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Nonparabolic Asymptotic Limits of Solutions of the Heat Equation on $\mathbb{R}^N$

Thierry Cazenave,[1] Flávio Dickstein,[2] and Fred B. Weissler[3,4]

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In this paper, we construct solutions $u(t,x)$ of the heat equation on $\mathbb{R}^N$ such that $t^{\frac{\mu}{2}} u(t, xt^{\beta})$ has nontrivial limit points in $C_0(\mathbb{R}^N)$ as $t \to \infty$ for certain values of $\mu > 0$ and $\beta > 1/2$. We also show the existence of solutions of this type for nonlinear heat equations.

KEY WORDS: Heat equation; asymptotic behavior; rescaling.

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1. INTRODUCTION

In this paper, we study the long time behavior of solutions of the heat equation

$$u_t - \Delta u = 0$$

in $\mathbb{R}^N$ with respect to nonparabolic rescalings. In the analysis of the asymptotic behavior of global solutions (in time) of parabolic evolution equations in $\mathbb{R}^N$, one often encounters solutions which decay to 0 as $t \to \infty$. In the case where this decay is power-like, the finer asymptotic behavior is often studied after applying a rescaling. There are two ways to

[1] Laboratoire Jacques-Louis Lions, UMR CNRS 7598, B.C. 187, Université Pierre et Marie Curie, 4, place Jussieu, 75252 Paris Cedex 05, France. E-mail: cazenave@ccr.jussieu.fr
[2] Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, 21944-970 Rio de Janeiro, RJ, Brazil. E-mail: flavio@labma.ufrj.br
[3] LAGA UMR CNRS 7539, Institut Galilée–Université Paris XIII, 99, Avenue J.-B. Clément, 93430 Villetaneuse, France. E-mail: weissler@math.univ-paris13.fr
[4] To whom correspondence should be addressed.

carry out such a rescaling. On the one hand, one considers the behavior as $t \to \infty$ of a spatially rescaled version of the solution, for example

$$v(t,x) = t^{\frac{\mu}{2}} u(t, x\sqrt{t}),$$

where $u(t,x)$ is a solution. The hope is that the rescaled solution $v(t,\cdot)$ will converge to some function f. In this case, undoing the rescaling gives the first term of an asymptotic development of $u(t,\cdot)$ as $t \to \infty$, in terms of f. On the other hand, one can study the limit of space–time dilations of the solutions, for example

$$u_\lambda(t,x) = \lambda^\mu u(\lambda^2 t, \lambda x)$$

as $\lambda \to \infty$. If the transformation $u \mapsto u_\lambda$ leaves invariant the set of solutions, then any limit of the u_λ as $\lambda \to \infty$ should also be a solution. In this case, since $u_{\sqrt{t}}(1,x) = v(t,x)$, these two limiting procedures are formally equivalent. Indeed, if $v(t,\cdot) \to f$, then at least for $t \ge 1$ $u_\lambda(t,x) \to w(t,x)$ as $\lambda \to \infty$, where w is the solution with $w(1,\cdot) = f$.

The first of these two procedures was extended by the authors in [2–4] and by Vázquez and Zuazua [9] to allow different limits of $v(t,x)$ along different sequences $t_n \to \infty$, all with respect to the same rescaling. In [5–7] the authors proved the existence of solutions of the heat equation which have different rates of decay along different sequences $t_n \to \infty$. Such a solution admits different limits of $t^{\frac{\mu}{2}} u(t, x\sqrt{t})$ along different sequences $t_n \to \infty$ and for different values of μ.

The spatial dilation $x \mapsto x\sqrt{t}$ comes from the invariance properties of the heat equation: if $u(t,x)$ is a solution, then so is u_λ for all $\lambda > 0$. The purpose of the present paper is to investigate limits of $t^{\frac{\mu}{2}} u(t, xt^\beta)$ along sequences $t_n \to \infty$ where u is a solution of the heat equation and $\beta > 0$ is not necessarily equal to 1/2. Surprisingly, we find that certain solutions of the heat equation give rise to nontrivial limits of this sort. Also, in this case the corresponding space–time dilations $\lambda^\mu u(\lambda^2 t, \lambda^{2\beta} x)$ no longer leave the set of solutions invariant. Nonetheless, we also find nontrivial limits along sequences $\lambda_n \to \infty$. These limits are all with respect to the uniform topology on $\mathbb{R}^N$, which is the natural one to use because of the smoothing effect of the heat semigroup. Indeed, using weaker topologies, one can obtain limits which clearly are not related to the genuine asymptotic properties of the solution. See Remark 2.3.

At this point, we give precise definitions of the objects we study. The heat semigroup $(e^{t\Delta})_{t\ge 0}$ on $\mathbb{R}^N$ is given by

$$u(t,x) = e^{t\Delta} u_0(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \tag{1.1}$$

for $u_0 \in C_0(\mathbb{R}^N)$ and $t > 0$. We want to consider the limit points in $C_0(\mathbb{R}^N)$ of $t^{\frac{\mu}{2}} u(t, xt^{\beta})$ with $\mu, \beta > 0$ (and in particular $\beta \neq 1/2$). Therefore, we define

$$\omega^{\mu,\beta}(u_0) = \{f \in C_0(\mathbb{R}^N);\ \exists t_n \to \infty \text{ s.t. } \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0 \underset{n\to\infty}{\longrightarrow} f \text{ in } L^{\infty}(\mathbb{R}^N)\}, \tag{1.2}$$

where the dilation $\mathcal{D}^{\mu,\beta}_{\lambda}$ is given by

$$\mathcal{D}^{\mu,\beta}_{\lambda} w(x) = \lambda^{\mu} w(x\lambda^{2\beta}) \tag{1.3}$$

for all $\mu, \beta, \lambda > 0$, and so

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0 = t_n^{\frac{\mu}{2}} u(t_n, \cdot t_n^{\beta}). \tag{1.4}$$

We define the space-time dilation $\Gamma^{\mu,\beta}_{\lambda}$ by

$$[\Gamma^{\mu,\beta}_{\lambda} u_0](t, x) = \lambda^{\mu} u(\lambda^2 t, x\lambda^{2\beta}) \tag{1.5}$$

for all $\mu, \beta, \lambda > 0$ where $u(t, \cdot) = e^{t\Delta} u_0(\cdot)$, and the corresponding ω-limit set

$$\gamma^{\mu,\beta}(u_0) = \{h \in C((0,\infty), C_0(\mathbb{R}^N));\ \exists \lambda_n \to \infty \text{ s.t. } \Gamma^{\mu,\beta}_{\lambda_n} u_0 \underset{n\to\infty}{\longrightarrow} h \text{ in } L^{\infty}((\varepsilon, T) \times \mathbb{R}^N) \text{ for all } 0 < \varepsilon < T < \infty\}. \tag{1.6}$$

As mentioned above, if $\beta = 1/2$ the transformation $\Gamma^{\mu,\beta}_{\lambda}$ leaves invariant the set of solutions of the heat equation. This is equivalently expressed by the commutation relation

$$\mathcal{D}^{\mu,\frac{1}{2}}_{\lambda} e^{\tau\Delta} = e^{\tau\lambda^{-2}\Delta} \mathcal{D}^{\mu,\frac{1}{2}}_{\lambda} \tag{1.7}$$

for all $\mu, \lambda > 0$. For arbitrary $\beta > 0$, the following generalization of (1.7) holds:

$$\mathcal{D}^{\mu,\beta}_{\lambda} e^{\tau\Delta} = e^{\tau\lambda^{-4\beta}\Delta} \mathcal{D}^{\mu,\beta}_{\lambda} \tag{1.8}$$

for all $\tau, \mu, \beta, \lambda > 0$, as can verified by an elementary calculation. In addition,

$$[\Gamma^{\mu,\beta}_{\lambda} u_0](t, \cdot) = \mathcal{D}^{\mu,\beta}_{\lambda} [e^{\lambda^2 t\Delta} u_0] = e^{t\lambda^{2-4\beta}\Delta} [\mathcal{D}^{\mu,\beta}_{\lambda} u_0] \tag{1.9}$$

and in particular

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t}} e^{t\Delta} = e^{t^{1-2\beta}\Delta} \mathcal{D}^{\mu,\beta}_{\sqrt{t}} \tag{1.10}$$

for all $t, \mu, \beta, \lambda > 0$.

If $\beta < 1/2$, then formally $e^{t^{(1-2\beta)}\Delta} \to 0$ as $t \to \infty$. Thus, by (1.10), if $\mathcal{D}^{\mu,\beta}_{\sqrt{t}} u_0$ is bounded in some sense, the limit in (1.2) is 0. In other words, one expects that if $0 < \beta < 1/2$, then either $\omega^{\mu,\beta}(u_0) = \emptyset$ or else $\omega^{\mu,\beta}(u_0) = \{0\}$. This turns out to be true, see Theorem 4.1. On the other hand, if $\beta > 1/2$, then $t^{1-2\beta} \to 0$ as $t \to \infty$ so that formally $e^{t^{1-2\beta}\Delta} \to Id$ as $t \to \infty$. Thus, formally again, the limit points in (1.2) are the limit points of $\mathcal{D}^{\mu,\beta}_{\sqrt{t}} u_0$. The precise situation is more delicate, see Theorem 5.6 and its proof (and in particular formula (5.43)).

From another point of view, $w = \Gamma^{\mu,\beta}_{\lambda} u_0$ is a solution of

$$\partial_t w = \lambda^{2-4\beta} \Delta w.$$

Thus, by a formal passage to the limit, one would expect that if $h \in \gamma^{\mu,\beta}(u_0)$, then

$$\begin{cases} \Delta h = 0 & \text{if } \beta < 1/2, \\ \partial_t h = 0 & \text{if } \beta > 1/2. \end{cases}$$

Since the only harmonic function in $C_0(\mathbb{R}^N)$ is 0 one expects that if $\beta < 1/2$ then either $\gamma^{\mu,\beta}(u_0) = \emptyset$ or $\gamma^{\mu,\beta}(u_0) = \{0\}$. This turns out to be true, see Corollary 3.2. On the other hand, if $\beta > 1/2$ it follows formally that if $h \in \gamma^{\mu,\beta}(u_0)$, then h is constant in time. This is also true, see Proposition 3.1 (iii). The surprising fact, and this is the main point of the paper, is that if $\beta > 1/2$ then $\gamma^{\mu,\beta}(u_0)$ (and $\omega^{\mu,\beta}(u_0)$) can nonetheless be highly nontrivial. (See Theorems 5.1 and Corollary 6.3.) Furthermore, if $f \in \omega^{\mu,\beta}(u_0)$, $f \neq 0$ and $\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0 \to f$ in $C_0(\mathbb{R}^N)$, then for any $\beta' \neq \beta$, $\mathcal{D}^{\mu,\beta'}_{\sqrt{t_n}} e^{t_n \Delta} u_0$ has no limit points in $C_0(\mathbb{R}^N)$. It follows that the full asymptotic behavior of $e^{t\Delta} u_0$ is not completely described by the standard parabolic rescaling. See Remark 5.5 for a further discussion of this point.

