Journal of Dynamics and Differential Equations, Vol. 20, No. 2, June 2008 (© 2007)
DOI: 10.1007/s10884-007-9080-3

Periodic Solutions of the Elliptic Isosceles Restricted Three-body Problem with Collision*

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Received October 26, 2006

The elliptic isosceles restricted three-body problem with collision, is a restricted three-body problem where the primaries move having consecutive elliptic collisions and the infinitesimal mass is moving in the plane perpendicular to the primaries motion that passes through the center of mass of the primary system. Our purpose in this paper is to prove the existence of many families of periodic solutions using Continuation's method, where the perturbing parameter is related with the energy of the primaries. This work is merely analytic and uses symmetry conditions and appropriate coordinates.

KEY WORDS: Restricted three-body problem; isosceles restricted three-body problem; periodic solutions; symmetries; continuation's method.

MATHEMATICAL SUBJECT CLASSIFICATIONS: 70F15; 37N05; 70F07.

1. INTRODUCTION

We consider a special case of the restricted three-body problem in the space, the *elliptic isosceles restricted three body with collision*. In this problem the motion of the primaries with equal mass, $m_1 = m_2 = 1/2$, is an elliptic one-dimensional solution of the Kepler's problem. This restricted problem consists of describing the motion of an infinitesimal particle, having initial conditions and velocities symmetric on the plane which is perpendicular to the primaries motion and passes through the center of mass

* Partially supported by Dirección de Investigación UBB, 064608 3/RS.
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of the primary system. The dynamics of the primaries is periodic and contains an infinite number of collisions. It represents a periodic forcing in the system causing it to be non-conservative. This problem is called elliptic isosceles problem with collision because the three bodies form an isosceles triangle at any time, and the primaries are in elliptic motion with collision in a line symmetric with respect to the plane that contains the massless.

Clearly, this problem is invariant under rotations about the axis that contains the primaries, and thus the angular momentum, c, is a first integral. In this paper, we will restrict our attention to the case $c \neq 0$, because the case $c = 0$ (one-dimensional) was studied in [1, 7] and in [2]. In [7] the phase space is compactified and the flow is analyzed on their boundary. Thus, the phase space is separated into different regions depending on the kind of orbits. On the other hand, in [1] the non-regularization of the triple collision is proved and in [2] it is showed the existence of a Bernoulli shift as a subsystem of the Poincaré map defined near a loop formed by two heteroclinic solutions associated to two periodic orbits at infinity and symbolic dynamics techniques are used. To the case $c \neq 0$ we find in the literature the work of Puel [11]. In this article, the study is made in a numerical point of view and some periodic solutions are obtained by numerical continuation method. Llibre and Pasca in [6] considered a "model" of isosceles restricted three-body problem with collision, but it does not represent a real restricted three-body problem. Here periodic solutions are obtained by Continuation's method.

In this paper, using analytic methods we prove the existence of several families of periodic solutions. We use essentially two kind of arguments, one of them is the Continuation method and the other is the use of a convenient variable, the so called "comets variables" introduced in [8], which are useful in order to study the infinity. To apply the Continuation method we need to introduce a parameter associated with the motion of the primaries, namely, the energy. After that, we took advantage of the symmetries of the system of differential equations that defines the elliptic isosceles restricted three body with collision. And finally, since the system of differential equations is not analytic as function of the parameter introduced, we need to use Arenstorf's Theorem (see details in [3]). We obtain periodic solutions of the first and second kinds, according to the definition in [14]. Similar arguments have been considered in [4] and [5] in the study of a new class of periodic orbits in the three-dimensional elliptic restricted three-body problem in the case of equal masses of the primaries. In these works, the orbit of each primary is elliptic without collision, and the parameter is obviously the eccentricity of the primaries orbit. Another important difference of this problem and our model, is that our problem is invariant under rotations around the vertical axis, which is not true in the other case. As a consequence of this fact, we can obtain first and second species orbits, while in [5], they only obtain orbits of first-species.

This paper is organized as follows. In Section 2, we review the one dimensional two body problem with null angular momentum. Here we set the notations and important preliminary results that will be used throughout the paper. Also, we introduce one parameter ϵ, such that its square is inversely proportional to the negative energy of the primaries. In Section 3, we introduce the problem in cartesian coordinates and rotating coordinates and some preliminary results are shown. Section 4 is dedicated to put the problem in a convenient way, that is, using Legendre's polynomial it is possible to write our research problem as a perturbation of one Kepler problem, considering the perturbed parameter associated with the energy of the primaries. It is observed that the dependence of this parameter is not differentiable, so the Poincaré Continuation Method is not applicable here. In Sections 5 and 6 we prove, respectively, the existence of a symmetric periodic solution as continuation of convenient circular and elliptic orbits of the Kepler problem for a discrete sequence of values of the parameter (i.e., the energy of the primaries). The main tool is Arenstorf's Theorem. A relevant property of the potential associated to this problem (Proposition 2) permits us to prove the existence of periodic solutions with symmetries of first and second kind for any value of the parameter or simply the negative energy of the primaries. The main difference between the first species and second species orbits, is that in this last situation we need to introduce the time as new variable in order to avoid degeneracy of the periodicity system. In Section 7, it is proved the existence of periodic solutions which are far from the primaries not necessarily with symmetries. Finally, in Section 8, we include in an appendix the Continuation's method (Arenstorf's Theorem) when the dependence on the small parameter of the perturbed system is not analytic, and also important results about the behavior of the perturbed solutions close to solutions of the unperturbed system are proved, because of our direct use of them. All the results obtained in this work are analytic and in order to make this manuscript self-contained and to facilitate the lecture to the reader we decide to include several details.

We conclude this introduction remarking that the study of periodic orbits of a non-integrable dynamical system is a very useful tool to obtain information on the topology of the phase space. In the vicinity of a periodic orbit, the study of the phase space can be reduced to the study of fixed points and the invariant curves of a Poincaré map on a surface of section, together with their stability character, determine critically the topology of the problem. For this reason the computation of periodic orbits plays an important role in the study of dynamical systems.

2. THE ELLIPTIC-COLLISION MOTION IN THE TWO-BODY PROBLEM

We decide to write this introductory section, because in the literature we did not find a good reference related to the study of the elliptic-collision orbits of the one-dimensional Kepler problem. Let z_i be the distance between the center of mass and the primary m_i, where $i=1,2$ (we will parametrize the masses such that $m_1=\mu$, $m_2=1-\mu$ with $\mu \in (0, 1/2]$ and let $\rho = z_1 - z_2$ be the distance between the primaries. Since the center of mass is at the origin it follows that $z_1=(1-\mu)\rho$ and $z_2=-\mu\rho$. In these coordinates the distance between the primaries satisfies the equation

$$\ddot{\rho} = -\frac{1}{\rho^2}, \tag{1}$$

and the energy integral for the "primaries" is

$$h = \frac{1}{2}\dot{\rho}^2 - \frac{1}{\rho}. \tag{2}$$

During all this work we will assume that the primaries describe an elliptic collision motion, i.e., we will assume that the energy of the primaries is $h<0$. This equation is singular at the collisions $z=0$ and this singularity can be removed through the introduction of a new time s defined by

$$\mathrm{d}t = \rho\, \mathrm{d}s. \tag{3}$$

Denoting $' = \frac{\mathrm{d}}{\mathrm{d}s}$ the Eq. (1) in the new time becomes $\rho\,\rho'' - {\rho'}^2 + \rho = 0$ and using the energy integral given in (2) we obtain

$$\rho'' - 2\,h\,\rho - 1 = 0. \tag{4}$$

The solution of this equation is

$$\rho(s) = \frac{1}{2\,h}\,[b \cos(\sqrt{-2\,h}\,s - s_0) - 1],$$

where b and s_0 are constants. Without loss of generality we will assume that $s_0=0$. Assuming that $\rho(0)=0$ we have that $b=1$. Therefore,

$$\rho(s) = -\frac{1}{2\,h}\left[1 - \cos(\sqrt{-2\,h}\,s)\right].$$

The angle $E=\sqrt{-2\,h}\,s$ is said to be eccentric anomaly. We define a parameter $\epsilon \geq 0$ through the relation

$$\epsilon^2 = -\frac{1}{2\,h}. \tag{5}$$

Using ϵ and the eccentric anomaly we obtain

$$\rho(E) = \epsilon^2 [1 - \cos E]. \tag{6}$$

Integrating the Eq. (3) we have Kepler's equation

$$l = n(t - \tau) = E - \sin E, \tag{7}$$

where $n = \epsilon^{-3}$ is called mean motion and l is called mean anomaly. We consider the time of pericenter passage $\tau = 0$. The Eq. (7) and the expression (6) give us the position of the primaries at a prescribed time t_1. The main problem here is the solution of Kepler's equation. Observe that if we simultaneously add or subtract any multiple of 2π both l and E in Eq. (7) unchanged. This means that given l, we can bring it into the range $-\pi \leqslant l \leqslant \pi$. So, it is sufficient to analyze the solution of Kepler's equation in this interval. Moreover, the equation is unchanged if l and E are simultaneously replaced by $-l$ and $-E$, respectively. This means that E is an odd function of l and it is enough to solve the equation when $0 \leqslant l \leqslant \pi$. When $l = 0$, $E = 0$ and when $l = \pi$, $E = \pi$. So, the problem is reduced to the range $0 < l < \pi$. Doing the graphic of l versus E, where $0 \leqslant l \leqslant \pi$, we see that the values of E also lie in the range $0 \leqslant E \leqslant \pi$. Thus, by the Inverse Function Theorem, there is a unique $E(l)$ such that $l = E - \sin E$, $0 < l < \pi$ holds. In this way, the function E depends on the variables t and ϵ. Then, we can write

$$E = E(l) \equiv E(t, \epsilon) := E_\epsilon(t) \tag{8}$$

and it follows that the Eq. (6) can be seen of the following manner

$$\rho(t) = \epsilon^2 [1 - \cos E_\epsilon(t)]. \tag{9}$$

Accordingly we will write $E(l)$ or $E(t/\epsilon^3)$ instead of $E_\epsilon(t)$. The following lemma will show important properties of the eccentric anomaly

Lemma 1. *The eccentric anomaly function E is such that*

1. $E(nt + \hat{k}\pi) = E(nt) + \hat{k}\pi$, *where* $\hat{k} \in 2\mathbb{Z}$.
2. $\dfrac{\partial E_\epsilon}{\partial t} = \dfrac{1}{\epsilon^3 [1 - \cos E_\epsilon(t)]}$.
3. $\dfrac{\partial E_\epsilon}{\partial \epsilon} = -\dfrac{3[E_\epsilon(t) - \sin E_\epsilon(t)]}{\epsilon [1 - \cos E_\epsilon(t)]}$.
4. $E(-l) = -E(l)$.

Proof. To prove item 1, we add $\hat{k}\pi$, to both E and l in the Eq. (7), where $\hat{k} \in 2\mathbb{Z}$. Thus, $l + \hat{k}\pi = E(l) + \hat{k}\pi - \sin(E(l) + \hat{k}\pi)$. On the other hand, we have that $l + \hat{k}\pi = E(l + \hat{k}\pi) - \sin E(l + \hat{k}\pi)$. But, by the Inverse

Function Theorem, there is a unique $E(l)$ such that $l = E - \sin E$. So, we conclude that $E(l + \hat{k}\pi) = E(l) + \hat{k}\pi$. Therefore, since $l = nt$ it follows that

$$E(nt + \hat{k}\pi) = E(nt) + \hat{k}\pi.$$

Differentiating both sides in Kepler's equation (7) with respect to the variable t and ϵ, it is obtained item 2 and 3. To prove item 4, again by Kepler's equation (7) follows that $-l = E(-l) - \sin E(-l)$, and, on the other hand, we can write $-l = -E(l) + \sin E(l) = -E(l) - \sin(-E(l))$. So, analogously to item 1, we conclude that $E(-l) = -E(l)$. □

Remark 1. Here, we point out the following important facts that will be used in the future.

(1) From Lemma 1 item 1 we have that the function $\rho(t)$, given by (9) is $2\pi\epsilon^3$ periodic in t. In fact, by Lemma 1 it follows that

$$\begin{aligned}\rho(t) &= \epsilon^2[1 - \cos E_\epsilon(nt)] = \epsilon^2[1 - \cos E_\epsilon(nt + 2\pi)] \\ &= \epsilon^2\left[1 - \cos E_\epsilon\left(n\left(t + \frac{2\pi}{n}\right)\right)\right] = \rho\left(t + \frac{2\pi}{n}\right).\end{aligned}$$

(2) Again by Lemma 1, we have that $E(\hat{k}\pi) = \hat{k}\pi$, where $\hat{k} \in 2\mathbb{Z}$. In fact, if $t = 0$, by Lemma 1 item 1, and since $E(0) = 0$, it follows that $E(\hat{k}\pi) = \hat{k}\pi$ with $\hat{k} \in 2\mathbb{Z}$. Now, we observe that if $t = k\pi\epsilon^3$, where $k \in 2\mathbb{Z}$ it obtained that $E_\epsilon(t) = E(t/\epsilon^3) = E(k\pi) = k\pi$. In particular, if we take $T = 2k\pi\epsilon^3$, it is verified that $E(T/2) = k\pi$ (where k is an even integer).

(3) Moreover, by (7) we see that the function $E_\epsilon(t)$ is not defined at $\epsilon = 0$ and so, neither is the function $\rho(t)$ given in (9). However, we can extend in a continuous way $\rho(t)$ at $\epsilon = 0$ defining precisely $\rho(t)$ at $\epsilon = 0$ by $\lim_{\epsilon \to 0} \rho(t) = 0$. Note that this procedure is possible, because $[1 - \cos E_\epsilon(t)]$ is a bounded function.

3. FORMULATION OF THE PROBLEM

We assume that the primaries with masses m_1 and m_2 are moving along the z-axis and that their center of mass is fixed at the origin. Let $\rho = z_1 - z_2$ be the distance between the primaries. In this case, it is necessary to assume that $m_1 = m_2 = 1/2$. Let (x, y) be the coordinates of the test particle in the plane perpendicular to the primaries motion (which are contained in the z-axis) (see Fig. 1).

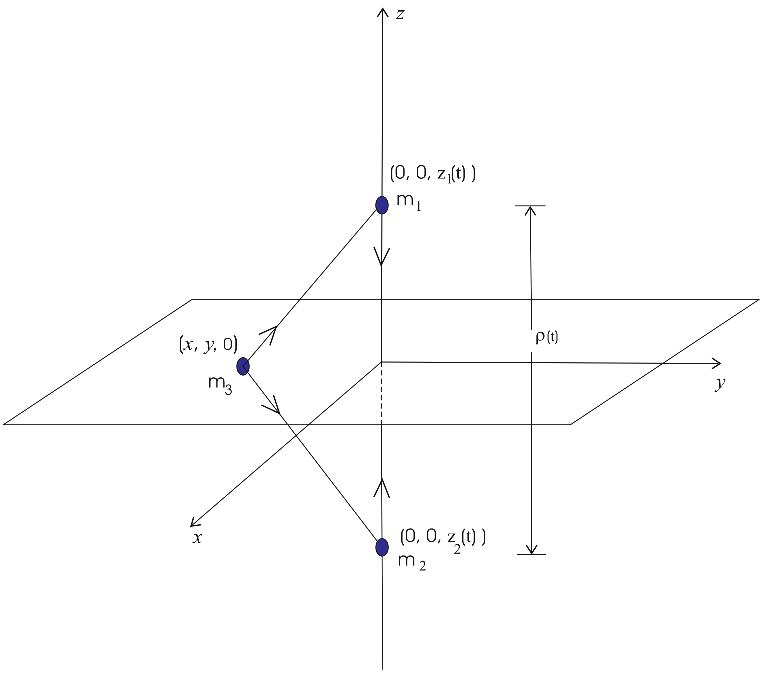

Figure 1. The elliptic isosceles restricted three body problem with collision.

With these coordinates, the equations of motion of the infinitesimal particle are

$$\begin{aligned} \frac{d^2x}{dt^2} &= V_x = -\frac{x}{\left(x^2+y^2+\frac{\rho^2(t)}{4}\right)^{3/2}}, \\ \frac{d^2y}{dt^2} &= V_y = -\frac{y}{\left(x^2+y^2+\frac{\rho^2(t)}{4}\right)^{3/2}}, \end{aligned} \tag{10}$$

because $z_1(t) = -z_2(t)$, where V is the potential function, which is given by

$$V = V(x, y; t) = \frac{1}{\sqrt{x^2 + y^2 + \frac{\rho^2(t)}{4}}}, \tag{11}$$

and V_x, V_y denote the partial derivatives of V with respect to x and y, respectively. The Hamiltonian function associated is

$$H = \frac{\dot{x}^2 + \dot{y}^2}{2} - V, \tag{12}$$

and thus, the system (10) can be written of the following way

$$\begin{aligned} \dot{x} &= p_x, \quad \dot{p_x} = -\frac{x}{\left(x^2+y^2+\frac{\rho^2(t)}{4}\right)^{3/2}}, \\ \dot{y} &= p_y, \quad \dot{p_y} = -\frac{y}{\left(x^2+y^2+\frac{\rho^2(t)}{4}\right)^{3/2}}, \end{aligned} \tag{13}$$

where $\rho(t) \equiv \rho(E(l(t))) := \rho(E_\epsilon(t)) = \epsilon^2[1 - \cos E_\epsilon(t)]$ as we see in the previous section. This system represents one Hamiltonian system with two and half degrees of freedom, because the periodicity in t (in fact, it is $2\pi\epsilon^3$ periodic). The Hamiltonian function is defined on the phase space

$$M = \{(x, y, p_x, p_y, t) \in \mathbb{R}^5 / (x, y, t) \neq (0, 0, 0 \mod 2\pi\epsilon^3))\},$$

the excised points from $\mathbb{R}^5$ correspond to singularity due to triple collision: $x = y = 0$ and $t = 0 \pmod{2\pi\epsilon^3}$.

