left[\sum_{l=0}^{+\infty}(a_{lj}\cos\frac{\omega l}{p}t + b_{lj}\sin\frac{\omega l}{p}t)\right]e_j$$

with

$$\|x\|^2_{L^2(0,pT)} = \frac{pT}{2}\sum_{j=1}^{n}\sum_{l=0}^{+\infty}(a_{lj}^2 + b_{lj}^2) < \infty$$

and

$$\|\dot{x}\|^2_{L^2(0,pT)} = \frac{pT}{2}(\frac{\omega}{p})^2\sum_{j=1}^{n}\sum_{l=0}^{+\infty}(a_{lj}^2 + b_{lj}^2)l^2 < \infty.$$

Clearly, $x \in X^*$ if and only if

$$x(t) = \sum_{j=1}^{n}\left(\sum_{l=1}^{+\infty} a_{lj}\sin\frac{\omega l}{p}t\right)e_j.$$

Lemma 2.1 ([29]) *If x is a critical point of J on X^*, then x is also a critical point of J on X. And the minimal period of x is an integer multiple of T.*

Lemma 2.2 *J is bounded from below and satisfies (PS) condition on X^* the definition of (PS) condition can be found in* ([19,23]).

Proof For any $x \in X^*$, we have

$$x(t) = \sum_{j=1}^{n}\left(\sum_{l=1}^{+\infty} a_{lj}\sin\frac{\omega l}{p}t\right)e_j$$

and

$$\int_0^{pT} |\dot{x}(t)|^2 dt = \frac{pT}{2}(\frac{\omega}{p})^2\sum_{j=1}^{n}\sum_{l=1}^{+\infty} a_{lj}^2 l^2.$$

So by $(\mathbf{F}_1)$,

$$\begin{aligned} J(x) &= \frac{1}{2}\frac{pT}{2}(\frac{\omega}{p})^2\sum_{j=1}^{n}\sum_{l=1}^{+\infty} a_{lj}^2 l^2 - \int_0^{pT} F(t,x(t))dt \\ &\geq \frac{1}{2}(\frac{\omega}{p})^2\|x\|^2_{L^2(0,pT)} - \int_0^{pT} F(t,x(t))dt \end{aligned}$$

By $(\mathbf{F}_2)$, for $\varepsilon \in (0, (\frac{\omega}{p})^2)$, there exists $M > 0$ such that

$$\frac{F(t,x)}{|x|^2} \leq \varepsilon + h, \quad \text{for } |x| > M.$$

Let $m = \sup_{\substack{|x|\leq M \\ t\in[0,pT]}} F(t,x)$. Then we have by $(\mathbf{F}_2)$,

$$J(x) \geq \frac{1}{2}(\frac{\omega}{p})^2\|x\|^2_{L^2(0,pT)} - \int_{\substack{t\in[0,pT] \\ |x|\leq M}} F(t,x(t))dt - \int_{\substack{t\in[0,pT] \\ |x|>M}} F(t,x(t))dt$$

$$\geq \frac{1}{2}(\frac{\omega}{p})^2\|x\|^2_{L^2(0,pT)} - \int\limits_{\substack{t\in[0,pT]\\|x|\leq M}} mdt - (\varepsilon+h)\int\limits_{\substack{t\in[0,pT]\\|x|>M}} |x|^2dt$$

$$\geq \frac{1}{2}((\frac{\omega}{p})^2 - 2\varepsilon - 2h)\|x\|^2_{L^2(0,pT)} - mpT \to +\infty \text{ as } \|x\|^2_{L^2(0,pT)} \to \infty.$$

It follows that J is bounded from below. From this we know that (PS) sequence must be bounded in $H^1(0, pT)$. Then by a standard argument, any (PS) sequence has a convergent subsequence [19,23]. The proof of Lemma 2.2 is finished.

