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Sets of Dynamical Systems with Various Limit Shadowing Properties*

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We study the C^1-interiors of sets of diffeomorphisms of a closed smooth manifold with various limit shadowing properties. It is shown that, for some natural analogs of the usual limit shadowing property, the corresponding C^1-interiors coincide with the set of Ω-stable diffeomorphisms. The same problem is considered for two-sided analogs of the limit shadowing property.

KEY WORDS: limit shadowing, hyperbolicity, transversality, structural stability.

Mathematics Subject Classification: 37C50

1. INTRODUCTION

The shadowing property of dynamical systems is now well-studied (see, for example, the monographs [1,2]). Consider a dynamical system generated by a homeomorphism f of a metric space (M, dist).

Fix $d > 0$. We say that a sequence $\xi = \{x_k \in M : k \in \mathbf{Z}\}$ is a d-pseudotrajectory of f if the inequalities

$$\mathrm{dist}(f(x_k), x_{k+1}) < d, \quad k \in \mathbf{Z} \tag{1}$$

hold. The shadowing property (usually abbreviated POTP, pseudoorbit tracing property) of f is formulated as follows: given $\varepsilon > 0$, there exists $d > 0$ with the following property: for any d-pseudotrajectory $\xi = \{x_k\}$, we can find a point $p \in M$ such that

$$\mathrm{dist}(f^k(p), x_k) < \varepsilon, \quad k \in \mathbf{Z}. \tag{2}$$

*To Pavol Brunovsky on the occasion of his 70th birthday.

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Let us assume that M is a closed smooth manifold. Denote by POTP the set of diffeomorphisms of M having the shadowing property (we denote the properties which we study by the same symbols as the sets of dynamical systems having these properties). We have no hope to characterize the set POTP in terms of properties standard for the classical theory of structural stability (such as hyperbolicity and transversality; see, for example, [3] for definitions) since if $f \in$ POTP, then any diffeomorphism topologically conjugate with f is in POTP as well [1] , while the above-mentioned properties are not preserved under homeomorphisms of the phase space.

The situation changes when we pass from the set POTP to its interior with respect to the C^1 topology, Int^1(POTP) (here and below, we denote by Int^1(P) the C^1-interior of any subset P of the space of diffeomorphisms $\mathrm{Diff}^1(M)$). It was shown by Sakai [4] that the set Int^1(POTP) coincides with the set $\mathcal{S}$ of structurally stable diffeomorphisms.

Later, several similar results were established for some other shadowing properties; let us mention, for example, that the set Int^1(OSP) coincides with the set $\mathcal{S}$ [5], where OSP is the set of diffeomorphisms of M having the orbital shadowing property (see the definition below).

In this paper, we study the structure of the C^1-interior for sets of diffeomorphisms having various limit shadowing properties.

We give the definitions below for a homeomorphism f of a metric space (M, dist); the consideration is restricted to the case of diffeomorphisms of a closed smooth manifold M when we work with the sets Int^1(P).

We say that f has the limit shadowing property (LmSP, [6]) if, for any sequence $\xi = \{x_k \in M : k \geqslant 0\}$ such that

$$\mathrm{dist}(f(x_k), x_{k+1}) \to 0, \quad k \to \infty \tag{3}$$

there exists a point p such that

$$\mathrm{dist}(f^k(p), x_k) \to 0, \quad k \to \infty. \tag{4}$$

Of course, one can define a "parallel" negative LmSP replacing $k \to \infty$ by $k \to -\infty$ in both relations (3) and (4) (and noting that

$$\mathrm{dist}(f(x_k), x_{k+1}) \to 0, \quad k \to -\infty \tag{5}$$

if and only if

$$\mathrm{dist}(f^{-1}(x_{k+1}), x_k) \to 0, \quad k \to -\infty.)$$

It is easy to transfer our results to such "parallel" properties.

In general, we know less about the limit shadowing property than about the usual one. For example, it is known that the POTP is C^0-generic in the space of homeomorphisms of any smooth closed manifold M [7], while no analog of this statement is known for the LmSP if $\dim M \geqslant 2$.

In [1, Theorem 1.4.1], it is shown that a diffeomorphism has the LmSP in a neighborhood of its hyperbolic set; we refer to this statement in the proof of Theorem 1 below (see also a related result in Proposition 11 of Chapter 11 in [8]).

Let us introduce several shadowing properties related to the LmSP. The first of these properties is, in a sense, an "orbital" analog of the LmSP.

For a point x, we denote by $O(x, f)$ its trajectory in the system f, i.e., the set

$$O(x, f) = \{f^k(x) : k \in \mathbf{Z}\}.$$

Recall that f has the orbital shadowing property (OSP, [5]) if, given $\varepsilon > 0$, there exists $d > 0$ with the following property: for any d-pseudotrajectory $\xi = \{x_k\}$, we can find a point $p \in M$ such that

$$\mathrm{dist}_H\left(\overline{\xi}, \overline{O(p, f)}\right) \leqslant \varepsilon, \tag{6}$$

where dist_H is the Hausdorff distance.

The sense of the "orbital" approach to shadowing is as follows: one studies not the pointwise closeness of the trajectory $O(p, f)$ and pseudotrajectory ξ expressed by inequalities (2), but their closeness as sets (where the indices of individual points $f^k(p)$ and x_k are irrelevant).

For a point $x \in M$, let us denote by $\omega(x)$ the ω-limit set of $O(x, f)$.

Similarly, for a sequence $\xi = \{x_k \in M : k \geqslant 0\}$, we denote by $\omega(\xi)$ the set of all limit points of ξ as $k \to \infty$. The following property is a natural "orbital" analog of the LmSP.

We say that f has the orbital LmSP (OLmSP) if, for any sequence $\xi = \{x_k \in M : k \geqslant 0\}$ satisfying relations (3), there exists a point p such that

$$\omega(\xi) = \omega(p). \tag{7}$$

Obviously, LmSP$\subset$OLmSP.

It is easy to see that the properties LmSP and OLmSP do not coincide, for example, for an irrational rotation of the circle (see the Appendix).

Now we consider "inverse" analogs of the LmSP. Dynamically, the shadowing property means the following: for any approximate trajectory we can find a close real trajectory.

It is natural to pose the "inverse" problem: given a dynamical system and a family of approximate trajectories, is it possible, for any real trajectory, to find a close approximate trajectory? The corresponding property (called the inverse shadowing property, ISP) was introduced in [9]. It is possible to consider various families of approximate trajectories. In [10], two classes of continuous methods were considered. In this paper, we consider one more class Θ_t close to the class Θ_s studied in [9] (for "one-sided limit shadowing" treated in Theorem 1 below, these classes coincide).

Fix $d>0$. A continuous d-method of the class Θ_t for a homeomorphism f is a sequence $\Psi=\{\psi_k : k\in\mathbf{Z}\}$, where any ψ_k is a continuous mapping $\psi_k : M\to M$ such that

$$\operatorname{dist}(\psi_k(x), f(x))<d, \quad k\geqslant 0, \text{ and } \operatorname{dist}(\psi_k(x), f^{-1}(x))<d, \quad k<0$$

for any $x\in M$. We say that a sequence $\xi=\{x_k\in M : k\in\mathbf{Z}\}$ is a pseudotrajectory generated by a d-method $\Psi=\{\psi_k\}$ (we write $\xi\in G\Psi$ in this case) if

$$x_{k+1}=\psi_k(x_k), \quad k\geqslant 0, \text{ and } x_{k-1}=\psi_{k-1}(x_k), \quad k\leqslant 0. \tag{8}$$

We say that f has the ISP if, given $\varepsilon>0$, there exists $d>0$ such that, for any $p\in M$ and any continuous d-method Ψ of the class Θ_t, we can find a pseudotrajectory $\xi\in G\Psi$ satisfying inequalities (2).

Now we define an "inverse" analog of the LmSP.

We say that a sequence $\Psi=\{\psi_k : k\geqslant 0\}$ is a positively convergent (p.c. below) method for a homeomorphism f if any ψ_k is a continuous mapping $\psi_k : M\to M$ and

$$\max_{x\in M}\operatorname{dist}(\psi_k(x), f(x))\to 0, \quad k\to\infty. \tag{9}$$

In the study of limit shadowing, we are mostly interested in the behavior of trajectories and pseudotrajectories as time goes to infinity and not in the behavior of their finite initial segments. Thus, we may restrict our attention to p.c. methods for which the values $\operatorname{dist}(\psi_k(x), f(x))$ are bounded from above by a preliminary fixed constant. On the other hand, standard topological reasons show that if f is a homeomorphism of a closed manifold M, then there exists a number $\Delta=\Delta(f)>0$ such that if ψ is a continuous mapping $M\to M$ and

$$\operatorname{dist}(\psi(x), f(x))<\Delta$$

then ψ maps M onto M. Thus, without loss of generality, we assume below that any mapping ψ_k in the definition of a p.c. method maps M onto M.

We say that a sequence $\xi=\{x_k\in M : k\geqslant 0\}$ is a pseudotrajectory generated by a p.c. method $\Psi=\{\psi_k\}$ (we write $\xi\in G\Psi$ in this case) if relations (8) hold.

We say that f has the inverse LmSP (ILmSP) if, for any $p \in M$ and any p.c. method $\Psi = \{\psi_k\}$ for f, there exists a pseudotrajectory $\xi \in G\Psi$ for which relation (4) holds.

Similarly, we say that f has the orbital inverse LmSP (OILmSP) if, for any $p \in M$ and any p.c. method $\Psi = \{\psi_k\}$ for f, there exists a pseudotrajectory $\xi \in G\Psi$ for which relation (7) holds.