Proposition 1. *The angular momentum c is constant along the solutions of the system (10).*

Proof. We know that c is given by $c = y\dot{x} - x\dot{y}$. So, we have $\frac{\mathrm{d}c}{\mathrm{d}t} = y\ddot{x} - x\ddot{y}$. Substituting the expressions of $\ddot{x}$ and $\ddot{y}$ given by (10), we obtain that $\frac{\mathrm{d}c}{\mathrm{d}t} = 0$. □

Since $\rho(t)$ depends on the parameter ϵ where $E_\epsilon(t)$ satisfies (7), we also can see the potential defined in (11) depending on (x, y, t, ϵ), i.e.,

$$V(\mathbf{q}, t, \epsilon) = -\frac{1}{\sqrt{\|\mathbf{q}\|^2 + \epsilon^4/4\ [1 - cos(E_\epsilon(t))]^2}}, \tag{14}$$

where $\mathbf{q} = (x, y)$. It follows immediately from definition of E and V that:

Proposition 2. *Assume that $\epsilon \neq 0$. Then, the following properties are true:*

1. $V(\epsilon^2\mathbf{q}, \epsilon^3\ t, \epsilon) = \epsilon^{-2}\ V(\mathbf{q}, t, 1)$,
2. $\nabla V(\epsilon^2\mathbf{q}, \epsilon^3\ t, \epsilon) = \epsilon^{-4}\ \nabla V(\mathbf{q}, t, 1)$,
3. $\mathbf{s}(t) = \epsilon^{-2}\mathbf{q}(\epsilon^3 t)$ *is a solution of* $\ddot{\mathbf{s}} = -\nabla V(\mathbf{s}, t, 1)$ *whether* $\mathbf{q}(t)$ *is the solution of* $\ddot{\mathbf{q}} = -\nabla V(\mathbf{q}, t, \epsilon)$ *and vice-versa, given* $\mathbf{s}(t)$ *a solution of* $\ddot{\mathbf{s}} = -\nabla V(\mathbf{s}, t, 1)$*, then* $\mathbf{q}(t) = \epsilon^2\mathbf{s}(t/\epsilon^3)$ *is solution of* $\ddot{\mathbf{q}} = -\nabla V(\mathbf{q}, t, \epsilon)$ *with* ϵ *arbitrary.*
4. *If the angular momentum of* $\mathbf{q}(t)$ *is c, then the angular momentum of* $\mathbf{s}(t)$ *is* $\tilde{c} = \epsilon^{-1}\ c$.

Proof. In order to prove the first item it is enough to observe that from (7), $E_\epsilon(\epsilon^3 t) = E_1(t)$, accordingly to the notation described in (8), since from (14)

$$V(\epsilon^2\mathbf{q}, \epsilon^3 t, \epsilon) = -\frac{1}{\epsilon^2}\frac{1}{\sqrt{\|\mathbf{q}\|^2 + 1/4\ [1 - \cos(E_\epsilon(\epsilon^3 t))]^2}}.$$

The second item is trivial. The third item is a direct consequence of the second item. The fourth item follows directly from definition of the angular momentum. □

For future purposes, it is important to write the isosceles symmetric problem with collision in rotating coordinates. Considering a rotating coordinate system $(\xi, \eta, p_\xi, p_\eta)$ with unitary angular velocity, which is a time symplectic transformation, the relation between the inertial systems and the synodical is given through the following manner:

$$\begin{pmatrix} \xi \\ \eta \\ p_\xi \\ p_\eta \end{pmatrix} = \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}. \tag{15}$$

The Hamiltonian function (12) in these new variables assumes the form

$$\hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}}, t, \epsilon) = \frac{\|\hat{\mathbf{p}}\|^2}{2} - \xi p_\eta + \eta p_\xi - \frac{1}{\sqrt{\|\hat{\mathbf{q}}\|^2 + \frac{\rho^2(t)}{4}}}, \tag{16}$$

where $\hat{\mathbf{q}} = (\xi, \eta)$ and $\hat{\mathbf{p}} = (p_\xi, p_\eta)$. Then, the Hamilton equations are

$$\begin{aligned} \dot{\xi} &= \eta + p_\xi, & \dot{p}_\xi &= p_\eta - \tfrac{\xi}{R^3}, \\ \dot{\eta} &= -\xi + p_\eta, & \dot{p}_\eta &= -p_\xi - \tfrac{\eta}{R^3}. \end{aligned} \tag{17}$$

3.1. Symmetries and Periodicity Conditions

The symmetries of the problem (13) are very useful to find symmetric periodic orbits, especially by means of the continuation method, as we will show in the next sections. The variable that determine the system (13) will be denoted by (x, y, p_x, p_y, t). Note that t determines E and vice-versa. Since, from Section 2, $E(-l) = -E(l)$, where $l = t/\epsilon^3$ and $\cos E(-l) = \cos E(l)$, it is easily verified that the equation of motion (10) possesses the following symmetry:

$$\begin{aligned}
I &: (x, y, p_x, p_y, t) \to (x, y, p_x, p_y, t)\\
S_1 &: (x, y, p_x, p_y, t) \to (x, -y, -p_x, p_y, -t)\\
S_2 &: (x, y, p_x, p_y, t) \to (-x, y, p_x, -p_y, -t)\\
S_3 &: (x, y, p_x, p_y, t) \to (x, y, -p_x, -p_y, -t)\\
S_4 &: (x, y, p_x, p_y, t) \to (-x, -y, p_x, p_y, -t)\\
S_5 &: (x, y, p_x, p_y, t) \to (x, -y, p_x, -p_y, t)\\
S_6 &: (x, y, p_x, p_y, t) \to (-x, y, -p_x, p_y, t)\\
S_7 &: (x, y, p_x, p_y, t) \to (-x, -y, -p_x, -p_y, t).
\end{aligned}$$

The above symmetries can be interpreted in the following way: let $\gamma(t)$ be a solution of the Hamiltonian system (13), then $S_i(\gamma(t))$ is another solution for $i \in \{1, 2, 3, 4, 5, 6, 7\}$. For $i \in \{1, 2, 3, 4, 5, 6, 7\}$ the orbit $\gamma(t)$ will be symmetric if and only if $S_i(\gamma(t)) = \gamma(t)$. Observe that the symmetries S_1 (time reverse) and S_5 correspond on the configuration space to a reflection with respect to the x-axis and the symmetries S_2 (time reverse) and S_6 correspond on the configuration space to a reflection with respect to the y-axis. We will use the anti-symplectic reflections symmetries S_1 and S_2 to obtain periodic orbits to our problem in the next sections. It is clear that the following Lagrangian subplanes on the space $\mathbb{R}^2 \times \mathbb{R}^2$ are invariant, more precisely, are a fixed set by the S_1 and S_2 symmetries, respectively:

$$L_1 = (x, 0, 0, p_y), \quad x, p_y \in \mathbb{R} \quad \text{and} \quad L_2 = (0, y, p_x, 0), \quad y, p_x \in \mathbb{R}.$$

Due to the symmetries, we can simplify the problem to find periodic orbits. In fact, some important properties of the symmetric orbits, whose proof is an immediate consequence of the Existence and Uniqueness Theorem for Ordinary Differential Equations, are expressed in the following proposition:

Proposition 3. *Let $\psi(t) = (x(t), y(t), p_x(t), p_y(t))$ be a solution of the system (13) and $E(t)$ the function defined by (7). Then:*

1. *If $(y(0), p_x(0)) = (0, 0)$, i.e., $\psi(0) \in L_1$ and $E(0) = 0$ and if $(y(T/2), p_x(T/2)) = (0, 0)$, i.e., $\psi(T/2) \in L_1$, where $(y(t), p_x(t)) \neq (0, 0), 0 < t < T/2$ and $E(T/2) = \hat{k}\pi$, where $\hat{k}$ is an even integer number, then $\psi(t)$ is a periodic solution of period T. These orbits are called S_1-symmetric periodic orbits of the elliptic isosceles restricted 3-body problem with collision.*
2. *If $(x(0), p_y(0)) = (0, 0)$, i.e., $\psi(0) \in L_2$ and $E(0) = 0$ and if $(x(T/2), p_y(T/2)) = (0, 0)$, i.e., $\psi(T/2) \in L_2$ where $(x(t), p_y(t)) \neq (0, 0), 0 < t < T/2$ and $E(T/2) = \hat{k}\pi$, then $\psi(t)$ is a periodic solution of period T. These orbits are called S_2-symmetric periodic orbits of the elliptic isosceles restricted 3-body problem with collision.*

3. *If* $(x(0), p_y(0)) = (0, 0)$*, i.e.,* $\psi(0) \in L_2$ *and* $E(0) = 0$ *and if* $(y(T/4), p_x(T/4)) = (0, 0)$*, i.e.,* $\psi(T/4) \in L_1$ *where* $(y(t), p_x(t)) \neq (0, 0), 0 < t < T/4$ *and* $E(T/4) = \hat{k}\pi$*, then* $\psi(t)$ *is a periodic solution of period* T*. These orbits are called doubly symmetric periodic orbits of the elliptic isosceles restricted 3-body problem with collision.*

Remark 2. Item 1 (resp. item 2) says that we need only to construct the half of one orbit that crosses the x-axis (resp. y-axis) orthogonally at two distinct points to get one symmetric periodic solution with respect to the x-axis (resp. y-axis). While, item 3 says that we need only to construct a quarter of one orbit that crosses the x-axis and the y-axis orthogonally to get one symmetric periodic solution with respect to the x-axis and y-axis.

Remark 3. The considerations about the function $E(t)$ are to control the position of the primaries when the infinitesimal particle pass through invariant subsets and, mainly, to satisfy the condition of commensurability of the periodic orbits. By Remark 1 item 2, it seen that to have $E(T/4) = \hat{k}\pi$, where $\hat{k}$ is an even integer, we must take $T = 4\hat{k}\pi\epsilon^3$. Analogously, to have $E(T/2) = \hat{k}\pi$, we must take $T = 2\hat{k}\pi\epsilon^3$. Observe that this choice (i.e., T) satisfies immediately the commensurability condition, in the sense that $\frac{T}{2\pi\epsilon^3}$ is a rational number.

In rotating coordinates, we will call $\hat{S}_1$ and $\hat{S}_2$ the symmetries corresponding to symmetries S_1 and S_2 in the inertial coordinates and in this way we will denote by $\hat{L}_1$ and $\hat{L}_2$ the invariant subsets corresponding to the invariant subsets in the inertial coordinates L_1 and L_2. So, for the system in rotating coordinates (17), it is obtained the following proposition, which is similar to Proposition 3:

Proposition 4. *Let* $\hat{\psi}(t) = (\xi(t), \eta(t), p_\xi(t), p_\eta(t))$ *be a solution of system (17) and* $E(t)$ *the function defined by (7). Then:*

1. *If* $(\eta(0), p_\xi(0)) = (0, 0)$*, i.e.,* $\hat{\psi}(0) \in \hat{L}_1$ *and* $E(0) = 0$ *and if* $(\eta(T/2), p_\xi(T/2)) = (0, 0)$*, i.e.,* $\hat{\psi}(T/2) \in \hat{L}_1$*, where* $(\eta(t), p_\xi(t)) \neq (0, 0), 0 < t < T/2$ *and* $E(T/2) = \hat{k}\pi$*, where* $\hat{k}$ *is an even integer number, then* $\hat{\psi}(t)$ *is a periodic solution of period* T*. These orbits are called* $\hat{S}_1$*-symmetric periodic orbits of the elliptic isosceles restricted 3-body problem with collision.*
2. *If* $(\xi(0), p_\eta(0)) = (0, 0)$*, i.e.,* $\hat{\psi}(0) \in \hat{L}_2$ *and* $E(0) = 0$ *and if* $(\xi(T/2), p_\eta(T/2)) = (0, 0)$*, i.e.,* $\hat{\psi}(T/2) \in \hat{L}_2$*, where* $(\xi(t), p_\eta(t)) \neq (0, 0), 0 < t < T/2$ *and* $E(T/2) = \hat{k}\pi$*, then* $\hat{\psi}(t)$ *is a periodic solution of period* T*. These orbits are called* $\hat{S}_2$*-symmetric periodic orbits of the elliptic isosceles restricted 3-body problem with collision.*

3. *If* $(\xi(0), p_\eta(0)) = (0,0)$, *i.e.,* $\hat{\psi}(0) \in \hat{L}_2$ *and* $E(0)=0$ *and if* $(\eta(T/4), p_\xi(T/4)) = (0,0)$, *i.e.,* $\hat{\psi}(T/4) \in \hat{L}_1$, *where* $(\eta(t), p_\xi(t)) \neq (0,0), 0 < t < T/4$ *and* $E(T/4) = \hat{k}\pi$, *then* $\hat{\psi}(t)$ *is a periodic solution of period* T. *These orbits are called doubly symmetric periodic orbits of the elliptic isosceles restricted 3-body problem with collision.*

4. THE PROBLEM IN A CONVENIENT WAY

The system (13) can be written as a perturbation of the Kepler problem, where the parameter of perturbation ϵ was introduced in Section 2 Eq. (5). But, since $E_\epsilon(t)$ (see Remark 1 item 3) is not defined for $\epsilon = 0$, it follows that neither is the Hamiltonian (12). In this way, the analysis is somewhat more delicate because the standard techniques, such as expansion in Taylor series, do not hold. To obtain an expansion of the system (13) in ϵ, we will use Legendre's polynomials (see more details in [13] pp. 102). We consider the following expression for the distance $R = \sqrt{x^2 + y^2 + \rho^2(t)/4}$

$$R = \|\mathbf{q}\| \sqrt{1 + w^2},$$

where $w = \dfrac{z_1}{\|\mathbf{q}\|} = \dfrac{\rho}{2\,\|\mathbf{q}\|}$. We assume that

$$\frac{\rho}{2\|\mathbf{q}\|} < 1, \tag{18}$$

so we can expand using Legendre's polynomials $1/R$ as power series in the variable w (for more details see [13]). In this way we obtain:

$$\frac{1}{R} = \frac{1}{\|\mathbf{q}\|} \left[1 + \sum_{j=1}^{\infty} P_j(0)\,(w)^j \right],$$

where $P_j(u)$ is jth-Legendre's polynomial. In particular, it is known that

$$P_0(u) = 1, \qquad P_3(u) = -\frac{3}{2}u + \frac{5}{2}u^3,$$

$$P_1(u) = u, \qquad P_4(u) = \frac{3}{8} - \frac{15}{4}u^2 + \frac{35}{8}u^4,$$

$$P_2(u) = -\frac{1}{2} + \frac{3}{2}u^2, \; P_5(u) = \frac{63}{2}u^5 - \frac{70}{8}u^3 + \frac{15}{8}u,$$

and the above series is convergent for $|w| < 1$ (see [13]). Using this expansion the Hamiltonian (12) becomes

$$H(\mathbf{q}, \mathbf{p}, t, \epsilon) = H_0(\mathbf{q}, \mathbf{p}) + \epsilon^4 H_1(\mathbf{q}, \mathbf{p}, t, \epsilon) + \epsilon^8 H_r(\mathbf{q}, \mathbf{p}, t, \epsilon), \tag{19}$$

where $\mathbf{q} = (x, y)$, $\mathbf{p} = (p_x, p_y)$ and

$$H_0(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\|\mathbf{p}\|^2 - \frac{1}{\|\mathbf{q}\|}, \tag{20}$$

$$H_1(\mathbf{q}, \mathbf{p}, t, \epsilon) = \frac{1}{8\|\mathbf{q}\|^3}[1 - \cos E_\epsilon(t)]^2, \tag{21}$$

$$H_r(\mathbf{q}, \mathbf{p}, t, \epsilon) = -\frac{3[1 - \cos E_\epsilon(t)]^4}{128\|\mathbf{q}\|^5} + \mathcal{O}(\epsilon^4). \tag{22}$$

Clearly H_0 represents the Hamiltonian of the Kepler problem in inertial coordinates. Similarly, the Hamiltonian function in rotating coordinates (16) becomes

$$\hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}}, t, \epsilon) = \hat{H}_0(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \epsilon^4 \hat{H}_1(\hat{\mathbf{q}}, \hat{\mathbf{p}}, t, \epsilon) + \epsilon^8 \hat{H}_r(\hat{\mathbf{q}}, \hat{\mathbf{p}}, t, \epsilon), \tag{23}$$

where

$$\hat{H}_0(\hat{\mathbf{q}}, \hat{\mathbf{p}}) = \frac{\|\hat{\mathbf{p}}\|^2}{2} - \xi p_\eta + \eta p_\xi - \frac{1}{\|\hat{\mathbf{q}}\|}, \tag{24}$$

$$\hat{H}_1(\hat{\mathbf{q}}, \hat{\mathbf{p}}, t, \epsilon) = \frac{1}{8\|\hat{\mathbf{q}}\|^3}[1 - \cos E_\epsilon(t)]^2, \tag{25}$$

$$\hat{H}_r(\hat{\mathbf{q}}, \hat{\mathbf{p}}, t, \epsilon) = -\frac{3[1 - \cos E_\epsilon(t)]^4}{128\|\hat{\mathbf{q}}\|^5} + \mathcal{O}(\epsilon^4). \tag{26}$$

Here $\hat{H}_0$ defines the Hamiltonian of the Kepler problem in rotating coordinates. Now, we can prove the following result:

Lemma 2. *The functions* $\epsilon H_1^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon)$ *and* $\epsilon H_r^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon)$ *can be defined in a continuous way at* $\epsilon = 0$.

Proof. The functions $H_1^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon)$ and $H_r^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon)$ are not defined at $\epsilon = 0$, because the function $E_\epsilon(t)$ is not, but they are bounded when ϵ tends to zero because of the term $\cos E_\epsilon(t)$. In this way, we can define $\epsilon H_1^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon)$ and $\epsilon H_r^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon)$ or extend them for all $\epsilon \geqslant 0$ in the following form

$$\epsilon H_1^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon) = \begin{cases} \epsilon H_1^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon) & \text{if } \epsilon \neq 0, \\ \lim_{\epsilon \to 0} \epsilon H_1^\mu(\mathbf{q}, \mathbf{p}, t, \epsilon) = 0 & \text{if } \epsilon = 0, \end{cases}$$

and

$$\epsilon H_r^\mu(\mathbf{q},\mathbf{p},t,\epsilon)=\begin{cases}\epsilon H_r^\mu(\mathbf{q},\mathbf{p},t,\epsilon) & \text{if } \epsilon\neq 0,\\ \lim_{\epsilon\to 0}\epsilon H_r^\mu(\mathbf{q},\mathbf{p},t,\epsilon)=0 & \text{if } \epsilon=0.\end{cases}$$

□

Similarly the same result holds for the functions $\epsilon\hat{H}_1^\mu(\hat{\mathbf{q}},\hat{\mathbf{p}},t,\epsilon)$ and $\epsilon\hat{H}_r^\mu(\hat{\mathbf{q}},\hat{\mathbf{p}},t,\epsilon)$.

Remark 4. By Lemma 2, the functions $\epsilon^4 H_1^\mu(\mathbf{q},\mathbf{p},t,\epsilon)$ and $\epsilon^4 H_r^\mu(\mathbf{q},\mathbf{p},t,\epsilon)$ are continuous with respect to $\epsilon\geqslant 0$ and it is easy to see that these functions are continuous with respect to $(\mathbf{q},\mathbf{p},t)$ for all $\mathbf{q}\neq\mathbf{0}$, $\mathbf{p}$ and t.