The rest of the paper is organized as follows. In Section 2, we show that nothing new for $\beta \neq 1/2$ is obtained if we allow only full limits in Definitions (1.2) and (1.6). (see Proposition 2.1.) It follows that allowing convergence along sequences $\lambda_n \to \infty$ or $t_n \to \infty$ is essential to our analysis. In Section 3, we study the interplay between the limits in Definitions (1.2) and (1.6). As a consequence, we show that if $0 < \beta < 1/2$, then $\gamma^{\mu,\beta}(u_0)$ is either empty or trivial. (see Corollary 3.2.) In Section 4, we conclude the study of the case $0 < \beta < 1/2$ by showing that $\omega^{\mu,\beta}(u_0)$ is either empty or trivial (see Theorem 4.1.). Sections 5 and 6 constitute the heart of the paper, where we study the case $\beta > 1/2$. In Theorem 5.1 and Corollary 6.3, we construct initial values $u_0 \in C_0(\mathbb{R}^N)$ for which $\gamma^{\mu,\beta}(u_0)$ and $\omega^{\mu,\beta}(u_0)$ are nontrivial for some values of $\mu \in (0, N)$ and $\beta > 1/2$. In Section 7, we give an analogous result for $\mu \geq N$ but only for oscillatory solutions (see Theorem 7.2.). Finally in Section 8, we show a result analogous to Theorem 5.1 for the nonlinear equation $u_t - \Delta u + |u|^\alpha u = 0$ (see Theorem 8.1).

2. LIMITS AS $\lambda \to \infty$

The purpose of this section is to prove that if we only allow limits as $\lambda \to \infty$ or as $t \to \infty$ in Definitions (1.6) and (1.2), then nothing new is obtained if $\beta \neq 1/2$. More precisely, we prove the following result.

Proposition 2.1. *Suppose $\mu, \beta > 0$, $\beta \neq 1/2$. Let $u_0, f \in C_0(\mathbb{R}^N)$ and let $h \in C((0,\infty), C_0(\mathbb{R}^N))$.*

(i) *If*

$$t^{\frac{\mu}{2}} u(t, \cdot t^{\beta}) \underset{t \to \infty}{\longrightarrow} f \tag{2.1}$$

in $C_0(\mathbb{R}^N)$, then $f = 0$.

(ii) *If*

$$\lambda^{\mu} u(\lambda^2 t, \cdot \lambda^{2\beta}) \underset{\lambda \to \infty}{\longrightarrow} h(t, \cdot) \tag{2.2}$$

in $C([\varepsilon, T], C_0(\mathbb{R}^N))$ for all $0 < \varepsilon < T < \infty$, then $h = 0$.

The proof uses the following lemma, which depends only on scaling properties.

Lemma 2.2. *Let $\beta, \mu > 0$ and $u \in C((0,\infty), C_0(\mathbb{R}^N))$. The following properties are equivalent.*

(i) *There exists $f \in C_0(\mathbb{R}^N)$ such that* (2.1) *holds.*
(ii) *There exists $h \in C((0,\infty), C_0(\mathbb{R}^N))$ such that* (2.2) *holds.*

In additon, if these properties are true, then $h(t,x) = t^{-\frac{\mu}{2}} f(xt^{-\beta})$ for all $t > 0$ and $x \in \mathbb{R}^N$.

Proof. Note first that property (ii) implies property (i) by setting $t = 1$ in property (ii).

Next, assume property (i) and define

$$u_\lambda(t,x) = \lambda^{\mu} u(\lambda^2 t, \lambda^{2\beta} x). \tag{2.3}$$

If we set $\tau = \lambda^2 t$, then

$$u_\lambda(t,x) = t^{-\frac{\mu}{2}} \tau^{\frac{\mu}{2}} u(\tau, \tau^{\beta}(xt^{-\beta})) \underset{\tau \to \infty}{\longrightarrow} t^{-\frac{\mu}{2}} f(xt^{-\beta}) \tag{2.4}$$

in $C_0(\mathbb{R}^N)$, uniformly for $t \in [\varepsilon, T]$. This concludes the proof. □

Proof of Proposition 2.1. By Lemma 2.2, it suffices to prove statement (ii). Let $h \in C((0,\infty), C_0(\mathbb{R}^N))$ and assume (2.2). Also by Lemma 2.2, we know that $h(t,x) = t^{-\frac{\mu}{2}} f(xt^{-\beta})$ with $f = h(1,\cdot)$. On the other hand, it follows from Proposition 3.1 below that h is independent of $t > 0$; and so, f must be homogeneous of degree $-\frac{\mu}{2\beta}$. Since $f \in C_0(\mathbb{R}^N)$, we conclude that $f = 0$ and thus $h = 0$. □

Remark 2.3. Nonzero limits in (2.1) for $\beta \neq 1/2$ can be obtained with respect to weaker topologies, but these limits have little to do with the asymptotic behavior of the solution. Indeed, fix $0 < \sigma < N$ and let $u_0(x) = |x|^{-\sigma}$. It follows that $u(t) = e^{t\Delta} u_0$ is given by $u(t,x) = t^{-\frac{\sigma}{2}} f(x/\sqrt{t})$ where $f = e^{\Delta} u_0$, so that

$$t^{\frac{\sigma}{2}} u(t, x\sqrt{t}) \equiv f(x). \tag{2.5}$$

Formula (2.5) clearly gives the decay rate and the spatial scaling which completely describe the asymptotic behavior of u.

On the other hand, fix $\mu > \sigma$ and set $\beta = \mu/2\sigma > 1/2$. Since $|x|^{\sigma} f(x) \to 1$ as $|x| \to \infty$, we see that

$$t^{\frac{\mu}{2}} u(t, xt^{\beta}) = t^{\frac{\mu-\sigma}{2}} f(xt^{\beta-\frac{1}{2}}) \underset{t\to\infty}{\longrightarrow} |x|^{-\sigma} \tag{2.6}$$

in $L^{\infty}(\{|x| > \varepsilon\})$ for every $\varepsilon > 0$. Formula (2.6) is misleading in that it suggests that the solution u decays like $t^{-\frac{\mu}{2}}$ for any $\mu > \sigma$.

Furthermore, if $0 < \beta < 1/2$, then

$$t^{\frac{\sigma}{2}} u(t, xt^{\beta}) = f(xt^{\beta-\frac{1}{2}}) \underset{t\to\infty}{\longrightarrow} f(0) \tag{2.7}$$

uniformly on compacts sets. Formula (2.7) represents a considerable loss of information in comparison with (2.5).

This example shows the importance of using the uniform topology in the Definition (1.2).

3. A RELATIONSHIP BETWEEN $\omega^{\mu,\beta}(u_0)$ AND $\gamma^{\mu,\beta}(u_0)$

In the previous section, we used scaling properties in order to obtain a relationship between the limit as $t \to \infty$ of $\mathcal{D}^{\mu,\beta}_{\sqrt{t}} e^{t\Delta} u_0$ and the limit as $\lambda \to \infty$ of $\Gamma^{\mu,\beta}_{\lambda} u_0$. In this section, we obtain a relationship between $\omega^{\mu,\beta}(u_0)$ and $\gamma^{\mu,\beta}(u_0)$ using the (quasi) invariance properties of the heat equation under the various scalings, as described by formulas (1.8) and (1.9).

Proposition 3.1. *Let* $u_0 \in C_0(\mathbb{R}^N)$, $\mu, \beta > 0$ *and let* $\omega^{\mu,\beta}(u_0)$ *and* $\gamma^{\mu,\beta}(u_0)$ *be defined by* (1.2) *and* (1.6), *respectively.*

(i) *If* $h \in \gamma^{\mu,\beta}(u_0)$*, then* $h(1) \in \omega^{\mu,\beta}(u_0)$.

(ii) *Let* $f \in \omega^{\mu,\beta}(u_0)$ *and suppose* $t_n \to \infty$ *is such that* $e^{t_n^{1-2\beta}\Delta}[\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u_0] \to f$ *in* $C_0(\mathbb{R}^N)$ *as* $n \to \infty$*. If we set* $\lambda_n = \sqrt{t_n}$*, then*

$$\Gamma^{\mu,\beta}_{\lambda_n} u_0(t,\cdot) \underset{n\to\infty}{\longrightarrow} g(t,\cdot) \tag{3.1}$$

in $C([1,T], C_0(\mathbb{R}^N))$ *for all* $1 < T < \infty$ *if* $\beta \geq 1/2$ *and in* $C([1+\varepsilon, T], C_0(\mathbb{R}^N))$ *for all* $1 < \varepsilon + 1 < T < \infty$ *if* $\beta < 1/2$*, where g is given by* $g(1) = f$ *and*

$$g(t) = \begin{cases} f & \text{if } \beta > 1/2, \\ e^{(t-1)\Delta} f & \text{if } \beta = 1/2, \\ 0 & \text{if } \beta < 1/2 \end{cases} \tag{3.2}$$

for $t > 1$.

(iii) *If* $h \in \gamma^{\mu,\beta}(u_0)$*, then*

$$\begin{cases} h(t) \equiv h(1) & \text{if } \beta > 1/2, \\ h(t+s) = e^{t\Delta} h(s), \quad s,t > 0 & \text{if } \beta = 1/2, \\ h(t) \equiv 0 & \text{if } \beta < 1/2. \end{cases} \tag{3.3}$$

Corollary 3.2. *Let* $u_0 \in C_0(\mathbb{R}^N)$ *and* $\mu > 0$*. If* $0 < \beta < 1/2$*, then either* $\gamma^{\mu,\beta}(u_0) = \emptyset$ *or else* $\gamma^{\mu,\beta}(u_0) = \{0\}$.

Proof of Proposition 3.1

(i) If $h \in \gamma^{\mu,\beta}(u_0)$, then there exist $\lambda_n \to \infty$ such that $[\Gamma^{\mu,\beta}_{\lambda_n} u_0](1,\cdot) \to h(1)$ in $C_0(\mathbb{R}^N)$. Letting $t_n = \lambda_n^2$ and applying formula (1.9), we deduce that $h(1) \in \omega^{\mu,\beta}(u_0)$.

(ii) By assumption, $e^{\lambda_n^{2-4\beta}\Delta}[\mathcal{D}^{\mu,\beta}_{\lambda_n} u_0] - f \to 0$ in $C_0(\mathbb{R}^N)$. Given $t > 1$ and applying $e^{(t-1)\lambda_n^{2-4\beta}\Delta}$, we deduce that $e^{t\lambda_n^{2-4\beta}\Delta}[\mathcal{D}^{\mu,\beta}_{\lambda_n} u_0] - e^{(t-1)\lambda_n^{2-4\beta}\Delta} f \to 0$ in $C_0(\mathbb{R}^N)$. Using (1.9), we obtain $\Gamma^{\mu,\beta}_{\lambda_n} u_0(t,\cdot) - e^{(t-1)\lambda_n^{2-4\beta}\Delta} f \to 0$. Property (ii) follows, since $e^{(t-1)\lambda_n^{2-4\beta}\Delta} f \to g(t)$ in $C_0(\mathbb{R}^N)$, uniformly for t in a compact subset of $(1,\infty)$ if $\beta < 1/2$ and uniformly for t in a bounded subset of $[1,\infty)$ if $\beta \geq 1/2$.

(iii) Let $h \in \gamma^{\mu,\beta}(u_0)$ and let $\lambda_n \to \infty$ be such that $[\Gamma^{\mu,\beta}_{\lambda_n} u_0] \to h$. Given any $s > 0$, it follows from (1.9) that $e^{s\lambda_n^{2-4\beta}\Delta}[\mathcal{D}^{\mu,\beta}_{\lambda_n} u_0] - h(s) \to 0$. Applying $e^{t\lambda_n^{2-4\beta}\Delta}$, with $t > 0$, we deduce that

$$e^{(t+s)\lambda_n^{2-4\beta}\Delta}[\mathcal{D}^{\mu,\beta}_{\lambda_n} u_0] - e^{t\lambda_n^{2-4\beta}\Delta} h(s) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.4}$$

On the one hand, the term on the left of (3.4) converges to $h(s+t)$ as $n \to \infty$. Since $e^{t\lambda_n^{2-4\beta}\Delta} h(s)$ converges to $h(s)$ if $\beta > 1/2$, to $e^{t\Delta}h(s)$ if $\beta = 1/2$ and to 0 if $\beta < 1/2$, the conclusion follows. $\square$

Remark 3.3. Given $f \in \omega^{\mu,\beta}(u_0)$, it is not clear if there exists $h \in \gamma^{\mu,\beta}(u_0)$ such that $h(1) = f$.