Clearly, ILmSP $\subset$ OILmSP; at the same time, the above-mentioned example of an irrational rotation of the circle shows that ILmSP $\neq$ OILmSP (we leave details to the reader).

The next group of considered properties is related to the notion of weak shadowing first considered in [9] (see also [5]).

Denote by $N(a, A)$ the a-neighborhood of a set $A \subset M$.

We say that f has the weak shadowing property (WSP) if, given $\varepsilon > 0$, there exists $d > 0$ with the following property: for any d-pseudotrajectory $\xi = \{x_k\}$, we can find a point $p \in M$ such that

$$\xi \subset N(\varepsilon, O(p, f)). \tag{10}$$

The problem of characterization of the set Int^1(WSP) was considered in [11, 12].

The "limit" analog of this property is as follows: we say that f has the weak LmSP (WLmSP) if, for any sequence $\xi = \{x_k : k \geqslant 0\}$ satisfying relations (3), we can find a point $p \in M$ such that

$$\omega(\xi) \subset \omega(p). \tag{11}$$

A close property (called the limit weak shadowing property) was studied in the recent paper [13].

Let us note here that $f \equiv \mathrm{id} \notin$ WLmSP for any manifold M with $\dim M \geqslant 1$, where id is the identity mapping, while $f \in$ WLmSP if f has a dense positive semitrajectory.

Now we define one more property considered in this paper.

We say that f has the weak inverse LmSP (WILmSP) if, for any $p \in M$ and any p.c. method $\Psi = \{\psi_k\}$ for f, there exists a pseudotrajectory $\xi \in G\Psi$ for which relation (11) holds.

Let us formulate the first result.

Theorem 1. *Any of the sets $Int^1(P)$, where $P = LmSp$, OLmSP, ILmSP, OILmSP, WILmSP, coincides with the set ΩS of Ω-stable diffeomorphisms.*

Remark 1. *This result indicates a difference between the sets of systems with weak limit and weak inverse limit shadowing properties.*

Mané had constructed in [14] a domain $\mathcal{M}$ in the space $Diff^1(T^3)$, where T^3 is the 3-dimensional torus, such that any diffeomorphism $f \in \mathcal{M}$

has a dense trajectory, $\Omega(f)=T^3$, *and* f *is not Anosov (hence,* f *is not* Ω*-stable). It is known (see [15]) that if* f *is a homeomorphism of a compact metric space* X *and* $\Omega(f)=X$, *then the existence of a dense trajectory implies the existence of a dense positive semitrajectory. By our comment above,* $\mathcal{M}\subset Int^1(WLmSP)$, *while* $\mathcal{M}\cap Int^1(WILmSP)=\emptyset$ *by Theorem 1.*

Now we pass to two-sided analogs of the LmSP (in this paper, we restrict our attention to two of such properties).

We say that f has the two-sided LmSP (TSLmSP) if there exists a number $d>0$ with the following property: if a sequence $\xi=\{x_k\in M : k\in\mathbf{Z}\}$ is a d-pseudotrajectory of f for which relations (3) and (5) hold, then there is a point p such that

$$\mathrm{dist}(f^k(p),x_k)\to 0,\quad |k|\to\infty. \tag{12}$$

It was noted in [1] that a diffeomorphism has the TSLmSP in a neighborhood of a hyperbolic set.

Remark 2. *In the global study of a dynamical system, it is unreasonable to consider a two-sided analog of the LmSP without restrictions on the values dist*$(f(x_k),x_{k+1})$. *Indeed, if a homeomorphism* f *has an attractor* A *and a repeller* B *with* $A\cap B=\emptyset$, *one may take a sequence* $\xi=\{x_k : k\in\mathbf{Z}\}$ *such that* $x_0\in A$, $x_k=f^{-1}(x_{k+1})$ *for* $k<0$, $x_1\in B$, *and* $x_{k+1}=f(x_k)$ *for* $k\geqslant 1$. *Of course, such a sequence* ξ *satisfies relations (3) and (5), but we cannot find a point* p *for which* $f^k(p)\to A$ *as* $k\to-\infty$ *and* $f^k(p)\to B$ *as* $k\to\infty$, *so that relation (12) cannot hold. Thus, it is natural to restrict the values of "jumps" for the pseudotrajectories considered (as was done in the definition above).*

To define a two-sided analog of the ILmSP, let us consider the following class of continuous methods. We say that a sequence $\Psi=\{\psi_k : k\in\mathbf{Z}\}$ is a two-sided convergent (ts.c. below) method for a homeomorphism f if any ψ_k is a surjective continuous mapping $\psi_k : M\to M$, relation (9) holds, and

$$\max_{x\in M}\mathrm{dist}(\psi_k(x),f^{-1}(x))\to 0,\quad k\to-\infty. \tag{13}$$

We say that a sequence $\xi=\{x_k\in M : k\in\mathbf{Z}\}$ is a pseudotrajectory generated by a ts.c. method $\Psi=\{\psi_k\}$ (we write $\xi\in G\Psi$ in this case) if relations (8) hold.

We say that a ts.c. method Ψ is a d-method if any sequence $\xi\in G\Psi$ is a d-pseudotrajectory of f.

We say that f has the two-sided ILmSP (TSILmSP) if there exists a constant $d>0$ with the following property: for any $p\in M$ and any ts.c. d-method Ψ, there exists a sequence $\xi\in G\Psi$ for which relation (12) holds.

The problem of description of the sets Int^1(TSLmSP) and Int^1 (TSILmSP) is really more complicated than the same problem for one-sided limit shadowing properties. We can prove the following general statement.

Theorem 2. *Any structurally stable diffeomorphism has the properties TSLmSP and TSILmSP.*

Since the set of structurally stable diffeomorphisms of any smooth closed manifold is open, Theorem 2 implies that the set $\mathcal{S}$ is a subset of both sets Int^1(TSLmSP) and Int^1(TSILmSP). Below we also establish the following result.

Theorem 3. *The sets $\mathcal{S}$, $Int^1(TSLmSP)$, and $Int^1(TSILmSP)$ are pairwise different.*

Proofs. We begin with the proof of Theorem 1.

First, we show that any Ω-stable diffeomorphism has the LmSP and ILmSP. Since LmSP $\subset$ OLmSP, ILmSP $\subset$ OILmSP $\subset$ WILmSP, and the set $\Omega\mathcal{S}$ of Ω-stable diffeomorphisms is C^1-open, in this case the set $\Omega\mathcal{S}$ belongs to any of the sets Int^1(P) mentioned in the statement of Theorem 1.

Lemma 1. $\Omega\mathcal{S} \subset LmSP$.

Proof. Let f be an Ω-stable diffeomorphism and let

$$\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_m$$

be the spectral decomposition of the nonwandering set $\Omega(f)$ into basic sets (see [3]).

Consider a sequence $\xi = \{x_k : k \geqslant 0\}$ for which relation (3) holds. We claim that there exists a basic set Ω_i such that

$$\mathrm{dist}(x_k, \Omega_i) \to 0, \quad k \to \infty. \tag{14}$$

For two different basic sets Ω_i and Ω_j, we write $\Omega_i \to \Omega_j$ if there exists a wandering point x such that

$$f^k(x) \to \Omega_i, \quad k \to -\infty, \quad \text{and} \quad f^k \to \Omega_j, \quad k \to \infty.$$

The graph with vertices corresponding to the basic sets and with edges corresponding to the relation $\to$ introduced above is usually called the phase diagram of f. Since f is Ω-stable, this graph does not contain cycles [3].

The following two statements are well known (cf. [12], Propositions 3.2 and 3.3):

(I) if Ω_i is a basic set, then for any neighborhood U_i of Ω_i we can find a neigborhood W_i of Ω_i such that if $f^k(x) \notin U_i$ for some $x \in W_i$ and $k > 0$, then $f^l(x) \notin W_i$ for $l \geqslant k$;

(II) there exist neighborhoods U_i of the basic sets Ω_i such that if $f^k(U_i) \cap U_j \neq \emptyset$ for some $k > 0$ and $i \neq j$, then there exist basic sets $\Omega_{i_1}, \ldots, \Omega_{i_l}$ such that

$$\Omega_i \to \Omega_{i_1} \to \ldots \to \Omega_{i_l} \to \Omega_j. \tag{15}$$

Let us fix disjoint neighborhoods U_i of the basic sets Ω_i for which statement (II) holds. There exists a number $b > 0$ and arbitrarily small neighborhoods W_i of the basic sets Ω_i such that

$$N(b, W_i) \subset U_i. \tag{16}$$

Consider the neighborhood $U = W_1 \cup \cdots \cup W_m$ of the set $\Omega(f)$ and the set of indices $\kappa := \{k \geqslant 0 : x_k \notin U\}$.

It is known that f has a global Lyapunov function (see [16]), i.e., a continuous function $V : M \to [0, +\infty)$ such that

$$V(f(x)) \leqslant V(x),\ x \in M, \quad \text{and} \quad V(f(x)) = V(x) \Leftrightarrow x \in \Omega(f).$$

The set $M' = M \setminus U$ is compact; the function $V(f(x)) - V(x)$ is continuous and negative in M'. Hence, there exists $a > 0$ such that

$$V(f(x)) - V(x) \leqslant -2a, \quad x \in M'.$$

If $k \in \kappa$, then

$$\begin{aligned} V(x_{k+1}) - V(x_k) &= (x_{k+1}) - V(f(x_k)) + V(f(x_k)) - V(x_k) \\ &\leqslant V(x_{k+1}) - V(f(x_k)) - 2a. \end{aligned}$$

Since V is uniformly continuous, $|V(x_{k+1}) - V(f(x_k))| \to 0$ as $k \to \infty$. Hence, there exists k_1 such that

$$V(x_{k+1}) - V(x_k) \leqslant -a, \quad k \in \kappa, k \geqslant k_1. \tag{17}$$

Let us say that a set $\{k, \ldots, l\} \subset \kappa$ of consecutive indices is a block of type (i, j), $i, j \in \{1, \ldots, m\}$, if $x_{k-1} \in W_i$ and $x_{l+1} \in W_j$; the number $l - k$ is called the length of the block. Since the Lyapunov function V is bounded, inequalities (17) imply that the length of any block $\{k, \ldots, l\}$ with $k \geqslant k_1$ is bounded from above by a value depending on the neighborhood U only.