4.1. The Problem in Delaunay–Poincaré Variables and Delaunay Variables

In order to continue Kepler's periodic orbits (the case $\epsilon=0$) to the case $\epsilon>0$, we need to study, the equations of motion given by the Hamiltonian function (19) in convenient variables. Here we will use the Delaunay–Poincaré and Delaunay variables (see details in [14]) which are symplectic transformations. Firstly, we introduce the Delaunay–Poincaré variables with the following choice

$$\begin{aligned} Q_1&=l+g, & P_1&=L,\\ Q_2&=-\sqrt{2(L-G)}\sin(g), & P_2&=\sqrt{2(L-G)}\cos(g), \end{aligned}\tag{27}$$

where $L=\sqrt{a}, a$ is the semimajor axis of the infinitesimal mass, G its angular momentum, l is the mean anomaly measured from pericenter, g is the argument of the pericenter. These variables are valid on circular orbits which occur at $L=G$. The circular orbits with $L=G$ correspond to $Q_2=P_2=0$. The Hamiltonian function (19) (inertial coordinates) in the variables (27) becomes

$$H(\mathbf{Q},\mathbf{P},t,\epsilon)=H_0(\mathbf{Q},\mathbf{P})+\epsilon^4 H_1(\mathbf{Q},\mathbf{P},t,\epsilon)+\epsilon^8 H_r(\mathbf{Q},\mathbf{P},t,\epsilon),\tag{28}$$

where $\mathbf{Q}=(Q_1,Q_2), \mathbf{P}=(P_1,P_2)$ and

$$H_0(\mathbf{Q},\mathbf{P})=-\frac{1}{2P_1^2},\tag{29}$$

$$H_1(\mathbf{Q},\mathbf{P},t,\epsilon)=\frac{1}{8\|\mathbf{q}\|^3}[1-\cos E_\epsilon(t)]^2,\tag{30}$$

$$H_r(\mathbf{Q}, \mathbf{P}, t, \epsilon) = -\frac{3[1 - \cos E_\epsilon(t)]^4}{128\|\mathbf{q}\|^5} + \mathcal{O}(\epsilon^4), \tag{31}$$

where $\mathbf{q}$ must be though as $\mathbf{q}$ in the new variable $\mathbf{Q}$ and $\mathbf{P}$. Also, it is verified that the Hamiltonian function (23) (synodical coordinates) in Delaunay–Poincaré variables described by (27) is

$$\hat{H}(\mathbf{Q}, \mathbf{P}, t, \epsilon) = \hat{H}_0(\mathbf{Q}, \mathbf{P}) + \epsilon^4 \hat{H}_1(\mathbf{Q}, \mathbf{P}, t, \epsilon) + \epsilon^8 \hat{H}_r(\mathbf{Q}, \mathbf{P}, t, \epsilon) \tag{32}$$

where

$$\hat{H}_0(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2P_1^2} - P_1 + \frac{Q_2^2 + P_2^2}{2}, \tag{33}$$

$$\hat{H}_1(\mathbf{Q}, \mathbf{P}, t, \epsilon) = \frac{1}{8\|\hat{\mathbf{q}}\|^3}[1 - \cos E_\epsilon(t)]^2, \tag{34}$$

and

$$\hat{H}_r(\mathbf{Q}, \mathbf{P}, t, \epsilon) = -\frac{3[1 - \cos E_\epsilon(t)]^4}{128\|\hat{\mathbf{q}}\|^5} + \mathcal{O}(\epsilon^4), \tag{35}$$

where as before $\hat{\mathbf{q}}$ denotes $\hat{\mathbf{q}}$ in the new variables.

Now we introduce another second set of variables which are not defined on circular orbits, but are very convenient to characterize elliptic orbits. These variables are given by the Delaunay elements (l, g, L, G)

$$\begin{aligned} \overline{Q}_1 &= l, & \overline{P}_1 &= L, \\ \overline{Q}_2 &= g, & \overline{P}_2 &= G, \end{aligned} \tag{36}$$

where l, g, L, G were previously defined. These coordinates, which are called Delaunay coordinates, are defined on the elliptic domain of the Kepler problem. The elliptic domain is the open set on $\mathbb{R}^4$ which is filled with elliptic solutions of the Kepler problem. In this way, these coordinates are not valid in a neighborhood of the circular orbits of the Kepler's problem. Now, we point out to the Hamiltonian function (23) (synodical coordinates) in the variables described in (36), which is

$$\hat{H}(\mathbf{Q}, \mathbf{P}, t, \epsilon) = \hat{H}_0(\mathbf{Q}, \mathbf{P}) + \epsilon^4 \hat{H}_1(\mathbf{Q}, \mathbf{P}, t, \epsilon) + \epsilon^8 \hat{H}_r(\mathbf{Q}, \mathbf{P}, t, \epsilon), \tag{37}$$

where $\mathbf{Q} = (\overline{Q}_1, \overline{Q}_2)$, $\mathbf{P} = (\overline{P}_1, \overline{P}_2)$ and

$$\hat{H}_0(\mathbf{Q}, \mathbf{P}) = -\frac{1}{2\overline{P}_1^2} - \overline{P}_2, \tag{38}$$

$$\hat{H}_1(\mathbf{Q},\mathbf{P},t,\epsilon)=\frac{1}{8\|\hat{\mathbf{q}}\|^3}[1-\cos E_\epsilon(t)]^2, \tag{39}$$

and

$$\hat{H}_r(\mathbf{Q},\mathbf{P},t,\epsilon)=-\frac{3[1-\cos E_\epsilon(t)]^4}{128\|\hat{\mathbf{q}}\|^5}+\mathcal{O}(\epsilon^4), \tag{40}$$

where we denote $\hat{\mathbf{q}}(\mathbf{Q},\mathbf{P})$ by $\hat{\mathbf{q}}$.

The Delaunay–Poincaré and Delaunay variables make easier the localization of the infinitesimal particle in the space. So, in the below propositions we give sufficient conditions in order that the test particle lies in the invariant subset L_1 or L_2 and in the invariant subsets $\hat{L}_1$ or $\hat{L}_2$.

Proposition 5. *Let* $\mathbf{Q}=(Q_1,Q_2)$, $\mathbf{P}=(P_1,P_2)$ *be the Delaunay–Poincaré variables defined in (27). The subset of* $\mathbb{R}^4$*:*

1. $\mathcal{L}_1: Q_1=0\,(mod\,\pi),\quad Q_2=0,$ *gives us points in the invariant subset* L_1 *or* $\hat{L}_1$.
2. $\mathcal{L}_2: Q_1=\frac{\pi}{2}\,(mod\,\pi),\quad P_2=0,$ *gives us points in the invariant subset* L_2 *or* $\hat{L}_2$.

Analogously,

Proposition 6. *Let* $\mathbf{Q}=(\overline{Q}_1,\overline{Q}_2)$, $\mathbf{P}=(\overline{P}_1,\overline{P}_2)$ *be the Delaunay variables defined in (36). The subset of* $\mathbb{R}^4$*:*

1. $\hat{\mathcal{L}}_1: \overline{Q}_1=0\,(mod\,\pi),\quad \overline{Q}_2=0\,(mod\,\pi),$ *gives us points in the invariant subset* L_1 *or* $\hat{L}_1$.
2. $\hat{\mathcal{L}}_2: \overline{Q}_1=0\,(mod\,\pi),\quad \overline{Q}_2=\frac{\pi}{2}\,(mod\,\pi),$ *gives us points in the invariant subset* L_2 *or* $\hat{L}_2$.

5. PERIODIC SOLUTIONS OF FIRST KIND

In this section we will prove the existence of periodic doubly-symmetric (S_1 and S_2 symmetric) solutions; S_2 symmetric solutions and S_1 symmetric solutions as continuation of convenient circular orbits of the Kepler problem via Arenstorf's Theorem 8.1 (see Appendix) for suitable values of the parameter ϵ.

5.1. Doubly-symmetric Periodic Solutions as Continuation of Circular Kepler's Orbits

In this case we consider the elliptic isosceles restricted three-body problem with collision in inertial coordinates and then we write the problem in Delaunay–Poincaré variables (27) (i.e., the system associated with Hamiltonian function (28)).

The equations of motion (13) in Delaunay–Poincaré variables given by (27) can be written as

$$\dot{\mathbf{Z}} = \mathbf{F}(\mathbf{Z}), \tag{41}$$

where $\mathbf{Z} = (Q_1, Q_2, P_1, P_2)$, $\mathbf{F}(\mathbf{Z}) = \mathbf{F}_0(\mathbf{Z}) + \epsilon^4 \mathbf{F}_1(\mathbf{Z}, t, \epsilon) + \epsilon^8 \mathbf{F}_r(\mathbf{Z}, t, \epsilon)$.

$$\mathbf{F}_0(\mathbf{Z}) = (P_1^{-3}, 0, 0, 0), \tag{42}$$

$$\mathbf{F}_1(\mathbf{Z}, t, \epsilon) = \left(\frac{\partial H_1}{\partial P_1}, \frac{\partial H_1}{\partial P_2}, -\frac{\partial H_1}{\partial Q_1}, -\frac{\partial H_1}{\partial Q_2} \right), \tag{43}$$

with $H_1(\mathbf{Q}, \mathbf{P}, t, \epsilon)$ given by (30) and finally,

$$\mathbf{F}_r(\mathbf{Z}, t, \epsilon) = \left(\frac{\partial H_r}{\partial P_1}, \frac{\partial H_r}{\partial P_2}, -\frac{\partial H_r}{\partial Q_1}, -\frac{\partial H_r}{\partial Q_2} \right), \tag{44}$$

with $H_r(\mathbf{Q}, \mathbf{P}, t, \epsilon)$ given by (31). Note that the system (41) is non-autonomous and $2\pi\epsilon^3$-periodic in t, and it is defined on the phase space

$$\Omega = \left\{ (\mathbf{Z}, t, \epsilon) \in \mathbb{R}^4 \times \mathbb{R} \times \mathbb{R} \ / \ P_1 \neq 0, \mathbf{q} \neq \mathbf{0} \right\}.$$

Now we will characterize doubly-symmetric orbits of the Kepler problem (42), whose equations of motion are

$$\begin{aligned} \dot{Q}_1 &= P_1^{-3}, & \dot{P}_1 &= 0, \\ \dot{Q}_2 &= 0, & \dot{P}_2 &= 0. \end{aligned} \tag{45}$$

The solutions of this system are given by

$$\begin{aligned} Q_1^{(0)}(t) &= P_{10}^{-3} t + Q_{10}, & P_1^{(0)}(t) &= P_{10}, \\ Q_2^{(0)}(t) &= Q_{20}, & P_2^{(0)}(t) &= P_{20}, \end{aligned} \tag{46}$$

for initial conditions $(Q_{10}, Q_{20}, P_{10}, P_{20})$ at $t = 0$. Now, we consider solutions of the Kepler problem (45) with initial conditions $\mathbf{z}_0^* \in \mathcal{L}_2$ when $t = 0$ and $E(0) = 0$, with

$$\mathbf{z}_0^* = \left(\left(m + \frac{1}{2} \right) \pi, Q_2^*, P_1^*, 0 \right),$$

where m is an integer that we will choose, without loss of generality, either 0 or 1, and P_1^*, Q_2^* are constants to be determined. These solutions denoted by $\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*)$ are of the form:

$$\begin{aligned} Q_1^{(0)}(t, \mathbf{z}_0^*) &= (P_1^*)^{-3} t + (m + \tfrac{1}{2})\pi, & P_1^{(0)}(t) &= P_1^*, \\ Q_2^{(0)}(t, \mathbf{z}_0^*) &= Q_2^*, & P_2^{(0)}(t) &= 0. \end{aligned}$$

By Proposition 3 item 3 and Proposition 5 item (i) we can obtain doubly symmetric orbit to the Kepler's problem (45), if we may solve the set of two equations at time $t = T/4$ in three unknowns:

$$Q_1^{(0)}\left(\frac{T}{4}, \mathbf{z}_0^*\right) = (P_1^*)^{-3}\frac{T}{4} + \left(m + \frac{1}{2}\right)\pi = (m + \tilde{m} + 1)\pi, \quad Q_2^{(0)}\left(\frac{T}{4}, \mathbf{z}_0^*\right) = Q_2^* = 0,$$

where $\tilde{m}$ is an integer. The second equation is satisfied by $Q_2^* = 0$ (note that with this choice it will be a circular orbit). Taking $P_1^{-3} = s$, where s is positive real constant, we have from the first equation that

$$T = 4\frac{\tilde{m} + \frac{1}{2}}{s}\pi. \tag{47}$$

On the other hand, by Remark 3, the conditions $E(0) = 0$ and $E(T/4) = \hat{k}\pi$, where $\hat{k}$ is a convenient even integer and in this way we must take

$$T = 4\hat{k}\pi\epsilon^3. \tag{48}$$

So, combining (47) and (48) we have $\hat{k} = (\tilde{m} + 1/2)ks^{-1}$, when we have considered $\epsilon^3 = 1/k$, $k, \tilde{m}$ a positive integers and s a positive real constant. Since $\hat{k}$ is an even positive integer, we will verify this condition assuming that for example, k is an even positive integer and $(\tilde{m} + 1/2)s^{-1}$ is a positive integer or either $(\tilde{m} + 1/2)s^{-1}$ is an even positive integer. To facilitate our approach we will choose the second option. In this way the value of s will be

$$s = \frac{\tilde{m} + 1/2}{2\tilde{s}},$$

where $\tilde{s} \in \mathbb{N}$. So, we have $\mathbf{Z}^{(0)}(T/4, \mathbf{z}_0^*) = ((m + \tilde{m} + 1)\pi,\ 0,\ (\frac{\tilde{m}+1/2}{2\tilde{s}})^{-1/3}, 0)$ which is in $\mathcal{L}_1$ and $E(T/4) = \hat{k}\pi$, where $\hat{k} = 2\tilde{s}k$. Therefore, the solution $\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*)$ is a doubly-symmetric circular orbit with radius $(\frac{\tilde{m}+1/2}{2\tilde{s}})^{-2/3}$ and period $T = 8\pi\tilde{s}$ on the plane xy.

Let $\mathcal{C} \subset \mathbb{R}^4$ be a compact neighborhood of $\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*)$ without singularities and $\mathbf{z}_0^*$ such that $\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*)$ remains bounded and bounded away from the singularities. In this way, we describe the following lemma:

Lemma 3. *The functions* $\mathbf{F}_0(\mathbf{Z})$, $\epsilon^4 \mathbf{F}_1(\mathbf{Z}, t, \epsilon)$ *and* $\epsilon^8 \mathbf{F}_r(\mathbf{Z}, t, \epsilon)$, *given in (41), together with all their derivatives with respect to* $\mathbf{Z}$ *are continuous at* $\mathcal{C}$.

Proof. It is easy to see that the function $\mathbf{F}_0$ is continuous on $\mathcal{C}$. The functions $\epsilon^4 \mathbf{F}_1$ and $\epsilon^8 \mathbf{F}_r$ are defined for all $(\mathbf{Z}, t, \epsilon) \in \Omega$ and are continuous on $\mathcal{C}$, because are rational functions in $\mathbf{Z}$ and the term $\|\mathbf{q}\|$ is a function continuous in $\mathcal{C}$ (see Lemma 7 in Appendix). With relation to the variables ϵ to $\epsilon \neq 0$ and t, these functions are clearly continuous. For see this, it is sufficient to observe that the term $\cos E_\epsilon(t)$ is a continuous function in t and in ϵ, for $\epsilon \neq 0$. Note that for $\epsilon = 0$, by Remark 4, we define the functions $\epsilon^4 \mathbf{F}_1$ and $\epsilon^8 \mathbf{F}_r$ at $\epsilon = 0$ in such a way that they are continuous at $\epsilon = 0$. So, we have that $\mathbf{F}_0, \epsilon^4 \mathbf{F}_1$ and $\epsilon^8 \mathbf{F}_r$ are continuous in $\mathcal{C}$. With relation to the derivatives of these functions with respect to $\mathbf{Z}$, since $\mathbf{F}_0, \epsilon^4\mathbf{F}_1$ and $\epsilon^8\mathbf{F}_r$ are rational functions with respect to $\mathbf{Z}$ and the term $\|\mathbf{q}\|$ have all derivatives continuous (see Lemma 7), it follows that there are the derivatives of $\mathbf{F}_0, \epsilon^4\mathbf{F}_1$ and $\epsilon^8\mathbf{F}_r$ with respect to $\mathbf{Z}$ in $\mathcal{C}$ in any order and are continuous. □

As a consequence by the above lemma we have that given the solution $\mathbf{Z}(t, z_0^*)$ of the system $\dot{\mathbf{Z}} = \mathbf{F}_0(\mathbf{Z})$ and a compact neighborhood of this solution $\mathcal{C}$, we have that $\mathbf{F}_0, \epsilon^4\mathbf{F}_1$ and $\epsilon^8\mathbf{F}_r$ together with all their derivatives with respect to $\mathbf{Z}$ are bounded at $\mathcal{C}$. In particular, since $\mathbf{F}_0$ is in C^1, it follows that $\mathbf{F}_0$ is Lipschitz in $\mathcal{C}$ (we can restrict the compact neighborhood if necessary). In this way, we can use the estimates obtained in Lemmas 5 and 6 described in the Appendix.

Now, we will proceed to study the perturbed system. We look for initial conditions in a neighborhood of $\mathbf{z}_0^*$, of the type

$$\mathbf{z}_0 = \left(\left(m + \frac{1}{2} \right) \pi,\ \triangle Q_2, s^{-1/3} + \triangle P_1, 0 \right),$$

where m can be either 0 or 1 and $s = \frac{\tilde{m}+1/2}{2\tilde{s}}$ with $\tilde{m}, \tilde{s} \in \mathbb{N}$, in such a way that a solution $\mathbf{Z}(t, \mathbf{z}_0)$ of (41), with $\epsilon \neq 0$ small enough, let be a doubly symmetric orbit. By Lemma 5 of the Appendix, we have that the solution $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ of (41) is given by

$$\begin{aligned}
Q_1(t,\mathbf{z}_0,\epsilon) &= \left(((\tilde{m}+1/2)/2\tilde{s})^{-1/3}+\triangle P_1\right)^{-3}t+\left(m+\tfrac{1}{2}\right)\pi \\
&\qquad\qquad\qquad\qquad\qquad\qquad +\epsilon^4 Q_1^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8)\\
Q_2(t,\mathbf{z}_0,\epsilon) &= \triangle Q_2 \qquad\qquad\qquad\qquad\quad +\epsilon^4 Q_2^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8)\\
P_1(t,\mathbf{z}_0,\epsilon) &= s^{-1/3}+\triangle P_1 \qquad\qquad\qquad +\epsilon^4 P_1^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8)\\
P_2(t,\mathbf{z}_0,\epsilon) &= 0 \qquad\qquad\qquad\qquad\qquad\quad +\epsilon^4 P_2^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8).
\end{aligned} \tag{49}$$

By Proposition 3 item 3 and Proposition 5 item (i), to obtain a doubly symmetric orbit of the system (41), the two equations below

$$Q_1(T/4,\mathbf{z}_0,\epsilon)=(m+\tilde{m}+1)\pi, \quad Q_2(T/4,\mathbf{z}_0,\epsilon)=0, \tag{50}$$

must be satisfied. Defining $\Phi(\mathbf{X},P)=(\Phi_1(\mathbf{X},P),\Phi_2(\mathbf{X},P))=(Q_1(\mathbf{X},P)-(\tilde{m}+m+1)\pi,\ Q_2(\mathbf{X},P))=(Q_1(T/4,\mathbf{z}_0,\epsilon)-(\tilde{m}+m+1)\pi,\ Q_2(T/4,\mathbf{z}_0,\epsilon))$, where $\mathbf{X}=(\triangle Q_2,\triangle P_1), P=\epsilon$ and $T/4=2\pi\tilde{s}$, the system (50) is equivalent to

$$\begin{aligned}
\Phi_1(\mathbf{X},P) &= (s^{-1/3}+\triangle P_1)^{-3}\frac{T}{4}-\left(\tilde{m}+\frac{1}{2}\right)\pi+\epsilon^4 Q_1^{(1)}\left(\frac{T}{4},\mathbf{z}_0,\epsilon\right)+\mathcal{O}(\epsilon^8)=0\\
\Phi_2(\mathbf{X},P) &= \triangle Q_2+\epsilon^4 Q_2^{(1)}\left(\frac{T}{4},\mathbf{z}_0,\epsilon\right)+\mathcal{O}(\epsilon^8)=0,
\end{aligned} \tag{51}$$

called periodicity equations.