4. THE CASE $\beta < 1/2$

In this section, we show the same result as in Corollary 3.2 but for $\omega^{\mu,\beta}(u_0)$ instead of $\gamma^{\mu,\beta}(u_0)$. Surprisingly this seems to require a different type of argument.

Theorem 4.1. *Let $u_0 \in C_0(\mathbb{R}^N)$ and $\mu > 0$. If $0 < \beta < 1/2$ and $\omega^{\mu,\beta}(u_0)$ is defined by* (1.2)*, then either $\omega^{\mu,\beta}(u_0) = \emptyset$ or else $\omega^{\mu,\beta}(u_0) = \{0\}$.*

Proof. Suppose $f \in \omega^{\mu,\beta}(u_0)$ so that

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta} u_0 \to f \tag{4.1}$$

in $\mathcal{S}'(\mathbb{R}^N)$ for some sequence $t_n \to \infty$. It follows that

$$\mathcal{F}(\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta} u_0) \to \widehat{f} \tag{4.2}$$

in $\mathcal{S}'(\mathbb{R}^N)$. Using (1.8) with $\tau = t_n$ and $\lambda = \sqrt{t_n}$, we see that

$$\begin{aligned} \mathcal{F}(\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta} u_0) &= e^{-4\pi^2 t_n^{(1-2\beta)} |\cdot|^2} \mathcal{F}(\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u_0) \\ &= e^{-4\pi^2 t_n^{(1-2\beta)} |\cdot|^2} t_n^{\frac{\mu}{2} - N\beta} \widehat{u_0}(\cdot / t_n^{\beta}). \end{aligned} \tag{4.3}$$

Therefore, given any $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\langle \mathcal{F}(\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta} u_0), \varphi \rangle = \langle \widehat{u_0}, t_n^{\frac{\mu}{2}} e^{-4\pi^2 t_n |\cdot|^2} \varphi(\cdot t_n^{\beta}) \rangle. \tag{4.4}$$

We now deduce from (4.2), (4.4), and Lemma 9.1 that $\operatorname{supp} \widehat{f} \subset \{0\}$. Thus, f is a polynomial, so that $f = 0$ since $f \in C_0(\mathbb{R}^N)$. $\square$

Remark 4.2. Note that both the cases $\omega^{\mu,\beta}(u_0) = \gamma^{\mu,\beta}(u_0) = \{0\}$ and $\omega^{\mu,\beta}(u_0) = \gamma^{\mu,\beta}(u_0) = \emptyset$ can be achieved. Indeed, given $\mu > 0$ let $\theta \in \mathcal{S}(\mathbb{R}^N)$ and let the integer k be sufficiently large so that $N + k > 2\mu$. Letting $u_0 = \partial^k_{x_1}\theta$, we have $\sup_{t>0} t^{\frac{N+k}{2}} \|e^{t\Delta}u_0\|_{L^\infty} < \infty$, so that $t^{\mu}\|e^{t\Delta}u_0\|_{L^\infty} \to 0$ as $t \to \infty$. One easily concludes that $\omega^{\mu,\beta}(u_0) = \gamma^{\mu,\beta}(u_0) = \{0\}$ for all $\beta > 0$. On the other hand, let $0 < \nu < N$ and let $u_0 = e^{\Delta}|\cdot|^{-\nu}$. It follows that $t^{\frac{\nu}{2}}\|e^{t\Delta}u_0\|_{L^\infty} \to c > 0$ as $t \to \infty$. Therefore, $\omega^{\mu,\beta}(u_0) = \gamma^{\mu,\beta}(u_0) = \emptyset$ for all $\beta > 0$ and $\mu > \nu$.

5. THE CASE $\beta > 1/2$: A FIRST RESULT

In this section, we give a relatively simple example of how $\omega^{\mu,\beta}(u_0)$ and $\gamma^{\mu,\beta}(u_0)$ can be nontrivial, see Theorem 5.1. Our construction is similar to the one used in [5] except that, unlike in [5], the initial value u_0 satisfies $\sup |x|^\sigma |u_0(x)| < \infty$ for some $0 < \sigma < N$. Furthermore, we show in Theorem 5.2 that if also $\inf |x|^\sigma |u_0(x)| > 0$, then the kind of phenomenon described in Theorem 5.1 is not possible. Finally, we show in Theorem 5.6 (see also Remark 5.5) that the non parabolic asymptotic behavior of $e^{t\Delta}u_0$ cannot be obtained from the parabolic asymptotic behavior.

Theorem 5.1. *Fix $\beta > 1/2$ and $0 < \mu < N$. Given $f \in \mathcal{S}(\mathbb{R}^N)$, $f \geq 0$, there exists $u_0 \in C_0(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, $u_0 \geq 0$ with the following properties.*

(i) *There exists a sequence $t_n \to \infty$ such that*

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta} u_0 \underset{n\to\infty}{\longrightarrow} f \tag{5.1}$$

in $C_0(\mathbb{R}^N)$. In other words, $f \in \omega^{\mu,\beta}(u_0)$.

(ii) *There exists a sequence $\lambda_n \to \infty$ such that*

$$\Gamma^{\mu,\beta}_{\lambda_n} u_0 \underset{n\to\infty}{\longrightarrow} h \tag{5.2}$$

in $L^\infty((\varepsilon, T) \times \mathbb{R}^N)$ for all $0 < \varepsilon < T < \infty$, where $h(t) \equiv f$. In other words, $h \in \gamma^{\mu,\beta}(u_0)$.

(iii) *There exists $0 < c_1 < \infty$ such that $|x|^{\frac{\mu}{2\beta}} |u_0(x)| \leq c_1$ for all $x \in \mathbb{R}^N$.*

(iv) *If f is radially symmetric and nonincreasing, then u_0 is radially symmetric and decreasing.*

Proof. Let $f \in \mathcal{S}(\mathbb{R}^N)$, $f \geq 0$ and set

$$\ell = \|f\|_{L^1}, \quad M = \|f\|_{L^\infty}, \quad \sigma = \frac{\mu}{2\beta} \in (0, N). \tag{5.3}$$

Next, we let $a_0 \geq 1$ be large enough so that

$$e^x \geq (2x)^N \quad \text{for} \quad x \geq a_0 \geq 1 \tag{5.4}$$

and we define the sequence $(a_j)_{j\geq 1}$ by

$$\begin{cases} a_1 = a_0, \\ a_{j+1} = \exp(\frac{a_j}{\sigma}), \quad j \geq 1. \end{cases} \tag{5.5}$$

Applying inductively the relation

$$a_{j+1} = \exp\left(\frac{a_j}{\sigma}\right) > \exp\left(\frac{a_j}{N}\right) \geq 2a_j, \tag{5.6}$$

where we used (5.5) and (5.4), we deduce that

$$a_j \uparrow \infty, \quad a_j \geq 2^{j-1} \tag{5.7}$$

for all $j \geq 1$. We now let $u_0 \geq 0$ be defined by

$$u_0(x) = \sum_{j=1}^{\infty} e^{-a_{j-1}} f(x/a_j) = \sum_{j=1}^{\infty} a_j^{-\sigma} f(x/a_j) \geq 0. \tag{5.8}$$

Given any multi-index α, we have

$$\|\partial^\alpha f(\cdot/a_j)\|_{L^\infty} = a_j^{-|\alpha|} \|\partial^\alpha f\|_{L^\infty} \leq \|\partial^\alpha f\|_{L^\infty}$$

by (5.7). Therefore, it follows from (5.7) that the series in (5.8) is normally convergent in $C_{\mathrm{b}}^m(\mathbb{R}^N)$ for all $m \geq 0$, so that $u_0 \in C^\infty(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

Given $t > 0$ and $\lambda_n > 0$, we write using (1.9)

$$[\Gamma_{\lambda_n}^{\mu,\beta} u_0](t, \cdot) = e^{t\lambda_n^{2-4\beta}\Delta}(u^n + v^n + w^n), \tag{5.9}$$

where

$$\begin{cases} u^n = \lambda_n^\mu \sum_{j=1}^{n-1} e^{-a_{j-1}} f(x\lambda_n^{2\beta}/a_j), \\ v^n = \lambda_n^\mu e^{-a_{n-1}} f(x\lambda_n^{2\beta}/a_n), \\ w^n = \lambda_n^\mu \sum_{j=n+1}^{\infty} e^{-a_{j-1}} f(x\lambda_n^{2\beta}/a_j). \end{cases} \tag{5.10}$$

We have by (5.3) and (5.6)

$$\|u^n\|_{L^1} \leq \lambda_n^\mu \sum_{j=1}^{n-1} \|f(\cdot\lambda_n^{2\beta}/a_j)\|_{L^1} = \ell\lambda_n^{\mu-2N\beta} \sum_{j=1}^{n-1} a_j^N \leq n\ell\lambda_n^{\mu-2N\beta} a_{n-1}^N. \tag{5.11}$$

Also,

$$\|w^n\|_{L^\infty} \leq M\lambda_n^\mu \sum_{j=n+1}^{\infty} e^{-a_{j-1}} = M\lambda_n^\mu \sum_{j=n}^{\infty} e^{-a_j}. \tag{5.12}$$

We deduce from (5.11) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta} u^n\|_{L^\infty} \leq t^{-\frac{N}{2}} \lambda_n^{-N(1-2\beta)} \|u^n\|_{L^1} \leq n\ell t^{-\frac{N}{2}} \lambda_n^{-(N-\mu)} a_{n-1}^N \tag{5.13}$$

and from (5.12) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta}w^n\|_{L^\infty} \le \|w^n\|_{L^\infty} \le M\lambda_n^{\mu}\sum_{j=n}^{\infty} e^{-a_j}. \tag{5.14}$$

We now prove Property (ii) and we let

$$\lambda_n = a_n^{1/2\beta}. \tag{5.15}$$

It follows from (5.10) and (5.5) that

$$v^n = f. \tag{5.16}$$

Moreover, we deduce from (5.13), (5.15), and (5.5) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta}u^n\|_{L^\infty} \le n\ell t^{-\frac{N}{2}} a_n^{-\frac{N-\mu}{2\beta}} a_{n-1}^N = n\ell t^{-\frac{N}{2}} e^{-\frac{N-\mu}{\mu}a_{n-1}} a_{n-1}^N.$$

Since $\mu < N$ and $a_n \uparrow \infty$, we conclude that

$$\|e^{t\lambda_n^{2-4\beta}\Delta}u^n\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0 \tag{5.17}$$

uniformly for $t \ge \varepsilon$, for any fixed $\varepsilon > 0$. Next, we deduce from (5.14) and (5.15) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta}w^n\|_{L^\infty} \le Ma_n^{\sigma}\sum_{j=n}^{\infty} e^{-a_j} \le M\sum_{j=n}^{\infty} a_j^{\sigma} e^{-a_j} \underset{n\to\infty}{\longrightarrow} 0, \tag{5.18}$$

where we used (5.7) in the last two relations. Given $T > 0$, we deduce from (5.15) and (5.16) that

$$e^{t\lambda_n^{2-4\beta}\Delta}v^n = e^{t\lambda_n^{2-4\beta}\Delta}f \underset{n\to\infty}{\longrightarrow} h(t) \tag{5.19}$$

in $C_0(\mathbb{R}^N)$, uniformly for $t \in [0,T]$. Property (ii) follows from (5.9), (5.17), (5.18), and (5.19).