Since the lengths of blocks are bounded, relations (3) imply that there exists a number $k_2 \geqslant k_1$ such that for any block $\{k, \dots, l\}$ with $k \geqslant k_2$, the inequalities

$$\operatorname{dist}(f^n(x_{k-1}), x_{k+n-1}) < b, \quad 0 \leqslant n \leqslant l-k+2 \tag{18}$$

hold, where the number b has property (16).

It follows that if $\{k, \dots, l\}$ is a block of type (i, j) such that $k \geqslant k_2$ and $i \neq j$, then $f^{l-k+2}(x_{k-1}) \in U_j$; hence, the phase diagram contains a chain

$$\Omega_i \to \dots \to \Omega_j$$

of the form (15).

It is easy to see that if $\{k', \dots, l'\}$ is the first block of type (i', j') to the right of $\{k, \dots, l\}$ (this means that $k' > k$) such that $i' \neq j'$, then $i' = j$ and the phase diagram contains a chain

$$\Omega_i \to \dots \to \Omega_j \to \dots \to \Omega_{j'}.$$

Since the phase diagram of f contains no cycles, $j' \notin \{i, j\}$. Thus, moving to the right and passing blocks of type (i, j) with $i \neq j$, we decrease the set from which elements of pairs (i, j) can be taken. It follows that the number of blocks of any type (i, j) with $i \neq j$ is finite.

Hence, there exist indices $i \in \{1, \dots, m\}$ and $k_3 > 0$ such that any block $\{k, \dots, l\}$ with $k \geqslant k_3$ is of type (i, i) (this means that if $k > k_3$, then $x_k \notin W_j$ with $j \neq i$).

The neighborhood W_i can be chosen arbitrarily small; hence, our reasoning above shows that if relations (3) are satisfied, then there exists an index $i \in \{1, \dots, m\}$ with the following property:

(III) for any neighborhood W of the basic set Ω_i, there exists $k' > 0$ such that if $\kappa' := \{k \geqslant k' : x_k \notin W\}$, then the length of any block of κ' (defined as above with W_i, W_j replaced by W) is bounded from above.

Now we fix an arbitrary neighborhood U of the basic set Ω_i and find neighborhoods U', W', W of Ω_i and a number $b > 0$ such that

$$N(b, W) \subset W' \quad \text{and} \quad N(b, U') \subset U.$$

In addition, we take the neighborhood W' so small that statement (I) above holds for the pair (W', U').

Property (III) implies that there exists a number k_0 such that if $\{k, \dots, l\}$ is a block of the set κ' with $k > k_0$, then inequalities (18) hold. In this case, $x_{k-1} \in W \subset W'$ and $f^{l-k+2}(x_{k-1}) \in W'$. If there exists $n \in \{k, \dots, l\}$ such that $x_n \notin U$, then $f^{n-k+1}(x_{k-1}) \notin U'$, and we get a contradiction.

Thus, $x_k \in U$ for $k > k_0$. We have proved that if relations (3) are satisfied, then there exists an index $i \in \{1, \dots, m\}$ for which relation (14) holds.

Now the statement of Lemma 1 follows from Theorem 1.4.1 of [1]. $\square$

Lemma 2. $\Omega S \subset ILmSP$.

Proof. Fix a point $p \in M$ and a p.c. method $\Psi = \{\psi_k\}$ for a diffeomorphism $f \in \Omega S$.

There exists a basic set Ω_i such that $f^k(p) \to \Omega_i$ as $k \to \infty$.

Precisely the same reasoning as in the proof of Theorem 1.1 in [9] shows that there exists a neighborhood U of the basic set Ω_i with the following property:

(IV) there exist positive constants d_0, L_0 such that if ϕ_k, $k \in \mathbf{Z}$, are continuous mappings of M with

$$\sup_{x \in U} \operatorname{dist}(f(x), \phi_k(x)) \leqslant d \leqslant d_0 \tag{19}$$

and $f^k(r) \in U$ for $k \geqslant 0$, then there exists a sequence $\{y_k\}$ such that $y_{k+1} = \phi_k(y_k)$ and $\operatorname{dist}(f^k(r), y_k) \leqslant L_0 d$.

(In [10], the case of a structurally stable diffeomorphism is considered, but the case of a neighborhood of a hyperbolic set is treated similarly, see also Lemma 6 below.)

It follows from the stable manifold theorem that, reducing the neighborhood U if necessary, we may as well assume that the following statement holds:

(V) there exists a constant $\Delta > 0$ such that if $f^k(q_1), f^k(q_2) \in U$ for $k \geqslant 0$ (for $k \leqslant 0$) and

$$\operatorname{dist}(f^k(q_1), f^k(q_2)) \leqslant \Delta \quad \text{for} \quad k \geqslant 0$$

(for $k \leqslant 0$), then

$$\operatorname{dist}(f^k(q_1), f^k(q_2)) \to 0 \quad \text{as} \quad k \to \infty$$

(as $k \to -\infty$, respectively).

Since $f^k(p) \to \Omega_i$, there exists an index k_0 such that $f^k(p') \in U$ for $k \geqslant 0$, where $p' = f^{k_0}(p)$, and the mappings $\phi_k = \psi_{k-k_0}$ satisfy inequalities (19) with

$$d = \min\left(d_0, \frac{\Delta}{2L_0}\right),$$

where Δ is fixed in statement (V).

By property (IV), there exists a sequence y_k such that $y_{k+1} = \phi_k(y_k)$ and

$$\operatorname{dist}(f^k(p'), y_k) \leqslant \Delta/2, \quad k \geqslant 0. \tag{20}$$

Since Ψ is a p.c. method, $\text{dist}(f(y_k), y_{k+1}) \to 0$ as $k \to \infty$. We have shown in Lemma 1 that $f \in \text{LmSP}$; hence, there exists a point z such that

$$\text{dist}(f^k(z), y_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{21}$$

(Of course, we may assume that $f^k(z), y_k \in U$ for $k \geqslant 0$.)

Find an index $k_1 > 0$ such that

$$\text{dist}(f^k(z), y_k) \leqslant \Delta/2, \quad k \geqslant k_1. \tag{22}$$

By the choice of Δ, inequalities (20) and (22) imply that

$$\text{dist}(f^k(z), f^k(p')) \to 0 \quad \text{as} \quad k \to \infty.$$

Combining this relation with (21), we see that

$$\text{dist}(f^k(p'), y_k) \to 0 \quad \text{as} \quad k \to \infty.$$

It remains to set $x_{k_0+k} = y_k$ for $k \geqslant 0$ and take successfully $x_{k_0-1} \in \psi_{k_0-1}^{-1}(y_0)$, $x_{k_0-2} \in \psi_{k_0-2}^{-1}(x_{k_0-1})$, and so on (recall that the mappings ψ_k are assumed to be surjective). Lemma 2 is proved. $\square$

To complete the proof of Theorem 1, now we are going to show that any of the sets $\text{Int}^1(\text{P})$ mentioned in its statement consists of Ω-stable diffemorphisms.

Let us say that a diffeomorphism f of a closed smooth manifold has the property HP if any of its periodic points is hyperbolic; denote $\mathcal{F} = \text{Int}^1(\text{HP})$. It was shown by Hayashi and Aoki in [17,18] that $\mathcal{F} \subset \Omega\mathcal{S}$.

Lemma 3. *$\mathit{Int}^1(\mathit{WILmSP}) \subset \mathcal{F}$.*

Proof. To obtain a contradiction, assume that there exists a diffeomorphism $f \in \text{Int}^1(\text{WILmSP}) \setminus \mathcal{F}$. In this case, there exists a C^1-neighborhood W of f such that $W \subset \text{Int}^1(\text{WILmSP})$ and W contains a diffeomorphism g having a nonhyperbolic periodic point p (let m be the period of p; we denote $p_i = g^i(p), i = 0, \dots, m-1$).

Let us fix a C^1-neighborhood $W' \subset W$ of g. First, we assume that the derivative $Dg^m(p)$ has an eigenvalue 1.

Obviously, we can find a diffeomorphism $h \in W'$ (by the choice of W' and W, h has the WILmSP) with the following properties:

(h1) $p_i = h^i(p), i = 0, \dots, m-1$ (i.e., p is a periodic point of h of period m);

(h2) there exist disjoint neighborhoods U_i of the points p_i with local coordinates y_i such that

(h2.1) p_i is the origin in the coordinates $y_i, i = 0, \dots, m-1$;

(h2.2) coordinates y_i can be represented as $y_i=(v_i, w_i)$, where the vector v_i is one-dimensional and the vector w_i is $(n-1)$-dimensional (we denote by n the dimension of M), so that the restriction $h_i = h|_{U_i}$ is given by the formula

$$h_i(y_i) = (r_i v_i, B_i w_i), \quad i = 0, \dots, m-1,$$

where the B_i are $(n-1)\times(n-1)$ matrices, and

$$r_0 \cdots r_{m-1} = 1; \tag{23}$$

(h2.3) there exists a number $a>0$ independent of i such that if $V_i = U_i \cap \{y_i : |v_i| \leqslant a, |w_i| \leqslant a\}$, then

$$h(V_i) \subset U_{i+1}, \quad i = 0, \dots, m-1$$

(as usual, we set $p_m = p_0$ and $U_m = U_0$).