Remark 5. By Lemma 5 in the Appendix the functions

$$\epsilon^4\left[Q_1^{(1)}\left(\frac{T}{4},\mathbf{z}_0,\epsilon\right)+\mathcal{O}(\epsilon^4)\right] := \epsilon^4 g_1(\mathbf{z}_0,\epsilon)$$
$$\epsilon^4\left[Q_2^{(1)}\left(\frac{T}{4},\mathbf{z}_0,\epsilon\right)+\mathcal{O}(\epsilon^4)\right] := \epsilon^4 g_2(\mathbf{z}_0,\epsilon)$$

are uniformly bounded as ϵ approaches to zero, so we can define them to be continuous at $\epsilon=0$ by taking its limit, which is zero.

By the previous Remark 5, we can extend $\Phi(\mathbf{X},P)$ to the case $P=\epsilon=0$ by defining

$$\begin{aligned}
\Phi_1(\mathbf{X},0) &= \left(((\tilde{m}+1/2)/2\tilde{s})^{-1/3}+\triangle P_1\right)^{-3}\frac{T}{4}-\left(\tilde{m}+\frac{1}{2}\right)\pi\\
\Phi_2(\mathbf{X},0) &= \triangle Q_2.
\end{aligned}$$

Thus, it is clear that $\Phi(\mathbf{0},0)=0$ with $T/4=2\pi\tilde{s}$. With the previous notation, we have the following result analogous to Proposition 7 given in the Appendix. Since this kind of arguments will be used frequently during this work, we will give all the details of the proof in order to does not repeat it the other cases.

Lemma 4. *Let B^* be a ball around $\mathbf{X}=\mathbf{0}$ and B a region containing $P=0$. The function $\Phi(\mathbf{X},P)$ is differentiable, with respect to $\mathbf{X}\in B^*$ for every $P\in B$, and satisfies the next three properties*

1. $|(D_{\mathbf{X}}\Phi)^{-1}(\mathbf{0},0)| \leqslant b$
2. $|D_{\mathbf{X}}\Phi(\mathbf{X},P)-D_{\mathbf{X}}\Phi(0,0)| \leqslant c(\|\mathbf{X}\|+\epsilon^4)$
3. $\|\Phi(\mathbf{0},\epsilon)\| \leqslant d\epsilon^4$

where b,c,d are constants independent of ϵ.

Proof. Firstly, we will prove the differentiability of Φ. Each term $\epsilon^4 g_j$ inside the coordinate functions Φ_j, for $j=1,2$ respectively, are differentiable at $\epsilon=0$. In fact, by the process of the limit, the partial derivatives at $\epsilon=0$ ar zero. Observe that the partial derivatives $\epsilon^4\dfrac{\partial g_j}{\partial\triangle P_1}$, $\epsilon^4\dfrac{\partial g_j}{\partial\triangle Q_2}$, $j=1,2$ exist and are continuous for each $\epsilon\neq 0$ because the functions $\mathbf{F}_0$, $\epsilon^4\mathbf{F}_1$ and $\epsilon^8\mathbf{F}_r$ are differentiable with respect to $\mathbf{Z}$ and all their derivatives with respect to $\mathbf{Z}$ are continuous in $\mathcal{C}$. So, by basic theory of Ordinary Differential Equation, the solution $\mathbf{Z}(t,\mathbf{z}_0)$ has the same properties. At $\epsilon=0$, Lemma 6 in the Appendix allows us to define these partial derivatives in a continuous way. So, we have that the function $\Phi(\mathbf{X},\epsilon)$ is differentiable, for every ϵ, with respect to $\mathbf{X}\in B^*$. Now, we will proceed to prove the properties 1, 2 and 3. It is verified that

$$D_{\mathbf{X}}\Phi(\mathbf{X},\epsilon)=\begin{pmatrix}0+\mathcal{O}(\epsilon^4) & -3(((\tilde{m}+1/2)/2\tilde{s})^{-1/3}+\triangle P_1)^{-4}\frac{T}{4}+\mathcal{O}(\epsilon^4)\\ 1+\mathcal{O}(\epsilon^4) & 0+\mathcal{O}(\epsilon^4)\end{pmatrix}, \tag{52}$$

where $T/4=2\pi\tilde{s}$. Then, evaluating at $\mathbf{X}=\mathbf{0}$ and $\epsilon=0$, it follows that

$$D_{\mathbf{X}}\Phi(\mathbf{0},0)=\begin{pmatrix}0 & -6\left(\frac{\tilde{m}+1/2}{2}\right)^{4/3}\pi\tilde{s}^{-1/3}\\ 1 & 0\end{pmatrix}. \tag{53}$$

Thus, $det D_{\mathbf{X}}\Phi(\mathbf{0},0)=-6\left(\frac{\tilde{m}+1/2}{2}\right)^{4/3}\pi\tilde{s}^{-1/3}\neq 0$ and therefore, there exists $(D_{\mathbf{X}}\Phi)^{-1}(\mathbf{0},0)$, then item 1 holds taking a convenient positive constant b. In order to prove 2, by the triangle inequality

$$|D_{\mathbf{X}}\Phi(\mathbf{X},P)-D_{\mathbf{X}}\Phi(\mathbf{0},0)| \leqslant |D_{\mathbf{X}}\Phi(\mathbf{X},P)-D_{\mathbf{X}}\Phi(\mathbf{X},0)|+|D_{\mathbf{X}}\Phi(\mathbf{X},0) - D_{\mathbf{X}}\Phi(\mathbf{0},0)|,$$

and as the matrix $D_{\mathbf{X}}\Phi(\mathbf{X},P)-D_{\mathbf{X}}\Phi(\mathbf{X},0)$ is given by partial derivatives of the functions $\epsilon^4 g_1$ and $\epsilon^4 g_2$, and these partial derivatives are continuous, it follows that the norm $|D_{\mathbf{X}}\Phi(\mathbf{X},P)-D_{\mathbf{X}}\Phi(\mathbf{X},0)|$ is bounded by, say, $c_1\epsilon^4$, where c_1 is a positive constant. To prove that the second norm in the above sum is bounded, we note that the components of $D_{\mathbf{X}}\Phi(\mathbf{X}, 0)$ are twice differentiable with respect to $\mathbf{X}$ and, applying the Mean Value Inequality, we obtain that the second norm is bounded, say, by $c_2\|\mathbf{X}\|$, for $\mathbf{X}$ in a compact neighborhood, where c_2 is a positive constant. So, item 2 follows taking $c = Max\{c_1, c_2\}$. To prove item 3, observe that

$$\Phi(\mathbf{0}, \epsilon) = \left(\epsilon^4 Q_1^{(1)} + \mathcal{O}(\epsilon^8), \epsilon^4 Q_2^{(1)} + \mathcal{O}(\epsilon^8)\right),$$

and using Lemma 5 in the Appendix, we see that there is a positive constant d such that $\|\Phi(\mathbf{0}, \epsilon)\| \leqslant d\epsilon^4$. □

Theorem 1. *Consider the equations of motion (41) for the elliptic isosceles restricted three-body problem with collision, where the primaries move in an elliptic-collision orbit with energy $h = -1/2\ \epsilon^{-2}$. If $\epsilon = k^{-1/3}$ for k a large enough positive integer, then there exist initial conditions for the infinitesimal body such that its motion is a doubly-symmetric periodic solution, near a Keplerian circular orbit on the xy-plane whose period is $8\pi\tilde{s}$, where $\tilde{s} \in \mathbb{N}$.*

Proof. Firstly, it is enough to observe that under the hypotheses of the theorem and previous computations, we are in position apply Arenstorf's Theorem to the periodicity equation (50) for ϵ in a sufficiently small interval with the auxiliary function $g(\mathbf{X}, P) = \mathbf{X} - [D_{\mathbf{X}}\Phi(\mathbf{X}_0, 0)]^{-1}\ \Phi(\mathbf{X}, P)$ (as in the proof of Proposition 7). Thus, there exist families (indexed by m, $\tilde{m}$, $\tilde{s}$) to one-parameter (ϵ) of initial conditions $\mathbf{X}_{m,\tilde{m},\tilde{s}}(\epsilon)$ such that $\Phi(\mathbf{X}_{m,\tilde{m},\tilde{s}}(\epsilon), \epsilon) = 0$. Secondly, in order to have periodic solutions of the elliptic isosceles restricted three-body problem with collision (41) and consequently of system (13), the conditions in (50) must be satisfied simultaneously with $E = \hat{k}\pi$, where $\hat{k}$ is even integer. Thus, for each $\epsilon = k^{-1/3}$, where k is a large enough positive integer, we have initial conditions whose associated solution is $8\pi\tilde{s}$-periodic and doubly symmetric. □

Remark 6. **(1)** For $\epsilon = k^{-1/3}$, where k is a large enough positive integer we have the relation $\hat{k} = 2\tilde{s}k$, $\quad \tilde{s} \in \mathbb{N}$.
(2) Since the period of the continued orbit is $T = 8\pi\tilde{s}$, the commensurability relation give us $T/T_p = 4\tilde{s}k$, where T_p denotes the period of the primaries or the period of the perturbed system. In this way, we have that the common period of the motion of the three bodies together is $T = 4\tilde{s}kT_p$, and so, while the infinitesimal body completes one revolution, the primaries completed $4\tilde{s}k$ revolutions or encounters.

We will finish this section with the following important results, which are direct consequences of Proposition 2 item 3 and Theorem 1:

Corollary 1. *There are doubly-symmetric periodic solutions for the elliptic isosceles restricted three-body problem with collision (41), where the primaries move in an elliptic-collision orbit with energy $h = -1/2$ near a Keplerian circular orbit on the xy-plane whose period is large enough.*

Proof. Let $\psi(t)$ be a solution of system (41) given by Theorem 1 with energy of the primaries $h = -1/2\ \epsilon^{-2}$. Then, by Proposition 2 item 3, we have that $s(t) = \frac{1}{\epsilon^2}\psi(\epsilon^3 t)$ is a solution of the system (41) where now the energy of the primaries is $h = -1/2$. Note that its period is $8\pi\tilde{s}k$, which is too large if k is too large. □

Another consequence is:

Corollary 2. *There are doubly symmetric periodic solutions which cut orthogonally any two perpendicular lines at the origin for the elliptic isosceles restricted three-body problem with collision (41), where the primaries move in an elliptic-collision orbit with energy $h = -1/2\ \epsilon^{-2}$ near a Keplerian circular orbit on the xy-plane.*

Proof. This result follows immediately since our problem is invariant under rotations around the z-axis. □

Using Corollary 1 and again Proposition 2 item 3, we can prove the existence of doubly-symmetric periodic solutions for any value of the parameter ϵ. Thus,

Theorem 2. *There are doubly-symmetric periodic solutions for the elliptic isosceles restricted three-body problem with collision (41), where the primaries move in an elliptic-collision orbit for any fixed energy $h < 0$ near a Keplerian circular orbit on the xy-plane.*

5.2. $\hat{S}_2$-symmetric Periodic Solutions

Here, we prove the existence of S_2-symmetric periodic orbits, using similar arguments of the doubly symmetric periodic solutions. In this case, it will be convenient to use the problem in rotating coordinates and after that, put it in Delaunay–Poincaré variables, i.e., the Hamiltonian (32). But, similar results can be obtained using the problem in original variables (inertial coordinates), but the calculus are more tedious. Thus, the equation of motion is

$$\dot{\mathbf{Z}} = \hat{F}(\mathbf{Z}) = \hat{\mathbf{F}}_0(\mathbf{Z}) + \epsilon^4\hat{\mathbf{F}}_1(\mathbf{Z}, t, \epsilon) + \epsilon^8\hat{\mathbf{F}}_r(\mathbf{Z}, t, \epsilon), \tag{54}$$

where $\mathbf{Z}=(Q_1, Q_2, P_1, P_2)$,

$$\hat{\mathbf{F}}_0(\mathbf{Z})=(P_1^{-3}-1, P_2, 0, -Q_2), \tag{55}$$

and

$$\hat{\mathbf{F}}_1(\mathbf{Z}, t, \epsilon) = \left(\frac{\partial \hat{H}_1}{\partial P_1}, \frac{\partial \hat{H}_1}{\partial P_2}, -\frac{\partial \hat{H}_1}{\partial Q_1}, -\frac{\partial \hat{H}_1}{\partial Q_2}\right),$$
$$\hat{\mathbf{F}}_r(\mathbf{Z}, t, \epsilon) = \left(\frac{\partial \hat{H}_r}{\partial P_1}, \frac{\partial \hat{H}_r}{\partial P_2}, -\frac{\partial \hat{H}_r}{\partial Q_1}, -\frac{\partial \hat{H}_r}{\partial Q_2}\right),$$

where $\hat{H}_1$ and $\hat{H}_r$ are given respectively by (34) and (35). Now, we will characterize the $\hat{S}_2$ symmetric circular of the Kepler's problem

$$\begin{array}{ll} \dot{Q}_1 = P_1^{-3}-1, & \dot{P}_1 = 0, \\ \dot{Q}_2 = P_2, & \dot{P}_2 = -Q_2, \end{array} \tag{56}$$

whose solutions with initial conditions $(Q_{10}, Q_{20}, P_{10}, P_{20})$ at $t=0$ are described by

$$\begin{array}{ll} Q_1^{(0)}(t) = ((P_{10})^{-3}-1)t + Q_{10}, & P_1^{(0)}(t) = P_{10}, \\ Q_2^{(0)}(t) = Q_{20}\cos t + P_{20}\sin t, & P_2^{(0)}(t) = -Q_{20}\sin t + P_{20}\cos t. \end{array} \tag{57}$$

We consider a solution of the Kepler's problem (56) with initial conditions $\mathbf{z}_0^* \in \mathcal{L}_2$ when $t=0$ and $E(0)=0$, with

$$\mathbf{z}_0^* = \left(\left(m+\frac{1}{2}\right)\pi, Q_2^*, P_1^*, 0\right).$$

where m is an integer that, without loss of generalities, can be taken as 0 or 1, and Q_2^*, P_1^* are constants to be determined. So, we have that this solution is written in the following manner

$$\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*) = \left((P_1^*)^{-3}-1)t + \left(m+\frac{1}{2}\right)\pi,\ Q_2^*\cos t,\ P_1^*, -Q_2^*\sin t\right).$$

Since, we want that this solution be S_2-symmetric, by Proposition 4 item 2 and Proposition 5, item (ii), it is sufficient to solve, at time $t=T/2$, the system

$$Q_1^{(0)}(T/2, z_0^*) = ((P_1^*)^{-3}-1)\frac{T}{2} + \left(m+\frac{1}{2}\right)\pi = \left(m+\frac{1}{2}+\tilde{m}\right)\pi$$
$$P_2^{(0)}(T/2, z_0^*) = -Q_2^*\sin\frac{T}{2} = 0,$$

where $\tilde{m}$ is an integer. The second equation is satisfied taking $Q_2^*=0$ and so, with this choice, we obtain a circular orbit. By the first equation we obtain $T/2=\tilde{m}\pi[(P_1^*)^{-3}-1]^{-1}$. Then, taking $(P_1^*)^{-3}=s$, where s is a positive real constant, we have that

$$\frac{T}{2}=\frac{\tilde{m}\pi}{s-1}. \tag{58}$$

By Remark 3, we must have $E(0)=0$ and $E(T/2)=\hat{k}\pi$ and by Proposition 1 item 2, we must take

$$T=2\hat{k}\pi\epsilon^3, \tag{59}$$

where $\hat{k}$ is an even positive integer. By Eqs. (58) and (59), it follows that

$$\hat{k}=\frac{\tilde{m}}{\epsilon^3(s-1)}, \tag{60}$$

where s is a positive real constant ($s\neq 1$). Since, for future applications, we will want that $\sin T/2\neq 0$, it is necessary to make more restrictions about $T/2$. So, we also must assume

$$\frac{\tilde{m}}{s-1}=\frac{1}{2}\ (mod\ 1). \tag{61}$$

Observe that by (60) and (61) it follows that $\hat{k}=\dfrac{2\tilde{s}+1}{2\epsilon^3}$, where $\tilde{s}\in\mathbb{N}$. So, since $\hat{k}$ is a positive even integer, we have that

$$\epsilon^3=\frac{1}{4k}, \quad \text{with} \quad k\in\mathbb{N}.$$

Therefore, it follows that $\mathbf{Z}^{(0)}(t,\mathbf{z}_0^*)$ is a circular orbit of radius $s^{-2/3}$, where $s=\frac{2\tilde{m}}{2\tilde{s}+1}+1$ with $\tilde{m},\tilde{s}\in\mathbb{N}$ in the xy plane whose period is $T=(2\tilde{s}+1)\pi$, where $\tilde{s}\in\mathbb{N}$.

