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The Effect of Freezing and Discretization to the Asymptotic Stability of Relative Equilibria

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In this paper we prove nonlinear stability results for the numerical approximation of relative equilibria of equivariant parabolic partial differential equations in one space dimension. Relative equilibria are solutions which are equilibria in an appropriately comoving frame and occur frequently in systems with underlying symmetry. By transforming the PDE into a corresponding PDAE via a freezing ansatz [2] the relative equilibrium can be analyzed as a stationary solution of the PDAE. The main result is the fact that nonlinear stability properties are inherited by the numerical approximation with finite differences on a finite equidistant grid with appropriate boundary conditions. This is a generalization of the results in [14] and is illustrated by numerical computations for the quintic complex Ginzburg Landau equation.

KEY WORDS: General evolution equations; equivariance; stability; Lie groups; partial differential algebraic equations; unbounded domains; finite differences; asymptotic stability.

AMS SUBJECT CLASSIFICATIONS: 65M99; 35K57.

1. INTRODUCTION

The purpose of this paper is to analyze numerical methods for the approximation of relative equilibria of parabolic systems in one space dimension

$$u_t = Au_{xx} + f(u, u_x)$$

which are equivariant w.r.t. the action of a finite dimensional Lie group. Relative equilibria are solutions of partial differential equations which are equilibria in an appropriately comoving frame. A basic class is formed by

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traveling waves which are solutions of the form $u(x,t)=\bar{v}(x-\bar{\lambda}t)$, where $\bar{v}$ is the wave form and $\bar{\lambda}$ the velocity. Then $\bar{v}$ is a stationary solution in a frame which is translated with the velocity of the wave, i.e.

$$0=A\bar{v}''+f(\bar{v},\bar{v}')+\bar{\lambda}\bar{v}'.$$

Since in general $\bar{\lambda}$ is unknown as well, we use the ansatz $u(x,t)=v(x-\gamma(t),t)$, $\lambda(t)=\gamma'(t)$, which leads to the partial differential algebraic equation (PDAE)

$$\begin{aligned} v_t &= Av_{xx}+f(v,v_x)+\lambda v_x, \\ 0 &= \langle \hat{v}', v-\hat{v}\rangle_{\mathcal{L}_2}, \end{aligned} \tag{1}$$

where $\hat{v}$ is a given function with $\bar{v}-\hat{v}\in\mathcal{H}^2$ and $\langle\cdot,\cdot\rangle_{\mathcal{L}_2}$ denotes the $\mathcal{L}_2$-inner product. Now $(\bar{v},\bar{\lambda})$ is a stationary solution of (1). The last equation is a phase condition which compensates for the additional degree of freedom which has been introduced by adding λ as an time-dependent variable. In the general case a similar ansatz leads to a PDAE where the algebraic conditions are related to extra solution components that determine the transformation into the comoving frame. In this paper we analyze the nonlinear stability of the stationary solution $(\tilde{v},\tilde{\lambda})$ of the DAE which one obtains after truncation of the PDAE to a finite interval and discretization with finite differences. The existence and approximation properties of $(\tilde{v},\tilde{\lambda})$ has been dealt with in [15]. Delicate analysis for $h\to 0$ and $J\to\mathbb{R}$ reveals that stability is preserved for h small, J large enough and appropriately chosen boundary conditions. To this end we prove a uniform stability estimate of the form

$$\|v(t)-\tilde{v}\|_{\mathcal{H}_h^1}+\|\mu(t)-\tilde{\mu}\|\leqslant \text{const e}^{-\nu t}, \quad \nu>0,$$

where $\|\cdot\|_{\mathcal{H}_h^1}$ denotes the discrete analogue of the Sobolevnorm $\|\cdot\|_{\mathcal{H}^2}$. Here resolvent estimates comprise the main technical challenge. This is an overall justification of the freezing method in [2] and is in accordance with the numerical results in [14].

The paper is organized as follows: In Section 1.1 we give a short introduction to the method of freezing relative equilibria [2, 14] and state conditions which ensure the asymptotic stability with asymptotic phase of these solutions. In Section 2 we introduce the finite difference approximation and state the main stability result Theorem 2.8 for the solution of the discretized equations. It is proven in Section 3 by using resolvent estimates which are proven in Section 4. Finally we illustrate the theory by numerical results for the cubic-quintic Ginzburg-Landau equation in Section 5 and we show by a counterexample that some of our assumptions on the boundary operators are sharp.

1.1. Equivariant Evolution Equations

In the following we extend the transformation into the comoving frame given in the introduction for traveling waves to the abstract framework developed in [15] that covers the approaches in [2, 3, 12, 14]. Although the main theorem in Section 2 is formulated for the special case of a PDE in a way which independent of this general approach we think it is instructive to see the derivation of the equations there.

Consider an evolutionary equation on a manifold M which is modelled over a Banach space X

$$u_t = F(u), \quad u(0) = u^0, \tag{2}$$

where $F: N \to TM$ is a vector field which maps a submanifold N modeled over a dense subspace $Y \subset X$ onto the tangent bundle TM of M. For our main stability result (see Section 2) we will either have Banach spaces $X = M$, $Y = N$ or affine spaces $M = \tilde{v} + X$, $N = \tilde{v} + Y$ for some $\tilde{v}: \mathbb{R} \to \mathbb{R}^m$. In these cases the tangent spaces always satisfy $T_u M = X$, $T_v N = Y$ for all $u \in M, v \in N$.

We assume that (2) is equivariant w.r.t. a finite dimensional (possibly noncompact) Lie group G which acts on M via

$$a: G \times M \to M, \quad (\gamma, u) \mapsto a(\gamma)u,$$

where

$$a(\gamma_1 \circ \gamma_2) = a(\gamma_1)a(\gamma_2), \quad a(\mathbb{1}) = I, \quad \mathbb{1} = \text{unit element in } G,$$

which has a tangent action Ta in TM, i.e $Ta(\gamma): T_v M \to T_{a(\gamma)v} M$.

Equivariance means that the following relation holds

$$\begin{aligned} a(\gamma)(N) &\subset N \quad \forall \gamma \in G, \\ F(a(\gamma)u) &= a(\gamma)F(u) \quad \forall u \in N, \ \gamma \in G. \end{aligned}$$

We assume that for any $v \in X$ the map

$$a(\cdot)v: G \to X, \ \ \gamma \mapsto a(\gamma)v$$

is continuous and it is continuously differentiable for any $v \in N$ with derivative

$$da(\gamma)v: T_\gamma G \to T_{a(\gamma)v} M, \quad \lambda \mapsto [da(\gamma)v]\lambda.$$

Here we use $T_\gamma G$ to denote the tangent space of G at γ. Note that in general we can neither expect the action a to be differentiable nor the map $\gamma \mapsto a(\gamma)u$ to be differentiable for any fixed $u \in M$.

Using the ansatz $u = a(\gamma(t))v$ and $\gamma_t(t) = dL_{\gamma(t)}(\mathbb{1})\mu$, where μ lies in the Lie algebra $T_{\mathbb{1}}G$, and dL_γ denotes the derivative of the left translation $L_\gamma : g \mapsto \gamma \circ g$, equation (2) is transformed into (cf. [2, 11, 14])

$$v_t = F(v) - [da(\mathbb{1})v]\mu. \tag{3}$$

The following is a constructive definition of relative equilibria which is appropriate from a numerical point of view [2].

Definition 1.1. A solution $\bar{u}$ of (2) is called a relative equilibrium if it has the form $\bar{u}(t) = a(\bar{\gamma}(t))\bar{v}$ where $\bar{\gamma} : [0, \infty) \to G$ is a smooth curve satisfying $\bar{\gamma}(0) = \mathbb{1}$ and $\bar{v}$ does not depend on time.

Note that usually the whole group orbit $\mathcal{O}(\bar{v}) = \{a(\gamma)\bar{v},\ \gamma \in G\}$ is called a relative equilibrium if it is invariant under the semi-flow [3, 8]. For our purpose it is more convenient to select a special time orbit within this group orbit.

1.2. Parabolic Equations

In the following we consider a special case of (2), namely an equivariant parabolic PDE,

$$u_t = Au_{xx} + f(u, u_x), \qquad x \in \mathbb{R},\ t > 0,\ u(x,t) \in \mathbb{R}^m, \tag{4}$$

where $A \in \mathbb{R}^{m,m}$ is a positive definite matrix. We make the following technical assumption to f which includes nonlinearities of the form uu_x.

Hypothesis 1.2. *Let* $\bar{f}(u, u')(x) = f(u(x), u'(x))$ *and* $f \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ *be of the form*

$$f(u, v) = f_1(u)v + f_2(u), \quad f_1 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^{m,m}),\ f_2 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m)$$

where f_1, f_2, f_1', f_2' *are globally Lipschitz.*

We choose a function $\tilde{v} : \mathbb{R} \to \mathbb{R}^m$ such that $A\tilde{v}'' + f(\tilde{v}, \tilde{v}') \in \mathcal{L}_2$ and define $M = \tilde{v} + \mathcal{L}_2$, $N = \tilde{v} + \mathcal{H}^2$. Then $F : \tilde{v} + \mathcal{H}^2 \to \mathcal{L}_2$ in (2) reads

$$F(u) = Au'' + \bar{f}(u, u').$$

We choose a basis $\{e^1, \ldots, e^p\}$ in the Lie algebra $T_{\mathbb{1}}G$, where p is the dimension of G, write $\mu = \sum_{i=1}^p \mu_i e^i$ and define $S^i(v) = -da(\mathbb{1})ve^i$. Then (3) reads

$$v_t = Av_{xx} + S(v)\mu + f(v, v_x) \tag{5}$$

where we use the short notation $S(v)\mu = \sum_{i=1}^{p} S^i(v)\mu_i$. In the rest of the paper we assume that the operators S^i are linear differential operators of order $\leqslant 1$ which can be written as

$$S^i(v)(x) = S_0^i v(x) + S_1^i v'(x), \quad S_{0,1}^i \in \mathbb{R}^{m,m}.$$

In order to compensate for the additional p degrees of freedom which are obtained by introducing the parameter $\mu \in \mathbb{R}^p$, a phase condition of the form

$$0 = \langle S^i(\hat{v}), v - \hat{v} \rangle, \quad i = 1, \ldots, p.$$

is added, where $\hat{v} \neq 0$ is a given reference function with $\hat{v} - \bar{v} \in \mathcal{H}^1$. This leads together with (5) to the PDAE

$$\begin{aligned} v_t &= A v_{xx} + \sum_{i=1}^{p} \mu_i (S_0^i v + S_1^i v_x) + f(v, v_x) \\ 0 &= \langle S^i(\hat{v}), v - \hat{v} \rangle. \end{aligned} \tag{6}$$

Let $(\bar{v}, \bar{\mu}) \in \tilde{v} + \mathcal{H}^2 \times \mathbb{R}^p$ be the stationary solution of (6) with

$$\lim_{x \to \pm\infty} \bar{v}(x) = \bar{v}_{\pm}. \tag{7}$$

From the condition $\bar{v} \in \tilde{v} + \mathcal{H}^2$ we obtain the condition $S^i(\bar{v}) \in \mathcal{L}_2$ for $i = 1, \ldots, p$. The concrete choice of $\tilde{v}$ will be given in the following examples:

Example 1.3. Let $\tilde{v}$ be a function with $\|\hat{v}(x) - v_{\pm}\| \leqslant \text{const } e^{\pm \varrho x}$ where $f(v_{\pm}, 0) = 0$. Consider the shift action of $G = \mathbb{R}$, i.e. $[a(\gamma)u](x) = u(x - \gamma)$ on $M = \tilde{v} + \mathcal{L}_2 \supset N = \tilde{v} + \mathcal{H}^2$. Then we have $[da(\mathbb{1})v]e^1 = -v_x$ i.e. $S_1^1 = I$, $S_0^1 = 0$ and (6) reads

$$\begin{aligned} v_t &= A v_{xx} + \lambda v_x + f(v, v_x), \\ 0 &= \langle \hat{v}', v - \hat{v} \rangle_{\mathcal{L}_2}. \end{aligned}$$

The relative equilibria are traveling waves $\bar{u}(x,t) = \bar{v}(x - \bar{\lambda} t)$ with stationary points $\lim_{x \to \pm\infty} \bar{v}(x) = v_{\pm}$.

Example 1.4. Consider (4) for $\tilde{v} = 0$, i.e. for $M = \mathcal{L}_2$ and $N = \mathcal{H}^2$. Let the Lie group be $G = S^1 \times \mathbb{R}$ with $(\rho, \tau) = \gamma \in G$ and $(\rho, \tau) \circ (\tilde{\rho}, \tilde{\tau}) = (\rho + \tilde{\rho}, \tau + \tilde{\tau})$. Let the action $a : G \times \mathcal{L}_2 \to \mathcal{L}_2$ be given for $u : \mathbb{R} \to \mathbb{R}^2$ by

$$[a(\gamma)u](x) = R_{-\rho} u(x - \tau), \quad R_\rho = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix}.$$

Then we have $[da(\mathbb{1})v]e^1 = -v_x, [da(\mathbb{1})v]e^2 = -R_{\frac{\pi}{2}}v$, i.e. $S_1^1 = I,\ S_0^2 = R_{\frac{\pi}{2}}$, $S_0^1 = S_1^2 = 0$ and (6) reads with $\mu_\tau = \tau_t,\ \mu_\rho = \rho_t$

$$v_t = Av_{xx} + \mu_\tau v_x + \mu_\rho R_{\frac{\pi}{2}} v + f(v, v_x),$$
$$0 = \langle \hat{v}', v - \hat{v}\rangle_{\mathcal{L}_2}, \quad 0 = \langle R_{\frac{\pi}{2}} v, v - \hat{v}\rangle_{\mathcal{L}_2}.$$

The relative equilibria are rotating and traveling waves $\bar{u}(x,t) = R_{-\bar{\mu}_\rho t}\bar{v}(x - \bar{\mu}_\tau t)$. Note that, if $\bar{v}$ is a front, i.e. $\bar{v}_- \neq \bar{v}_+$, then $\bar{v}$ and $R_{\frac{\pi}{2}}\bar{v}$ are not in $\mathcal{L}_2$. In this case, considering a rotating front, the condition $S^2(\bar{v}) = R_{\frac{\pi}{2}}\bar{v} \in \mathcal{L}_2$ is not satisfied and the stability result of this paper cannot be applied.

We are interested in the asymptotic stability of $(\bar{v}, \bar{\mu})$ which is defined as follows.

Definition 1.5 (Asymptotic stability)**.** The stationary solution $(\bar{v}, \bar{\mu})$ of (6) is asymptotically stable, if $\forall \epsilon > 0,\ \exists \delta > 0$ such that for all solutions (v, μ) of (6) with $\|\mu(0) - \bar{\mu}\| + \|v(\cdot, 0) - \bar{v}\| \leqslant \delta$:

$$\|\mu(t) - \bar{\mu}\| + \|v(\cdot, t) - \bar{v}\| \begin{cases} \leqslant \epsilon & \forall t \geqslant 0 \\ \to 0 & \text{for } t \to \infty. \end{cases}$$

Remark 1.6. Note that by the freezing ansatz the well known notion of asymptotic stability with asymptotic phase for $\bar{u}$ is converted into asymptotic stability for $(\bar{v}, \bar{\mu})$.

The stability of the PDAE solution $(\bar{v}, \bar{\mu})$ is determined by the spectrum of the linearization $\Lambda : \mathcal{H}^2 \to \mathcal{L}_2$ of the r.h.s. of (5) w.r.t. v at $(\bar{v}, \bar{\mu})$ which is given by

$$\Lambda v = Av'' + Bv' + Cv, \quad \text{where} \tag{8}$$

$$B(x) = D_2 f(\bar{v}(x), \bar{v}'(x)) + \sum_{i=1}^{p} \bar{\mu}_i S_1^i, \quad C(x) = D_1 f(\bar{v}(x), \bar{v}'(x)) + \sum_{i=1}^{p} \bar{\mu}_i S_0^i.$$

Assumption (7) implies with the properties of A that $\lim_{x\to\pm\infty} \bar{v}'(x) = 0$. Thus Λ converges for $x \to \pm\infty$ to constant coefficient operators

$$\Lambda_\pm v = Av'' + B_\pm v' + C_\pm v, \qquad B_\pm = \lim_{x \pm\infty} B(x), C_\pm = \lim_{x \pm\infty} C(x).$$

Our standing assumption in this paper is the following: The operator Λ defined in (8) satisfies the usual conditions which guarantee asymptotic stability with asymptotic phase for $\bar{u}$ [7, 17]:

Hypothesis 1.7 (Eigenvalue condition). *The functions* $S^i(\bar{v}) \in \mathcal{L}_2,\ i = 1, \dots, p$ *are linearly independent and span the null space of* $\Lambda : \mathcal{H}^2 \to \mathcal{L}_2$, *i.e.*

$$\mathcal{N}(\Lambda) = \operatorname{span}\{S^1(\bar{v}), \dots, S^p(\bar{v})\}.$$

The eigenvalue zero is semi-simple and there exists $\beta > 0$ *such that there are no other isolated eigenvalues* s *of finite multiplicity with* $\operatorname{Re} s \geqslant -\beta$.

Hypothesis 1.8 (Spectral condition). *There exist* $\sigma > 0,\ \beta > 0$, *such that for* s *with* $\operatorname{Re} s \geqslant -\beta$ *the solutions* λ *of the quadratic eigenvalue problems*

$$\det(\lambda^2 A + \lambda B_\pm + C_\pm - sI) = 0$$

satisfy: $|\operatorname{Re}\lambda| \geqslant \sigma$.

Example 1.9. For Example 1.4 the operator Λ reads

$$\Lambda v = Av'' + (\mu_\tau I + D_2 f(\bar{v}, \bar{v}'))v' + (\mu_\rho R_{\frac{\pi}{2}} + D_1 f(\bar{v}, \bar{v}'))v$$

and its null space is spanned by $\bar{v}'$ and $R_{\frac{\pi}{2}}\bar{v}$.

Note, that for the excluded case of a rotating front, the continuous spectrum of Λ touches the imaginary axis. Therefore even in the continuous case the usual stability theory which relies on a spectral gap cannot be applied.

2. NUMERICAL APPROXIMATION

2.1. DAE Formulation

In order to compute numerical approximations of $(\bar{v}, \bar{\mu})$ we define a discrete interval

$$J = [n_-, n_+] = \{n \in \mathbb{Z} :\ n_- \leqslant n \leqslant n_+,\ \text{ where } n_\pm \in \mathbb{Z} \cup \{\pm\infty\}\}$$

and a corresponding equidistant grid with grid size $h > 0$

$$J_h = \{x_n :\ x_n = nh,\ n \in J\}.$$

We denote the Banach space of sequences in $\mathbb{R}^m$ which are indexed by J provided with the supremum norm $\|z\|_\infty = \sup_{n\in J}\|z_n\|$ by $\ell_\infty^J(\mathbb{R}^m)$ and write $J_h \to \mathbb{R}$ if $h \to 0$ and simultaneously $h \cdot \min\{-n_-, n_+\} \to \infty$, i.e. $\pm n_\pm$ grows faster than h decreases, so that $[hn_-, hn_+] \to \mathbb{R}$.

If necessary, we embed each $u \in \ell_\infty^J(\mathbb{R}^m)$ in $\ell_\infty(\mathbb{R}^m)$ by setting $u_n = 0$ for $n \in \mathbb{Z}\backslash J$ without further notice. If no confusion is possible we drop the argument $\mathbb{R}^m$ and write just ℓ_∞^J and ℓ_∞. Let the standard finite difference operators on the extended grid

$$\hat{J}_h = \{x_n : \ x_n = nh, \ \ n \in \hat{J} = [n_- - 1, n_+ + 1]\}$$

be given by $\delta_0 : \ell_\infty^{\hat{J}} \to \ell_\infty^J$, $\quad \delta_+ : \ell_\infty^{[n_-, n_+ + 1]} \to \ell_\infty^J$, $\quad \delta_- : \ell_\infty^{[n_- - 1, n_+]} \to \ell_\infty^J$, where

$$(\delta_0 v)_n = \frac{1}{2h}(v_{n+1} - v_{n-1}), \quad (\delta_+ v)_n = \frac{1}{h}(v_{n+1} - v_n), \quad (\delta_- v)_n = \frac{1}{h}(v_n - v_{n-1}).$$

Then for sequences $u, v \in \ell_\infty^J(\mathbb{R}^m)$, $J = [n_-, n_+]$ we define the inner product and discrete Sobolev norms by

$$\langle u, v \rangle_{J_h} = \sum_{n=n_-}^{n_+} h u_n^T v_n, \qquad \|u\|_{\mathcal{L}_{2,h}} = \sqrt{\langle u, u \rangle_{J_h}},$$

$$\|u\|_{\mathcal{H}_h^1} = \|u\|_{\mathcal{L}_{2,h}} + \|\delta_+ u\|_{\mathcal{L}_{2,h}}, \qquad \|u\|_{\mathcal{H}_h^2} = \|u\|_{\mathcal{H}_h^1} + \|\delta_+\delta_- u\|_{\mathcal{L}_{2,h}}.$$

Discretizing (6) and adding linear boundary conditions

$$\mathcal{B}v = P_- v_{n_-} + Q_-(\delta_0 v)_{n_-} + P_+ v_{n_+} + Q_+(\delta_0 v)_{n_+}, \quad P_\pm, Q_\pm \in \mathbb{R}^{2m,m}$$

leads to the differential algebraic equation (DAE)

$$v_n' = A(\delta_+\delta_- v)_n + \hat{S}_n(v)\mu + f(v_n, \delta_0 v_n), \quad n \in J, \ t > 0 \tag{9a}$$

$$0 = \mathcal{B}v - \eta, \tag{9b}$$

$$0 = \langle \hat{S}^i(\hat{v}_{|J_h}), v - \hat{v}_{|J_h} \rangle_{J_h}, \quad i = 1, \ldots, p, \tag{9c}$$

where $\hat{S}_n^i(v) = S_0^i v_n + S_1^i(\delta_0 v)_n \in \mathbb{R}^m$ and $\hat{S}_n(v)\mu = \sum_{i=1}^p \mu_i \hat{S}_n^i(v)$. This system is a DAE of differentiation index 2 [6].