Thus, we assume that the derivative $Dh^m(p)$ has an eigenvalue 1 of multiplicity 1 and that h is linear in any of the neighborhoods U_i. Of course, this can be achieved by an arbitrarily C^1-small perturbation of g and a proper choice of the neighborhoods U_i.

Since h is a diffeomorphism, there exists a number $r>0$ such that

$$|r_j \cdots r_k| \geqslant r \tag{24}$$

for any finite set of distinct indices $j, \dots, k \in \{0, \dots, m-1\}$.

Let us construct a p.c. method $\Psi = \{\psi_k\}$ as follows. We set $\psi_0 = \mathrm{id}$; for $k \geqslant 1$, we define ψ_k in the sets V_i by the formula

$$\psi_k(y_i) = \left(r_i v_i + \operatorname{sign} v_i \frac{r_i}{k}, B_i w_i \right).$$

Clearly, it is possible to construct the mappings ψ_k so that they are continuous on M and analogs of relations (9) hold with f replaced by h.

Since h has the WILmSP, there exists a pseudotrajectory $\xi = \{x_k\}$ generated by the method Ψ for which equality (7) holds with the point p having properties (h1) and (h2) (of course, in this case $\omega(p)$ is the ω-limit set of the trajectory $O(p,h)$; in fact, this set consists of the points $p_0, \dots, p_{m-1}$).

Let us show that there exist indices $k_0 \geqslant 1$ and $i \in \{0, \dots, m-1\}$ such that

$$x_k \in V_{k'}, \quad k \geqslant k_0, \text{ where } k' = i + k \ (\mathrm{mod}\ m). \tag{25}$$

Indeed, since analogs of relations (9) for h and property (h2.3) hold, there exists k_1 such that

$$\psi_k(V_j) \subset U_{j+1}, \quad k \geqslant k_1, \ j = 0, \dots, m-1. \tag{26}$$

Equality (7) implies that there exists $k_0 \geqslant k_1$ such that

$$x_k \in V_0 \cup \cdots \cup V_{m-1}, \quad k \geqslant k_0.$$

Since the neighborhoods U_i are disjoint, the above relation combined with inclusions (26) proves that if $k \geqslant k_0$ and $x_k \in V_j$, then $x_{k+1} = \psi_k(x_k) \in V_{j+1}$ (of course, we set $V_m = V_0$). This proves inclusions (25).

For definiteness, let us assume that $i = 0$ and denote $z_k = x_{k+k_0}$ for $k \geqslant 0$. Thus, $z_k \in V_{k'}$, where $k' = k(\text{mod } m)$. Represent $z_k = (v_k, w_k)$ in the local coordinates of $V_{k'}$.

For $1 \leqslant k < l$, denote

$$\sigma(k,l) = \sum_{i=k}^{l} \frac{1}{i}.$$

Obviously, $\sigma(k,l) \to \infty$ if k is fixed and $l \to \infty$.

By the definition of the mappings ψ_k,

$$|v_1| = |r_0 v_0| + \frac{|r_0|}{k_0}, \quad |v_2| = |r_1 r_0 v_0| + \frac{|r_1 r_0|}{k_0} + \frac{|r_1|}{k_0 + 1}, \ldots,$$

$$|v_{m-1}| = |r_{m-2} \cdots r_0 v_0| + \frac{|r_{m-2} \cdots r_0|}{k_0} + \cdots + \frac{|r_{m-2}|}{k_0 + m - 2}$$

and

$$|v_m| = |v_0| + \frac{1}{k_0} + \cdots + \frac{|r_{m-1}|}{k_0 + m - 1} \tag{27}$$

(we apply relation (23)).

Taking into account inequalities (24), we deduce from relation (27) that

$$|v_m| \geqslant |v_0| + r\sigma(k_0, k_0 + m - 1).$$

Similarly, we see that if $j \to \infty$, then

$$|v_{jm}| \geqslant |v_0| + r\sigma(k_0, k_0 + jm - 1) \to \infty,$$

and we obtain a contradiction with inclusions (25).

If $Dg^m(p)$ has an eigenvalue $\lambda = -1$, then the reasoning is almost the same as above in the case of $\lambda = 1$. We take a diffeomorphism $h \in \text{WILmSP}$ with properties (h1) and (h2) and note that relation (23) is replaced by the relation

$$(r_0 \cdots r_{m-1})^2 = 1.$$

We take the same convergent method Ψ as above and note that relation (27) is replaced by the relation

$$|v_{2m}| = |v_0| + \frac{1}{k_0} + \cdots + \frac{|r_{m-1}|}{k_0 + 2m - 1}.$$

The rest of the proof in this case is the same as above.

Now we assume that $Dg^m(p)$ has a complex eigenvalue λ that is a root of 1, i.e., $\lambda^\nu = 1$ for some natural ν. We take a diffeomorphism $h \in$ WILmSP with the same property (h1) as in the first case and property (h2) modified as follows: property (h2.2) is replaced by

(h2.2′) if the coordinate y_i is represented as $y_i = (v_i, w_i)$, where the vector v_i is two-dimensional, the vector w_i is $(n-2)$-dimensional, and $v_i = (\rho_i \cos\theta_i, \rho_i \sin\theta_i)$, with $\rho_i \geqslant 0$ and $\theta_i \in [0, 2\pi)$, then the restriction $h_i = h|_{U_i}$ is given by the formula

$$h_i(y_i) = (r_i\rho_i, \theta_i + \chi_i (\mathrm{mod}\ 2\pi), B_i w_i), \quad i = 0, \ldots, m-1, \tag{28}$$

where the B_i are $(n-2) \times (n-2)$ matrices, $r_i > 0$,

$$(r_0 \cdots r_{m-1})^\nu = 1 \tag{29}$$

and

$$\nu(\chi_0 + \cdots + \chi_{m-1}) = 0 \pmod{2\pi}. \tag{30}$$

In this case, we construct a p.c. method $\Psi = \{\psi_k\}$ as follows. We set $\psi_0 = $id; for $k \geqslant 1$, the mappings ψ_k are defined by the formula

$$\psi_k(y_i) = \left(r_i\rho_i + \frac{r_i}{k}, \theta_i + \chi_i (\mathrm{mod}\ 2\pi), B_i w_i\right)$$

in the sets V_i, continuous on M, and satisfy analogs of relations (9) with f replaced by h.

If there exists a pseudotrajectory $\xi = \{x_k\}$ generated by the method Ψ for which equality (7) holds, then the same reasoning as in the first case shows there exist indices k_0 and $i \in \{0, \ldots, m-1\}$ such that relations (25) hold.

We again assume that $i = 0$, denote $z_k = x_{k+k_0}$ for $k \geqslant 0$, and represent $z_k = (\rho_k, \chi_k, w_k)$ in the local coordinates of $V_{k'}$.

We fix a number $r > 0$ such that

$$r_j^{\mu_j} \cdots r_0^{\mu_0} \geqslant r, \quad 0 \leqslant j \leqslant m-1,\ 0 \leqslant \mu_j \leqslant \nu. \tag{31}$$

By the definition of the mappings ψ_k,

$$\rho_1 = r_0\rho_0 + \frac{r_0}{k_0}, \quad \rho_2 = r_1 r_0 \rho_0 + \frac{r_1 r_0}{k_0} + \frac{r_1}{k_0+1}, \ldots$$

and

$$\rho_{\nu m} = \rho_0 + \frac{1}{k_0} + \cdots + \frac{r_{m-1}}{k_0 + \nu m - 1}.$$

Hence,

$$\rho_{\nu m} \geqslant \rho_0 + r\sigma(k_0, k_0 + \nu m - 1),$$

where r satisfies inequalities (31) (we take into account that the numerators of fractions in the expression for $\rho_{\nu m}$ belong to the set of numbers listed in the left-hand sides of inequalities (31)).

Similarly, we see that if $j \to \infty$, then

$$\rho_{j\nu m} \geqslant \rho_0 + r\sigma(k_0, k_0 + j\nu m - 1) \to \infty$$

and we obtain a contradiction with inclusions (25).

Finally, if $Dg^m(p)$ has an eigenvalue λ with $|\lambda| = 1$ that is not a root of 1 (i.e., $\lambda = \exp(i\theta)$ with an irrational real θ/π), we can find a diffeomorphism $h \in W'$ (recall that W' is a neighborhood of g belonging to WILmSP) with the following properties: p is a periodic point of h of period m, $Dh^m(p)$ has an eigenvalue λ that is a root of 1, and h satisfies conditions (h1) and (h2) (with (h2.2) replaced by (h2.2′)). For this purpose, it is enough to approximate g by a diffeomorphism h' with representations (28) in the neighborhoods U_i, and then to perturb h' by an arbitrarily small change of r_0 and χ_0 to satisfy conditions (29) and (30) (here we take into account that ν can be taken as large as we want). Thus, this case is reduced to the previous one. Lemma 3 is proved. □

Thus, $\mathrm{Int}^1(\mathrm{WILmSP}) \subset \Omega\mathcal{S}$. Since ILmSP⊂OILmSP⊂WILmSP, the statement of Theorem 1 is proved for P=ILmSP, OILmSP, WILmSP.

Since LmSP⊂OLmSP, to complete the proof of Theorem 1, it is enough to prove the following statement.