Now, we will consider the perturbed problem (54) and we look for initial conditions in a neighborhood of $\mathbf{z}_0^*$ of type

$$\mathbf{z}_0=\left(\left(m+\frac{1}{2}\right)\pi,\triangle Q_2,s^{-1/3}+\triangle P_1,0\right),$$

where $s=\frac{2\tilde{m}}{2\tilde{s}+1}+1$ in such a way that the solution $\mathbf{Z}(t,\mathbf{z}_0,\epsilon)$ of (54), with $\epsilon\neq 0$ small enough, is a S_2-symmetric periodic orbit. We know that the solution of Kepler's problem (55) with initial condition $\mathbf{z}_0$ is given by

$$\begin{aligned}\mathbf{Z}^{(0)}(t,\mathbf{z}_0)=(((s^{-1/3}+\triangle P_1)^{-3}-1)t+\left(m+\frac{1}{2}\right)\pi,\\ \triangle Q_2\cos t,s^{-1/3}+\triangle P_1,-\triangle Q_2\sin t).\end{aligned}$$

Again by Lemma 5 of the Appendix, we have that the solution $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ of (54) is given by

$$\begin{aligned}
Q_1(t,\mathbf{z}_0,\epsilon) &= ((s^{-1/3}+\triangle P_1)^{-3}-1)t+(m+\tfrac{1}{2})\pi &&+\epsilon^4 Q_1^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8)\\
Q_2(t,\mathbf{z}_0,\epsilon) &= \triangle Q_2\cos t &&+\epsilon^4 Q_2^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8)\\
P_1(t,\mathbf{z}_0,\epsilon) &= s^{-1/3}+\triangle P_1 &&+\epsilon^4 P_1^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8)\\
P_2(t,\mathbf{z}_0,\epsilon) &= -\triangle Q_2\sin t &&+\epsilon^4 P_2^{(1)}(t,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^8).
\end{aligned} \tag{62}$$

To obtain S_2-symmetric periodic orbits, by Proposition 4, item 2, and Proposition 5, item (ii) it is enough to solve, at time $t=T/2$, where we take $T/2=\pi/2\ (2\tilde{s}+1)$, with $\tilde{s}\in\mathbb{N}$ the following periodicity equations:

$$\begin{aligned}
Q_1(T/2,\mathbf{z}_0,\epsilon) &= ((s^{-1/3}+\triangle P_1)^{-3}-1)\frac{T}{2}+\left(m+\frac{1}{2}\right)\pi+\epsilon^4[Q_1^{(1)}(T/2,\mathbf{z}_0,\epsilon)\\
&\qquad +\mathcal{O}(\epsilon^4)]=\left(m+\frac{1}{2}+\tilde{m}\right)\pi \qquad (63)\\
P_2(T/2,\mathbf{z}_0,\epsilon) &= -\triangle Q_2\sin\frac{T}{2}+\epsilon^4[P_1^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]=0.
\end{aligned}$$

Let be $\Phi(\mathbf{X},P)=(\Phi_1(\mathbf{X},P),\Phi_2(\mathbf{X},P))=(Q_1(\mathbf{X},P)-(m+\frac{1}{2}+\tilde{m})\pi,$ $P_2(\mathbf{X},P))=(Q_1(T/2,\mathbf{z}_0,\epsilon)-(m+\frac{1}{2}+\tilde{m})\pi, P_2(T/2,\mathbf{z}_0,\epsilon))$, where $\mathbf{X}=(\triangle Q_2,\triangle P_1)$, $P=\epsilon$. Thus, the periodicity equations are given by

$$\begin{aligned}
\Phi_1(\mathbf{X},P) &= ((s^{-1/3}+\triangle P_1)^{-3}-1)\tfrac{T}{2}-\tilde{m}\pi &&+\epsilon^4[Q_1^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]=0\\
\Phi_2(\mathbf{X},P) &= -\triangle Q_2\sin\tfrac{T}{2} &&+\epsilon^4[P_1^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]=0.
\end{aligned} \tag{64}$$

Similarly to the pervious case, we can extend $\Phi(\mathbf{X},P)$ to the case $\epsilon=0$ by defining

$$\begin{aligned}
\Phi_1(\mathbf{X},0) &= ((s^{-1/3}+\triangle P_1)^{-3}-1)\tfrac{T}{2}-\tilde{m}\pi\\
\Phi_2(\mathbf{X},0) &= -\triangle Q_2\sin\tfrac{T}{2},
\end{aligned}$$

and by the choosing of T and s, it follows that $\Phi(\mathbf{0},0)=0$. Here, we easily verify that

$$D_{\mathbf{X}}\Phi(\mathbf{X},\epsilon)=\begin{pmatrix} 0+\mathcal{O}(\epsilon^4) & -3\frac{\pi}{2}(2\tilde{s}+1)\left(\left(\dfrac{2\tilde{m}}{2\tilde{s}+1}+1\right)^{-1/3}+\triangle P_1\right)^{-4}+\mathcal{O}(\epsilon^4)\\ 1+\mathcal{O}(\epsilon^4) & 0+\mathcal{O}(\epsilon^4)\end{pmatrix},$$

and so,

$$D_{\mathbf{X}}\Phi(\mathbf{0},0)=\begin{pmatrix} 0 & -\frac{3\pi}{2}(2\tilde{m}+1)^{4/3}(2\tilde{s}+1)^{-1/3} \\ \pm 1 & 0 \end{pmatrix}.$$

It is a simple exercise to verify that an analogous result to Lemma 4 is valid in this case. Thus, now we can use the Arenstorf's Theorem to solve the Eq. (63). In this way, we have the following theorem:

Theorem 3. *Consider the equations of motion (54) for the elliptic isosceles restricted three-body problem with collision, where the primaries move in an elliptic-collision orbit with energy $h=1/2\ \epsilon^{-2}$. If $\epsilon=(4k)^{-1/3}$ for k a large enough positive integer, then there exist initial conditions for the infinitesimal body such that its motion is a $\hat{S}_2$-symmetric periodic solution, near a Keplerian circular orbit on the xy-plane whose period is $(2\tilde{s}+1)\pi$ with $\tilde{s}\in\mathbb{N}$.*

Proof. Initially we observe that under the hypotheses of the theorem, we are in position to apply Arenstorf's Theorem to the periodicity Eq. (63) for ϵ in a sufficiently small interval. Thus, there exist families (indexed by $m,\tilde{m},\tilde{s}$) to one-parameter (ϵ) of initial conditions $\mathbf{X}_{m,\tilde{m},\tilde{s}}(\epsilon)$ such that $\Phi(\mathbf{X}_{m,\tilde{m},\tilde{s}}(\epsilon),\epsilon)=0$. After that, in order to have periodic solutions of the elliptic isosceles restricted three-body problem with collision (13), the conditions in (63) must be satisfied simultaneously with $E=\hat{k}\pi$, where $\hat{k}$ is an even integer. Thus, for each $\epsilon=(4k)^{-1/3}$, where $k\in\mathbb{N}$, we have initial conditions whose associated solution is $(2\tilde{s}+1)\pi$-periodic and $\hat{S}_2$-symmetric. □

Remark 7. **(1)** For $\epsilon=(4k)^{-1/3}$, where $k\in\mathbb{N}$ and $s=2\tilde{m}(2\tilde{s}+1)^{-1}+1$, where $\tilde{m},\tilde{s}\in\mathbb{N}$ we have the relation $\hat{k}=2k(2\tilde{s}+1)$.
(2) Since the period of the continued orbit in inertial coordinates is $T=2(2\tilde{s}+1)\pi$, where $\tilde{s}\in\mathbb{N}$ and the system (13) defining the elliptic isosceles restricted three-body problem with collision is $T_p=2\pi\epsilon^3$, with $\epsilon^3=1/4k$, the commensurability relation give us $T/T_p=4k(2\tilde{s}+1)$. Notice that T_p is also the period of the primaries. In this way, we have that the period of the motion of the three bodies together is $T=4k(2\tilde{s}+1)T_p$, and so, while the infinitesimal body completes one revolution, the primaries have completed $4k(2\tilde{s}+1)$ revolutions or encounters.
(3) Similar results to Corollary 1, 2 and Theorem 2 are also true in this case.

5.3. $\hat{S}_1$-symmetric Periodic Solutions

In order to get $\hat{S}_1$-symmetric orbits of the elliptic isosceles restricted three-body problem with collision, we will use the same coordinates as the case of $\hat{S}_2$ symmetry. Similar results also can be obtained in the orginal variables. At this time, we will consider a solution of Kepler's problem (55) with initial conditions $\mathbf{z}_0^* \in \mathcal{L}_1$ when $t=0$ and $E(0)=0$, with

$$\mathbf{z}_0^* = (m\pi, 0, P_1^*, P_2^*),$$

where m is an integer that, without loss of generality, we can take as 0 or 1, and P_1^*, P_2^* are constants to be determined. So, we have that this solution is written in the following manner

$$\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*) = ((P_1^*)^{-3} - 1)t + m\pi,\ P_2^* \sin t,\ P_1^*,\ P_2^* \cos t).$$

We want this solution to be the S_1-symmetric of Kepler's problem (55). So by Proposition 4, item 1 and Proposition 5, item (i) it is enough to solve, at time $t = T/2$, the system

$$\begin{aligned} Q_1^{(0)}(T/2, z_0^*) &= ((P_1^*)^{-3} - 1)\tfrac{T}{2} + m\pi &&= (m + \tilde{m})\pi \\ Q_2^{(0)}(T/2, z_0^*) &= P_2^* \sin \tfrac{T}{2} &&= 0, \end{aligned}$$

where $\tilde{m}$ is an integer. The second equation is satisfied taking $P_2^* = 0$ (note that with this choice, this orbit will be circular), and by the first equation we obtain $T/2 = \tilde{m}\pi[(P_1^*)^{-3} - 1]^{-1}$. So, taking $(P_1^*)^{-3} = s$, where s is a positive real constant, we have that

$$\frac{T}{2} = \frac{\tilde{m}\pi}{s-1}. \tag{65}$$

Considering Remark 3, it is necessary to have $E(0) = 0$ and $E(T/2) = \hat{k}\pi$ and by Remark 1 item 2, it is seen that we must take

$$T = 2\hat{k}\pi\epsilon^3, \tag{66}$$

where $\hat{k}$ is a even positive integer. From Eqs. (65) and (66), it follows that

$$\hat{k} = \frac{\tilde{m}}{\epsilon^3(s-1)}, \tag{67}$$

where s is a positive real constant. Since we will demand $\sin T/2 \neq 0$, we take

$$\frac{\tilde{m}}{s-1} = \frac{1}{2}(mod\ 1). \tag{68}$$

In this way, given $\tilde{m}$ (to facilitate the calculus we take $\tilde{m}\in\mathbb{N}$), we choose $s\in\mathbb{R}_+$ such that $\tilde{m}(s-1)^{-1}=\tilde{s}+1/2$, where $\tilde{s}\in\mathbb{N}$. So, we have that

$$\frac{T}{2}=\frac{2\tilde{s}+1}{2}\pi,\ \tilde{s}\in\mathbb{N}, \tag{69}$$

and by (67) with $\tilde{m}(s-1)^{-1}=\tilde{s}+1/2$ it follows that $\hat{k}=(\tilde{s}+1/2)\epsilon^{-3}$, where $\tilde{s}\in\mathbb{N}$. So, since $\hat{k}$ is a positive even integer we take ϵ such that

$$\epsilon^3=\frac{1}{4k},\ \text{with}\, k\in\mathbb{N}.$$

Therefore, it follows that $\mathbf{Z}^{(0)}(t,\mathbf{z}_0^*)$ is a circular orbit of radius $s^{-2/3}$ on the plane xy whose period is $T=(2\tilde{s}+1)\pi$, where $\tilde{s}\in\mathbb{N}$.

In order to get $\hat{S}_1$-symmetric periodic solutions of the perturbed system (54), we look for an initial condition in a neighborhood of $\mathbf{z}_0^*$ of the type

$$\mathbf{z}_0=(m\pi,0,s^{-1/3}+\triangle P_1,\triangle P_2),$$

where $s=2\tilde{m}(2\tilde{s}+1)^{-1}+1$, $\tilde{s}\in\mathbb{N}$. The solution of the Kepler problem (55), with initial condition $\mathbf{z}_0$, is given by

$$\mathbf{Z}^{(0)}(t,\mathbf{z}_0)=(((s^{-1/3}+\triangle P_1)^{-3}-1)t+m\pi,\quad \triangle P_2\sin t,s^{-1/3}+\triangle P_1,\triangle P_2\cos t).$$

To obtain S_1-symmetric periodic orbits, by Proposition 4, item 1 and Proposition 5, item (i), it is sufficient to solve, at time $t=T/2$, where we take $T/2=(\tilde{s}+1/2)\pi$, with $\tilde{s}\in\mathbb{N}$, the following periodicity equations:

$$\begin{aligned} Q_1(T/2,\mathbf{z}_0,\epsilon)&=((s^{-1/3}+\triangle P_1)^{-3}-1)\frac{T}{2}+m\pi + \epsilon^4[Q_1^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]\\ &= (m+\tilde{m})\pi\\ Q_2(T/2,\mathbf{z}_0,\epsilon)&=\triangle P_2\sin\frac{T}{2} + \epsilon^4[Q_2^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]\\ &=0. \end{aligned} \tag{70}$$

Let be $\Phi(\mathbf{X},P)=(\Phi_1(\mathbf{X},P),\Phi_2(\mathbf{X},P))=(Q_1(\mathbf{X},P)-(m+\tilde{m})\pi, Q_2(\mathbf{X},P))$ $=(Q_1(T/2,\mathbf{z}_0,\epsilon)-(m+\tilde{m})\pi, Q_2(T/2,\mathbf{z}_0,\epsilon))$, where $\mathbf{X}=(\triangle P_1,\triangle P_2)$, $P=\epsilon$, so the periodicity equations are equivalent to the system

$$\begin{aligned} \Phi_1(\mathbf{X},P)&=((s^{-1/3}+\triangle P_1)^{-3}-1)\frac{T}{2}-\tilde{m}\pi + \epsilon^4[Q_1^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]\\ &=0\\ \Phi_2(\mathbf{X},P)&=\triangle P_2\sin\frac{T}{2} + \epsilon^4[Q_2^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]\\ &=0. \end{aligned} \tag{71}$$

Note that similarly to the previous cases, we can extend $\Phi(\mathbf{X}, P)$ to the case $\epsilon = 0$ by defining

$$\Phi_1(\mathbf{X}, 0) = ((s^{-1/3} + \triangle P_1)^{-3} - 1)\frac{T}{2} - \tilde{m}\pi$$
$$\Phi_2(\mathbf{X}, 0) = \triangle P_2 \sin\frac{T}{2},$$

and by the choosing of T and s, it follows that $\Phi(\mathbf{0}, 0) = 0$. Here, we verify that

$$D_{\mathbf{X}}\Phi(\mathbf{X}, \epsilon) = \begin{pmatrix} -3(s^{-1/3} + \triangle P_1)^{-4}\frac{T}{2} + \mathcal{O}(\epsilon^4) & 0 + \mathcal{O}(\epsilon^4) \\ 0 + \mathcal{O}(\epsilon^4) & \sin\frac{T}{2} + \mathcal{O}(\epsilon^4) \end{pmatrix},$$

where $T = (2\tilde{s} + 1)\pi$ and $s = \frac{2\tilde{m}}{2\tilde{s}+1} + 1, \tilde{s} \in \mathbb{N}$. So, it follows that

$$D_{\mathbf{X}}\Phi(\mathbf{0}, 0) = \begin{pmatrix} -\frac{3\pi}{2}(s)^{4/3}(2\tilde{s} + 1) & 0 \\ 0 & \pm 1 \end{pmatrix},$$

It is verified an analogous result to Lemma 4. Now, we can use the Arenstorf's Theorem to solve the equations in (70). In this way we can prove in a similar way to Theorem 1 the following result:

Theorem 4. *Consider the equations of motion (54) for the elliptic isosceles restricted three-body problem with collision, where the primaries move in an elliptic-collision orbit with energy* $h = 1/2\ \epsilon^{-2}$. *If* $\epsilon = (4k)^{-1/3}$ *for* k *a large enough positive integer, then there exist initial conditions for the infinitesimal body such that its motion is a* $\hat{S}_1$*-symmetric periodic solution, near a Keplerian circular orbit on the* xy*-plane whose period is* $(2\tilde{s} + 1)\pi$ *with* $\tilde{s} \in \mathbb{N}$.

Remark 8. **(1)** Similar facts to the ones given in Remark 6 are valid to this symmetry, and analogous results to Corollary 1, 2 and Theorem 2 are also true in this case.

6. PERIODIC SOLUTIONS OF SECOND KIND

In this section we will prove the existence of periodic doubly-symmetric ($\hat{S}_1$ and $\hat{S}_2$ symmetric) solutions; $\hat{S}_2$ symmetric solutions and $\hat{S}_1$ symmetric solutions as continuation of convenient elliptic orbits of the Kepler problem via Arenstorf's Theorem 8.1 for appropriate values of the parameter ϵ. As it is clear more ahead, the main difference between this case with the first kind case, is that here we need to introduce time as a new variable in order to avoid degeneracy of the periodicity system. This kind of arguments have been used, for example, by Schmidt in [12].

To study this case we will use the elliptic isosceles restricted three-body problem with collision in rotating coordinates followed by the

Delaunay variables (36) $(\overline{Q}_1, \overline{Q}_2, \overline{P}_1, \overline{P}_2)$. Thus, the equations of motion are given by

$$\dot{\mathbf{Z}} = \hat{\mathbf{F}}(\mathbf{Z}), \tag{72}$$

where $\mathbf{Z} = (\overline{Q}_1, \overline{Q}_2, \overline{P}_1, \overline{P}_2)$, $\hat{\mathbf{F}}(\mathbf{Z}) = \hat{\mathbf{F}}_0(\mathbf{Z}) + \epsilon^4 \hat{\mathbf{F}}_1(\mathbf{Z}, t, \epsilon) + \epsilon^8 \hat{\mathbf{F}}_r(\mathbf{Z}, t, \epsilon)$.

$$\hat{\mathbf{F}}_0(\mathbf{Z}) = (\overline{P}_1^{-3}, -1, 0, 0), \tag{73}$$

$$\hat{\mathbf{F}}_1(\mathbf{Z}, t, \epsilon) = \left(\frac{\partial \hat{H}_1}{\partial \overline{P}_1}, \frac{\partial \hat{H}_1}{\partial \overline{P}_2}, -\frac{\partial \hat{H}_1}{\partial \overline{Q}_1}, -\frac{\partial \hat{H}_1}{\partial \overline{Q}_2} \right),$$
$$\hat{\mathbf{F}}_r(\mathbf{Z}, t, \epsilon) = \left(\frac{\partial \hat{H}_r}{\partial \overline{P}_1}, \frac{\partial \hat{H}_r}{\partial \overline{P}_2}, -\frac{\partial \hat{H}_r}{\partial \overline{Q}_1}, -\frac{\partial \hat{H}_r}{\partial \overline{Q}_2} \right),$$

where $\hat{H}_1$ and $\hat{H}_r$ are given by (30) and (31), respectively. Note that a similar result of Lemma 3 holds here. So, in this way we can also use the estimates of Lemmas 5 and 6.