We assume that the boundary conditions are partitioned into a Dirichlet and Neumann part, i.e. the matrices $(P_\pm, Q_\pm) \in \mathbb{R}^{2m,2m}$ have the following structure

$$(P_\pm, Q_\pm) = \begin{pmatrix} P_\pm^N & Q_\pm^N \\ P_\pm^D & 0 \end{pmatrix}, \qquad P_\pm^N, Q_\pm^N \in \mathbb{R}^{k,m}, \ \ P_\pm^D \in \mathbb{R}^{2m-k,m}$$

and the matrix $(Q_- Q_+)$ is of rank $r \in [0, 2m]$. This induces the following splitting of the boundary conditions (9b) into one part that does depend on the external variables $v_{n_- -1}, v_{n_+ +1}$ and one part that depends on the values at the inner grid points $v_{n_-}, \ldots, v_{n_+}$ only:

$$\mathcal{B}^N v = P_-^N v_{n_-} + Q_-^N \delta_0 v_{n_-} + P_+^N v_{n_+} + Q_+^N \delta_0 v_{n_+} = \eta^N, \tag{10a}$$

$$\mathcal{B}^D v_{|J_h} = P_-^D v_{n_-} + P_+^D v_{n_+} = \eta^D. \tag{10b}$$

Note that initial values v^0, μ^0 are called consistent if they solve the algebraic constraints (9b),(9c) as well as the equations

$$\begin{aligned} 0 &= \mathcal{B}^D(A\delta_+\delta_- v + \hat{S}(v)\mu + f(v,\delta_0 v)), \\ 0 &= \langle \hat{S}(\hat{v}), A\delta_+\delta_- v + \hat{S}(v)\mu + f(v,\delta_0 v)\rangle_{J_h}, \end{aligned} \tag{11}$$

which are obtained by differentiating (10b),(9c) w.r.t. time t and inserting (9a).

Define $\pi : \ell_\infty^{\hat{J}}(\mathbb{R}^m) \to \ell_\infty^J(\mathbb{R}^m)$ as the restriction operator onto J by

$$\pi : \ (u_{n_- -1}, \ldots, u_{n_+ +1}) \mapsto (u_{n_-}, \ldots, u_{n_+}).$$

Then (9) can be written in the form

$$\begin{aligned} (\pi v)' &= f_{\text{diff}}(v,\lambda), \quad v(0) = v^0, \lambda(0) = \lambda^0 \\ 0 &= f_{\text{alg}}(v,\lambda), \end{aligned} \tag{12}$$

where $f_{\text{diff}} : \ell_\infty^{\hat{J}}(\mathbb{R}^m) \times \mathbb{R}^p \to \ell_\infty^J(\mathbb{R}^m)$, $f_{\text{alg}} : \ell_\infty^{\hat{J}}(\mathbb{R}^m) \times \mathbb{R}^p \to \mathbb{R}^{2m+1}$.

The proper notion of a solution of (12) is the following

Definition 2.1. A function $(v,\lambda) : [0,\tau) \to \ell_\infty^{\hat{J}}(\mathbb{R}^m) \times \mathbb{R}^p$ is called a solution of (12) in $(0,\tau)$, $\tau \in \mathbb{R} \cup \{\infty\}$ if

(1) $f_{\text{diff}}(v(\cdot),\lambda(\cdot)) : [0,\tau) \to \ell_\infty^J$ is continuous
(2) $(v,\lambda) : [0,\tau) \to \ell_\infty^{\hat{J}}(\mathbb{R}^m) \times \mathbb{R}^p$ is continuous
(3) $(\pi v)'(t)$ exists, $(\pi v)'(t) = f_{\text{diff}}(v(t),\lambda(t)) \in \ell_\infty^J(\mathbb{R}^m)$ for $t \in (0,\tau)$, and $(v(0),\lambda(0)) = (v^0,\lambda^0)$
(4) $f_{\text{alg}}(v(t),\lambda(t)) = 0 \ \forall t \in [0,\tau)$.

2.2. Main Result

The main result of this paper is the following discrete stability theorem for the stationary solution $(\bar{v},\bar{\mu})$ of (9a). The existence of such a solution for large enough J and small h has been proven in Theorem 2.6 in [15] together with the convergence estimate

$$\|\bar{v}_{|J_h} - \tilde{v}\|_{\mathcal{H}_h^2} + \|\bar{\mu} - \tilde{\mu}\| \leqslant \text{const } (h^2 + e^{-\alpha h \min\{-n_-, n_+\}}). \tag{13}$$

Before we can state the stability result Theorem 2.8 we have to collect the necessary hypotheses on the boundary conditions and the phase condition.

We assume that $\hat{v} : \mathbb{R} \to \mathbb{R}^m$ is a given template function and define the following class $\mathcal{E}_\varrho(I, \mathbb{R}^{m,p})$ of functions:

Definition 2.2. We define a function $g: I \to \mathbb{R}^{m,p}$, $I \subset \mathbb{R}$ to be in $\mathcal{E}_\varrho(I, \mathbb{R}^{m,p})$ if there exists $K > 0$ such that for all $x \in I$:

$$\|g(x)\| \leqslant K\mathrm{e}^{-\varrho|x|} \quad \text{and} \quad \|g'(x)\| \leqslant K\mathrm{e}^{-\varrho|x|}.$$

Hypothesis 2.3 (phase condition). *Assume that* $S(\hat{v}) \in \mathcal{E}_\varrho(\mathbb{R}, \mathbb{R}^{m,p})$ *and the* $p \times p$ *matrix*

$$\langle S(\hat{v}), S(\bar{v})\rangle_{\mathcal{L}_2} = \int_{\mathbb{R}} [S(\hat{v})](x)^T [S(\bar{v})](x)\, dx.$$

is nonsingular.

The following determinant condition is needed for resolvent estimates in a compact region for the continuous operator restricted to finite intervals [1]. It allows to control the growing terms for $x \to \pm\infty$ of the solution to the resolvent equation. Since for bounded $|s|$ we rely on the solution of the corresponding problem for the continous system we have to employ the same condition here.

Definition 2.4. Define

$$\mathcal{D}(s) = \det\left(\begin{pmatrix} P_- & Q_- \end{pmatrix}\begin{pmatrix} Y_-^s(s) \\ Y_-^s(s)\Sigma_-^s(s)\end{pmatrix} \begin{pmatrix} P_+ & Q_+ \end{pmatrix}\begin{pmatrix} Y_+^u(s) \\ Y_+^u(s)\Sigma_+^u(s)\end{pmatrix}\right)$$

where $Y_-^s(s), Y_+^u(s) \in \mathbb{R}^{m,m}$ and $\Sigma_-^s(s), \Sigma_+^u(s) \in \mathbb{R}^{m,m}$ solve the quadratic eigenvalue problems

$$AY\Sigma^2 + B_\pm Y\Sigma + (C_\pm - sI)Y = 0$$

with $\mathrm{Re}\,\sigma(\Sigma_\pm^s(s)) < 0$ and $\mathrm{Re}\,\sigma(\Sigma_\pm^u(s)) > 0$.

Hypothesis 2.5 (boundary conditions). *The boundary condition* (9b) *is satisfied at the stationary points* $\bar{v}_\pm$, *i.e.* $\eta = P_-\bar{v}_- + P_+\bar{v}_+$ *and there exist* $\beta, C > 0$ *such that* $\mathcal{D}(s) \neq 0$ *if* $|s| \leqslant C$ *and* $\mathrm{Re}\, s > -\beta$.

In order to obtain resolvent estimates for large $|s|$ we have to employ a truly discrete condition, which ensures that a certain z dependent matrix is uniformly invertible for z in a special region of $\mathbb{C}$.

If $\delta > 0$ is chosen such that $|\arg\mu| < \frac{\pi}{2} - \delta \quad \forall \mu \in \sigma(A^{-1})$ then there exists $C > 0$ such that the following matrix function is well defined

$$\Delta(z) = \begin{cases} \frac{1}{(1+|z|^2)^{\frac{1}{2}}}(I + z^2A^{-1})^{\frac{1}{2}}A^{-\frac{1}{2}}, & |\arg(z)| \leqslant \frac{\pi}{4} + \frac{\delta}{3} \\ \frac{z}{(1+|z|^2)^{\frac{1}{2}}}(\frac{1}{z^2}I + A^{-1})^{\frac{1}{2}}A^{-\frac{1}{2}}, & |z| \geqslant C. \end{cases} \tag{14}$$

Then we can formulate the following hypothesis.

Hypothesis 2.6. *Assume that there exists $C>0$ such that the matrices*

$$\Gamma_z = \begin{pmatrix} Q_-^N \Delta(z) & -Q_+^N \Delta(z) \\ P_-^D & P_+^D \end{pmatrix} \tag{15}$$

have uniformly bounded inverses for

$$z \in \mathbb{C}: \ \arg(z) \leqslant \frac{\pi}{4} + \frac{\delta}{3} \qquad or \qquad |z| \geqslant C.$$

This hypothesis is used in Section 4 to prove resolvent estimates which are needed in Section 3. The uniformity conditions in Hypotheses 2.5 and 2.6 seem rather technical and in fact hard to check, but the following remark shows that Hypotheses 2.5 and 2.6 are strongly related to another condition which stems from the continous problem that can be checked easily.

Remark 2.7. The following statements are equivalent

(1) Γ_z has a uniformly bounded inverse for all $|\arg z| \leqslant \frac{\pi}{4} + \frac{\delta}{3}$ and for $|z| \geqslant C$.

(2) The matrices

$$\Gamma_0 = \begin{pmatrix} Q_-^N A^{-\frac{1}{2}} & -Q_+^N A^{-\frac{1}{2}} \\ P_-^D & P_+^D \end{pmatrix} \text{ and } \Gamma_\infty = \begin{pmatrix} Q_-^N A^{-1} & -Q_+^N A^{-1} \\ P_-^D & P_+^D \end{pmatrix}$$

are nonsingular and Γ_z is nonsingular for $|\arg z| \leqslant \frac{\pi}{4} + \frac{\delta}{3}$, $z \neq 0$.

The nonsingularity of Γ_0 corresponds to the corresponding condition (see Theorem 2.1 in [1]) which is necessary for resolvent estimates for large $|s|$ for the continuous operator which is restricted to a finite interval. The nonsingularity of Γ_∞ will also be used in Section 3 to reduce the DAE to a corresponding ODE the stability of which can then be discussed. Moreover, one can show that $\det(\Gamma_0) \neq 0$ implies $D(s) \neq 0$ for all large s (see the corresponding remark in Section 5 of [1]).

For the boundary conditions which are used in the numerical computations such as Neumann, Dirichlet and periodic boundary conditions, Hypothesis 2.6 is always satisfied.

Note that Hypothesis 2.5 is crucial as the following example shows: For a traveling wave solution $\bar{v}$ of scalar equation

$$u_t = u_{xx} + f(u)$$

which moves with velocity $\bar{\lambda}$ we consider boundary conditions, which are a homotopy between Neumann and Dirichlet conditions, i.e.

$$P_- = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \quad Q_- = \begin{pmatrix} 1-\alpha \\ 0 \end{pmatrix}, \quad Q_+ = \begin{pmatrix} 0 \\ 1-\alpha \end{pmatrix}, \qquad \alpha \in [0, 1].$$

Then condition 2.5 reads

$$\det\begin{pmatrix}\alpha+(1-\alpha)\nu_-^s(s) & 0\\ 0 & \alpha+(1-\alpha)\nu_+^u(s)\end{pmatrix}\neq 0,$$

where $\nu_\pm^{s,u}$ denotes the stable and unstable spatial eigenvalue respectively, i.e. the roots of the characteristic equation

$$\nu^2+\bar{\lambda}\nu+f'(\bar{v}_\pm)-s=0.$$

Thus Hypothesis 2.5 is violated for $\alpha\in(0,1)$ with

$$s(\alpha)=\left(\frac{\alpha}{\alpha-1}\right)^2+\frac{\bar{\lambda}\alpha}{\alpha-1}+f'(\bar{v}_\pm).$$

In this case the value $s(\alpha)$ is a spurious eigenvalue which is created by the boundary conditions. If it is positive then it affects stability. We will illustrate this effect in Section 5.

Now we can state the main result of this paper.

Theorem 2.8. *Assume that Hypotheses 2.3,2.5,2.6 hold.*

Then there exist $h_0>0, T>0$ such that for $h<h_0$, $\mp hn_\pm>T$ the stationary solution $(\tilde{v},\tilde{\mu})\in\ell_\infty^J(\mathbb{R}^m)\times\mathbb{R}^p$ of (9a) *is asymptotically stable.*

More precisely, there exist $\nu,\rho,h_0,T>0$ such that for $h<h_0$, $\mp hn_\pm>T$ the following statements hold if $\mathrm{e}^{-\alpha T}<c\sqrt{h}$ for some $c>0$, where α denotes the constant in Hypothesis 2.3:

For each consistent initial value $(v^0,\mu^0)\in\ell_\infty^{\hat{J}}(\mathbb{R}^m)\times\mathbb{R}^p$ (i.e. (9b), (9c), (11) *are satisfied) with $\|v^0-\tilde{v}\|_{\mathcal{H}_h^1}\leqslant\rho$, there exists a unique solution (v,μ) of* (9) *with initial condition $(v(0),\mu(0))=(v^0,\mu^0)$ which obeys for some $\nu>0$ the estimate*

$$\|v(t)-\tilde{v}\|_{\mathcal{H}_h^1}+\|\mu(t)-\tilde{\mu}\|\leqslant\text{const }\mathrm{e}^{-\nu t}.\tag{16}$$

Remark 2.9. Combining estimate (16) with (13) we obtain for $h>h_0$, $\pm n_\pm>T$ and a sufficiently large $\tau_0>0$:

$$\|v(t)-\bar{v}\|_{\mathcal{H}_h^2}+\|\mu(t)-\bar{\mu}\|\leqslant\text{const }(\mathrm{e}^{-\nu t}+h^2+\mathrm{e}^{-\alpha h\min\{-n_-,n_+\}})\quad\forall t>\tau_0.$$

Note that similar estimates hold for $\|\cdot\|_\infty$ (see [14, 15]).

Remark 2.10. We will show later in Lemma 3.3 that if one prescribes the initial value v^0 on the grid J and if the so called essential conditions (9c),(10b) are satisfied, then the external points $v^0_{n_--1}, v^0_{n_++1}$ of v^0 and the initial parameter μ^0 can be chosen in such a way, that (v^0,μ^0) solves (9b), (9c), (11).

Theorem 2.8 will be proven at the end of the next section, in the beginning of which we give a short outline of the main steps of the proof.

It mainly relies on resolvent estimates for the linearized operator, which (after reduction to an ODE) can be used to prove stability estimates. Moreover, we make use of the fact that the linearized operator in the continuous case is sectorial and there is a gap between the essential spectrum and the zero eigenvalues. This gap is used here to derive resolvent estimates for the discretized system in a similar way as has been carried out for the continuous system in [1, 14]. The main tool are exponential dichotomies combined with linearization at the asymptotic states. We expect that part of this analysis can still be used for special patterns in higher dimensions.

3. STABILITY OF THE NONLINEAR SYSTEM

System (9) has the special structure of an initial boundary value problem with an additional constraint. Therefore we will reduce the algebraic constraints directly and try to follow the spirit of the semigroup approach which has been used to prove asymptotic stability with asymptotic phase for relative equilibria of the continuous system [7].

To this end in Section 3.1 we transform (9) into a semilinear equation with stationary solution zero and prove a stability result for this system in Section 3.4. This is achieved by reducing the DAE to a corresponding ODE in Section 3.2 and proving exponential estimates for the solution operator of the corresponding linear equation in Section 3.3. These estimates can be concluded from an integral representation using resolvent estimates which will be shown in Section 4.

3.1. The Semilinear Equation

Let $(\tilde{v}, \tilde{\lambda})$ be the stationary solution of (9) and insert $w = v - \tilde{v}$, $\mu = \lambda - \tilde{\lambda}$ into (9) to obtain

$$w_n' = (\tilde{\Lambda} w)_n + \hat{S}_n(\tilde{v})\mu + \varphi_n(w, \mu), \quad n \in J \tag{17a}$$

$$0 = \mathcal{B} w \tag{17b}$$

$$0 = \langle \hat{S}(\hat{v}), w \rangle_{J_h}, \tag{17c}$$

where $\tilde{\Lambda} : \ell_\infty^{\hat{J}} \to \ell_\infty^{J}$, $(\tilde{\Lambda} v)_n = A(\delta_+ \delta_- v)_n + \tilde{B}_n (\delta_0 v)_n + \tilde{C}_n v_n$,

$$\tilde{B}_n = D_2 f(\tilde{v}_n, (\delta_0 \tilde{v})_n) + \sum_{i=1}^{p} \tilde{\lambda}_i S_1^i, \qquad \tilde{C}_n = D_1 f(\tilde{v}_n, (\delta_0 \tilde{v})_n) + \sum_{i=1}^{p} \tilde{\lambda}_i S_0^i$$

and $\varphi : \ell_\infty^{\hat{J}} \times \mathbb{R}^p \to \ell_\infty^J$, $\varphi_n(v,\mu) = \hat{\omega}_n(v) + \hat{S}_n(v)\,\mu$, with

$$\begin{aligned}\hat{\omega}_n(v) &= f(\tilde{v}_n + v_n, \delta_0\tilde{v}_n + \delta_0 v_n) - f(\tilde{v}_n, \delta_0\tilde{v}_n) - D_1 f(\tilde{v}_n, \delta_0\tilde{v}_n)v_n \\ &\quad - D_2 f(\tilde{v}_n, \delta_0\tilde{v}_n)\delta_0 v_n.\end{aligned}$$

Using the notations $\Psi = \hat{S}(\hat{v})$, $\Phi = \hat{S}(\tilde{v})$ stability of $(\tilde{v}, \tilde{\lambda})$ is now equivalent to the stability of zero as a solution of (17) which we rewrite using the operator π and (10) as follows:

$$\pi v' = \tilde{\Lambda} v + \Phi\mu + \varphi(v,\mu), \tag{18a}$$

$$0 = \mathcal{B}^N v, \tag{18b}$$

$$0 = \mathcal{B}^D \pi v, \tag{18c}$$

$$0 = \langle \Psi, \pi v\rangle_{J_h}. \tag{18d}$$

For the semilinear equation (18) the consistency conditions (11) read

$$\begin{aligned} 0 &= \mathcal{B}^D(\tilde{\Lambda} v + \Phi\mu + \varphi(v,\mu)), \\ 0 &= \langle \Psi, \tilde{\Lambda} v + \Phi\mu + \varphi(v,\mu)\rangle_{J_h}. \end{aligned} \tag{19}$$

For $(v,\mu) \in \ell_\infty^J \times \mathbb{R}$ we use the notation

$$B_\delta^{\mathcal{H}_h^1}((v,\mu)) = \{(u,\lambda) \in \ell_\infty^{\hat{J}} \times \mathbb{R} :\ \|v-u\|_{\mathcal{H}_h^1} + \|\mu - \lambda\| \leqslant \delta\}$$

and define the space of consistent initial conditions by

$$\ell_{\mathrm{co}}^J = \{(v,\mu) \in \ell_\infty^{\hat{J}} \times \mathbb{R}^p :\ (v,\mu) \text{ satisfies (18b)–(18d), (19)}.$$

The main assumptions on φ are summarized in the following hypothesis.

Hypothesis 3.1. *Assume that* $\varphi : \ell_\infty^{\hat{J}} \times \mathbb{R}^p \to \ell_\infty^J$ *satisfies* $\varphi(0,0) = 0$ *and that there exist* $\rho_0, h_0, T > 0$ *such that for* $h < h_0$, $\pm n_\pm h > T$ *for all* $(v,\mu), (u,\lambda) \in B_\rho^{\mathcal{H}_h^1}(0)$, *with* $\rho < \rho_0$, *the uniform estimates*

$$\|\varphi(v,\mu) - \varphi(u,\lambda)\|_{\mathcal{L}_{2,h}} \leqslant \text{const}\ (\|v-u\|_{\mathcal{H}_h^1} + \max(\|v\|_{\mathcal{H}_h^1}, \|u\|_{\mathcal{H}_h^1})\|\mu - \lambda\|) \tag{20}$$

$$\|\varphi(v,\mu)\|_{\mathcal{L}_{2,h}} \leqslant \text{const}\ \rho(\|v\|_{\mathcal{H}_h^1} + \|\mu\|) \tag{21}$$

hold, with constants which are independent of J *and* h.

The main result of this section is the following stability theorem for the zero solution of the DAE (18).

Theorem 3.2. *Let Λ satisfy Hypotheses 1.7,1.8 and let φ satisfy Hypothesis 3.1. Assume further that $\Psi = \hat{S}(\hat{v})$, where $\hat{v}$ satisfies Hypothesis 2.3 and that the boundary conditions satisfy Hypotheses 2.5,2.6.*

Then there exist $h_0 > 0, T > 0$, such that for $h < h_0$, $\mp hn_\pm > T$ the stationary solution $0 \in \ell_\infty^{\hat{J}} \times \mathbb{R}$ of (17) *is asymptotically stable.*

More precisely, there exist $\rho, h_0, T > 0$ such that for $h < h_0$, $\mp hn_\pm > T$ with $\mathrm{e}^{-\alpha T} < c\sqrt{h}$ for some $c > 0$, where α denotes the constant in Hypothesis 2.3, the following statements hold.