Property (i) follows from Property (ii) by setting $t = 1$ and $t_n = \lambda_n^2$. Property (iii) follows from Lemma 9.3 and the definition of u_0. Finally, Property (iv) follows from formula (5.8). □

At first sight it may seem that Property (iii) in Theorem 5.1 is not relevant to the study of $\omega^{\mu,\beta}(u_0)$. On the other hand, the following theorem shows that if this property is strengthened to give a lower bound as well as an upper bound, then $\omega^{\mu,\beta}(u_0)$ is either empty or trivial for all $\beta \ne 1/2$.

Theorem 5.2. *Let $u_0 \in C_0(\mathbb{R}^N)$ and suppose there exist $0 < \sigma < N$ and $0 < c_0 < c_1 < \infty$ such that*

$$c_0 \leq |x|^\sigma u_0(x) \leq c_1 \tag{5.20}$$

for all $x \in \mathbb{R}^N$. If $\mu, \beta > 0$ are such that there exists $f \in \omega^{\mu,\beta}(u_0)$, $f \not\equiv 0$, then $\mu = \sigma$ and $\beta = 1/2$.

Proof. Let $t_n \to \infty$ be such that

$$t_n^{\frac{\mu}{2}} u(t_n, \cdot t_n^\beta) \underset{n\to\infty}{\longrightarrow} f \tag{5.21}$$

in $C_0(\mathbb{R}^N)$. It follows from the results of Cazenave et al. [2] that there exists a subsequence, which we still denote by $(t_n)_{n\geq 1}$, and $\widetilde{f} \in C_0(\mathbb{R}^N)$, $\widetilde{f} > 0$, such that

$$t_n^{\frac{\sigma}{2}} u(t_n, \cdot t_n^{1/2}) \underset{n\to\infty}{\longrightarrow} \widetilde{f} \tag{5.22}$$

in $C_0(\mathbb{R}^N)$. (It is the lower inequality in (5.20) which guarantees that $\widetilde{f} > 0$.) We deduce from (5.21) and (5.22) by taking the L^∞ norm that $\sigma = \mu$. It follows from (5.22) that

$$t_n^{\frac{\sigma}{2}} u(t_n, x t_n^\beta) - \widetilde{f}(x t_n^{\beta - \frac{1}{2}}) \underset{n\to\infty}{\longrightarrow} 0 \tag{5.23}$$

uniformly on $\mathbb{R}^N$. Since $\widetilde{f} \in C_0(\mathbb{R}^N)$ and $\widetilde{f} > 0$, this is compatible with (5.21) only if $\beta = 1/2$. □

Remark 5.3. Let u_0 be given by formula (5.8). Calculations similar to those used in the proof of Theorem 5.1 show that

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u_0 \to f \tag{5.24}$$

in $\mathcal{S}'(\mathbb{R}^N)$ where $t_n = \lambda_n^2$. Setting $s_n = t_n^{2\beta}$ and $\sigma = \frac{\mu}{2\beta}$, this becomes

$$\mathcal{D}^{\sigma,\frac{1}{2}}_{\sqrt{s_n}} u_0 \to f \tag{5.25}$$

in $\mathcal{S}'(\mathbb{R}^N)$. Since $\sup |x|^\sigma |u_0(x)| < \infty$, it follows from Proposition 3.8 (i) in [2] that

$$\mathcal{D}^{\sigma,\frac{1}{2}}_{\sqrt{s_n}} e^{s_n \Delta} u_0 = e^\Delta \mathcal{D}^{\sigma,\frac{1}{2}}_{\sqrt{s_n}} u_0 \to e^\Delta f \tag{5.26}$$

in $C_0(\mathbb{R}^N)$. This implies that $e^\Delta f \in \omega^{\sigma,1/2}(u_0)$.

Remark 5.4. Given $0 < \sigma < N$, there exist $u_0 \in C_0(\mathbb{R}^N)$ with $\sup |x|^\sigma |u_0(x)| < \infty$, $f \in C_0(\mathbb{R}^N)$, $f \neq 0$ and a sequence $s_n \to \infty$ such that

$$\mathcal{D}^{\sigma, \frac{1}{2}}_{\sqrt{s_n}} u_0 \to f \tag{5.27}$$

in $\mathcal{S}'(\mathbb{R}^N)$ and

$$\mathcal{D}^{\sigma, \frac{1}{2}}_{\sqrt{s_n}} e^{s_n \Delta} u_0 \to e^{\Delta} f \tag{5.28}$$

in $C_0(\mathbb{R}^N)$, see Proposition 3.8 (i) in [2]. If $\sigma < \mu < N$, $\beta = \mu/2\sigma > 1/2$ and

$$t_n = s_n^{1/2\beta} \tag{5.29}$$

then (5.27) becomes

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u_0 \to f \tag{5.30}$$

in $\mathcal{S}'(\mathbb{R}^N)$. We deduce from (1.8) and (5.30) that

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0 = e^{t_n^{(1-2\beta)} \Delta} \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u_0 \to f \tag{5.31}$$

in $\mathcal{S}'(\mathbb{R}^N)$. In this way, we may obtain convergence of $\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0$ to a function of $C_0(\mathbb{R}^N)$, but we do not know if the convergence is in $C_0(\mathbb{R}^N)$.

Remark 5.5. The previous two remarks suggest the possibility that if $u_0, f \in C_0(\mathbb{R}^N)$ satisfy $\sup |x|^\sigma (|u_0(x)| + |f(x)|) < \infty$, then the following two limits are equivalent

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0 \to f,$$
$$\mathcal{D}^{\sigma, \frac{1}{2}}_{\sqrt{s_n}} e^{s_n \Delta} u_0 \to e^{\Delta} f$$

in $C_0(\mathbb{R}^N)$ as $n \to \infty$, where $s_n = t_n^{2\beta}$ and $\beta = \mu/2\sigma$. If true, this would mean that the asymptotic behavior with respect to the nonparabolic rescaling (1.4) is somehow equivalent to the asymptotic behavior with respect to a parabolic rescaling. In fact, these limits are *not* equivalent, as the following theorem shows. Thus, and we emphasize this point, consideration of $\omega^{\mu,\beta}(u_0)$ with $\beta > 1/2$ is essential to the understanding of the asymptotic behavior of $e^{t\Delta} u_0$ as $t \to \infty$.

Theorem 5.6. *Let* $0 < \sigma < \mu < N$ *and set* $\beta = \mu/2\sigma > 1/2$. *Given* $f \in C_0(\mathbb{R}^N)$, $f \geq 0$ *such that*

$$f(0) > 0 \tag{5.32}$$

and

$$\sup_{x \in \mathbb{R}^N} |x|^\sigma f(x) < \infty \tag{5.33}$$

there exist $u_0 \in C_0(\mathbb{R}^N)$ *satisfying* $u_0 \geq 0$ *and* $\sup |x|^\sigma u_0(x) < \infty$ *and a sequence* $(t_n)_{n \geq 0}$, $t_n \uparrow \infty$ *such that*

$$\mathcal{D}^{\sigma, \frac{1}{2}}_{\sqrt{s_n}} e^{s_n \Delta} u_0 \to e^\Delta f \tag{5.34}$$

in $C_0(\mathbb{R}^N)$ *with* $s_n = t_n^{2\beta}$ *and*

$$\| \mathcal{D}^{\mu, \beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0 \|_{L^\infty} \to \infty \tag{5.35}$$

as $n \to \infty$.

Proof. By (5.33), we may assume without loss of generality that

$$\sup_{x \in \mathbb{R}^N} |x|^\sigma f(x) \leq 1. \tag{5.36}$$

Furthermore, it follows from (5.32) that there exist $\varepsilon, \delta > 0$ such that

$$f(x) \geq \varepsilon \quad \text{if } |x| \leq \delta. \tag{5.37}$$

Define inductively the sequences $(a_k)_{k \geq 0}$ and $(b_k)_{k \geq 0}$ as follows. Fix $1 < a_0 < b_0$ such that $a_0^{2\beta} > b_0$ and let

$$a_j = a_{j-1}^{2\beta}, \qquad b_j = b_{j-1}^{2\beta}. \tag{5.38}$$

It follows easily that $a_n < b_n < a_{n+1}$ for all $n \geq 0$. Set

$$t_n = (a_n b_n)^{1/2\beta}. \tag{5.39}$$

Consider a sequence $(\varphi_k)_{k \geq 0} \subset C_0(\mathbb{R}^N)$ such that

$$\begin{cases} 0 \leq \varphi_k \leq 1, & \\ \varphi_k(x) = 1 & \text{if } |x| \in [a_k, b_k], \\ \operatorname{supp} \varphi_k \cap \operatorname{supp} \varphi_j = \emptyset & \text{if } k \neq j. \end{cases} \tag{5.40}$$

We define

$$u_0(x) = \sum_{k \geq 0} \varphi_k(x) \mathcal{D}^{\mu, \beta}_{t_k^{-1/2}} f(x). \tag{5.41}$$

Using (5.40), (5.36), it is clear that $u_0 \in C_0(\mathbb{R}^N)$ and $u_0 \geq 0$. Note that by (5.33), $\mathcal{D}^{\mu,\beta}_{t_k^{-1/2}} f(x) \leq |x|^{-\sigma}$ for all $k \geq 0$. Thus $u_0(x) \leq |x|^{-\sigma}$, so that

$$\mathcal{D}^{\mu,\beta}_{\lambda} u_0(x) \leq |x|^{-\sigma} \tag{5.42}$$

for all $\lambda > 0$. We next show that

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u_0 \to f \tag{5.43}$$

in $\mathcal{S}'(\mathbb{R}^N)$, for t_n given by (5.39). Indeed, decompose

$$\begin{aligned} \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u_0 &= \sum_{k<n} \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} (\varphi_k \mathcal{D}^{\mu,\beta}_{t_k^{-1/2}} f) + \varphi_n(\cdot t_n^{\beta}) f + \sum_{k>n} \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} (\varphi_k \mathcal{D}^{\mu,\beta}_{t_k^{-1/2}} f) \\ &:= v^n + w^n + z^n. \end{aligned} \tag{5.44}$$

We write

$$v^n(x) = \sum_{k<n} \varphi_k(x t_n^{\beta}) \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} \mathcal{D}^{\mu,\beta}_{t_k^{-1/2}} f(x).$$

If $|x| > (a_n/b_n)^{1/2}$, then $|x| t_n^{\beta} > (a_n/b_n)^{1/2} (a_n b_n)^{1/2} = a_n$. Using (5.40), this implies that $\varphi_k(x t_n^{\beta}) = 0$ if $k < n$. It follows that $\operatorname{supp} v^n \subset \{|x| \leq (a_n/b_n)^{1/2}\}$. Since $a_n/b_n \to 0$ as $n \to \infty$ and $v^n(x) \leq |x|^{-\sigma}$, see (5.42), we obtain that

$$v^n \underset{n\to\infty}{\longrightarrow} 0 \tag{5.45}$$

in $\mathcal{S}'(\mathbb{R}^N)$. Next, we have

$$z^n(x) = \sum_{k>n} \varphi_k(x t_n^{\beta}) \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} \mathcal{D}^{\mu,\beta}_{t_k^{-1/2}} f(x).$$

If $|x| < (b_n/a_n)^{1/2}$. then $|x| t_n^{\beta} < (b_n/a_n)^{1/2} (a_n b_n)^{1/2} = b_n$, so that $\varphi_k(x t_n^{\beta}) = 0$ if $k > n$. Thus $\operatorname{supp} z^n \subset \{|x| \geq (b_n/a_n)^{1/2}\}$. Since $b_n/a_n \to \infty$ as $n \to \infty$ and $z^n(x) \leq |x|^{-\sigma}$, see (5.42), we obtain that

$$z^n \underset{n\to\infty}{\longrightarrow} 0 \tag{5.46}$$

in $C(\mathbb{R}^N)$. Finally, we observe that $w^n(x) = \varphi_n(x t_n^{\beta}) f(x) = f(x)$ if $|x| t_n^{\beta} \in [a_n, b_n]$, that is, if $(a_n/b_n)^{1/2} \leq |x| \leq (b_n/a_n)^{1/2}$. Using that $a_n/b_n \to 0$ and (5.36), we conclude that

$$w^n \underset{n\to\infty}{\longrightarrow} f \tag{5.47}$$

in $\mathcal{S}'(\mathbb{R}^N)$. The convergence (5.43) now follows from (5.44) to (5.47). This means (see Remark 5.3) that $\mathcal{D}^{\sigma,1/2}_{\sqrt{s_n}} u_0 \to f$ in $\mathcal{S}'(\mathbb{R}^N)$, which implies (5.34) (see Remark 5.4).