Now by usual argument J achieves its minimum on X^*. Let $x_0 \in X^*$ be such that

$$J(x_0) = \min_{x\in X^*} J(x). \tag{2.6}$$

Proof of Theorem 2.1 Let $x_0 \in X^*$ be defined as in (2.6). The proof of Theorem 2.1 will be finished if we can prove that the minimal period of x_0 is pT. For the sake of contradiction, let the minimal period of x_0 be pT/q, for some integer $q > 1$. In view of Lemma 2.1, we know that q is a factor of p, and so $q \geq s_p$. By Fourier expansion,

$$x_0(t) = \sum_{j=1}^{n}\left(\sum_{l=1}^{+\infty} a_{lj} \sin\frac{q\omega l}{p}t\right)e_j.$$

Define $x_1(t)$ and $x_2(t)$ as follows:

$$x_1(t) = \sum_{j=1}^{n}\left(\sum_{s=1}^{+\infty} a_{\frac{ps}{q}j} \sin\omega st\right)e_j,$$

and

$$x_2(t) = x_0(t) - x_1(t).$$

Then x_1 is T-periodic and $x_1 \perp x_2$. It is also easy to see that $x_2 \perp F'_x(t,0)$ and $\dot{x}_1 \perp \dot{x}_2$. Thus we have by $(\mathbf{F}_1)$

$$\begin{aligned}
J(x_0) &= J(x_1+x_2) = \frac{1}{2}\int_0^{pT} |\dot{x}_1(t)+\dot{x}_2(t)|^2dt - \int_0^{pT} F(t, x_1(t)+x_2(t))dt \\
&= \frac{1}{2}\int_0^{pT} (|\dot{x}_1(t)|^2 + |\dot{x}_2(t)|^2)dt - \int_0^{pT} < F'_x(t,0), x_1(t)+x_2(t) > dt \\
&\quad - \int_0^{pT} \left[F(t, x_1(t)+x_2(t)) - < F'_x(t,0), x_1(t)+x_2(t) >\right]dt \\
&\geq \frac{1}{2}\int_0^{pT} (|\dot{x}_1(t)|^2 + |\dot{x}_2(t)|^2)dt - \int_0^{pT} < F'_x(t,0), x_1(t) > dt
\end{aligned}$$

$$
\begin{aligned}
&-\frac{A}{2}\int_0^{pT} |x_1(t)+x_2(t)|^2dt \\
&\ge \frac{1}{2}\omega^2\int_0^{pT} |x_1(t)|^2dt + \frac{1}{2}(\frac{q\omega}{p})^2\int_0^{pT} |x_2(t)|^2dt \\
&-\|F_x'(t,0)\|_{L^2(0,pT)}\cdot\|x_1\|_{L^2(0,pT)} \\
&-\frac{1}{2}A\left(\|x_1\|^2_{L^2(0,pT)}+\|x_2\|^2_{L^2(0,pT)}\right) \\
&= \frac{1}{2}((\frac{q\omega}{p})^2-A)\|x_2\|^2_{L^2(0,pT)}+\frac{p}{2}(\omega^2-A)\|x_1\|^2_{L^2(0,T)} \\
&-p\|F_x'(t,0)\|_{L^2(0,T)}\cdot\|x_1\|_{L^2(0,T)} \\
&\ge \frac{p}{2}(\omega^2-A)\|x_1\|^2_{L^2(0,T)}-p\|F_x'(t,0)\|_{L^2(0,T)}\cdot\|x_1\|_{L^2(0,T)}. \qquad (2.7)
\end{aligned}
$$

On the other hand, let

$$\bar{x}(t)=\sqrt{\delta}\sin\frac{\omega}{p}t\cdot e_1.$$

Then $\bar{x}$ is pT-periodic with minimal period pT. Since $F_x'(t,0)$ is T-periodic, we get

$$\int_0^{pT} F_x'(t,0)\cdot\bar{x}(t)dt=0.$$

By $(\mathbf{F}_1)$, we have for any $x\in\mathbf{R}^n$,

$$F(t,x)-F_x'(t,0)\cdot x\ge\frac{A}{2}|x|^2-\frac{\beta}{2}|x|^4.$$

Hence,

$$
\begin{aligned}
J(\bar{x}) &\le \frac{1}{2}\int_0^{pT}|\dot{\bar{x}}(t)|^2dt-\int_0^{pT}F_x'(t,0)\cdot\bar{x}(t)dt-\frac{A}{2}\int_0^{pT}|\bar{x}(t)|^2dt+\frac{\beta}{2}\int_0^{pT}|\bar{x}(t)|^4dt \\
&= \frac{1}{2}\delta(\frac{\omega}{p})^2\cdot\frac{pT}{2}-\frac{A}{2}\cdot\delta\cdot\frac{pT}{2}+\frac{\beta}{2}\delta^2\cdot\frac{3pT}{8} \\
&= -\frac{pT}{4}[A-(\frac{\omega}{p})^2]\delta+\frac{3pT}{16}\cdot\beta\cdot\delta^2
\end{aligned}
$$