For each initial value $(v^0, \mu^0) \in \ell_{\mathrm{co}}^J$ with $\|v^0\|_{\mathcal{H}_h^1} + \|\mu^0\| < \rho$ there exists a unique solution (v, μ) of (17)*. This solution obeys for some $\nu > 0$ the estimate*

$$\|v(t)\|_{\mathcal{H}_h^1} + \|\mu(t)\| \leqslant \text{const } \mathrm{e}^{-\nu t} \quad \forall t \geqslant 0. \tag{22}$$

We first show that Theorem 3.2 implies the stability result Theorem 2.8.

Proof of Theorem 2.8. For $\varphi(v, \mu) = \hat{\omega}(v) + \hat{S}(v)\mu$, we prove that Hypothesis 3.1 is satisfied.

Hypothesis 1.2 implies that f_1', f_2' are globally bounded and

$$D_1 f(u, w) = f_1'(u)(w, \cdot) + f_2'(u), \quad D_2 f(u, w) = f_1(u),$$

for $u, w, \delta_u, \delta_w \in \mathbb{R}^m$

$$\begin{aligned} &\|D_1 f(u + \delta_u, w + \delta_w) - D_1 f(u, w)\| \leqslant \text{const } (\|\delta_u\| + \|\delta_w\|), \\ &\|D_2 f(u + \delta_u, w + \delta_w) - D_2 f(u, w)\| \leqslant \text{const } \|\delta_u\|. \end{aligned} \tag{23}$$

Thus we obtain for $v, u \in B_\rho^{1,\infty}(0)$

$$\begin{aligned} \|\hat{\omega}_n(v) - \hat{\omega}_n(u)\| = \|&f(\tilde{v}_n + v_n, \delta_0\tilde{v}_n + \delta_0 v_n) - f(\tilde{v}_n + u_n, \delta_0\tilde{v}_n + \delta_0 u_n) \\ &- D_1 f(\tilde{v}_n, \delta_0\tilde{v}_n)(v_n - u_n) - D_2 f(\tilde{v}_n, \delta_0\tilde{v}_n)(\delta_0 v_n - \delta_0 u_n)\| \\ \leqslant &\text{const } (\|v_n - u_n\| + \|v_n - u_n\| \|\delta_0 v_n\| + \|u_n\| \|\delta_0 (v - u)_n\|) \end{aligned}$$

This implies for all $(v, \mu), (u, \lambda) \in B_\rho^{\mathcal{H}_h^1}(0)$ using Hypothesis 3.1 and the Sobolev imbedding $\|v\|_\infty \leqslant \text{const } \|v\|_{\mathcal{H}_h^1}$

$$
\begin{aligned}
\|\hat{\omega}(v)-\hat{\omega}(u)\|_{\mathcal{L}_{2,h}}^2 &= \sum_{n=n_-}^{n_+} h\|\hat{\omega}_n(v)-\hat{\omega}_n(u)\|^2 \\
&\leqslant \text{const } \left(\sum_{n=n_-}^{n_+} h\|v_n-u_n\|^2 + \|\delta_0 v\|_\infty^2 \sum_{n=n_-}^{n_+} h\|v_n-u_n\|^2 \right. \\
&\quad \left. + \|u\|_\infty^2 \sum_{n=n_-}^{n_+} h\|\delta_0(v-u)_n\|^2\right) \\
&\leqslant \text{const } (\|v-u\|_{\mathcal{L}_{2,h}}^2 + \|v-u\|_{\mathcal{H}_h^1}^2 \|v\|_{\mathcal{H}_h^1}^2 + \|u\|_{\mathcal{H}_h^1}^2 \|v-u\|_{\mathcal{H}_h^1}^2) \\
&\leqslant \text{const } \|v-u\|_{\mathcal{H}_h^1}^2.
\end{aligned}
$$

Furthermore, (23) leads for $\|v\|_{1,\infty} \leqslant \rho$ to

$$
\begin{aligned}
\|\hat{w}_n(v)\| &\leqslant \|f(\tilde{v}_n+v_n, \delta_0\tilde{v}_n+\delta_0 v_n) - f(\tilde{v}_n, \delta_0\tilde{v}_n) \\
&\qquad - D_1 f(\tilde{v}_n, \delta_0\tilde{v}_n)v_n - D_2 f(\tilde{v}_n, \delta_0\tilde{v}_n)\delta_0 v_n\| \\
&\leqslant \int_0^1 \|[D_1 f(\tilde{v}_n+tv_n, \delta_0\tilde{v}_n+t\delta_0 v_n) - D_1 f(\tilde{v}_n, \delta_0\tilde{v}_n)]v_n\|\,dt \\
&\quad + \int_0^1 \|[D_2 f(\tilde{v}_n+tv_n, \delta_0\tilde{v}_n+t\delta_0 v_n) - D_2 f(\tilde{v}_n, \delta_0\tilde{v}_n)]\delta_0 v_n\|\,dt \\
&\leqslant \text{const } \int_0^1 t(\|v_n\|+\|\delta_0 v_n\|)\|v_n\|\,dt + \int_0^1 t\|v_n\|\,\|\delta_0 v_n\|\,dt \\
&\leqslant \text{const } (\|v_n\|+\|\delta_0 v_n\|)\|v_n\|.
\end{aligned}
$$

This implies for $\|v\|_{\mathcal{H}_h^1} \leqslant \rho$

$$
\begin{aligned}
\|\hat{w}(v)\|_{\mathcal{L}_{2,h}}^2 &\leqslant \text{const } \sum_{n=n_-}^{n_+} h(\|v_n\|+\|\delta_0 v_n\|)^2\|v_n\|^2 \\
&\leqslant \text{const } \|v\|_\infty^2 h \sum_{n=n_-}^{n_+} (\|v_n\|+\|\delta_0 v_n\|)^2 \\
&\leqslant \text{const } \|v\|_{\mathcal{H}_h^1}^2 \|v\|_{\mathcal{H}_h^1}^2 \leqslant \text{const } \rho^2\|v\|_{\mathcal{H}_h^1}^2.
\end{aligned}
$$

These estimates show together with

$$
\begin{aligned}
\|\mu\hat{S}(v)-\lambda\hat{S}(u)\|_{\mathcal{L}_{2,h}} &\leqslant \text{const } (\|v\|_{\mathcal{H}_h^1}\|\mu-\lambda\| + \|v-u\|_{\mathcal{H}_h^1}\|\lambda\|) \\
&\leqslant \text{const } \rho(\|v-u\|_{\mathcal{H}_h^1} + \|\mu-\lambda\|)
\end{aligned}
$$

and $\varphi(0,0)=0$ that Hypothesis 3.1 holds. Finally, (v^0, μ^0) satisfies (17b), (17c) and (19) if and only if (u^0, λ^0) satisfies (9b),(9c) and (11). □

3.2. Reduction to an ODE

In the following we will use equations (18b), (19) to reduce system (18) to an ODE in the subspace

$$\ell^J_{\text{ess}} = \{u \in \ell^J_\infty(\mathbb{R}^m) :\ \mathcal{B}^D u = 0,\ \ \langle \Psi, u\rangle_{J_h} = 0\}.$$

where the essential algebraic conditions (18c),(18d) are satisfied.

We will show in Lemma 3.4, that there exists $\delta > 0$ such that for each $u^0 \in \ell^J_{\text{ess}}$ with $\|u^0\| \leqslant \delta$, there exists a unique extension $(v^0, \mu^0) \in \ell^J_{\text{co}}$ which satisfies $\pi v^0 = u^0$.

The following lemma states conditions under which a consistent $(v, \mu) \in \ell^{\hat{J}}_\infty \times \mathbb{R}^p$ can be uniquely determined from a given $u \in \ell^J_{\text{ess}}$ with $\pi v = u$. Here only the limiting case $|z| \to \infty$ of Hypothesis 2.6 is needed.

The proofs of the following two lemmas and the corollary are given in the appendix.

Lemma 3.3. *For each $u \in \ell^J_{\text{ess}}$ and each $r \in \ell^J_\infty$ there exists a unique extension $(v, \mu) \in \ell^{\hat{J}}_\infty \times \mathbb{R}^p$ such that $\pi v = u$,* (18b) *and*

$$\begin{aligned} 0 &= \mathcal{B}^D(\tilde{\Lambda} v + \Phi\mu + r), \\ 0 &= \langle \Psi, \tilde{\Lambda} v + \Phi\mu + r\rangle_{J_h} \end{aligned} \tag{24}$$

hold. The map $(u, r) \mapsto (v, \mu)$ is linear in u and r. Moreover with the notation

$$v = M_v u + R_v r, \qquad \mu = M_\mu u + R_\mu r,$$

where $M_v, R_v : \ell^J_\infty \to \ell^{\hat{J}}_\infty$, $M_\mu, R_\mu : \ell^J_\infty \to \mathbb{R}^p$, we obtain the estimate

$$\|R_v r\|_{\mathcal{H}^2_h} + \|R_\mu r\| \leqslant \text{const}\ \|r\|_{\mathcal{L}_{2,h}}. \tag{25}$$

The following Lemma guarantees the solvability of the equations (18b), (19) which define a transformation $\ell^J_{\text{ess}} \ni u \to (v, \mu) \in \ell^{\hat{J}}_\infty \times \mathbb{R}^p$.

Lemma 3.4. *Let the assumptions of Theorem* (3.2) *hold.*

Then there exist $c, h_0, T > 0$ such that for all $h < h_0, \pm h n_\pm > T$ with $\mathrm{e}^{-\alpha T} > c\sqrt{h}$ the following statements hold.

For each $u \in \ell^J_{\text{ess}}$ there exists a unique extension $\ell^{\hat{J}}_\infty \times \mathbb{R}^p \ni (v, \mu) = (T_v(u), T_\mu(u))$ such that $\pi v = u$, $T_v(0) = 0, T_\mu(0) = 0$ and (18b), (19) *hold.*

Moreover, we have the following estimates.

$$\|T_v(u_1) - T_v(u_2)\|_{\mathcal{L}_{2,h}} + \|T_\mu(u_1) - T_\mu(u_2)\| \leqslant \text{const}\ \|u_1 - u_2\|_{\mathcal{H}^1_h} \tag{26a}$$

$$\|T_v(u)\|_{\mathcal{L}_{2,h}} + \|T_\mu(u)\| \leqslant \text{const}\ \|u\|_{\mathcal{H}^1_h}. \tag{26b}$$

We will use the above transformations T_v, T_μ to reduce the DAE (18) to an equivalent ODE in ℓ^J_{ess}

$$u' = \tilde{\Lambda}_p u + \tilde{\varphi}(u), \quad u(0) = u^0 \tag{27}$$

where

$$\tilde{\Lambda}_p : \ell^J_{\mathrm{ess}} \to \ell^J_{\mathrm{ess}}, \;\; u \mapsto (\tilde{\Lambda} M_v + \Phi M_\mu) u$$

and

$$\tilde{\varphi}(u) = \tilde{\Lambda}(T_v(u) - M_v u) + \Phi(T_\mu(u) - M_\mu u) + \varphi(T_v(u), T_\mu(u)). \tag{28}$$

The properties of $\tilde{\varphi}$ are an immediate consequence of Lemma 3.5:

Corollary 3.5. *The nonlinearity $\tilde{\varphi}$ satisfies*

$$\|\tilde{\varphi}(u) - \tilde{\varphi}(v)\|_{\mathcal{L}_{2,h}} \leqslant \text{const } \|u - v\|_{\mathcal{H}^1_h},$$

and for each $\sigma > 0$ there exists $\rho > 0$ such that

$$\|\tilde{\varphi}(u)\|_{\mathcal{L}_{2,h}} \leqslant \sigma \|u\|_{\mathcal{H}^1_h}, \quad \textit{if } \|u\|_{\mathcal{H}^1_h} \leqslant \rho.$$

Remark 3.6. Note that if $\varphi : \ell^J_\infty \times \mathbb{R}^p \to \ell^J_\infty$ does not depend on (v, μ), i.e. $\varphi(v, \mu) = r \in \ell^J_\infty$ then the transformation $\varphi \to \tilde{\varphi}$ is just a projection $\tilde{\varphi} = \Pi r \in \ell^J_{\mathrm{ess}}$, where

$$\Pi r = (\tilde{\Lambda} R_v + \Phi R_\mu + I) r. \tag{29}$$

The following Lemma shows the equivalence of (27) and (18).

Lemma 3.7. *Assume the same as in Theorem 3.2. Then there exist $h_0, T > 0$ such that for $h < h_0$, $\pm n_\pm h > T$ we have the following equivalence.*

For each $\rho > 0$ there exists a $\delta > 0$ such that if $u \in \mathcal{C}([0, \tau), \ell^J_{\mathrm{ess}} \cap B^{\mathcal{H}^1_h}_\delta(0))$ is a solution of (27) on $(0, \tau)$ with $u(0) = u^0$ then $(v(t), \mu(t)) = (T_v(u(t)), T_\mu(u(t))) \in \mathcal{C}([0, t), \ell^{\hat{J}}_\infty \times \mathbb{R}^p)$ is a solution of (18) on $(0, \tau)$ with $v(0) = T_v(u^0)$, $\mu(0) = T_\mu(u^0)$ and $\|v(t)\|_{\mathcal{H}^1_h} + \|\mu(t)\| \leqslant \rho$.

Conversely, there exists $\rho > 0$ such that if $(v(t), \mu(t)) \in \mathcal{C}([0, t), \ell^{\hat{J}}_\infty \times \mathbb{R}^p)$ is a solution of (18) on $(0, \tau)$ with $(v(0), \mu(0)) = (v^0, \mu^0) \in \ell^J_{\mathrm{co}}$ and $\|v(t)\|_{\mathcal{H}^1_h} + \|\mu(t)\| \leqslant \rho$, then $u = \pi v$ is a solution of (27) with $\|u(t)\|_{\mathcal{H}^1_h} < \rho$.

Proof. Let $(v(t), \mu(t))$ a solution of (18) for consistent initial values $(v^0, \mu^0) \in \ell^J_{\mathrm{co}}$ on $(0, \tau)$. Then differentiating (18c), (18d) w.r.t. time we obtain by (18a) that $(v(t), \mu(t))$ solves (19) for $t \in (0, \tau)$. For $u = \pi v$ we can insert $v = T_v(u),\ \mu = T_\mu(u)$ into (18a) to obtain

$$\begin{aligned} u' = \pi v' &= \tilde{\Lambda} v + \Phi\mu + \varphi(v, \mu) \\ &= \tilde{\Lambda} T_v(u) + \Phi T_\mu(u) + \varphi(T_v(u), T_\mu(u)) = \tilde{\Lambda}_p u + \tilde{\varphi}(u). \end{aligned}$$

Conversely, if u solves the reduced ODE (27) then Lemma 3.4 implies that $v(t) = T_v(u(t)), \mu(t) = T_\mu(u(t))$ is a solution of (18) in $B^{\mathcal{H}^1_h}_\rho(0) \subset \ell^{\hat{J}}_\infty \times \mathbb{R}^p$ for some $\rho > 0$ in the sense of in the sense of Definition 2.1. □

Note that it is sufficient to consider (27) in ℓ^J_{ess}. Thus we have reduced the bordered system (18) to an ODE (27) in ℓ^J_{ess} which is then solved as usual via the "variation of constants" formula

$$u(t) = \Sigma_p(t) u^0 + \int_0^t \Sigma_p(t-s)\, \tilde{\varphi}(u(s))\, ds. \tag{30}$$

Here the operator $\Sigma_p(t)$ is defined via the Dunford integral

$$\Sigma_p(t) = \frac{1}{2\pi i} \oint_\Gamma \mathrm{e}^{st} (sI - \tilde{\Lambda}_p)^{-1}\, ds$$

and Γ is a closed curve which encloses the spectrum of $\tilde{\Lambda}_p$.

3.3. Estimates of the Solution Operator

In order to obtain stability estimates for (27) estimates on $\Sigma_p(t)$ are required which are proven using resolvent estimates in different regions of $\mathbb{C}$. These are given in the following lemma which will be proved in Section 4.

Lemma 3.8. *There exist $\alpha > 0$, $\phi \in (\frac{\pi}{2}, \pi)$, $C > 0$ such that $s \in \rho(\Lambda)$ if $|s| > Ch^{-2}$ or $|\arg(s+\alpha)| \leqslant \phi$, $s \neq -\alpha$. Furthermore, for all $r \in \ell^J_\infty$ the resolvent $u = (sI - \tilde{\Lambda}_p)^{-1} \Pi r \in \ell^J_{\mathrm{ess}}$ with Π the projection defined in* (29)*, can be estimated by*

$$\|u\|_{\mathcal{L}_{2,h}} \leqslant \frac{\mathrm{const}}{|s+\alpha|} \|r\|_{\mathcal{L}_{2,h}}, \qquad \|u\|_{\mathcal{H}^1_h} \leqslant \frac{\mathrm{const}}{\sqrt{|s+\alpha|}} \|r\|_{\mathcal{L}_{2,h}}. \tag{31}$$

Lemma 3.9. *Let Λ satisfy Hypotheses 1.7, 1.8 and assume that Hypothesis 2.3 holds.*

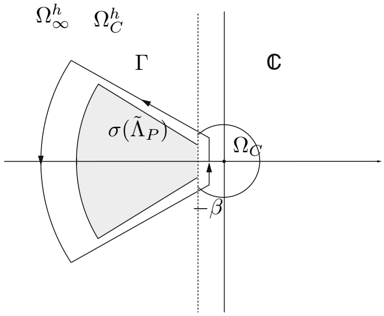

Figure 1. Path of integration.

Then there exist $h_0, T, K > 0$ *such that for all* $h < h_0$ *and* $\pm n_\pm h > T$ *the solution operator* $\Sigma(t)$ *can be estimated by*

$$\|\Sigma_p(t)r\|_{\mathcal{L}_{2,h}} \leqslant K\mathrm{e}^{-\alpha t}\|r\|_{\mathcal{L}_{2,h}}, \qquad \|\Sigma_p(t)r\|_{\mathcal{H}_h^1} \leqslant K\mathrm{e}^{-\alpha t}\frac{1}{\sqrt{t}}\|r\|_{\mathcal{L}_{2,h}}.$$

Proof. We introduce the following notation for a function $g : \Gamma \to [0, \infty)$, where $\Gamma = \{\gamma(\xi) : \xi \in [0, l]\}$ is a closed curve

$$\oint_\Gamma g(z)|dz| := \int_0^l g(\gamma(\xi))|\gamma'(\xi)|\,d\xi.$$

Note that we can take a path Γ around the eigenvalues of $\tilde{\Lambda}_p$ where $\mathrm{Re}\, s < 0\ \forall s \in \Gamma$ (see Fig. 1). We denote the resolvent by $G(s) = (sI - \tilde{\Lambda}_p)^{-1}$ and obtain for $r \in \ell_{\mathrm{ess}}^J$ with (31) for $t > 0$ the following:

$$\begin{aligned}
\|\Sigma_p(t)\,r\|_{\mathcal{L}_{2,h}} &= \left\|\frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{st}G(s)r\,ds\right\|_{\mathcal{L}_{2,h}} = \left\|\frac{1}{2\pi i}\oint_{\Gamma-\alpha} \mathrm{e}^{st}G(s)r\,ds\right\|_{\mathcal{L}_{2,h}} \\
&= \left\|\frac{1}{2\pi i}\oint_\Gamma \mathrm{e}^{(s-\alpha)t}G(s-\alpha)r\,ds\right\|_{\mathcal{L}_{2,h}} \\
&\leqslant \frac{1}{2\pi}\mathrm{e}^{-\alpha t}\oint_\Gamma |\mathrm{e}^{st}|\|G(s-\alpha)r\|_{\mathcal{L}_{2,h}}|ds| \\
&\leqslant \frac{1}{2\pi}\mathrm{e}^{-\alpha t}\oint_\Gamma \left|\frac{\mathrm{e}^\lambda}{t}\right|\|G(\frac{\lambda}{t}-\alpha)r\|_{\mathcal{L}_{2,h}}|d\lambda| \\
&\leqslant \text{const } \mathrm{e}^{-\alpha t}\|r\|_{\mathcal{L}_{2,h}}\oint_\Gamma \frac{|\mathrm{e}^\lambda|}{|\lambda|}|d\lambda| \\
&\leqslant K\mathrm{e}^{-\alpha t}\|r\|_{\mathcal{L}_{2,h}}.
\end{aligned}$$

Here we have used the fact that we can move the curve Γ to the left up to $\Gamma-\alpha$ for $\alpha<\beta$ small enough without changing the integral. Along the rays this is the standard estimate for sectorial operators (see [9, 7]). Along the arc $\gamma(\xi)=Re^{i\xi}$, $\xi\in[\frac{\pi}{2}+\delta, \frac{3\pi}{2}-\delta]$ we obtain

$$\int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} R|\mathrm{e}^{tRe^{i\xi}}|\,\|G(Re^{i\xi})r\|_{\mathcal{L}_{2,h}}\,d\xi \leqslant \|r\|_{\mathcal{L}_{2,h}}\int_{\frac{\pi}{2}+\delta}^{\frac{3\pi}{2}-\delta} R\mathrm{e}^{tR\cos(\xi)}\frac{1}{R}\,d\xi < \frac{\pi}{2}\|r\|_{\mathcal{L}_{2,h}}.$$

In a similar way we obtain

$$\|\Sigma_p(t)\,r\|_{\mathcal{H}_h^1} \leqslant K\mathrm{e}^{-\alpha t}\frac{1}{\sqrt{t}}\|r\|_{\mathcal{L}_{2,h}}. \qquad \square$$

3.4. Local Existence, Uniqueness and Stability

In this section we prove the solvability of the integral equation (30) together with some estimates. Note that the existence of a solution of (27) follows from standard ODE theory.