Lemma 4. *$Int^1(OLmSP) \subset \mathcal{F}$.*

Proof. To get a contradiction, let us assume that there exists a diffeomorphism $f \in \mathrm{Int}^1(\mathrm{OLmSP}) \setminus \mathcal{F}$. In this case, there exists a C^1-neighborhood W of f such that $W \subset \mathrm{Int}^1(\mathrm{OLmSP})$ and W contains a diffeomorphism g having a nonhyperbolic periodic point P. □

To clarify the main idea, let us treat in detail the simplest case where P is a fixed point of g and the derivative $Dg(P)$ has an eigenvalue $\lambda = 1$ (below, we explain how to treat the remaining cases).

Let us fix a C^1-neighborhood $W' \subset W$ of g and a diffeomorphism $h \in W'$ (by the choice of W' and W, h has the OLmSP) with the following properties:

(h3) P is a fixed point of h;

(h4) there exists a neighborhood U of the point P with local coordinate y such that

(h4.1) P is the origin in the coordinate y;

(h4.2) the coordinate y can represented as $y = (v, w)$, where the vector v is one-dimensional and the vector w is $(n-1)$-dimensional, so that the restriction $h' = h|_U$ is given by the formula

$$h'(y) = (v, Bw),$$

where B is a $(n-1) \times (n-1)$ matrix;

(h4.3) there exists a number $a > 0$ such that if $V = U \cap \{y : |v| \leqslant a, |w| \leqslant a\}$, then

$$h(V) \subset U.$$

Thus, we assume that h is linear in the neighborhood U and the derivative $Dh(P)$ has an eigenvalue 1 of multiplicity 1. Of course, this can be achieved by an arbitrarily C^1-small perturbation of g and a proper choice of the neighborhood U.

Now we perturb the diffeomorphism h as follows: we take a number $c \in (0, a/3)$ and consider a mapping s such that $s = h$ outside U, and the restriction $s' = s|_V$ is given by the formula

$$s'(y) = (t(v), Bw),$$

where $t(v)$ is a C^1 function with the following properties:

(t1) $t(v) = v$ if either $|v| \leqslant c$ or $|v| \geqslant 3c$;

(t2) $t(v) > v$ if $c < |v| < 3c$.

Obviously, if c is small enough, we may construct s so that s is a diffeomorphism and $s \in W'$ (hence, $s \in$ OLmSP).

Let us consider the following pseudotrajectory $\xi = \{x_k\}$ of s: $x_0 = p$ (i.e., the origin of U), $x_1 = (c, 0)$ (here and below, $x_k = (v_k, 0)$), and the values v_k, $k \geqslant 2$, are defined recursively by the formula

$$v_{k+1} = v_k + \frac{b_k c}{k},$$

where the numbers $b_k \in \{1, -1\}$ are chosen by the following rule: $b_1 = \cdots = b_{k(1)-1} = -1$, where $k(1)$ is such that $v_k \in [-c, c]$ for $k \in (1, k(1))$ and $v_{k(1)} < -c$; $b_{k(1)+1} = \cdots = b_{k(2)} = 1$, where $k(2)$ is such that $v_k \in$

$(v_{k(1)}, c]$ for $k \in (k(1), k(2))$ and $v_{k(2)} > c$, and so on; for a natural number m, $b_{k(2m)} = \cdots = b_{k(2m+1)-1} = -1$, where $k(2m+1)$ is such that $v_k \in [-c, v_{k(2m)})$ for $k \in (k(2m), k(2m+1) - 1)$ and $v_{k(2m+1)} < -c$; $b_{k(2m+1)} = \cdots = b_{k(2m+2)-1} = 1$, where $k(2m+2)$ is such that $v_k \in (v_{k(2m+1)}, c]$ for $k \in (k(2m+1), k(2m+2) - 1)$ and $v_{k(2m+2)} > c$.

It is easy to see that the analog of relation (3) holds for the sequence ξ (with f replaced by s) and that $\omega(\xi) = [-c, c] \times \{w = 0\}$. By assumption, $s \in$ OLmSP; hence, there exists a point p such that the set $\omega(p)$ (the ω-limit set of $O(p, s)$) satisfies equality (7).

The set $V' = (-2c, 2c) \times \{|w| < a\}$ is a neighborhood of $\omega(\xi)$. Hence, there exists an index k_0 such that $s^k(p) \in V'$ for $k \geqslant k_0$. Let $s^k(p) = (q_k, w_k)$. Then $q_{k+1} = t(q_k)$ for $k \geqslant k_0$. It follows from properties (t1) and (t2) that if $q_k \in (-2c, 2c)$ for all large k, then $q_k \to c' \in [-c, c]$ as $k \to \infty$. Hence, equality (7) cannot hold.

The contradiction obtained completes the consideration of the case where g has a fixed point P such that the derivative $Dg(P)$ has an eigenvalue 1. The cases where g has a periodic point P of period m such that the derivative $Dg^m(P)$ has an eigenvalue 1 or -1 are treated similarly (we may perturb the corresponding diffeomorphism s in the same way as above to obtain a pseudotrajectory ξ whose ω-limit set contains a nondegenerate segment, while the ω-limit set for any exact trajectory in a neighborhood of $\omega(\xi)$ is a periodic orbit).

If g has a periodic point P of period m such that the derivative $Dg^m(P)$ has a complex eigenvalue λ that is a root of 1, we pass to polar coordinates and construct (by a similar kind of perturbations that influence the polar radius only) a pseudotrajectory ξ such that its ω-limit set intersects all concentric circles of small radius in the 2-dimensional real plane $\mathcal{P}$ passing through P and corresponding to λ, while the intersecton of the ω-limit set for any exact trajectory with $\mathcal{P}$ in a neighborhood of $\omega(\xi)$ belongs to a single circle. If λ is not a root of 1, we perturb the diffeomorphism once more (see the end of the proof of Lemma 3). Details are left to the reader. $\square$

Thus, Theorem 1 is proved. Theorem 2 is a corollary of the following Lemmas 5 and 6.

Lemma 5. *If $f \in \mathcal{S}$, then $f \in$ TSLmSP.*

Proof. Since f is structurally stable, there exists a number $\epsilon > 0$ and neighborhoods U_i of the basic sets Ω_i of f such that statement (V) (see the proof of Lemma 2) holds for the neighborhoods U_i with $\Delta = 2\epsilon$. Reducing ϵ, if necessary, we may assume that there exist neighborhoods V_i of the basic sets Ω_i such that $N(\epsilon, V_i) \subset U_i$.

Since f is structurally stable, it has the POTP (see [1]). Hence, there exists a number $d>0$ such that if $\xi=\{x_k\}$ is a d-pseudotrajectory of f, then there exists a point p for which relation (2) holds.

We claim that this d has the property formulated in the definition of the TSLmSP.

Indeed, consider a d-pseudotrajectory ξ of f for which relations (3) and (5) hold. Find a point p for which relation (2) holds.

Since f is structurally stable, it is Ω-stable as well. It is shown in the proof of Lemma 1 that there exists a basic set Ω_i of f and a point q such that $f^k(q)\to\Omega_i$ and $\mathrm{dist}(f^k(q),x_k)\to 0$ as $k\to\infty$.

There exists a number k_1 such that $x_k\in V_i$ (hence, $f^k(p)\in U_i$), $f^k(q)\in U_i$, and $\mathrm{dist}(f^k(q),f^k(p))<\Delta$ for $k\geqslant k_1$.

By statement (V), $\mathrm{dist}(f^k(q),f^k(p))\to 0$ as $k\to\infty$. We see that $\mathrm{dist}(x_k,f^k(p))\to 0$ as $k\to\infty$. Similarly, one shows that $\mathrm{dist}(x_k,f^k(p))\to 0$ as $k\to-\infty$. Thus, relations (12) hold. The lemma is proved. □

Lemma 6. *If $f\in\mathcal{S}$, then $f\in TSILmSP$.*

Proof. First, we show that if $f\in\mathcal{S}$, then f has the ISP with respect to continuous methods of the class Θ_t considered in this paper (the proof is almost parallel to that given in [10] for methods of the class Θ_s, so we only indicate the main differences). We denote below by T_pM the tangent space of M at p.

Since $f\in\mathcal{S}$, there exist $C>0$, $\lambda\in(0,1)$ such that, for any $p\in M$, the trajectory $O(p,f)$ has a (C,λ)-structure (see [1]). This means that if $p_k=f^k(p)$, then there exist projections P_k, Q_k in $T_{p_k}M$ such that if $S_k=P_kT_{p_k}M$ and $U_k=Q_kT_{p_k}M$, then

$$T_{p_k}M=S_k\oplus U_k,\quad \|P_k\|,\|Q_k\|\leqslant C,$$

$$Df(p_k)S_k\subset S_{k+1},\quad Df^{-1}(p_k)U_k\subset U_{k-1},$$

$$\left|Df^m(p_k)v\right|\leqslant C\lambda^m|v|,\ v\in S_k,\ m\geqslant 0,$$

$$\left|Df^m(p_k)v\right|\leqslant C\lambda^{-m}|v|,\ v\in U_k,\ m\leqslant 0.$$

We set

$$L_0=C^2\frac{1+\lambda}{1-\lambda}$$

and fix $N\geqslant 1$ such that $||Df(r)||\leqslant N$ for $r\in M$.

We note that the problem can be "linearized" similarly to [10], so we work below in the corresponding "local" coordinates.