It remains to show (5.35). Set $\tau_n = t_n^{1-2\beta}$. We have, using (1.8),

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta} u_0 \ge \mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta}(\varphi_{n-1}\mathcal{D}^{\mu,\beta}_{t_{n-1}^{-1/2}} f) = e^{\tau_n\Delta}\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}}(\varphi_{n-1}\mathcal{D}^{\mu,\beta}_{t_{n-1}^{-1/2}} f). \quad (5.48)$$

Note that by (5.38)–(5.39) $a_n/t_n^\beta \to 0$ as $n\to\infty$ so that $a_{n-1}t_n^{-\beta} < \delta t_{n-1}^\beta t_n^{-\beta}$ for n large, where δ is as in (5.37). Since $\varphi_{n-1}(x t_n^\beta) = 1$ if $a_{n-1}t_n^{-\beta} \le |x| \le \delta t_{n-1}^\beta t_n^{-\beta}$ by (5.40), we have

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}}(\varphi_{n-1}\mathcal{D}^{\mu,\beta}_{t_{n-1}^{-1/2}} f)(x) = (t_n t_{n-1}^{-1})^{\frac{\mu}{2}} f(x(t_n/t_{n-1})^\beta) \ge \varepsilon t_{n-1}^{(2\beta-1)\frac{\mu}{2}}$$

if $a_{n-1}t_n^{-\beta} \le |x| \le \delta t_{n-1}^\beta t_n^{-\beta}$, where we used (5.37). Therefore,

$$\begin{aligned} & e^{\tau_n\Delta}\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}}(\varphi_{n-1}\mathcal{D}^{\mu,\beta}_{t_{n-1}^{-1/2}} f)(0) \\ & \quad \ge \varepsilon(4\pi\tau_n)^{-\frac{N}{2}} t_{n-1}^{(2\beta-1)\frac{\mu}{2}} \int_{\{a_{n-1}t_n^{-\beta} < |y| < \delta t_{n-1}^\beta t_n^{-\beta}\}} e^{-\frac{|y|^2}{4\tau_n}}\,dy. \end{aligned}$$

Since $a_{n-1} < \delta t_{n-1}^\beta/2$ if n is large and $t_{n-1}^{2\beta} = t_n^{2\beta}\tau_n$, we obtain that

$$\begin{aligned} \|\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta}u_0\|_{L^\infty} & \ge \varepsilon(4\pi\tau_n)^{-\frac{N}{2}} t_{n-1}^{(2\beta-1)\frac{\mu}{2}} e^{-\frac{\delta^2}{4}} |\{a_{n-1}t_n^{-\beta} < |y| < \delta t_{n-1}^\beta t_n^{-\beta}\}| \\ & \ge \eta \tau_n^{-\frac{N}{2}} t_{n-1}^{(2\beta-1)\frac{\mu}{2}} \frac{(\delta t_{n-1}^\beta - a_{n-1})^N}{t_n^{N\beta}} \ge \eta\left(\frac{\delta}{2}\right)^N t_{n-1}^{(2\beta-1)\frac{\mu}{2}} \end{aligned}$$

for some constant $\eta > 0$. Thus $\|\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n\Delta}u_0\|_{L^\infty} \to \infty$ as $n\to\infty$. This completes the proof. □

6. GENERALIZATION

In this section, we show how to construct $u_0 \in C_0(\mathbb{R}^N)$ so that $\omega^{\mu,\beta}(u_0)$ and $\gamma^{\mu,\beta}(u_0)$ have much richer structures than described in Theorem 5.1.

Theorem 6.1. *Fix a countable subset S of $(0, N)$ and a countable subset E of $\mathcal{S}(\mathbb{R}^N)$. There exists $u_0 \in C_0(\mathbb{R}^N)$ with the following properties.*

(i) *For all*

$$\beta \ge \frac{1}{2}, \quad 0 < \mu < N \quad (6.1)$$

such that

$$\frac{\mu}{2\beta} \in S \quad (6.2)$$

and all $\psi \in E$, there exists a sequence $t_n \to \infty$ such that

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} e^{t_n \Delta} u_0 \underset{n\to\infty}{\longrightarrow} \psi_\beta \tag{6.3}$$

in $C_0(\mathbb{R}^N)$, where

$$\psi_\beta = \begin{cases} \psi & \text{if } \beta > 1/2, \\ e^{\Delta}\psi & \text{if } \beta = 1/2. \end{cases} \tag{6.4}$$

In other words, $\psi_\beta \in \omega^{\mu,\beta}(u_0)$ for all μ, β satisfying (6.1)–(6.2) *and all $\psi \in E$.*

(ii) *For all μ, β satisfying* (6.1)–(6.2) *and all $\psi \in E$, there exists a sequence $\lambda_n \to \infty$ such that*

$$\Gamma^{\mu,\beta}_{\lambda_n} u_0 \underset{n\to\infty}{\longrightarrow} \Psi_\beta \tag{6.5}$$

in $L^\infty((\varepsilon, T) \times \mathbb{R}^N)$ for all $0 < \varepsilon < T < \infty$, where

$$\Psi_\beta(t, \cdot) = \begin{cases} \psi & \text{if } \beta > 1/2, \\ e^{t\Delta}\psi & \text{if } \beta = 1/2 \end{cases} \tag{6.6}$$

for all $t > 0$. In other words, $\Psi_\beta \in \gamma^{\mu,\beta}(u_0)$ for all μ, β satisfying (6.1)–(6.2) *and all $\psi \in E$.*

(iii) *Set*

$$\mathcal{N} = \{ce^{\Delta}\partial^\alpha \delta_0;\ c \in \mathbb{R}, \alpha \in \mathbb{R}^N\}, \tag{6.7}$$

where δ_0 is the Dirac mass at 0. *For all $0 < \mu < N$ and all $f \in \mathcal{N}$, there exists a sequence $t_n \to \infty$ such that*

$$\mathcal{D}^{\mu,\frac{1}{2}}_{\sqrt{t_n}} e^{t_n \Delta} u_0 \underset{n\to\infty}{\longrightarrow} f \tag{6.8}$$

in $C_0(\mathbb{R}^N)$. In other words, $\mathcal{N} \subset \omega^{\mu,\frac{1}{2}}(u_0)$ for all $0 < \mu < N$.

(iv) *Set*

$$\begin{aligned} \mathcal{M} = \{h \in C((0,\infty), C_0(\mathbb{R}^N));\\ \exists c \in \mathbb{R}, \exists \alpha \in \mathbb{R}^N, h(t)\\ = ce^{t\Delta}\partial^\alpha \delta_0 \text{ for all } t > 0\}, \end{aligned} \tag{6.9}$$

where δ_0 is the Dirac mass at 0. *For all $0 < \mu < N$ and all $h \in \mathcal{M}$, there exists a sequence $\lambda_n \to \infty$ such that*

$$\Gamma^{\mu,\frac{1}{2}}_{\lambda_n} u_0 \underset{n\to\infty}{\longrightarrow} h \tag{6.10}$$

in $L^\infty((\varepsilon, T) \times \mathbb{R}^N)$ for all $0 < \varepsilon < T < \infty$. In other words, $\mathcal{M} \subset \gamma^{\mu,\frac{1}{2}}(u_0)$ for all $0 < \mu < N$.

For the proof of Theorem 6.1, we will use the following lemma.

Lemma 6.2. *Let S be a countable subset of $(0, N)$ and let E be a countable subset of $\mathcal{S}(\mathbb{R}^N)$ such that $E \ni 0$. It follows that there exist a sequence $(\sigma_j)_{j\geq 1} \subset S$ and a sequence $(\theta_j)_{j\geq 1} \subset E$ such that the following properties are satisfied.*

(i) *Every element $(r, \psi) \in S \times E$ occurs infinitely often in the sequence $(\sigma_j, \theta_j)_{j\geq 1}$.*

(ii) *The sequence $(\theta_j)_{j\geq 1} \subset E$ satisfies*

$$\max\{\|\theta_j\|_{L^1}, \|\theta_j\|_{L^\infty}\} \leq j, \tag{6.11}$$

for all $j \geq 1$.

Proof. Since the set $S \times E$ is countable, it easily follows that there exists a sequence $(\sigma_j)_{j\geq 1} \subset S$ and a sequence $(\widetilde{\theta}_j)_{j\geq 1} \subset E$ such that every element $(r, \psi) \in S \times E$ occurs infinitely often in the sequence $(\sigma_j, \widetilde{\theta}_j)_{j\geq 1}$. We then define

$$\theta_j = \begin{cases} \widetilde{\theta}_j & \text{if } \widetilde{\theta}_j \text{ satisfies (6.11)}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

Since $0 \in E$, we see that $(\theta_j)_{j\geq 1} \subset E$ and it follows from (6.12) that $(\theta_j)_{j\geq 1}$ satisfies (6.11) for all $j \geq 1$. Thus Property (ii) is satisfied. Next, let $(r, \psi) \in S \times E$ and let $j_k \to \infty$ be such that $(\sigma_{j_k}, \widetilde{\theta}_{j_k}) = (r, \psi)$ for all $k \geq 1$. If $j_k \geq \max\{\|\psi\|_{L^1}, \|\psi\|_{L^1}\}$, we deduce from (6.12) that $\widetilde{\theta}_{j_k} = \theta_{j_k}$. Thus $(\sigma_{j_k}, \theta_{j_k}) = (r, \psi)$ for all sufficiently large k, so that Property (i) is satisfied. □

Proof of Theorem 6.1 Fix a function

$$\begin{cases} \varphi \in \mathcal{S}(\mathbb{R}^N), \\ \varphi \geq 0, \\ \|\varphi\|_{L^1} = 1 \end{cases} \tag{6.13}$$

and consider the countable set

$$F = \bigcup_{\alpha \in \mathbb{R}^N} \{\partial^\alpha \varphi\}. \tag{6.14}$$

We observe that without loss of generality, we may assume that

$$F \cup \{0\} \subset E. \tag{6.15}$$

Consider two sequences of $(\sigma_j)_{j\geq 1} \subset S$ and $(\theta_j)_{j\geq 1} \subset E$ as given by Lemma 6.2. Next, we let a_0 be large enough so that

$$e^x \geq x^{2N} \quad \text{for} \quad x \geq a_0 \geq 1. \tag{6.16}$$

We fix the sequence $(a_j)_{j\geq 1}$ by

$$\begin{cases} a_1 = a_0, \\ a_{j+1} = \exp\left(\dfrac{a_j}{\sigma_{j+1}}\right), \quad j \geq 1. \end{cases} \tag{6.17}$$

Applying inductively the relation

$$a_{j+1} = \exp\left(\frac{a_j}{\sigma_{j+1}}\right) > \exp\left(\frac{a_j}{N}\right) \geq 2a_j, \tag{6.18}$$

where we used (6.17), the fact that $S \subset (0, N)$ and (6.16), we deduce that

$$a_j \uparrow \infty, \quad a_j \geq 2^{j-1} \tag{6.19}$$

for all $j \geq 1$. We define $u_0 \geq 0$ by

$$u_0(x) = \sum_{j=1}^{\infty} e^{-a_{j-1}} \theta_j(x/a_j). \tag{6.20}$$

It follows from (6.11) and (6.19) that

$$\sum_{j=1}^{\infty} e^{-a_{j-1}} \|\theta_j(x/a_j)\|_{L^\infty} \leq \sum_{j=1}^{\infty} j e^{-a_{j-1}} < \infty$$

so that the series in (6.20) is normally convergent in $C_0(\mathbb{R}^N)$. Thus $u_0 \in C_0(\mathbb{R}^N)$.