Lemma 3.10. *Assume the same as in Lemma 3.7. There exists $h_0, T>0$ such that for $h<h_0$, $\pm hn_\pm>T$ the following statements hold:*

For each $\rho>0$ there exist $\delta>0$ such that for each $u^0\in\ell^J_{\mathrm{ess}}$ with $\|u^0\|_{\mathcal{L}_{2,h}}<\delta$ there exists $\tau(h,T)>0$ such that a unique solution of (27) *exists on* $(0,\tau(h,T))$ *and $\|u(t)\|_{\mathcal{H}_h^1}\leqslant\rho$ for $t\in[0,\tau(h,T))$.*

Proof. For each fixed h, $J=[n_-,n_+]$ we use the fact that there exist $C_1(h,J), C_2(h,J)$ with

$$C_1(h,J)\|u\| \leqslant \|u\|_{\mathcal{L}_{2,h}} \leqslant C_2(h,J)\|u\|.$$

By Lemma 3.4 there exists $\rho>0$ such that for $\|u\|_{\mathcal{H}_h^1}<\rho$ the map $\tilde{\varphi}$ is Lipschitz. Thus we can apply the standard Picard-Lindelöf theorem in $\mathbb{R}^{n_+-n_-+1}$ to obtain the existence of a solution of (27) for $[0,\tau(h,J))$. We can further achieve that $\|u\|\leqslant C_2(h,J)^{-1}\rho$ in $[0,\tau(h,T))$ such that $\|u\|_{\mathcal{L}_{2,h}}\leqslant\rho$ for all $t\in[0,\tau(h,T))$. $\square$

The stability of zero as a solution of the reduced system (27) is the usual Lyapunov type estimate. We repeat it here, since we are interested not only in the stability of the solution of a single DAE but we aim at a uniform stability estimate for a whole family of solutions of DAEs corresponding to discretizations with different h and T.

Lemma 3.11. *Assume the same as in Theorem 3.2.*

Then there exist $\rho, h_0, T > 0$ *such that for any* $h < h_0$, $\pm n_\pm h > T$ *and any consistent initial condition* $u^0 \in \ell^J_{\text{ess}}$ *with* $\|u^0\|_{\mathcal{H}^1_h} \leqslant \rho$ *the following holds: There exists a unique solution* u *of* (27) *which can be estimated by*

$$\|u(t)\|_{\mathcal{H}^1_h} \leqslant \text{const } \mathrm{e}^{-\nu t}, \quad \nu > 0, \qquad \forall t \geqslant 0. \tag{32}$$

where all constants are independent of h, T.

Proof. We choose $\nu \in (0, \alpha)$ and $\sigma > 0$ so small that

$$K\sigma \int_0^\infty \frac{\mathrm{e}^{-(\alpha-\nu)s}}{\sqrt{s}}\, ds \leqslant \frac{1}{2}.$$

Using Corollary 3.5 we choose $\delta > 0$ such that $\tilde{\varphi} : \ell^J_{\text{ess}}(\mathbb{R}^m) \to \ell^J_{\text{ess}}(\mathbb{R}^m)$ satisfies

$$\|\tilde{\varphi}(u)\|_{\mathcal{L}_{2,h}} \leqslant \sigma \|u\|_{\mathcal{H}^1_h} \quad \text{for } \|u\|_{\mathcal{H}^1_h} \leqslant \delta.$$

Then for each h, J we find by Lemma 3.10 some $\rho > 0$ such that for $u^0 \in \ell^J_{\text{ess}}$ with $\|u^0\|_{\mathcal{H}^1_h} \leqslant \rho$ a solution u of (27) exists on $(0, \tau(h, J))$ with $\|u(t)\|_{\mathcal{H}^1_h} \leqslant \delta$ for $t \in [0, \tau(h, J)$. With (30) and the estimates in Lemma 3.9 we obtain

$$\begin{aligned}
\|u(t)\|_{\mathcal{H}^1_h} &\leqslant \|\Sigma_p(t)u^0\|_{\mathcal{H}^1_h} + \int_0^t \|\Sigma_p(t-s)\tilde{\varphi}(u(s))\|_{\mathcal{H}^1_h}\, ds \\
&\leqslant K\mathrm{e}^{-\alpha t}\|u^0\|_{\mathcal{H}^1_h} + K\int_0^t \frac{1}{\sqrt{t-s}}\mathrm{e}^{-\alpha(t-s)}\|\tilde{\varphi}(u(s))\|_{\mathcal{L}_{2,h}}\, ds \\
&\leqslant \frac{\delta}{4} + K\sigma \int_0^\infty \frac{1}{\sqrt{s}}\mathrm{e}^{-\alpha s}\, ds\ \ \|u\|^\tau_{\mathcal{H}^1_h} \leqslant \frac{3}{4}\delta.
\end{aligned}$$

Since the ODE (27) is autonomous, this leads to $\tau(h, J) = \infty$ using the usual arguments. From this the existence of u in $(0, \infty)$ follows with $\|u(t)\|_{\mathcal{H}^1_h} < \delta$ for all $t \in [0, \infty)$ and small enough h and large enough T. It remains to prove the exponential estimate. Define $n(t) = \sup_{s\in[0,t]}\{\mathrm{e}^{\nu s}\|u(s)\|_{\mathcal{H}^1_h}\}$ then

$$\begin{aligned}
\|u(t)\|_{\mathcal{H}^1_h}\mathrm{e}^{\nu t} &\leqslant K\mathrm{e}^{(\nu-\alpha)t}\|u^0\|_{\mathcal{H}^1_h} + K\sigma\int_0^t \frac{1}{\sqrt{t-s}}\mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{\nu t}\|u(s)\|_{\mathcal{H}^1_h}\ ds \\
&\leqslant K\|u^0\|_{\mathcal{H}^1_h} + K\sigma\int_0^t \frac{1}{\sqrt{t-s}}\mathrm{e}^{(\nu-\alpha)(t-s)}\mathrm{e}^{\nu s}\|u(s)\|_{\mathcal{H}^1_h}\ ds \\
&< K\|u^0\|_{\mathcal{H}^1_h} + \frac{1}{4}n(t).
\end{aligned}$$

Taking the supremum on both sides gives $n(t) < 4K\|u^0\|_{\mathcal{H}_h^1} < \delta$ for $t \geqslant 0$ and we obtain (32). $\square$

Now the stability Theorem 3.2 follows easily.

Proof of Theorem 3.2. For each $\delta > 0$ there exists $\rho > 0$ such that for any $(v^0, \mu^0) \in \ell_{\text{co}}^J$ with $\|v^0\|_{\mathcal{H}_h^1} + \|\mu^0\| < \rho$ we have $u^0 = \pi v^0 \in \ell_{\text{ess}}^J$ and $\|u^0\|_{\mathcal{H}_h^1} \leqslant \delta$. By Lemma 3.11 we obtain a solution u of (27) on $(0, \infty)$ which satisfies (32). Then Lemma 3.7 implies that

$$v(t) = T_v(u(t)), \qquad \mu(t) = T_\mu(u(t))$$

solves (18) with $v(0) = T_v(u^0) = v^0$, $\mu(0) = T_\mu(u^0) = \mu^0$. Moreover, it follows from (26b),(32) that (v, μ) can be estimated by (22). $\square$

4. RESOLVENT ESTIMATES

We prove resolvent estimates in the regions $\Omega_C, \Omega_C^h, \Omega_\infty^h$ (cf. Fig. 2) for the discretized system. To this end we transform the resolvent equation for the projected operator $\tilde{\Lambda}_p$ back into a bordered equation. This is accomplished by reintroducing the algebraic variables. A direct application of Lemma 3.3 leads to the following equivalence.

$$\Omega_C : \ |s| < C, \ \text{Re}\, s \geq -\beta$$

$$\Omega_C^h : \ |s| \in (C, \frac{C}{h^2}], \quad |\arg(s)| \leq \frac{\pi}{2} + \frac{2}{3}\delta$$

$$\Omega_\infty^h : \ |s| > \frac{C}{h^2}$$

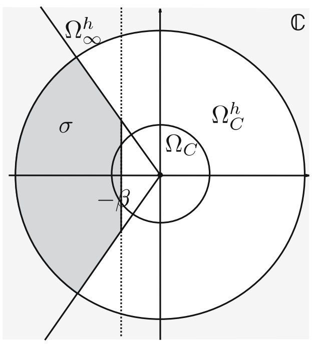

Figure 2. Regions for resolvent estimates

Lemma 4.1. *Let $r \in \ell_\infty^J$, then $u \in \ell_{\text{ess}}^J$ solves*

$$(sI - \tilde{\Lambda}_P)u = \Pi r \tag{33}$$

and

$$v = M_v u + R_v r, \quad \mu = M_\mu u + R_\mu r$$

if and only if the pair $(v, \mu) \in \ell_{\text{co}}^J$ is a solution of the bordered system

$$(sI - \tilde{\Lambda})v - \Phi\mu = r \tag{34a}$$

$$\mathcal{B}v = 0 \tag{34b}$$

$$\langle \Psi, \pi v \rangle_h = 0. \tag{34c}$$

The main result of this section are the following estimates

Theorem 4.2. *There exist $h_0, T > 0$ such that for each $h < h_0$, $\pm n_\pm > T$ there exists for each $s \in \Omega_C \cup \Omega_C^h \cup \Omega_\infty^h$ and each $r \in \ell_\infty^J$ a solution u of* (33) *which can be estimated by*

$$\|u\|_{\mathcal{L}_{2,h}} \leqslant \text{const } \|r\|_{\mathcal{L}_{2,h}}, \quad s \in \Omega_C$$

$$|s|^2 \|u\|_{\mathcal{L}_{2,h}}^2 + |s| \|u\|_{\mathcal{H}_h^1}^2 \leqslant \text{const } \|r\|_{\mathcal{L}_{2,h}}^2, \quad s \in \Omega_C^h \cup \Omega_\infty^h$$

with a constant which does not depend on h and T.

This implies immediately Lemma 3.8 which has been used in the previous section.

For s in a compact set, a similar method as in the proof of the approximation Theorem 2.6 in [15] can be used. For $s \in \Omega_C$ a solution of (34) can be constructed directly by using the continuous system. For large $|s|$ a different approach is necessary, since the discrete resolvent equation (35) cannot be related to corresponding continuous systems uniformly in s. In that case the solutions for the resolvent equation are constructed directly by a similar method as in [1].

4.1. Compact Subsets

Lemma 4.3. *Let the same assumptions as in the previous lemma hold. Then for each $C > 0$ there exist $h_0, T > 0$ such that for each $h < h_0$, $\pm n_\pm > T$ the following holds. For $s \in \Omega_C$ and $r \in \ell_\infty^J$ the resolvent equation* (34) *has a unique solution $(v, \mu) \in \ell_\infty^{\hat{J}} \times \mathbb{R}^p$ which satisfies the following uniform estimate in s*

$$\|v\|_{\mathcal{H}_h^2} + \|\mu\| \leqslant \text{const } \|r\|_{\mathcal{L}_{2,h}}.$$

The proof is along the same lines as the proof for the existence of the eigenvalue zero for the discretized equations in [15] and can be found in [14], so we omit it here.

4.2. $|s|$ Large

The main result of this subsection is a resolvent estimate for the solution w of

$$(\tilde{\Lambda} - sI)w = r, \tag{35a}$$

$$\mathcal{B}w = \eta. \tag{35b}$$

Using a solution of (35) the existence of which will be proven in Lemma 4.5 we can construct a solution of (34).

Lemma 4.4. *For $s \in \Omega_C^h \cup \Omega_\infty^h$ there exists a solution $(v, \mu) \in \ell_\infty^{\hat{J}} \times \mathbb{R}^p$ of* (34) *which satisfies*

$$\|v\|_{\mathcal{H}_h^1} + \|\mu\| \leqslant \text{const } \|r\|_{\mathcal{L}_{2,h}}.$$

The main work of this section is the proof of the following lemma:

Lemma 4.5. *Consider the resolvent equation* (35) *with diagonalizable $A > 0$ and assume that Hypothesis 2.6 holds.*

Then C can be chosen such that there exist $T > 0$, $h_0 > 0$ such that for $h < h_0$ and $\pm hn_\pm > T$ and $s \in \Omega_C^h \cup \Omega_\infty^h$ the following holds. The resolvent equation (35a) *with boundary conditions* (35b) *possesses for each $r \in \ell_\infty^J(\mathbb{C}^m)$ and each $\eta = (\eta^N, \eta^D)^T \in \mathbb{C}^k \times \mathbb{C}^{2m-k}$ a unique solution $w \in \ell_\infty^{\hat{J}}(\mathbb{C}^m)$. Furthermore, w can be estimated by*

$$|s|^2\|w\|_{\mathcal{L}_{2,h}}^2 + |s|\|w\|_{\mathcal{H}_h^1}^2 \leqslant \text{const } (\|r\|_{\mathcal{L}_{2,h}}^2 + |s|\|\eta^N\|^2 + |s|^2\|\eta^D\|^2), \quad s \in \Omega_C^h$$

$$|s|^2\|\pi w\|_{\mathcal{L}_{2,h}}^2 + |s|\|\pi w\|_{\mathcal{H}_h^1}^2 \leqslant \text{const } (\|r\|_{\mathcal{L}_{2,h}}^2 + |s|\|\eta^N\|^2 + |s|^2\|\eta^D\|^2), \quad s \in \Omega_\infty^h$$

Before we continue with the proofs of Lemmas 4.4 and 4.5 we show that Theorem 4.2 follows directly from the preceding estimates.

Proof of Theorem 4.2. Using $\pi v = u$ we obtain from Lemma 4.3 and Lemma 4.4 with Lemma 4.1 the asserted estimates. □

Proof of Lemma 4.4. For $s \in \Omega_C^h \cup \Omega_\infty^h$ we can solve equation (34a),(34b) using Lemma 4.5 by taking $\Phi\mu$ to the right hand side. We denote its solution operator with $\mathcal{G}$ and obtain by inserting $v = \mathcal{G}(r + \Phi\mu)$ into (34c)

$$\mu = -\langle \Psi, \mathcal{G}\Phi\rangle^{-1}\langle \Psi, \mathcal{G}r\rangle$$

which leads to $v = \mathcal{Q}\mathcal{G}r$ where the projector $\mathcal{Q}$ is defined by

$$\mathcal{Q}w = w - \mathcal{G}\Phi\ \langle\Psi, \mathcal{G}\Phi\rangle^{-1}\langle\Psi, w\rangle.$$

In order to estimate μ and $\mathcal{Q}$ we need a bound of $\|\langle\Psi, \mathcal{G}\Phi\rangle^{-1}\|$. Use $\Phi = \mathcal{G}\Lambda\Phi - s\mathcal{G}\Phi = \mathcal{G}\epsilon - s\mathcal{G}\Phi$ and multiply with Ψ from the left. Then $\langle\Psi, \mathcal{G}\epsilon\rangle - \langle\Psi, \Phi\rangle = s\langle\Psi, \mathcal{G}\Phi\rangle$ and $\|\epsilon\| \to 0$ as $J_h \to \mathbb{R}$ imply the invertibility of $\langle\Psi, \mathcal{G}\Phi\rangle$ for $\pm n > T, h < h_0$ as well as

$$\|\langle\Psi, \mathcal{G}\Phi\rangle^{-1}\| \leqslant \text{const}\ |s|\|\langle\Psi, \Phi\rangle\|^{-1} \leqslant \text{const}\ |s|.$$

This implies with the estimates in Lemma 4.5 for $\mathcal{G}$

$$\|\mathcal{Q}w\|_{\mathcal{L}_{2,h}} \leqslant \text{const}\ \|w\|_{\mathcal{L}_{2,h}} \qquad \text{and} \qquad \|\mathcal{Q}w\|_{\mathcal{H}^1_h} \leqslant \text{const}\ \|w\|_{\mathcal{H}^1_h}.$$

Thus we obtain again with Lemma 4.5

$$\|v\|_{\mathcal{L}_{2,h}} \leqslant \text{const}\ \frac{1}{|s|}\|r\|_{\mathcal{L}_{2,h}} \qquad \text{and} \qquad \|v\|_{\mathcal{H}^1_h} \leqslant \text{const}\ \frac{1}{\sqrt{|s|}}\|r\|_{\mathcal{L}_{2,h}}. \qquad \square$$

Before we start with a series of Lemmas which are needed for the proof of Lemma 4.5, we give a short outline: We use exponential dichotomies for the discrete and the continuous system, for references see [10, 4] in a similar way as in [18, 1]. Equation (35) is transformed to first order via the scaled transformation $z_n = (w_n, \frac{1}{\rho}\delta_- w_n)$. The transformed system is approximated by constant coefficient operators $\hat{L}(s, \rho)z_n = z_{n+1} - \hat{M}(s, \rho)z_n$, for small h and large ρ. The matrices $\hat{M}(s, \rho)$ are hyperbolic for $s \in \Omega^h_C \cup \Omega^h_\infty$ which implies that $\hat{L}(s, \rho)$ has exponential dichotomies on $\mathbb{Z}$. In order to obtain estimates for the solution of the corresponding boundary value problem for large ρh we need to take into account the structure of the right hand side of the transformed system.

In order to simplify the presentation we restrict ourselves to diagonalizable A. Using a pretransform with a matrix U that diagonalizes A and using the fact that Hypothesis 2.6 is invariant under this transformation we assume w.l.o.g. that $A \in \mathbb{C}^{m,m}$ is diagonal. Transformation to first order via $z_n = (w_n, \frac{1}{\rho}\delta_- w_n),\ n = n_-, \ldots, n_+ + 1$, for some $\rho > 0$ leads to the equation

$$N_n(\rho)z_{n+1} - K_n(s, \rho)z_n = \hat{r}_n, \quad n \in J = [n_-, n_+] \tag{36a}$$
$$R(\rho)z = \hat{\eta} \tag{36b}$$

where

$$N_n(\rho)=\begin{pmatrix} I & -h\rho I \\ 0 & E_n^+ \end{pmatrix}, \quad K_n(s,\rho)=\begin{pmatrix} I & 0 \\ \frac{h}{\rho}(sI-C_n) & E_n^- \end{pmatrix}, \quad E_n^{\pm}=A\pm\frac{h}{2}B_n,$$

$$R(\rho)z=B_-(\rho)z_{n_-}+\hat{B}_-z_{n_-+1}+B_+(\rho)z_{n_+}+\hat{B}_+z_{n_++1}$$

and

$$\hat{r}_n=\begin{pmatrix} 0 \\ \frac{h}{\rho}r_n \end{pmatrix}, \quad B_{\pm}(\rho)=\begin{pmatrix} \frac{1}{\rho}P_{\pm}^N & \frac{1}{2}Q_{\pm}^N \\ P_{\pm}^D & 0 \end{pmatrix}, \quad \hat{B}_{\pm}=\begin{pmatrix} 0 & \frac{1}{2}Q_{\pm}^N \\ 0 & 0 \end{pmatrix}, \quad \hat{\eta}=\begin{pmatrix} \frac{1}{\rho}\eta^N \\ \eta^D \end{pmatrix}.$$

For h small enough we can invert $N_n(\rho)$ to obtain the explicit formulation of (36a)

$$(\tilde{L}(s,\rho)z)_n=\frac{h}{\rho}\begin{pmatrix} h\rho I \\ I \end{pmatrix}E_n^{+-1}r_n, \quad n\in J \tag{37}$$

where

$$(\tilde{L}(s,\rho)z)_n=z_{n+1}-M_n(s,\rho)z_n,$$

$$M_n(s,\rho)=N_n(\rho)^{-1}K_n(s,\rho)=\begin{pmatrix} I+h^2E_n^{+-1}(sI-C_n) & h\rho E_n^{+-1}E_n^- \\ \frac{h}{\rho}E_n^{+-1}(sI-C_n) & E_n^{+-1}E_n^- \end{pmatrix}. \tag{38}$$

In order to obtain solutions of (37), (36b) we will use the following constant coefficient difference equation, given by

$$(\hat{L}(s,\rho)z)_n=\frac{h}{\rho}\begin{pmatrix} h\rho I \\ I \end{pmatrix}r_n, \quad n\in J \tag{39}$$

where

$$(\hat{L}(s,\rho)z)_n=z_{n+1}-\hat{M}(s,\rho)z_n, \tag{40}$$

$$\hat{M}(s,\rho)=\hat{N}(\rho)^{-1}\hat{K}(s,\rho)=I+h\rho\begin{pmatrix} \frac{h}{s}\rho A^{-1} & I \\ \frac{s}{\rho^2}A^{-1} & 0 \end{pmatrix} \tag{41}$$

and

$$\hat{N}(\rho)=\begin{pmatrix} I & -h\rho I \\ 0 & A \end{pmatrix}, \quad \hat{K}(s,\rho)=\begin{pmatrix} I & 0 \\ \frac{h}{\rho}sI & A \end{pmatrix}.$$

As we will show later, $\hat{L}(s, \sqrt{|s|})$ is a small perturbation of $\tilde{L}(s, \sqrt{|s|})$ for $|s|$ large. In the following we define $\rho = \sqrt{|s|}$ and set $s = \rho^2 e^{2i\theta}$. Then we obtain

$$\hat{M}(s, \rho) = I + h\rho \begin{pmatrix} h\rho e^{2i\theta} A^{-1} & I \\ e^{2i\theta} A^{-1} & 0 \end{pmatrix}. \tag{42}$$

We will prove that the matrices $\hat{M}(s, \rho)$ are hyperbolic for $s \in \Omega_C^h$ and $s \in \Omega_\infty^h$. Then $\hat{L}(s, \rho)$ possesses an exponential dichotomy on $\mathbb{Z}$, which will be used to construct a solution of (39), (36b).