6.1. Doubly-symmetric Periodic Solutions as Continuation of Elliptic Kepler's Orbits

As in the circular case, initially we will show how to obtain doubly symmetric elliptic orbits of the Kepler problem associated to (73) using symmetries in the Delaunay variables. The general solution of this system is given by

$$\begin{aligned} \overline{Q}_1^{(0)}(t) &= \overline{P}_{10}^{-3} t + \overline{Q}_{10}, & \overline{P}_1^{(0)}(t) &= \overline{P}_{10}, \\ \overline{Q}_2^{(0)}(t) &= -t + \overline{Q}_{20}, & \overline{P}_2^{(0)}(t) &= \overline{P}_{20}, \end{aligned} \tag{74}$$

for initial conditions $(\overline{Q}_{10}, \overline{Q}_{20}, \overline{P}_{10}, \overline{P}_{20})$ at $t = 0$. Now, we consider a solution of Kepler's problem with initial conditions $\mathbf{z}_0^* \in \mathcal{L}_2$ when $t = 0$ and $E(0) = 0$, with

$$\mathbf{z}_0^* = \left(m\pi, \left(i + \frac{1}{2} \right) \pi, \overline{P}_1^*, \overline{P}_2^* \right),$$

where m and i are integers that we are going to choose, without loss of generality, to be 0 or either 1, and $\overline{P}_1^*, \overline{P}_2^*$ are constants to be determined. Thus, this solution is of the form:

$$\begin{aligned} \overline{Q}_1^{(0)}(t, \mathbf{z}_0^*) &= (\overline{P}_1^*)^{-3} t + m\pi, & \overline{P}_1^{(0)}(t) &= \overline{P}_1^*, \\ \overline{Q}_2^{(0)}(t, \mathbf{z}_0^*) &= -t + \left(\tfrac{1}{2} + i\right)\pi, & \overline{P}_2^{(0)}(t) &= \overline{P}_2^*. \end{aligned}$$

By Proposition 4, item 3 and Proposition 6 item (i), these elliptic orbits will be doubly symmetric orbit to Kepler's problem (73), if we may solve the set of two equations in two unknowns:

$$\begin{aligned}\overline{Q}_1^{(0)}(T/4,\mathbf{z}_0^*) &= (\overline{P}_1^*)^{-3}\tfrac{T}{4} + m\pi = (m+\tilde{m})\pi \\ \overline{Q}_2^{(0)}(T/4,\mathbf{z}_0^*) &= -T/4 + \left(\tfrac{1}{2}+i\right)\pi = (1+i+j)\pi,\end{aligned} \tag{75}$$

where $\tilde{m}, j$ are integers. The second equation is satisfied taking

$$\frac{T}{4} = -\left(j+\frac{1}{2}\right)\pi, \tag{76}$$

where $j \in \mathbb{Z}_-$. And substituting (76) in the first equation of (75), it follows that

$$(\overline{P}_1^*)^{-3} = -\frac{\tilde{m}}{j+1/2}, \tag{77}$$

where $\tilde{m} \in \mathbb{Z}_+$. Observe that for instance we have some restrictions in $\overline{P}_2^*$. Namely, $\overline{P}_2^* \neq \overline{P}_1^*$ because the orbit is elliptic but not circular and as $\overline{P}_2$ is the angular momentum $\overline{P}_2^*$ must be also not null. By Remark 3, we will want also that $E(0)=0$ and $E(T/4)=\hat{k}\pi$, where $\hat{k}$ is a convenient even integer and in this way we must take

$$T = 4\hat{k}\pi\epsilon^3. \tag{78}$$

A comparison between equation (76) and (78) shows that

$$\hat{k} = -2k(2j+1),$$

where $\epsilon^3 = (4k)^{-1}$, k a positive integer and $j \in \mathbb{Z}_-$. So, we have

$$\mathbf{Z}^{(0)}(T/4,\mathbf{z}_0^*) = \left((m+\tilde{m})\pi,\ (i+j+1)\pi,\ \left(-\frac{\tilde{m}}{j+1/2}\right)^{-1/3},\overline{P}_2^*\right),$$

where $\overline{P}_2^*$ is a real positive constant different from $\overline{P}_1^* = (-\frac{\tilde{m}}{j+1/2})^{-1/3}$. By construction, $\mathbf{Z}^{(0)}(T/4,\mathbf{z}_0^*)$ lies in $\mathcal{L}_1$ and $E(T/4)=\hat{k}\pi$, where $\hat{k}=-2k(2j+1)$. Therefore, the solution $\mathbf{Z}^{(0)}(t,\mathbf{z}_0^*)$ is a doubly symmetric elliptic orbit of the Kepler problem with period $T=-2(2j+1)\pi$ ($j \in \mathbb{Z}_-$) on the plane xy.

Next, we will analyze the perturbed system (72). Firstly, we take initial conditions in a neighborhood of $\mathbf{z}_0^*$ of the type

$$\mathbf{z}_0 = \left(m\pi,\ \left(i+\frac{1}{2}\right)\pi,\ \left(-\frac{\tilde{m}}{j+1/2}\right)^{-1/3} + \triangle\overline{P}_1,\ \overline{P}_2^* + \triangle\overline{P}_2\right),$$

in such a way that a solution $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ of (72), with small enough $\epsilon \neq 0$ be a doubly symmetric orbit of second kind. By Lemma 5, we have that the solution $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ of (72) is given by

$$\begin{aligned}
\overline{Q}_1(t, \mathbf{z}_0, \epsilon) &= (\overline{P}_1^* + \triangle \overline{P}_1)^{-3} t + m\pi && + \epsilon^4 \overline{Q}_1^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8) \\
\overline{Q}_2(t, \mathbf{z}_0, \epsilon) &= -t + (i + \tfrac{1}{2})\pi && + \epsilon^4 \overline{Q}_2^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8) \\
\overline{P}_1(t, \mathbf{z}_0, \epsilon) &= \overline{P}_1^* + \triangle \overline{P}_1 && + \epsilon^4 \overline{P}_1^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8) \\
\overline{P}_2(t, \mathbf{z}_0, \epsilon) &= \overline{P}_2^* + \triangle \overline{P}_2 && + \epsilon^4 \overline{P}_2^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8),
\end{aligned} \tag{79}$$

where $\overline{P}_1^*$ is given in (77). Calling $\Phi(\mathbf{X}, P) = (\Phi_1(\mathbf{X}, \mathbf{P}), \Phi_2(\mathbf{X}, \mathbf{P})) = (\overline{Q}_1(\mathbf{X}, \mathbf{P}) - (\tilde{m} + m)\pi, \overline{Q}_2(\mathbf{X}, \mathbf{P}) - (i + j + 1)\pi) = (\overline{Q}_1(T/4, \mathbf{z}_0, \epsilon) - (\tilde{m} + m)\pi, \overline{Q}_2(T/4, \mathbf{z}_0, \epsilon) - (i + j + 1)\pi)$, where $\mathbf{X} = (T, \triangle \overline{P}_1), \mathbf{P} = (\epsilon, \triangle \overline{P}_2)$. By Proposition 4, item 3 and Proposition 6 item (ii), to obtain doubly symmetric orbit of the system (72), it is sufficient to solve the two below equations in two unknowns,

$$\begin{aligned}
\Phi_1(\mathbf{X}, \mathbf{P}) &= (\overline{P}_1^* + \triangle \overline{P}_1)^{-3} \tfrac{T}{4} - \tilde{m}\pi && + \epsilon^4 [\overline{Q}_1^{(1)}(T/4, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^4)] = 0 \\
\Phi_2(\mathbf{X}, \mathbf{P}) &= -T/4 - (j + 1/2)\pi && + \epsilon^4 [\overline{Q}_2^{(1)}(T/4, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^4)] = 0,
\end{aligned} \tag{80}$$

called periodicity equations. It is clear that we can extend $\Phi(\mathbf{X}, \mathbf{P})$ to the case $\mathbf{P} = \mathbf{0}$ by defining

$$\begin{aligned}
\Phi_1(\mathbf{X}, \mathbf{0}) &= (\overline{P}_1^* + \triangle \overline{P}_1)^{-3} \tfrac{T}{4} - \tilde{m}\pi \\
\Phi_2(\mathbf{X}, \mathbf{0}) &= -T/4 - (j + 1/2)\pi.
\end{aligned}$$

Thus, $\Phi(\mathbf{X}_0, \mathbf{0}) = 0$ when $\mathbf{X}_0 = (-2(2j + 1)\pi, 0)$. In this case, the matrix $D_{\mathbf{X}}\Phi(\mathbf{X}, \mathbf{P})$ is given by

$$D_{\mathbf{X}}\Phi(\mathbf{X}, \mathbf{P}) = \begin{pmatrix} \frac{1}{4}(\overline{P}_1^* + \triangle \overline{P}_1)^{-3} + \mathcal{O}(\epsilon^4) & -3(P_1^* + \triangle \overline{P}_1)^{-4} \frac{T}{4} + \mathcal{O}(\epsilon^4) \\ -\frac{1}{4} + \mathcal{O}(\epsilon^4) & 0 + \mathcal{O}(\epsilon^4) \end{pmatrix},$$

where $\overline{P}_1^* = (-\tilde{m}/(j + 1/2))^{-1/3}$. So, it follows that

$$D_{\mathbf{X}}\Phi(\mathbf{X}_0, 0) = \begin{pmatrix} \frac{1}{4}\left(-\frac{\tilde{m}}{j+1/2}\right)^{-3} & \frac{3}{2}\left(-\frac{\tilde{m}}{j+1/2}\right)^{4/3}(2j + 1)\pi \\ -1/4 & 0 \end{pmatrix},$$

whose determinant is not null by the appropriate conditions. Thus, we can obtain a similar result of Lemma 4. Therefore, now we are in position to apply Arenstorf's theorem, to solve the system of equations (80).

Theorem 5. *Consider the equations of motion (72) for the elliptic isosceles restricted three-body problem with collision, where the primaries move in an elliptic-collision orbit with energy* $h=1/2\ \epsilon^{-2}$. *If* $\epsilon=(4k)^{-1/3}$ *for* k *a large enough positive integer, then there exist initial conditions for the infinitesimal body such that its motion is a doubly-symmetric periodic solution, near a Keplerian elliptic orbit on the* xy*-plane whose period is close to* $-2(2j+1)\pi$ *with* $j\in\mathbb{Z}_{-}$.

Proof. As in Theorem 1, first we observe that under the hypotheses of the theorem we can apply Arenstorf's Theorem to the periodicity equation (80) for ϵ in a sufficiently small interval with $\mathbf{X}=(T,\triangle\overline{P}_1)$ and $P=(\epsilon,\triangle\overline{P}_2)$. Thus, there are families (indexed by $m,\tilde{m},j,i,\overline{P}_2^*$) to two-parameter $(\epsilon,\triangle\overline{P}_2)$ of initial conditions $\mathbf{X}_{m,\tilde{m},j,i,\overline{P}_2^*}(\epsilon,\triangle\overline{P}_2)$ such that $\Phi(\mathbf{X}_{m,\tilde{m},j,i,\overline{P}_2^*}(\epsilon,\triangle\overline{P}_2),\epsilon,\triangle\overline{P}_2)=0$. In order to have periodic solutions of the elliptic isosceles restricted three-body problem with collision (72), the conditions in (80) must be satisfied simultaneously with the commensurability condition $T/2\pi\epsilon^3\in\mathbb{Q}$. So, since $T=T(\epsilon,\triangle\overline{P}_2)$ and $\epsilon^3=(4k)^{-1}$, we take $k\in\mathbb{N}$ and $\triangle\overline{P}_2\in\mathbb{R}$ such that $T((4k)^{-1/3},\triangle\overline{P}_2)/2\pi\epsilon^3=2kT((4k)^{-1/3},\triangle\overline{P}_2)/\pi\in\mathbb{Q}$. Thus, for these $\epsilon=(4k)^{-1/3}$, where k is a large enough positive integer, and $\triangle\overline{P}_2\in\mathbb{R}$ we have some initial conditions that give origin to a solution of (72) which are $T=T(\epsilon,\triangle\overline{P}_2)$-periodic and doubly symmetric. □

Remark 9. *Analogous considerations to Remark 6 are valid to this case and similar results to Corollary 1, 2 and Theorem 2 are also true in this situation.*

6.2. $\hat{S}_2$-symmetric Periodic Solutions as Continuation of Elliptic Orbits

We will prove the existence of $\hat{S}_2$-symmetric periodic solutions of second kind to the perturbed system (72). Again, the first argument is to obtain $\hat{S}_2$-elliptic orbits of Kepler's problem given by the function (73), and we consider solutions with initial conditions $\mathbf{z}_0^*\in\mathcal{L}_2$ when $t=0$ and $E(0)=0$, with

$$\mathbf{z}_0^*=\left(m\pi,\left(i+\frac{1}{2}\right)\pi,\overline{P}_1^*,\overline{P}_2^*\right),$$

where m, i are integers that we will choose, without loss of generality, 0 or either 1 and $\overline{P}_1^*,\overline{P}_2^*$ are constants to be determined. By Proposition 4, item 2 and Proposition 6 item (ii), we can obtain elliptic $\hat{S}_2$-symmetric orbit to the Kepler problem (73), if we may solve the set of two equations at time $t=T/2$ in three unknowns:

$$\begin{aligned}\overline{Q}_1^{(0)}(T/2) &= (\overline{P}_1^*)^{-3}\tfrac{T}{2} + m\pi = (m+\tilde{m})\pi \\ \overline{Q}_2^{(0)}(T/2) &= -T/2 + \left(\tfrac{1}{2}+i\right)\pi = \left(i+j+\tfrac{1}{2}\right)\pi,\end{aligned} \tag{81}$$

where $\tilde{m}, j$ are integers. The second equation is satisfied by

$$\frac{T}{2} = -j\pi, \tag{82}$$

where $j \in \mathbb{Z}_-$. And substituting (82) in the first equation of (81), it follows that

$$(\overline{P}_1^*)^{-3} = -\frac{\tilde{m}}{j}, \tag{83}$$

where $\tilde{m} \in \mathbb{Z}_+$. By Remark 3, we must have $E(0)=0$ and $E(T/2)=\hat{k}\pi$, where $\hat{k}$ is a convenient even integer and in this way we must take

$$T = 2\hat{k}\pi\epsilon^3. \tag{84}$$

So, combining (82) and (84) we have $\hat{k} = -2kj$, with $\epsilon^3 = (2k)^{-1}$, k is a positive integer and $j \in \mathbb{Z}_-$. So, we have

$$\mathbf{Z}^{(0)}(T/2, \mathbf{z}_0^*) = \left((m+\tilde{m})\pi,\ \left(i+j+\frac{1}{2}\right)\pi,\ \left(-\frac{\tilde{m}}{j}\right)^{-1/3}, \overline{P}_2^*\right),$$

where $\overline{P}_2^*$ is a real positive constant differ of $\overline{P}_1^* = (-\tilde{m}/j)^{-1/3}$. So, by construction, $\mathbf{Z}^{(0)}(T/2, \mathbf{z}_0^*)$ lies on $\mathcal{L}_2$ and $E(T/2) = \hat{k}\pi$, where $\hat{k} = -2kj$. Therefore, the solution $\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*)$ is a $\hat{S}_2$-symmetric elliptic orbit with period $T = -2j\pi$ ($j \in \mathbb{Z}_-$) on the plane xy.

For the perturbed system, we will look for initial conditions in a neighborhood of $\mathbf{z}_0^*$, in the form

$$\mathbf{z}_0 = \left(m\pi,\ \left(i+\frac{1}{2}\right)\pi, (-\tilde{m}/j)^{-1/3} + \triangle\overline{P}_1, \overline{P}_2^* + \triangle\overline{P}_2\right),$$

in such a way that a solution $\mathbf{Z}(t, \mathbf{z}_0)$ of (72), with a small enough $\epsilon \neq 0$, be a $\hat{S}_2$-symmetric orbit of second kind. Again, by Lemma 5 we have that the solution $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ of (72) is given by

$$\begin{aligned}\overline{Q}_1(t,\epsilon,\mathbf{z}_0) &= (\overline{P}_1^* + \triangle\overline{P}_1)^{-3}t + m\pi &&+ \epsilon^4\overline{Q}_1^{(1)}(t,\mathbf{z}_0,\epsilon) + \mathcal{O}(\epsilon^8) \\ \overline{Q}_2(t,\epsilon,\mathbf{z}_0) &= -t + \left(i+\tfrac{1}{2}\right)\pi &&+ \epsilon^4\overline{Q}_2^{(1)}(t,\mathbf{z}_0,\epsilon) + \mathcal{O}(\epsilon^8) \\ \overline{P}_1(t,\epsilon,\mathbf{z}_0) &= \overline{P}_1^* + \triangle\overline{P}_1 &&+ \epsilon^4\overline{P}_1^{(1)}(t,\mathbf{z}_0,\epsilon) + \mathcal{O}(\epsilon^8) \\ \overline{P}_2(t,\epsilon,\mathbf{z}_0) &= \overline{P}_2^* + \triangle\overline{P}_2 &&+ \epsilon^4\overline{P}_2^{(1)}(t,\mathbf{z}_0,\epsilon) + \mathcal{O}(\epsilon^8),\end{aligned} \tag{85}$$

where $\overline{P}_1^*$ is given in (83). Let $\Phi(\mathbf{X},P)=(\Phi_1(\mathbf{X},\mathbf{P}),\Phi_2(\mathbf{X},\mathbf{P}))=(\overline{Q}_1(T/2,\triangle\overline{P}_1,\epsilon,\triangle\overline{P}_2)-(\tilde{m}+m)\pi,\overline{Q}_2(T/2,\triangle\overline{P}_1,\epsilon,\triangle\overline{P}_2)-(i+j+\frac{1}{2})\pi)$, where $\mathbf{X}=(T,\triangle\overline{P}_1),\mathbf{P}=(\epsilon,\triangle\overline{P}_2)$. By Proposition 4, item 2 and Proposition 6 item (ii), to obtain $\hat{S}_2$-symmetric orbit of the system (72), it is sufficient to solve

$$\begin{aligned}\Phi_1(\mathbf{X},\mathbf{P})&=(\overline{P}_1^*+\triangle\overline{P}_1)^{-3}\tfrac{T}{2}-\tilde{m}\pi+\epsilon^4[\overline{Q}_1^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]=0\\ \Phi_2(\mathbf{X},\mathbf{P})&=-T/2-j\pi+\epsilon^4[\overline{Q}_2^{(1)}(T/2,\mathbf{z}_0,\epsilon)+\mathcal{O}(\epsilon^4)]=0.\end{aligned}\tag{86}$$

By the same reasons of the previous cases (see Remark 5) we can extend $\Phi(\mathbf{X},\mathbf{P})$ to the case $\mathbf{P}=\mathbf{0}$ by defining

$$\begin{aligned}\Phi_1(\mathbf{X},\mathbf{0})&=(\overline{P}_1^*+\triangle\overline{P}_1)^{-3}\tfrac{T}{2}-\tilde{m}\pi\\ \Phi_2(\mathbf{X},\mathbf{0})&=-T/2-j\pi.\end{aligned}$$

Thus, it is clear that $\Phi(\mathbf{X}_0,\mathbf{0})=0$, where $\mathbf{X}_0=(-2j\pi,0)$. Here, we have

$$D_{\mathbf{X}}\Phi(\mathbf{X},\mathbf{P})=\begin{pmatrix}\frac{1}{2}(\overline{P}_1^*+\triangle\overline{P}_1)^{-3}+\mathcal{O}(\epsilon^4) & -3(\overline{P}_1^*+\triangle\overline{P}_1)^{-4}\frac{T}{2}+\mathcal{O}(\epsilon^4)\\ -\frac{1}{2}+\mathcal{O}(\epsilon^4) & 0+\mathcal{O}(\epsilon^4)\end{pmatrix},$$

where $\overline{P}_1^*=(-\tilde{m}/j)^{-1/3}$. So, it follows that

$$D_{\mathbf{X}}\Phi(\mathbf{X}_0,0)=\begin{pmatrix}-\frac{1}{2}\frac{\tilde{m}}{j} & 3\pi(-\tilde{m})^{4/3}j^{-1/3}\\ -1/2 & 0\end{pmatrix},$$

whose determinant is not null. Then, we can prove a similar result as in Lemma 4. Now we are in a position to apply Arenstorf's theorem to solve the system of equations (86) using the same arguments as in Theorem 5.