Given $0 < \mu < N$, $\beta \geq 1/2$, $t > 0$ and $\lambda_n > 0$, we write using (1.9)

$$[\Gamma_{\lambda_n}^{\mu,\beta} u_0](t, \cdot) = e^{t\lambda_n^{2-4\beta}\Delta}(u^n + v^n + w^n), \tag{6.21}$$

where

$$\begin{cases} u^n = \lambda_n^{\mu} \displaystyle\sum_{j=1}^{n-1} e^{-a_{j-1}} \theta_j(x\lambda_n^{2\beta}/a_j), \\ v^n = \lambda_n^{\mu} e^{-a_{n-1}} \theta_n(x\lambda_n^{2\beta}/a_n), \\ w^n = \lambda_n^{\mu} \displaystyle\sum_{j=n+1}^{\infty} e^{-a_{j-1}} \theta_j(x\lambda_n^{2\beta}/a_j). \end{cases} \tag{6.22}$$

We have by (6.11),

$$\|u^n\|_{L^1} \leq \lambda_n^{\mu} \sum_{j=1}^{n-1} \|\theta_j(\cdot\lambda_n^{2\beta}/a_j)\|_{L^1} \leq \lambda_n^{\mu-2N\beta} \sum_{j=1}^{n-1} j a_j^N \leq n^2 \lambda_n^{\mu-2N\beta} a_{n-1}^N \tag{6.23}$$

and

$$\|w^n\|_{L^\infty} \le \lambda_n^\mu \sum_{j=n+1}^{\infty} j e^{-a_{j-1}} = \lambda_n^\mu \sum_{j=n}^{\infty} (j+1) e^{-a_j}. \tag{6.24}$$

We deduce from (6.23) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta} u^n\|_{L^\infty} \le t^{-\frac{N}{2}} \lambda_n^{-N(1-2\beta)} \|u^n\|_{L^1} \le n^2 t^{-\frac{N}{2}} \lambda_n^{-(N-\mu)} a_{n-1}^N \tag{6.25}$$

and from (6.24) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta} w^n\|_{L^\infty} \le \|w^n\|_{L^\infty} \le \lambda_n^\mu \sum_{j=n}^{\infty} (j+1) e^{-a_j}. \tag{6.26}$$

We now prove Property (ii). Assume (6.1)–(6.2) and set $r = \mu/2\beta \in S$. We let

$$\lambda_n = a_n^{\frac{1}{2\beta}}. \tag{6.27}$$

It follows from (6.22), (6.17), and (6.27) that

$$v^n = a_n^{\frac{\mu}{2\beta} - \sigma_n} \theta_n \tag{6.28}$$

for all $n \ge 1$. Moreover, it follows from (6.25),(6.27), and (6.17) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta} u^n\|_{L^\infty} \le n^2 t^{-\frac{N}{2}} a_n^{-\frac{N-\mu}{2\beta}} a_{n-1}^N = n^2 t^{-\frac{N}{2}} e^{-\frac{N-\mu}{2\beta\sigma_n} a_{n-1}} a_{n-1}^N. \tag{6.29}$$

Since $\sigma_n \le N$ and $a_n \uparrow \infty$, we conclude that

$$\|e^{t\lambda_n^{2-4\beta}\Delta} u^n\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0 \tag{6.30}$$

uniformly for $t \ge \varepsilon$ for any fixed $\varepsilon > 0$. Next, (6.26), (6.27), and (6.19) yield

$$\|e^{t\lambda_n^{2-4\beta}\Delta} w^n\|_{L^\infty} \le a_n^{\frac{\mu}{2\beta}} \sum_{j=n}^{\infty} (j+1) e^{-a_j} \le \sum_{j=n}^{\infty} (j+1) a_j^{\frac{\mu}{2\beta}} e^{-a_j} \underset{n\to\infty}{\longrightarrow} 0 \tag{6.31}$$

Let $\psi \in E$ and let $(n_k)_{k\ge 1}$ be a sequence of integers going to infinity such that $\sigma_{n_k} = r$ and $\theta_{n_k} = \psi$ for all $k \ge 1$. We deduce from (6.28) that

$$e^{t\lambda_{n_k}^{2-4\beta}\Delta} v^{n_k} = e^{t\lambda_{n_k}^{2-4\beta}\Delta} \psi \underset{n\to\infty}{\longrightarrow} \Psi_\beta(t,\cdot) \tag{6.32}$$

in $C_0(\mathbb{R}^N)$, uniformly for $t \ge 0$ and bounded. Property (ii) follows from (6.21) and (6.30)–(6.32).

We next prove Property (iv), and so $\beta = 1/2$. Using (6.15), we fix a multi-integer α and we let $(n_k)_{k\geq 1}$ be a sequence of integers going to infinity such that

$$\sigma_{n_k} \leq \frac{\mu}{2}, \quad \theta_{n_k} = \partial^\alpha \varphi \tag{6.33}$$

for all $k \geq 1$. We then let $c > 0$ and define λ_k by

$$c\lambda_k^{N-\mu} = a_{n_k}^{N-\sigma_{n_k}}. \tag{6.34}$$

It follows from (6.22), (6.17), (6.33), and (6.34) that

$$v^{n_k} = \lambda_k^\mu a_{n_k}^{-\sigma_{n_k}} \partial^\alpha \varphi(\cdot \lambda_k a_{n_k}^{-1}) = c\lambda_k^N a_{n_k}^{-N} \partial^\alpha \varphi(\cdot \lambda_k a_{n_k}^{-1}). \tag{6.35}$$

Set

$$d_k = \lambda_k a_{n_k}^{-1} \tag{6.36}$$

so that by (6.35)

$$v^{n_k} = c\partial^\alpha [d_k^N \varphi(d_k \cdot)]. \tag{6.37}$$

It follows from (6.34), (6.33), and (6.19) that

$$d_k = c^{-\frac{1}{N-\mu}} a_{n_k}^{\frac{\mu-\sigma_{n_k}}{N-\mu}} \geq c^{-\frac{1}{N-\mu}} a_{n_k}^{\frac{\mu}{2(N-\mu)}} \underset{k\to\infty}{\longrightarrow} \infty. \tag{6.38}$$

Therefore, we deduce from (6.37), (6.38), and Lemma 9.2 that

$$e^{t\Delta} v^{n_k} \underset{k\to\infty}{\longrightarrow} \partial^\alpha e^{t\Delta}\delta_0 \tag{6.39}$$

in $C_0(\mathbb{R}^N)$, uniformly in $t \geq \varepsilon$ for all $\varepsilon > 0$. Next, it follows from (6.25), (6.34), (6.17), and (6.19) that

$$\|e^{t\Delta} u^{n_k}\|_{L^\infty} \leq c n_k^2 t^{-\frac{N}{2}} e^{-\frac{N-\sigma_{n_k}}{\sigma_{n_k}} a_{n_k - 1}} a_{n_k-1}^N \underset{k\to\infty}{\longrightarrow} 0 \tag{6.40}$$

uniformly in $t \geq \varepsilon$ for all $\varepsilon > 0$. Moreover, we deduce from (6.26), (6.34), (6.17), and (6.19) that

$$\begin{aligned} \|e^{t\Delta} w^{n_k}\|_{L^\infty} &\leq c^{-\frac{\mu}{N-\mu}} a_{n_k}^{\frac{\mu(N-\sigma_{n_k})}{N-\mu}} \sum_{j=n_k}^\infty (j+1) e^{-a_j} \\ &\leq c^{-\frac{\mu}{N-\mu}} \sum_{j=n_k}^\infty (j+1) a_j^{\frac{\mu(N-\sigma_{n_k})}{N-\mu}} e^{-a_j} \longrightarrow_{k\to\infty} 0. \end{aligned} \tag{6.41}$$

Property (iv) follows from (6.21), (6.39), (6.40), and (6.41).

Finally, Properties (i) and (iii) follow from Properties (ii) and (iv) by setting $t = 1$ and $t_n = \lambda_n^2$. □

Corollary 6.3. *Given any countable subset S of $(0, N)$, there exists $u_0 \in C_0(\mathbb{R}^N)$ with the following properties.*

(i) $\omega^{\mu,\beta}(u_0) = C_0(\mathbb{R}^N)$ *and* $\gamma^{\mu,\beta}(u_0) = \mathcal{E}_\beta$ *for all* μ, β *satisfying* (6.1)–(6.2), *where*

$$\mathcal{E}_\beta = \begin{cases} \{h \in C((0,\infty), C_0(\mathbb{R}^N));\ h(t) \equiv h(1)\} & \text{if } \beta > 1/2, \\ \{h \in C((0,\infty), C_0(\mathbb{R}^N));\ h(t+s) = e^{t\Delta}h(s) \forall s,t > 0\} & \text{if } \beta = 1!/2. \end{cases} \tag{6.42}$$

(ii) $\mathcal{N} \subset \omega^{\mu,\frac{1}{2}}(u_0)$ *and* $\mathcal{M} \subset \gamma^{\mu,\frac{1}{2}}(u_0)$ *for all* $0 < \mu < N$, *where* $\mathcal{N}$ *and* $\mathcal{M}$ *are defined by* (6.7) *and* (6.9), *respectively.*

Proof. Let $E \subset \mathcal{S}(\mathbb{R}^N)$ be a countable dense subset of $C_0(\mathbb{R}^N)$ which also satisfies (6.15), and consider u_0 given by Theorem 6.1 applied with this set E. Property (ii) follows from Properties (iii) and (iv) of Theorem 6.1. Since E is dense in $C_0(\mathbb{R}^N)$ and $\omega^{\mu,\beta}$ is clearly closed in $C_0(\mathbb{R}^N)$, the first statement of Property (i) follows from Property (i) of Theorem 6.1. Suppose now $\beta > 1/2$. Note that $\gamma^{\mu,\beta}(u_0)$ is a closed subset of $C((0,\infty), C_0(\mathbb{R}^N))$ for the topology of uniform convergence on $(\varepsilon, T) \times \mathbb{R}^N$ for all $0 < \varepsilon < T < \infty$. Since E is dense in $C_0(\mathbb{R}^N)$, the second statement of Property (i) follows from Property (ii) of Theorem 6.1. Finally, suppose $\beta = 1/2$ and let

$$\mathcal{V} = \{h \in C((0,\infty), C_0(\mathbb{R}^N));\ \exists \psi \in C_0(\mathbb{R}^N), h(t) \equiv e^{t\Delta}\psi\}. \tag{6.43}$$

Using Property (ii) of Theorem 6.1, we see that $\mathcal{V} \subset \gamma^{\mu,\frac{1}{2}}(u_0)$. Since $\mathcal{V}$ is clearly dense in $\mathcal{E}_{\frac{1}{2}}$ (approximate h by $h(\cdot + \varepsilon)$) and $\gamma^{\mu,\frac{1}{2}}(u_0)$ is closed, we deduce that $\gamma^{\mu,\frac{1}{2}}(u_0) = \mathcal{E}_{\frac{1}{2}}$. This completes the proof. □

7. THE CASE $\mu \geq N$

It is well-known that positive solutions of the heat equation do not decay faster than $t^{-\frac{N}{2}}$, so that $\gamma^{\mu,\beta}(u_0) = \emptyset$ and $\omega^{\mu,\beta}(u_0) = \emptyset$ if $\mu > N$ and $u_0 \geq 0$, $u_0 \not\equiv 0$. See Proposition 7.1. On the other hand, oscillatory solutions can decay faster, allowing the possibility of nontrivial asymptotic limits. This situation is the subject of Theorem 7.2.