Lemma 4.6. *Consider*

$$M = I + \kappa N(\kappa), \quad \text{where } N(\kappa) = \begin{pmatrix} \kappa S & I \\ S & 0 \end{pmatrix}$$

with $\kappa > 0$, and $S \in \mathbb{C}^{m,m}$ a nonsingular diagonal matrix. Then there exist $\delta, C > 0$ such that the following holds: If either ($\kappa \leqslant C$ and $\arg(\sigma(S)) \leqslant \pi - \delta$) or $\kappa > C$ then M is a hyperbolic matrix with m stable eigenvalues $\nu_{s,i}$ and m unstable eigenvalues $\nu_{u,i}$, $i = 1, \dots, m$. Moreover, there exist $\alpha, a > 0$, $\epsilon \in (0, C]$ such that for $i = 1, \dots, m$, the following estimates hold:

$$\begin{aligned} &a\kappa^2 \geqslant |\nu_{u,i}| \geqslant \alpha\kappa^2, && \frac{a}{\kappa^2} \leqslant |\nu_{s,i}| \leqslant \frac{\alpha}{\kappa^2} && \textit{for } \kappa > C \\ &|\nu_{u,i}| \geqslant 1 + \alpha, && |\nu_{s,i}| \leqslant \frac{1}{1+\alpha} && \textit{for } \kappa \in [\epsilon, C],\ \arg(\sigma(S)) \leqslant \pi - \delta \\ &|\nu_{u,i}| \geqslant 1 + \alpha\kappa, && |\nu_{s,i}| \leqslant \frac{1}{1+\alpha\kappa} && \textit{for } \kappa \in (0, \epsilon),\ \arg(\sigma(S)) \leqslant \pi - \delta. \end{aligned}$$

Proof. Let $\mu \in \mathbb{C}$ be an eigenvalue of S with eigenvector u. Then λ is an eigenvalue of $N(\kappa)$ with eigenvector v if and only if λ is a solution of

$$\lambda^2 - \lambda\kappa\mu - \mu = 0 \tag{43}$$

and $v = \begin{pmatrix} \lambda S^{-1} v \\ u \end{pmatrix}$. The solutions of (43) are given by

$$\lambda_\pm = \begin{cases} \frac{1}{2}\left(\kappa\mu \pm \sqrt{\kappa^2\mu^2 + 4\mu}\right), & \text{if } \kappa > 0,\ |\arg\mu| \leqslant \pi - \delta, \\ \frac{\kappa\mu}{2}\left(1 \pm \sqrt{1 + \frac{4}{\mu\kappa^2}}\right), & \text{if } \kappa > C. \end{cases} \tag{44}$$

Note that both definitions coincide on the common domain of definition, and that

$$\lambda_+ - \lambda_- = \begin{cases} \sqrt{\kappa^2\mu^2 + 4\mu} & \text{if } \kappa > 0,\ |\arg\mu| \leqslant \pi - \delta, \\ \frac{\kappa\mu}{2}\sqrt{1 + \frac{4}{\mu\kappa^2}} & \text{if } \kappa > C \end{cases}$$

implies an lower estimate

$$|\lambda_+ - \lambda_-| \geqslant \text{const } \max(\kappa, 1). \tag{45}$$

The eigenvalues $\nu_\pm$ of M are given by $\nu_\pm = 1 + \kappa\lambda_\pm$. From $\lambda_-\lambda_+ = -\mu$, $\lambda_- + \lambda_+ = \kappa\mu$ and (43) we obtain $1 + \kappa\lambda_- = (1 + \kappa\lambda_+)^{-1}$. We consider $\nu_\pm$ for κ in three different regions:

1. Large κ:
Use the expansion $\sqrt{1+z} = 1 + \frac{z}{2} + \mathcal{O}(z^2)$ to obtain

$$|1 + \kappa\lambda_+| = |1 + \frac{\mu\kappa^2}{2}\left(1 + \sqrt{1 + \frac{4}{\mu\kappa^2}}\right)| \geqslant \alpha\kappa^2 \text{ if } \kappa > C.$$

This implies $|\nu_{u,i}| \geqslant \alpha\kappa^2$, as well as $|\nu_{s,i}| < \frac{1}{\alpha\kappa^2}$ for $\kappa > C$, $i = 1, \ldots, m$.

2. Small κ, $|\arg\mu| \leqslant \pi - \delta$
For small κ and $|\arg\mu| \leqslant \pi - \delta$ we have the expansion

$$1 + \kappa\lambda_+ = 1 + \frac{\kappa^2\mu}{2} + \kappa\sqrt{\mu}\sqrt{1 + \frac{\kappa^2\mu}{4}} = 1 + \kappa\sqrt{\mu} + \mathcal{O}(\kappa^2).$$

From $|\arg\mu| \leqslant \pi - \delta$ we obtain $\operatorname{Re}\sqrt{\mu} > 0$ and hence $|\nu_{u,i}| \geqslant 1 + \alpha\kappa$, $|\nu_{s,i}| \leqslant \frac{1}{1+\alpha\kappa}$ for some $\alpha > 0$ and $\kappa \in (0, \epsilon)$.

3. κ **in the compact set** $\kappa \in [\epsilon, C]$, $|\arg\mu| \leqslant \pi - \delta$
Let $\kappa > 0$, $|\arg\mu| \leqslant \pi - \delta$. In particular $\operatorname{Re}\mu > 0$. Then $\operatorname{Re}\sqrt{\kappa^2\mu^2 + 4\mu} \geqslant 0$ by definition. Hence $\operatorname{Re}\lambda_+ = \operatorname{Re}\frac{\kappa\mu}{2} + \operatorname{Re}\sqrt{\kappa^2\mu^2 + 4\mu} \geqslant \operatorname{Re}\frac{\kappa\mu}{2} \geqslant c\kappa$ for some $c > 0$. Therefore $\operatorname{Re}(1 + \kappa\lambda_+) \geqslant 1 + c\kappa^2$ and $|1 + \kappa\lambda_+| > 1$. Since κ varies in a compact interval the Lemma is proved.

□

By application of the previous Lemma with $S = e^{2i\theta}A^{-1}$ and $\kappa = \rho h$ we obtain that the constant coefficient operators $\hat{L}(s, \rho)$ possess an exponential dichotomy on $\mathbb{Z}$ if $s \in \Omega_C^h \cup \Omega_\infty^h$ as the following corollary shows.

Corollary 4.7. *Assume that* $A \in \mathbb{C}^{m,m}$ *is diagonal and positive definite. Then there exist* $C, \epsilon, \delta > 0$ *such that the operators* $\hat{L}(s, \rho)$ *possess exponential dichotomies on* $\mathbb{Z}$ *if* $s = \rho^2 e^{2i\theta} \in \Omega_C^h \cup \Omega_\infty^h$. *The dichotomy data are*

(K, β, P), where K is independent of ρ and h, and for some $\alpha > 0$

$$\begin{aligned}
&\beta = \ln(\alpha(\rho h)^2) && \textit{for } \rho > \frac{C}{h},\\
&\beta = \ln(1+\alpha) && \textit{for } \rho \in \left[\frac{\epsilon}{h}, \frac{C}{h}\right],\ |\theta| \leqslant \frac{\pi}{4} + \frac{\delta}{3},\\
&\beta = \ln(1+\alpha\rho h) && \textit{for } \rho \in [C, \frac{\epsilon}{h}],\ |\theta| \leqslant \frac{\pi}{4} + \frac{\delta}{3}
\end{aligned}$$

and the projector P is given by

$$P = \begin{pmatrix} (\Lambda_s - \Lambda_u)^{-1}\Lambda_s & -(\Lambda_s - \Lambda_u)^{-1} \\ -\Lambda_u(\Lambda_s - \Lambda_u)^{-1}\Lambda_s & \Lambda_s(\Lambda_s - \Lambda_u)^{-1} \end{pmatrix}. \tag{46}$$

Here Λ_s and Λ_u are defined by

$$\Lambda_s = \mathrm{diag}(\lambda_{-,i})_{i=1,\dots,m}, \quad \Lambda_u = \mathrm{diag}(\lambda_{+,i})_{i=1,\dots,m}$$

where $\lambda_{\pm,i}$ are defined for each $i = 1, \dots, m$ by (44) *with $\mu = \mu_i \in \sigma(A^{-1})$.*

Proof. Denote the eigenvalues of A^{-1} by $r\mathrm{e}^{-2i\phi}$, then the eigenvalues of $\mathrm{e}^{2i\theta}A^{-1}$ are given by $r\mathrm{e}^{2i(\theta-\phi)}$ and for $|\theta| < \frac{\pi}{4} + \frac{\delta}{3}$ and $|2\phi| \leqslant \frac{\pi}{2} - \delta$ we obtain $2|\theta - \phi| < \pi - \frac{\delta}{3}$. Application of Lemma 4.6 with $S = \mathrm{e}^{2i\theta}A^{-1}$ implies that the matrix $\hat{M}(s, \rho)$ given by 42 is hyperbolic for $|\theta| < \frac{\pi}{4} + \frac{\delta}{3}$. Furthermore, the m stable eigenvalues $\nu_{s,i} = 1 + h\rho\lambda_{s,i}$ and the m unstable eigenvalues $\nu_{u,i} = \nu_{s,i}^{-1}$, $i = 1, \dots, m$ can be estimated using Lemma 4.6 by

$$\begin{aligned}
&|\nu_{u,i}| \geqslant \alpha(\rho h)^2, && |\nu_{s,i}| \leqslant \frac{\alpha}{(\rho h)^2}, && \text{for } \rho > \frac{C}{h}\\
&|\nu_{u,i}| \geqslant 1+\alpha, && |\nu_{s,i}| \leqslant \frac{1}{1+\alpha}, && \text{for } \rho \in [\frac{\epsilon}{h}, \frac{C}{h}]\\
&|\nu_{u,i}| \geqslant 1+\alpha\rho h, && |\nu_{s,i}| \leqslant \frac{1}{1+\alpha\rho h}, && \text{for } \rho \in [C, \frac{\epsilon}{h}].
\end{aligned} \tag{47}$$

The matrices $\hat{M}(s, \rho)$ can be transformed to diagonal form via $TD = \hat{M}(s, \rho)T$ with

$$D = \begin{pmatrix} D_s & 0 \\ 0 & D_s^{-1} \end{pmatrix}, \qquad D_s = I + \kappa\Lambda_s, \quad D_u = I + \kappa\Lambda_u$$

and

$$T = \begin{pmatrix} -I & -I \\ \Lambda_u & \Lambda_s \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} (\Lambda_s - \Lambda_u)^{-1} & 0 \\ 0 & (\Lambda_s - \Lambda_u)^{-1} \end{pmatrix} \begin{pmatrix} -\Lambda_s & -I \\ \Lambda_u & I \end{pmatrix}. \tag{48}$$

Note the relations

$$\begin{aligned}\Lambda_u \Lambda_s &= \Lambda_s \Lambda_u = -S, \quad \Lambda_s + \Lambda_u = \kappa S, \quad D_u = D_s^{-1}, \\ \Lambda_u D_s &= -\Lambda_s, \quad \Lambda_s = \frac{1}{\kappa}(D_s - I).\end{aligned} \tag{49}$$

From this the existence of an exponential dichotomy on $\mathbb{Z}$ for the constant coefficient operators $\hat{L}(s,\rho)$ follows by Remark 2.5 in [10] with data (K, β, P) where $\beta = -\ln \nu_s$, $\nu_s \in (\max_{i=1,\dots,m} |\nu_{s,i}|, 1)$ and P is defined in (46). □

Using the exponential dichotomy we can construct directly a solution of (39) in the usual way [10].

Lemma 4.8. *For $s \in \Omega_C^h \cup \Omega_\infty^h$ exist $h_0, T > 0$ such that for $h < h_0, \pm n_\pm h > T$ and for each $r \in \ell_\infty^J(\mathbb{C}^m)$ there exists a unique solution $\tilde{z} \in \ell_\infty^{\hat{J}}(\mathbb{C}^{2m})$ of the boundary value problem*

$$\begin{aligned}(\hat{L}(s,\rho)z)_n &= \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} r_n, \quad n \in J \\ P z_{n_-} &= \rho_- \in \mathcal{R}(P) \\ (I-P) z_{n_+} &= \rho_+ \in \mathcal{R}(I-P)\end{aligned}$$

where P is the dichotomy projector defined in (46)*. The solution has the form*

$$\tilde{z}_n = z_n^{\text{hom}} + \hat{z}_n(r), \quad n \in J, \qquad \tilde{z}_{n_+ + 1} = \hat{M} \tilde{z}_{n_+} + \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} r_{n_+}, \ \textit{where} \tag{50}$$

$$z_n^{\text{hom}} = \hat{\mathcal{S}}(n, n_-)\rho_- + \hat{\mathcal{S}}(n, n_+)\rho_+, \quad \hat{\mathcal{S}}(n,m) = \hat{M}(s,\rho)^{n-m} \ \textit{and} \tag{51}$$

$$\begin{aligned}\hat{z}_n(r) = \frac{h}{\rho} \Bigg(&\sum_{m=n_-}^{n-1} \hat{\mathcal{S}}(n, m+1) P \begin{pmatrix} h\rho I \\ I \end{pmatrix} r_m \\ &- \sum_{m=n}^{n_+ - 1} \hat{\mathcal{S}}(n, m+1)(I-P) \begin{pmatrix} h\rho I \\ I \end{pmatrix} r_m \Bigg).\end{aligned} \tag{52}$$

In order to obtain the necessary estimates of $\hat{z}$, especially for the case $h\rho > C$, we have to take into account the special structure of the right hand side. Therefore we diagonalize equation (50) using the transformation T given in (48). For $w_n = T^{-1} z_n$ equation (39) reads

$$w_{n+1} - \begin{pmatrix} D_s & 0 \\ 0 & D_s^{-1} \end{pmatrix} w_n = \frac{h}{\rho} T^{-1} \begin{pmatrix} h\rho I \\ I \end{pmatrix} r_n, \quad n \in J = [n_-, n_+].$$

In order to be able to distinguish estimates in the different components we introduce the following vector norm notation. For $z=(u,v)\in\mathbb{R}^m\times\mathbb{R}^m$, $\|z\|_{\text{vec}}=\begin{pmatrix}n_u\\ n_v\end{pmatrix}$ means $\|u\|=n_u$, $\|v\|=n_v$ and $\|z\|_{\text{vec}}\leqslant\begin{pmatrix}c_u\\ c_v\end{pmatrix}$ means the componentwise estimates $\|u\|\leqslant c_u$ and $\|v\|\leqslant c_v$. With this notation we obtain the following estimates for $\hat{\mathcal{S}}$.

Lemma 4.9. *Let $|\sigma(D_s)|<\nu_s<1$. Then the following holds.*

$$\left\|\hat{\mathcal{S}}(n,m+1)P\ \begin{pmatrix}h\rho I\\ I\end{pmatrix}\right\|_{\text{vec}}\leqslant\frac{c}{\max(\rho h,1)}\begin{pmatrix}\nu_s\\ \frac{1}{\rho h}(1-\nu_s)\end{pmatrix}\nu_s^{n-m-1},\ n\geqslant m$$
$$\left\|\hat{\mathcal{S}}(n,m+1)(I-P)\ \begin{pmatrix}h\rho I\\ I\end{pmatrix}\right\|_{\text{vec}}\leqslant\frac{c}{\max(\rho h,1)}\begin{pmatrix}1\\ \frac{1}{\rho h}(1-\nu_s)\end{pmatrix}\nu_s^{m-n},\quad n<m \tag{53}$$

and

$$\begin{aligned}\|\hat{\mathcal{S}}(n,n_-)T_-\|_{\text{vec}}&\leqslant\begin{pmatrix}\nu_s\\ \frac{1}{\rho h}(1-\nu_s)\end{pmatrix}\nu_s^{n-n_--1},\\ \|\hat{\mathcal{S}}(n,n_+)T_+\|_{\text{vec}}&\leqslant\begin{pmatrix}1\\ \frac{1}{\rho h}(1-\nu_s)\end{pmatrix}\nu_s^{n_+-n},\end{aligned} \tag{54}$$

where $T=(T_-,\ T_+)$ with T defined by (48)*.*

Proof. With

$$\hat{\mathcal{S}}(n,m)=TD^{n-m}T^{-1},\quad P=TE^sT^{-1},\quad E^s=\begin{pmatrix}I&0\\0&0\end{pmatrix} \tag{55}$$

we obtain using $D_s=I+h\rho\Lambda_s$

$$\begin{aligned}\hat{\mathcal{S}}(n,m+1)P\ \begin{pmatrix}h\rho I\\ I\end{pmatrix}&=T\begin{pmatrix}D_s^{n-m-1}&0\\0&0\end{pmatrix}T^{-1}\begin{pmatrix}h\rho I\\ I\end{pmatrix}\\ &=\begin{pmatrix}I\\ -\Lambda_u\end{pmatrix}D_s^{n-m}(\Lambda_s-\Lambda_u)^{-1}\\ &=\begin{pmatrix}D_s\\ \frac{1}{\rho h}(D_s-I)\end{pmatrix}D_s^{n-m-1}(\Lambda_s-\Lambda_u)^{-1}\end{aligned}$$

and similarly

$$\hat{\mathcal{S}}(n,m+1)(I-P)\ \begin{pmatrix}h\rho I\\ I\end{pmatrix}=\begin{pmatrix}-I\\ \frac{1}{\rho h}(D_s-I)\end{pmatrix}D_s^{m-n}(\Lambda_s-\Lambda_u)^{-1}.$$

This implies the estimates (53). Similarly with (45)

$$\hat{\mathcal{S}}(n,n_-)T_- = \begin{pmatrix} -D_s \\ \frac{1}{\rho h}(D_s - I) \end{pmatrix} D_s^{n-n_--1} \quad \text{and}$$
$$\hat{\mathcal{S}}(n,n_+)T_+ = \begin{pmatrix} -I \\ \frac{1}{\rho h}(D_s - I) \end{pmatrix} D_s^{n_+-n}$$

lead to (54). □

The special solution $\hat{z}(r)$ from (52) is estimated in the following Lemma.

Lemma 4.10. *For $s \in \Omega_C^h \cap \Omega_\infty^h$ exist $h_0, T > 0$ such that for $h < h_0, \pm n_\pm h > T$ for each $r \in \ell_\infty^J(\mathbb{C}^m)$ the solution $\hat{z}(r) \in \ell_\infty^J(\mathbb{C}^{2m})$ given by* (52) *can be estimated by*

$$\|\hat{z}(r)\|_{\mathcal{L}_{2,h}} \leqslant \text{const } \frac{1}{\rho^2}\|r\|_{\mathcal{L}_{2,h}}. \tag{56}$$

Moreover, we obtain

$$\|\hat{M}\hat{z}_{n_+}(r)\|_{\text{vec}} \leqslant \text{const } \begin{pmatrix} h^2 + \frac{h}{\rho} + \frac{1}{\rho^2} \\ \frac{h}{\rho} + \frac{1}{\rho^2} \end{pmatrix} \|r\|_\infty. \tag{57}$$

Proof. Using the estimates (53) we obtain for $\hat{z}(r) = (\hat{u}, \hat{v})$ with $\nu_s < 1$

$$\|\hat{u}_n\| \leqslant \frac{ch}{\max(\rho h, 1)\rho} \sum_{m=n_-}^{n_+-1} \nu_s^{-|n-m|}\|r_m\| \leqslant c_u \frac{1+\nu_s}{1-\nu_s}\|r\|_\infty, \quad n \in J, \tag{58}$$

for some $c_u > 0$. The estimate

$$c_u \frac{1+\nu_s}{1-\nu_s} \leqslant \frac{c}{\rho^2} \tag{59}$$

which follows from (47) with some generic constant $c > 0$ implies

$$\|\hat{u}_n\| \leqslant \frac{c}{\rho^2}\|r\|_\infty, \quad \forall n \in J.$$

Using the second coordinate of (53) we obtain

$$\|\hat{v}_n\| \leqslant \frac{c(1-\nu_s)}{\rho^2 \max(\rho h, 1)} \left(\sum_{m=n_-}^{n-1} \nu_s^{n-m-1}\|r_m\| + \sum_{m=n}^{n_+-1} \nu_s^{m-n}\|r_m\| \right) \leqslant \frac{c}{\rho^2}\|r\|_\infty. \tag{60}$$

The $\mathcal{L}_{2,h}$ estimate is similar to the estimate in Lemma 3.6 in [15]. From (58) we find

$$\begin{aligned}
\|\hat{u}_n\|^2 &\leqslant c_u^2 \left(\sum_{m=n_-}^{n_+-1} \nu_s^{-|n-m|} \|r_m\| \right)^2 \leqslant c_u^2 \sum_{m=-\infty}^{\infty} \nu_s^{-|n-m|} \sum_{m=n_-}^{n_+-1} \nu_s^{-|n-m|} \|r_m\|^2 \\
&\leqslant c_u^2 \frac{1+\nu_s}{1-\nu_s} \sum_{m=n_-}^{n_+-1} \nu_s^{-|n-m|} \|r_m\|^2 \ \leqslant \frac{cc_u}{\rho^2} \sum_{m=n_-}^{n_+-1} \nu_s^{-|n-m|} \|r_m\|^2,
\end{aligned}$$

which implies by summation over all $n \in J$ with (59)

$$\begin{aligned}
\|\hat{u}\|_{\mathcal{L}_{2,h}}^2 &= \sum_{n=n_-}^{n_+} h \|\hat{u}_n\|^2 \leqslant \frac{ch}{\rho^2} c_u \sum_{n=n_-}^{n_+} \sum_{m=n_-}^{n_+-1} \nu_s^{-|n-m|} \|r_m\|^2 \\
&\leqslant \frac{ch}{\rho^2} c_u \sum_{m=n_-}^{n_+-1} \|r_m\|^2 \sum_{n=n_-}^{n_+} \nu_s^{-|n-m|} \\
&\leqslant \frac{ch}{\rho^2} c_u \frac{1+\nu_s}{1-\nu_s} \sum_{m=n_-}^{n_+-1} \|r_m\|^2 \leqslant \left(\frac{c}{\rho^2}\right)^2 h \sum_{m=n_-}^{n_+-1} \|r_m\|^2 = \left(\frac{c}{\rho^2}\right)^2 \|r_m\|_{\mathcal{L}_{2,h}}^2.
\end{aligned}$$

Similarly, (60) implies with $c_v = (\rho^2 \max(\rho h, 1))^{-1}$

$$\begin{aligned}
\|\hat{v}_n\|^2 &\leqslant cc_v^2 (1-\nu_s)^2 \left[\left(\sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|r_m\| \right)^2 + \left(\sum_{m=n}^{n_+-1} \nu_s^{m-n} \|r_m\| \right)^2 \right] \\
&\leqslant cc_v^2 (1-\nu_s)^2 \left[\sum_{m=-\infty}^{n-1} \nu_s^{n-m-1} \sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|r_m\|^2 \right. \\
&\qquad \left. + \sum_{m=n}^{\infty} \nu_s^{m-n} \sum_{m=n}^{n_+-1} \nu_s^{m-n} \|r_m\|^2 \right] \\
&\leqslant cc_v^2 (1-\nu_s)^2 \left[\frac{1}{1-\nu_s} \sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|r_m\|^2 + \frac{1}{1-\nu_s} \sum_{m=n}^{n_+-1} \nu_s^{m-n} \|r_m\|^2 \right] \\
&\leqslant cc_v^2 (1-\nu_s) \left[\sum_{m=n_-}^{n-1} \nu_s^{n-m-1} \|r_m\|^2 + \sum_{m=n}^{n_+-1} \nu_s^{m-n} \|r_m\|^2 \right]
\end{aligned}$$

which leads to

$$
\begin{aligned}
\|\hat{v}\|^2_{\mathcal{L}_{2,h}} = \sum_{n=n_-}^{n_+} h\|\hat{v}_n\|^2 &\leqslant cc_v^2(1-\nu_s)h \sum_{n=n_-}^{n_+}\left[\sum_{m=n_-}^{n-1} \nu_s^{n-m-1}\|r_m\|^2\right.\\
&\left.+\sum_{m=n}^{n_+-1} \nu_s^{m-n}\|r_m\|^2\right]\\
&\leqslant cc_v^2(1-\nu_s)h \sum_{m=n_-}^{n_+-1}\|r_m\|^2\left[\sum_{n=m+1}^{n_+}\nu_s^{n-m-1}+\sum_{m=n_-}^{m}\nu_s^{m-n}\right]\\
&\leqslant cc_v^2 h\sum_{m=n_-}^{n_+-1}\|r_m\|^2 = \frac{c}{\rho^4}\|r\|^2_{\mathcal{L}_{2,h}}.
\end{aligned}
$$

Finally the estimate (57) follows from the definition of $\hat{M}$ in (42)

$$
\begin{aligned}
\|\hat{M}\hat{z}_{n+}(r)\|_{\mathrm{vec}} &\leqslant \mathrm{const}\ \begin{pmatrix}(1+(\rho h)^2)\|\hat{u}_{n_+}\|+\rho h\|\hat{v}_{n_+}\|\\ \rho h\|\hat{u}_{n_+}\|+\|\hat{v}_{n_+}\|\end{pmatrix}\\
&\leqslant \mathrm{const}\ \begin{pmatrix}h^2+\frac{h}{\rho}+\frac{1}{\rho^2}\\ \frac{h}{\rho}+\frac{1}{\rho^2}\end{pmatrix}\|r\|_\infty.
\end{aligned}
$$

□

Inserting the ansatz (50) for $\tilde{z}$ into the boundary conditions (36b) we obtain the following lemma.