Theorem 6. *Consider the equations of motion (72) for the elliptic isosceles restricted three-body problem with collision, where the primaries move in an elliptic-collision orbit with energy $h=1/2\ \epsilon^{-2}$. If $\epsilon=(2k)^{-1/3}$ for k a large enough positive integer, then there exist initial conditions for the infinitesimal body such that its motion is a $\hat{S}_2$-symmetric periodic solution, near a Keplerian elliptic orbit on the xy-plane whose period is close to $-2j\pi$ with $j\in\mathbb{Z}_-$.*

Remark 10. Similar facts to the Remark 6 are valid to this symmetry and to this case, we also have that an analogous Corollary 1, 2 and Theorem 2 are also true in this case.

6.3. $\hat{S}_1$-symmetric Periodic Solutions as Continuation Of Elliptic Orbits

We are going to characterize the solutions of the Kepler problem (73) with initial conditions $\mathbf{z}_0^* \in \mathcal{L}_1$ when $t=0$ and $E(0)=0$,

$$\mathbf{z}_0^* = (m\pi, i\pi, \overline{P}_1^*, \overline{P}_2^*),$$

where m, i are integers that we will choose, without loss of generality, to be either 0 or 1, and $\overline{P}_1^*, \overline{P}_2^*$ are constants to be determined. These solutions are of the form:

$$\begin{aligned} \overline{Q}_1^{(0)}(t, \mathbf{z}_0^*) &= (\overline{P}_1^*)^{-3} t + m\pi, & \overline{P}_1^{(0)}(t, \mathbf{z}_0^*) &= \overline{P}_1^* \\ \overline{Q}_2^{(0)}(t, \mathbf{z}_0^*) &= -t + i\pi, & \overline{P}_2^{(0)}(t, \mathbf{z}_0^*) &= \overline{P}_2^*. \end{aligned}$$

By Proposition 4, item 1 and Proposition 6 item (i), we can obtain a $\hat{S}_1$-symmetric elliptic orbit to the Kepler problem (73), if we can solve the set of two equations at time $t = T/2$ in three unknowns:

$$\begin{aligned} \overline{Q}_1^{(0)}(T/2, \mathbf{z}_0^*) &= (\overline{P}_1^*)^{-3} \tfrac{T}{2} + m\pi = (m + \tilde{m})\pi \\ \overline{Q}_2^{(0)}(T/2, \mathbf{z}_0^*) &= -T/2 + i\pi = (i + j)\pi, \end{aligned} \tag{87}$$

where $\tilde{m}, j$ are integers. The second equation is satisfied by

$$\frac{T}{2} = -j\pi, \tag{88}$$

where $j \in \mathbb{Z}_-$. Substituting (88) in the first equation of (87), it follows that

$$(\overline{P}_1^*)^{-3} = -\frac{\tilde{m}}{j}, \tag{89}$$

where $\tilde{m} \in \mathbb{Z}_+$. Observe that we have some freedom in choosing $\overline{P}_2^*$, because we have only the restrictions $\overline{P}_2^* \neq \overline{P}_1^*$ and as $\overline{P}_2$ is the angular momentum, $\overline{P}_2^*$ must be not null. By Remark 3, we will want also that $E(0) = 0$ and $E(T/2) = \hat{k}\pi$, where $\hat{k}$ is a convenient even integer, and in this way we must take

$$T = 2\hat{k}\pi\epsilon^3. \tag{90}$$

So, combining (88) and (90) we have

$$\hat{k} = -2kj,$$

with $\epsilon^3 = (2k)^{-1}$, k a positive integer and $j \in \mathbb{Z}_-$. So, we have

$$\mathbf{Z}^{(0)}(T/2, \mathbf{z}_0^*) = \left((m+\tilde{m})\pi,\ (i+j)\pi,\ \left(-\frac{\tilde{m}}{j}\right)^{-1/3}, \overline{P}_2^* \right),$$

where $\overline{P}_2^*$ is a real positive constant different from $\overline{P}_1^* = (-\tilde{m}/j)^{-1/3}$. So, by construction, $\mathbf{Z}^{(0)}(T/2, \mathbf{z}_0^*)$ lies in $\mathcal{L}_1$ and $E(T/2) = \hat{k}\pi$, where $\hat{k} = -2kj$. Therefore, the solution $\mathbf{Z}^{(0)}(t, \mathbf{z}_0^*)$ is a $\hat{S}_1$-symmetric elliptic orbit with period $T = -2j\pi$ $(j \in \mathbb{Z}_-)$ in the xy plane.

Now, we will proceed to study the perturbed system. We will look for initial conditions in a neighborhood of $\mathbf{z}_0^*$, of type

$$\mathbf{z}_0 = (m\pi,\ i\pi,\ (-\tilde{m}/j)^{-1/3} + \triangle\overline{P}_1, \overline{P}_2^* + \triangle\overline{P}_2),$$

in such a way that a solution $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ of (72), with small enough $\epsilon \neq 0$, be a $\hat{S}_1$-symmetric orbit of second kind.

By Lemma 5 we have that the solution $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ of (72) is given by

$$\begin{aligned}
\overline{Q}_1(t, \mathbf{z}_0, \epsilon) &= (\overline{P}_1^* + \triangle\overline{P}_1)^{-3} t + m\pi + \epsilon^4 \overline{Q}_1^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8)\\
\overline{Q}_2(t, \mathbf{z}_0, \epsilon) &= -t + i\pi + \epsilon^4 \overline{Q}_2^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8)\\
\overline{P}_1(t, \mathbf{z}_0, \epsilon) &= \overline{P}_1^* + \triangle\overline{P}_1 + \epsilon^4 \overline{P}_1^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8)\\
\overline{P}_2(t, \mathbf{z}_0, \epsilon) &= \overline{P}_2^* + \triangle\overline{P}_2 + \epsilon^4 \overline{P}_2^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^8),
\end{aligned} \tag{91}$$

where $\overline{P}_1^*$ is given in (89). Defining, $\Phi(\mathbf{X}, P) = (\Phi_1(\mathbf{X}, \mathbf{P}), \Phi_2(\mathbf{X}, \mathbf{P})) = (\overline{Q}_1(T/2, \mathbf{z}_0, \epsilon) - (\tilde{m} + m)\pi,\ \overline{Q}_2(T/2, \mathbf{z}_0, \epsilon) - (i + j)\pi)$, where $\mathbf{X} = (T, \triangle\overline{P}_1)$, $\mathbf{P} = (\epsilon, \triangle\overline{P}_2)$, and by Proposition 4, item 1 and Proposition 6 item (i), to obtain a $\hat{S}_1$-symmetric orbit of the system (72), it is enough to solve,

$$\begin{aligned}
\Phi_1(\mathbf{X}, \mathbf{P}) &= (\overline{P}_1^* + \triangle\overline{P}_1)^{-3} \tfrac{T}{2} - \tilde{m}\pi + \epsilon^4 [\overline{Q}_1^{(1)}(T/2, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^4)] = 0\\
\Phi_2(\mathbf{X}, \mathbf{P}) &= -T/2 - j\pi + \epsilon^4 [\overline{Q}_2^{(1)}(T/2, \mathbf{z}_0, \epsilon) + \mathcal{O}(\epsilon^4)] = 0.
\end{aligned} \tag{92}$$

By the same reasons of the previous cases (see Remark 5) we can extend $\Phi(\mathbf{X}, \mathbf{P})$ to the case $\mathbf{P} = \mathbf{0}$ by defining

$$\begin{aligned}
\Phi_1(\mathbf{X}, \mathbf{0}) &= (s^{-1/3} + \triangle\overline{P}_1)^{-3} \tfrac{T}{2} - \tilde{m}\pi\\
\Phi_2(\mathbf{X}, \mathbf{0}) &= -\tfrac{T}{2} - j\pi.
\end{aligned}$$

Thus, it is clear that $\Phi(\mathbf{X}_0, \mathbf{0}) = 0$, where $\mathbf{X}_0 = (-2j\pi, 0)$. It is verified that

$$D_{\mathbf{X}}\Phi(\mathbf{X}, \mathbf{P}) = \begin{pmatrix} \frac{1}{2}(\overline{P}_1^* + \triangle\overline{P}_1)^{-3} + \mathcal{O}(\epsilon^4) & -3(\overline{P}_1^* + \triangle\overline{P}_1)^{-4} \frac{T}{2} + \mathcal{O}(\epsilon^4)\\ -\frac{1}{2} + \mathcal{O}(\epsilon^4) & 0 + \mathcal{O}(\epsilon^4) \end{pmatrix},$$

where $\overline{P}_1^* = (-\tilde{m}/j)^{-1/3}$. So, it follows that

$$D_{\mathbf{X}}\Phi(\mathbf{X}_0, 0) = \begin{pmatrix} -\frac{1}{2}\frac{\tilde{m}}{j} & -3\pi\tilde{m}^{4/3} j^{-1/3}\\ -1/2 & 0 \end{pmatrix},$$

whose determinant is not null. Easily we prove a similar result as in Lemma 4. Thus, we can apply Arenstorf's theorem to solve the system of Eqs. (92) in a similar way of Theorem 5.

Theorem 7. *Consider the equations of motion (72) for the elliptic isosceles restricted three-body problem with collision, where the primaries move in an elliptic-collision orbit with energy* $h = 1/2\ \epsilon^{-2}$. *If* $\epsilon = (2k)^{-1/3}$ *for k a large enough positive integer, then there exist initial conditions for the infinitesimal body such that its motion is a* $\hat{S}_1$*-symmetric periodic solution, near a Keplerian elliptic orbit on the* xy*-plane whose period is close to* $-2j\pi$ *with* $j \in \mathbb{Z}_-$.

Remark 11. Similar consequences of Remark 6 are valid to this symmetry and an adapted version of Corollary 1, 2 and Theorem 2 are also true in this case.

7. PERIODIC ORBITS FAR FROM THE PRIMARIES NOT NECESSARILY SYMMETRIC

In this section using similar arguments to [9] in Chapter 9 or equivalently to Section E of Chapter VI in [8] (both applied to an autonomous Hamiltonian system) or in [6] (applied to a periodic Hamiltonian system), we will consider the situation when the infinitesimal particle is far from the primaries. Without loss of generality we will assume that $\epsilon = 1$, then system (13) is 2π-periodic.

Theorem 8. *There exist two one-parameter families of nearly circular large periodic solutions of the elliptic collision isosceles restricted 3-body problem whose period is nearly* 2π. *These orbits tend to infinity.*

Proof. The proof is based in similar arguments to the ones used in [8] pp. 161 in the planar circular restricted three-body problem. But, we need to provide the arguments of the proof in our case because they are problems from different nature.

Now, we will consider the situation when the infinitesimal body is far from the primaries in which case it will be called the *comet*. To consider orbits close to infinity, scale the position variables by μ^{-2} and momentum variables by μ, where μ is a positive parameter.

Like in the case of the planar restricted 3-body problem, we cannot prove the continuation of the periodic solutions in fixed coordinates because the system in these coordinates is very degenerated. For this reason, we consider a rotating coordinate system (ξ, η) obtained through the symplectic transformation (15). In this way we have that the equations of motion in the new variables are given by

$$\dot{\xi} = p_\xi + \eta,\ \dot{p}_\xi = p_\eta - \frac{\xi}{(\xi^2 + \eta^2 + \frac{\rho^2(t)}{4})^{3/2}}$$
$$\dot{\eta} = p_\eta - \xi,\ \dot{p}_\eta = -p_\xi - \frac{\eta}{(\xi^2 + \eta^2 + \frac{\rho^2(t)}{4})^{3/2}},$$

and the new Hamiltonian function is

$$H(\xi, \eta, p_\xi, p_\eta, t) = \frac{p_\xi^2 + p_\eta^2}{2} + \eta p_\xi - \xi p_\eta - \frac{1}{\sqrt{\xi^2 + \eta^2 + \frac{\rho^2(t)}{4}}}. \tag{93}$$

The "comet variables" $(\underline{x}, \underline{y}, p_{\underline{x}}, p_{\underline{y}})$ are introduced by

$$\xi = \mu^{-2}\underline{x},\ \eta = \mu^{-2}\underline{y},\ p_\xi = \mu\, p_{\underline{x}},\ p_\eta = \mu\, p_{\underline{y}}.$$

Since this change is μ-symplectic, the Hamiltonian function in the comet variables is

$$\mathcal{H} = \mathcal{H}(\underline{\mathbf{q}}, \underline{\mathbf{p}}, t, \mu) = -\underline{\mathbf{q}}^T K \underline{\mathbf{p}} + \mu^3 \left(\frac{\|\underline{\mathbf{p}}\|^2}{2} - \frac{1}{\sqrt{\|\underline{\mathbf{q}}\|^2 + \mu^4 \frac{\rho^2(t)}{4}}} \right), \tag{94}$$

where $\underline{\mathbf{q}} = (\underline{x}, \underline{y}), \underline{\mathbf{p}} = (p_{\underline{x}}, p_{\underline{y}})$.

Since $\mathcal{H}$ is an analytic function in μ, expanding it as a Taylor series around $\mu = 0$, we have that

$$\mathcal{H} = -\underline{\mathbf{q}}^T K \underline{\mathbf{p}} + \mu^3 \left(\frac{\|\underline{\mathbf{p}}\|^2}{2} - \frac{1}{\|\underline{\mathbf{q}}\|} \right) + \mathcal{O}(\mu^7). \tag{95}$$

Observe that when μ is very small, the infinitesimal body is close to infinity and by the Hamiltonian (95) near to infinity the Coriolis force dominates. Introducing symplectic polar coordinates (r, θ, R, Θ) given by

$$\underline{x} = r\cos\theta,\ \underline{y} = r\sin\theta$$

$$p_{\underline{x}} = R\cos\Theta - \frac{\Theta}{r}\sin\theta,\ p_{\underline{y}} = R\sin\theta + \frac{\Theta}{r},$$

the Hamiltonian (95) becomes

$$\underline{\mathcal{H}} = \underline{\mathcal{H}}(r, \theta, R, \Theta, t, \mu) = -\Theta + \mu^3 \left(\frac{1}{2} \left[R^2 + \frac{\Theta^2}{r^2} - \frac{1}{r} \right) + \mathcal{O}(\mu^7) \right].$$

So, the equations of motion are given by

$$\begin{aligned} \dot{r} &= \mu^3 R + \mathcal{O}(\mu^7), & \dot{R} &= \tfrac{\mu^3 \Theta^2}{r^3} - \tfrac{\mu^3}{r^2} + \mathcal{O}(\mu^7), \\ \dot{\theta} &= -1 + \tfrac{\mu^3 \Theta}{r^2} + \mathcal{O}(\mu^7), & \dot{\Theta} &= 0 + \mathcal{O}(\mu^7). \end{aligned} \tag{96}$$

The parameter μ is inversely proportional to the square root of the distance of the infinitesimal body from the primaries. Thus, as $\mu \to 0$, this distance goes to infinity and the form of the differential equation (96) degenerates. We cannot, therefore, use perturbation methods which rely on solving the differential equation when $\mu = 0$. Instead, we need to obtain solutions for μ in a deleted neighborhood of $\mu = 0$, and to do this, we need to approximate solutions to this system of differential equations and good estimates. Also, since we are looking for periodic solutions far from the primaries and therefore of long period, we need these approximate solutions for large values of t and small values of μ.

Now, we will consider the equation of the first approximation by dropping the μ^7 term, i.e., consider the equations

$$\begin{aligned} \dot{r} &= \mu^3 R, & \dot{R} &= \tfrac{\mu^3 \Theta^2}{r^3} - \tfrac{\mu^3}{r^2}, \\ \dot{\theta} &= -1 + \tfrac{\mu^3 \Theta}{r^2}, & \dot{\Theta} &= 0, \end{aligned} \tag{97}$$

which are the equations of motion of the Kepler problem in the scale rotating polar. Omitting these terms gives us a system where Θ is a first integral, so let $\Theta = \pm\sqrt{2}c$, where $c \in \mathbb{Z}$. The circular orbit $\psi(t)$ given by $r(t) = c^2, R = 0$ is a periodic solution of (97) with period $2\pi c^3\ (-c^3 \pm \sqrt{2}\ \mu^3)^{-1}$. Observe that Θ is also a first integral to the full problem, so we can reduce the dimension of the space phase in one unit and moreover introducing the Poincaré map in a level surface of the angular momentum Θ and so, we will reduce it in one more unit. In this way, when we linearize the system (97), we will work only in the variables r and R. By linearizing the r and R equations about $\psi(\tau)$ we have

$$\dot{r} = \mu^3 R, \quad \dot{R} = -\frac{4\mu^3}{c^6}.$$

This system has solutions of the form $exp(\pm\frac{2\mu^3}{c^3})$, and so the non-trivial multipliers of the circular orbits of (97) are $exp(\frac{\pm 2\pi\mu^3\ i}{-c^3 \pm \sqrt{2}\ \mu^3}) = \mp\frac{2\pi}{c^3}\mu^3\ i + \mathcal{O}(\mu^6)$.