Proposition 7.1. *Let* $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \geq 0$, $u_0 \not\equiv 0$. *If there exist* $v \in C_0(\mathbb{R}^N)$, $s > 0$, $\mu_0 \geq N$, $\beta_0 > 0$ *and a sequence* $\lambda_n \to \infty$ *such that*

$$\Gamma_{\lambda_n}^{\mu_0,\beta_0} u_0(s,\cdot) \underset{n\to\infty}{\longrightarrow} v \tag{7.1}$$

in $C_0(\mathbb{R}^N)$, *then* $u_0 \in L^1(\mathbb{R}^N)$, $\mu_0 = N$ *and* $\beta_0 = 1/2$. *In particular,* $\gamma^{\mu,\beta}(u_0) = \emptyset$ *and* $\omega^{\mu,\beta}(u_0) = \emptyset$ *if* $\mu > N$ *or if* $\mu = N$ *and* $\beta \neq 1/2$.

Proof. It follows from (7.1) and formula (1.9) that $\mathcal{D}_{\lambda_n}^{\mu_0,\beta_0} e^{\lambda_n^2 s\Delta} u_0(0) \le v(0)+1$ for n sufficiently large; and so,

$$v(0)+1 \ge \lambda_n^{\mu_0-N}(4\pi s)^{-\frac{N}{2}} \int e^{-\frac{|y|^2}{4\lambda_n^2 s}} u_0(y)\,dy.$$

Letting $n \longrightarrow \infty$, we conclude by monotone convergence that $\mu_0 = N$ and $u_0 \in L^1(\mathbb{R}^N)$. It is then well-known that $\mathcal{D}_{\sqrt{t}}^{N,\frac{1}{2}} e^{t\Delta} u_0 \to (4\pi)^{-N/2}\|u_0\|_{L^1} e^{-\frac{|\cdot|^2}{4}}$ in $C_0(\mathbb{R}^N)$ as $t \to \infty$. (See Herraiz [8].) One then concludes that $\beta_0 = 1/2$ and that $\gamma^{N,\beta}(u_0) = \emptyset$ and $\omega^{N,\beta}(u_0) = \emptyset$ if $\beta \ne 1/2$ (see the proof of Theorem 5.2.). This completes the proof. □

Theorem 7.2. *Given $\mu \ge N$ and $\beta \ge 1/2$, there exist $u_0 \in C_0(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, a function $f \in C_0(\mathbb{R}^N)$, $\theta \not\equiv 0$, and a sequence $t_n \to \infty$ such that*

$$\Gamma_{\lambda_n}^{\mu,\beta} u_0 \underset{n\to\infty}{\longrightarrow} h \tag{7.2}$$

in $C([\varepsilon, T], C_0(\mathbb{R}^N))$ for all $0 < \varepsilon < T < \infty$, where $h(t) \equiv f$ if $\beta > 1/2$ and $h(t) \equiv e^{t\Delta} f$ if $\beta = 1/2$. In particular, $h \in \gamma^{\mu,\beta}$ and $h(1) \in \omega^{\mu,\beta}$.

Proof. The proof is similar to that of Theorem 5.1. Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$, $\varphi \not\equiv 0$ and let the integer k be sufficiently large so that

$$N + k \ge 2\mu. \tag{7.3}$$

Let α be a multi-index such that $|\alpha| = k$ and set $f = \partial^\alpha \varphi$, so that

$$\sup_{t>0} t^{\frac{N+k}{2}} ||e^{t\Delta} f||_{L^\infty} < +\infty. \tag{7.4}$$

Set

$$M = \|f\|_{L^\infty}, \quad \sigma = \frac{\mu}{2\beta} \tag{7.5}$$

and let $a_0 \ge 1$ be large enough so that

$$e^x \ge (2x)^\sigma \quad \text{for} \quad x \ge a_0 \ge 1 \tag{7.6}$$

We define the sequence $(a_j)_{j\ge 1}$ by

$$\begin{cases} a_1 = a_0, \\ a_{j+1} = \exp\left(\dfrac{a_j}{\sigma}\right), \quad j \ge 1. \end{cases} \tag{7.7}$$

Applying inductively the relation

$$a_{j+1} = \exp\left(\frac{a_j}{\sigma}\right) \ge 2a_j, \tag{7.8}$$

where we used (7.7) and (7.6), we deduce that

$$a_j \uparrow \infty, \quad a_j \ge 2^{j-1} \tag{7.9}$$

for all $j \ge 1$. We now define

$$u_0(x) = \sum_{j=1}^{\infty} e^{-a_{j-1}} f(x/a_j) = \sum_{j=1}^{\infty} a_j^{-\sigma} f(x/a_j). \tag{7.10}$$

Given any multi-index α, we have

$$\|\partial^\alpha f(\cdot/a_j)\|_{L^\infty} = a_j^{-|\alpha|} \|\partial^\alpha f\|_{L^\infty} \le \|\partial^\alpha f\|_{L^\infty}$$

by (7.9). Therefore, it follows from (7.9) that the series in (7.10) is normally convergent in $C_{\mathrm{b}}^m(\mathbb{R}^N)$ for all $m \ge 0$, so that $u_0 \in C^\infty(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

Given $t > 0$ and $\lambda_n > 0$, we write using (1.9)

$$[\Gamma_{\lambda_n}^{\mu,\beta} u_0](t, \cdot) = e^{t\lambda_n^{2-4\beta}\Delta}(u^n + v^n + w^n), \tag{7.11}$$

where

$$\begin{cases} u^n = \lambda_n^\mu \displaystyle\sum_{j=1}^{n-1} e^{-a_{j-1}} f(x\lambda_n^{2\beta}/a_j), \\ v^n = \lambda_n^\mu e^{-a_{n-1}} f(x\lambda_n^{2\beta}/a_n), \\ w^n = \lambda_n^\mu \displaystyle\sum_{j=n+1}^{\infty} e^{-a_{j-1}} f(x\lambda_n^{2\beta}/a_j). \end{cases} \tag{7.12}$$

We let

$$\lambda_n = a_n^{\frac{1}{2\beta}}. \tag{7.13}$$

It follows from (7.12) and (7.7) that

$$v^n = f \tag{7.14}$$

so that

$$e^{t\lambda_n^{2-4\beta}\Delta} v^n = e^{t\lambda_n^{2-4\beta}\Delta} f \underset{n\to\infty}{\longrightarrow} h \tag{7.15}$$

in $C_0(\mathbb{R}^N)$, uniformly for $t \ge 0$ and bounded. Next, we deduce from (7.12), (7.13), and (7.5) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta} w^n\|_{L^\infty} \le \|w^n\|_{L^\infty} \le M a_n^\sigma \sum_{j=n}^{\infty} e^{-a_j} \le M \sum_{j=n}^{\infty} a_j^\sigma e^{-a_j} \underset{n\to\infty}{\longrightarrow} 0, \tag{7.16}$$

where we used (7.9) in the last two relations. Next, it follows from (7.12), (7.13), (1.8), and (7.7) that

$$e^{t\lambda_n^{2-4\beta}\Delta}u^n = \sum_{j=1}^{n-1} a_j^{\sigma} e^{-a_{j-1}} \mathcal{D}^{\mu,\beta}_{(a_n/a_j)^{1/2\beta}} e^{ta_n^{1/\beta}a_j^{-2}\Delta} f$$

so that

$$\begin{aligned}\|e^{t\lambda_n^{2-4\beta}\Delta}u^n\|_{L^\infty} &\le \sum_{j=1}^{n-1} a_n^{\sigma} e^{-a_{j-1}} \|e^{ta_n^{\frac{1}{\beta}}a_j^{-2}\Delta} f\|_{L^\infty} \\ &\le C\sum_{j=1}^{n-1} a_n^{\sigma} e^{-a_{j-1}} (ta_n^{\frac{1}{\beta}}a_j^{-2})^{-\frac{N+k}{2}} \\ &\le Ct^{-\frac{N+k}{2}} \sum_{j=1}^{n-1} a_n^{-\sigma} e^{-a_{j-1}} a_j^{N+k}, \end{aligned} \tag{7.17}$$

where we used (7.4), (7.3), and (7.7). We deduce from (7.17) that

$$\|e^{t\lambda_n^{2-4\beta}\Delta}u^n\|_{L^\infty} \le Ct^{-\frac{N+k}{2}} a_n^{-\sigma} a_{n-1}^{N+k} = Ct^{-\frac{N+k}{2}} e^{-a_{n-1}} a_{n-1}^{N+k}. \tag{7.18}$$

Since $a_n \uparrow \infty$, we conclude that

$$\|e^{t\lambda_n^{2-4\beta}\Delta}u^n\|_{L^\infty} \underset{n\to\infty}{\longrightarrow} 0 \tag{7.19}$$

uniformly for $t \ge \varepsilon$, for any fixed $\varepsilon > 0$. The result now follows from (7.11), (7.15), (7.16), and (7.19). □

8. NONLINEAR HEAT EQUATIONS

In this section, we consider the nonlinear heat equation

$$\begin{cases} u_t - \Delta u + |u|^{\alpha} u = 0, \\ u(0,x) = U_0(x) \end{cases} \tag{8.1}$$

in $\mathbb{R}^N$, where $\alpha > 0$. We show the following analogue of Theorem 5.1.

Theorem 8.1. *Let*

$$\alpha > \frac{2}{N} \tag{8.2}$$

and

$$\frac{2}{\alpha} < \sigma < N. \tag{8.3}$$

Fix

$$\frac{1}{2} < \beta < \frac{1}{2}\min\left\{\frac{N}{\sigma}, 1+\alpha-\frac{2}{\sigma}\right\} \tag{8.4}$$

and set

$$\mu = 2\beta\sigma. \tag{8.5}$$

Let $f \in \mathcal{S}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ *and let* $u_0 \in C_0(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, $u_0 \geq 0$ *be given by Theorem* 5.1. *If* u *is the solution of the Eq.* (8.1) *with the initial value* $U_0 = u_0$, *then there exists a sequence* $t_n \to \infty$ *such that*

$$\mathcal{D}^{\mu,\beta}_{\sqrt{t_n}} u(t_n) \underset{n\to\infty}{\longrightarrow} f \tag{8.6}$$

in $C_0(\mathbb{R}^N)$.