Now we are going to choose some positive number δ such that

$$
\begin{aligned}
&-\frac{pT}{4}[A-(\frac{\omega}{p})^2]\delta+\frac{3pT}{16}\cdot\beta\cdot\delta^2\le\frac{p}{2}(\omega^2-A)\|x_1\|^2_{L^2(0,T)} \\
&-p\|F_x'(t,0)\|_{L^2(0,T)}\cdot\|x_1\|_{L^2(0,T)}, \qquad (2.8)
\end{aligned}
$$

which will prove $J(\bar{x})\le J(x_0)$.

If $J(\bar{x})=J(x_0)$, then $\bar{x}$ is a critical point of J on X^* with minimal period pT, this finished the proof. If $J(\bar{x})<J(x_0)$, it is clearly a contradiction to the assumption of x_0. In order to prove (2.8), let $\delta=q\|x_1\|_{L^2(0,T)}$. Then (2.8) becomes

$$-\frac{T}{4}[A-(\frac{\omega}{p})^2]q+\frac{3T}{16}\cdot\beta\cdot q^2\|x_1\|_{L^2(0,T)}\leq\frac{1}{2}(\omega^2-A)\|x_1\|_{L^2(0,T)}$$
$$-\|F_x'(t,0)\|_{L^2(0,T)}, \tag{2.9}$$

Taking

$$q=\frac{2(A-(\frac{\omega}{p})^2)}{3\beta\|x_1\|_{L^2(0,T)}},$$

(2.9) becomes

$$-\frac{T(A-(\frac{\omega}{p})^2)^2}{12\beta\|x_1\|_{L^2(0,T)}}\leq\frac{1}{2}(\omega^2-A)\|x_1\|_{L^2(0,T)}-\|F_x'(t,0)\|_{L^2(0,T)}. \tag{2.10}$$

That is

$$\|F_x'(t,0)\|_{L^2(0,T)}\leq\frac{1}{2}(\omega^2-A)\|x_1\|_{L^2(0,T)}+\frac{T(A-(\frac{\omega}{p})^2)^2}{12\beta\|x_1\|_{L^2(0,T)}},$$

which is true under the assumption (2.2). And therefore, the proof is now complete.

In [29], the following assumptions were made:

($\mathbf{V}_0$) $F(t,x)\in C^2(\mathbf{R}\times\mathbf{R}^n,\mathbf{R})$ is T-periodic in t, and for any $(t,x)\in\mathbf{R}\times\mathbf{R}^n$,

$$F(-t,-x)=F(t,x);$$

($\mathbf{V}_1$) There exist constant $B>0$, $A>\tilde{A}>0$ such that

$$(F_{xx}(t,x)\eta,\eta)\leq\frac{A}{2}|\eta|^2,\quad\forall(t,x)\in\mathbf{R}\times\mathbf{R}^n,\eta\in\mathbf{R}^n$$

and

$$(F_{xx}(t,x)\eta,\eta)\geq\frac{\tilde{A}}{2}|\eta|^2,\quad\forall|x|\leq B,t\in\mathbf{R},\eta\in\mathbf{R}^n;$$

($\mathbf{V}_2$)

$$\lim_{|u|\to\infty}\frac{F(t,u)}{|u|^2}=0,\ \text{ uniformly in } t.$$

The following result was proved in [29].