Lemma 4.11. *Assume Hypothesis 2.6. Then for $s\in\Omega_C^h\cup\Omega_\infty^h$ exist $h_0, T>0$ such that the following holds. If $h<h_0$ and $\pm hn_\pm>T$ then for each $r\in\ell_\infty^J(\mathbb{C}^m)$ there exists a unique solution $\tilde{z}\in\ell_\infty^{[n_-,n_++1]}(\mathbb{C}^{2m})$ of* (39) *which satisfies the boundary conditions* (36b)*, i.e.*

$$
R(\rho)z=\hat{\eta}=\begin{pmatrix}\frac{1}{\rho}\eta^N\\ \eta^D\end{pmatrix}. \tag{61}
$$

Morevoer, $\tilde{z}$ can be estimated as follows

$$
\|\tilde{z}\|_{\mathcal{L}_{2,h}}\leqslant \mathrm{const}\ \left(\frac{1}{\rho}\|\eta^N\|+\|\eta^D\|+\frac{1}{\rho^2}\|r\|_{\mathcal{L}_{2,h}}\right), \quad \textit{for } s\in\Omega_C^h, \tag{62}
$$

$$
\|\tilde{z}_{|[n_-+1,\dots,n_+]}\|_{\mathcal{L}_{2,h}}\leqslant \mathrm{const}\ \left(\frac{1}{\rho}\|\eta^N\|+\|\eta^D\|+\frac{1}{\rho^2}\|r\|_{\mathcal{L}_{2,h}}\right), \quad \textit{for } s\in\Omega_\infty^h. \tag{63}
$$

Proof. Inserting the ansatz (50) into the boundary condition (61) one obtains

$$\begin{aligned}
&B_-(\rho)(\rho_- + \hat{\mathcal{S}}(n_-,n_+)\rho_+) + \hat{B}_-(\hat{\mathcal{S}}(n_-+1,n_-)\rho_- + \hat{\mathcal{S}}(n_-+1,n_+)\rho_+)\\
&\quad + B_+(\rho)(\hat{\mathcal{S}}(n_+,n_-)\rho_- + \rho_+) + \hat{B}_+\hat{M}(\hat{\mathcal{S}}(n_+,n_-)\rho_- + \rho_+)\\
&\quad = \hat{\eta} - \Bigg(B_-(\rho)\hat{z}_{n_-}(r) + \hat{B}_-\hat{z}_{n_-+1}(r) + B_+(\rho)\hat{z}_{n_+}(r) + \hat{B}_+\Bigg[\hat{M}\hat{z}_{n_+}(r)\\
&\quad + \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} r_{n_+}\Bigg]\Bigg).
\end{aligned}$$

This equation has to be solved for ρ_- and ρ_+. We can write $\rho\pm = T_\pm\xi_\pm$, $\xi_\pm \in \mathbb{C}^m$ where $T = (T_- \;\; T_+)$. After rearranging terms we obtain from the previous equation

$$R_\rho(\xi_-,\xi_+) + \Delta R_\rho(\xi_-,\xi_+) = \hat{\eta} - F_\rho(r), \tag{64}$$

where

$$\begin{aligned}
R_\rho(\xi_-,\xi_+) &= B_-(\rho)T_-\xi_- + \hat{B}_-\hat{\mathcal{S}}(n_-+1,n_-)T_-\xi_- + B_+(\rho)T_+\xi_+ + \hat{B}_+\hat{M}T_+\xi_+\\
\Delta R_\rho(\xi_-,\xi_+) &= \Big(B_-(\rho)\hat{\mathcal{S}}(n_-,n_+) + \hat{B}_-\hat{\mathcal{S}}(n_-+1,n_+)\Big)\,T_+\xi_+\\
&\quad + (B_+(\rho) + \hat{B}_+\hat{M})\hat{\mathcal{S}}(n_+,n_-)T_-\xi_-\\
F_\rho(r) &= B_-(\rho)\hat{z}_{n_-}(r) + \hat{B}_-\hat{z}_{n_-+1}(r) + B_+(\rho)\hat{z}_{n_+}(r)\\
&\quad + \hat{B}_+\left[\hat{M}\hat{z}_{n_+}(r) + \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} r_{n_+}\right].
\end{aligned}$$

With (55) and the relations $\hat{M} = TDT^{-1}$, $T^{-1}T_- = \binom{I}{0}$, $T^{-1}T_+ = \binom{0}{I}$, $TD = \begin{pmatrix} -D_s & -D_s^{-1} \\ \Lambda_u D_s & \Lambda_s D_s^{-1} \end{pmatrix}$ and $\Lambda_u(I + D_s) = \Lambda_u - \Lambda_s$ wich is implied by (49) these terms can be calculated as follows:

$$\begin{aligned}
R_\rho(\xi_-,\xi_+) &= \begin{pmatrix} \frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N \\ P_-^D & 0 \end{pmatrix} T_-\xi_- + \begin{pmatrix} 0 & \frac{1}{2}Q_-^N \\ 0 & 0 \end{pmatrix} TD \begin{pmatrix} I \\ 0 \end{pmatrix} \xi_-\\
&\quad + \begin{pmatrix} \frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N \\ P_+^D & 0 \end{pmatrix} T_+\xi_+ + \begin{pmatrix} 0 & \frac{1}{2}Q_+^N \\ 0 & 0 \end{pmatrix} TD \begin{pmatrix} 0 \\ I \end{pmatrix} \xi_+ = \mathcal{B}\begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix},
\end{aligned}$$

where

$$\mathcal{B} = -\begin{pmatrix} \frac{1}{\rho}P_-^N - \frac{1}{2}Q_-^N(\Lambda_u - \Lambda_s) & \frac{1}{\rho}P_+^N + \frac{1}{2}Q_+^N(\Lambda_u - \Lambda_s) \\ P_-^D & P_+^D \end{pmatrix}.$$

From (44) we get with $z=\frac{1}{2}\rho h e^{i\theta}$, $\delta(\theta,z)=2e^{i\theta}(1+|z|^2)^{\frac{1}{2}}$ and the definition of $\Delta(z)$ in (14)

$$\begin{aligned}\Lambda_u-\Lambda_s&=\begin{cases}((\rho h e^{2i\theta})A^{-1}+4I)^{\frac{1}{2}}e^{i\theta}A^{-\frac{1}{2}}, & \text{if } \rho h>0,\ |\theta|\leqslant\frac{\pi}{4}+\frac{\delta}{3},\\ \rho h e^{2i\theta}A^{-1}(1+\frac{4}{(\rho h)^2}e^{-2i\theta}A)^{\frac{1}{2}}, & \text{if } \rho h>C\end{cases}\\ &=\delta(\theta,z)\Delta(z).\end{aligned}$$

With these notations the matrix $\mathcal{B}$ reads $\mathcal{B}=\mathcal{S}\mathcal{B}_s$ where

$$\mathcal{S}=\begin{pmatrix}-\delta(\theta,z)I_r & 0\\ 0 & -I_{2m-r}\end{pmatrix}, \tag{65}$$

and

$$\mathcal{B}_s=\begin{pmatrix}\frac{2}{\rho\delta(\theta,z)}P_-^N+Q_-^N\Delta(z) & \frac{2}{\rho\delta(\theta,z)}P_+^N-Q_+^N\Delta(z)\\ P_-^D & P_+^D\end{pmatrix}.$$

From Hypothesis 2.6 and the definition of Ω_C^h and Ω_∞^h we obtain that

$$\hat{\mathcal{B}}_s=\begin{pmatrix}Q_-^N\Delta(z) & -Q_+^N\Delta(z)\\ P_-^D & P_+^D\end{pmatrix}$$

has a uniformly bounded inverse. From $c_1\max(1,|z|)\leqslant|\delta(\theta,z)|\leqslant c_2$ $\max(1,|z|)$ we find

$$\frac{1}{|\delta(\theta,z)|}\leqslant c\min\left(1,\frac{1}{\rho h}\right)\leqslant c. \tag{66}$$

Therefore the difference $\|\mathcal{B}_s-\hat{\mathcal{B}}_s\|$ can be estimated by

$$\|\mathcal{B}_s-\hat{\mathcal{B}}_s\|\leqslant\frac{2}{\rho|\delta(\theta,z)|}\left(\|P_-^N\|+\|P_+^N\|\right)\leqslant\frac{c}{\rho}$$

which tends to zero as $\rho\to\infty$. Choosing C in the definition of Ω_C^h large enough, we obtain $\|\mathcal{B}^{-1}\|\leqslant c$.

For the error term ΔR_ρ we get

$$\begin{aligned}\Delta R_\rho(\xi_-,\xi_+)&=(B_-(\rho)\hat{\mathcal{S}}(n_-,n_+)+\hat{B}_-\hat{\mathcal{S}}(n_-+1,n_+))T_+\xi_+\\ &\quad+(B_+(\rho)+\hat{B}_+\hat{M})\hat{\mathcal{S}}(n_+,n_-)T_-\xi_-=\Delta\mathcal{B}\begin{pmatrix}\xi_-\\ \xi_+\end{pmatrix},\end{aligned}$$

where

$$\Delta\mathcal{B}=\mathcal{B}\begin{pmatrix}0 & D_s^{(n_+-n_-)}\\ D_s^{(n_+-n_-)} & 0\end{pmatrix}=\mathcal{S}\mathcal{B}_s\begin{pmatrix}0 & D_s^{(n_+-n_-)}\\ D_s^{(n_+-n_-)} & 0\end{pmatrix}.$$

Here $\mathcal{S}$ denotes the scaling matrix defined in (65). Furthermore $\nu_s^{(n_+-n_-)}$ vanishes as $n_+-n_-\to\infty$ and

$$\|\mathcal{B}_s\|\leqslant c\left(\frac{1}{\rho|\delta(\theta,z)|}+\|\Delta(z)\|\right)\leqslant c$$

implies that $\Delta\mathcal{B}_s=\mathcal{S}^{-1}\Delta\mathcal{B}$ vanishes as $n_+-n_-\to\infty$. The right hand side of (64) can be rewritten as follows:

$$\begin{aligned}F_\rho(r)=&\begin{pmatrix}\frac{1}{\rho}P_-^N & \frac{1}{2}Q_-^N\\ P_-^D & 0\end{pmatrix}\hat{z}_{n_-}(r)+\begin{pmatrix}0 & \frac{1}{2}Q_-^N\\ 0 & 0\end{pmatrix}\hat{z}_{n_-+1}(r)+\begin{pmatrix}\frac{1}{\rho}P_+^N & \frac{1}{2}Q_+^N\\ P_+^D & 0\end{pmatrix}\hat{z}_{n_+}(r)\\ &+\begin{pmatrix}0 & \frac{1}{2}Q_+^N\\ 0 & 0\end{pmatrix}\left(\hat{M}\hat{z}_{n_+}(r)+\begin{pmatrix}h^2I\\ \frac{h}{\rho}I\end{pmatrix}r_{n_+}\right)\\ =&\begin{pmatrix}\frac{1}{\rho}P_-^N\hat{u}_{n_-}+\frac{1}{2}Q_-^N(\hat{v}_{n_-}+\hat{v}_{n_-+1})+\frac{1}{2}Q_+^N(\gamma_v+\frac{h}{\rho}r_{n_+})+\frac{1}{\rho}P_+^N\hat{u}_{n_+}\\ P_-^D\hat{u}_{n_-}+P_+^D\hat{u}_{n_+}\end{pmatrix}\end{aligned}$$

where we used the notation $\hat{M}\hat{z}_{n_+}(r)=(\gamma_u,\gamma_v)^T$. Using (58), (60), (57) we obtain

$$\|F_\rho(r)\|_{\text{vec}}\leqslant c\begin{pmatrix}\frac{1}{\rho^2}+\frac{h}{\rho}\\ \frac{1}{\rho^2}\end{pmatrix}\|r\|_\infty.$$

Then the scaled version of $F_\rho(r)$ can be estimated by

$$\begin{aligned}\left\|\begin{pmatrix}\frac{1}{\delta(\theta,z)}I_r & 0\\ 0 & I_{2m-r}\end{pmatrix}F_\rho(r)\right\|&\leqslant c\left(\min\left(1,\frac{1}{\rho h}\right)\left(\frac{1}{\rho^2}+\frac{h}{\rho}\right)+\frac{1}{\rho^2}\right)\|r\|_\infty\\ &\leqslant\frac{c}{\rho^2}\|r\|_\infty.\end{aligned}$$

Equation (64) is equivalent to

$$(\mathcal{B}_s+\Delta\mathcal{B}_s)\begin{pmatrix}\xi_-\\ \xi_+\end{pmatrix}=\begin{pmatrix}-\frac{1}{\rho\delta(\theta,z)}\eta^N\\ \eta^D\end{pmatrix}+\begin{pmatrix}\frac{1}{\delta(\theta,z)}I_r & 0\\ 0 & I_{2m-r}\end{pmatrix}F_\rho(r),$$

thus we can estimate the solution (ξ_-,ξ_+) of (64) using (66) by

$$\|(\xi_-,\xi_+)\|\leqslant c\left(\frac{1}{\rho}\|\eta^N\|+\|\eta^D\|+\frac{1}{\rho^2}\|r\|_\infty\right). \tag{67}$$

The solution $z^{\mathrm{hom}} = (u^{\mathrm{hom}}, v^{\mathrm{hom}})$ defined in (51) can be estimated using (54) as follows. The estimates

$$\left\| \hat{\mathcal{S}}(n, n_-)\rho_- \right\|_{\mathrm{vec}} = \|\hat{\mathcal{S}}(n, n_-)T_-\xi_-\|_{\mathrm{vec}} \leqslant \begin{pmatrix} \nu_s \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{n-n_--1} \|\xi_-\|,$$

$$\left\| \hat{\mathcal{S}}(n, n_+)\rho_+ \right\|_{\mathrm{vec}} = \|\hat{\mathcal{S}}(n, n_+)T_+\xi_+\|_{\mathrm{vec}} \leqslant \begin{pmatrix} 1 \\ \frac{1}{\rho h}(1-\nu_s) \end{pmatrix} \nu_s^{n_+-n} \|\xi_+\|$$

imply for all $n \in J$

$$\|u_n^{\mathrm{hom}}\| \leqslant c(\nu_s^{n-n_-}\|\xi_-\| + \nu_s^{n_+-n}\|\xi_-\|) \leqslant c(\|\xi_+\| + \|\xi_+\|) \tag{68}$$

and for $n \in \hat{J} = [n_- + 1, n_+]$

$$\|v_n^{\mathrm{hom}}\| \leqslant c\frac{1-\nu_s}{\rho h}\left(\nu_s^{n-n_--1}\|\xi_-\| + \nu_s^{n_+-n}\|\xi_+\|\right) \leqslant c(\|\xi_-\| + \|\xi_+\|). \tag{69}$$

From (47) and Lemma 4.6 we obtain

$$\|v_{n_-}^{\mathrm{hom}}\| \leqslant c\frac{1-\nu_s}{\rho h}\left(\nu_s^{-1}\|\xi_-\| + \nu_s^{n_+-n_-}\|\xi_+\|\right) \leqslant c(\max(1, \rho h)\|\xi_-\| + \|\xi_+\|). \tag{70}$$

The estimates (68) and (56) lead for $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$ defined in (50) for all $n \in J$ to

$$\begin{aligned} \|\tilde{u}_n\| &\leqslant \|u_n^{\mathrm{hom}}\| + \|\hat{z}\|_\infty \leqslant c\left(\|\xi_-\| + \|\xi_+\| + \frac{1}{\rho^2}\|r\|_\infty\right) \\ &\leqslant c\left(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|r\|_\infty\right) \end{aligned}$$

and for $n \in J_h = [n_- + 1, n_+]$ to

$$\begin{aligned} \|\tilde{v}_n\| &\leqslant \|v_n^{\mathrm{hom}}\| + \|\hat{z}\|_\infty \leqslant c(\|\xi_-\| + \|\xi_+\| + \frac{1}{\rho^2}\|r\|_\infty) \\ &\leqslant c\left(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|r\|_\infty\right). \end{aligned}$$

Finally $\frac{1}{\rho h}(\nu_s^{-1} - 1) \leqslant c\max(1, \rho h)$ implies with (70)

$$\|\tilde{v}_{n_-}\| \leqslant \|v_{n_-}^{\mathrm{hom}}\| + \|\hat{z}\|_\infty \leqslant c\max(1, \rho h)\left(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|r\|_\infty\right)$$

and using

$$\left\|\hat{M}z_{n+}^{\mathrm{hom}}\right\|_{\mathrm{vec}} \leqslant c\left(\begin{pmatrix} (\rho h)^2\nu_s^{n_+-n_-} \\ (1-\nu_s)\nu_s^{n_+-n_--1} \end{pmatrix}\|\xi_-\| + \begin{pmatrix} (\rho h)^2\nu_s^{n_+-n_-} \\ (1-\nu_s)\nu_s^{n_+-n_-} \end{pmatrix}\|\xi_+\|\right)$$

with $n_+ - n_- > 1$ we end up with

$$\|\hat{M} z_{n+}^{\text{hom}}\| \leqslant c(\|\xi_-\| + \|\xi_+\|). \tag{71}$$

By (47) we obtain for $\rho \in (C, \frac{C}{h}]$ the estimate $\frac{h}{1-\nu_s^2} < c$ as well as $\frac{h}{1-\nu_s^2} < h$ for $\rho h > C$. This leads to

$$\begin{aligned}\|u^{\text{hom}}\|_{\mathcal{L}_{2,h}}^2 &\leqslant c\left(\sum_{n=n_-}^{n_+} h\nu_s^{2(n-n_-)}\|\xi_-\|^2 + \sum_{n=n_-}^{n_+} h\nu_s^{2(n_+-n)}\|\xi_+\|^2\right)\\ &\leqslant c\frac{h}{1-\nu_s^2}(\|\xi_-\|^2 + \|\xi_+\|^2) \leqslant c(\|\xi_-\|^2 + \|\xi_+\|^2). \end{aligned}\tag{72}$$

In the restricted interval $J_h = [n_- + 1, n_+]$ we obtain in the same way

$$\begin{aligned}\|v_{|J_h}^{\text{hom}}\|_{\mathcal{L}_{2,h}}^2 &\leqslant c\left(\sum_{n=n_-+1}^{n_+} h\frac{(1-\nu_s)^2}{(\rho h)^2}\nu^{2(n-n_--1)}\|\xi_-\|^2 + \sum_{n=n_-+1}^{n_+} h\nu^{2(n_+-n)}\|\xi_+\|^2\right),\\ &\leqslant c\left(\|\xi_-\|^2 + \|\xi_+\|^2\right)\end{aligned}\tag{73}$$

and with Lemma 4.6 we arrive at

$$\begin{aligned}\|v^{\text{hom}}\|_{\mathcal{L}_{2,h}}^2 &\leqslant ch\left(\frac{1-\nu_s}{(\rho h)^2\nu_s^2(1+\nu_s)}\|\xi_-\|^2 + \frac{1}{1-\nu_s^2}\|\xi_+\|^2\right)\\ &\leqslant c\left(\max(1, (\rho h)^2)\|\xi_-\|^2 + \|\xi_+\|^2\right).\end{aligned}\tag{74}$$

Using (56), (57), (72), (74) and (67) we obtain (62) with $\rho h < C$

$$\begin{aligned}\|\tilde{z}\|_{\mathcal{L}_{2,h}} &\leqslant \|\hat{z}\|_{\mathcal{L}_{2,h}} + \|z^{\text{hom}}\|_{\mathcal{L}_{2,h}} + \sqrt{h}(\|\hat{M} z_{n_+}^{\text{hom}}\| + \|\hat{M}\hat{z}_{n_+}\|)\\ &\leqslant c(\frac{1}{\rho^2}\|r\| + \max(1, \rho h)\|\xi_-\| + \|\xi_+\| + (h^2 + \frac{h}{\rho} + \frac{1}{\rho^2})\|r\|_{\mathcal{L}_{2,h}})\\ &\leqslant c\left(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|r\|_{\mathcal{L}_{2,h}}\right).\end{aligned}$$

In the same way (56), (72), (73) and (67) lead to (63). □

Remark 4.12. The restriction to J_h in (63) is necessary, since from (57), (70) and (71) we obtain for $s \in \Omega_\infty^h$ only

$$\|\tilde{z}\|_{\mathcal{L}_{2,h}} \leqslant c\max(1, (\rho h)^2)\left(\frac{1}{\rho}\|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2}\|r\|_{\mathcal{L}_{2,h}}\right).$$

From the above estimates the invertibility of (37), (36b) follows from a regular perturbation argument.