For any point $r \in M$ and a vector w, we represent

$$f(r+w) = f(r) + Df(r)w + G(r, w)$$

and

$$f^{-1}(r+w) = f^{-1}(r) + Df^{-1}(r)w + H(r, w)$$

and note that the following statements hold: $G(r, 0) = 0$, $H(r, 0) = 0$, and there exists $\delta > 0$ such that if $|w|, |w'| \leqslant \delta$, then

$$|J(r, w) - J(r, w')| \leqslant \frac{1}{2NL_0}|w - w'| \text{ for } J = G, H.$$

Let us fix a point $p \in M$ and a d-method $\Psi = \{\psi_k : k \in \mathbf{Z}\}$ of the class Θ_t with $2NL_0 d < \delta$; we denote $p_k = f^k(p)$ and want to find a sequence x_k such that the points x_k are close to p_k, $x_{k+1} = \psi_k(x_k), k \geqslant 0$, and $x_{k-1} = \psi_{k-1}(x_k), k \leqslant 0$.

If we represent $x_k = p_k + v_k$, then we obtain the following equations for the unknown vectors v_k:

$$v_{k+1} = Df(p_k)v_k + G(p_k, v_k) + \psi_k(p_k + v_k) - f(p_k + v_k), \quad k \geqslant 0$$

and

$$v_{k-1} = Df^{-1}(p_k)v_k + H(p_k, v_k) + \psi_k(p_k + v_k) - f^{-1}(p_k + v_k), \quad k \leqslant 0. \tag{32}$$

Obviously, relations (32) are equivalent to the relations

$$v_k = Df(p_{k-1})v_{k-1} - Df(p_{k-1})\left[H(p_k, v_k) + \psi_k(p_k + v_k) - f^{-1}(p_k + v_k)\right].$$

Thus, our problem is reduced to the search for a sequence $V = \{v_k : k \in \mathbf{Z}\}$ such that

$$v_{k+1} = Df(p_k)v_k + z_k(V),$$

where

$$z_k(V) = G(p_k, v_k) + \psi_k(p_k + v_k) - f(p_k + v_k), \quad k \geqslant 1$$

and

$$z_k(V) = -Df(p_{k-1})\left[H(p_k, v_k) + \psi_k(p_k + v_k) - f^{-1}(p_k + v_k)\right], \quad k \leqslant 0.$$

For a sequence $W = \{w_k\}$, we set

$$||W||_\infty := \sup_k |w_k|.$$

The above-mentioned properties of the functions G and H imply that if $||V||_\infty \leqslant 2NL_0d$, then

$$||Z(V)||_\infty \leqslant \frac{1}{2NL_0}||V||_\infty + Nd.$$

Now the proof of the existence of the required sequence V repeats literally the corresponding proof of Theorem 1 in [10] for the class Θ_s.

The rest of the proof of Lemma 6 repeats the reasoning in the proof of Lemma 5 since the constructed sequence x_k satisfies relations (3) and (5). □

To prove Theorem 3, we first note that we denote by $\mathcal{S}$ the set of all structurally stable diffeomorphisms (on all closed smooth manifolds); the symbols Int^1(TSLmSP) etc have a similar meaning. Thus, to show that the sets considered are different, it is enough to show that there exists a manifold for which the corresponding sets of diffeomorphisms are different.

For this purpose, we consider a special class $\mathcal{P}$ of Ω-stable diffeomorphisms of the 2-dimensional torus T^2; diffeomorphisms of that class played an important role in the study of the weak shadowing property (see [19]).

We say that f is a diffeomorphism of the class $\mathcal{P}$ if the following conditions hold.

($\mathcal{P}$1) The nonwandering set of f consists of four hyperbolic fixed points (we denote them o, s, q, r) such that s is a sink (an asymptotically stable fixed point), o is a source (an asymptotically stable fixed point for f^{-1}), and q, r are saddles. We denote by $W^s(p)$ and $W^u(p)$ the stable and unstable manifolds of a fixed point p, respectively.

($\mathcal{P}$2) $W^s(r)\setminus\{r\} \subset W^u(o)$ and $W^u(q)\setminus\{q\} \subset W^s(s)$.

Lemma 7. *Let f be a diffeomorphism of the class $\mathcal{P}$. If $W^s(q) \cap W^u(o) \neq \emptyset$ and $W^u(r) \cap W^s(s) \neq \emptyset$, then f has the TSLmSP.*

Proof. If $W^u(r) \cap W^s(q) = \emptyset$, then the one-dimensional stable and unstable manifolds do not intersect; hence, the transversality condition is satisfied automatically, and the diffeomorphism f is structurally stable. In this case, the statement of our lemma follows from Theorem 2. □

Otherwise, the phase diagram Φ of f (see the proof of Lemma 1) contains the connection $r \to q$. By the definition of the class $\mathcal{P}$, Φ contains the connections $o \to s$, $o \to r$, and $q \to s$. By the assumption of our lemma, Φ contains the connections $o \to q$ and $r \to s$.

Let us say that a pseudotrajectory $\xi = \{x_k\}$ is of type (P, P'), where $P, P' \in \{s, o, q, r\}$, if $x_k \to P$ as $k \to -\infty$ and $x_k \to P'$ as $k \to \infty$.

It follows from Proposition 3.3p of [12] that there exist neighborhoods U_s, U_o, U_q, U_r of the fixed points s, o, q, r and a number $d > 0$ such that if

P, $P' \in \{s, o, q, r\}$ and there exists a d-pseudotrajectory $\xi = \{x_k\}$ such that $x_k \in U_P$ and $x_l \in U_{P'}$ for some $l > k$, then Φ contains a connection $P \to \ldots \to P'$.

Our diffeomorphism f is Ω-stable, hence Φ does not contain cycles [3]. Thus, Φ does not contain connections $s \to \cdots \to o$, $r \to \cdots \to o$, $q \to \cdots \to o$, $s \to \cdots \to q$, and $s \to \cdots \to r$.

Hence, for the above-mentioned d, f does not have d-pseudotrajectories of the types (s, o), (r, o), (q, o), (s, q), (s, r).

Now it follows from our description of the diagram Φ that if ξ is a d-pseudotrajectory satisfying relations (3) and (5) (and hence having some type (P, P') by Lemma 1), then either $P = P'$ or Φ contains the connection $P \to P'$. In both cases, there exists a point p for which relation (12) holds. The lemma is proved. □

Remark 3. *It is easy to see that the conditions of Lemma 7 are not only sufficient but also necessary.*

To prove this, let us first show that if

$$W^s(q) \cap W^u(o) = \emptyset \tag{33}$$

then

$$W^u(r) \cap W^s(s) = \emptyset \tag{34}$$

as well. To simplify notation, let us denote by G and G' the connected components of $W^s(q) \setminus \{q\}$ and by H and H' the connected components of $W^u(r) \setminus \{r\}$, respectively. If relation (33) holds, then $G \subset W^u(r)$. Note that G is a path connected subset of $W^u(r)$ not containing r; hence, either $G \subset H$ or $G \subset H'$. Assume that the first inclusion holds. It is well known (see [3]) that there exists an injective continuous mapping $a : \mathbf{R} \to H$ such that $a(\mathbf{R}) = H$ and $a(t) \to r$ as $t \to -\infty$. Take a point $x \in H \cap G$ and note that $a^{-1}(f^k(x)) \to -\infty$ as $k \to -\infty$ and $a^{-1}(f^k(x)) \to \infty$ as $k \to \infty$. Since G is a path connected subset of H, it follows that $G = H$. The same reasoning shows that if relation (33) holds, then either $G' = H$ (but this is impossible since $G \cap G' = \emptyset$) or $G' = H'$.

Thus, if relation (33) holds, then we have a "double" saddle connection: connected components of $W^s(q) \setminus \{q\}$ coincide with connected components of $W^u(r) \setminus \{r\}$.

In this case, f does not have the TSLmSP. Indeed, for any $d > 0$, we can construct a d-pseudotrajectory $\xi = \{x_k\}$ such that $x_k \in W^s(r), k < 0$, $x_k \to o, k \to -\infty$, $x_k \in W^u(r), k \geqslant 0$, and $x_k \to q, k \to \infty$. At the same time, since relation (33) holds, we cannot find a point p such that $f^k(p) \to o, k \to \infty$, and $f^k(p) \to q, k \to \infty$.

The case of relation (34) is considered similarly.

Lemma 8. *Int*^{1}*(TSLmSP)*$\setminus$*TSILmSP*$\neq \emptyset$.

Proof. Let us consider the following diffeomorphism of the class $\mathcal{P}$. Represent the torus T^2 as the rectangle $[-2, 2] \times [-2, 2]$ with identified opposite sides and with coordinates (y, z). We assume that the saddle point q is the origin, its unstable manifold coincides with $\{0\} \times (-2, 2)$, and the sink s is represented by the points $(0, -2)$ and $(0, 2)$. In addition, we assume that f is linear in the square $Q = [-1, 1] \times [-1, 1]$:

$$f(y, z) = (y/2, 2z), \quad (y, z) \in Q$$

and $W^s(q) \cap Q = [-1, 1] \times \{0\}$.

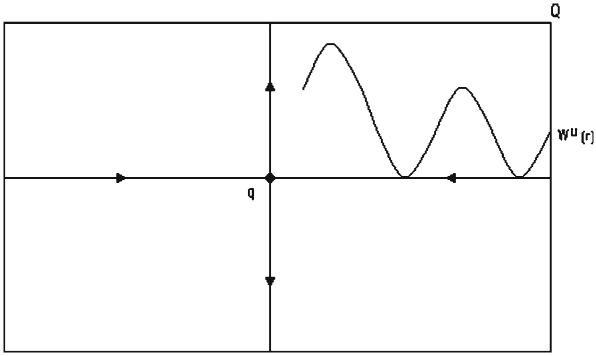

Figure 1. To the proof of Lemma 8.