Theorem A *Let F satisfy ($\mathbf{V}_0$)-($\mathbf{V}_2$) and ($\mathbf{F}_3$). Suppose that*

$$\frac{\omega^2}{\tilde{A}}<p^2<\frac{\omega^2 s_p^2}{A} \tag{2.11}$$

and

$$\|F_x'(t,0)\|^2_{L^2(0,T)}<\frac{\pi B^2}{\omega}\left(\frac{s_p^2\omega^2}{p^2}-A\right)\left(\tilde{A}-\frac{\omega^2}{p^2}\right). \tag{2.12}$$

Then (1.1) has at least one periodic solution with minimal period pT.

By a similar argument to that of Theorem 2.1, we have

Theorem 2.2 *Under the assumptions ($\mathbf{V}_0$), ($\mathbf{V}_1$), ($\mathbf{F}_2$) and ($\mathbf{F}_3$). Suppose that*

$$\frac{\omega^2}{\tilde{A}}<p^2\leq\frac{\omega^2 s_p^2}{A} \tag{2.11$'$}$$

and

$$\|F_x'(t,0)\|^2_{L^2(0,T)} \le \frac{\pi B^2}{\omega}(\omega^2 - A)(\tilde{A} - \frac{\omega^2}{p^2}). \tag{2.12$'$}$$

Then (1.1) has at least one periodic solution with minimal period pT.

Clearly, $(2.11)'$ and $(2.12)'$ improve (2.11) and (2.12) respectively.

3 Applications to the Pendulum Equations

Next, we apply Theorem 2.1 to the classical pendulum Eq. 1.3.

In [29], authors obtained the following result.

Theorem B *Assume that f is T-periodic with minimal period T and odd in t, and that for an integer $p > 1$,*

$$\frac{\omega^2}{A} < p^2 < \frac{\omega^2 s_p^2}{A} \tag{3.1}$$

and

$$\|f\|^2_{L^2(0,T)} < \frac{\pi}{\omega}\left(\frac{s_p^2\omega^2}{p^2} - A\right)\left(2A(1-\cos\bar{\delta}) - \frac{\bar{\delta}^2\omega^2}{p^2}\right) \tag{3.2}$$

where $\omega = \frac{2\pi}{T}$, s_p is the least prime factor of p, $\bar{\delta}$ is the root of $\sin\delta = (\omega^2/Ap^2)\delta$ in the interval $(0,\pi)$. Then (1.3) has at least one periodic solution with minimal period pT.

Notice the fact that $\bar{\delta} \to \pi$ as $p \to \infty$, the next corollary was immediately obtained .

Corollary 3.1 ([29]) *If* (3.1) *and* (3.2) *are replaced by*

$$0 < A < \omega^2 \tag{3.3}$$

and

$$\|f\|^2_{L^2(0,T)} < \frac{4\pi A}{\omega}(\omega^2 - A) \tag{3.4}$$

respectively, then there exists a $P > 0$ such that, for any prime integer $p > P$, (1.3) *has at least one periodic solution with minimal period pT.*

In this case, $F(t,x) = A(1-\cos x) - f(t)\cdot x$, We have

$$F_x'(t,0) = f(t).$$

and

$$\max\{0, \frac{A}{2}x^2 - \frac{A}{24}x^4\} \le F(t,x) - F_x'(t,0)\cdot x = A(1-\cos x) \le \frac{A}{2}x^2.$$

We have immediately the following result.

Theorem 3.1 *Assume that $f(t)$ is T-periodic with minimal period T. IF* (2.1) *holds and*

$$\|f\|^2_{L^2(0,T)} \le \frac{8\pi(\omega^2 - A)(A - (\frac{\omega}{p})^2)^2}{\omega A}. \tag{3.5}$$

Then (1.3) *has at least one periodic solution with minimal period pT.*

Let $p \to \infty$, (3.5) reduces to

$$\|f\|^2_{L^2(0,T)} \leq \frac{8\pi A(\omega^2 - A)}{\omega}, \tag{3.6}$$

which also improves (3.4).

It is easy to prove by combining the technique used in [29] with the method in this paper that (3.2) may be improved by

$$\|f\|^2_{L^2(0,T)} \leq \frac{\pi}{\omega}(\omega^2 - A)\left(2A(1-\cos\bar{\delta}) - \frac{\bar{\delta}^2\omega^2}{p^2}\right) \tag{3.7}$$

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