Lemma 4.13. *Let $A \in \mathbb{R}^{m,m}$ be diagonalizable and positive definite and assume Hypothesis 2.6. Then there exist $C, h_0, T > 0$, such that for $s \in \Omega_C^h \cup \Omega_\infty^h$ and $h < h_0$, $\pm n_\pm h > T$ the following holds. For each $r \in \ell_\infty^J(\mathbb{C}^m)$, there exists a unique solution $z \in \ell_\infty^{[n_-, n_+ + 1]}(\mathbb{C}^m)$ of* (37), (36b) *which can be estimated by*

$$\|z\|_{\mathcal{L}_{2,h}} \leqslant \text{const} \left(\frac{1}{\rho} \|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2} \|r\|_{\mathcal{L}_{2,h}} \right), \quad \textit{for } s \in \Omega_C^h \tag{75}$$

$$\|z_{|[n_-+1, n_+]}\|_{\mathcal{L}_{2,h}} \leqslant \text{const} \left(\frac{1}{\rho} \|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2} \|r\|_{\mathcal{L}_{2,h}} \right), \quad \textit{for } s \in \Omega_\infty^h. \tag{76}$$

Proof. Write (37) as

$$z_{n+1} - \hat{M}(s, \rho) z_n = \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} E_n^{+^{-1}} r_n + (M_n(s, \rho) - \hat{M}(s, \rho)) z_n, \qquad n \in J$$

and define the space

$$S = \Big\{ (\hat{r}, \hat{\eta}) \in \ell_\infty^{[n_-, n_+ + 1]}(\mathbb{C}^{2m}) \times \mathbb{R}^{2m} : \\ \hat{r}_n = \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} r_n, \quad n \in [n_-, n_+ + 1], \quad r \in \ell_\infty^{[n_-, n_+ + 1]}(\mathbb{C}^m) \Big\}$$

equipped with the norm

$$\left\| (\hat{r}, \hat{\eta}) \right\|_{\mathcal{L}_{2,h}}^* = \frac{1}{\rho} \|\eta^N\| + \|\eta^D\| + \frac{1}{\rho^2} \|r\|_{\mathcal{L}_{2,h}}, \quad \hat{\eta} = \begin{pmatrix} \frac{1}{\rho} \eta^N \\ \eta^D \end{pmatrix}, \eta^N \in \mathbb{R}^k, \eta^D \in \mathbb{R}^{2m-k}.$$

Then Lemma 4.11 implies that the operators $\hat{\Lambda}_\rho : \ell_\infty^{[n_-, n_+ + 1]} \to S$ which are given by $\hat{\Lambda}_\rho = \begin{pmatrix} \hat{L}(s, \rho) \\ R(\rho) \end{pmatrix}$ where $\hat{L}(s, \rho)$, $R(\rho)$ are defined in (40), (36b), are nonsingular for $s \in \Omega_C^h \cup \Omega_\infty^h$ with a uniform bound for the inverse for $s \in \Omega_C^h$. Using (38), (41) we obtain for $z_n = (u_n, v_n)$

$$(M_n(s, \rho) - \hat{M}(s, \rho)) z_n = \begin{pmatrix} h^2 I \\ \frac{h}{\rho} I \end{pmatrix} \\ \times \left[(s(E_n^{+^{-1}} - A^{-1}) - C_n) u_n + \left(\frac{\rho}{h} (E_n^{+^{-1}} E_n^- - I) \right) v_n \right].$$

Combinining this with the error estimate

$$\frac{1}{\rho^2}\|(s(E_n^{+^{-1}} - A^{-1}) - C_n)u_n + (\frac{\rho}{h}(E_n^{+^{-1}} E_n^- - I))v_n\| \leqslant c(h + \frac{1}{\rho^2} + \frac{1}{\rho})\|z_n\|$$

implies for $\rho > C$

$$\left\| \begin{pmatrix} \tilde{L}(s,\rho) - \hat{L}(s,\rho) \\ 0 \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\eta} \end{pmatrix} \right\|_{\mathcal{L}_{2,h}}^{*} \leqslant c(h + \frac{1}{\rho})\|r\|_{\mathcal{L}_{2,h}}.$$

Taking h small and ρ large and using $\|E_n^{+^{-1}}\| \leqslant c$ we find that the system (37), (36b) has a unique solution for $s \in \Omega_C^h$ which can be estimated by (75). In a similar way we obtain the existence of a unique solution of (37),(36b) for $s \in \Omega_\infty^h$ which satisfies the estimate (76). □

Proof of Lemma 4.5. Lemma 4.5 follows directly from Lemma 4.13 using $\|\delta_- w\|_{\mathcal{L}_{2,h}} = \|\delta_+ w\|_{\mathcal{L}_{2,h}}$ which implies

$$\|w\|_{\mathcal{L}_{2,h}}^2 + \frac{1}{\rho^2}\|\delta_+ w\|_{\mathcal{L}_{2,h}}^2 \leqslant \text{const } (\|u\|_{\mathcal{L}_{2,h}}^2 + \|v\|_{\mathcal{L}_{2,h}}^2).$$

□

5. NUMERICAL EXAMPLES

5.1. Cubic Quintic Ginzburg Landau Equation

We choose the cubic quintic Ginzburg Landau equation [13, 16, 5]

$$u_t = au_{xx} + \delta u + g(u), \quad g(u) = \beta|u|^2 u + \gamma|u|^4 u, \;\; \delta \in \mathbb{R}, \;\; a, \beta, \gamma \in \mathbb{C}. \quad (77)$$

as a numerical example.

This equation shows a variety of coherent structures, like stable pulse solutions, fronts, sources, sinks. Moreover, there are parameter regimes where the behavior is intrinsically chaotic. For certain parameter values, this equation possesses stable rotating pulses and unstable pulses, as well as rotating and traveling fronts. Depending on the choice of initial conditions a different type of solution is selected. The real version of (77) which we use for numerical computations has the equivariance properties given in Example 1.4.

For the parameter set $a = 1$, $\delta = -0.1$, $\beta = 3 + i$, $\gamma = -2.75 + i$, which has been used in [13], we found numerically a stable pulse with rotational velocity $\mu_\rho \approx -1.30$ as well as a rotating front. Here we used a grid size $h = 0.1$ and Dirichlet boundary conditions for the pulse and Neumann

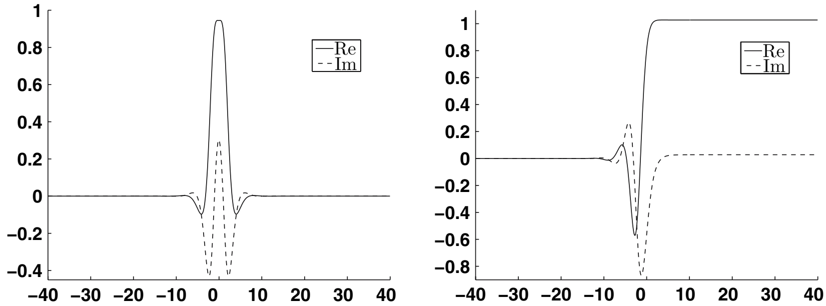

Figure 3. QCGL, pulse and front.

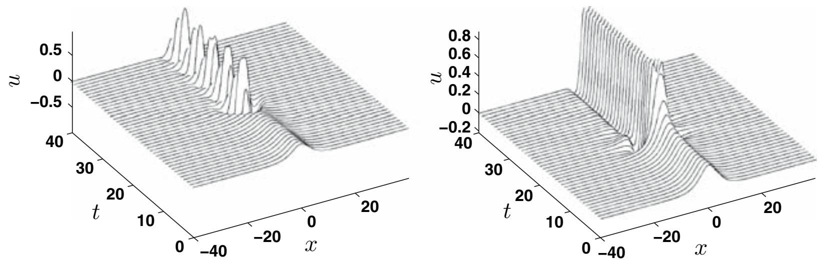

Figure 4. QCGL, rotating vs. frozen pulse.

boundary conditions for the front on the interval $[-40, 40]$. These solutions are depicted on Fig. 3.

The time evolution of the real part of the stable pulse is compared for the frozen and the rotating system in Fig. 4 on the interval $J = [-40, 40]$ with grid size $h = 0.1$ and Neumann boundary conditions.

After a transient phase until $t \approx 15$, the rotating pulse rotates with a fixed rotational velocity $\bar{\mu}_\rho$. In contrast, the frozen pulse is stabilized. The comparison of the rotating and traveling with the frozen front in Fig. 5 shows a similar situation. The frozen wave stabilizes quickly, whereas the non-frozen front continues to rotate and travels out of the computational domain at $t \approx 60$. As is shown in Fig. 6 the parameter μ_ρ converges to a fixed velocity $\bar{\mu}_\rho$ whereas the translational speed μ_τ stays at zero for the pulse and in case of the front the parameters μ_τ and μ_ρ converge to the same translational and rotational velocity that are observed in the non-frozen system. The rate of this convergence is displayed in Fig. 7, where the time evolution of the difference to the stationary solution of (9) is shown. The error $|\tilde{\mu}_* - \mu_*(t)|$ for $* \in \{\tau, \rho\}$ in the parameters μ_τ, μ_ρ is displayed as well as the error in the profile of the wave $\|\tilde{u} - u(t)\|_\infty$.

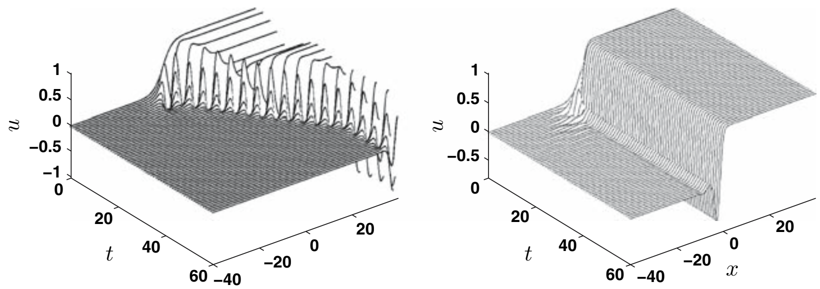

Figure 5. QCGL, rotating vs. frozen front.

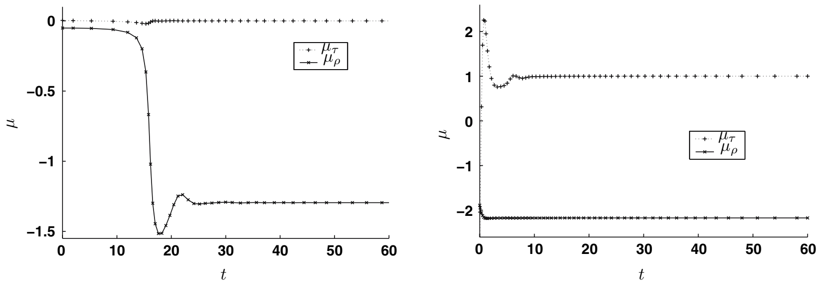

Figure 6. QCGL, time evolution of μ_ρ, μ_τ for pulse (left) and front (right).

Note that Theorem 2.8 is not applicable to the rotating front. In this case $R_{\frac{\pi}{2}}\bar{v}$ is not in $\mathcal{L}_2$ (cf. Example 1.4). Nevertheless, the numerical computations displayed in Fig. 7 suggest it to be true even in that case.

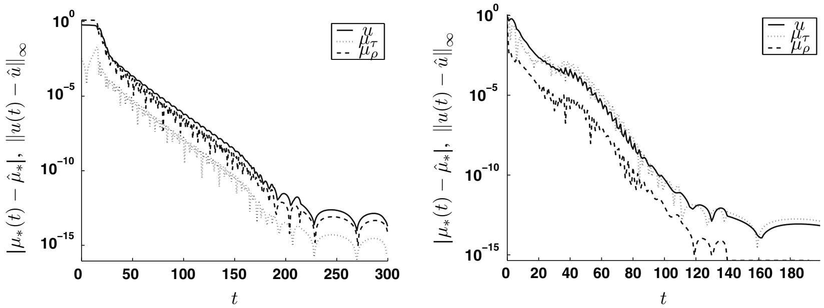

Figure 7. QCGL, time evolution of errors for pulse (left) and front (right).

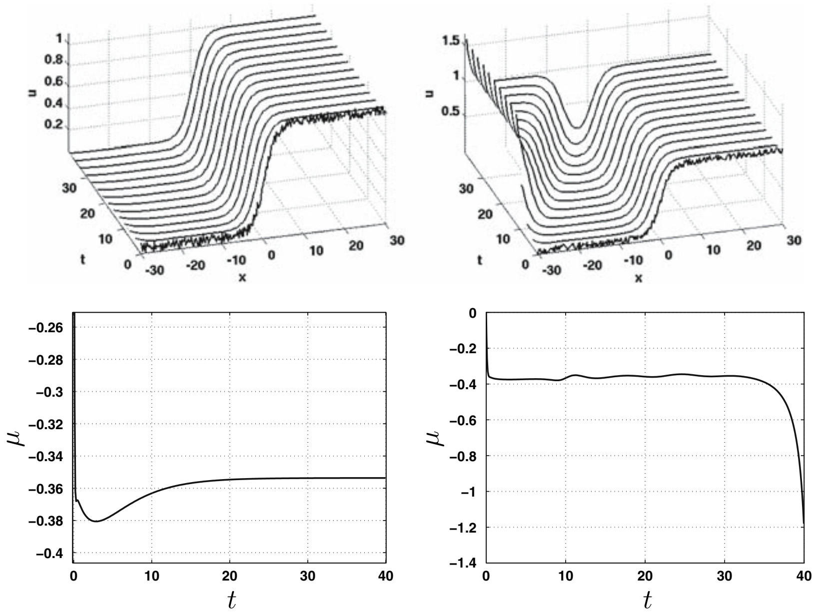

Figure 8. Nagumo, time evolution of u and μ for $\alpha = 0.2$ (left) vs. $\alpha = 0.3$ (right).

5.2. A Counterexample for the Nagumo Equation

We illustrate the necessity of the boundary condition 2.5 at the scalar Nagumo equation

$$u_t = u_{xx} + u(1-u)(u - \tfrac{1}{4}), \quad u(x,t) \in \mathbb{R}, \ x \in \mathbb{R}, \ t > 0. \tag{78}$$

An explicit traveling wave solution which connects the stationary points $u_- = 0$, $u_+ = 1$ is given by

$$\bar{v}(x) = \left(1 + e^{\frac{-x}{\sqrt{2}}}\right)^{-1}, \quad \bar{\lambda} = -\frac{\sqrt{2}}{4}. \tag{79}$$

For $a = 0.25$ we have $s(\alpha) > 0$ for approximately $\alpha > 0.26$. In Fig. 8 the time-evolution of the solution (v, μ) of the frozen PDAE is compared for values below and above this critical value of t. One can see clearly the effect of the instability created by the spurious unstable eigenvalue. This is not an effect of the freezing and the occurs in the same way for the non-frozen PDE.

6. APPENDIX

Lemma 6.1 (Summation by parts). *With the notation*

$$\langle u, v\rangle_{r,s} = h\sum_{n=r}^{s} u_n^H v_n, \qquad \|u\|_{r,s}^2 = \langle u, u\rangle_{r,s}$$

we have

$$\langle u, \delta_+ v\rangle_{r,s} = -\langle \delta_- u, v\rangle_{r+1,s+1} + u_{s+1}^H v_{s+1} - u_r^H v_r. \tag{80}$$

Proof of Lemma 3.3. Let $u \in \ell_{\mathrm{ess}}^J$ be given and set $v = (v_{n_- -1}, u_{n_-}, \dots, u_{n_+}, v_{n_+ +1})$. It remains to compute the external points $v_{n_- -1}, v_{n_+ +1}$ and μ from the equations (18b), (24) which read

$$\begin{aligned}
0 &= P_-^N v_{n_-} + Q_-^N \delta_0 v_{n_-} + P_+^N v_{n_+} + Q_+^N \delta_0 v_{n_+}\\
0 &= P_-^D(\tilde\Lambda v_{n_-} + \Phi_{n_-}\mu + r_{n_-}) + P_+^D(\tilde\Lambda v_{n_+} + \Phi_{n_+}\mu + r_{n_+})\\
0 &= \langle \Psi, \tilde\Lambda v\rangle_{J_h} + \langle \Psi, \Phi\rangle_{J_h}\mu + \langle \Psi, r\rangle_{J_h}.
\end{aligned}$$

We use the relation

$$\delta_+\delta_- v_n = \frac{2}{h}(\delta_0 v_n + \delta_- v_n) = \frac{2}{h}(-\delta_0 v_n + \delta_+ v_n) \tag{81}$$

as well as the definition of $\tilde\Lambda$ in (17a) to obtain the equivalent system for $w = (w_-, w_+) = (\delta_0 v_{n_-}, \delta_0 v_{n_+}) \in \mathbb{R}^{2m}$ and $\mu \in \mathbb{R}^p$

$$\mathcal{M}\begin{pmatrix} w\\ \mu\end{pmatrix} = \mathcal{R}^u u + \mathcal{R}^r r \tag{82}$$

where

$$\mathcal{M} = \begin{pmatrix} Q_-^N & Q_+^N & 0\\ -P_-^D(A - \frac{h}{2}B_{n_-}) & P_+^D(A + \frac{h}{2}B_{n_+}) & \frac{h}{2}(P_-^D\Phi_{n_-} + P_+^D\Phi_{n_+})\\ -\Psi_{n_-}^T(A - \frac{h}{2}B_{n_-}) & \Psi_{n_+}^T(A + \frac{h}{2}B_{n_+}) & \frac{1}{2}\langle \Psi, \Phi\rangle_{J_h}\end{pmatrix},$$

$$\mathcal{R}^u u = \begin{pmatrix} -P_-^N u_{n_-} - P_+^N u_{n_+}\\ -P_-^D A\delta_+ u_{n_-} - P_+^D A\delta_- u_{n_+} - \frac{h}{2}(P_-^D C_{n_-} u_{n_-} + P_+^D C_{n_+} u_{n_+})\\ -\Psi_{n_-}^T(A\delta_+ u_{n_-} + \frac{h}{2}C_{n_-}u_{n_-}) - \Psi_{n_+}^T(A\delta_- u_{n_+} + \frac{h}{2}C_{n_+}u_{n_+}) - \frac{h}{2}\sum_{n=n_-+1}^{n_+-1}\tilde\Psi_n^T\tilde\Lambda u_n\end{pmatrix},$$

$$\mathcal{R}^r r = -\frac{1}{2}\begin{pmatrix} 0\\ h(P_-^D r_{n_-} + P_+^D r_{n_+})\\ \langle \Psi, r\rangle_{J_h}\end{pmatrix}.$$

For $J_h \to \mathbb{R}$ the matrix $\mathcal{M}$ converges to

$$\hat{\mathcal{M}} = \begin{pmatrix} Q_-^N & Q_+^N & 0 \\ -P_-^D A & P_+^D A & 0 \\ -(S(\hat{v})(x_{n_-}))^T A & (S(\hat{v}(x_{n_+})))^T A & \frac{1}{2}\langle S(\hat{v}), S(\bar{v})\rangle_{\mathcal{L}_2} \end{pmatrix}$$

which is invertible due to condition (15) and the invertibility of the $p \times p$ matrix $\langle S(\hat{v}), S(\bar{v})\rangle$ which is ensured by Hypothesis 2.3. Therefore the solution $(\hat{w}, \hat{\mu})$ of $\hat{\mathcal{M}}(w, \mu)^T = \mathcal{R}^r r$ (i.e. $u \equiv 0$) can be estimated by

$$\|\hat{w}\| \leqslant \text{const } h(\|r_{n_-}\| + \|r_{n_+}\|) \leqslant \text{const } h\|r\|_\infty \tag{83}$$

and we obtain the same estimate for $w = (w_-, w_+)$ with a different constant. Together with the relations

$$v_{n_- - 1} = -2hw_- + u_{n_- + 1} = -2hw_-, \qquad v_{n_+ + 1} = 2hw_+ + u_{n_+ - 1} = 2hw_+$$

this implies

$$\|v_{n_- - 1}\| + \|v_{n_+ + 1}\| \leqslant \text{const } h\|w\| \leqslant \text{const } h^2\|r\|_\infty. \tag{84}$$

Furthermore, the relation

$$\delta_+ v_{n_+} = 2\delta_0 v_{n_+} - \delta_+ u_{n_+ - 1} = 2w_+, \quad \delta_+ v_{n_- - 1} = \delta_- v_{n_-} = 2w_- \tag{85}$$

leads for $u \equiv 0$ with (83) to

$$\|\delta_+ v\|_\infty \leqslant \text{const } h\|r\|_\infty. \tag{86}$$

Similarly by (81) we find

$$\begin{aligned} \delta_+\delta_- v_{n_-} &= \frac{2}{h}(-w_- + \delta_+ u_{n_-}) = -\frac{2}{h} w_-, \\ \delta_+\delta_- v_{n_+} &= \frac{2}{h}(w_+ - \delta_+ u_{n_+ - 1}) = \frac{2}{h} w_+, \end{aligned}$$

which implies with (83)

$$\|\delta_+\delta_- v\|_\infty \leqslant \text{const } \|r\|_\infty.$$

Together with (84),(86) this leads to (25). □

For the proof of Lemma 3.4 we use the uniform contraction principle in the following form.