We assume that the phase diagram of f has the connections $r \to q$ and $r \to s$ and the intersection of the unstable manifold $W^u(r)$ with $[0, 1] \times [-2, 2]$ is a curve belonging to $[0, 1] \times [0, 2]$ and having points of one-sided tangency with $W^s(q)$ (see Fig. 1). It is also assumed that

$$R := (-1/2, 1/2) \times (-2, 0) \cup (-1/2, 1/2) \times (0, 2) \subset W^s(s) \tag{35}$$

and if $(y, z) \in R$ and $f(y, z) = (y', z')$, then

$$|y'| \leqslant |y| \quad \text{and } |z'| > |z|. \tag{36}$$

We claim that f does not have the TSILmSP. To get a contradiction, assume that f has the TSILmSP and fix the corresponding $d > 0$. Let us consider the following ts.c. d-method Ψ. Denote $Q' = (-1/2, 1/2) \times$

$(-1/2, 1/2)$ and define mappings ψ_k as follows: $\psi_k = f^{-1}$ for $k < 0$, $\psi_0 = f$, and if $k \geqslant 1$, then $\psi_k = f$ outside Q' and

$$\psi_k(y, z) = (y/2, 2z + g(y, z)/k), \quad (y, z) \in Q',$$

where g is continuous and $0 < g(y, z) < d/2$ in Q'.

Since $W^u(r) \cap W^s(q) \neq \emptyset$, there exists a pseudotrajectory $\xi = \{x_k\} \in G\Psi$ such that $x_k \to q$ as $k \to \infty$ and $x_k \to r$ as $k \to -\infty$.

The definition of Ψ and properties (35) and (36) imply that $x_k \notin R \setminus Q'$ for all k (otherwise, $x_{l+1} = f(x_l)$ for $l \geqslant k$, and $x_l \to s$ as $l \to \infty$). Hence, there exists an index l such that $x_k \in Q'$ for $k \geqslant l$ and $x_k \notin Q'$ for $k < l$. Let $x_k = (y_k, z_k)$.

Let us consider the following two possible cases.

Case 1. $l \geqslant 1$. If $z_l \geqslant 0$, then

$$z_{l+1} = 2z_l + g(y_l, z_l)/l > z_l, \quad z_{l+2} > z_{l+1}, \ \ldots$$

and we get a contradiction with the relation $z_k \to 0, k \to \infty$.

Hence, $z_l < 0$. It remains to note that $x_l = f(x_{l-1})$, $x_{l-1} \in Q$ (hence, $z_{l-1} < 0$) and $x_k = f(x_{k-1})$ for $k \leqslant l-1$ (hence, $x_{l-1} \in W^u(r)$, and $z_{l-1} \geqslant 0$). The contradiction obtained shows that case 1 is impossible.

Case 2. $l < 1$. In this case, $\operatorname{sign} z_l = \operatorname{sign} z_{l+1} = \cdots = \operatorname{sign} z_0$, and this case is reduced to the previous one.

Thus, we have proved that f does not have the TSILmSP.

It is easy to see that the assumptions of Lemma 7 are satisfied for any diffeomorphism C^1-close to f. Hence, $f \in \operatorname{Int}^1(\text{TSLmSP})$. The lemma is proved. □

To complete the proof of Theorem 3, let us consider a diffeomorphism f of the class $\mathcal{P}$ with the same behavior in the square Q but with a different position of $W^u(r)$ in Q (see Fig. 2).

Consider a point $p \in W^u(r) \cap W^s(q) \cap Q$ and assume that the connected component Π of $W^u(r) \cap W^s(q)$ containing p is a subset of $\operatorname{Int} Q$. Denote by π and π' the endpoints of Π (if Π is a point, then $\pi = \pi' = p$). Let us say that the component Π is two-sided if any neighborhoods V and V' of the points π and π' (in the inner topology of $W^u(r)$) contain points $\rho = (Y, Z)$ and $\rho' = (Y', Z')$, respectively, such that $YY' < 0$; i.e., the points ρ and ρ' lie to different sides of the line $y = 0$ (note that any component shown in Fig. 2 is two-sided).

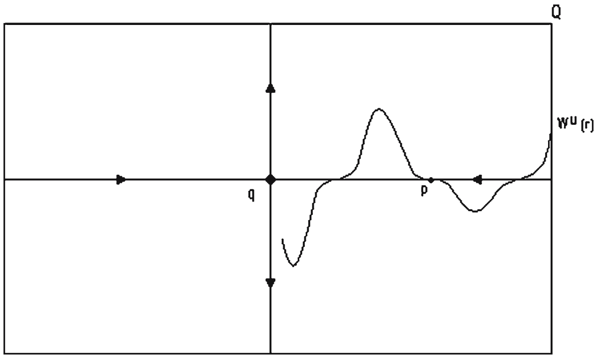

Figure 2. To the proof of Lemma 9.

Lemma 9. *If the considered diffeomorphism f of the class $\mathcal{P}$ has a two-sided component Π, then $f \in TSILmSP$.*

Proof. Clearly, it is enough to show that there exists $d > 0$ with the following property: if Ψ is a ts.c. d-method and P, P' is a pair of fixed points of f such that $W^u(P) \cap W^s(P') \neq \emptyset$, then there exists a pseudotrajectory $\{x_k\} \in G\Psi$ of type (P, P') (see Lemma 7).

If either $P = P'$ or (P, P') is one of the pairs $(o, s), (o, r), (o, q), (q, s), (r, s)$, then obviously f has a (C, λ)-structure at $O(p, f)$, hence the required fact is proved by the same reasoning as Lemma 6.

Thus, it remains to consider the case $(P, P') = (r, q)$. Take a point p belonging to a two-sided component Π and set $p_k = f^k(p)$.

First we describe the main technical construction applied in the proof (cf. a similar construction in [20]). For a continuous mapping $\psi : M \to M$, denote

$$\mathrm{Dist}(f, \psi) = \max_{z \in T^2} \mathrm{dist}(f(z), \psi(z)).$$

Assume that there exists a number $\lambda > 1$ and positive numbers Δ, Δ_1 with the following property (HC).

(HC) For any point p_k, $k \geqslant 0$, we can introduce coordinates (t, v) centered at p_k such that the corresponding "inner" metrics determined by these coordinates are equivalent to dist (with constants independent of k) and if

$$P_k = \{|t| \leqslant \Delta, |v| \leqslant \Delta\} \quad \text{and} \quad P'_k = \{|t| \leqslant \Delta_1, |v| \leqslant \Delta_1\}$$

then (HCI) if $(t, v) \in P_k$ and $f(t, v) = (t', v') \in P_{k+1}$, then $t't \geqslant 0$, $v'v \geqslant 0$, $|t'| \leqslant \lambda^{-1}|t|$, and $|v'| \geqslant \lambda|v|$;

(HCII) if $a \in (0, \Delta]$ and we denote by $C^s(a, k)$ the segment of $W^s(q)$ of length $2a$ centered at p_k (where the length is measured in the "inner" metric of P_k), then $C^s(\Delta, k) = P_k \cap \{v = 0\}$;

(HCIII) there exists a number $d_0 > 0$ such that if ψ is a continuous mapping of T^2 with $\mathrm{Dist}(f, \psi) < d_0$, then $\psi(P_k) \subset P'_{k+1}$.

For simplicity of notation, we assume that all of the introduced "inner" metrics coincide with dist.

Let $\{\psi_k : k \geqslant 0\}$ be a sequence of continuous mappings of T^2. For $m \geqslant 0$, denote

$$\Psi_m = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_0.$$

Fix $\epsilon, \beta < \Delta$ and introduce the sets $R_k := R_k(\beta, \epsilon) := \{|t| \leqslant \beta, |v| \leqslant \epsilon\}$, $R_k^+ := \{|t| \leqslant \beta, v = \epsilon\}$, $R_k^- := \{|t| \leqslant \beta, v = -\epsilon\}$, and $R_k^* := R_k^+ \cup R_k^-$, $k \geqslant 0$.

Consider a continuous curve $L : [0, 1] \to R_k$. We say that the curve L is a vertical arc in R_k if $\{L(0), L(1)\} \subset R_k^*$ and the endpoints $L(0), L(1)$ belong to different "horizontal" sides R_k^+, R_k^- of R_k.

Lemma 10. *Assume that f has property (HC). For any $\epsilon, \beta > 0$, there exists a number $d_1 > 0$ with the following property. If $\{\psi_k : k \geqslant 0\}$ is a sequence of continuous mappings of T^2 such that Dist$(f, \psi_k) < d_1$, then the set*

$$X = \{z \in R_0 = R_0(\beta, \epsilon) : \Psi_{m-1}(z) \in R_m, m > 0\}$$

is a closed subset of R_0 that intersects any vertical arc in R_0.

Proof. Since the mappings ψ_k are continuous and the sets R_k are closed, the set X is obviously closed.

To show that X intersects any vertical arc in R_0, consider a continuous mapping ψ such that $\mathrm{Dist}(f, \psi) < d_1$, where $d_1 < d_0$ (see property (HCIII)),

$$\lambda\epsilon - d_1 > \epsilon, \qquad \text{and } \frac{\beta}{\lambda} + d_1 < \beta. \tag{37}$$

Let $L : [0, 1] \to R_k$ be a vertical arc in R_k with endpoints $L(0), L(1)$. We claim that $l = \psi(L)$ contains a vertical arc in R_{k+1}. First we note that $l \subset P'_{k+1} \cap \{|t| \leqslant \beta\}$. The inclusion $l \subset P'_{k+1}$ follows from property (HCIII); the second desired inclusion follows from inequalities (37): if $(t', v') = \psi(t, v) \in l$, then

$$|t'| \leqslant \frac{\beta}{\lambda} + \mathrm{dist}(f(t, v), \psi(t, v)) < \frac{\beta}{\lambda} + d_1 < \beta.$$

If (t', v') is one of the points $l_0 = \psi(L(0))$ and $l_1 = \psi(L(1))$, then we deduce from the inequalities

$$|v'| \geqslant \lambda\epsilon - d_1 > \epsilon$$

that $l_0, l_1 \notin R_{k+1}$.