Proof. It follows from the calculations of Lemma 5.1 in [3] that if the function $v \in C([0,\infty), C_0(\mathbb{R}^N))$ satisfies

$$M := \sup_{t>0, x\in\mathbb{R}^N} (1+t+|x|^2)^{\frac{\sigma}{2}} |v(t,x)| < \infty \tag{8.7}$$

then for every

$$\gamma < \min\{N-\sigma, \sigma\alpha - 2\} \tag{8.8}$$

there exists C such that

$$(1+t+|x|^2)^{\frac{\sigma}{2}} \left| \int_0^t e^{(t-s)\Delta} |v|^\alpha v(s,x)\, ds \right| \leq C M^{\alpha+1} (1+t)^{-\frac{\gamma}{2}} \tag{8.9}$$

for all $t \geq 0$ and $x \in \mathbb{R}^N$. Note that $|u(t)| \leq e^{t\Delta}|u_0|$ by the maximum principle. Since

$$\sup_{t>0, x\in\mathbb{R}^N} (1+t+|x|^2)^{\frac{\sigma}{2}} |e^{t\Delta} u_0(x)| \leq C \sup_{x\in\mathbb{R}^N} (1+|x|^2)^{\frac{\sigma}{2}} |u_0(x)| \tag{8.10}$$

by Corollary 8.5 in [1], we see that u satisfies (8.7). Since

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} |u|^\alpha u(s)\, ds$$

we deduce from (8.9), (8.8), and (8.4) that

$$t^{\frac{\mu}{2}} \| u(t) - e^{t\Delta} u_0 \|_{L^\infty} \underset{t\to\infty}{\longrightarrow} 0 \tag{8.11}$$

and the conclusion follows from Theorem 5.1. □

Remark 8.2. Consider the equation

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, \\ u(0,x) = U_0(x). \end{cases} \tag{8.12}$$

If $\alpha \le 2/N$, then all positive solutions of (8.12) blow up in finite time. If $\alpha > 2/N$, then it follows easily from (8.9), (8.10) and an easy contraction mapping argument that if $\|(1+|\cdot|^2)^{\frac{\sigma}{2}} U_0\|_{L^\infty} \le \varepsilon$ with $\varepsilon > 0$ sufficiently small, then the solution of (8.12) is global and satisfies (8.11). Therefore, if we consider f and u_0 as in Theorem 5.1 and if we let $U_0 = \eta u_0$ with $\eta > 0$ sufficiently small, then $u(t)$ satisfies (8.6).

APPENDIX

Lemma 9.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ satisfy* $\operatorname{supp}\varphi \not\ni \{0\}$ *and let $a, \nu > 0$. If $0 < \beta < 1/2$, then $t^\nu e^{-at|\cdot|^2}\varphi(\cdot t^\beta) \to 0$ in $\mathcal{S}(\mathbb{R}^N)$ as $t \to \infty$.*

Proof. Suppose

$$\varphi(\xi) = 0 \quad \text{for } |\xi| \le a. \tag{9.1}$$

Let m be a non-negative integer and α a multi-index. We have

$$|\xi|^m |D^\alpha[t^\nu e^{-at|\xi|^2}\varphi(\xi t^\beta)]| \le C t^\nu |\xi|^m \sum_{\alpha_1+\alpha_2=\alpha} |D^{\alpha_1}[e^{-at|\xi|^2}]|\,|D^{\alpha_2}[\varphi(\xi t^\beta)]|. \tag{9.2}$$

On the one hand,

$$|D^{\alpha_2}[\varphi(\xi t^\beta)]| = t^{|\alpha_2|\beta}|D^{\alpha_2}\varphi(\xi t^\beta))| \tag{9.3}$$

and on the other hand $|D^{\alpha_1}[e^{-at|\xi|^2}]|$, is estimated by a sum of terms of the form

$$t^\gamma |\xi|^\sigma e^{-at|\xi|^2} \tag{9.4}$$

with $0 \le \gamma, \sigma \le |\alpha_1|$. Thus $|\xi|^m |D^\alpha[t^\nu e^{-at|\xi|^2}\varphi(\xi t^\beta)]|$ is estimated by a sum of terms of the form

$$t^{\nu+\gamma+|\alpha_2|\beta}|\xi|^{m+\sigma}e^{-at|\xi|^2}|D^{\alpha_2}\varphi(\xi t^\beta))|. \tag{9.5}$$

Setting $y = \xi t^\beta$ and using (9.1), we are led to estimate terms of the form

$$A = \sup_{|y|\ge a} t^p |y|^q e^{-at^{1-2\beta}|y|^2}|D^{\alpha_2}\varphi(y))| \tag{9.6}$$

with $p, q \geq 0$. The last term in (9.6) is bounded in y, so that

$$A \leq C \sup_{|y|\geq a} t^p |y|^q e^{-at^{1-2\beta}|y|^2} \underset{t\to\infty}{\longrightarrow} 0 \tag{9.7}$$

since $\beta < 1/2$. This completes the proof. $\square$

Lemma 9.2. *Let $\varphi \in L^1(\mathbb{R}^N)$ satisfy*

$$\int_{\mathbb{R}^N} \varphi = 1 \tag{9.8}$$

let $(d_n)_{n\geq 1} \subset (0,\infty)$ satisfy $d_n \to \infty$, and let δ_0 be the Dirac mass at 0. If $\varphi_n(x) = d_n^N \varphi(d_n x)$, then

$$e^{t\Delta}\varphi_n \underset{n\to\infty}{\longrightarrow} e^{t\Delta}\delta_0 \tag{9.9}$$

in $C_0(\mathbb{R}^N)$, uniformly in $t \geq \eta$ for all $\eta > 0$. In addition, for every multi-index α with $|\alpha| \leq m$,

$$\partial^\alpha e^{t\Delta}\varphi_n \underset{n\to\infty}{\longrightarrow} \partial^\alpha e^{t\Delta}\delta_0 \tag{9.10}$$

in $C_0(\mathbb{R}^N)$, uniformly in $t \geq \eta$ for all $\eta > 0$.

Proof. Given $\psi \in \mathcal{S}(\mathbb{R}^N)$,

$$\int \varphi_n(x)\psi(x)\,dx = \int \varphi(x)\psi(x/d_n)\,dx \underset{n\to\infty}{\longrightarrow} \psi(0)\int \varphi = \psi(0)$$

so that $\varphi_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^N)$; and so

$$e^{t\Delta}\varphi_n \underset{n\to\infty}{\longrightarrow} e^{t\Delta}\delta_0 \text{ in } \mathcal{S}'(\mathbb{R}^N) \qquad \text{for all } t > 0. \tag{9.11}$$

In addition, $\|\varphi_n\|_{L^1} = \|\varphi\|_{L^1}$, so that by parabolic regularity,

$$\sup_{t>0}\left\{ t^{\frac{N}{2}} \|e^{t\Delta}\varphi_n\|_{L^\infty} + t^{\frac{N+1}{2}} \|\nabla e^{t\Delta}\varphi_n\|_{L^\infty} + t^{\frac{N+2}{2}} \|\partial_t e^{t\Delta}\varphi_n\|_{L^\infty} \right\} \leq C\|\varphi\|_{L^1}. \tag{9.12}$$

It follows in particular from (9.11), (9.12) and Ascoli's theorem that

$$e^{t\Delta}\varphi_n \underset{n\to\infty}{\longrightarrow} e^{t\Delta}\delta_0 \text{ in } C([\eta,\infty) \times \{|x| \leq R\}) \tag{9.13}$$

for all $\eta, R > 0$. Also,

$$\begin{aligned}(4\pi t)^{\frac{N}{2}} |e^{t\Delta}\varphi_n(x)| &= \left| \int_{\{|y|<1\}} e^{-\frac{|x-y|^2}{4t}} \varphi_n(y) + \int_{\{|y|>1\}} e^{-\frac{|x-y|^2}{4t}} \varphi_n(y) \right| \\ &\leq \sup_{|y|<1} e^{-\frac{|x-y|^2}{4t}} + \int_{\{|y|>d_n\}} \varphi.\end{aligned} \tag{9.14}$$

We easily deduce from (9.14) that $|e^{t\Delta}\varphi_n(x)| \to 0$ as $|x| \to \infty$, uniformly in $n \geq 1$ and $t \geq \eta > 0$. This, together with (9.13), implies (9.9). Now (9.10) follows immediately from (9.9) and parabolic regularity. □

Lemma 9.3. *Let* $(a_j)_{j\geq 1} \subset (1, \infty)$, $j_0 \in \mathbb{R}$ *satisfy*

$$a_j \leq \varepsilon a_{j+1} \tag{9.15}$$

for some $0 < \varepsilon < 1$ *and for all* $j \geq j_0$. *Given* $\theta \in \mathcal{S}(\mathbb{R}^N)$ *and* $\sigma > 0$, *set*

$$u(x) = \sum_{j=1}^{\infty} a_j^{-\sigma} \theta(x/a_j). \tag{9.16}$$

It follows that $u \in C_0(\mathbb{R}^N)$ *and* $|\cdot|^\sigma u(\cdot) \in L^\infty(\mathbb{R}^N)$.

Proof. Note that by (9.15),

$$a_j \geq \varepsilon^{-(j-j_0)} a_{j_0} \tag{9.17}$$

for all $j > j_0$. In particular, $\sum a_j^{-\sigma} < \infty$, so that the series in (9.16) is normally convergent in $L^\infty(\mathbb{R}^N)$. Therefore $u \in C_0(\mathbb{R}^N)$. Next, we write

$$|x|^\sigma u(x) = \sum_{j\geq 1} \psi(x/a_j), \tag{9.18}$$

where $\psi(y) = |y|^\sigma \theta(y)$. Clearly, it suffices to show that there exists a constant C such that

$$\sum_{j\geq j_0} \psi(x/a_j) \leq C \tag{9.19}$$

for all $x \in \mathbb{R}^N$, so we may assume without loss of generality that $j_0 = 1$. Considering $y_j = x/a_j$, this is equivalent to

$$\sum_{j\geq 1} \psi(y_j) \leq C \tag{9.20}$$

for all sequences $(y_j)_{j\geq 0} \subset \mathbb{R}^N$ such that

$$|y_{j+1}| \leq \varepsilon |y_j|. \tag{9.21}$$

To prove (9.20), let $(y_j)_{j\geq 0} \subset \mathbb{R}^N$ satisfy (9.21) and set $k = \min\{j \geq 0;\ |y_j| \leq 1\}$, so that

$$\begin{cases} |y_j| \leq \varepsilon^{j-k}, & j \geq k, \\ |y_j| \geq \varepsilon^{-(k-j-1)}, & j < k. \end{cases} \tag{9.22}$$

We write

$$\sum_{j\geq 1}\psi(y_j)=\sum_{j\geq k}\psi(y_j)+\sum_{j<k}\psi(y_j). \tag{9.23}$$

From (9.22) we deduce that

$$\sum_{j\geq k}\psi(y_j)\leq \|\theta\|_{L^\infty}\sum_{\ell\geq 0}\varepsilon^{\ell\sigma}=\|\theta\|_{L^\infty}(1-\varepsilon^{\sigma})^{-1}. \tag{9.24}$$

Take now $A>0$ such that $|\psi(y)|\leq A|y|^{-1}$ for all $|y|\geq 1$. Using (9.22) we obtain

$$\sum_{j<k}\psi(y_j)\leq A\sum_{j<k}|y_j|^{-1}\leq A\sum_{\ell\geq 0}\varepsilon^{\ell-1}=A\varepsilon^{-1}(1-\varepsilon)^{-1}. \tag{9.25}$$

Then (9.20) follows from (9.23)–(9.25). □

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