Theorem 6.2 *Let X, Y be Banach spaces and $F : (X \times Y) \supset B_\varrho(0) \times B_\delta(0) \to Y$ be a continuous mapping, which satisfies the following estimates for $q \in [0, 1)$:*

$$\|F(x, y_1) - F(x, y_2)\| \leqslant q\|y_1 - y_2\| \quad \forall x \in B_\rho(0), \ \ y_1, y_2 \in B_\delta(0) \tag{87}$$

$$\|F(x, 0)\| \leqslant \delta(1-q) \quad \forall x \in B_\rho(0) \tag{88}$$

Then for each $x \in B_\rho(0)$ there exists a unique fixed point $\bar{y} = g(x)$ of $F(x, \cdot)$, i.e. $F(x, g(x)) = g(x)$ and the following estimate holds

$$\begin{aligned} \|y_1 - y_2\| \ &\leqslant \frac{1}{1-q}\|y_1 - F(x, y_1) - (y_2 - F(x, y_2))\| \\ &\forall x \in B_\rho(0), y_1, y_2 \in B_\delta(0). \end{aligned} \tag{89}$$

Note that (89) implies the continuity of g in $B_\rho(0)$, since

$$\begin{aligned} \|g(x_1) - g(x_2)\| &\leqslant \frac{1}{1-q}\|g(x_1) - F(x_1, g(x_1)) - (g(x_2) - F(x_1, g(x_2)))\| \\ &= \frac{1}{1-q}\|F(x_2, g(x_2)) - F(x_1, g(x_2))\|. \end{aligned} \tag{90}$$

Proof of Lemma 3.4. Let $u \in \ell_\infty^J$ be given and set $v = (v_{n_--1}, u_{n_-}, \ldots, u_{n_+}, v_{n_++1})$. It remains to compute the external points v_{n_--1}, v_{n_++1} and μ from the equations (18b), (19) which read

$$\begin{aligned} 0 &= P_-^N v_{n_-} + Q_-^N \delta_0 v_{n_-} + P_+^N v_{n_+} + Q_+^N \delta_0 v_{n_+} \\ 0 &= P_-^D(\tilde{\Lambda} v_{n_-} + \Phi_{n_-}\mu + \varphi_{n_-}(v, \mu)) + P_+^D(\tilde{\Lambda} v_{n_+} + \Phi_{n_+}\mu + \varphi_{n_+}(v, \mu)) \\ 0 &= \langle \Psi, \tilde{\Lambda} v + \Phi\mu + \varphi(v, \mu)\rangle_{J_h} \end{aligned} \tag{91}$$

Define the map $\chi : \ell_\infty^J \times \mathbb{R}^{2m} \to \ell_\infty^{\hat{J}}, (u, w) \mapsto v, \ w = (w_-, w_+)$ by

$$v_n = u_n, \ \ n = n_-, \ldots, n_+, \quad v_{n_--1} = -2hw_- + u_{n_-+1}, \quad v_{n_++1} = 2hw_+ + u_{n_+-1}.$$

Then $\delta_0 v_{n_\pm} = w_\pm$ and we obtain

$$\|\chi(u, w) - \chi(u, z)\|_{\mathcal{L}_{2,h}} \leqslant ch\sqrt{h}\|w - z\|.$$

The relation (85) leads to

$$\|\chi(u, w) - \chi(u, z)\|_{\mathcal{H}_h^1} \leqslant c\sqrt{h}\|w - z\|, \tag{92}$$

as well as

$$\|\chi(u,w)\|_{\mathcal{H}_h^1} \leqslant c(\|u\|_{\mathcal{H}_h^1} + h\|w\|). \tag{93}$$

In the same way as in the proof of Lemma 3.3 we obtain with (81) the following system which is equivalent to (91).

$$\mathcal{M}\begin{pmatrix} w \\ \mu \end{pmatrix} = \mathcal{R}^u u + g(u,w,\mu), \tag{94}$$

where $\mathcal{M}$, $\mathcal{R}^u$ are given by (82) and (cf. $\mathcal{R}^r$ in (82))

$$g(u,w,\mu) = -\frac{1}{2}\begin{pmatrix} 0 \\ h(P_-^D \varphi_{n_-}(\chi(u,w),\mu) + P_+^D \varphi_{n_+}(\chi(u,w),\mu)) \\ \langle \Psi, \varphi(\chi(u,w),\mu)\rangle_{J_h} \end{pmatrix}.$$

For $h < h_0 \pm hn_\pm > T$ the matrix $\mathcal{M}$ is nonsingular and we can define G: $\ell_\infty^J \times \mathbb{R}^{2m} \times \mathbb{R}^p \to \mathbb{R}^{2m} \times \mathbb{R}^p$ by

$$G(u,w,\mu) = \mathcal{M}^{-1}(\mathcal{R}^u u + g(u,w,\mu)),$$

the fixed point of which is a solution of (94). To apply the parametrized contraction mapping Theorem 6.2 we have to verify (87),(88). From (21),(93) we obtain

$$\|\varphi(\chi(u,0),0)\|_{\mathcal{L}_{2,h}} \leqslant c\rho\|\chi(u,0)\|_{\mathcal{H}_h^1} \leqslant c\rho\|u\|_{\mathcal{H}_h^1} \tag{95}$$

which implies

$$\sqrt{h}\|\varphi(\chi(u,0),0)\|_\infty \leqslant \|\varphi(\chi(u,0),0)\|_{\mathcal{L}_{2,h}} \leqslant c\rho\|u\|_{\mathcal{H}_h^1} \tag{96}$$

as well as with Cauchy Schwartz, Hypothesis 2.3 and (93)

$$\|\langle \Psi, \varphi(\chi(u,0),0)\rangle_{J_h}\| \leqslant c\|\chi(u,0)\|_{\mathcal{L}_{2,h}} \leqslant c\rho\|u\|_{\mathcal{H}_h^1}. \tag{97}$$

Using (20) we obtain with (92) and (93)

$$\begin{aligned} &\|\varphi(\chi(u,w),\mu) - \varphi(\chi(u,z),\lambda)\|_{\mathcal{L}_{2,h}} \\ &\quad \leqslant c(\|\chi(u,w) - \chi(u,z)\|_{\mathcal{H}_h^1} + \max(\|\chi(u,w)\|_{\mathcal{H}_h^1}, \|\chi(u,z)\|_{\mathcal{H}_h^1})\|\mu-\lambda\|) \\ &\quad \leqslant c(\sqrt{h}\|w-z\| + (\|u\|_{\mathcal{H}_h^1} + h\max(\|w\|,\|z\|))\|\mu-\lambda\|) \end{aligned} \tag{98}$$

Equation (98) leads for $\|u\|_{\mathcal{H}_h^1} < \rho$ to

$$\|\varphi(\chi(u,w),\mu) - \varphi(\chi(u,z),\lambda)\|_{\mathcal{L}_{2,h}} \leqslant c(\sqrt{h} + \rho + h\delta)(\|w-z\| + \|\mu-\lambda\|)$$

as well as for $\|u\|_{\mathcal{H}_h^1} \leqslant \sqrt{h}\|u\|_{1,\infty} < \sqrt{h}\rho$ to

$$\|\varphi(\chi(u,w),\mu) - \varphi(\chi(u,z),\lambda)\|_{\mathcal{L}_{2,h}} \leqslant c(\sqrt{h}(1+\rho+\delta)(\|w-z\| + \|\mu-\lambda\|).$$

Thus (95), (96), (97) imply for $\|u\|_{\mathcal{H}_h^1} \leqslant \rho$

$$\begin{aligned}\|g(u,0,0)\| &\leqslant h(\|\varphi_{n_-}(\chi(u,0),0)\| + \|\varphi_{n_+}(\chi(u,0),0)\| \\ &\quad + \|\langle \Psi, \varphi(\chi(u,0),0)\rangle_{J_h}\|) \leqslant c\rho\|u\|_{\mathcal{H}_h^1}\end{aligned} \tag{99}$$

as well as for $\|u\|_{1,\infty} \leqslant \rho$

$$\|g(u,0,0)\| \leqslant c\rho\|u\|_{1,\infty}. \tag{100}$$

Similarly, with (98) we find

$$\begin{aligned}\|g(u,w,\mu) - g(u,z,\lambda)\| &\leqslant c(h\|\varphi(\chi(u,w),\mu) - \varphi(\chi(u,z),\lambda)\|_\infty \\ &\quad + \|\langle \Psi, \varphi(\chi(u,w),\mu) - \varphi(\chi(u,z),\lambda)\rangle_{J_h}\|) \\ &\leqslant c\|\varphi(\chi(u,w),\mu) - \varphi(\chi(u,z),\lambda)\|_{\mathcal{L}_{2,h}}.\end{aligned} \tag{101}$$

It remains to estimate $\|\mathcal{R}^u u\|$: The summation by parts formula (80)

$$\begin{aligned}\langle \Psi, A\delta_-\delta_+ u\rangle_{n_-+1,n_+-1} &= -\langle \delta_+\Psi, A\delta_+ u\rangle_{n_-,n_+-2} + \Psi_{n_-}^T A(\delta_+ u)_{n_-} \\ &\quad - \Psi_{n_+-1}^T A(\delta_+ u)_{n_+-1}\end{aligned}$$

leads for $J_h = [n_- + 1, n_+ - 1]$ with

$$\langle \Psi, \tilde{\Lambda} u\rangle_{J_h} = \langle \Psi, A\delta_-\delta_+ u\rangle_{J_h} + \langle \Psi, B\delta_0 u\rangle_{J_h} + \langle \Psi, Cu\rangle_{J_h}$$

to

$$\|\langle \Psi_{|J_h}, \tilde{\Lambda} u\rangle_{J_h}\| \leqslant c\|u\|_{1,\infty}. \tag{102}$$

Using Hypothesis 2.3 for $\pm hn_\pm > T$ we find

$$\|\langle \Psi_{|J_h}, \tilde{\Lambda} u\rangle_{J_h}\| \leqslant c(\|u\|_{\mathcal{H}_h^1} + h^{-\frac{1}{2}} e^{-\alpha T}\|\delta_+ u\|_{\mathcal{L}_{2,h}}) \leqslant c(1 + h^{-\frac{1}{2}} e^{-\alpha T})\|u\|_{\mathcal{H}_h^1}.$$

This implies with the definition of $\mathcal{R}^u$ in (82) and (102)

$$\|\mathcal{R}^u u\| \leqslant c(\|u\|_{1,\infty} + \|\langle \Psi_{|J_h}, \tilde{\Lambda} u\rangle_{J_h}\|) \leqslant c\|u\|_{1,\infty}$$

as well as

$$\begin{aligned}\|\mathcal{R}^u u\| &\leqslant c(h^{-\frac{1}{2}} e^{-\alpha T}\|\delta_+ u\|_{\mathcal{L}_{2,h}} + \sqrt{h}\|u\|_{\mathcal{L}_{2,h}} + \|\langle \Psi_{|J_h}, \tilde{\Lambda} u\rangle_{J_h}\|) \\ &\leqslant c(1 + h^{-\frac{1}{2}} e^{-\alpha T})\|u\|_{\mathcal{H}_h^1}.\end{aligned}$$

For $\|u\|_{1,\infty} \leqslant \rho$ we obtain with (100)

$$\|G(u,0,0)\| \leqslant c(\|u\|_{1,\infty} + \|g(u,0,0)\|) \leqslant c(1+\rho)\|u\|_{1,\infty} \leqslant c_0\rho$$

and similarly, if $h^{-\frac{1}{2}}e^{-\alpha T} < c_2$ for $\|u\|_{\mathcal{H}_h^1} \leqslant \rho$ with (99)

$$\|G(u,0,0)\| \leqslant c(\|u\|_{\mathcal{H}_h^1} + \|g(u,0,0)\|) \leqslant c(1+\rho)\|u\|_{\mathcal{H}_h^1} \leqslant c_0\rho$$

For $(w,\mu),(z,\lambda) \in B_\delta(0) \subset \mathbb{R}^{2m+1}$ equation (101) leads for $\|u\|_{1,\infty} \leqslant \rho$ or $\|u\|_{\mathcal{H}_h^1} \leqslant \rho$ to

$$\|G(u,w,\mu) - G(u,z,\lambda)\| \leqslant c_1(\sqrt{h} + \rho + h\delta)(\|\mu - \lambda\| + \|w - z\|).$$

Choosing $h, \delta < 1$ so small that $\sqrt{h} + (\frac{1}{2c_0} + h)\delta < \frac{1}{c_1}$ and $\rho < \min(1, \frac{\delta}{2c_0})$ we can apply Theorem 6.2 with $q = \frac{1}{2}$. This yields a unique solution $(\bar{w},\bar{\mu}) \in B_\delta(0)$ of (94). Equation (90) implies with the continuity of G estimate (26a) which implies with $T_v(0)=0$, $T_\mu(0)=0$ (26b). $\square$

Proof of Corollary 3.5. Using the definition of $T_v(\cdot)$, $T_\mu(\cdot)$ and M_v, M_μ and subtracting (24) from (19) we obtain that $v^\Delta = T_v(u) - M_v u$, $\mu^\Delta = T_\mu(u) - M_\mu u$ solves $\pi v^\Delta = 0$ and

$$\begin{aligned}
0 &= \mathcal{B}^N v^\Delta\\
0 &= \mathcal{B}^D(\Lambda v^\Delta + \Phi\mu^\Delta + \varphi(T_v(u), T_\mu(u))),\\
0 &= \langle \Psi, \Lambda v^\Delta + \Phi\mu^\Delta + \varphi(T_v(u), T_\mu(u))\rangle_{J_h}.
\end{aligned}$$

Application of estimate (25) in Lemma 3.3 to (v^Δ, μ^Δ) leads to

$$\|T_v(u) - M_v u\|_{\mathcal{H}_h^2} + \|T_\mu(u) - M_\mu u\| \leqslant c\|\varphi(T_v(u), T_\mu(u))\|_{\mathcal{L}_{2,h}}.$$

Thus we have for $\tilde{\varphi}$ defined in (28) by (26b) and (21)

$$\begin{aligned}
\|\tilde{\varphi}(u)\|_{\mathcal{L}_{2,h}} &\leqslant \|\tilde{\Lambda}(T_v(u) - M_v u)\|_{\mathcal{L}_{2,h}} + \|\Phi(T_\mu(u) - M_\mu u)\|_{\mathcal{L}_{2,h}}\\
&\quad + \|\varphi(T_v(u), T_\mu(u))\|_{\mathcal{L}_{2,h}}\\
&\leqslant c\|\varphi(T_v(u), T_\mu(u))\|_{\mathcal{L}_{2,h}} \leqslant c\rho(\|T_v(u)\|_{\mathcal{L}_{2,h}} + \|T_\mu(u)\|)
\end{aligned}$$

which leads to

$$\|\tilde{\varphi}(u)\|_\infty \leqslant c\rho\|u\|_{1,\infty}$$

as well as for $h^{-\frac{1}{2}}e^{-\alpha T} < c_2$ to

$$\|\tilde{\varphi}(u)\|_{\mathcal{L}_{2,h}} \leqslant c\rho\|u\|_{\mathcal{H}_h^1}.$$

In the same way we obtain for $u_1, u_2 \in \ell^J_{\text{ess}}$ that $v^\Delta = T_v(u_1) - M_v u_1 - (T_v(u_2) - M_v u_2)$, $\mu^\Delta = T_\mu(u_1) - M_\mu u_1 - (T_\mu(u_2) - M_\mu u_2)$ solves $\pi v^\Delta = 0$ and

$$\begin{aligned}
0 &= \mathcal{B}^N v^\Delta \\
0 &= \mathcal{B}^D(\tilde{\Lambda} v^\Delta + \Phi\mu^\Delta + \varphi(T_v(u_1), T_\mu(u_1)) - \varphi(T_v(u_2), T_\mu(u_2))), \\
0 &= \langle \Psi, \tilde{\Lambda} v^\Delta + \Phi\mu^\Delta + \varphi(T_v(u_1), T_\mu(u_1)) - \varphi(T_v(u_2), T_\mu(u_2))\rangle_{J_h}.
\end{aligned}$$

Again, application of estimate (25) in Lemma 3.3 to (v^Δ, μ^Δ) implies

$$\begin{aligned}
&\|T_v(u_1) - M_v u_1 - (T_v(u_2) - M_v u_2)\|_{\mathcal{H}^2_h} + \|T_\mu(u_1) - M_\mu u_1 - (T_\mu(u_2) - M_\mu u_2)\| \\
&\qquad \leqslant c\|\varphi(T_v(u_1), T_\mu(u_1)) - \varphi(T_v(u_2), T_\mu(u_2))\|_{\mathcal{L}_{2,h}}.
\end{aligned}$$

Thus we obtain with (26a) and (20)

$$\begin{aligned}
\|\tilde{\varphi}(u_1) - \tilde{\varphi}(u_2)\|_{\mathcal{L}_{2,h}} &\leqslant \|\tilde{\Lambda}(T_v(u_1) - M_v u_1 - (T_v(u_2) - M_v u_2))\|_{\mathcal{L}_{2,h}} \\
&\quad + \|\Phi(T_\mu(u_1) - M_\mu u_1 - (T_\mu(u_2) - M_\mu u_2))\|_{\mathcal{L}_{2,h}} \\
&\quad + \|\varphi(T_v(u_1), T_\mu(u_1)) - \varphi(T_v(u_2), T_\mu(u_2))\|_{\mathcal{L}_{2,h}} \\
&\leqslant c\|\varphi(T_v(u_1), T_\mu(u_1)) - \varphi(T_v(u_2), T_\mu(u_2))\|_{\mathcal{L}_{2,h}} \\
&\leqslant c\|u_1 - u_2\|_{\mathcal{H}^1_h}.
\end{aligned}$$

□

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REFERENCES

1. Beyn, W.-J., and Lorenz, J. (1999). Stability of traveling waves: dichotomies and eigenvalue conditions on finite intervals. *Numer. Funct. Anal. Optim.* **20**(3–4), 201–244.
2. Beyn, W.-J., and Thümmler, V. (2004). Freezing solutions of equivariant evolution equations. *SIAM J. Appl. Dynam. Syst.* **3**(2), 85–116.
3. Chossat, P., and Lauterbach, R. (2000). *Methods in Equivariant Bifurcations and Dynamical Systems*, Vol. 15 of *Advanced Series in Nonlinear Dynamics*, World Scientific Publishing Co. Inc., River Edge, NJ.
4. Coppel, W. A. (1978). *Dichotomies in Stability Theory*, Vol. 629 of *Lecture notes in mathematics*. Springer-Verlag, Berlin, Heidelberg, New York.
5. de Mottoni, P., and Schatzman, M. (1993). Asymptotics of the Thual-Fauve pulse. In *Asymptotic and numerical methods for partial differential equations with critical parameters (Beaune, 1992)*, Vol. 384 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pp. 225–239. Kluwer Acad. Publ., Dordrecht.

6. Hairer, E., and Wanner, G. (1991). *Solving Ordinary Differential Equations. II*, Vol. 14 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1991. Stiff and differential-algebraic problems.
7. Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations*, Vol. 840 of *Lecture notes in mathematics*. Springer-Verlag, Berlin, Heidelberg, New York.
8. Marsden, J. E., and Ratiu, T. S. (1999). *Introduction to Mechanics and Symmetry*, Vol. 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition. A basic exposition of classical mechanical systems.
9. Miklavčič, M. (1998). *Applied Functional Analysis and Partial Differential Equations*. World Scientific Publishing Co. Inc., River Edge, NJ.
10. Palmer, K. J. (1988). Exponential dichotomies, the shadowing lemma and transversal homoclinic points. In *Dynamics Reported*, Vol. 1, pp. 265–306. B. G. Teubner, Stuttgart.
11. Rowley, C. W., Kevrekidis, I. G., Marsden, J. E., and Lust, K. (2003). Reduction and reconstruction for self-similar dynamical systems. *Nonlinearity* **16**(4), 1257–1275.
12. Rowley, C. W., and Marsden, J. E. (2000). Reconstruction equations and the Karhunen-Loève expansion for systems with symmetry. *Phys. D* **142**(1–2), 1–19.
13. Thual, O., and Fauve, S. (1988). Localized structures generated by subcritical instabilities. *J. Phys. France* **49**, 1829–1833.
14. Thümmler, V. (2005). *Numerical Analysis of the Method of Freezing Traveling Waves*. Ph.D thesis, Dept. of Mathematics, Bielefeld University.
15. Thümmler, V. (2006). *Numerical Approximation of Relative Equilibria for Equivariant PDEs*. Preprint no. 05-017 of the CRC 701, Bielefeld University.
16. van Saarloos, W., and Hohenberg, P. C. (1992). Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations. *Phys. D* **56**(4), 303–367.
17. Volpert, A. I., Volpert, V. A., and Volpert, V. A. (1994). *Traveling wave solutions of parabolic systems*, Vol. 140 of *Translations of mathematical monographs*. American Mathematical Society, Providence, Rhode Island.
18. Zou, Y., and Beyn, W.-J. (2003). On manifolds of connectings orbits in discretizations of dynamical systems. *Nonlinear Anal. TMA* **52**, 1499–1520.