Let $(t(s), v(s))$, $s \in [0, 1]$, be a parametrization of L. Consider the function $v'(s)$, where $(t'(s), v'(s)) = \psi(t(s), v(s))$.

Assume, for definiteness, that $L(0) = (t_0, \epsilon)$ (i.e., $L(0) \in R_k^+$). As was shown above, $v'(0) > \epsilon$ and $v'(1) < -\epsilon$ in this case. Since the function $v'(s)$ is continuous, there exists a segment $[s_0, s_1] \subset (0, 1)$ such that $v'(s_0) = \epsilon$, $v'(s_1) = -\epsilon$, and $v'(s) \in (-\epsilon, \epsilon)$ for $s \in (s_0, s_1)$. Thus, $L'(s) = \psi(L(s)) \in R_{k+1}, s \in [s_0, s_1]$, is the desired vertical arc.

Now we consider an arbitrary sequence of continuous mappings $\{\psi_k : k \geqslant 0\}$ of T^2 such that $\mathrm{Dist}(f, \psi_k) < d_1$ and a vertical arc $L_0 : [0, 1] \to R_0$. It was shown above that there exists a segment $\sigma_1 \subset (0, 1)$ such that $L_1 = \psi_0(L_0(\sigma_1))$ is a vertical arc in R_1. Applying the same reasoning once more, we find a segment $\sigma_2 \subset \sigma_1$ such that

$$L_2 = \Psi_1(L_0(\sigma_2)) = \psi_1 \circ \psi_0(L_0(\sigma_2))$$

is a vertical arc in R_2, and so on. Obviously, any point $z \in L_0$ corresponding to a point of the infinite intersection of embedded segments

$$[0, 1] \supset \sigma_1 \supset \sigma_2 \supset \ldots$$

belongs to the set X.

The lemma is proved. □

Recall that $p = p_0$ belongs to a two-sided component of $W^u(r) \cap W^s(q)$.

Obviously, there exist numbers $\lambda, \Delta, \Delta_1$ and an index k_1 such that property (HC) is satisfied for $k \geqslant k_1$.

Fix a neighborhood Π_0 of Π in $W^s(q)$ and let Π_1 be the closure of Π_0. We may assume that $f^{k_1}(\Pi_1)$ is a subset of $C^s(\Delta, k_1)$ (see property (HCII)).

Since k_1 is finite, we can introduce coordinates in closed neighborhoods P'_k of the points $p_0, \ldots, p_{k_1-1}$ so that property (HC) is satisfied for $k \geqslant 0$ and the corresponding "inner" metrics are equivalent to dist (with multipliers depending on λ, Δ, k_1, etc; for example, one may take $P'_{k_1-1} = f^{-1}(P'_{k_1})$, modify the "induced" coordinates by proper expansions and contractions, and so on). We denote these metrics by dist as well. In addition, we assume that $\Delta < 1/2$ and $\Pi_1 \subset \mathrm{Int} P_0$.

For two points $z, z' \in W^u(r)$, let us denote by $[z, z']_r$ the segment of $W^u(r)$ joining z and z'.

By the definition of a two-sided component, we can find points $\rho, \rho' \in W^u(r) \cap P_0$ such that the projection of $[\rho, \rho']_r$ (with respect to inner coordinates of P_0) belongs to Π_0 and if $\rho = (t, v)$ and $\rho' = (t', v')$, then $vv' < 0$.

There exists a number $a > 0$ such that $\rho, \rho' \notin R'$, where $R' = \Pi_1 \times [-a, a]$. Now we find points $\chi, \chi' \in W^u(r) \cap R'$ with the same properties as ρ, ρ' and such that the segments $[\rho, \chi]_r$ and $[\chi', \rho']_r$ do not intersect Π_1.

Fix $\epsilon > 0$ such that

$$\operatorname{dist}(z, \Pi_1) > 2\epsilon, \quad z \in [\rho, \chi]_r \cup [\chi', \rho']_r. \tag{38}$$

Taking into account that the inner metrics in P'_k are equivalent to dist and applying the same reasoning as above, we can find a number $d_2 > 0$ such that if $\{\psi_k : k \geqslant 0\}$ is a sequence of continuous mappings of T^2 such that $\operatorname{Dist}(f, \psi_k) < d_2$, then the set $R^* = \Pi_1 \times [-\epsilon, \epsilon]$ contains a closed subset X such that $\Psi_{m-1}(z) \in P_m, m > 0$, and X intersects any vertical arc in R^* (and, of course, the same is true for any vertical arc in R').

Now we repeat the same procedure with the points $p_k, k \leqslant 0$, mappings $\psi_k, k < 0$, and the set $[\chi, \chi']_r$ instead of Π_1.

As a result, we can get a "rectangle" $S_0 \subset R'$ with "horizontal" axis $[\chi, \chi']_r$, with "horizontal" sides S^1, S^2, "parallel" to $[\chi, \chi']_r$, and with "vertical" sides S^3, S^4 containing the points χ, χ'. We take S_0 such that the sides S^3, S^4 belong to the ϵ-neighborhoods of the points χ, χ' (so that these sides do not intersect R^* by inequalities (38)).

For this rectangle S_0, we can find numbers $\Delta' \in (0, 1/2)$ and $d_3 > 0$ such that if $\operatorname{Dist}(f^{-1}, \psi_k) < d_3$ for $k < 0$, then S_0 contains a closed set Y with the following properties: if $z \in Y$, then

$$\psi_m \circ \cdots \circ \psi_{-1}(z) \in N(\Delta', p_m), \quad m < 0$$

and Y intersects any arc in S_0 that joins the sides S^1 and S^2.

Denote $X' = X \cap S_0$. Any arc L in S_0 that joins the sides S^3 and S^4 can be extended to a vertical arc L' in R' by adding parts of the sides S^3, S^4 and of the segments $[\rho, \chi]_r, [\chi', \rho']_r$. Such an arc L' intersects X. Inequalities (38) and the choice of S^3, S^4 imply that $L' \cap X = L \cap X \in X'$. Now we refer to the following topological statement (one can find a proof in [20]).

Lemma 11. *Let K be a square in the plane with sides K_1, K_2, K_3, K_4 such that the sides K_1 and K_3 are opposite. If A (B) is a closed subset of K such that any arc in K joining K_1 and K_3 (any arc in K joining K_2 and K_4) intersects A (respectively, B), then $A \cap B \neq \emptyset$.*

Let $d = \min(d_2, d_3)$. Take a ts.c. d-method Ψ and construct the corresponding sets X' and Y. By Lemma 11 applied to S_0, $X' \cap Y \neq \emptyset$. If we take the pseudotrajectory $\xi \in G\Psi$ with $x_0 \in X' \cap Y$, then $\operatorname{dist}(x_k, p_k) \leqslant 1/2$ by the choice of Δ and Δ'. Hence, ξ is of type (r, q). Lemma 9 is proved. □

To complete the proof of Theorem 3, it remains to note that

- if a diffeomorphism $f \in \mathcal{P}$ has a two-sided component of $W^u(r) \cap W^s(q)$, then any diffeomorphism g, C^1-close to f, has the same property;
- there exist such diffeomorphisms f that are not structurally stable.

APPENDIX

Let $M = S^1$ be the circle with coordinate $x \in [0, 1)$ and let $f(x) = x + a \pmod 1$, where a is irrational. To show that f does not have the LmSP, consider any sequence $\xi = \{x_k \in S^1 : k \geqslant 0\}$ such that

$$x_{k+1} = f(x_k) + d_k \pmod 1,$$

where $d_k > 0$, $d_k \to 0$ as $k \to \infty$, and the series $\sum d_k$ diverges. Obviously, for any point $p \in S^1$, the inequality $|f^k(p) - x_k| > 1/4$ holds for an infinite set of indices $k > 0$, so that relation (4) does not hold. Thus, $f \notin$ LmSP.

At the same time, for any sequence $\xi = \{x_k \in S^1 : k \geqslant 0\}$ for which condition (3) holds, $\omega(\xi) = S^1$. Indeed, take any point $r \in S^1$ and any positive ε. Let U be the ε-neighborhood of r. Since f is an irrational rotation, there exists $N > 0$ such that

$$\{f^n(x) : 0 \leqslant n \leqslant N\} \cap U \neq \emptyset$$

for any $x \in S^1$. Take an index l such that

$$\sum_{k=l}^{l+N} |f(x_k) - x_{k+1}| < \varepsilon.$$

There exists $n \in [0, N]$ such that $f^n(x_l) \in U$; in this case,

$$\begin{aligned}|x_{l+n} - f^n(x_l)| &\leqslant |x_{l+n} - f(x_{l+n-1})| + |f(x_{l+n-1}) - f^2(x_{l+n-2})| \\ &\quad + \cdots |f^{n-1}(x_{l+1}) - f^n(x_l)| < \varepsilon\end{aligned}$$

(we take into account that the Lipschitz constant of f equals 1). Hence, $|r - x_{l+n}| < 2\varepsilon$. This shows that $\omega(\xi) = S^1$. Since $\omega(p) = S^1$ for any $p \in S^1$, $f \in$ OLmSp.

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