Consider the Poincaré map in a level surface of the angular momentum Θ about the circular orbit $\psi(\tau)$. Let u be the coordinate in this surface, with $u = 0$, corresponding to the circular orbit when $\mu = 0$. The Poincaré map has a fixed point at the origin up to the μ^3 terms and it is the identity up the terms of order μ^2, and at μ^3 there is a term whose Jacobian has eigenvalues $\mp\frac{2\pi}{c^3}\ i$. That is, the Poincaré map is of the form $P(u)=u+\mu^3 p(u)+\mathcal{O}(\mu^6)$, where $p(0)=0$, $\partial p(0)/\partial u$ has eigenvalues $\mp\frac{2\pi}{c^3}\ i$. So, $\partial p(0)/\partial u$ is non-singular. Applying the Implicit Function Theorem to $G(u,\mu)=(P(u)-u)/\mu^3=p(u)+\mathcal{O}(\mu^3)$. Since $G(0,0)=0$ and $\partial G(0,0)/\partial u=\partial p(0)/\partial u\neq 0$, there is a smooth function $\underline{u}(\mu)$ such that $G(\underline{u}(\mu),\mu)=0$ for all sufficiently small μ. So, the two solutions can be continued from the equations in (97) to the full equations, where the $\mathcal{O}(\mu^7)$ terms are included. In the scale variables, these solutions have period T approximated to 2π. Thus, we conclude the proof of the theorem. □

8. APPENDIX

8.1. Continuation Method with Non-regular Dependency on the Parameter

Let U be an open domain in $\mathbb{R}^n$, V an open domain in $\mathbb{R}^k$, $\mathbf{X}\in U$ and $\mathbf{P}=(\epsilon, y_1, y_2, \ldots, y_{k-1})$. Poincaré's method of analytic continuation reduces to solving a system of equations $\Phi(\mathbf{X},\mathbf{P})$, with $\Phi(\mathbf{X}_0,0)=0$ for $\mathbf{X}$ as function of $\mathbf{P}$. If the system is analytic or differentiable enough and $D_{\mathbf{X}}\Phi(\mathbf{X}_0,0)\neq 0$, then the Implicit Function Theorem guarantees the existence of such a solution. There are cases, however, where the function Φ is not differentiable with respect to $\mathbf{P}$, so the Implicit Function Theorem cannot be applied. Arenstorf proved a result that dropped this problem. The differenciability with respect to $\mathbf{P}$ is not required and instead of this condition, he uses that the function Φ satisfies some mild regularity conditions. Arenstorf's Fixed Point Theorem is as follows:

Theorem A (Arenstorf's Theorem). *We assume W and V to be Banach spaces with elements $\mathbf{X}$ and $\mathbf{P}$ respectively. Let $\mathbf{g}$ be a mapping from the product space $W\times V$ into W, given be $(\mathbf{X},\mathbf{P})\to\mathbf{g}(\mathbf{X},\mathbf{P})\in W$, and defined for $\mathbf{X}$ in a ball B^* around $\mathbf{X}=\mathbf{X}_0\in W$ and $\mathbf{P}$ in a region B of V containing $\mathbf{P}=0$, with $\mathbf{g}(\mathbf{X}_0,0)=\mathbf{X}_0$, and $B^*=\{\mathbf{X}\in W/\|\mathbf{X}-\mathbf{X}_0\|\leqslant\alpha^*, \alpha^*>0\}$. If, for every $\mathbf{P}\in B, \mathbf{g}$ is differentiable with respect to $\mathbf{X}\in B^*$ and*

$$|D_{\mathbf{X}}\mathbf{g}(\mathbf{X},\mathbf{P})|\leqslant\zeta^*\leqslant\frac{1}{2}\, on\ B^*\times B,$$

(where the norm of the linear operator $D_{\mathbf{X}}g(\mathbf{X},\mathbf{P}P)$ is the sup norm) and if

$$\|\mathbf{g}(\mathbf{X}_0,\mathbf{P})-\mathbf{X}_0\|\leqslant\frac{\alpha^*}{2}, on B,$$

then there exists a function $\mathbf{X}(\mathbf{P})$ *with*

$$\mathbf{g}(\mathbf{X}(P),\mathbf{P})=\mathbf{X}(\mathbf{P}),\mathbf{X}(\mathbf{P})\in B^{*}\, for \mathbf{P}\in B,\mathbf{X}(0)=\mathbf{X}_0.$$

In [3] the reader can find the proof of this theorem. We observe that neither continuity of $\mathbf{g}$ in $\mathbf{X},\mathbf{P}$ together, nor continuity of $D_{\mathbf{X}}\mathbf{g}$ at all is required here in contrast to the usual formulation and derivation of the Implicit Function Theorem.

Remark 12. It is possible to check from the proof of Arenstorf's Theorem ([3]) that if the function $\mathbf{g}$ is continuous in $(\mathbf{X},\mathbf{P})$ then the function $\mathbf{X}(\mathbf{P})$ is also continuous in $\mathbf{P}$ in appropriate neighborhood.

Applications of the Arenstorf theorem were given by Cors et al in [4, 5] and Meyer and Howison in [10]. They showed the existence of symmetric periodic orbits in the restricted three-body problem, in contrast with others results which generally use the Poincaré Continuation Method, or more precisely, the Implicit Function Theorem to obtain periodic orbits. The problem of differentiability of the equations of periodic in the problems of continuation of periodic orbits, to some extent, is dropped by Arenstorf's theorem.

By mean of this theorem it can be seen that a sufficient condition for the existence of a solution of $\Phi(\mathbf{X},\mathbf{P})=0$, in a neighborhood of $\mathbf{X}=\mathbf{X}_0$, $\mathbf{P}=0$ is that the determinant of $D_{\mathbf{X}}\Phi(\mathbf{X}_0,0)$ does not vanish together with some regularity conditions, as stated in the following proposition. Its proof follows in a similar way to the proof of proposition 2 in [5], but since we will use this kinds of arguments through this paper we decide to include the proof with all the details.

Proposition 7. *Let U be an open domain in $\mathbb{R}^n$, V a region of $\mathbb{R}^k$ containing $\mathbf{P}=0$. Let $\mathbf{X}\in U$, $\mathbf{P}=(\nu,y_1,\ldots,y_{k-1})\in V$ and $\Phi:U\times V\to\mathbb{R}^n$ with $\Phi(\mathbf{X}_0,0)=\mathbf{0}$, differentiable, for every $\mathbf{P}\in V$, with respect to $\mathbf{X}\in U$, and $D_{\mathbf{X}}\Phi(\mathbf{0},0)$ non-singular. Assume that there exist positive constants b,c,d such that for $\mathbf{X}\in U$*

1. $|(D_{\mathbf{X}}\Phi(\mathbf{X}_0,0))^{-1}|\leqslant b$,
2. $|D_{\mathbf{X}}\Phi(\mathbf{X},\mathbf{P})-D_{\mathbf{X}}\Phi(\mathbf{X}_0,0)|\leqslant c\,(\|\mathbf{X}-\mathbf{X}_0\|+\nu)$,
3. $\|\Phi(\mathbf{X}_0,\mathbf{P})\|\leqslant d\,\nu$.

Then there exist a function $\mathbf{X}(P)\in U$, defined for $\mathbf{P}\in V'\subset V$, such that $\Phi(\mathbf{X}(\mathbf{P}),\mathbf{P})=0$ and $\mathbf{X}(0)=\mathbf{X}_0$.

Proof. Let $\alpha^{*}=\dfrac{d}{2c(db+1/2)}$, $\beta=\dfrac{1}{4bc(db+1/2)}$ and let the auxiliary function

$$g(\mathbf{X},\mathbf{P})=\mathbf{X}-(D_{\mathbf{X}}\phi(\mathbf{X}_0,0))^{-1}\phi(\mathbf{X},\mathbf{P}). \tag{98}$$

Since by hypothesis $\Phi(\mathbf{X}, P)$ is differentiable with respect to $\mathbf{X}\in U$ for every $P\in V$, we have that $g(\mathbf{X},\mathbf{P})$ is differentiable, for every $\mathbf{P}\in V$, with respect to $\mathbf{X}\in U$. Now, by hypothesis 1 and 2 we have

$$\begin{aligned}\|D_{\mathbf{X}}g(\mathbf{X},\mathbf{P})\| &= \|I-(D_{\mathbf{X}}\Phi(\mathbf{X}_0,0))^{-1}D_{\mathbf{X}}\Phi(\mathbf{X},\mathbf{P})\|\\ &\leqslant \|(D_{\mathbf{X}}\Phi(\mathbf{X}_0,0))^{-1}\|\,\|D_{\mathbf{X}}\Phi(\mathbf{X},\mathbf{P})-D_{\mathbf{X}}\Phi(\mathbf{0},0)\|\\ &\leqslant bc(\|\mathbf{X}-\mathbf{X}_0\|+\nu)\leqslant bc(\alpha^*+\beta)=\frac{1}{2},\end{aligned}$$

considering $\mathbf{X}\in U$ and ν such that $\|\mathbf{X}-\mathbf{X}_0\|\leqslant\alpha^*$ and $\nu\leqslant\beta$, here I represents the identity matrix.

On the other hand, by hypothesis 1 and 3, for $\nu\leqslant\beta$ the following inequality holds,

$$\|g(\mathbf{X}_0,\mathbf{P})-\mathbf{X}_0\|=\|(D_{\mathbf{X}}\Phi(\mathbf{X}_0,0))^{-1}\Phi(\mathbf{X}_0,\mathbf{P})\|\leqslant bd\nu\leqslant bd\beta=\frac{1}{2}\alpha^*.$$

Therefore, g satisfies the hypothesis of Arenstorf's Fixed Point Theorem and there exists a neighborhood of the origin $V'\subset V$ and a function $\mathbf{X}(\mathbf{P})\in U$ such that $\Phi(\mathbf{X}(\mathbf{P}),\mathbf{P})=0$ for $\mathbf{P}\in V'$. □

This result will be used to show the existence of periodic solutions, when the infinitesimal body is at great distance from the primaries and the perturbation can be seen as a fast periodic forcing. But, for this we will need of good long term estimate such that the hypothesis of the Proposition 7 were obtained. So, in the next section it will be shown results about approximation of solutions of the perturbed system.

8.2. Approximation of Solutions of the Perturbed System

Consider the following differential equation

$$\dot{\mathbf{Z}}=\mathbf{F}(\mathbf{Z},t,\epsilon), \tag{99}$$

where $\mathbf{Z}\in\mathbb{R}^n$, $t\in\mathbb{R}$, $\epsilon\in\mathbb{R}_+$ and

$$\mathbf{F}(\mathbf{Z},t,\epsilon)=\mathbf{F}_0(\mathbf{Z})+\epsilon^l\mathbf{F}_1(\mathbf{Z},t,\epsilon)+\epsilon^{l+r}\mathbf{F}_r(\mathbf{Z},t,\epsilon),$$

where $l>0, r>0$. This case is slightly different to the case considered in [5].

Let $\mathbf{z}_0$ be an initial condition such that $\mathbf{Z}^{(0)}(t,\mathbf{z}_0)$ is a solution of $\dot{\mathbf{Z}}=\mathbf{F}_0(\mathbf{Z})$ which remains bounded and bounded away from the singularities. Let $\mathcal{C}\subset\mathbb{R}^6$ be a compact neighborhood of $\mathbf{Z}^{(0)}(t,\mathbf{z}_0)$ without singularities and let the functions $\mathbf{F}_0, \epsilon^l\mathbf{F}_1, \epsilon^{l+r}\mathbf{F}_r$ be continuous for $\mathbf{Z}\in\mathcal{C}, \epsilon\in[0,\epsilon_1]$, $t\in\mathbb{R}$. Furthermore, suppose that $\mathbf{F}_0, \mathbf{F}_1$ and $\mathbf{F}_r$ together with all their derivatives with respect to $\mathbf{Z}$ are bounded on $\mathcal{C}$ by a positive constant, say, c_1 independent of ϵ. So, in particular, we have that $\mathbf{F}_0$ is Lipschitz with respect to the variable $\mathbf{Z}$ with , say, a positive constant c_2. In what follows, the maximum norm for vectors $\mathbf{v} \in \mathbb{R}^6$ and the usual norm of the supreme on the unit ball for linear operators will be used. The next two lemma, as seen in [5], show that the solution of Eq. (99) can be written as the solution of $\dot{\mathbf{Z}} = \mathbf{F}_0(\mathbf{Z})$ plus terms which are of order ϵ^l, and the same is true about its partial derivatives with respect to the initial conditions.

Lemma 5. *Let $\mathbf{z}_0$ be the initial condition such that $\mathbf{Z}^{(0)}(t, \mathbf{z}_0)$ is a solution of*

$$\dot{\mathbf{Z}} = \mathbf{F}_0(\mathbf{Z}),$$

which remains bounded and bounded away from the singularities. Let $\mathbf{Z}(t, \mathbf{z}_0, \epsilon)$ be a solution of the system (99) with initial condition $\mathbf{z}_0$. Then, it is verified that

$$\mathbf{Z}(t, \mathbf{z}_0, \epsilon) = \mathbf{Z}^{(0)}(t, \mathbf{z}_0) + \epsilon^l \mathbf{Z}^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathbf{Z}_r(t, \mathbf{z}_0, \epsilon),$$

where $\mathbf{Z}^{(1)}(t, \mathbf{z}_0, \epsilon)$ is the solution of the variational equation

$$\dot{\mathbf{Z}}^{(1)}(t, \mathbf{z}_0, \epsilon) = DF_0(\mathbf{Z}^{(0)}(t, \mathbf{z}_0))\mathbf{Z}^{(1)}(t, \mathbf{z}_0, \epsilon) + \mathbf{F}_1(\mathbf{Z}^{(0)}(t, \mathbf{z}_0), t, \epsilon), \quad (100)$$

which is bounded on $\mathcal{C}$ with initial condition $\mathbf{Z}^{(1)}(0, \mathbf{z}_0, \epsilon) = 0$, where $DF(\cdot)$ is the matrix whose entries are the partial derivatives of F with respect to the $\mathbf{Z}$ variable, and $\mathbf{Z}_r(t, \mathbf{z}_0, \epsilon)$ is $\mathcal{O}(\epsilon^{l+s})$, where $s = Min\{l, r\}$ in a finite interval of time.

Proof. The prove is analogous to Lemma 3 in [5]. □

The next lemma shows that similar bounds hold for the partial of $\mathbf{Z}_1$ and $\mathbf{Z}_r$ with respect to $\mathbf{Z}$.

Lemma 6. *Let $\mathbf{Z}_r(t, \mathbf{z}_0, \epsilon)$ and $\mathbf{Z}_1(t, \mathbf{z}_0, \epsilon)$be as Lemma 5. Then $D_{\mathbf{z}_0}\mathbf{Z}_r(t, \mathbf{z}_0, \epsilon)$ is $\mathcal{O}(\epsilon^{l+s})$, where $s = \min\{l, r\}$ and $D_{\mathbf{z}_0}\mathbf{Z}_1(t, \mathbf{z}_0, \epsilon)$ is $\mathcal{O}(\epsilon^l)$. for $t \in [0, T^*]$.*

Proof. The proof is analogous to Lemma 4 in [5]. □

8.3. Relation between the Variables

We have the following result:

Lemma 7. *The expression of $\|\mathbf{q}\|$, which coincides with the expression of $\|\hat{\mathbf{q}}\|$, in the Delaunay–Poincaré variables (27), obtained as power series in Q_2, P_1, P_2 near $Q_2=0$, $P_1=s^{-1/3}$, $P_2=0$ is given by*

$$\begin{aligned}\|\mathbf{q}\| &= s^{-2/3}+s^{-1/2}Q_2\sin Q_1 - s^{-1/2}P_2\cos Q_1 + 2s^{-1/3}\left(P_1-s^{-1/3}\right)\\ &\quad + s^{-1/3}Q_2^2\cos^2 Q_1 + \frac{3}{2}s^{-1/6}Q_2\left(P_1-s^{-1/3}\right)\sin Q_1\\ &\quad + 2s^{-1/3}Q_2P_2\sin Q_1\cos Q_1 + \left(P_1-s^{-1/3}\right)^2 - \frac{3}{2}s^{-1/6}\left(P_1-s^{-1/3}\right)\\ &\quad \times P_2\cos Q_1 + s^{-1/3}P_2^2\sin Q_1^2 + \mathcal{O}\left(\|X\|^3\right),\end{aligned}$$

where $X=(Q_2, P_1-s^{-1/3}, P_2)=(\triangle Q_2, \triangle P_1, \triangle P_2)$. And the expression of $\|\hat{\mathbf{q}}\|$ in the Delaunay variables (36), obtained as power series in P_1, P_2 near $P_1=s^{-1/3}$, $P_2=P_2^*$, where $s\in\mathbb{Z}$ and P_2^* is a non-zero real constant that differ of $s^{-1/3}$, is given by

$$\begin{aligned}\|\hat{\mathbf{q}}\| &= s^{-2/3}-\left(s^{-2/3}-\left(\overline{P}_2^*\right)^2\right)\sin\overline{Q}_1 - s^{-1/3}\sqrt{s^{-2/3}+\left(\overline{P}_2^*\right)^2}\cos Q_1\\ &\quad + \left(2s^{-1/3}-\left(\frac{s^{-2/3}}{\sqrt{s^{-2/3}+\left(\overline{P}_2^*\right)^2}}+\sqrt{s^{-2/3}+\left(\overline{P}_2^*\right)^2}\right)\cos\overline{Q}_1 - 2s^{-1/3}\sin Q_1\right)\\ &\quad \times\left(\overline{P}_1-s^{-1/3}\right)+\left(-\frac{vw}{\sqrt{s^{-2/3}+\left(\overline{P}_2^*\right)^2}}\cos\overline{Q}_1+2w\sin\overline{Q}_1\right)\left(\overline{P}_2-\overline{P}_2^*\right)\\ &\quad + \left(\left(-\frac{3s^{-1/3}}{2\sqrt{s^{-2/3}+\left(\overline{P}_2^*\right)^2}}+\frac{s^{-1}}{2\left(s^{-2/3}+\left(\overline{P}2^*\right)^2\right)^{3/2}}\right)\cos\overline{Q}_1+1-\sin\overline{Q}_1\right)\\ &\quad \times\left(\overline{P}_1-s^{-1/3}\right)^2+\left(\frac{s^{-2/3}\overline{P}_2^*}{\left(s^{-2/3}+\left(\overline{P}_2^*\right)^2\right)^{3/2}}-\frac{\overline{P}_2^*}{\sqrt{s^{-2/3}+\left(\overline{P}_2^*\right)^2}}\right)\end{aligned}$$

$$\times \cos \overline{Q}_1 \left(\overline{P}_1 - s^{-1/3}\right)\left(\overline{P}_2 - \overline{P}_2^*\right)$$

$$+ \left(\left(-\frac{s^{-1/3}}{2\sqrt{s^{-2/3} + \left(\overline{P}_2^*\right)^2}} + \frac{s^{-1/3}\left(\overline{P}_2^*\right)^2}{2\left(s^{-2/3} + \left(\overline{P}_2^*\right)^2\right)^{3/2}} \right) \cos \overline{Q}_1 + \sin \overline{Q}_1 \right)$$

$$\times \left(\overline{P}_2 - \overline{P}_2^*\right)^2 + \mathcal{O}\left(\|X\|^3\right),$$

where $X = \left(\overline{P}_1 - s^{-1/3}, \overline{P}_2 - \overline{P}_2^*\right) = (\triangle P_1, \triangle P_2)$.

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